Some properties relating to the Mittag-Leffler function of two variables

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Abstract

An attempt is made here to study the Mittag-Leffler function with two variables. Its various properties including integral and operational relationships with other known Mittag-Leffler functions of one variable, pure and differential recurrence relations, Euler transform, Laplace transform, Mellin transform, Whittaker transform, Mellin-Barnes integral representation, and its relationship with Wright hypergeometric function are investigated and established. Also, properties of the Mittag-Leffler function of two variables associated with fractional calculus operators are considered.

Keywords:Mittag-Leffler type functions; Fractional calculus operators; Integral transforms; Integral relations; operational relations.
2010 Mathematics Subject Classification. 33E12, 65R10, 26A33

1. Introduction and definitions

The Mittag-Leffler function has gained importance and popularity due to its applications in the solution of fractional order differential equations and fractional order integral equations. Also, the Mittag-Leffler function plays an important role in various branches of applied mathematics and engineering sciences, such as chemistry, biology, statistics, thermodynamics, mechanics, quantum physics, informatics, signal processing. Besides this, the Mittag-Leffler function of several variables appears in the solution of certain boundary value problems involving fractional integro-differential equations of Volterra type [1], initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation [2], and initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients[3]. In the usual notation $\Gamma(x)$ for the Gamma function and $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, n \geq 0, x \neq 0, -1, -2, ...,$ the Pochhammer symbol, Mittag-Leffler introduced the function $E_\alpha(z)$ in the form[4]:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, (\alpha > 0, z \in \mathbb{C}) . \tag{1.1}$$

Here and in the following, let $\mathbb{C}, \mathbb{R}^+, \mathbb{N}$ and $\mathbb{Z}_0^-$ denote the sets of complex numbers, positive real numbers, positive integers, and non-positive integers, respectively. Also let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. By increasing the number of parameters, the Mittag-Leffler function (1.1) has been extended and investigated by several workers, for example, see ([5-13]). In [14]

Wiman introduced a generalization of $E_\alpha(x)$ of two parameters as follows:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}, \tag{1.2}$$

$$(\alpha,\beta\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0).$$

Prabhakar [9] introduced the function $E^{\gamma}_{\alpha.\beta}(x)$ of three parameters

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!}, \tag{1.3}$$

$$(\alpha,\beta,\gamma\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0).$$

Further, three interesting unifications and generalizations of the function $E_\alpha(x)$ considered by Skukla and Prajapati[16]

$$E^{\gamma,q}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{nq} z^n}{\Gamma(\alpha n+\beta)n!}, \tag{1.4}$$

$$(\alpha,\beta,\gamma\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,q\in(0,1)),$$

Salim [11]

$$E^{\gamma,\delta}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)(\delta)_n}, \tag{1.5}$$

$$(\alpha,\beta,\gamma\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\delta>0),$$

and Salim and Faraj[10]

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{nq} z^n}{\Gamma(\alpha n+\beta)(\delta)_{np}}, \tag{1.6}$$

$$(\alpha,\beta,\gamma\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\delta>0,(p,q)>0,q\le\Re(\alpha)+p).$$

For the purpose of our present study, we recall the following Mellin-Barnes integral representation for the function $E^{\gamma}_{\alpha,\beta}(x)$[9]

$$E^{\gamma}_{\alpha,\beta}(z)=\frac{1}{2\pi i}\frac{1}{\Gamma(\gamma)}\int_L\frac{\Gamma(s)\Gamma(\gamma-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}ds, \tag{1.7}$$

$$\left(|\mathrm{arg}z|<\pi,\alpha\in\mathbb{R}^+,\gamma\in\mathbb{C}\diagdown\mathbb{Z}_0^-,\Re(\beta)>0\right).$$

Here, for any $c\in\mathbb{R}^+$ with $0<c<\Re(\gamma),L$ is a contour which is an infinite vertical line beginning at $c-i\infty$ and ending at $c+i\infty$. Then it is found that the contour L can separate the poles $s=-k(k\in\mathbb{N}_0)$ of $\Gamma(s)$ and those $s=\gamma+k(k\in\mathbb{N}_0)$ of $\Gamma(\gamma-s)$ to its left and right sides, respectively.

An interesting generalization of the Mittag-Leffler function $E_\alpha(z)$ to several variables has been suggested by Luchko and Gorenflo [15], who used it for solving linear fractional differential equations with constant coefficients by the operational method:

$$E_{(\alpha_1,\cdots,\alpha_m),\beta}(z_1,\cdots,z_m)=\sum_{k=0}^{\infty}\sum_{l_1+\cdots+l_m=k}\frac{(k;l_1,\cdots,l_m)z_1^{l_1}\cdots z_m^{l_m}}{\Gamma(\beta+\sum_{i=1}^m\alpha_i l_i)},$$

$$l_1>0,\cdots,l_m>0.$$

Here $(k;l_1,\cdots,l_m)$ denotes the multinomial coefficients $(k;l_1,\cdots,l_m):=\frac{k!}{l_1\times\cdots\times l_m!}$ with $k=\sum_{j=1}^m\ l_j$.

In this work , we aim to investigate and study the following special case of the Mittag-Leffler function of several variables $E_{(\alpha_1,\cdots,\alpha_m),\beta}(z_1,\cdots,z_m)$:

$$E_{\alpha,\beta,\gamma}(z_1,z_2)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{z_1^m\ z_2^{n-m}}{\Gamma(\alpha m+(n-m)\gamma+\beta)}, \tag{1.8}$$

where

$$(\ \Re(\beta)>\Re(\gamma)>\Re(\alpha)>0).$$

In view of the identity[15]

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}A(n,m)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A(n+m,m),$$

the function $E_{\alpha,\beta,\gamma}(z_1,z_2)$ can be written in the more elegant form

$$E_{\alpha,\beta,\gamma}(z_1,z_2)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n+m}{m}\frac{z_1^m z_2^n}{\Gamma(\alpha m+n\gamma+\beta)}, \tag{1.9}$$

or equivalently

$$E_{\alpha,\beta,\gamma}(z_1,z_2)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(n+m)!z_1^m z_2^n}{n!m!\Gamma(\alpha m+n\gamma+\beta)}. \tag{1.10}$$

It may of interest to point out that the series representation (1.9), in particular yields the following relationships:
$(i) For\ n\mapsto 0,\beta\mapsto 1$, we get

$$E_{\alpha,1,\gamma}(z,0)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m+1)}=E_{\alpha}(z), \tag{1.11}$$

$(ii) For\ n\mapsto 0$,we get

$$E_{\alpha,\beta,\gamma}(z,0)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m+\beta)}=E_{\alpha,\beta}(z), \tag{1.12}$$

$(iii) For\ m\mapsto 0$,we get

$$E_{\alpha,\beta,\gamma}(0,z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\gamma n+\beta)}=E_{\gamma,\beta}(z). \tag{1.13}$$

Since

$$\frac{(m+n)!}{m!n!}=\frac{(n+1)_m}{m!}=\frac{(m+1)_n}{n!},$$

we infer from (1.10) and (1.3) that

$$E_{\alpha,\beta,\gamma}(z_1,z_2)=\sum_{m=0}^{\infty}E_{\gamma,\alpha m+\beta}^{m+1}(z_2)\ z_1^m, \tag{1.14}$$

and

$$E_{\alpha,\beta,\gamma}(z_1,z_2)=\sum_{n=0}^{\infty}E_{\alpha,\gamma n+\beta}^{n+1}(z_1)\ z_2^n. \tag{1.15}$$

Formulas (1.14) and (1.15) are very useful in obtaining other needed properties for the function $E_{\alpha,\beta,\gamma}(z_1,z_2)$. Note that the function $E_{\alpha,\beta}^{\gamma}(z)$ is a special case of the Wright generalized hypergeometric function ${}_p\Psi_q$ (see [17]):

$$E_{\alpha,\beta}^{\gamma}(z)=\ \frac{1}{\Gamma(\gamma)}{}_1\Psi_1\left[z\left|\begin{array}{c}(\gamma,1)\\(\beta,\alpha)\end{array}\right.\right], \tag{1.16}$$

where (see[16]):

$$ {}_p\Psi_q \left[\begin{array}{ccc} (\alpha_1, A_1), ..., (\alpha_p, A_p); & & \\ & & z \\ (\beta_1, B_1), ..., (\beta_q, B_q); & & \end{array} \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j n)} \frac{z^n}{n!}, \tag{1.17} $$

and the coefficients $A_1, \cdots, A_p$ and $B_1, \cdots, B_q$ are positive real numbers such that

$$ 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0. $$

Hence the series expansions (1.14) and (1.15), can be rewritten in the forms

$$ E_{\alpha,\beta,\gamma}(z_1, z_2) = \sum_{m=0}^{\infty} {}_1\Psi_1 \left[z_2 \middle| \begin{array}{c} (m+1, 1) \\ (\alpha m + \beta, \gamma) \end{array} \right] \frac{z_1^m}{\Gamma(m+1)}, \tag{1.18} $$

and

$$ E_{\alpha,\beta,\gamma}(z_1, z_2) = \sum_{n=0}^{\infty} {}_1\Psi_1 \left[z_1 \middle| \begin{array}{c} (n+1, 1) \\ (\gamma n + \beta, \alpha) \end{array} \right] \frac{z_2^n}{\Gamma(n+1)}, \tag{1.19} $$

respectively.

Motivated essentially by the success of the applications of the Mittag-Leffler functions in many areas of science and engineering, we aim in this work at investigating certain properties and formulas involving the Mittag-Leffler function of two variables and three parameters (1.8). The organization of the paper is as follows: In Section 2, we give a number of integral representations of the Mittag-Leffler function $E_{\alpha,\beta,\gamma}(z_1, z_2)$ in terms of known elementary functions and drive some integral formulas involving $E_{\alpha,\beta,\gamma}(z_1, z_2)$. Also, we establish operational connections formulas for $E_{\alpha,\beta,\gamma}(z_1, z_2)$ with other Mittag-Leffler functions with one variable. In Section 3, we obtain differential and pure recurrence relations for the function $E_{\alpha,\beta,\gamma}(z_1, z_2)$. In Section 4, we discuss some useful integral transforms like Euler transform, Laplace transform, Mellin transform and Whittaker transform.

2. Integral and operational relations

In many situations, an integral representation of the Mittag-Leffler function is more convenient to use than its series representation. First of all, we establish an integral representation for $E_{\alpha,\beta,\gamma}(z_1, z_2)$ that is derived directly from Hankle representation of Gamma function $\Gamma(z)$ (see e.g.[8] and [19]):

$$ \frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C e^t t^{-z}\, dt, \tag{2.1} $$

where the path of integration is a simple loop beginning and ending at $-\infty$ and encircling the origin in the positive direction.

Theorem 2.1. Let $\Re(\beta) > \Re(\gamma) > \Re(\alpha) > 0$, then

$$ E_{\alpha,\beta,\gamma}(z_1, z_2) = \frac{1}{2\pi i} \int_C e^t t^{-\beta} \, (1 - z_1 t^{\alpha} - z_2\, t^{\gamma})^{-1} \, dt, \tag{2.2} $$

Proof. By Letting $z = \beta + \alpha m + \gamma n$ in (2.1) and multiplying both sides by

$$ \frac{(m+n)!\ z_2^m\ z_1^n}{m!n!},\ m, n = 0, 1, 2, ..., $$

we get

$$\frac{(m+n)!\ z_2^m\ z_1^n}{m!\ n!\ \Gamma(\alpha m+\gamma n+\beta)}=\frac{1}{2\pi i}\int_C e^t t^{-(\beta+\alpha m+\gamma n)}\frac{(m+n)!\ z_2^m\ z_1^n}{m!n!}\ dt. \tag{2.3}$$

Now, using (1.10) and the results

$$(m+n)!=(1)_{m+n}\text{ and }(1-x-y)^{-a}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}\ x^m\ y^n}{m!n!},$$

we led finally to the desired result (2.2).□

Next, we express $E_{\alpha,\beta,\gamma}(z_1,z_2)$ as the Mellin-Barnes type integral.

Theorem 2.2 Let $\Re(\beta)>\Re(\gamma)>\Re(\alpha)>0$ be satisfied then $E_{\alpha,\beta,\gamma}(z_1,z_2)$ is represented via the double Mellin-Barnes-type integral as

$$\begin{aligned}&E_{\alpha,\beta,\gamma}(z_1,z_2)\\&=\frac{1}{(2\pi i)^2}\int_{L_t}\int_{L_s}\frac{\Gamma(s)\Gamma(t)\Gamma(1-s-t)(-z_1)^{-t}\ (-z_2)^{-s}}{\Gamma(\beta-\gamma s-\alpha t)}ds\ dt.\end{aligned} \tag{2.4}$$

where $\{|\arg(z_1)|,|\arg(z_2)|\}<\pi$; and we assume that the counter L_s is in the s-plane and runs from $c-i\infty$ to $c+i\infty$ and the counter L_t is in the t-plane and runs from $C-i\infty$ to $C+i\infty$.

Proof.

Starting from the assertion(1.14) and replacing $E_{\alpha,\beta}^{\gamma}(z)$ by its Mellin-Barnes integral representation (1.7), we get

$$\begin{aligned}E_{\alpha,\beta,\gamma}(z_1,z_2)&=\sum_{m=0}^{\infty}\left\{\frac{1}{2\pi i}\int_{L_s}\frac{\Gamma(s)\Gamma(m-s+1)(-z_2)^{-s}}{\Gamma(\beta+\alpha m-\gamma s)}ds\right\}\frac{z_1^m}{\Gamma(m+1)}\\&=\frac{1}{2\pi i}\int_{L_s}\Gamma(s)\Gamma(1-s)(-z_2)^{-s}\left\{\sum_{m=0}^{\infty}\frac{(1-s)_m}{\Gamma(\beta+\alpha m-\gamma s)}\frac{z_1^m}{m!}\right\}ds.\end{aligned}$$

Now, considering the definition of $E_{\alpha,\beta}^{\gamma}(z)$ in (1.3), we obtain

$$E_{\alpha,\beta,\gamma}(z_1,z_2)=\frac{1}{2\pi i}\int_{L_s}\Gamma(s)\Gamma(1-s)(-z_2)^{-s}E_{\alpha,\beta-\gamma s}^{1-s}(z_1)ds.$$

Again, on replacing $E_{\alpha,\beta-\gamma s}^{1-s}(z_1)$ by its Mellin-Barnes integral representation (1.7), we led to

$$\begin{aligned}&E_{\alpha,\beta,\gamma}(z_1,z_2)\\&=\frac{1}{(2\pi i)^2}\int_{L_t}\int_{L_s}\frac{\Gamma(s)\Gamma(t)\Gamma(1-s-t)(-z_1)^{-t}\ (-z_2)^{-s}}{\Gamma(\beta-\gamma s-\alpha t)}ds\ dt,\end{aligned}$$

which is the desired result□.

Next, we derive a number of integral formulas involving $E_{\alpha,\beta,\gamma}$.

Theorem 2.3 Let $\Re(\beta)>\Re(\gamma)>\Re(\alpha)>0$, then

$$\frac{1}{\Gamma(\lambda)}\int_0^1 t^{\beta-1}(1-t)^{\lambda-1}E_{\alpha,\beta,\gamma}(z_1t^{\alpha},z_2t^{\gamma})dt=E_{\alpha,\beta+\lambda,\gamma}(z_1,z_2), \tag{2.5}$$

$$\frac{1}{\Gamma(\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{\beta-1}E_{\alpha,\beta,\gamma}(z_1(1-t)^{\alpha},z_2(1-t)^{\gamma})dt = E_{\alpha,\beta+\lambda,\gamma}(z_1,z_2), \tag{2.6}$$

$$\int_0^z t^{\beta-1}E_{\alpha,\beta,\gamma}(w_1t^{\alpha},w_2t^{\gamma})dt = z^{\beta}\ E_{\alpha,\beta+1,\gamma}(w_1z^{\alpha},w_2z^{\gamma}), \tag{2.7}$$

$$\int_0^{\infty} e^{-st}\ t^{\beta-1}E_{\alpha,\beta,\gamma}(z_1t^{\alpha},z_2t^{\gamma})dt = s^{-\beta}\left(1-\frac{z_1}{s^{\alpha}}-\frac{z_2}{s^{\gamma}}\right)^{-1}, \tag{2.8}$$

$$\frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\beta}E_{\alpha,\beta,\gamma}(w_1(s-t)^{\alpha},w_2(s-t)^{\gamma})ds$$
$$= (x-t)^{\beta+\delta-1}E_{\alpha,\beta+\delta,\gamma}(w_1(x-t)^{\alpha},w_2(x-t)^{\gamma}), \tag{2.9}$$

$$\int_0^x t^{\beta-1}(x-t)^{\lambda-1}\ \hat{D}_s^{-1}\{\ E_{\alpha,\beta,\gamma}(w_1s(x-t)^{\alpha},w_2s(x-t)^{\gamma})$$
$$\times\ E_{\alpha,\lambda,\gamma}(w_1st^{\alpha},w_2st^{\gamma})\} = sx^{\beta+\lambda-1}E_{\alpha,\beta+\lambda,\gamma}(w_1sx^{\alpha},w_2sx^{\gamma}). \tag{2.10}$$

Proof.
We have
$\int_0^1 t^{\beta-1}(1-t)^{\lambda-1}E_{\alpha,\beta,\gamma}(z_1t^{\alpha},z_2t^{\gamma})dt$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{z_1^m\ z_2^{n-m}}{\Gamma(\alpha m+(n-m)\gamma+\beta)}\int_0^1 t^{\beta+\alpha m+\gamma(n-m)-1}(1-t)^{\lambda-1}dt$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{z_1^m\ z_2^{n-m}\Gamma(\alpha m+(n-m)\gamma+\beta)\Gamma(\lambda)}{\Gamma(\alpha m+(n-m)\gamma+\beta)\Gamma(\alpha m+(n-m)\gamma+\beta+\lambda)}$$
$$= \Gamma(\lambda)E_{\alpha,\beta+\lambda,\gamma}(z_1,z_2),$$

which is the proof of (2.5). The proof of assertion (2.6) is similar to that of (2.5).
We have
$\int_0^z t^{\beta-1}E_{\alpha,\beta,\gamma}(w_1t^{\alpha},w_2t^{\gamma})dt$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{w_1^m\ w_2^{n-m}}{\Gamma(\alpha m+(n-m)\gamma+\beta)}\int_0^z t^{\beta+\alpha m+\gamma(n-m)-1}dt$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{w_1^m\ w_2^{n-m}}{\Gamma(\alpha m+(n-m)\gamma+\beta-1)}$$
$$= z^{\beta}E_{\alpha,\beta-1,\gamma}(w_1z^{\beta},w_2z^{\gamma}),$$

which is the proof of (2.7).
We have
$\int_0^{\infty} e^{-st}\ t^{\beta-1}E_{\alpha,\beta,\gamma}(z_1t^{\alpha},z_2t^{\gamma})dt$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{w_1^m\ w_2^{n-m}}{\Gamma(\alpha m+(n-m)\gamma+\beta)}\int_0^{\infty} e^{-st}t^{\beta+\alpha m+\gamma(n-m)}dt$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}w_1^m\ w_2^{n-m}\ s^{\beta+\alpha m+\gamma(n-m)}$$
$$= s^{\beta}\left(1-\frac{z_1}{s^{\alpha}}-\frac{z_2}{s^{\gamma}}\right)^{-1},$$

which is the proof of (2.8).
Let $u=\frac{s-t}{x-t}$ in the left-hand side of the assertion (2.9), then

$\frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\beta}E_{\alpha,\beta,\gamma}(w_1(s-t)^{\alpha},w_2(s-t)^{\gamma})ds$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\delta)}\int_0^1 (x-t)^{\delta-1}(1-t)^{\delta-1}(x-t)^{\beta-1}u^{\beta-1}(x-t)\\
&\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!\ w_1^m\ w_2^n\ u^{\alpha m+\gamma n}\ (x-t)^{\alpha m+\gamma n}}{m!n!\Gamma(\alpha m+\gamma n+\beta)}\\
&= \frac{1}{\Gamma(\delta)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!\ w_1^m\ w_2^n\ u^{\alpha m+\gamma n}\ (x-t)^{\alpha m+\gamma n+\beta+\delta-1}}{m!n!\Gamma(\alpha m+\gamma n+\beta)}\\
&\times \int_0^1 (1-u)^{\delta-1}u^{\alpha m+\gamma n+\beta-1}du\\
&= \frac{1}{\Gamma(\delta)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!\ w_1^m\ w_2^n\ u^{\alpha m+\gamma n}\ (x-t)^{\alpha m+\gamma n+\beta+\delta-1}}{m!n!\Gamma(\alpha m+\gamma n+\beta)}\\
&\times \frac{\Gamma(\delta)\Gamma(\alpha m+\gamma n+\beta)}{\Gamma(\alpha m+\gamma n+\beta+\delta)}.
\end{aligned}
$$

Simplification the above equation yields (2.9).
Starting from the left-hand side of assertion (2.10) and using the definition (1.8), we get

$$
\begin{aligned}
&\int_0^x t^{\beta-1}(x-t)^{\lambda-1}\ \hat{D}_s^{-1}\{\ E_{\alpha,\beta,\gamma}(w_1 s(x-t)^{\alpha},w_2 s(x-t)^{\gamma})\times\ E_{\alpha,\lambda,\gamma}(w_1 st^{\alpha},w_2 st^{\gamma})\}\,dt\\
&= \sum_{m,n,p,q=0}^{\infty}\frac{(m+n)!(p+q)!w_1^{m+p}w_2^{n+q}\left(\hat{D}_s^{-1}s^{m+n+p+q}\right)}{m!n!p!q!\Gamma(\alpha m+\gamma n+\beta)\Gamma(\alpha p+\gamma q+\lambda)}\\
&\times \int_0^x t^{\alpha p+\gamma q+\beta-1}(x-t)^{\alpha m+\gamma n+\lambda-1}dt\\
&= \sum_{m,n,p,q=0}^{\infty}\frac{(m+n)!(p+q)!w_1^{m+p}w_2^{n+q}(m+n+p+q)!x^{\alpha m+\alpha p+\gamma n+\gamma q+\lambda+\beta-1}}{m!n!p!q!(m+n+p+q+1)!\Gamma(\alpha m+\alpha p+\gamma n+\gamma q+\beta+\lambda)}\\
&= \sum_{m,n,p,q=0}^{\infty}\frac{(m+n-p-q)!(p+q)!w_1^m w_2^n(m+n)!x^{\alpha m+\gamma n+\lambda+\beta-1}}{(m-p)!(n-q)!p!q!(m+n+1)!\Gamma(\alpha m+\gamma n+\beta+\lambda)}\\
&= \sum_{m=0}^{\infty}\sum_{p=0}^{m}\binom{m}{p}\frac{(1)_{m-p}(1)_p\ w_1^m\ w_2^n}{m!}\\
&\times \sum_{n=0}^{\infty}\sum_{p=0}^{n}\binom{n}{q}\frac{(1+m-p)_{m-p}(1+p)_q\ w_1^m\ w_2^n x^{\alpha m+\gamma n+\lambda+\beta-1}}{n!\Gamma(\alpha m+\gamma n+\beta+\lambda)}.
\end{aligned}
$$

Now, if we use the formula (see[20]):

$$(a+b)_m=\sum_{k=0}^{m}\binom{m}{k}(a)_k(b)_{m-k},$$

and simplify, we obtain

$$
\begin{aligned}
&\int_0^x t^{\beta-1}(x-t)^{\lambda-1}\ \hat{D}_s^{-1}\{\ E_{\alpha,\beta,\gamma}(w_1 s(x-t)^{\alpha},w_2 s(x-t)^{\gamma})\\
&\times\ E_{\alpha,\lambda,\gamma}(w_1 st^{\alpha},w_2 st^{\gamma})\}\,dt = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{w_1^m\ w_2^n\ x^{\alpha m+\gamma n+\lambda+\beta-1}s^{m+n+1}}{m!n!\Gamma(\alpha m+\gamma n+\beta+\lambda)},
\end{aligned}
$$

which is in view of the definition (1.10) gives us the desired result(2.10)□.
Now, we turn to some operational relations for the Mittag-Leffler function of two variables. Let us note that in mathematical treatises on fractional differential equations the Riemann-Liouville approach to the notion of the fractional derivative of order $\mu(\mu \geq 0)$ is normally used (see [21]):

$$\left(\hat{D}^{\nu} f\right)(x) := \left(\frac{d}{dx}\right)^{m} (J^{m-\nu}\ f)(x), m-1 < \nu \leq m, m \in \mathbb{N}, x > 0, \tag{2.11}$$

where

$$(J^{\nu}\ f)(x) := \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1}\ f(t)dt,\ \nu > 0, x > 0,$$
$$(J^0 f)(x) := f(x), x > 0,$$

is the Riemann-Liouville fractional integral of order ν.
For the case when $f(x) = x^a, a > -1$, we get the results

$$\hat{D}_z^{\nu} z^a = \frac{\Gamma(a+1)}{\Gamma(a-\nu+1)} z^{a-\nu}, \tag{2.12}$$

where $\hat{D}_z^{\nu} = \left(\frac{d}{dz}\right)^{\nu}, z > 0, a > -1\ ,\ \nu \geq 0$ is not restricted to integer values. Also, we recall the following relations from [25,p.74-75]. For $m-1 \leq p < m, n-1 \leq q < n$, we have

$$\hat{D}_t^p \hat{D}_t^q f(t) = \hat{D}_t^q \hat{D}_t^p f(t) = \hat{D}_t^{p+q} f(t) \Leftrightarrow f^{(j)}(0) = 0, \tag{2.13}$$

and

$$\hat{D}_t^p \hat{D}_t^q f(t) = \hat{D}_t^{p+q} - \sum_{j=1}^{n} \left[\hat{D}_t^{q-j} f(t)\right]_{t=0} \frac{(t-a)^{-p-j}}{\Gamma(1-p-j)}. \tag{2.14}$$

By exploiting the results (2.11) to (2.14), we can derive the following operational connections for the function $E_{\alpha,\beta,\gamma}(z_1, z_2)$ with the Mittag-Leffler functions in (1.1) to (1.6) with one variable.
Theorem 2.4. Let $\Re(\beta) > \Re(\gamma) > \Re(\alpha) > 0, 0 < \beta < 1, 0 < \delta < 1,\ \Re(\delta) > \Re(\lambda)p > 0, q > 0$, then

$$t^{1-\beta} E_{\alpha,\beta,\gamma}(z_1 t^{\alpha}, z_2 t^{\gamma})$$
$$= \hat{D}_t^{\beta-1} E_{\alpha} \left(z_2 \left(1 + \frac{z_1}{z_2} \hat{D}_t^{\gamma-\alpha}\right) \hat{D}_t^{\alpha-\gamma} t^{\alpha}\right), \tag{2.15}$$

$$t^{\beta-1} E_{\alpha,\beta,\gamma}(z_1 t^{\alpha}, z_2 t^{\gamma})$$
$$= E_{\alpha,\beta} \left(z_2 \left(1 + \frac{z_1}{z_2} \hat{D}_t^{\gamma-\alpha}\right) \hat{D}_t^{\alpha-\gamma} t^{\alpha}\right) \{t^{\beta-1}\}, \tag{2.16}$$

$$t^{\beta-1} u^{\delta-1} E_{\alpha,\beta,\gamma}(z_1 t^{\alpha} u, z_2 t^{\gamma} u)$$
$$= \Gamma(\delta) \hat{D}_u^{1-\delta} \left\{E_{\alpha,\beta}^{\delta} \left(z_2 \left(1 + \frac{z_1}{z_2} \hat{D}_t^{\gamma-\alpha}\right) \hat{D}_t^{\alpha-\gamma} t^{\alpha} u\right)\right\} \{t^{\beta-1}\}, \tag{2.17}$$

$$t^{\beta-1} u^{\delta-1} E_{\alpha,\beta,\gamma}(z_1 t^{\alpha} u^q, z_2 t^{\gamma} u^p)$$
$$= \Gamma(\delta) \hat{D}_u^{1-\delta} \left\{E_{\alpha,\beta}^{\delta,q} \left(z_2 \left(1 + \frac{z_1}{z_2} \hat{D}_t^{\gamma-\alpha}\right) \hat{D}_u^{1-q} \hat{D}_t^{\alpha-\gamma} t^{\alpha} u\right)\right\} \{t^{\beta-1}\}, \tag{2.18}$$
$$t^{\beta-1} u^{\lambda-1} E_{\alpha,\beta,\gamma}(z_1 t^{\alpha} u, z_2 t^{\gamma} u)$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\delta)} \hat{D}_u^{\delta-\lambda} \left\{ E_{\alpha,\beta}^{\lambda,\delta} \left(z_2 \left(1 + \frac{z_1}{z_2} \hat{D}_t^{\gamma-\alpha} \right) \hat{D}_t^{\alpha-\gamma} t^{\alpha} u \right) \right\} \{t^{\beta-1} u^{\delta-1}\}, \tag{2.19}$$

$$t^{\beta-1} u^{\lambda-1} E_{\alpha,\beta,\gamma}(z_1 t^{\alpha} u^q, z_2 t^{\gamma} u^q)$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\delta)} \hat{D}_u^{\delta-\lambda} \left\{ E_{\alpha,\beta,p}^{\lambda,\delta,q} \left(z_2 \left(1 + \frac{z_1}{z_2} \hat{D}_t^{\gamma-\alpha} \right) \hat{D}_u^{q-p} \hat{D}_t^{\alpha-\gamma} t^{\alpha} u^q \right) \right\} \{t^{\beta-1} u^{\delta-1}\}. \tag{2.20}$$

Proof. Let I denote the right-hand side of (2.15). Making use of the definition (1.1), we get

$$I = \sum_{k=0}^{\infty} \frac{z_2^k}{\Gamma(\alpha k + 1)} \hat{D}_t^{1-\beta} \left(1 + \frac{z_1}{z_2} \hat{D}_t^{\gamma-\alpha} \right)^k \hat{D}_t^{\alpha k - \gamma k} t^{\alpha k},$$

which on using the relations (see [16,p.23,Equation 23 and p.356, Equation 11]), gives us

$$I = \sum_{k=0}^{\infty} \sum_{s=0}^{k} \frac{z_1^s z_2^{k-s} k!}{s!(k-s)!\Gamma(\alpha k + 1)} \hat{D}_t^{1-\beta} \left(\hat{D}_t^{\gamma s - \alpha s} \left(\hat{D}_t^{\alpha k - \gamma k} \left(t^{\alpha k} \right) \right) \right). \tag{2.21}$$

Now, in view of relations (2.13) and (2.14) , we get

$$\left(\hat{D}_t^{\gamma s - \alpha s} \left(\hat{D}_t^{\alpha k - \gamma k} \left(t^{\alpha k} \right) \right) \right)$$

$$= \left(\hat{D}_t^{\gamma s - \alpha s + \alpha k - \gamma k} \left(t^{\alpha k} \right) - \sum_{j=0}^{s} \left[\hat{D}_t^{\alpha k - \gamma k - j} (t^{\alpha k}) \right]_{t=0} \frac{(t-a)^{\alpha s - \gamma s - j}}{\Gamma(1 + \alpha s - \gamma s - j)} \right).$$

Note that, the function $t^{\alpha k}$ has a sufficient number of continuous derivatives, then the conditions

$$\left[\hat{D}_t^{\alpha k - \gamma k - j} \left(t^{\alpha k} \right) \right]_{t=0} = 0, \; j = 0, 1, 2, \cdots,$$

are equivalent to

$$f^{(j)}(0) = 0, \; j = 0, 1, 2, \cdots, \max(m, n),$$

that is

$$\left[\left(t^{\alpha k} \right)^{(j)} \right]_{t=0} = 0, \; j = 0, 1, 2, \cdots \max(m, n),$$

Hence, we get

$$\left(\hat{D}_t^{\gamma s - \alpha s} \left(\hat{D}_t^{\alpha k - \gamma k} \left(t^{\alpha k} \right) \right) \right) = \left(\hat{D}_t^{\gamma s - \alpha s + \alpha k - \gamma k} \left(t^{\alpha k} \right) \right). \tag{2.22}$$

By repeating the same steps leading to (2.22) and using (2.12), we find that

$$\hat{D}_t^{1-\beta} \left(\hat{D}_t^{\gamma s - \alpha s + \alpha k - \gamma k} \left(t^{\alpha k} \right) \right) = \hat{D}_t^{1-\beta+\gamma s - \alpha s + \alpha k - \gamma k} \left(t^{\alpha k} \right)$$

$$= \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha s + \gamma(k-s) + \beta)} \; t^{\alpha s + \gamma(k-s) + \beta - 1}. \tag{2.23}$$

Now, by inserting the result (2.23) in (2.21) and simplify, we obtain

$$I = \sum_{k=0}^{\infty} \sum_{s=0}^{k} \frac{z_1^s z_2^{k-s} k! \; t^{\alpha s + \gamma(k-s) + \beta - 1}}{\Gamma(\alpha s + \gamma(k-s) + \beta) s!(k-s)!},$$

which in view of the definition (1.8) gives us the left-hand side of assertion (2.15) and this completes the proof of (2.15). Similarly one can prove the formulas (2.16) to (2.20)□.

3. Recurrence relations

By the definitions (1.8) to (1.10) and by employing the operator (2.12), we derive the following interesting differential relations.

Theorem 3.1. If $\Re(\beta) > \Re(\gamma) > \Re(\alpha) > 0$, then

$$\hat{D}^p_{z_1}\{E_{\alpha,\beta,\gamma}(z_1,z_2)\} = \Gamma(p+1)\sum_{m=0}^{\infty}(p+1)_m\; E^{m+p+1}_{\gamma,\beta+\alpha(m+p)}(z_2)\;\frac{z_1^m}{m!} \tag{3.1}$$

$$\hat{D}^p_{z_1}\{E_{\alpha,\beta,\gamma}(z_1,z_2)\} = \Gamma(p+1)\sum_{n=0}^{\infty}(p+1)_n E^{n+p+1}_{\alpha,\beta+\gamma n+\alpha p}(z_1)\;\frac{z_2^n}{n!} \tag{3.2}$$

$$\hat{D}^p_{z_2}\{E_{\alpha,\beta,\gamma}(z_1,z_2)\} = \Gamma(p+1)\sum_{m=0}^{\infty}(p+1)_m\; E^{m+p+1}_{\gamma,\beta+\alpha m+\gamma p}(z_2)\;\frac{z_1^m}{m!} \tag{3.3}$$

$$\hat{D}^p_{z_2}\{E_{\alpha,\beta,\gamma}(z_1,z_2)\} = \Gamma(p+1)\sum_{n=0}^{\infty}(p+1)_n\; E^{n+p+1}_{\alpha,\beta+\gamma(n+p)}(z_1)\;\frac{z_2^n}{n!} \tag{3.4}$$

$$\hat{D}^p_t\left\{t^{\beta-1}E_{\alpha,\beta,\gamma}(z_1t^{\alpha},z_2t^{\gamma})\right\} = t^{\beta-p-1}\; E_{\alpha,\beta-p,\gamma}(z_1t^{\alpha},z_2t^{\gamma}) \tag{3.5}$$

$$\hat{D}^{-p}_t\{E_{\alpha,\beta,\gamma}(z_1t^{\alpha},z_2t^{\gamma})\} = t^{\beta+p-1}\; E_{\alpha,\beta+p,\gamma}(z_1t^{\alpha},z_2t^{\gamma}) \tag{3.6}$$

Proof.
From definitions (1.10) and (2.12), we find that

$$\begin{aligned}
&\hat{D}^p_{z_1}\{E_{\alpha,\beta,\gamma}(z_1,z_2)\}\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!z_1^{m-p}z_2^n}{(m-p)!n!\Gamma(\alpha m+\gamma n+\beta)}\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n+p)!z_1^m z^n}{m!n!\Gamma(\alpha m+\alpha p+\gamma n+\beta)}\\
&=\Gamma(p+1)\sum_{m=0}^{\infty}\frac{(p+1)_m}{m!}\left\{E^{m+p+1}_{\gamma,\beta+\alpha(m+p)}(z_2)\right\}z_1^m,
\end{aligned}$$

which is the result (3.1). Similarly, one can prove the results (3.2), (3.3), and (3.4). Next, from(1.10), we have

$$\begin{aligned}
&\hat{D}^p_t\left\{t^{\beta-1}E_{\alpha,\beta,\gamma}(z_1t^{\alpha},z_2t^{\gamma})\right\}\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!z_1^m z_2^n}{m!n!\Gamma(\alpha m+\gamma n+\beta)}\;\hat{D}^p_t t^{\beta+\alpha m+\gamma n-1}\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!z_1^m z_2^n}{m!n!\Gamma(\alpha m+\gamma n+\beta+p)}\;t^{\beta+\alpha m+\gamma n-p-1}\\
&=t^{\beta-p-1}\;E_{\alpha,\beta-p,\gamma}(z_1t^{\alpha},z_2t^{\gamma}),
\end{aligned}$$

which is the right-hand side of the assertion (3.5). For the proof of (3.6), we refer to the proof of (3.5)□.

Next, we derive some differential recurrence relations for $E_{\alpha,\beta,\gamma}(z_1,z_2)$.

Theorem 3.2. If $\Re(\beta) > \Re(\gamma) > \Re(\alpha) > 0$, then

$$E_{\alpha,\beta,\gamma}(z_1^{\alpha},z_2^{\gamma})$$

$$= \beta\ E_{\alpha,\beta+1,\gamma}(z_1^{\alpha}, z_2^{\gamma}) + z_1 \frac{d}{dz_1}\ E_{\alpha,\beta+1,\gamma}(z_1^{\alpha}, z_2^{\gamma}) + z_2 \frac{d}{dz_2}\ E_{\alpha,\beta+1,\gamma}(z_1^{\alpha}, z_2^{\gamma}). \tag{3.7}$$

$$E_{\alpha,\beta,\gamma}(z_1, z_2)$$

$$= \beta\ E_{\alpha,\beta+1,\gamma}(z_1, z_2) + z_1 \alpha \frac{d}{dz_1}\ E_{\alpha,\beta+1,\gamma}(z_1, z_2) + z_2 \gamma\ \frac{d}{dz_2}\ E_{\alpha,\beta+1,\gamma}(z_1, z_2). \tag{3.8}$$

$$E_{\alpha,\beta,\gamma}(z_1 t^{\alpha}, z_2 t^{\gamma}) = \beta\ E_{\alpha,\beta+1,\gamma}(z_1 t^{\alpha}, z_2 t^{\gamma}) + t \frac{d}{dt}\ E_{\alpha,\beta+1,\gamma}(z_1 t^{\alpha}, z_2 t^{\gamma}). \tag{3.9}$$

Proof.
From the right-hand side of formula (3.7), we have

$$\beta\ E_{\alpha,\beta+1,\gamma}(z_1^{\alpha}, z_2^{\gamma}) + z_1 \frac{d}{dz_1}\ E_{\alpha,\beta+1,\gamma}(z_1^{\alpha}, z_2^{\gamma}) + z_2 \frac{d}{dz_2}\ E_{\alpha,\beta+1,\gamma}(z_1^{\alpha}, z_2^{\gamma})$$

$$\begin{aligned}
&= \beta\ \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{z_1^{\alpha m}\ z_2^{\gamma(n-m)}}{\Gamma(\alpha m + \gamma(n-m) + \beta + 1)} \\
&+ \sum_{n=0}^{\infty} \sum_{m=0}^{n} \alpha m \binom{n}{m} \frac{z_1^{\alpha m}\ z_2^{\gamma(n-m)}}{\Gamma(\alpha m + \gamma(n-m) + \beta + 1)} \\
&+ \sum_{n=0}^{\infty} \sum_{m=0}^{n} \gamma(n-m) \binom{n}{m} \frac{z_1^{\alpha m}\ z_2^{\gamma(n-m)}}{\Gamma(\alpha m + \gamma(n-m) + \beta + 1)} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{z_1^{\alpha m}\ z_2^{\gamma(n-m)}}{\Gamma(\alpha m + \gamma(n-m) + \beta + 1)} (\alpha m + \gamma(n-m) + \beta) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{z_1^{\alpha m}\ z_2^{\gamma(n-m)}}{\Gamma(\alpha m + \gamma(n-m) + \beta)},
\end{aligned}$$

which is the left-hand side of formula (3.7) and then the proof is completed.
Next, we have

$$\beta\ E_{\alpha,\beta+1,\gamma}(z_1, z_2) + z_1 \alpha \frac{d}{dz_1}\ E_{\alpha,\beta+1,\gamma}(z_1, z_2) + z_2 \gamma\ \frac{d}{dz_2}\ E_{\alpha,\beta+1,\gamma}(z_1, z_2)$$

$$\begin{aligned}
&= \beta\ \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{z_1^{m}\ z_2^{(n-m)}}{\Gamma(\alpha m + \gamma(n-m) + \beta + 1)} \\
&+ \sum_{n=0}^{\infty} \sum_{m=0}^{n} \alpha m \binom{n}{m} \frac{z_1^{m}\ z_2^{(n-m)}}{\Gamma(\alpha m + \gamma(n-m) + \beta + 1)} \\
&+ \sum_{n=0}^{\infty} \sum_{m=0}^{n} \gamma(n-m) \binom{n}{m} \frac{z_1^{m}\ z_2^{(n-m)}}{\Gamma(\alpha m + \gamma(n-m) + \beta + 1)} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{z_1^{m}\ z_2^{(n-m)}}{\Gamma(\alpha m + \gamma(n-m) + \beta + 1)} (\alpha m + \gamma(n-m) + \beta) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{z_1^{m}\ z_2^{(n-m)}}{\Gamma(\alpha m + \gamma(n-m) + \beta)},
\end{aligned}$$

which is the left-hand side of formula (3.8) and then the proof is completed.
Finally, consider the right-hand side of formula (3.9), we have

$$\beta\ E_{\alpha,\beta+1,\gamma}(z_1t^{\alpha},z_2t^{\gamma})+t\,\tfrac{d}{dt}\ E_{\alpha,\beta+1,\gamma}(z_1t^{\alpha},z_2t^{\gamma})$$

$$=\beta\ \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{z_1^m t^{\alpha m}\ z_2^{n-m}t^{\gamma(n-m)}}{\Gamma(\alpha m+\gamma(n-m)+\beta+1)}$$

$$+\sum_{n=0}^{\infty}\sum_{m=0}^{n}(\alpha m+\gamma(n-m))\binom{n}{m}\frac{z_1^m t^{\alpha m}\ z_2^{n-m}t^{\gamma(n-m)}}{\Gamma(\alpha m+\gamma(n-m)+\beta+1)}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{z_1^m t^{\alpha m}\ z_2^{n-m}t^{\gamma(n-m)}}{\Gamma(\alpha m+\gamma(n-m)+\beta+1)}(\alpha m+\gamma(n-m)+\beta)$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{z_1^m t^{\alpha m}\ z_2^{n-m}t^{\gamma(n-m)}}{\Gamma(\alpha m+\gamma(n-m)+\beta)},$$

which is the left-hand side of formula (3.9) and then the proof is completed.□
Further, we derive three pure recurrence relations for $E_{\alpha,\beta,\gamma}(z_1,z_2)$. In order to establish two of these relations, we need the following known result(see [9] and [22]):

$$x\ E^{\rho}_{\alpha,\beta}(x)=E^{\rho}_{\alpha,\beta-\alpha}(x)-E^{\rho-1}_{\alpha,\beta-\alpha}(x). \tag{3.12}$$

Theorem 3.3. If $\Re(\beta)>\Re(\gamma)>\Re(\alpha)>0$, then

$$z_2E_{\alpha,\beta,\gamma}(z_1,z_2)=E_{\alpha,\beta-\gamma,\gamma}(z_1,z_2)-z_1E_{\alpha,\beta+\alpha-\gamma,\gamma}(z_1,z_2), \tag{3.13}$$

$$z_1E_{\alpha,\beta,\gamma}(z_1,z_2)=E_{\alpha,\beta-\alpha,\gamma}(z_1,z_2)-z_2E_{\alpha,\beta-\alpha+\gamma,\gamma}(z_1,z_2), \tag{3.14}$$

$$z_1E_{\alpha,\beta+\alpha,\gamma}(z_1,z_2)+z_2E_{\alpha,\beta+\gamma,\gamma}(z_1,z_2)=E_{\alpha,\beta,\gamma}(z_1,z_2). \tag{3.15}$$

Proof.
We have (see (1.14)):

$$E_{\alpha,\beta,\gamma}(z_1,z_2)=\sum_{m=0}^{\infty}E^{m+1}_{\gamma,\alpha m+\beta}(z_2)\ z_1^m.$$

On multiplying both the sides of the above equation by z_2 and using (3.12), we get

$$z_2E_{\alpha,\beta,\gamma}(z_1,z_2)=\sum_{m=0}^{\infty}\left\{E^{m+1}_{\gamma,\alpha m+\beta-\gamma}(z_2)-E^{m}_{\gamma,\alpha m+\beta-\gamma}(z_2)\right\}\ z_1^m$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+1)_n z_1^m z_2^n}{n!\Gamma(\alpha m+\gamma n+\beta)}-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m)_n z_1^m z_2^n}{n!\Gamma(\alpha m+\gamma n+\beta)}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!z_1^m z_2^n}{m!n!\Gamma(\alpha m+\gamma n+\beta)}-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n-1)!\ z_1^m z_2^n}{(m-1)!n!\Gamma(\alpha m+\gamma n+\beta)}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!z_1^m z_2^n}{m!n!\Gamma(\alpha m+\gamma n+\beta)}-z_1\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!\ z_1^m z_2^n}{(m)!n!\Gamma(\alpha m+\gamma n+\beta-\gamma)},$$

which is the left-hand side of the formula (3.13) and the proof is completed.
If instead of (1.14) we use (1.15), we can easily prove formula (3.14) and we skip the details.
Next, we have

$$z_1E_{\alpha,\beta+\alpha,\gamma}(z_1,z_2)+z_2E_{\alpha,\beta+\gamma,\gamma}(z_1,z_2)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!z_1^{m+1}z_2^n}{m!n!\Gamma(\alpha m+\gamma n+\beta+\alpha)}+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!z_1^{m}z_2^{n+1}}{m!n!\Gamma(\alpha m+\gamma n+\beta+\gamma)}$$
$$= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n-1)!z_1^{m}z_2^n}{(m-1)!n!\Gamma(\alpha m+\gamma n+\beta)}+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{(m+n-1))!z_1^{m}z_2^{n}}{m!(n-1)!\Gamma(\alpha m+\gamma n+\beta)}$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(m+n-1)!z_1^{m}z_2^n}{(m-1)!(n-1)!\Gamma(\alpha m+\gamma n+\beta)}\left\{\frac{1}{m}+\frac{1}{n}\right\}$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(m+n-1)!z_1^{m}z_2^n}{(m-1)!(n-1)!\Gamma(\alpha m+\gamma n+\beta)}\left\{\frac{(m+n)}{mn}\right\}$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(m+n)!z_1^{m}z_2^n}{(m)!(n)!\Gamma(\alpha m+\gamma n+\beta)}$$

which is the left-hand side of formula (3.15) and the proof is completed□.

4.Integral transforms

In this section, we establish some useful integral transforms like Euler transform, Laplace transform, Mellin transform, Whittaker transform. First, we introduce a Mellin integral -type for the function $E_{\alpha,\beta,\gamma}(z_1,z_2)$. Recall that, the double Mellin transform of the function $f(x,y)$ is defined by [23]:

$M=\{f(x,y):s,t\}$

$$= \int_0^{\infty}\int_0^{\infty} f(x,y)x^{s-1}y^{t-1}ds\ dt = f^*(s,t), \Re(s)>0, \Re(t)>0, \tag{4.1}$$

then

$$f(x,y)=\frac{1}{2\pi i}\int_L\int_L f^*(s,t)x^{-s}y^{-t}dsdt. \tag{4.2}$$

By using the above definition, we prove the following result.

Theorem 4.1.(*Double Mellin transform*). If $\Re(\beta)>\Re(\gamma)>\Re(\alpha)>0$, then

$$\int_0^{\infty}\int_0^{\infty} u_1^{s-1}u_2^{t-1}E_{\alpha,\beta,\gamma}(-w_1u_1,-w_2u_2)dsdt = \frac{\Gamma(s)\Gamma(t)\Gamma(1-s)\Gamma(1-s-t)}{w_1^s w_2^t\Gamma(\beta-\alpha t-\gamma s)\Gamma(1-t)}. \tag{4.3}$$

Proof.

In the Mellin-Barnes integral representation (2.4), let $z_1=-w_1u_1$ and $z_2=-w_2u_2$, we get

$E_{\alpha,\beta,\gamma}(-w_1u_1,-w_2u_2)$

$$= \frac{1}{(2\pi i)^2}\int_L\int_L \frac{\Gamma(s)\Gamma(1-s)\Gamma(t)\Gamma(1-s-t)(w_1u_1)^{-t}\ (w_2u_2)^{-s}}{\Gamma(\beta-\gamma s-\alpha t)\Gamma(1-t)}dsdt$$
$$= \frac{1}{(2\pi i)^2}\int_L\int_L f^*(s,t)u_1^{-t}\ u_2^{-s}dsdt,$$

where

$$f^*(s,t)=\frac{\Gamma(s)\Gamma(1-s)\Gamma(t)\Gamma(1-s-t)}{w_1^t\ w_2^s\Gamma(\beta-\gamma s-\alpha t)\Gamma(1-t)}.$$

Hence from the definition of the Mellin-Barnes integral transform (4.1) and (4.2), we get the result (4.3)□.

Next, we aim to obtain Whittaker transform for the function $E_{\alpha,\beta,\gamma}(z_1,z_2)$.
First, we recall the definition of Whittaker function $W_{k,\mu}(z)$(see [16,p.39(24)]):

$$W_{k,\mu}(z)=z^{\mu+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right)\ {}_1F_1\left(\mu-k+1/2;2\mu+1;z\right),$$

where ${}_1F_1$ is Kummer's function[16], and the integral formula:

$$\int_0^\infty e^{\frac{-t}{2}}t^{\nu-1}\ W_{\lambda,\mu}(t)dt=\frac{\Gamma(\frac{1}{2}+\mu+\nu)\Gamma(\frac{1}{2}-\mu+\nu)}{\Gamma(1-\lambda+\nu)},$$

$$\Re(\nu\pm\mu)>-\frac{1}{2}.$$

Theorem 4.2.(*Whittaker transform*).If $\Re(\beta)>\Re(\gamma)>\Re(\alpha)>0$, then

$$\int_0^\infty e^{\frac{-pt}{2}}t^{\rho-1}\ W_{\lambda,\mu}(pt)\ \times\ E_{\alpha,\beta,\gamma}(w_1t^\delta,w_2t^\sigma)\ dt$$

$$=p^{-\rho}\sum_{m=0}^\infty {}_3\Psi_2\left[\begin{array}{cc}(m+1,1),(\frac{1}{2}+\delta m+\rho+\mu,\sigma),(\frac{1}{2}+\delta m+\rho-\mu,\sigma); & \\ & w_2/p^\sigma\\ (\beta+\alpha m,\gamma),(1-\delta m+\rho-\lambda,\sigma); & \end{array}\right]\frac{(w_1/p^\delta)^m}{m!}. \tag{4.4}$$

Proof.
Let $pt=u$ in the left-hand side of assertion (4.4),we get

$$\frac{1}{p}\int_0^\infty e^{\frac{-u}{2}}\left(\frac{u}{p}\right)^{\rho-1}\ W_{\lambda,\mu}(u)\ E_{\alpha,\beta,\gamma}(w_1(u/p)^\delta,w_2(u/p)^\sigma)\ dt$$

$$=p^{-\rho}\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{(n+m)!(w_1/p^\delta)^m(w_2/p^\rho)^n}{n!m!\Gamma(\alpha m+\gamma n+\beta)}\times\int_0^\infty u^{\delta m+\sigma n+\rho-1}e^{\frac{-u}{2}}\ W_{\lambda,\mu}(u)du$$

$$=p^{-\rho}\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{(n+m)!(w_1/p^\delta)^m(w_2/p^\rho)^n}{n!m!\Gamma(\alpha m+\gamma n+\beta)}$$

$$\times\frac{\Gamma(\frac{1}{2}+\delta m+\sigma n+\rho+\mu)\Gamma(\frac{1}{2}+\delta m+\sigma n+\rho-\mu)}{\Gamma(1+\delta m+\sigma n+\rho-\lambda)},$$

which in view of (1.17), yields the assertion (4.4). □

Theorem 4.3.(*Euler transform*). If $\Re(\beta)>\Re(\gamma)>\Re(\alpha)>0$, then

$$\int_0^1\int_0^1\ z_1^{a_1-1}(1-z_1)^{b_1-1}z_2^{a_2-1}(1-z_2)^{b_2-1}\ E_{\alpha,\beta,\gamma}(x_1z^\sigma,x_2z_2^\delta)dz_1dz_2$$

$$=\Gamma(b_1)\Gamma(b_2)\sum_{m=0}^\infty {}_2\Psi_2\left[\begin{array}{cc}(m+1,1),(a_2,\delta); & \\ & x_2\\ (\beta+\alpha m,\gamma),(a_2+b_2,\delta); & \end{array}\right]\frac{\Gamma(a_1+\sigma m)x_1^m}{\Gamma(a_1+b_1+\sigma m)m!}. \tag{4.5}$$

Proof.
Denote the left-hand side of equation (4.5) by I, then from the definition (1.10), we get

$$I=\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{(m+n)!x_1^mx_2^n}{m!n!\Gamma(\alpha m+\gamma n+\beta)}\times\int_0^1\ z_1^{a_1+m\sigma-1}(1-z_1)^{b_1-1}dz_1$$

$$\times\int_0^1\ z_2^{a_2+n\delta-1}(1-z_2)^{b_2-1}dz_2.$$

Now, by using the Beta function (see e.g. [24]):

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

we obtain

$$I=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!\Gamma(\sigma m+a_1)\Gamma(b_1)\Gamma(\delta n+a_2)\Gamma(b_2)x_1^m x_2^n}{m!n!\Gamma(\alpha m+\gamma n+\beta)\Gamma(\sigma m+a_1+b_1)\Gamma(\delta n+a_2+b_2)},$$

which in view of (1.17), yields the assertion(4.5).□

Theorem 4.4.(*Laplace transform*).If $\Re(\beta)>\Re(\gamma)>\Re(\alpha)>0$, then

$$\int_0^{\infty}\int_0^{\infty} z_1^{a_1-1}e^{-s_1z_1}z_2^{a_2-1}e^{-s_2z_2}\ E_{\alpha,\beta,\gamma}(x_1z_1^{\sigma_1},x_2z_2^{\sigma_2})dz_1dz_2$$

$$=s_1^{-a_1}s_2^{-a_2}\sum_{m=0}^{\infty}{}_2\Psi_1\left[\begin{array}{cc}(m+1,1),(a_2,\sigma_2) & \\ & x_2s_2^{-\sigma_2}\\ (\beta+\alpha m,\gamma); & \end{array}\right]\frac{\Gamma(a_1+\sigma_1 m)(x_1s_1^{-\sigma_1})^m}{m!}. \quad (4.6)$$

Proof.
Denote the left-hand side of equation (4.6) by I, then from the definition (1.10), we get

$$I=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!x_1^m x_2^n}{m!n!\Gamma(\alpha m+\gamma n+\beta)}\times\int_0^{\infty} z_1^{a_1+\sigma_1 m-1}e^{-s_1z_1}dz_1\times\int_0^{\infty} z_2^{a_2+\sigma_2 n-1}e^{-s_2z_2}dz_2.$$

Now, by using the integral representation of Gamma function[24]

$$a^{-z}\Gamma(z)=\int_0^{\infty}t^{z-1}e^{-at}dt,$$

we obtain

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!\Gamma(\sigma_1 m+a_1)\Gamma(\sigma_2 n+a_2)s_1^{-(a_1+\sigma_1 m)}s_2^{-(a_2+\sigma_2 n)}x_1^m x_2^n}{m!n!\Gamma(\alpha m+\gamma n+\beta)},$$

which in view of (1.17), yields the assertion (4.6).□

5.Conclusions

The principal object of this work is to present a natural further step toward the mathematical properties concerning the Mittag-Leffler function with two variables $E_{\alpha,\beta,\gamma}(z_1,z_2)$. In this regard, we obtained several properties for $E_{\alpha,\beta,\gamma}(z_1,z_2)$ including integral representations, operational relations, differential and pure recurrence relations, Euler transform, Mellin transform and Whittaker transform. We conclude this investigation by remarking the suggested schema in the derivation of the results can applied to find other new formulas for the Mittag-Leffler function with several variable $E_{(\alpha_1,\cdots,\alpha_m),\beta}(z_1,\cdots,z_m)$. Therefore, the properties of the Mittag-Leffler function with two variables $E_{\alpha,\beta,\gamma}(z_1,z_2)$ assume noticeable importance.

Acknowledgments
We thank the anonymous reviewers for their valuable suggestions, which made the presentation of the paper more readable.

Disclosure statement

No potential conflict of interest was reported by the authors.

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