Stochastic Optimisation Problems of Online Selection under Constraints

by

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Declaration

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List of publications

Many of the ideas, chapters, and sections of this thesis are based on manuscripts which are either published or in preparation, these are listed below.

- A. Gnedin and A. Seksenbayev. Diffusion approximations in the online increasing subsequence problem. `ArXiv e-print 2001.02249` (short version published in the proceedings of 31st International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms)

- A. Gnedin and A. Seksenbayev. Asymptotics and renewal approximation in the online selection of increasing subsequence. `ArXiv e-print 1904.11213` (Revised version submitted to *Bernoulli*)

- A. Seksenbayev. Refined asymptotics in the online selection of an increasing subsequence. `ArXiv e-print 1808.06300` (Revised version submitted to *Acta Applicandae Mathematica*)

Abstract

This thesis deals with several closely related, but subtly different problems in the area of sequential stochastic optimisation. A joint property they share is the online constraint that is imposed on the decision-maker: once she observes an element, the decision whether to accept or reject it should be made immediately, without an option to recall the element in future. Observations in these problems are random variables, which take values in either $\mathbb{R}$ or in $\mathbb{R}^d$, following known reasonably well-behaving continuous distributions.

The stochastic nature of observations and the online condition shape the optimal selection policy. Furthermore, the latter indeed depends on the latest information and is updated at every step. The optimal policies may not be easily described. Even for a small number of steps, solving the optimality recursion may be computationally demanding. However, a detailed investigation yields a range of easily-constructible suboptimal policies that asymptotically perform as well as the optimal one. We aim to describe both optimal and suboptimal policies and study properties of the random processes that arise naturally in these problems.

Specifically, in this thesis we focus on the sequential selection of the longest increasing subsequence in discrete and continuous time introduced by Samuels and Steele [55], the quickest sequential selection of the increasing subsequence of a fixed size recently studied by Arlotto et al. [3], and the sequential selection under a sum constraint introduced by Coffman et al. [26].

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Chapter 1

Introduction

This dissertation deals with several *Markov decision problems*. In these problems the decision-maker is faced with a task of choosing the course of action at each step of a finite or an infinite time horizon while unravelling more information. Each decision is immediate and terminal. The stochastic nature of the information forces the decision-maker to think probabilistically when assessing what the best course of action is.

In general, Markov decision problems involve optimising an objective function, sometimes subject to constraints added to the informational constraints above. A key paradigm used in the study of sequential optimisation problems is *dynamic programming*. In essence, it describes the solution to the optimisation task by breaking it down into smaller sub-tasks until the sub-task in focus is trivial. The *value functions*, measuring the optimal expected reward at each step, are then recovered by recursive calculation.

In this work we focus on the maximisation and minimisation problems subject to the *online* constraint. At each step of the selection process, the decision-maker is to choose between obtaining an immediate reward but reducing the sample of future admissible observations and forfeiting the immediate reward in hopes of maximising the future expected payoff. For example, a big part of this thesis is dedicated to the problem of choosing the longest increasing subsequence from a finite sequence of random variables

[55]. In this case the goal is to maximise the value function, which is the expected length of the selected subsequence. Observing each consequent element of the sequence, the decision-maker may choose to keep the element, thus immediately incrementing the length of the subsequence by one but constraining the future choices to be larger than this element. Alternatively, she may decide to discard the element without the possibility of retrieval. Finding the right balance between these lies at the heart of the optimisation problem.

The longest increasing subsequence problem is intimately connected with a range of well-known Markov decision problems. One example is the much-studied stochastic bin-packing problem [26]. Noticing the equivalence between the uniform case of the bin-packing problem and the longest increasing subsequence made the distributional results derived in the latter's study directly applicable to the former. Although the equivalence breaks for other distributions, it is not unreasonable to conjecture that similar limiting theorems hold for the more general bin-packing problem.

Like many other dynamic programming problems, results obtained in the study of the longest increasing subsequence are often used by computer scientists in the analysis of algorithms. One such example is the application to sorting algorithms. Specifically, to the instances when an 'almost' sorted list is required to be built dynamically, and the longest increasing subsequence takes a role of a 'sortedness' measure [34, 46, 62].

Our research answers several main questions:

- **How does the optimal value function behave in the long run?** This is a natural question in the context of Markov decision problems since most of the optimality recursions do not accept a closed-form solution. We significantly refine the existing asymptotic expansions in several sequential selection problems by virtue of a *comparison method.*

- **Are there any 'good' suboptimal strategies that are easy to describe?** The optimal strategies are often computationally complex and challenging to describe. One way of assessing their performance is by comparing it to the performance of more accessible suboptimal strategies. Throughout this work we assess the value functions of the suboptimal strategies introduced in the literature and construct new well-performing suboptimal selection policies.

- **Given the stochastic nature of the decision problems in focus, what are the statistical properties of the reward?** This question concerns the distributional properties of the total reward as a random variable. We considerably improve the asymptotic estimates of the variance via deriving the equations for the second moments of the total reward functions and subjecting them to the asymptotic comparison method. To obtain the central limit results, we reformulate the decision problems as optimisation of the control function of a piecewise deterministic Markov process and compare its asymptotic behaviour to that of a renewal process.

- **What are the limiting properties of the stochastic processes arising in the Markov decision problems?** The last major topic tackled in this work is the functional convergence of the random processes in the sequential longest increasing subsequence selection. Working directly with the infinitesimal generator of the suitably scaled *running maximum* and *length* processes, we prove the functional convergence while navigating around the singularities at the end of the horizon. We explicitly compute the cross-covariance matrix of the limiting two-dimensional process and show that the weak convergence holds for a particular class of suboptimal strategies.

1.1 Overview of the key definitions and concepts

For a more convenient and enjoyable readthrough, in this section we lay out the main concepts appearing throughout the whole dissertation in an informal fashion. A technical definition of each concept is given separately before each respective chapter, as definitions vary slightly from one problem to another.

The elements $X_1, X_2, \ldots$ of a (possibly infinite) random sample are referred to as *items*, *atoms*, or *observations*. By an *admissible observation* we mean an item whose size fits the current optimisation constraint, e.g. in the increasing subsequence problem it has to be larger than the latest selection.

In every dynamic programming problem, the *optimality equation* plays a central role. Often in the literature related to online selection problems, it is referred to as the *Bellman equation* in honour of R.E. Bellman, who formalised the optimality principle [11]. Another essential concept in optimisation problems is the *value function*; this measures the optimal expected value of the objective function given the current state of the problem, e.g. in the increasing subsequence selection, it is the expected length of a subsequence given the last selection size and the number of remaining observations. The optimality equation defines the value function of the problem.

Several types of selection policies are frequently referred to in this work. Naturally, first is the *optimal policy*; this is the policy that achieves the optimisation goal, e.g. the maximal expected length of the selected subsequence, or the minimum expected time to select a subsequence of a specific size. Clearly, in each problem the optimal policy is intrinsically related to the optimality equation.

The name of the class of *suboptimal policies* speaks for itself. Often the performance of suboptimal policies is used to derive a lower bound to the optimal performance since the former are usually easier to work with.

The policies described in this work are of the *threshold* type: at any given point in time, the decision-maker accepts an observation if its size is within a specified interval. We call this interval an *acceptance interval*, or an *acceptance window*, and the function measuring the size of this interval — a *threshold function*.

We differentiate between types of selection policies based on the form of the threshold function. The first and most straightforward type is a *stationary policy.* Its threshold functions remain constant, i.e. they do not depend on the last selection size or the remaining length of the sample. Stationary policies are the simplest in terms of definition, but, as we shall subsequently see, can be very powerful.

Another more general class of Markovian policies are *variable-threshold policies.* Threshold functions of a policy with a variable threshold are dependent on both the latest selection size and the time of the current observation. As we will learn later, the optimal policy belongs to the class of variable-threshold policies.

1.2 The problems in focus and motivation

In Chapter 2 we present several vital lemmas that form a basis for our *asymptotic comparison* method. The method is potent in the context of the Markov decision problems and is applied repeatedly throughout this dissertation. On a high level, the asymptotic comparison method is based on the idea of sandwiching the solution to the optimality equation between two carefully chosen approximating functions that satisfy corresponding inequalities. The optimality equations' inherent properties allow us to compute highly refined asymptotic expansions of the optimal value functions via multiple iterations of the method applied with more precise approximating functions. Moreover, the method does not rely on properties specific to the optimal value function equations but instead utilises tools applicable to a broader range of near-optimal selection strategies.

In Chapter 3 we consider several Markov decision problems in the discrete-time setting. The optimisation problem in the focus of Section 3.1 can be stated as follows: the decision-maker aims to maximise the expected length of an increasing subsequence selected from a finite random sample in an online fashion. In the sequel we distinguish between two versions of the problem: a *discrete-time* problem, where the selection is commenced from a sample of a fixed size $n \in \mathbb{N}$; and a *continuous-time* problem where the observations arrive with unit-rate Poisson process over a horizon of fixed length $t \in \mathbb{R}_+$.

Traditionally, these problems are studied separately; however, an intimate connection between the two is always noted in the literature. Our results highlight this connection once again. We substantially refine the asymptotic expansions of the optimal value function with the help of the comparison method. Morever, the comparison method is also utilised for assessing the expected performance of a suboptimal selection policy which is not far from optimality.

In Section 3.2 we turn to a closely related problem of minimising the expected sample size required to choose an increasing subsequence of fixed size $k \in \mathbb{N}$. Although the comparison method is not directly applicable to the optimality recursion in the quickest selection problem, a similar approach proves to be fruitful. Considerably refining the existing asymptotic estimates of the optimal performance, we construct several well-performing suboptimal selection policies.

In Section 3.3 we study a so-called *stochastic bin-packing* problem, where the goal is to choose as many elements from a random sample subject to a sum constraint. Once again appealing to the comparison method, we obtain a significantly more accurate asymptotic estimate of the value function.

In Chapter 4 we turn to the continuous-time problems. In Section 4.1 we focus on the increasing subsequence selection from a sequence of d-dimensional observations. Transforming the optimality equation into a convolution-type equation, we represent the selection problem in terms of a piecewise deterministic Markov process. Combining the comparison method with classical coupling argument, we obtain an extremely precise asymptotic estimate of the value function that goes beyond the $O(1)$-term and compute the asymptotic expansion of the variance of the length of the increasing subsequence. Moreover, utilising the renewal-type behaviour of the transformed Markov process, we show a distributional convergence of the length of the selected subsequence to a normal random variable.

In Section 4.2 we discuss the implications of the results of Section 4.1 to the more classical one-dimensional problem. We extend the results to a certain class of suboptimal selection policies specific to the one-dimensional setting and perform numerical simulations.

We dedicate Section 4.3 to the continuous-time stochastic knapsack or the bin-packing problem. By applying the methods from Section 4.1 to the bin-packing, we derive precise asymptotic expansions of the mean and the variance of the number of packed items, and obtain a central limit theorem. Section 4.4 follows the same steps in the context of the continuous-time interval parking problems.

Finally, in Chapter 5 we investigate the limiting behaviour of the running maximum process $X(t)$ and the last selection process $L(t)$ in the continuous-time longest increasing subsequence selection. Working our way around the singularities near $t = 1$, we prove that the joint process $(X(t), L(t))$ converges to a Gaussian diffusion consisting of a Brownian bridge and an Ornstein-Uhlenbeck process. Explicitly calculating the cross-covariance matrix, we derive the functional limits for the compensators and demonstrate how our results add clarity to Bruss and Delbaen's [20] functional central limit theorem.

Since this dissertation is comprised of investigations of several problems, a review of existing results is completed for each problem separately at the beginning of the respective section. However, in the rest of the introductory chapter, we survey the literature related to the classical discrete-time increasing subsequence problem, which is the main focus of our manuscript.

1.3 The offline setting

We refer to an optimisation problem with a fully observable horizon as an *offline* problem. The offline variant of the longest increasing subsequence problems is Ulam's question first posed it in [61] as an example of an exercise that is appropriate to be approached with the Monte-Carlo simulation method. We briefly go through the important contributions here; however, the detailed account of the developments in the longest increasing subsequence can be found, for example, in [54].

The first successful attempt at an analytic investigation was published by Hammersley in his seminal paper [35]. Firstly, using the Erdős-Szekeres theorem, he proved that for any permutation $\sigma_n \in S_n$, the length of the longest ascending subsequence of a permutation of n numbers $l(\sigma_n)$ satisfies

$$l(\sigma_n) > \sqrt{n}, \quad \text{for all } n \geq 2.$$

Picking a permutation σ_n uniformly at random, define L_n to be the expected length of its longest subsequence $L_n := \mathbb{E}\, l(\sigma_n)$. Via an ingenious use of the planar Poisson process and subadditivity, Hammersley proved that $\lim_{n\to\infty} l(\sigma_n)/\sqrt{n}$ exists, and, moreover,

$$\frac{L_n}{\sqrt{n}} \xrightarrow{p} c, \quad n \to \infty,$$

where c is an absolute constant. Indeed, through symmetry it is clear that the length of a descending subsequence has the same statistical properties. Thus, from now on in this work, we will only consider increasing subsequences.

In 1977, Logan and Shepp [41] and Vershik and Kerov [63] independently obtained the following result

$$\frac{l(\sigma_n)}{\sqrt{n}} \to 2, \quad \text{as } n \to \infty. \tag{1.1}$$

Alternative methods were used to obtain (1.1) in [1, 38, 56].

The next considerable refinement was achieved by Odlyzko and Rains [50], who showed that the following limit exists

$$\lim_{n\to\infty} \frac{L_n - 2\sqrt{n}}{n^{1/6}} = c_1$$

and conjectured that the constant $c_1 \approx -1.758$ based on the numerical simulations.

Finally, the investigation culminated in Baik, Deift and Johansson's result [8]

$$L_n = 2\sqrt{n} + c_1 n^{1/6} + o(n^{1/6}), \quad n \to \infty,$$

where the value of a constant $c_1 = -1.77108\ldots$ is obtained by numerically solving the Painlevé II equation. Furthermore, they showed the following surprising convergence in distribution

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{l(\sigma_n) - 2\sqrt{n}}{n^{1/6}} \le x\right) = F_{TW}(x), \quad \text{for all } x \in \mathbb{R},$$

where $F_{TW}(x)$ is the cumulative distribution function of the Tracy-Widom probability law, which was first introduced in work related to the random matrix theory [60].

1.4 The online setting

When it comes to *online* selection problems, we study the properties of subsequences selected by a non-anticipating policy from a sequence of random items $X_1, X_2, \ldots, X_n$ sampled independently from a known, reasonably well-behaved continuous distribution F. In contrast to the Ulam-Hammersley problem, the sequence is revealed one item at a time, and the decision to accept or reject X_i at time i is immediately terminal.

1.5 The longest increasing subsequence selection in discrete time

In 1981, Samuels and Steele [55] introduced the classical problem of online selection of the longest increasing subsequence. The decision-maker in their setup is sequentially inspecting elements from a finite random sample $X_1, \ldots, X_n$, where $X_i \sim F$ are independent. Suppose the i-th observation is of size $x \in \mathbb{R}_+$; then, intending to construct the longest possible increasing subsequence, the decision-maker has to decide whether to accept the item, restricting all future selections to be larger than x, or to reject the item, without the possibility of recall. Any continuous distribution F can be translated into Uniform$[0,1]$ with a monotone transformation; thus, it is sufficient to assume without loss of generality that $X_i \sim$ Uniform$[0,1]$. From now on we adhere to this assumption, though acknowledging the instance when useful properties were derived by Arlotto et al. [4] in their investigation of the case of exponentially distributed random variables.

Let $v_n : \mathbb{N} \to \mathbb{R}_+$ be the maximal expected length of an increasing subsequence chosen in an online fashion. Samuels and Steele obtained the following asymptotic result

$$v_n \sim \sqrt{2n}, \quad \text{as } n \to \infty. \tag{1.2}$$

Notation. *Here and hereafter, the asymptotic equivalence relation $\sim$ is used for asymptotic expansions written without estimate of the remainder, e.g.*

$$f(t) \sim f_1(t) + f_2(t) + \cdots + f_k(t), \quad \text{as } t \to \infty,$$

means that $f_{i+1}(t) = o(f_i(t))$ for $1 \le i < k$.

The asymptotic (1.2) was obtained in three steps. Firstly, the square-root order was established using subadditivity, similarly to Hammersley's argument in [35]. Secondly, a simple suboptimal strategy was used to obtain a lower bound on v_n. Finally, deriving certain regularity conditions of the solution to the optimality equation, a sufficient lower bound was proved.

Comparing (1.1) and (1.2), we can see that the ratio $2 : \sqrt{2}$ reflects the advantage of a 'prophet' with a full overview of a sequence $X_1, \dots, X_n$ over the rational but nonclairvoyant 'gambler' learning the sequence and making decisions in real time. A surprising feature is that this ratio is finite. For comparison, a naive decision-maker selecting every successive record obtains an expected length of the subsequence $\sum_{i=1}^n 1/i \sim \log n$, which is much more modest than both the prophet and the gambler (see [22] for details on record sequences).

Another significant discovery is Samuels and Steele's suboptimal selection policy that chooses every consecutive item within $\sqrt{2/n}$ above the latest selection. Remarkably, this simple policy achieves (1.2).

In subsequent work the asymptotics (1.2) were refined as

$$\sqrt{2n} + O(\log n) \le v_n \le \sqrt{2n}, \quad n \to \infty. \tag{1.3}$$

We should note that the log-remainder term here on the left is negative. The upper bound first appeared in the context of Bruss and Robertson's study [21] of the maximal expected number of elements whose sum does not exceed a specified value. Later, this upper bound was obtained in a completely different fashion in Gnedin's investigation [33] of online selection from a sample of random size.

The $O(\log n)$-term in the lower-bound was first hinted at by Bruss and Delbaen's result obtained in the study of the continuous-time version of the problem [19]. This continuous-time analogue of Samuels and Steele's original problem is the main focus of Chapter 3. Although the two problems are very similar, Bruss and Delbaen's lower bound was not immediately applicable to v_n. A bridge between the two problems was constructed by Arlotto et al. [4]; they derived the lower bound for v_n and used it to prove the central limit theorem for the length of an optimally chosen subsequence L_n,

$$\frac{\sqrt{3}\,\{L_n - \sqrt{2n}\}}{(2n)^{1/4}} \xrightarrow{d} \mathcal{N}(0,1), \quad \text{as } n \to \infty.$$

The tightest lower-bound on v_n known today was derived by Arlotto et al. [5]

$$\sqrt{2n} - 2\log n - 2 \le v_n.$$

To narrow the gap, Arlotto et al. assessed a considerably more involved online selection policy, which has the size of acceptance interval dependent on both the number of remaining observations $n - i$ and the size of the last selection. Based on an extensive numerical simulation, Arlotto et al. also suggested that their policy is, in fact, within a constant off optimality.

Chapter 2

Asymptotic comparison

In this chapter we present the key lemmas forming a basis of our method of *asymptotic comparison*. This method proves to be very powerful when applied in the context of several sequential selection problems, and we use it on numerous occasions throughout this thesis.

We utilise the comparison method to approximate asymptotically the solutions to difference and differential equations satisfying particular monotonicity properties. The procedure is reminiscent of a familiar method of successive estimation of the solution to the differential equations (see, for example, [18], Section 9.1). The following lemmas should cover most of our needs.

We will consider sequences of functions $f_n : \mathbb{D} \to \mathbb{R}$ on some set $\mathbb{D}$ with $\sup_{z \in \mathbb{D}} |f_n(z)| < \infty$, so each f_n is bounded for $n = 0, 1, \ldots$. For every $n \geq 0, z \in \mathbb{D}$ let $G_{n+1}(f_0, \ldots, f_n)(z)$ be a functional possessing the following properties

(i) *Shift-invariance:* $G_{n+1}(f_0 + c, \ldots, f_n + c)(z) = G_{n+1}(f_0, \ldots, f_n)(z) + c$ for any constant c,

(ii) *Monotonicity:* if $\widehat{f}_0 \geq f_0, \ldots, \widehat{f}_n \geq f_n$, then $G_{n+1}(\widehat{f}_0, \ldots, \widehat{f}_n)(z) \geq G_{n+1}(f_0, \ldots, f_n)(z)$.

For example, we will consider the optimality equation in the longest increasing subsequence problem. With

$$G_{n+1}(f_n)(z) := zf_n(z) + \int_z^1 \max\{f_n(x)+1, f_n(z)\}\,\mathrm{d}x$$

and $\mathbb{D} := [0,1]$, the equation becomes $f_{n+1}(z) = G_{n+1}(f_n)(z),\ f_0(z) = 0$. The operator G_n satisfies both conditions (i) and (ii). It satisfies the former inherently from definition and the latter by monotonicity of the value functions specific to the problem. Intuitively, the shift-invariance indicates that if we were to shift the starting number of selections by a constant, the resulting sequence of value functions would shift precisely by this constant.

Now, suppose space $\mathbb{D}$ is equipped with *size* functions $g_n : \mathbb{D} \to \mathbb{R}$. We may say (n, z) is sufficiently large meaning that $g_n(z) > c$ for suitable constant c. We do not require $z \to g_n(z)$ to be bounded for given n, so sufficiently large may refer to small n. The role of function $g_n(z)$ is to determine the range of limit regimes for (n, z) as $n \to \infty$.

Definition 1. *For a given space $\mathbb{D}$ equipped with a size function $g_n(z)$, we say that a sequence of functions (f_n) is* locally bounded from above *if*

$$\sup_{(n,z):g_n(z)\le c} f_n(z) < \infty,$$

for every $c > 0$, locally bounded from below *if $(-f_n)$ is locally bounded from above, and* locally bounded *if $|f_n|$ is locally bounded from above.*

Assume we have a sequence of functions $f_0, f_1, \ldots$, such that

(a) $f_n(z)$ are locally bounded,

(b) $f_{n+1}(z) = G_{n+1}(f_0, \ldots, f_n)(z)$ for all $n \ge 0, z \in \mathbb{D}$.

The idea is to compare a solution to (b) to a sequence asymptotically satisfying an analogous inequality. This is where the monotonicity property (ii) comes into play. In simplest terms, it ensures that if an approximating sequence $\widehat{f}_n$ is above f_n, it will continue growing at least as quickly as f_n.

Lemma 1. *Suppose a sequence of functions $\widehat{f}_0, \widehat{f}_1, \ldots$ is locally bounded from below and $\widehat{f}_{n+1}$ satisfies $\widehat{f}_{n+1} \geq G_{n+1}(\widehat{f}_0, \ldots, \widehat{f}_n)(z)$ when (n, z) is sufficiently large. Then, the difference $f_n(z) - \widehat{f}_n(z)$ is bounded from above uniformly for all n and z. Similarly, if $\widehat{f}_n$ are locally bounded from above and $\widehat{f}_{n+1} \leq G_{n+1}(\widehat{f}_0, \ldots, \widehat{f}_n)(z)$ when (n, z) is sufficiently large, then $f_n(z) - \widehat{f}_n(z)$ is bounded from below uniformly for all n and z.*

Proof. Adding a constant if necessary and using the shift-invariance property (i) of the functional, we can reduce to the case $\widehat{f}_n(z) > 0$.

If the claim of the lemma is not true, then for every constant $c > 0$ we can find n_0, z_0 such that $f_{n_0+1}(z_0) - \widehat{f}_{n_0+1}(z_0) > c$. Choose the minimal such $n_0 = n_0(c)$, then

$$f_j(z) \leq \widehat{f}_j(z) + c \quad \text{for } j \leq n_0,\ z \in \mathbb{D}.$$

From assumption (a) and local boundedness of (f_n) one can deduce $n_0 = n_0(c) \to \infty$ as $c \to \infty$. To that end, observe that

$$f_{n_0+1}(z_0) > \widehat{f}_{n_0+1}(z_0) + c > c.$$

Since (f_n) are locally bounded, we have $g_{n_0+1}(z_0) \geq f_{n_0+1}(z_0) \geq c$, so we can choose c large enough to achieve $G_{n_0+1}(\widehat{f}_0, \ldots, \widehat{f}_{n_0})(z_0) < \widehat{f}_{n_0+1}(z_0)$. Therefore, appealing to shift invariance, we obtain

$$G_{n_0+1}(\widehat{f}_0 + c, \ldots, \widehat{f}_{n_0} + c)(z_0) = G_{n_0+1}(\widehat{f}_0, \ldots, \widehat{f}_{n_0})(z_0) + c \leq \widehat{f}_{n_0+1}(z_0) + c < f_{n_0+1}(z_0).$$

However, by the choice of n_0, z_0 and the monotonicity property (ii), we have

$$f_{n_0+1}(z_0) = G_{n_0+1}(f_0, \ldots, f_{n_0})(z_0) \leq G_{n_0+1}(\widehat{f}_0, \ldots, \widehat{f}_{n_0})(z_0),$$

which is a contradiction. The second part of the lemma can be proved analogously. □

To derive similar results for the differential equations, we first need to prove the following elementary lemma.

Lemma 2. *Suppose* $f \in C^1(\mathbb{R}_+)$ *satisfies* $\limsup_{t\to\infty} f(t) = \infty$*. Then there exists an arbitrarily large* $x > 0$*, such that for some* t

(a) $f(s) < f(t) = x$ *for* $0 \leq s \leq t$*,*

(b) $f'(t) > 0$.

Proof. Let $g(t) := \max_{s\in[0,t]} f(s)$ be the running maximum of $f(t)$. For $x > f(0)$ let

$$l(x) = \min\{t : g(t) = x\}, \quad r(x) = \max\{t : g(t) = x\},$$

which are well defined because g is nondecreasing and by the assumption satisfies $g(t) \to \infty$ as $t \to \infty$. So $l(x) \leq r(x)$ and $f(r(x)) = f(l(x)) = x$. If neither $f'(l(x)) > 0$, nor $f'(r(x)) > 0$, then $g'(t) = 0$ for $l(x) \leq t \leq r(x)$. Now if the latter holds for all sufficiently large x, then $g'(t) = 0$ for all large enough t, but this is only possible if f is bounded from above, which is a contradition. □

Now, consider functions $f \in C^1[0,\infty)$ and suppose a functional $\mathcal{I}_z f = \mathcal{I}_z(f|_{[0,z]})$ possesses the following properties

(i) *Shift-invariance:* $\mathcal{I}_z(f + c) = \mathcal{I}_z f$ for any constant c,

(ii) *Monotonicity:* if $\widehat{f}(s) \geq f(s)$ for all $0 \leq s \leq z$, then $\mathcal{I}_z\widehat{f} \geq \mathcal{I}_z f$.

For functions f satisfying the equation $f'(z) = \mathcal{I}_z f$, we have the following result.

Lemma 3. *Let the function* f *be a solution to* $f'(z) = \mathcal{I}_z f$ *and suppose* $\widehat{f}$ *satisfies* $\widehat{f}'(z) \geq \mathcal{I}_z\widehat{f}$ *for all sufficiently large* z*. Then* $\sup_{z\in\mathbb{R}_+} (f(z) - \widehat{f}(z)) < \infty$*. Likewise, if*

$\widehat{f}'(z) \le \mathcal{I}_z \widehat{f}$ *for all sufficiently large* z, *then* $\inf_{z\in\mathbb{R}_+} (f(z) - \widehat{f}(z)) > -\infty$.

Proof. Assume to the contrary that $\widehat{f}'(z) \ge \mathcal{I}_z \widehat{f}$ for $z > z_0$ but the difference $f(z) - \widehat{f}(z)$ is not bounded from above. Then, by Lemma 2, we can find a constant c large enough to achieve that $z_1 := \min\{z : f(z) = \widehat{f}(z) + c\}$ satisfies $z_1 > z_0$ and $f'(z_1) > \widehat{f}'(z_1)$. By the properties (i) and (ii) of the operator $\mathcal{I}$, we have

$$f'(z_1) = \mathcal{I}_{z_1} f \le \mathcal{I}_{z_1}(\widehat{f} + c) = \mathcal{I}_{z_1} \widehat{f} \le \widehat{f}'(z_1).$$

However, this is a contradiction since $f'(z_1) > (\widehat{f} + c)'(z_1) = \widehat{f}'(z_1)$. The second part of the lemma is argued similarly. □

For example, we will consider equations of the form

$$f'(z) = \int_0^z (f(z-y) + 1 - f(z))_+ \, \mu(z, \mathrm{d}y),$$

where $\mu(z, \mathrm{d}y)$ is some probability measure. Lemma 3 is then applied with

$$\mathcal{I}_z f := \int_0^z (f(z-y) + 1 - f(z))_+ \, \mu(z, \mathrm{d}y)$$

and a suitable choice of an approximating function $\widehat{f}$.

Chapter 3

Discrete-time selection problems

3.1 The longest increasing subsequence selection

We formalise now the definitions of main concepts and the notation used in this chapter. Let $X_i,\ i = 1, \ldots, n$ be independent distributional copies of $X \sim \text{Uniform}[0, 1]$.

Definition 2. *An* online selection policy *is a collection of stopping times* $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots)$

(i) *adapted to the sequence of sigma-fields* $\mathcal{F}_i = \sigma\{X_1, X_2, \ldots, X_i\},\ 1 \le i < \infty$,

(ii) *satisfying* $\tau_1 < \tau_2 < \ldots$.

An online policy $\boldsymbol{\tau}$ *is called* admissible *in the increasing subsequence problem if it also satisfies*

(iii) $X_{\tau_1} < X_{\tau_2} < \ldots$.

Definition 3. *Let us set* $\tau_0 := 0$ *and* $X_{\tau_0} := 0$, *and let* $\{h_m(z) : 1 \le m \le n\}$ *be a sequence of* threshold functions *satisfying* $h_{n-i}(z) \le 1 - z$ *for all* $i = 0, \ldots, n-1$, *where* z *is the size of the last selection made so far. Then, a* threshold policy, *uniquely characterised by its corresponding threshold functions, is an online selection policy, which* *has its stopping times defined recursively as*

$$\tau_j = \min\left\{\tau_{j-1} < i \leq n : X_i \in \left[X_{\tau_{j-1}}, X_{\tau_{j-1}} + h_{n-i+1}(X_{\tau_{j-1}})\right]\right\}$$

with the convention that $\min\emptyset = \infty$*. Clearly, for a threshold policy to be admissible, its threshold functions must be positive.*

A somewhat cumbersome stopping times notation above defines a simply described selection strategy that at time i when $n-i$ observations remain to be inspected and the size of the last selection is z, accepts the observation X_i if and only if it falls within the acceptance interval

$$z \leq X_i \leq z + h_{n-i+1}(z).$$

Depending on the form of the threshold functions, we differentiate between the types of threshold policies.

Definition 4. *A threshold policy is* stationary *if its threshold functions are of the form* $h_{n-i}(z) = \min\{c, 1-z\}$ *for all* $i = 0, \ldots, n-1$*, where* c *is a fixed constant.*

Definition 5. *Let* $v^{(h)}_{n-i}(z) : [0,1] \to \mathbb{R}_+$ *denote the expected length of an increasing subsequence built by a threshold policy with threshold functions* $h_{n-i}(z)$*. We call* $v^{(h)}_{n-i}$ *the* value functions *of this threshold policy.*

Definition 6. *Let* $L_n(\boldsymbol{\tau_z})$ *be the number of selections made by an admissible policy* $\boldsymbol{\tau_z}$*. Then, the* optimal value function $v_n(z)$ *is defined by taking a supremum over all admissible policies* $v_n := \sup_{\boldsymbol{\tau_z}} \mathbb{E}L_n(\boldsymbol{\tau_z})$*.*

We mark the value functions of suboptimal threshold policies with a superindex indicating the corresponding threshold functions. In contrast, the optimal value functions have no superindex.

3.1.1 The optimality equation

Assume we have $n+1$ elements remaining to inspect and the last selected element so far is of size z. Then, the optimality equation is a recursion [4, 5, 55]

$$v_{n+1}(z) = z\ v_n(z) + \int_z^1 \max\{v_n(x)+1, v_n(z)\}\,\mathrm{d}x, \quad v_0(z) = 0. \tag{3.1}$$

We record here a trivial fact that $v_1(z) = \mathbb{P}(X_1 \geq z) = 1-z$ and note the *shift-invariance* of the solution: $v_n(z)+c$ also satisfies (3.1) for any constant c.

The intuition behind (3.1) is as follows. At the current state of selection, with probability z the next observed item is inadmissible, therefore leaving the expected length at $v_n(z)$. On the other hand, if the observation is admissible, then the dynamic programming principle prescribes the decision-maker to choose whatever provides a larger expected length of a sequence: keeping the item and thus increasing the length by 1, or discarding it and thus leaving it at $v_n(z)$ by the optimal continuation. Averaging over the uniformly distributed observation yields the integral on the right-hand side of (3.1). Observe that $v_n(z)$ can also be viewed as the maximum expected length of an increasing subsequence chosen from N items, with $N \stackrel{d}{=} \mathrm{Bin}(n, 1-z)$ (see Gnedin [33], p. 945 and Samuels and Steele [55], p. 1083).

Let $h_n(z) : [0,1] \to [0,1]$ be the solution to

$$v_n(z+x) + 1 = v_n(z) \tag{3.2}$$

if $v_n(z) > 1$, and $h_n(z) = 1-z$ otherwise. The value function $v_n(z)$ is monotonically decreasing in z. This can be shown by the following coupling argument. Assume we have $z_1 < z_2$ and wish to compare $v_n(z_1)$ to $v_n(z_2)$. In the problem with the last selection size z_1, the class of admissible policies includes all of the admissible policies in the more restrictive case with the last selection size z_2. Moreover, there are additional admissible policies in the former case that are not available in the latter. Thus, referring to the

Definition 6, we can conclude that $v_n(z_1) \geq v_n(z_2)$ ([4], Lemma 3 provides a technical proof of the monotonicity based on the analysis of the optimality equation). Therefore, the integrand in (3.1) is equal to $v_n(x)+1$ on the interval $[z, z+h_n(z)]$. On the remaining interval $[z+h_n(z), 1]$, the integrand assumes value $v_n(z)$.

This provides the form of the optimal selection policy: accept the observation x if it falls into the acceptance window $[z, z+h_n(z)]$. From (3.2) it can be seen that the acceptance window is updated dynamically with every observation. Thus, the optimal policy indeed belongs to the class of policies with a variable acceptance window, and we call functions $h_n(z)$ the *optimal threshold functions*. Note, the equation (3.2) has a solution only when $v_n(z) > 1$. This has a logical interpretation: when $v_n(z) \leq 1$, the decision-maker should select every admissible observation whatever its value as this provides the largest expected payoff.

3.1.2 Asymptotic expansion of the optimal value function

In the sequel we work directly with the optimality equation (3.1) to refine the expansion (1.2). The plan is to exploit the asymptotic comparison method, for which we laid the foundation in Chapter 2.

Given a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}_+$ introduce a forward-difference operator Δ and an integral operator $\mathcal{P}$ acting on $f_n(z)$ as

$$\Delta f_n(z) := f_{n+1}(z) - f_n(z), \quad \mathcal{P} f_n(z) := \int_z^1 (f_n(x) + 1 - f_n(z))_+ \mathrm{d}x,$$

respectively. With this notation, the optimality equation (3.1) assumes the form

$$\Delta v_n(z) = \mathcal{P} v_n(z), \quad v_0(z) = 0. \tag{3.3}$$

We specialise Lemma 1 to obtain the following corollary.

Corollary 1. *If for $n(1-z)$ large enough, $\Delta f_n(z) > \mathcal{P} f_n(z)$, then the difference $v_n(z) -$*

$f_n(z)$ is bounded from above uniformly in n and z. Likewise, if $\Delta f_n(z) < \mathcal{P} f_n(z)$ when $n(1-z)$ is large, then $v_n(z) - f_n(z)$ is bounded from below uniformly in n and z.

Proof. The result is obtained by applying Lemma 1 with $G_{n+1}(f_n)(z) := f_n(z) + \mathcal{P} f_n(z)$, and the size function $g_n(z) := n(1-z)$. □

We apply Corollary 1 to compare $v_n(z)$ with a sequence of suitable test functions. With every iteration we choose an approximating function that refines the asymptotic expansion of $v_n(z)$.

Notation. *Because we work with the expansions of $v_n(z)$ when (n, z) is large enough, introduce for convenience $\widehat{n} := n(1-z)$.*

To obtain the principal asymptotics, consider the test function

$$v_n^{(0)}(z) := \gamma_0 \sqrt{n(1-z)} = \gamma_0 \sqrt{\widehat{n}},$$

where $\gamma_0 \in \mathbb{R}_+$ is a parameter. Expanding for large $\widehat{n}$, we obtain

$$\Delta v_n^{(0)}(z) \sim \gamma_0 \frac{1-z}{2\sqrt{\widehat{n}}}. \tag{3.4}$$

On the other hand, using the change of variable $y := (x-z)/(1-z)$, we can write the integral as

$$\begin{aligned}
\mathcal{P} v_n^{(0)}(z) &= (1-z) \int_0^1 \left(\gamma_0 \sqrt{\widehat{n} - \widehat{n} y} - \gamma_0 \sqrt{\widehat{n}} + 1 \right)_+ \mathrm{d}y \\
&= (1-z) \int_0^{h_n^{(0)}(z)} \left(\gamma_0 \sqrt{\widehat{n} - \widehat{n} y} - \gamma_0 \sqrt{\widehat{n}} + 1 \right) \mathrm{d}y,
\end{aligned}$$

where $h_n^{(0)}(z)$ is the solution to

$$\gamma_0 \sqrt{\widehat{n}(1-x)} - \gamma_0 \sqrt{\widehat{n}} + 1 = 0.$$

For $\widehat{n} \to \infty$, we have

$$h_n^{(0)}(z) \sim \frac{2}{\gamma_0 \sqrt{\widehat{n}}}, \quad \widehat{n} \to \infty. \tag{3.5}$$

Using Taylor expansion in y around 0 to estimate the integrand yields

$$1 - \gamma_0 \sqrt{\widehat{n}}\, \frac{y}{2} + O(\widehat{n}^{-1/2});$$

hence, integrating and using (3.5), we arrive at

$$\mathcal{P}v_n^{(0)}(z) \sim \frac{1-z}{\gamma_0 \sqrt{\widehat{n}}}, \quad \widehat{n} \to \infty. \tag{3.6}$$

The match between (3.4) and (3.6) occurs for $\gamma_0 = \sqrt{2}$; therefore, we have, for $n(1-z)$ large enough,

$$\Delta v_n^{(0)}(z) > \mathcal{P}v_n^{(0)}(z), \quad \text{when } \gamma_0 > \sqrt{2},$$
$$\Delta v_n^{(0)}(z) < \mathcal{P}v_n^{(0)}(z), \quad \text{when } \gamma_0 < \sqrt{2}.$$

Applying Corollary 1, we see that

$$\limsup_{n(1-z)\to\infty} \left(v_n(z) - \gamma_0 \sqrt{n(1-z)}\right) < \infty.$$

In light of this,

$$\limsup_{n(1-z)\to\infty} \frac{v_n(z)}{\sqrt{n(1-z)}} \le \gamma_0, \quad \text{for } \gamma_0 > \sqrt{2}.$$

Consequently,

$$\limsup_{n(1-z)\to\infty} \frac{v_n(z)}{\sqrt{n(1-z)}} \le \sqrt{2}. \tag{3.7}$$

A parallel argument with $\gamma_0 < \sqrt{2}$ yields

$$\liminf_{n(1-z)\to\infty} \frac{v_n(z)}{\sqrt{n(1-z)}} \ge \sqrt{2}. \tag{3.8}$$

Combining (3.7) with (3.8), we obtain

$$v_n(z) \sim \sqrt{2n(1-z)}, \quad n(1-z) \to \infty.$$

For a better approximation we consider the test function

$$v_n^{(1)}(z) = \sqrt{2n(1-z)} + \gamma_1 \log\left(n(1-z)+1\right)$$

with $\gamma_1 \in \mathbb{R}$. We choose $\log(\widehat{n}+1)$ over $\log \widehat{n}$ to avoid the annoying singularity at 0. The forward difference becomes

$$\Delta v_n^{(1)}(z) = \sqrt{2n(1-z)} \left(\left(1 + \frac{1}{n}\right)^{1/2} - 1 \right) + \gamma_1 \log\left(1 + \frac{1}{n(1-z)+1} + \frac{1}{n}\right).$$

Using Taylor expansion with a remainder yields

$$\Delta v_n^{(1)}(z) = \frac{1-z}{\sqrt{2\widehat{n}}} + \gamma_1 \frac{1-z}{\widehat{n}+1} + O(\widehat{n}^{-3/2}), \quad \widehat{n} \to \infty. \tag{3.9}$$

On the other hand, using a substitution $y = (x-z)/(1-z)$ it can be deduced that

$$\begin{aligned} &\mathcal{P} v_n^{(1)}(z) = \\ &= (1-z) \int_0^1 \left(\sqrt{2\widehat{n}} \left((1-y)^{1/2} - 1 \right) + \gamma_1 \log\left(\widehat{n}(1-y)+1\right) - \gamma_1 \log\left(\widehat{n}+1\right) + 1 \right)_+ \mathrm{d}y \\ &= (1-z) \int_0^{h_{\widehat{n}}^{(1)}(z)} \left(\sqrt{2\widehat{n}} \left((1-y)^{1/2} - 1 \right) + \gamma_1 \log\left(\widehat{n}(1-y)+1\right) - \gamma_1 \log\left(\widehat{n}+1\right) + 1 \right) \mathrm{d}y, \end{aligned} \tag{3.10}$$

where $h_n^{(1)}(z)$ solves

$$\sqrt{2\widehat{n}} \left((1-y)^{1/2} - 1 \right) + \gamma_1 \log\left(\widehat{n}(1-y)+1\right) - \gamma_1 \log\left(\widehat{n}+1\right) + 1 = 0. \tag{3.11}$$

For $\widehat{n} \to \infty$,

$$h_n^{(1)}(z) = \sqrt{\frac{2}{\widehat{n}}} - \left(\frac{1}{2} + 2\gamma_1\right)\frac{1}{\widehat{n}+1} + O(\widehat{n}^{-3/2}). \tag{3.12}$$

Actually, we only need the first term of (3.12) to obtain the expansion of $\mathcal{P}v_n^{(1)}(z)$ with the desired accuracy. This is down to the fact that order $O(\widehat{n}^{-1})$-term in (3.12) contributes only $O(\widehat{n}^{-3/2})$ to $\mathcal{P}v_n^{(1)}(z)$. Indeed, keeping $\widehat{n}$ as a parameter, let us view the integral on the third line of (3.10) as a function of its upper limit

$$I(h) := (1-z)\int_0^h \left(\sqrt{2\widehat{n}}\left((1-y)^{1/2}-1\right) + \gamma_1 \log\left(\frac{\widehat{n}(1-y)+1}{\widehat{n}+1}\right) + 1\right) \mathrm{d}y.$$

In view of (3.11), $h_1 := h_n^{(1)}(z)$ is a stationary point of the integrand. Expanding at h_1 with a remainder we get for some $\zeta \in [0,1]$

$$\begin{aligned} I(h_1+\varepsilon) - I(h_1) &= I'(h_1)\,\varepsilon + I''(h_1+\zeta\varepsilon)\,\frac{\varepsilon^2}{2} \\ &= (1-z)\left(\frac{\sqrt{2\widehat{n}}}{2\sqrt{1-(h_1+\zeta\varepsilon)}} - \frac{\gamma_1}{1-(h_1+\zeta\varepsilon)}\right)\frac{\varepsilon^2}{2}. \end{aligned}$$

Now, letting $\widehat{n} \to \infty$ and setting $\varepsilon := O(\widehat{n}^{-1})$ we obtain

$$I(h_1+\varepsilon) - I(h_1) = O(\widehat{n}^{-3/2}),$$

as claimed. In light of this, integrating and expanding we obtain

$$\mathcal{P}v_n^{(1)}(z) \sim \frac{1-z}{\sqrt{2\widehat{n}}} - \left(\gamma_1 + \frac{1}{6}\right)\frac{1-z}{\widehat{n}+1}, \quad \widehat{n} \to \infty. \tag{3.13}$$

Expansions (3.9) and (3.13) match at $\gamma_1 = -1/12$. Thus, another application of Corollary 1 gives us the refinement

$$v_n(z) \sim \sqrt{2n(1-z)} - \frac{1}{12}\log\left(n(1-z)\right), \quad n(1-z) \to \infty.$$

We need one more iteration to bound the remainder. Consider the test functions

$$v_n^{(2)}(z) = \sqrt{2n(1-z)} - \frac{1}{12}\log\left(n(1-z)+1\right) + \gamma_2 \frac{1}{\sqrt{n(1-z)+1}}, \quad \gamma_2 \in \mathbb{R}.$$

For $\widehat{n} \to \infty$, we obtain the expansion

$$\Delta v_n^{(2)}(z) \sim \frac{1-z}{\sqrt{2\widehat{n}}} - \frac{1-z}{12(\widehat{n}+1)} + \frac{1-z}{(\widehat{n}+1)^{3/2}}\left(-\frac{\gamma_2}{2} - \frac{(1-z)\sqrt{2}}{8}\right), \tag{3.14}$$

uniformly in $z \in [0,1)$, and, with some more effort, for the integral operator

$$\mathcal{P}v_n^{(2)}(z) \sim \frac{1-z}{\sqrt{2\widehat{n}}} - \frac{1-z}{12(\widehat{n}+1)} + \frac{1-z}{(\widehat{n}+1)^{3/2}}\left(\frac{\gamma_2}{2} + \frac{35\sqrt{2}}{144} - \frac{\sqrt{2}}{4}\right). \tag{3.15}$$

Since $z \in [0,1)$, we have

$$-\frac{\gamma_2}{2} - \frac{\sqrt{2}}{8} \leq -\frac{\gamma_2}{2} - \frac{(1-z)\sqrt{2}}{8} < -\frac{\gamma_2}{2}. \tag{3.16}$$

Appealing to (3.14), (3.15) and the first inequality in (3.16), we conclude that for large $n(1-z)$

$$\Delta v_n^{(2)}(z) > \mathcal{P}v_n^{(2)}(z), \quad \text{for } \gamma_2 \leq \frac{\sqrt{2}}{144} - \frac{\sqrt{2}}{8};$$

hence, by Corollary 1, $v_n(z) - v_n^{(2)}(z)$ for such γ_2 is bounded from above. On the other hand, exploiting the second inequality in (3.16), we derive that for large $n(1-z)$

$$\Delta v_n^{(2)}(z) < \mathcal{P}v_n^{(2)}(z), \quad \text{for } \gamma_2 \geq \frac{\sqrt{2}}{144};$$

thus, by Corollary 1 $v_n(z) - v_n^{(2)}(z)$ for such γ_2 is bounded from below. However, since the last term of $v_n^{(2)}(z)$ is already bounded, it follows readily that

$$v_n(z) = \sqrt{2n(1-z)} - \frac{1}{12}\log\left(n(1-z)\right) + O(1), \quad n(1-z) \to \infty. \tag{3.17}$$

The main result of this section is the special case of (3.17) with $z = 0$.

Theorem 1. *The maximum expected length of an increasing subsequence chosen in an online fashion* $v_n = v_n(0)$ *satisfies*

$$v_n = \sqrt{2n} - \frac{1}{12}\log n + O(1), \quad n \to \infty. \tag{3.18}$$

Theorem 1 refines the most recent expansion (1.3) significantly. However, our comparison method does not anywhere use the initial condition $v_0(z) = 0$. Thus, the expansion is limited by order of the effect of the shift in the initial condition, which is indeed $O(1)$.

3.1.3 A variable-threshold policy

As was demonstrated in the previous section, the optimal selection strategy is the variable-threshold policy with threshold functions $h_{n-i+1}(z)$ solving

$$v_{n-i}(z + x) + 1 = v_{n-i}(z), \quad v_{n-i}(z) > 1.$$

However, there are good policies that can be defined more simply. For example, the *stationary policy* of Samuels and Steele [55] has constant threshold functions independent of the remaining sample size. It accepts every observation that exceeds the last selection by no more than $\sqrt{2/n}$. Setting threshold functions $\widetilde{h}_{n-i+1}(z) := \min\{\sqrt{2/n}, 1 - z\}$ for all $i = 1, 2, \ldots, n$ describes this strategy completely. The minimum ensures that the policy is feasible.

Remarkably, this uncomplicated policy achieves asymptotic optimality up to the leading order term of the expected performance. The intuition behind the choice of this acceptance interval can be demonstrated by working out the *mean-constraint* bound on v_n. From Definition 3 it follows that

$$v_n = \mathbb{E}\sum_{i=1}^{n} \mathbb{1}\left(X_i \in [Z_{i-1}, Z_{i-1} + h_{n-i+1}(Z_{i-1})]\right) = \sum_{i=1}^{n} \mathbb{E}h_{n-i+1}(Z_{i-1}), \tag{3.19}$$

where Z_i is the last selected element at observation i, with the convention $Z_0 = 0$.

Moreover, we have

$$\mathbb{E}\left(Z_i - Z_{i-1} | Z_{i-1}\right) = \frac{\left(h_{n-i+1}(Z_{i-1})\right)^2}{2};$$

thus, taking the expectation and telescoping gives

$$\frac{1}{2}\sum_{i=1}^{n} \mathbb{E}\left(\left(h_{n-i+1}(Z_{i-1})\right)^2\right) = \mathbb{E}Z_n \leq 1, \tag{3.20}$$

where the last bound comes as a trivial consequence of the state space of Z_n. However, (3.20) is a relaxation of the more restrictive constraint $Z_n \leq 1$ in the increasing subsequence selection. Thus, solving the optimisation problem of the form

$$\sum_{i=1}^{n} \varphi_i \to \max, \quad \text{subject to} \quad \sum_{i=1}^{n} \varphi_i^2 \leq 2, \tag{3.21}$$

where φ_i is a sequence of variables with $\varphi_i \geq 0,\ i = 1, \ldots, n$ yields an upper bound for v_n. The unique solution $\varphi_i^* = \sqrt{2/n},\ i = 1, \ldots, n$ with the corresponding optimal value $\sqrt{2n}$ is easily obtained via convex optimisation.

By Jensen's inequality,

$$\left(\mathbb{E}h_{n-i+1}(Z_{i-1})\right)^2 \leq \mathbb{E}\left(h_{n-i+1}(Z_{i-1})\right)^2;$$

therefore, given (3.20), the sequence of variables $y_i = \mathbb{E}h_{n-i+1}(Z_{i-1}),\ i = 1, \ldots, n$ is a feasible solution to the optimisation problem (3.21). So, from (3.19),

$$v_n \leq \sqrt{2n}. \tag{3.22}$$

Note, the optimal solution φ_i^* was used by Samuels and Steele as a value for the threshold functions of their stationary policy. A delicate point here is that the constant threshold functions may not fit the feasiblity constraint $h_{n-i}(z) \leq 1 - z$. One needs to consider a greedy selection policy for when the feasibility constraint is violated by the constant threshold.

A considerably more sophisticated policy was introduced by Arlotto et al. [5]. In contrast to the stationary policy of Samuels and Steele, the acceptance window here is variable. The acceptance criterion for this policy is

$$z < x \leq z + \overline{h}_{n-i+1}(z),$$

where

$$\overline{h}_{n-i+1}(z) = \min\left\{\sqrt{\frac{2(1-z)}{n-i}}, 1-z\right\} \quad \text{for } i = 1, \dots, n. \tag{3.23}$$

Observe that normalising (3.23) leads to the acceptance condition

$$0 < \frac{x-z}{1-z} < \sqrt{\frac{2}{(n-i)(1-z)}},$$

i.e. the threshold functions are similar to Samuels and Steele's with one exception: Arlotto et al. used the expected number of remaining admissible observations $(n-i)(1-z)$ in the calculation of threshold functions.

The value functions corresponding to Arlotto et al.'s policy satisfy the recursion

$$v_{n+1}^{(\overline{h})}(z) - v_n^{(\overline{h})}(z) = \int_z^{z+\overline{h}_n(z)} \left(v_n^{(\overline{h})}(x) - v_n^{(\overline{h})}(z) + 1\right) \mathrm{d}x, \quad v_0^{(\overline{h})}(z) = 0.$$

By virtue of Lemma 1, this equation can be analysed analogously to (3.3), leading to the same expansion as in (3.17)

$$v_n^{(\overline{h})}(z) = \sqrt{2n(1-z)} - \frac{1}{12}\log n(1-z) + O(1).$$

Taken together with (3.17) this settles the conjecture in [5].

Theorem 2. *As n gets large, the policy with threshold functions* (3.23) *has the expected* *performance* $v_n^{(\bar{h})} = v_n^{(\bar{h})}(0)$ *satisfying*

$$|v_n - v_n^{(\bar{h})}| = O(1), \quad n \to \infty.$$

3.2 The quickest increasing subsequence selection

We now turn to the quickest selection problem studied recently by Arlotto et al. [3]. In contrast to Samuels and Steele's problem, Arlotto et al. asked to find the minimum expected time β_k to choose a k-long increasing subsequence from an infinitely long sequence of random observations in an online fashion. Reformulating the goal as minimising the expected size of the random sample required to choose a said sequence highlights a natural duality to the Samuels-Steele problem.

Via an analytical tour de force, Arlotto et al. [3] obtained the following bounds on the optimal value function

$$\frac{k^2}{2} \leq \beta_k \leq \frac{k^2}{2} + O(k \log k), \quad \text{for all } k \geq 2, \tag{3.24}$$

and the following asymptotic expansion for the variance, as $k \to \infty$,

$$\mathrm{Var}(\tau_k) = \frac{k^3}{3} + O(k^2 (\log k)^\alpha), \quad \text{for all } \alpha > 2.$$

The quickest selection problem of Arlotto et al. [3] is equivalent to a special case of the sum constraint problem of Chen et al. [24] with the uniformly distributed observations and a unit sum constraint (see Example 2 on p. 541 for the $k^2/2$ asymptotics and Mallows et al. [43] for a multidimensional extension). So the principal asymptotics $k^2/2$ can be read off from this earlier work. Coffman et al. [26] in Section 6 showed that the same asymptotics $k^2/2$ also occur in the *offline* quickest selection problem.

Definition 7. *Fix* $z \in (0, 1)$. *The optimal value function* $\beta_k(z)$ *in the quickest selection problem is the minimal expected size of the sample required to select a monotone* *subsequence of length k with items above z only.*

Note that $\beta_k = \beta_k(0)$. In stopping time notation the optimal value function $\beta_k = \inf_{\boldsymbol{\tau}} \mathbb{E}\tau_k$, where the infinum is taken over all admissible online selection policies $\boldsymbol{\tau}$.

Definition 8. *Let $h_{k-i},\ 0 \leq h_{k-i} \leq 1$ be a sequence of threshold functions. An online selection policy in the quickest selection problem is called* self-similar *if it amounts to the following rule: when at some stage there are $k-i$ items yet to be chosen and the last selection value is z, then the observation of size x should be selected if and only if*

$$0 < \frac{x-z}{1-z} < h_{k-i}.$$

Earlier, in the study of closely-related selection of a k-long subsequence as quickly as possible under a fixed sum constraint, Chen et al. [24] demonstrated the necessary asymptotic conditions for the stationary strategy to be optimal up to the term of principal order. Chen et al.'s threshold functions $\widetilde{h}_k$ satisfy $\widetilde{h}_k \sim 2/k$ and result in the expected time of selection $\beta_k^{(\widetilde{h})} \sim k^2/2,\ k \to \infty$. In Section 3.2.5 we explicitly construct a quasi-stationary selection policy that meets these criteria.

Our quasi-stationary policy from Section 3.2.5 has two regimes. A second more conservative regime kicks in once the running maximum crosses a certain barrier. This is reminiscent of a two-stage selection strategy described in the proof of Theorem 9 in [26] in context of the quickest selection under a sum constraint.

3.2.1 The optimality equation

The inter-arrival times between consecutive items that fall in $[z, 1]$ are independent and distributed geometrically with success probability $(1-z)$. Therefore, with a minute thought we see that the value function $\beta_k(z)$ satisfies the identity

$$\beta_k(z) = \frac{\beta_k}{1-z} \tag{3.25}$$

in the problem with k choices to make. Arguing by induction on $k = 1, 2, \ldots$ we arrive at the recursive optimality equation

$$\beta_{k+1} = 1 + \mathbb{E} \min \left\{ \beta_{k+1}, \frac{\beta_k}{1-X} \right\}, \tag{3.26}$$

where the initial condition is $\beta_1 = 1$ and X is uniform on $[0, 1]$. With $k + 1$ items to choose, the next observation increments the total time by 1 no matter if it is selected or discarded, thus explaining the unit increment on the right-hand side. The second term on the right-hand side is dictated by the optimality principle, which requires us to choose an action that minimises the expected time. The options are: skip the observation X, thus leaving the expected time of selection unchanged at β_{k+1}, or accept the observation X, thus selecting the increasing subsequence from a thinned sequence of items above X. Referring to the identity (3.25), one can obtain the second term in (3.26).

Lemma 4. *The optimal value function β_k satisfies the recursive equation*

$$\beta_{k+1} - \beta_k - \beta_k \log \left(\frac{\beta_{k+1}}{\beta_k} \right) = 1, \tag{3.27}$$

initialised with $\beta_1 = 1$.

Proof. We have $\beta_k/(1 - x) \leq \beta_{k+1}$ if and only if $x \leq 1 - \beta_k/\beta_{k+1} =: h_k$. Thus, it is optimal to choose the observation X only when $X \leq h_k$. From this we can rewrite (3.26) as

$$\beta_{k+1} = 1 + (1 - h_k)\, \beta_{k+1} + \int_0^{h_k} \frac{\beta_k}{1-x} \, \mathrm{d}x, \quad \beta_1 = 1.$$

Integrating and permuting yields (3.27). □

The optimal strategy is self-similar. If at stage j some k items are yet to be chosen, the last selection was z, and the observed item is $X_j = x$ then the item should be selected if and only if

$$0 < \frac{x-z}{1-z} < h_k.$$

Arlotto et al. derived (3.24) by analysing the optimality recursion in the following form ([3], Lemma 3)

$$\beta_{k+1} = \min_{t\in[0,1]} \left(\frac{1}{t} - \frac{\beta_k}{t} \log(1-t) \right), \quad \beta_1 = 1. \tag{3.28}$$

The recursion (3.28) is, in fact, equivalent to the recursion (3.27). Indeed, we find that the minimising value of $t = t^*$ satisfies

$$\log(1-t^*) = \frac{1}{\beta_k} - \frac{t^*}{1-t^*}. \tag{3.29}$$

Substituting (3.29) into (3.28) yields the optimal solution $t^* = 1 - \beta_k/\beta_{k+1}$. Plugging the optimal solution into (3.28) and rearranging gives (3.27).

Arlotto et al. also proved that the function $k \to \beta_k$ is convex and the optimal threshold functions satisfy

$$h_k = \frac{2}{k} + O(k^{-2}\log k), \quad k \to \infty. \tag{3.30}$$

3.2.2 Preparation for the asymptotic analysis

The quickest selection problem is a decision problem with infinite horizon. Still, a version of the comparison method turns out to be useful here too. In this section we develop the tools necessary to apply the 'comparison-like' approach to (3.27).

Recursion (3.27) possesses several useful analytical properties. To begin with, define a function $G : \mathbb{R}_+^2 \to \mathbb{R}$ as

$$G(x,y) := y - x - x\log\frac{y}{x}.$$

In terms of G, (3.27) becomes

$$G(\beta_k, \beta_{k+1}) = 1, \quad \beta_1 = 1. \tag{3.31}$$

Recursion (3.31) taken together with the condition

$$1 = \beta_1 < \beta_2 < \dots$$

defines the sequence β_k uniquely, as seen from the next lemma. For $x \geq 0$ define $g(x)$ as a solution to

$$G(x, g(x)) = 1.$$

This function $g(x)$ has two branches, and we are interested in the upper branch.

Lemma 5. *The function g has a branch that lies entirely in the domain $\mathcal{D} = \{(x, y) : x + 1 < y,\ x > 0\}$.*

Proof. Calculating the partial derivatives

$$\frac{\partial G}{\partial x} = \log \frac{x}{y}, \quad \frac{\partial G}{\partial y} = 1 - \frac{x}{y},$$

we see that, if $G(x_0, y_0) = 1$, then, by the Implicit Function Theorem, in the vicinity of (x_0, y_0) there is a uniquely defined function $g(x)$ with

$$g'(x) = -\frac{\partial G}{\partial x} \Big/ \frac{\partial G}{\partial y} = \frac{-\log(x/y)}{1 - x/y}$$

provided $x_0 \neq y_0$. If, furthermore, $0 < x_0 < y_0$, then this function has derivative $g'(x) > 1$, since

$$-\log z > 1 - z, \quad \text{for } 0 < z < 1,$$

where $z = x/y$. Thus, if there is one such point $(x_0, y_0) \in \mathcal{D}$, then there is a branch $g(x) : \mathbb{R}_+ \to \mathbb{R}_+$ with $(x, g(x)) \in \mathcal{D}$. $\square$

In particular, we can pick $(x_0, y_0) = (1, y_0)$, where $y_0 = 3.146\ldots$ solves

$$y - \log y = 2.$$

Note that $g(0+) = 1$, but $g'(0+) = \infty$.

From now on we only consider the branch of $g(x)$ defined in Lemma 5. In these terms

$$\beta_{k+1} = g(\beta_k), \quad \beta_1 = 1.$$

That is, the sequence of optimal value functions β_k is obtained as iterations of g, starting with $\beta_1 = 1$. So $\beta_2 = g(1)$, $\beta_3 = g(g(1))$, etc. We wish to find now the asymptotic behaviour of g for large x.

Lemma 6. *A function $g(x)$ possesses the following asymptotic expansion*

$$g(x) = x + \sqrt{2x} + \frac{2}{3} + \frac{\sqrt{2}}{18\sqrt{x}} + O(x^{-1}), \quad x \to \infty. \tag{3.32}$$

Proof. Dividing both sides of $G(x, y) = 1$ by x yields

$$\frac{y}{x} - 1 - \log\frac{y}{x} = \frac{1}{x}.$$

Note that taking limits on both sides gives

$$\lim_{x\to\infty} \frac{y}{x} = 1 + \lim_{x\to\infty} \log\frac{y}{x}, \quad \text{where } y = g(x).$$

Performing a change of variables

$$z = \frac{y}{x} - 1, \quad w = \frac{1}{x},$$

we arrive at

$$z - \log(1+z) = w. \tag{3.33}$$

Because $\limsup y/x < \infty$ and $a = 1$ is the unique solution to $a = 1 + \log a$, we may conclude that

$$\frac{y}{x} \to 1.$$

In light of this, we may investigate (3.33) in the vicinity of $z = w = 0$. The function $w(z)$ is analytic within a unit circle. Thus, expanding a logarithm yields a series representation of $w(z)$

$$w(z) = \frac{z^2}{2} - \frac{z^3}{3} + \frac{z^4}{4} - \dots. \tag{3.34}$$

Since $w'(0) = 0$, the inverse function has an algebraic branch point at 0 of order 1 (see [45] for definition). The inverse $z(w)$ is representable as Puiseux series in powers of $w^{1/2}$, with coefficients that can be calculated recursively. From the first two terms of series (3.34) we obtain

$$z(w) = \sqrt{2}w^{1/2} + O(w).$$

Plugging $z(w) = \sqrt{2}w^{1/2} + a_0 w + o(w)$, where a_0 is a constant coefficient, into (3.34) yields

$$\sqrt{2}a_0 w^{3/2} - \frac{2\sqrt{2}}{3}w^{3/2} + O(w^2) = 0,$$

which provides us with a refinement

$$z(w) = \sqrt{2}w^{1/2} + \frac{2}{3}w + O(w^{3/2}).$$

Another iteration of the method with $z(w) = \sqrt{2}w^{1/2} + \frac{2}{3}w + a_1 w^{3/2} + o(w^{3/2})$, a_1

constant, results in the expansion

$$z(w) = \sqrt{2}w^{1/2} + \frac{2}{3}w + \frac{\sqrt{2}}{18}w^{3/2} + O(w^2).$$

Translating this back in terms of variables x, y, we obtain the desired asymptotic expansion

$$y = g(x) = x + \sqrt{2x} + \frac{2}{3} + \frac{\sqrt{2}}{18\sqrt{x}} + O(x^{-1}), \quad x \to \infty.$$

□

Suppose now that (x_k) is a sequence of iterations

$$x_{k+1} = g(x_k), \quad k = 1, 2, \dots$$

with some $x_1 > 0$. Since $x_{k+1} > x_k + 1$, we have $x_{k+1} > x_0 + k$, and so $x_k \to \infty$, as $k \to \infty$. Thus,

$$x_{k+1} = x_k + \sqrt{2x_k} + \frac{2}{3} + o(1), \quad k \to \infty. \tag{3.35}$$

To derive the leading asymptotic term from (3.35) we only need

$$x_{k+1} - x_k \sim \sqrt{2x_k}, \quad \text{as } k \to \infty. \tag{3.36}$$

The idea is to compare x_k with a solution of the analogous differential equation

$$f'(x) = \sqrt{2f(x)},$$

which satisfies

$$\int_{f(t)}^{f(t+1)} \frac{\mathrm{d}u}{\sqrt{2u}} = 1. \tag{3.37}$$

Equation (3.37), in turn, yields

$$\sqrt{2f(t+1)} - \sqrt{2f(t)} = 1.$$

An application of the mean value theorem leads to

$$\sqrt{2x_{k+1}} - \sqrt{2x_k} = \frac{x_{k+1} - x_k}{\sqrt{2\tilde{x}_k}},$$

where $x_k < \tilde{x}_k < x_{k+1}$. Hence,

$$\frac{x_{k+1} - x_k}{\sqrt{2x_{k+1}}} \le \sqrt{2x_{k+1}} - \sqrt{2x_k} \le \frac{x_{k+1} - x_k}{\sqrt{2x_k}}.$$

Recalling that $\lim_{k\to\infty} x_{k+1}/x_k = 1$ and the asymptotics (3.36), we obtain

$$\sqrt{2x_k} \sim k, \quad k \to \infty,$$

and, therefore,

$$x_k \sim \frac{k^2}{2}. \tag{3.38}$$

The recursion $x_{k+1} = g(x_k)$ is shift-invariant because any consequent term x_{k+1} of the sequence is a function of x_k only, independent of k. Thus, we are interested in how the shift in the initial condition affects the sequence for large k.

Lemma 7. *For any sequence (x_k) solving the recursion $G(x_k, x_{k+1}) = 1$, it holds that*

$$|x_k - \beta_k| = O(k), \quad k \to \infty.$$

Proof. If $x_1 = \beta_1$ the sequences are identical and the assertion trivial. We shall examine the case $x_1 > \beta_1$ only, as the opposite case is treated similarly.

Given the monotonicity of $g(x)$, we have that $x_k > \beta_k$ for all $k \in \mathbb{N}$. Thus, it suffices to prove that there exists a positive constant c such that $x_k - \beta_k \leq ck$ for all k.

Since β_k is an increasing sequence, we can find a finite k_0 such that $\beta_{k_0} > x_1$. Having identified the point of the inequality change k_0, we know that, by monotonicity of $g(x)$, the elements $\beta_{k_0+1}, \beta_{k_0+2}, \ldots$ dominate $x_2, x_3, \ldots$ respectively. Observe that

$$\beta_{k_0} = \beta_1 + \sum_{i=1}^{k_0} \Delta\beta_i.$$

Hence, comparing x_k to the shifted sequence yields

$$x_k - \left(\beta_k + \sum_{i=k}^{k+k_0} \Delta\beta_i\right) < 0, \quad \text{for all } k.$$

Whence the upper bound

$$x_k - \beta_k < \sum_{i=1}^{k_0} \Delta\beta_i.$$

The asymptotic expansion (3.35) together with (3.38) implies $\Delta\beta_k = O(k)$; therefore, allowing us to choose $c = k_0$. □

Before stating an analogue of Lemma 1, we need to highlight the following monotonicity property of G.

Lemma 8. *Let* $0 < u < v$. *If* $G(u, v) > 1$ *and* $u > x$, *then* $v > g(x)$. *Analogously, if* $G(u, v) < 1$ *and* $u < x$, *then* $v < g(x)$.

Proof. From

$$g'(u) = \frac{-\log(u/v)}{1 - u/v} > 0$$

we have $g(u) > g(x)$. Then, $G(u, g(u)) = 1$ and $G(u, v) > 1$ imply $v > g(u)$ by monotonicity of G that follows from

$$\frac{\partial G}{\partial v} = 1 - \frac{u}{v} > 0.$$

Hence $v > g(x)$. The second part of the lemma can be proved by a symmetric argument.

□

Now, we state and prove the analogue of Lemma 1 in the quickest selection problem.

Lemma 9. *Let (x_k) be an increasing sequence such that $G(x_k, x_{k+1}) > 1$ (or, equivalently, $x_{k+1} > g(x_k)$) for all sufficiently large k. Then for some constant $c > 0$*

$$\beta_k - x_k < ck, \quad k \in \mathbb{N}.$$

Similarly, if $G(x_k, x_{k+1}) < 1$ (or, equivalently, $x_{k+1} < g(x_k)$) for all sufficiently large k, then for some $c > 0$

$$x_k - \beta_k < ck, \quad k \in \mathbb{N}.$$

Proof. Assume to the contrary that for arbitrarily large $c_0 \in \mathbb{R}_+$ there exists $k_0 \in \mathbb{N}$ such that

$$\begin{aligned} \beta_k - x_k &< c_0 k, \quad \text{for } k < k_0, \\ \beta_k - x_k &\geq c_0 k, \quad \text{for } k \geq k_0. \end{aligned} \tag{3.39}$$

Choosing c_0 large ensures $x_k > g(x_k)$ for $k \geq k_0$.

Now, it is easy to see that $\beta_{k_0} < x_{k_0}$ leads to a contradiction with the second inequality in (3.39); thus, we only consider the case $\beta_{k_0} > x_{k_0}$. Introducing a sequence y_k that satisfies $G(y_k, y_{k+1}) = 1$ and $y_{k_0} = x_{k_0}$, we have

$$x_k > y_k, \quad k \geq k_0 + 1. \tag{3.40}$$

Moreover, by Lemma 7, there exists a positive constant c_1 such that

$$\beta_k - y_k < c_1 k, \quad k \in \mathbb{N}.$$

Let $c_2 := c_0 \vee c_1$. Then we can find a $k_1 \leq k_0, k_1 \in \mathbb{N}$ such that

$$\beta_k - x_k \geq c_2 k, \quad k \geq k_1, \tag{3.41}$$

and

$$\beta_k - y_k < c_2 k, \quad k \in \mathbb{N}.$$

Recalling (3.40) yields

$$x_k > \beta_k - c_2 k, \quad k \geq k_0 + 1,$$

which contradicts (3.41). The second part of the lemma can be proved similarly. $\square$

With this result in our toolbox, we are fully equipped to refine the asymptotic expansion of β_k.

3.2.3 Refined asymptotic expansion of the optimal value function

The order of the next term of expansion of β_k is readily suggested by the upper bound in (3.24). However, to strengthen the hypothesis, we provide a heuristic argument based on the natural duality between this problem and the original problem of selecting the longest increasing subsequence.

Firstly, let us lay out the duality in mean-constraint bounds. Recall the upper bound (3.22) derived by relaxing the optimisation constraint to the mean-constraint. The analogous approach was also applied to the quickest selection problem ([3], Section 3.1) yielding a dual inequality

$$\beta_k > \frac{k^2}{2}.$$

We are taking a step further in exploring the connection between the two problems.

Obtaining the asymptotic inverse of (3.18) suggests that

$$\beta_k = \frac{k^2}{2} + \frac{k \log k}{6} + O(k), \quad \text{as } k \to \infty.$$

Thus, heuristics hint at the refinement of order $O(k \log k)$. In view of this, we choose the first approximating sequence $x_k^{(0)}$

$$x_k^{(0)} := \frac{k^2}{2} + \omega_0 k \log k,$$

where ω_0 is a parameter. Recalling the expansion (3.32), we obtain

$$g(x_k^{(0)}) = x_k^{(0)} + \sqrt{2x_k^{(0)}} + \frac{2}{3} + O(k^{-1}) = \frac{k^2}{2} + \omega_0 k \log k + k + \omega_0 \log k + \frac{2}{3} + o(1),$$

as $k \to \infty$. Therefore,

$$x_{k+1}^{(0)} - g(x_k^{(0)}) = -\frac{1}{6} + \omega_0 + o(1), \quad k \to \infty.$$

Straightforwardly it follows that, for k large enough,

$$\begin{aligned} x_{k+1}^{(0)} &> g(x_k^{(0)}), \quad \text{when } \omega_0 > \frac{1}{6} \\ x_{k+1}^{(0)} &< g(x_k^{(0)}), \quad \text{when } \omega_0 < \frac{1}{6}. \end{aligned} \tag{3.42}$$

Combining the inequalities (3.42) with Lemma 9 produces the following result

Corollary 2. *As $k \to \infty$,*

$$\beta_k \sim \frac{k^2}{2} + \frac{k \log k}{6}. \tag{3.43}$$

To bound the remainder, we need only one more successive approximation. Choose a test function of the form

$$x_k^{(1)} = \frac{k^2}{2} + \frac{k \log k}{6} + \omega_1 (\log k)^2,$$

where ω_1 is a constant. On the one hand, we have, as $k \to \infty$,

$$x_{k+1}^{(1)} = \frac{k^2}{2} + \frac{k \log k}{6} + k + \omega_1 (\log k)^2 + \frac{\log k}{6} + \frac{2}{3} + \frac{2\omega_1 \log k}{k} + O(k^{-1}).$$

On the other hand, taking all four terms of expansion (3.32),

$$g(x_k^{(1)}) = \frac{k^2}{2} + \frac{k \log k}{6} + k + \omega_1 (\log k)^2 + \frac{\log k}{6} + \frac{2}{3} + \frac{\omega_1 (\log k)^2}{k} - \frac{(\log k)^2}{72k} + O(k^{-1}).$$

Hence,

$$\begin{aligned} x_{k+1}^{(1)} - g(x_k^{(1)}) > 0, &\quad \text{when } \omega_1 < \frac{1}{72}, \\ x_{k+1}^{(1)} - g(x_k^{(1)}) < 0, &\quad \text{when } \omega_1 > \frac{1}{72}. \end{aligned}$$

Recall that the shift in the initial condition of the optimality recursion (3.27) results in the order $O(k)$ change to the solution. Since the comparison to the approximating sequence $(x_k^{(1)})$ provides a refinement of smaller order, we may bound the remainder in the expansion (3.43).

Theorem 3. *The minimum expected time required to select an increasing sequence of length k satisfies the following asymptotic expansion*

$$\beta_k = \frac{k^2}{2} + \frac{k \log k}{6} + O(k), \quad k \to \infty. \tag{3.44}$$

Corollary 3. *The optimal threshold h_k satisfies the following refined asymptotic expansion*

$$h_k = \frac{2}{k} - \frac{\log k}{3k^2} + O(k^{-2}), \quad k \to \infty. \tag{3.45}$$

Proof. Unfortunately, the direct computation of $h_k = 1 - \beta_k/\beta_{k+1}$ from (3.44) does not yield meaningful results due to the $O(k)$ remainder in the expansion. However, [3], Lemma 7 provides an asymptotic expression for the optimal threshold functions in terms of the optimal value function

$$h_k = \sqrt{\frac{2}{\beta_k}}(1 + O(k^{-1/2})), \quad k \to \infty. \tag{3.46}$$

Using the one-term asymptotic expansion $\beta_k \sim k^2/2$, Arlotto et al. computed that $h_k = 2/k + O(k^{-2}\log k),\ k \to \infty$. Plugging in the refined asymptotics (3.44) into (3.46) and applying binomial theorem leads to the more refined expansion (3.45). □

3.2.4 Numerical approximation of the $O(k)$-term

The comparison method adapted to the quickest increasing subsequence is not geared to capture the term resulting from the shift in the initial condition (the $O(k)$-term in 3.44). However, it is of interest to check if numerical simulations can be used to make an educated guess.

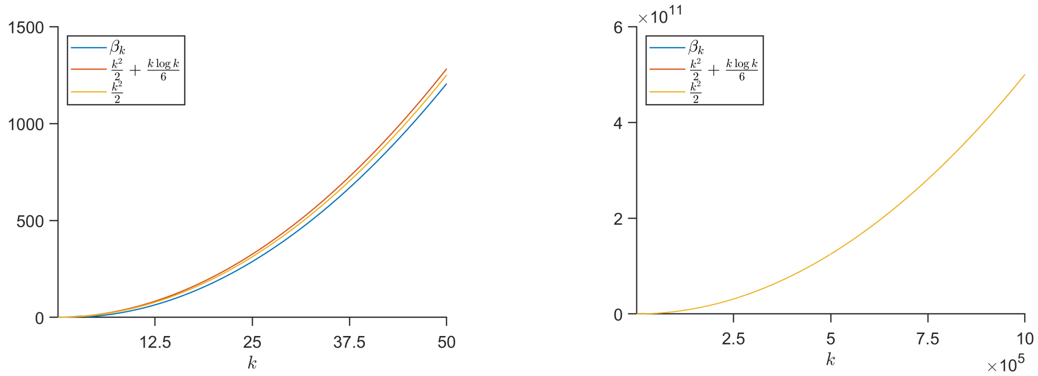
With some help of `SciPy`'s implementation of Brent's root-finding method (see [17], Chapter 3 for reference), we computed the sequence (β_k), $k = 1, \ldots, 10^6$ in `Python` by initialising the sequence with $\beta_1 = 1$. For $k = 2, \ldots, 10^6$, β_k was approximated by finding the root of the equation

$$G(\beta_{k-1}, x) - 1 = 0, \quad \text{subject to } x > \beta_{k-1}.$$

Define a residual function

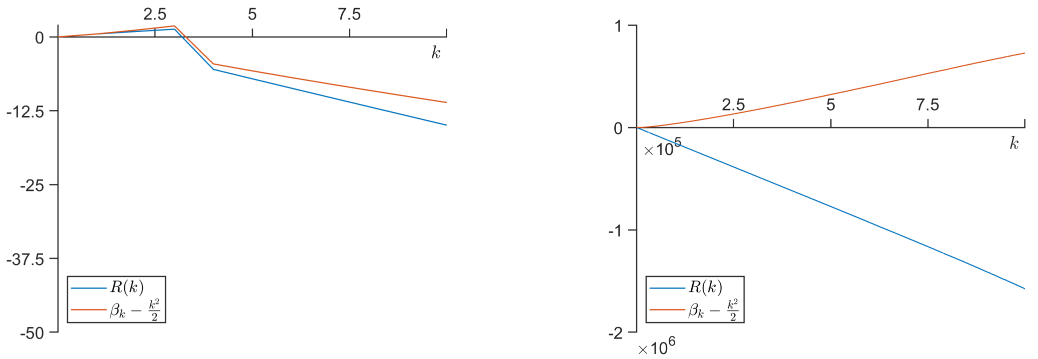
$$R(k) := \beta_k - \left(\frac{k^2}{2} + \frac{k\log k}{6}\right).$$

We plot below the fruitful results of the simulation.



(a) Small values of k (b) Large values of k

Figure 3.1: β_k in comparison to the one-term and the two-term expansions of itself



(a) Small values of k (b) Large values of k

Figure 3.2: Residual functions for different values of k

As expected, from Figures 3.1a and 3.1b we see that the leading term asymptotically dominates the lower order terms. Looking at Figures 3.2a and 3.2b, the residual $R(k)$ appears to be linear in k, at least for $k \geq 4$. This is in line with the remainder term in the expansion (3.44). Approximating the slope of $R(k)$ with a linear regression gives the relation of the form $R(k) = rk$ with $r = -1.5558$.

3.2.5 A quasi-stationary policy

In this section, we construct a simple *quasi-stationary* policy that, as k grows large, has the expected time of selection matching β_k up to the leading term of expansion.

We call it quasi-stationary because it has a second more conservative selection mode with a narrower acceptance window. However, the threshold functions in both regimes are independent of the remaining number of elements to be chosen, analogously to the stationary policy introduced by Samuels and Steele [55] in the longest increasing subsequence problem.

We define our policy by choosing the threshold functions $\widetilde{h}_i(z)$, $i = 1, \dots k$

$$\widetilde{h}_i(z) = \begin{cases} \eta, & \text{if } z < 1 - a(k) - \eta, \\ \frac{a(k)}{k}, & \text{if } z \geq 1 - a(k) - \eta, \end{cases}$$

where

$$\eta = \frac{2(1 - a(k))}{k(1 + a(k))}$$

and the function $a(k) : \mathbb{R}_+ \to [0, 1]$ is monotone decreasing in k. We fix $a(k)$ in the sequel.

The policy acts in two regimes. Firstly, we accept every consecutive observation within η above the last selected item. Secondly, when the last selection size gets above $1 - a(k) - \eta$, we abandon the initial rule and accept all admissible elements within an acceptance window of size $a(k)/k$. This is similar to the two-stage selection strategy used by Coffman et al. in the proof of Theorem 9 in [26].

The choice of η is inspired by the asymptotics of the optimal threshold (3.30). However, choosing $2/k$ exactly leads to a problem: with high probability the selection process will cross the $1 - \eta$ barrier, while there are $O(k^{1/2})$ elements yet to choose. Loosely speaking, as k gets large, the selection process with a constant window is governed by the central limit theorem. Although the expectation of the sum of k random variables distributed uniformly on $[0, 2/k]$ is 1, it has a standard deviation of $O(k^{-1/2})$. A way to overcome this issue is to decrease the threshold size so that the probability of reaching the barrier is low, but keep it large enough so that the expected time of selection remains unchanged up to the terms of a lower order. The task narrows down to choosing a suitable $a(k)$.

The value function $\beta_k^{(\widetilde{h})}$ corresponds to the expected performance of our quasi-stationary policy in the rest of this section.

Theorem 4. *The quasi-stationary policy with threshold functions $\widetilde{h}_i(z)$ is asymptotically optimal up to the leading term of the value function expansion, i.e.*

$$\beta_k^{(\widetilde{h})} \sim \frac{k^2}{2}, \quad \text{as } k \to \infty.$$

Proof. Let $(Z_j)_{j\in\mathbb{N}}$ denote the last selection process of the quasi-stationary policy. Introduce a hitting time ξ of the barrier $1 - a(k) - \eta$

$$\xi := \inf\{j : Z_j > 1 - a(k) - \eta\},$$

where we follow the convention $\inf \emptyset = \infty$. Moreover, let the stopping time ρ be defined as

$$\rho := \xi \wedge k.$$

In this notation, we can write $\beta_k^{(\widetilde{h})}$ out as follows

$$\beta_k^{(\widetilde{h})} = \mathbb{E}\tau_\rho + \mathbb{E}(\tau_k - \tau_\rho)^+. \tag{3.47}$$

Before the barrier is hit, the inter-selection times are independent and distributed identically as $\mathrm{Geom}(\eta)$, hence

$$\mathbb{E}\tau_\rho = \frac{\mathbb{E}\rho}{\eta};$$

moreover, by Wald's identity

$$\mathbb{E}\rho < \frac{1 - a(k) - \eta}{\eta/2}.$$

Consequently,

$$\mathbb{E}\tau_\rho < \frac{k^2(1 + a(k))^2}{2(1 - a(k))} - \frac{k(1 + a)}{(1 - a)}. \tag{3.48}$$

The second expectation in (3.47) is bounded by the expected time of selection in case the barrier is hit. Thus,

$$\mathbb{E}(\tau_k - \tau_\rho)^+ \leq \mathbb{P}(\rho < k) \ \mathbb{E}(\tau_k | \rho < k).$$

A rough upper-bound on $\mathbb{E}(\tau_k|\rho < k)$ suffices for our purposes

$$\mathbb{E}(\tau_k|\rho < k) < \frac{k^2}{a(k)}; \tag{3.49}$$

it follows from computing an expected time to select all k elements with a constant window $a(k)/k$. To get a grip on $\mathbb{P}(\rho < k)$ we first notice that

$$\mathbb{P}(\rho < k) = \mathbb{P}\left(Z_k > 1 - a(k) - \eta\right).$$

Introduce a renewal sequence (S_j) with inter-arrival times distributed uniformly on $[0, \eta]$. For $j < \rho$, this sequence is equivalent in distribution to the gaps between consecutive selections $Z_{j+1} - Z_j|\rho < j$. In light of this, we can write

$$\mathbb{P}(\rho < k) = \mathbb{P}\left(\sum_{j=1}^{k} S_j \geq 1 - a(k) - \eta\right). \tag{3.50}$$

Since we have

$$\mu = \mathbb{E}\left(\sum_{j=1}^{k} S_j\right) = \frac{k\eta}{2},$$

we can write the probability on the right-hand side of (3.50) in terms of μ as

$$\mathbb{P}\left(\sum_{j=1}^{k} S_j \geq 1 - a(k) - \eta\right) = \mathbb{P}\left(\sum_{j=1}^{k} S_j \geq (1+\epsilon)\mu\right),$$

where $\epsilon = a(k) - 2/k$. The probability in focus can be estimated from above by applying the Chernoff-Hoeffding inequality (see, for example, [12] for details)

$$\mathbb{P}\left(\sum_{j=1}^{k} S_j > (1+\epsilon)\mu\right) \leq \exp\left(-\frac{k\epsilon^2}{2}\right). \tag{3.51}$$

Thus, choosing $a(k) := k^{-1/2+\varepsilon}$, $0 < \varepsilon < 1/2$ makes sure the probability in (3.51) has an exponentially decreasing upper-bound

$$\mathbb{P}\left(\sum_{j=1}^{k} S_j > (1+\epsilon)\mu\right) \leq \exp\left(-\frac{k^{2\varepsilon}}{2}\right) + O(\exp(k^{-1/2+\varepsilon})). \tag{3.52}$$

With $a(k)$ finally fixed, from an upper bound (3.48) we have

$$\mathbb{E}\tau_\rho < \frac{k^2}{2} + O(k^{3/2+\varepsilon}). \tag{3.53}$$

Taking together (3.49), (3.52) and (3.53) yields

$$\beta_k^{(\widetilde{h})} < \frac{k^2}{2} + O(k^{3/2+\varepsilon}). \tag{3.54}$$

A sufficient lower-bound on $\beta_k^{(\widetilde{h})}$ follows from the inequality

$$\beta_k^{(\widetilde{h})} = \mathbb{E}\tau_k \geq \mathbb{E}(\tau_k|\rho > k) = \frac{k^2(1+a(k))}{2(1-a(k))}.$$

Plugging in the expression for $a(k)$ yields

$$\beta_k^{(\widetilde{h})} \geq \frac{k^2}{2} + O(k^{3/2+\varepsilon}), \quad k \to \infty. \tag{3.55}$$

At last, combining (3.54) with (3.55) leads to

$$\beta_k^{(\widetilde{h})} = \frac{k^2}{2} + O(k^{3/2+\varepsilon}), \quad k \to \infty,$$

and the result of Theorem 4 follows immediately. □

3.2.6 A self-similar policy

We shall construct next a suboptimal self-similar policy to closer approach optimality. Recall that a selection policy is self-similar if it chooses the observation of size x if and only if

$$0 < \frac{x-z}{1-z} < \overline{h}_k.$$

Let $\beta_k^{(\overline{h})}$ be the value functions of such strategy; then, decomposing at the first arrival yields

$$\beta_{k+1}^{(\overline{h})} = 1 + \mathbb{E}\left(\beta_{k+1}^{(\overline{h})}\,\mathbb{1}\left(X > \overline{h}_{k+1}\right) + \frac{\beta_k^{(\overline{h})}}{1-X}\,\mathbb{1}\left(X \leq \overline{h}_{k+1}\right)\right).$$

Computing the integral and rearranging yields

$$\beta_{k+1}^{(\overline{h})}\overline{h}_{k+1} + \beta_k^{(\overline{h})}\log(1-\overline{h}_{k+1}) = 1.$$

This is an inhomogeneous linear recursion, which can be solved explicitly in terms of $\overline{h}_k$'s by the method of variation of constants.

Introduce a self-similar suboptimal selection policy with thresholds

$$\overline{h}_k := \frac{2}{k+1}, \quad k \in \mathbb{N}. \tag{3.56}$$

Note that $\overline{h}_k < 1$ for $k > 1$, thus $\beta_k^{(\overline{h})} < \infty$ for all k. The recursion defining the value functions $\beta_k^{(\bar{h})}$ becomes

$$\beta_{k+1}^{(\bar{h})} = a_k \beta_k^{(\bar{h})} + b_k, \quad \beta_1^{(\bar{h})} = 1, \tag{3.57}$$

where

$$a_k = \left(\frac{k}{2} + 1\right) \log\left(1 + \frac{2}{k}\right), \quad b_k = \frac{k}{2} + 1\,.$$

The homogeneous equation (3.57) has the general solution of the form

$$y_{k+1} = a_1 \dots a_k \, y_1.$$

Taking two terms in the expansion of the logarithm we get

$$a_k = 1 + \frac{1}{k} + O\left(\frac{1}{k}\right),$$

which readily implies

$$y_k \sim c y_1 k, \quad k \to \infty,$$

for some constant $c > 0$. We see that y_k is about linear in the initial value y_1. Likewise, because the general solution is the sum of a particular solution and the general solution to the homogeneous equation, if we replace the initial value $\beta_1^{(\bar{h})} = 1$ in the inhomogeneous equation by $\beta_1^{(\bar{h})} + \theta,\ \theta \in \mathbb{R}$ the corresponding solution will change by about θck.

On the other hand, the equation (3.57) possesses inherent monotonicity properties required to apply the asymptotic comparison method (since $a_k > 0$). Checking that a test function satisfies the appropriate inequality for $k > k_0$, we adjust the initial value (resulting in the $O(k)$ deflection) for this k_0 to apply comparison in the already familiar way. We state without proof the counterpart of Lemma 9.

Lemma 10. *If a sequence (y_k) is such that $y_{k+1} > a_k y_k + b_k$ for k large enough, then $\beta_k^{(\bar{h})} - y_k < ck$. Similarly, if a sequence (y_k) is such that $y_{k+1} < a_k y_k + b_k$ for k large enough, then $\beta_k^{(\bar{h})} - y_k > ck$.*

Following the usual procedure, we choose test functions of the form

$$y_k = d_0 k^2 + d_1 k \log k + d_2 \log k, \quad d_0, d_1, d_2 \in \mathbb{R}.$$

The computation consists of three successive refinements, which we display explicitly in Appendix A.1. Matching coefficients and observing that the last term of y_k is of order $o(k)$, we obtain

$$\beta_k^{(\bar{h})} = \frac{k^2}{2} + \frac{k \log k}{6} + O(k).$$

This result, together with the expansion (3.44), allows us to obtain the following theorem.

Theorem 5. *The self-similar strategy with threshold functions* (3.56) *has the value functions* $\beta_k^{(\bar{h})}$ *satisfying*

$$|\beta_k - \beta_k^{(\bar{h})}| = O(k), \quad k \to \infty.$$

3.3 The selection under a sum constraint

The problem in focus of this section was first introduced by Coffman et al. [26] as a simplistic model for processes arising in storage optimisation. Having been studied extensively, it is also referred to in the literature as stochastic knapsack and stochastic bin packing problem. The decision-maker observes the sequence $X_i \sim F,\ i = 1, \dots, n$ of positive independent identically distributed random variables one at a time. Aiming to select as many elements as possible, she has to decide whether to keep the current observation or discard it without the possibility of recall (an online constraint). Another binding constraint is that the sum of selected X_i's cannot be larger than a given constant $C > 0$. One can think of X_i's as sizes of the objects that are to be efficiently packed into a one-dimensional storage unit of capacity C.

Let $v_n(C)$ be the maximal expected number of elements selected under the constraints stated above. Having made some fairly unrestrictive assumptions about the distribution law $F(x)$ of X_i's, Coffman et al. derived the following asymptotic

$$v_n(C) \sim \gamma_0^*(C^\alpha n)^{1/(\alpha+1)}, \quad \text{as } n \to \infty, \tag{3.58}$$

where

$$\gamma_0^* = \left(A\left(\frac{\alpha+1}{\alpha}\right)^\alpha\right)^{1/(\alpha+1)}$$

and the constants A and α come from the assumption that $F(x) \sim Ax^\alpha$, as $x \to 0$. The result above was derived in two steps: first, the upper-bound was obtained using Chernoff estimates [25]. Second, the lower-bound was derived by investigating a suboptimal selection policy with constant thresholds. This result was later generalised to arbitrary continuous distributions by Rhee and Talagrand [53].

Boshuizen and Kertz [15] studied a strongly related 'smallest fit' problem, where the question is how many order statistics of a fully-known sample $\{X_1, \ldots X_n\}$ can one fit into a knapsack of capacity C (this problem was first formulated by Bruss and Robertson [21]). The prophet with a full overview of a sequence in Coffman et al.'s bin-packing problem would pack the smallest items into the knapsack to maximise the number of items packed. Using this parallel Boshuizen and Kertz showed the joint convergence in distribution of the suitably normalised optimal offline count and a number of items packed by a good threshold policy in the online bin-packing.

Stanke [58] studied a multidimensional variant of the bin-packing problem. He derived the principal asymptotics of the maximal expected number of items packed by studying a multidimensional stationary policy with a simplicial section as an acceptance region.

Arlotto and Xie [6] studied the 'regret' size of the decision-maker playing by the online rules after discovering the whole sequence. Moreover, the number of observations in their setup is not deterministic — the items arrive according to a known Bernoulli process over n periods — and each selection yields a fixed positive reward r. Imposing fairly unrestrictive regularity conditions on the distribution F, they prove that the regret size is at most of the order $O(\log n)$.

To complete the literature review, we should mention a few other common variations of the knapsack problem. Assuming each selection occupies a unit space and yields random rewards leads to a famous multisecretary problem of Cayley [23]. Inversely, assuming equal rewards and random item sizes formulates the uniprocessor scheduling problem of Baruah et al. [9]. Finally, considering both selection and reward sizes to be stochastic constitutes to, among many other variants, the multisecretary problem of Nakai [49].

Our assumptions about the probability distribution of X_i's are based on Coffman et al. [26], but slightly more restrictive due to the technical reasons that become apparent later on. We consider $F(x)$ that admits an expansion $F(x) \sim Ax^{\alpha} + Bx^{\alpha+1} + o(x^{\alpha+1})$, as $x \to 0$, where $A, \alpha > 0$ are positive real numbers. An example of such F is a Beta(α, β) distribution.

3.3.1 The optimality equation

Let $v_n(C) : \mathbb{R}_+ \to \mathbb{R}_+$ be the optimal value functions under the capacity C and the sample size n. Following the reasoning outlined in [26], we provide an intuitive derivation of the optimality equation

$$v_{n+1}(C) = (1 - F(C))\, v_n(z) + \int_0^C \max\{v_n(C), 1 + v_n(C - x)\}\, \mathrm{d}F(x). \tag{3.59}$$

The decision-maker observes an element of size x when there are n more observations to be examined. The first term on the right-hand side of (3.59) comes from the possibility that the observed element violates the capacity constraint C. This happens with probability $1 - F(C)$. The second term is dictated by the optimality principle. If x is admissible, then we must choose whatever is larger: discarding the observation, which results in the expected value of $v_n(C)$, or accepting the observation, which results in the expected value of $1 + v_n(C - x)$.

As was the case in the increasing subsequence selection, the optimality equation (3.59) is invariant to the shift in the initial condition. That is, the optimal value function corresponds to the natural initial condition $v_0(C) = 0$, but any function of the form $v_n(C) + \mathit{const}$ is a solution too.

Define a *threshold* function $h_n(C)$ as

$$h_n(C) := \begin{cases} C, & \text{if } v_n(C) \le 1, \\ \text{unique solution to} & \\ v_n(C) = v_n(C-x) + 1, \quad 0 \le x \le C, & \text{if } v_n(C) \ge 1. \end{cases}$$

The value functions $v_n(C)$ are monotone increasing; thus, the uniquness of $h_n(C)$ is guaranteed. With this in mind, the optimality equation (3.59) can be written as

$$v_n(C) = (1 - F(h_n(C)))\, v_n(C) + \int_0^{h_n(C)} (1 + v_n(C-x))\, \mathrm{d}F(x), \quad v_0(C) = 0.$$

Therefore, the optimal selection policy in the bin-packing problem is of threshold type.

3.3.2 Asymptotic expansion of the optimal value function

The analytical approach to the optimality equation based on Lemma 1 proves to be fruitful in this problem too. Let us write the optimality recursion (3.59) in a more suggestive form

$$v_{n+1}(C) - v_n(C) = \int_0^C (v_n(C-x) + 1 - v_n(C))_+\, \mathrm{d}F(x). \tag{3.60}$$

With a help of Lemma 1, we are able to apply the asymptotic comparison method to obtain the approximate solution to (3.60) when the size function $g_n(z) := C^\alpha n$ is large enough.

Denote the operator on the right-hand side of (3.60) by $\mathcal{K}$

$$\mathcal{K}v_n(C) := \int_0^C (v_n(C-x) + 1 - v_n(C))_+ \,\mathrm{d}F(x).$$

We proceed now with the successive approximations method established in the earlier sections. Let us try functions of the form

$$v_n^{(0)}(C) = \gamma_0\,(C^\alpha n)^{1/(\alpha+1)}, \quad \gamma_0 \in \mathbb{R}_+. \tag{3.61}$$

First, let us investigate the asymptotic behaviour of the forward difference operator. For convenience introduce $\widehat{n} := C^\alpha n$ and write

$$\begin{aligned} \Delta v_n^{(0)}(C) &= \gamma_0 (C^\alpha(n+1))^{1/(\alpha+1)} - \gamma_0 (C^\alpha n)^{1/(\alpha+1)} \\ &= \gamma_0 \widehat{n}^{1/(\alpha+1)} \left(\left(1 + \frac{C^\alpha}{\widehat{n}}\right)^{1/(\alpha+1)} - 1 \right) \\ &= \frac{C^\alpha \gamma_0}{\alpha+1}\,\widehat{n}^{-\alpha/(\alpha+1)} + O(\widehat{n}^{-(2\alpha+1)/(\alpha+1)}), \quad \widehat{n} \to \infty, \end{aligned} \tag{3.62}$$

where the result comes from a straightforward application of the binomial theorem.

Second, by monotonicity of $v_n^{(0)}(C)$ in C, the operator $\mathcal{K}$ can be written as

$$\mathcal{K}v_n^{(0)}(C) = \int_0^{h_n^{(0)}(C)} (v_n^{(0)}(C-x) + 1 - v_n^{(0)}(C))\,\mathrm{d}F(x),$$

where $h_n^{(0)}(C)$ is the unique solution to

$$v_n^{(0)}(C-x) + 1 = v_n^{(0)}(C). \tag{3.63}$$

Now, from (3.61) and (3.63), one can compute

$$h_n^{(0)}(C) = \frac{C(\alpha+1)}{\alpha\gamma_0}\widehat{n}^{-1/(\alpha+1)} + O(\widehat{n}^{-2/(\alpha+1)}), \quad \widehat{n} \to \infty.$$

Hence,

$$\begin{aligned}\mathcal{K}v_n^{(0)}(C) &= \int_0^{h_n^{(0)}(C)} \left(\gamma_0 \widehat{n}^{1/(\alpha+1)} \left(\left(1-\frac{x}{C}\right)^{\alpha/(\alpha+1)} - 1\right) + 1\right) \mathrm{d}F(x) \\ &\sim \int_0^{h_n^{(0)}} \left(-\frac{\gamma_0 \alpha}{(\alpha+1)C}\widehat{n}^{1/(\alpha+1)} x + 1\right) A\alpha x^{\alpha-1}\mathrm{d}x \\ &\sim A\frac{C^\alpha(\alpha+1)^{\alpha-1}}{\gamma_0^\alpha \alpha^\alpha}\widehat{n}^{-\alpha/(\alpha+1)}, \quad \widehat{n}\to\infty.\end{aligned} \tag{3.64}$$

Combining (3.62) with (3.64), one can see that, for $C^\alpha k$ large enough,

$$\begin{aligned}\Delta v_n^{(0)}(C) &> \mathcal{K}v_n^{(0)}(C), \qquad \text{if } \gamma_0 > \gamma_0^*, \text{ and} \\ \Delta v_n^{(0)}(C) &< \mathcal{K}v_n^{(0)}(C), \qquad \text{if } \gamma_0 < \gamma_0^*.\end{aligned}$$

An application of Lemma 1 yields

$$v_n(C) \sim \gamma_0^* \left(C^\alpha n\right)^{1/(\alpha+1)}, \quad \text{as } C^\alpha n \to \infty.$$

Clearly, for any fixed $C > 0$, $C^\alpha n \to \infty$ as $n \to \infty$; therefore, the central result (3.58) of [26] follows from the above.

Let us move on to a more precise approximation

$$v_n^{(1)}(C) = \gamma_0^*\, \widehat{n}^{1/(\alpha+1)} + \gamma_1(C)\log(\widehat{n}+1).$$

There are two things to note here. Firstly, because the forward difference is taken with respect to n but the expansions are calculated in $\widehat{n}$, the log-term coefficient depends on C. Secondly, taking $\widehat{n}+1$ instead of $\widehat{n}$ helps us avoid the singularity at 0.

On the one hand, as $\widehat{n} \to \infty$,

$$\Delta v_n^{(1)}(C) \sim \left(\frac{A}{\alpha^\alpha(\alpha+1)}\right)^{1/(\alpha+1)} C^\alpha \widehat{n}^{-\alpha/(\alpha+1)} + \frac{C^\alpha \gamma_1(C)}{(\widehat{n}+1)}. \tag{3.65}$$

On the other hand,

$$\mathcal{K}v_n^{(1)}(C) = \int_0^{h_n^{(1)}(C)} (v_n^{(1)}(C-x) - v_n^{(1)}(C) + 1)\, \mathrm{d}F(x), \tag{3.66}$$

where $h_n^{(1)}(C)$ solves

$$v_n^{(1)}(C-x) - v_n^{(1)}(C) + 1 = 0.$$

By direct computation we find

$$h_n^{(1)}(C) = \frac{C(\alpha+1)}{\alpha\gamma_0^*}\,\widehat{n}^{-1/(\alpha+1)} + O(\widehat{n}^{-2/(\alpha+1)}), \quad \widehat{n}\to\infty. \tag{3.67}$$

Since $h^{(1)}(C)$ is a stationary point of the integrand in (3.66), it is enough to consider the principal term of (3.67) to obtain

$$\begin{aligned} \mathcal{K}v_n^{(1)}(C) \;&\sim\; \left(\frac{A}{\alpha^\alpha(\alpha+1)}\right)^{1/(\alpha+1)} C^\alpha \widehat{n}^{-\alpha/(\alpha+1)} \\ &+\; \left(-\alpha\,\gamma_1(C) - \frac{A\alpha - 2BC(\alpha-1)}{2A\alpha(\alpha+2)}\right)\frac{C^\alpha}{(\widehat{n}+1)}, \quad \widehat{n}\to\infty. \end{aligned} \tag{3.68}$$

Consider

$$\gamma_1^*(C) := -\frac{A\alpha - 2BC(\alpha+1)}{2A\alpha(\alpha+1)(\alpha+2)}.$$

Introduce functions

$$f(C) := -\frac{A\alpha - 2(B+\varepsilon)C(\alpha+1)}{2A\alpha(\alpha+1)(\alpha+2)}, \quad g(C) := -\frac{A\alpha - 2(B-\varepsilon)C(\alpha+1)}{2A\alpha(\alpha+1)(\alpha+2)}$$

with parameter $\varepsilon > 0$. Comparing (3.65) to the last line of (3.68), one can deduce

$$\begin{aligned} \Delta v_n^{(1)}(C) > \mathcal{K}v_n^{(1)}(C), &\quad \text{if } \gamma_1(C) = f(C), \text{ and} \\ \Delta v_n^{(1)}(C) < \mathcal{K}v_n^{(1)}(C), &\quad \text{if } \gamma_1(C) = g(C). \end{aligned}$$

Thus, taking limits as $\varepsilon \to 0$ and applying Lemma 1 yields

$$v_n(C) \sim \gamma_0^*(C^\alpha n)^{1/(\alpha+1)} + \gamma_1^*(C)\log(C^\alpha n), \quad \text{as } C^\alpha n \to \infty. \tag{3.69}$$

One should notice that the second term in the asymptotics of $F(x)$ contributed to the expansion (3.69) via the log-term. This forced us to make a more restrictive assumption regarding the distribution law of X_i's. As in [26], only the leading term assumption on F was needed for the principal asymptotics of the value function.

Theorem 6. *As $n \to \infty$, the maximal expected number $v_n(C)$ of items packed into a knapsack of capacity C in an online fashion assumes the following asymptotic expansion*

$$v_n(C) = \gamma_0^*(C^\alpha n)^{1/(\alpha+1)} + \gamma_1^*(C)\log n + O(1).$$

Proof of Theorem 6. Finally, consider one more approximating function

$$v_n^{(2)}(C) = \gamma_0^*\,\widehat{n}^{1/(\alpha+1)} + \gamma_1^*(C)\log(\widehat{n}+1) + \frac{\gamma_2}{(\widehat{n}+1)^{1/(\alpha+1)}}.$$

We have

$$\Delta v_n^{(2)}(C) \sim \frac{C^\alpha \gamma_0^*}{\alpha+1}\widehat{n}^{-\alpha/(\alpha+1)} + \frac{C^\alpha \gamma_1^*(C)}{(\widehat{n}+1)} + O(\widehat{n}^{-(\alpha+2)/(\alpha+1)}), \quad \widehat{n} \to \infty. \tag{3.70}$$

On the other hand,

$$\mathcal{K}v_n^{(2)}(C) = \int_0^{h_n^{(2)}} (v_n^{(2)}(C-x) - v_n^{(2)}(C) + 1)\,\mathrm{d}F(x), \tag{3.71}$$

where $h_n^{(2)}(C)$ solves

$$v_n^{(2)}(C-x) - v_n^{(2)}(z) + 1 = 0.$$

With $F(x) = Ax^\alpha + Bx^{\alpha+1} + o(x^{\alpha+1}),\ x \to 0$, we need a two-term expansion of $h_n^{(2)}(C)$

as $\widehat{n} \to \infty$; namely,

$$h_n^{(2)}(C) \sim \frac{C(\alpha+1)}{\alpha\gamma_0^*}\widehat{n}^{-1/(\alpha+1)} - \frac{C(\alpha+1)}{\alpha\gamma_0^{*2}}\left(\gamma_1^*(C)\,(\alpha+1) + \frac{1}{2\alpha}\right)(\widehat{n}+1)^{-2/(\alpha+1)}. \tag{3.72}$$

Now, plugging(3.72) into (3.71) yields

$$\mathcal{K}v_n^{(2)}(C) \sim \frac{C^\alpha\gamma_0^*}{\alpha+1}\widehat{n}^{-\alpha/(\alpha+1)} + \frac{C^\alpha\gamma_1^*(C)}{(\widehat{n}+1)} + o(\widehat{n}^{-1}), \quad \widehat{n} \to \infty. \tag{3.73}$$

From (3.70) and (3.73), for $\widehat{n}$ large enough, there exist constant $\widetilde{\gamma}_2$ and $\widehat{\gamma}_2$ such that

$$\Delta v_n^{(2)}(C) > \mathcal{K}v_n^{(2)}(C), \quad \text{if } \gamma_2 > \widetilde{\gamma}_2,$$
$$\Delta v_n^{(2)}(C) < \mathcal{K}v_n^{(2)}(C), \quad \text{if } \gamma_2 < \widehat{\gamma}_2.$$

Lemma 1, then, implies that the difference $v_n(C) - v_n^{(2)}(C)$ is bounded from above when $\gamma_2 > \widetilde{\gamma}_2$, and is bounded from below when $\gamma_2 < \widehat{\gamma}_2$ respectively. Since the third term of $v_n^{(2)}(C)$ is of order $o(1)$, we have

$$v_n(C) = \gamma_0^*(C^\alpha n)^{1/(\alpha+1)} + \gamma_1^*(C)\log(C^\alpha n) + O(1), \quad C^\alpha n \to \infty.$$

Absorbing the C^α term in the logarithm into the remainder yields the result of the theorem. □

3.3.3 Connection to the longest increasing subsequence selection

Coffman et al. [26] observed that the online selection of the longest monotone subsequence problem is a special case of the selection under a sum constraint with the distribution law of X_i's taken to be Uniform$[0,1]$ and the sum constraint set to $C = 1$. The equivalence can be proved by transforming the optimality equation (3.59) into the optimality equation (3.3). Indeed, one can write 3.59 with the $F(x) = x$ as follows

$$\Delta v_n(C) = \int_0^C (v_n(C-x) + 1 - v_n(C))_+ \mathrm{d}x = \int_0^C (v_n(y) + 1 - v_k(C))_+ \mathrm{d}y, \qquad (3.74)$$

where the second equality follows immediately by a straightforward substitution of the variable. With $C = 1$, equation (3.74) precisely matches the optimality equation in Samuels and Steele's longest monotone subsequence problem. However, Steele [59], Section 5 emphasised the interest in an explicit coupling between the two problems without resorting to the comparison of the optimality recursions (see [32] for this coupling in continuous time). We build this connection below.

We define the *running maximum* process $(X_i^{(\boldsymbol{\tau})}),\ i = 0, \dots, n$ driven by a given online selection policy $\boldsymbol{\tau}$ in the increasing subsequence selection to be a non-decreasing jump process with the initial condition $X_0^{(\boldsymbol{\tau})} = 0$ and the state space $[0, 1]$. The sequence of jumps (τ_k, x_k) forms an increasing chain in the partial order in two dimensions.

Similarly, we define the *partial sum* process $(S_i^{(\widetilde{\boldsymbol{\tau}})}),\ i = 0, \dots, n$ driven by an online selection policy $\widetilde{\boldsymbol{\tau}}$ in the bin-packing problem to be a non-decreasing jump process with the initial condition $S_0^{(\widetilde{\boldsymbol{\tau}})} = 0$ and the state space $[0, 1]$.

For the running maximum $(X_i^{(\boldsymbol{\tau})})$, we define an invertible random transformation $\varphi_{X^{(\boldsymbol{\tau})}}$ of $\{x \in \mathbb{Z} : 0 \le x \le n\} \times [0, 1]$, which maps $\boldsymbol{\tau}$ to an online selection policy $\widetilde{\boldsymbol{\tau}}$ in the bin-packing problem with the same path $X^{(\boldsymbol{\tau})} = S^{(\widetilde{\boldsymbol{\tau}})}$. This transformation is defined iteratively.

At each step $j = 0, \dots, n$ we shall have $\{x \in \mathbb{Z} : 0 \le x \le n\} \times [0, 1]$ and its duplicate obtained by a measure-preserving π_j with π_0 being the identity. Start with the two identical copies of the strip and a fixed path of the running maximum $X^{(\boldsymbol{\tau})}$. If a jump occurs at step $j > 0$, the strip $\pi_k(\{x \in \mathbb{Z} : j \le x \le n\} \times [x_{j-1}, 1])$ is subjected to a change, which is comprised of cutting at height $x_j - x_{j-1}$ horizontally and placing part $\pi_k(\{x \in \mathbb{Z} : j \le x \le n\} \times [x_{j-1}, x_j])$ atop of $\pi_k(\{x \in \mathbb{Z} : j \le x \le n\} \times [x_j, 1])$

while preserving the orientation. Then, the mapping π_{j+1} is the composition of π_j and this surgery, and we may define $\varphi_{X(\boldsymbol{\tau})}$ as the composition of all π_k's. The transformation $\varphi_{X(\boldsymbol{\tau})}$ sends the sequence of selections (τ_k, x_k) to $(\tau_k, x_k - x_{k-1})$, thus mapping the online selection policy $\boldsymbol{\tau}$ to $\widetilde{\boldsymbol{\tau}}$.

Given the equivalence between the two problems, from Theorem 6 we can obtain the asymptotic expansion of the value function $\widetilde{v}_n$ in the longest increasing subsequence problem.

Corollary 4. *The maximum expected length of a monotone subsequence that can be achieved by an online selection $\widetilde{v}_n$ satisfies*

$$\widetilde{v}_n = v_n(1) = \sqrt{2n} - \frac{1}{12}\log n + O(1), \quad \text{as } n \to \infty.$$

The asymptotic expansion above is of course in agreement with the result of Theorem 1, but the equivalence breaks for F other than uniform.

Chapter 4

Continuous-time selection problems

In this chapter we study a poissonised variant of the longest increasing subsequence problem. This variant was first mentioned by Samuels and Steele in [55], where they exploited it to prove the existence of the limit $\lim_{n\to\infty} v_n/\sqrt{n}$.

Suppose a sequence of independent random marks with given continuous distribution is observed at times of the unit-rate Poisson process. Each time a mark is observed, it can be selected or rejected, with every decision becoming immediately final. What is the maximum expected length $v(t)$ of increasing subsequence which can be selected over a given horizon t in an online fashion?

It is more convenient to work with the optimality equation in the poissonised selection problem since it is, in fact, a differential equation rather than a difference equation. Moreover, a stronger invariance property of the model results in a value function $v(t)$ depending on a single parameter, as compared to $v_n(z)$ in the discrete problem. The remaining horizon $(t-s)(1-x)$ here corresponds to the expected number of admissible observations $(n-i)(1-z)$ in the discrete-time problem.

In this chapter, by applying the comparison method, we derive refined asymptotic expansions of the expected value and the variance of the length of the longest increasing subsequence. We then represent the problem in terms of a controlled piecewise deterministic Markov process with decreasing paths. And, finally, with the aid of a renewal approximation, we give a novel proof of a central limit theorem for the length of the increasing subsequence selected under either the optimal strategy or a strategy sufficiently close to optimality.

We work in the setup first investigated by Baryshnikov and Gnedin [10], where the observations are d-dimensional vectors (for some fixed $d \in \mathbb{N}$). However, at the end of the chapter we discuss the implications for the more popular special case $d = 1$.

4.1 Multidimensional setting and the optimality equation

The d-dimensional problem was introduced by Baryshnikov and Gnedin [10]. Their main asymptotic result is

$$v(t) \sim \alpha_1^* t^{1/(d+1)}, \quad \text{as } t \to \infty, \tag{4.1}$$

with

$$\alpha_1^* = \frac{(d+1)}{(d+1)!^{1/(d+1)}}.$$

They also showed that the optimal value function in the discrete-time multidimensional problem has the same leading term of the asymptotic expansion. Moreover, they built a stationary strategy that achieves the optimality up to the principal order term in the discrete-time setting.

To build a multidimensional setting, we must first formalise useful notation and terminology. Bold symbols, from now on, represent d-dimensional vectors, bodies or stochastic processes with state spaces in $\mathbb{R}^d$. Most importantly, $\boldsymbol{x} \in [0,1]^d$ denotes a d-dimensional vector $(x^{(1)}, \ldots, x^{(d)})$. In line with this convention, $\mathbf{0}$ and $\mathbf{1}$ denote the all-zero and all-unit d-dimensional vectors, respectively.

Definition 9. *For d-dimensional vectors* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *we define relation* $\boldsymbol{x} < \boldsymbol{y}$ *component-wise:* $\boldsymbol{x} < \boldsymbol{y}$ *if and only if* $x^{(i)} < y^{(i)}$ *for all* $i = 1, \dots, d$.

Definition 10. *For d-dimensional vectors* $\boldsymbol{x}, \boldsymbol{y}$, *and* $\boldsymbol{z}$ *we define vector addition and subtraction operations as follows*

$$\boldsymbol{z} = \boldsymbol{x} \pm \boldsymbol{y} = (z^{(1)}, \dots, z^{(d)}), \quad \text{where } z^{(i)} = x^{(i)} \pm y^{(i)} \text{ for all } i = 1, \dots, d.$$

Definition 11. *For d-dimensional vectors* $\boldsymbol{x}, \boldsymbol{y},\ \boldsymbol{x} < \boldsymbol{y}$, *we call a d-dimensional interval* $\{\boldsymbol{z} : \boldsymbol{x} < \boldsymbol{z} < \boldsymbol{y}\}$ *a* box *and denote it by* $[\boldsymbol{x}, \boldsymbol{y}]$.

We also need to slightly adjust the formal definition of an online policy for the poissonised setup.

Definition 12. *Let* $(\boldsymbol{X_1}, T_1), (\boldsymbol{X_2}, T_2), \dots$ *be the atoms of a unit-rate Poisson point process, where* $T_1 < T_2 < \dots$ *are the arrival times. An* online selection policy *in the continuous-time increasing subsequence selection problem is a collection of stopping times* $\boldsymbol{\tau} = (\tau_1, \tau_2, \dots)$ *satisfying*

(i) *each* τ_i *is adapted to* $(\boldsymbol{X_1}, T_1), (\boldsymbol{X_2}, T_2), \dots$,

(ii) *each* τ_i *assumes values in the set* $\{T_j\}$,

(ii) $\tau_1 < \tau_2 < \dots$,

(iii) $\boldsymbol{X_{\tau_1}} < \boldsymbol{X_{\tau_2}} < \dots$.

Let $\boldsymbol{\Pi}$ be a random scatter of points in $[0, \infty) \times [0, 1]^d$ spread according to a homogeneuous Poisson point process with Lebesgue measure as intensity. The event $(s, \boldsymbol{x}) \in \boldsymbol{\Pi}$ that $\boldsymbol{\Pi}$ has an atom at $(s, \boldsymbol{x})$ is interpreted as an item $\boldsymbol{x}$ being observed at time s. A sequence of atoms $(s_1, \boldsymbol{x_1}), \dots, (s_k, \boldsymbol{x_k})$ is said to be increasing if $s_1 < \dots < s_k$ and $\boldsymbol{x_1} < \dots < \boldsymbol{x_k}$. We think of the configuration of points in a finite box $\boldsymbol{\Pi}_{|[0,s] \times [\boldsymbol{0}, \boldsymbol{1}]}$ as of information available to the decision-maker at time $s \geq 0$. The task is to maximise the expected length $v(t)$ of an increasing sequence over online selection strategies adapted to the aforementioned information.

Now, let $\boldsymbol{x}$ be the last selection at time s and let the function $p(\boldsymbol{x}) : [\boldsymbol{0}, \boldsymbol{1}] \to [0, 1]$ be the Euclidian volume of the box $[\boldsymbol{x}, \boldsymbol{1}]$

$$p(\boldsymbol{x}) = \prod_{i=1}^{d} (1 - x^{(i)}).$$

Then, from mapping the box $[(t-s), t] \times [\boldsymbol{x}, \boldsymbol{1}]$ on $[0, (t-s)\, p(\boldsymbol{x})] \times [\boldsymbol{0}, \boldsymbol{1}]$, it becomes apparent that the rest of selection can be represented as selecting an increasing subsequence from a Poisson process with intensity $(t-s)p(\boldsymbol{x})$. Thus, the optimal value function on the rest of selection is $v((t-s)\, p(\boldsymbol{x}))$. To derive the optimality equation, we recall the steps laid out in [10]. Suppose the first mark $\boldsymbol{x}$ is observed shortly after the start of the process at time $s \in [0, h]$. If $\boldsymbol{x}$ is selected, the mean length of selected subsequence gained by the optimal continuation is $1 + v((t-s)p(\boldsymbol{x}))$. If $\boldsymbol{x}$ is rejected, the optimal continuation yields $v(t-s)$. The dynamic programming principle prescribes to select $\boldsymbol{x}$ if and only if $1 + v((t-s)p(\boldsymbol{x})) \geq v(t-s)$, so the better action gives $\max\{1 + v((t-s)p(\boldsymbol{x})), v(t-s)\}$. Integrating over a random variable uniformly distributed on $[0, 1]^d$, we obtain a recursion

$$v(t) = (1-h)\, v(t) + h \int_{[\boldsymbol{0},\boldsymbol{1}]} \max\{v(tp(\boldsymbol{x})) + 1, v(t)\}\, \mathrm{d}\boldsymbol{x} + o(h).$$

Taking the limit as $h \to 0$ on both sides yields

$$v'(t) = \int_{[\boldsymbol{0},\boldsymbol{1}]} (v(tp(\boldsymbol{x})) - v(t) + 1)_+ \, \mathrm{d}\boldsymbol{x}, \tag{4.2}$$

complemented with the initial condition $v(\boldsymbol{0}) = 0$. One should note that (4.2) has a break point at $\alpha \in \mathbb{R}_+$, where $v(\alpha) = 1$; but, since we will be dealing mostly with asymptotics as t gets large, we can ignore this for now.

The dependence of the optimal value function on one parameter allows us to reduce the d-dimensional integral in (4.2) to a one-dimensional integral by making a substitution $p(\boldsymbol{x}) = 1 - \xi$

$$v'(t) = \int_0^1 (v(t(1-\xi)) - v(t) + 1)_+ \frac{|\log(1-\xi)|^{d-1}}{(d-1)!} \mathrm{d}\xi, \tag{4.3}$$

Finally, as means to complete a brief introduction, we provide here the definition of the analogue of a threshold policy in the multidimensional setup.

Definition 13. *A* Markovian *selection policy in a multidimensional continuous-time selection problem is an online selection policy that accepts an observation* $(s, \boldsymbol{x}) \in [0, t] \times [0, 1]^d$ *if and only if* $\boldsymbol{x} \subset \boldsymbol{y} + \boldsymbol{D}(t, s, \boldsymbol{y})$*, where* $\boldsymbol{y}$ *is the last selected item (with* $\boldsymbol{y} = \boldsymbol{0}$ *if no items were accepted). The d-dimensional region* $\boldsymbol{D}(t, s, \boldsymbol{y})$ *is called an* acceptance region *or an* acceptance window*.*

4.1.1 Suboptimal selection policies and the mean-constraint upper bound

From (4.2) and (4.3), it is clear that to solve the optimisation problem one needs to apply a Markovian selection policy with the acceptance window defined recursively as

$$\boldsymbol{D}(t, s, \boldsymbol{y}) = \{\boldsymbol{x} \in [\boldsymbol{0}, \boldsymbol{1}] : v((t-s)\, p(\boldsymbol{y} + \boldsymbol{x})) + 1 \geq v((t-s)\, p(\boldsymbol{y}))\}.$$

The acceptance region can be regarded as a multivariate control variable for the running maximum, which is a right-continuous Markov process $\boldsymbol{Y} = (\boldsymbol{Y}(s),\ 0 \leq s \leq t)$ starting with $\boldsymbol{Y}(0) = \boldsymbol{0}$, with piecewise constant paths increasing by positive jumps. At time s in state $\boldsymbol{y}$ a transition occurs at a rate equal to the Euclidian volume of $\boldsymbol{D}(t, s, \boldsymbol{y})$, and, given that $\boldsymbol{Y}$ jumps, the increment $\boldsymbol{Y}(s) - \boldsymbol{Y}(s-)$ is uniformly distributed in $\boldsymbol{D}$.

Intuitively, a large acceptance window steers $\boldsymbol{Y}$ from $\boldsymbol{0}$ to about $\boldsymbol{1}$ in just a few jumps. On the other hand, a small acceptance window makes the jumps rare, so the time resource expires before a substantial number of selections is made. The optimal acceptance region $\boldsymbol{D}$ yields the maximal expected number of jumps $v(t)$.

Analogously to the one-dimensional case, a *stationary* strategy in the multidimensional problem has the acceptance region of the form $\widetilde{\boldsymbol{D}}(t,s,\boldsymbol{y}) = \widetilde{\boldsymbol{D}}(t)$, depending neither on the time of observation nor on the running maximum, as long as $\boldsymbol{Y}$ does not reach the state $\boldsymbol{y}$ such that $\boldsymbol{y} + \widetilde{\boldsymbol{D}}(t) \not\subset [0,1]^d$. An example of the stationary strategy that achieves the maximal expected length up to the principal-order term was demonstrated by Baryshnikov and Gnedin [10]. This strategy involves choosing all subsequent observations $(s,\boldsymbol{x})$ that satisfy

$$\boldsymbol{x} - \boldsymbol{y} \in \widetilde{\boldsymbol{\Sigma}}, \quad \text{where } \widetilde{\boldsymbol{\Sigma}} = \{\boldsymbol{z} \in [0,1]^d : z^{(1)} + \ldots + z^{(d)} \leq \delta(t)\},$$

where they set $\delta(t) := ((d+1)!/t)^{1/(d+1)}$. Their choice of simplex as the shape for the stationary acceptance region hinged on the fact observed on p. 264 of [10], which we prove in details in the following lemma. We should note here that Stanke [58] showed the simplex to be the solution to a dual problem, while studying the selection of multidimensional vectors under a sum constraint. More specifically, he showed that the simplex is the shape maximising the volume for a given constraint on the maximal coordinate of the barycentre.

Lemma 11. *Of all bodies of fixed volume that lie in the positive orthant, the maximal coordinate of the barycentre is minimal for a coordinate simplex.*

Proof. First, recall that the standard coordinate simplex $\boldsymbol{\Sigma} \subset \mathbb{R}_+^d$ is the convex hull of 0 and d basis vectors. The volume of $\boldsymbol{\Sigma}$ is $1/d!$ and the barycenter coordinates are $((d+1)^{-1}, \ldots, (d+1)^{-1})$.

Let $\|x\|_p$ denote the l^p norm of $\boldsymbol{x} \in \mathbb{R}^d$; in particular, $\|x\|_\infty = \max_i |x^{(i)}|$. In general, under a *body* we shall mean a measurable set $\boldsymbol{E} \subset \mathbb{R}_+^d$ of finite Lebesgue measure (*volume*). In integral form, the volume and the barycentre of $\boldsymbol{E}$ are

$$\int_{\boldsymbol{E}} \mathrm{d}\boldsymbol{x} \quad \text{and} \quad \frac{\int_{\boldsymbol{E}} \boldsymbol{x}\,\mathrm{d}\boldsymbol{x}}{\int_{\boldsymbol{E}} \mathrm{d}\boldsymbol{x}},$$

respectively, where $\mathrm{d}\boldsymbol{x}$ is the element of Lebesgue measure and $\boldsymbol{x} \in \mathbb{R}^d$ is the identity function. Since we take the volume as a constraint, minimising the maximal coordinate of the barycentre is equivalent to the following variational problem:

$$\left\| \int_{\boldsymbol{E}} \boldsymbol{x} \mathrm{d}\boldsymbol{x} \right\|_{\infty} \to \min, \quad \text{subject to } \int_{\boldsymbol{E}} \mathrm{d}\boldsymbol{x} = \frac{1}{d!}, \quad \boldsymbol{E} \subset \mathbb{R}_+^d. \tag{4.4}$$

We have $\|z\|_1 \leq d\,\|z\|_\infty$, with equality sign when $|z_1| = \ldots = |z_d|$; thus, to prove that $\boldsymbol{\Sigma}$ is the solution to (4.4), it is sufficient to show that $\boldsymbol{\Sigma}$ solves

$$\left\| \int_{E} \boldsymbol{x} \mathrm{d}\boldsymbol{x} \right\|_{1} \to \min, \quad \text{subject to } \int_{E} \mathrm{d}\boldsymbol{x} = \frac{1}{d!}, \quad \boldsymbol{E} \subset \mathbb{R}_+^d. \tag{4.5}$$

Next, we observe that for both (4.4) and (4.5) it is sufficient to consider only star-shaped sets $\boldsymbol{E}$ with an apex at $\mathbf{0}$. We appeal to the intuitive proof that if some points are not seen from the origin, the barycentre can be moved closer to $\mathbf{0}$ by transporting some mass along the rays to fill in the holes. Similarly, it suffices to focus on star-shaped domains which have a nontrivial intersection with every positive ray.

Introduce the polar coordinates $r = \|\boldsymbol{x}\|$, $\boldsymbol{s} = \boldsymbol{x}/\|\boldsymbol{x}\|$ for all $\boldsymbol{x} \neq \mathbf{0}$, where $\boldsymbol{s}$ varies over $\boldsymbol{S}_+$, the intersection of the unit sphere with the positive orthant. In polar coordinates, a star-shaped domain has the form

$$\boldsymbol{E} = \{(r, \boldsymbol{s}) : 0 < r \leq \rho(\boldsymbol{s})\}$$

form some function $\rho : \boldsymbol{S}_+ \to \mathbb{R}_+$. In particular, $\rho(\boldsymbol{s}) = 1/\|\boldsymbol{s}\|_1$ corresponds to $\boldsymbol{\Sigma}$, because the equation $r\,\|s\|_1 = 1$ in Euclidian coordinates translates to $\|x\|_1 = 1$.

The volume and the barycentre can be represented in the integral form as

$$V(\rho) = \int_{\boldsymbol{S}_+} (\rho(\boldsymbol{s}))^d \sigma(\mathrm{d}\boldsymbol{s}) \quad \text{and} \quad m(\rho) = \frac{1}{V(\rho)} \int_{\boldsymbol{S}_+} \boldsymbol{s}(\rho(\boldsymbol{s}))^{d+1} \sigma(\mathrm{d}\boldsymbol{s})$$

respectively, where the second integral involves the identity function $\boldsymbol{s} = (s^{(1)}, \ldots, s^{(d)})$, and $\sigma(\mathrm{d}\boldsymbol{s})$ denotes the $(d-1)$-dimensional spherical Lebesgue measure with normalisation

$$\int_{\boldsymbol{S}_+} \sigma(\mathrm{d}\boldsymbol{s}) = \frac{\pi^{d/2}}{\Gamma(d/2+1)\,2^d}. \tag{4.6}$$

Therefore, (4.5) can be reduced to

$$\|V(\rho)\,m(\rho)\|_1 \to \min, \quad \text{subject to } V(\rho) = \frac{1}{d!},$$

where the function $\rho : \boldsymbol{S}_+ \to \mathbb{R}_+$ belongs to $L^{d+1}(\boldsymbol{S}_+, \sigma)$. The problem has Lagrange function

$$\mathcal{L}(\rho, \eta) = \int_{S_+} \left[(\rho(\boldsymbol{s}))^{d+1} \|\boldsymbol{s}\|_1 - \eta(\rho(\boldsymbol{s}))^d\right] \sigma(\mathrm{d}\boldsymbol{s}).$$

Hence, for $\eta > 0$, $\mathcal{L}(\rho, \eta)$ is uniquely minimised at

$$\rho(\boldsymbol{s}) = \frac{\eta d}{(d+1)\,\|\boldsymbol{s}\|_1}. \tag{4.7}$$

To have $V(\rho) = 1/d!$, we choose $\eta = (d+1)/d$, in which case (4.7) corresponds to the standard coordinate simplex. This completes the proof of the asserted extremal property of $\boldsymbol{\Sigma}$. □

The solution to (4.5) can be stated as a solution to an isoperimetric inequality

$$\sum_{i=1}^{d} \int_{\boldsymbol{E}} x^{(i)} \mathrm{d}\boldsymbol{x} \geq \frac{d(d!)^{1/d}}{d+1} \left(\int_{\boldsymbol{E}} \mathrm{d}\boldsymbol{x}\right)^{(d+1)/d}, \quad \boldsymbol{E} \subset \mathbb{R}_+^d,$$

where equality is achieved only for a coordinate simplex $c\boldsymbol{\Sigma}$, $c > 0$. For star-shaped regions, this is the same as the inequality

$$\sum_{i=1}^{d} \int_{\boldsymbol{S}_+} s^{(i)} (\rho(\boldsymbol{s}))^{d+1} \sigma(\mathrm{d}\boldsymbol{s}) \geq \frac{d\,(d!)^{1/d}}{d+1} \left(\int_{\boldsymbol{S}_+} (\rho(\boldsymbol{s}))^{d+1} \sigma(\mathrm{d}\boldsymbol{s})\right)^{(d+1)/d}.$$

Note, however, that with normalisation (4.6) the traditional $(d-1)$-dimensional spherical volume measure is not σ, but rather is proportionate $\widetilde{\sigma} = d\sigma$. If integration with $\widetilde{\sigma}$ is used, the constant in the inequality above should be replaced with

$$\frac{d^{(d-1)/d}(d!)^{1/d}}{d+1}.$$

Now, let us get back to the increasing subsequence selection. Set the acceptance region of a stationary policy $\boldsymbol{D}(t) := \widetilde{\boldsymbol{\Sigma}}$. Up to the first instance when the last selection $\boldsymbol{y}$ is such that $\boldsymbol{y} + \boldsymbol{D}(t) \notin [0,1]^d$, the running maximum process $\boldsymbol{Y}$ coincides with a compound Poisson process $\boldsymbol{S}$, characterised by the jump rate $V(\boldsymbol{D})$ and the $[\boldsymbol{0}, \boldsymbol{D}]$-uniform distribution of increments. For $t \to \infty$, $V(\boldsymbol{D}) \to 0$ but $tV(\boldsymbol{D}) \to \infty$. The number of jumps of $\boldsymbol{S}$ over the time horizon t is asymptotic to $tV(\boldsymbol{D})$, and the number of jumps until $\boldsymbol{S}$ passes the state $\boldsymbol{s} : \boldsymbol{s} + \boldsymbol{D} \notin [0,1]^d$ is asymptotic to $1/m^{(1)}(\boldsymbol{D})$ (note that by symmetry any coordinate $i = 1, \ldots, d$ can be picked). By monotonicity of $tV(\boldsymbol{D})$ and $1/m^{(1)}(\boldsymbol{D})$ in $\delta(t)$, the maximum $tV(\boldsymbol{D}) \wedge 1/m^{(1)}(D)$ is achieved precisely at $\delta(t) = ((d+1)!/t)^{1/(d+1)}$. This strategy maintains a balance between increasing on the marks and time scales so that the running maximum $\boldsymbol{Y}$ fluctuates roughly about the main diagonal of $[0,1]^d$.

Now, as to the compound Poisson process $\boldsymbol{S}$ controlled by $\boldsymbol{D}(t)$, the expected number of jumps is equal to

$$\frac{d+1}{(d+1)!^{1/(d+1)}}\, t^{1/(d+1)}.$$

We show that the upper-bound on $v(t)$ can be obtained by comparing $\boldsymbol{S}$ to the optimal chain with a weaker *mean-value* constraint. To that end, consider an online problem of selecting marks from the Poisson random measure in the unbounded domain $[0,t] \times [0,\infty)^d$, but with the restriction that the next observation $(s, \boldsymbol{x})$ may be selected if and only if $\boldsymbol{0} < \boldsymbol{x} - \boldsymbol{y} \leq \boldsymbol{1}$, where $\boldsymbol{y}$ is the state of the running maximum at the time instance s. Set the objective to maximise the number of selections subject to the constraint that the sum of the *mean* coordinates of the last selection does not exceed d.

Clearly, every strategy with selections made from $[0,t] \times [0,1]^d$ is also admissible in the extended scenario. In the extended scenario, the observations that satisfy the constraint arrive as a unit-rate Poisson process independent of the selected marks. If $\boldsymbol{y}$ is the last selection made before time s, the next (if any) observation is $\boldsymbol{y} + \boldsymbol{\xi}$, where $\boldsymbol{\xi} \stackrel{d}{=} \text{Uniform}[0,1]^d$ and independent of the previous marks. Let $(s_i, \boldsymbol{\xi_i})$ be the increasing sequence of observation times of these marks and $\boldsymbol{\xi_i}$ their associated uniform variables. The decisions at time s_i whether to accept or reject the observation can be represented by an indicator function adapted to the Poisson random measure within $[0, s_i] \times [0, \infty)^d$. The task comes down to choosing suitable acceptance regions $\boldsymbol{\Sigma_i}$

$$\mathbb{E}\left(\sum_i \mathbb{1}\left(\boldsymbol{\xi_i} \in \boldsymbol{\Sigma_i}\right)\right) \to \max, \quad \text{subject to } \; \mathbb{E}\left(\sum_i \sum_{j=1}^d \mathbb{1}\left(\boldsymbol{\xi_i} \in \boldsymbol{\Sigma_i}\right) \xi_i^{(j)}\right) \leq d.$$

The optimal solution is $\boldsymbol{\Sigma_i} = \boldsymbol{\Sigma}^*$, where $\boldsymbol{\Sigma}^*$ is the shape maximising the volume given the constraint on the sum of the coordinates of the barycentre. Finding the correct shape is precisely the variational problem solved by Lemma 11; therefore, the shape of the region we seek is the coordinate simplex. All that left is to work out the side δ^* of the simplex. Restating the optimisation problem in terms of $\boldsymbol{\Sigma}^*$ we obtain

$$tV(\boldsymbol{\Sigma}^*) \to \max, \quad \text{subject to } \; tV(\boldsymbol{\Sigma}^*) m^{(1)}(\boldsymbol{\Sigma}^*) \leq 1,$$

where the new constraint is obtained by noting that the barycentre of a coordinate simplex has coordinates equal along all axes. The constraint yields the optimal solution $\delta^* = \delta(t)$. Hence, the selected chain has the distribution equivalent to the compound process $\boldsymbol{S}$; therefore, the following upper-bound follows.

Lemma 12 (Mean-constraint bound)**.** *The maximal expected length of an increasing subsequence that can be selected from d-dimensional elements in continuous-time setting $v(t)$ satisfies the upper bound*

$$v(t) \leq \frac{(d+1)}{(d+1)!^{1/(d+1)}} t^{1/(d+1)}, \quad t \geq 0.$$

4.1.2 Refined asymptotic expansion of the optimal value function

Before we apply the comparison method to obtain a refined asymptotic expansion of $v(t)$, we linearise the optimality equation (4.3). Set $\mu(z,y)$ to

$$\mu(z,y) := z^{d-1}\,(1-y/z)^d\,|\log(1-y/z)|^{d-1}\,.$$

With the change of variables $u(z) := v(z^{d+1})$, the optimality equation (4.3) becomes

$$u'(z) = (d+1)z^d \int_0^z (u(z(1-\xi)^{1/(d+1)}) - u(z) + 1)_+ \frac{|\log(1-\xi)|^{d-1}}{(d-1)!}\,\mathrm{d}\xi, \quad u(0) = 0.$$

Now, substituting $z(1-\xi)^{1/(d+1)} =: z - y$ yields a convolution-type equation

$$u'(z) = \frac{(d+1)^{d+1}}{(d-1)!} \int_0^z (u(z-y) - u(z) + 1)_+\,\mu(z,y)\mathrm{d}y, \quad u(0) = 0. \tag{4.8}$$

Note that using the monotonicity property of the integrand, we can rewrite (4.9) as

$$u'(z) = \frac{(d+1)^{d+1}}{(d-1)!} \int_0^{\theta(z)} (u(z-y) - u(z) + 1)\ \mu(z,y)\mathrm{d}y, \tag{4.9}$$

where

$$\begin{cases} \theta(z) = z, & \text{for } u(z) \le 1, \\ \theta(z) \text{ solves } u(z-y) - u(z) + 1 = 0, & \text{otherwise.} \end{cases} \tag{4.10}$$

Equation (4.9) is a special case of the more general equation

$$w'(z) = \frac{(d+1)^{d+1}}{(d-1)!} \int_0^{\theta(z)} (w(z-y) + r(z) - w(z))\,\mu(z,y)\mathrm{d}y, \quad w(0) = b, \tag{4.11}$$

where $r(z)$ and $\theta(z)$ are given functions on $[0,\infty)$, $0 < \theta(z) \le z$, and b is a constant. Apart from more general inhomogeneous term and initial condition, a major difference between (4.9) and (4.8) is that the integrand need not be sign-definite, nor should $\theta(z)$ be a zero of the integrand.

Let $\mathcal{I}$ be the integral operator acting on functions $g \in C^1[0,\infty)$ as

$$\mathcal{I}g(z) := \frac{(d+1)^{d+1}}{(d-1)!}\int_0^{\theta(z)} (g(z-y) - g(z) + 1)\,\mu(z,y)\mathrm{d}y.$$

In this notation equation (4.8) becomes $u' = \mathcal{I}u,\ u(0) = 0$. We will now compare the solution to (4.8) with various test functions.

Let $u_1(z) := \alpha_1 z,\ \alpha_1 \in \mathbb{R}^+$. We have $u_1'(z) = \alpha_1$ and, with $\theta_1(z) := 1/\alpha_1$,

$$\mathcal{I}u_1(z) = \frac{(d+1)^{d+1}}{(d-1)!}\int_0^{\theta_1(z)} (u_1(z-y) + 1 - u_1(z))\,\mu(z,y)\mathrm{d}y \to \frac{(d+1)^d}{\alpha_1^d\, d!}, \quad z \to \infty.$$

The match $\alpha_1 = (d+1)^d/(\alpha_1^d\, d!)$ occurs at $\alpha_1 = \alpha_1^* := (d+1)/(d+1)!^{1/(d+1)}$; thus, by Lemma 3, $\limsup_{z\to\infty}(u(z) - u_1(z)) < \infty$ for $\alpha_1 > \alpha_1^*$ and therefore $\limsup_{z\to\infty} u(z)/z \le \alpha_1^*$. Likewise, the second part of the lemma yields $\liminf_{z\to\infty} u(z)/z \ge \alpha_1^*$. These bounds imply $u(z) \sim \alpha_1^* z$, which corresponds to the principal-term asymptotics (4.1) from [10].

We try next functions $u_2(z) := \alpha_1^* z + \alpha_2 \log(z+1),\ \alpha_2 \in \mathbb{R}$ (we take $\log(z+1)$ and not $\log z$ to avoid the unpleasant singularity at 0). Solving $u_2(z-y) + 1 - u_2(z) = 0$, for large z we get expansion

$$\theta_2(z) \sim \frac{1}{\alpha_1^*} - \frac{\alpha_2}{\alpha_1^{*2}(z+1)}. \tag{4.12}$$

We may proceed with only the first term in (4.12) since the second makes a negligible $O(z^{-2})$ contribution to $\mathcal{I}u_2(z)$. This is confirmed by the following lemma.

Lemma 13. *Let $\varepsilon(t) = o(\theta_2(z))$. Then, adding $\varepsilon(z)$ to $\theta_2(z)$ results in $O(\varepsilon^2)$-order shift in $\mathcal{I}u_2(t)$.*

Proof. Let us define a function $I(x)$ which takes the upper limit of the integral as an argument

$$I(x) = \frac{(d+1)^{d+1}}{(d-1)!}\int_0^{x} (u_2(z-y) - u_2(z) + 1)\ \mu(z,y)\mathrm{d}y.$$

Expanding I into Taylor series around θ_2 yields for some $\xi \in [0,1]$

$$I(\theta_2+\varepsilon)-I(\theta_2)=I'(\theta_2)\varepsilon+I''(\theta_2+\xi\varepsilon)\frac{\varepsilon^2}{2}.$$

The first term on the right-hand side vanishes since θ_2 is the stationary point of the integrand. For the second term, we expand the integrand into series as $z\to\infty$,

$$(u_2(z-y)-u_2(z)+1)\ \mu(z,y)=(1-\alpha_1^*)\,y^{d-1}+O(z^{-1}).$$

Now, using this expansion and the expansion (4.12), we can show that

$$I''(\theta_2+\xi\varepsilon)\frac{\varepsilon^2}{2}=(1-\alpha_1^*)(d-1)(\theta_2+\xi\varepsilon)^{d-2}\,\frac{\varepsilon^2}{2}=O(\varepsilon^2).$$

□

With Lemma 13 proved, we use the one-term expansion $\theta_2\sim 1/\alpha_1^*,\ z\to\infty$, to obtain

$$\mathcal{I}u_2(z)\sim\alpha_1^*-\left(d\alpha_2+\frac{d(d+1)}{2(d+2)}\right)\frac{1}{z+1},\quad z\to\infty.$$

With $u_2'(z)=\alpha_1^*+\alpha_2/(z+1)$, the match between $u_2'(z)$ and $\mathcal{I}u_2(z)$ occurs at $\alpha_2^*=-d/(2(d+2))$. It follows readily from Lemma 3 that $(u(z)-\alpha_1^*z)/(\log(z+1))\to\alpha_2^*$; that is

$$u(z)\sim\alpha_1^*z+\alpha_2^*\log z,\quad z\to\infty.$$

To further refine the approximation, we try

$$u_3(z):=\alpha_1^*z+\alpha_2^*\log(z+1)+\frac{\alpha_3}{z+1},\quad \alpha_3\in\mathbb{R}. \tag{4.13}$$

This time we actually need to calculate $\mathcal{I}$ up to the order $O(z^{-2})$-term. Recalling Lemma

13, one needs to obtain a two-term expansion

$$\theta_3(z) \sim \frac{1}{\alpha_1^*} - \frac{\alpha_2^*}{\alpha_1^{*2}(z+1)}, \quad z \to \infty. \tag{4.14}$$

Expanding the integrand and integrating, we get

$$\mathcal{I}u_3(z) \sim \alpha_1^* + \frac{\alpha_2^*}{z} + d\left(\alpha_2^* - \frac{(d^2+d+1)(d+1)!^{1/(d+1)}}{6(d+2)^2(d+3)} + \alpha_3\right)\frac{1}{z^2}, \quad z \to \infty.$$

To match with

$$u_3'(z) \sim \alpha_1^* + \frac{\alpha_2^*}{z} - \frac{\alpha_2^* + \alpha_3}{z^2}, \quad z \to \infty,$$

we must choose

$$\alpha_3 := \alpha_3^* - \alpha_2^*,$$

where

$$\alpha_3^* = \frac{(d^2+d+1)(d+1)!^{1/(d+1)}}{6(d+3)(d+2)^2(d+1)}.$$

Taking α_3 bigger or smaller than $\alpha_3^* - \alpha_2^*$ enables us to sandwich u. However, our comparison method based on Lemma 3 only yields

$$u(z) = \alpha_1^* z + \alpha_2^* \log z + O(1), \quad z \to \infty, \tag{4.15}$$

since the third term in (4.13) is already bounded. A different approach will be applied to show convergence of the $O(1)$ remainder.

4.1.3 A piecewise deterministic Markov process

By the self-similarity of the continuous-time problem, if $\boldsymbol{y}$ is the running maximum at time s, the distribution of the number of selections to follow only depends on the process past through $(t-s)p(\boldsymbol{y})$. This suggests merging the running maximum and the observation time into one parameter and studying its evolution. Adopting $z := ((t-s)p(\boldsymbol{y}))^{1/(d+1)}$ as a state variable and introducing an intrinsic time variable will lead us to a nearly homogeneous Markov process which we denote Z.

Let $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $0 < \theta(z) \leq z$; introduce, for $z \geq 0$, a normalising factor

$$\lambda(z) := \int_0^{\theta(z)} \mu(z,y)\,\mathrm{d}y \sim \frac{\theta(z)^d}{d} - \frac{\theta(z)^{d+1}}{2z} + O(z^{-2}), \quad z \to \infty.$$

The following rules define a piecewise deterministic Markov process Z on $[0,\infty)$ with continuous drift component and random instantaneous jumps:

(i) the process decreases continuously with unit speed,

(ii) the jumps are negative and occur at rate

$$\frac{(d+1)^{d+1}}{(d-1)!}\,\lambda(z), \quad \text{for } z > 0,$$

(iii) if a jump from state z occurs, the jump size has density $\mu(z,y)\,\mathrm{d}y$ with support $[0,\theta(z)]$,

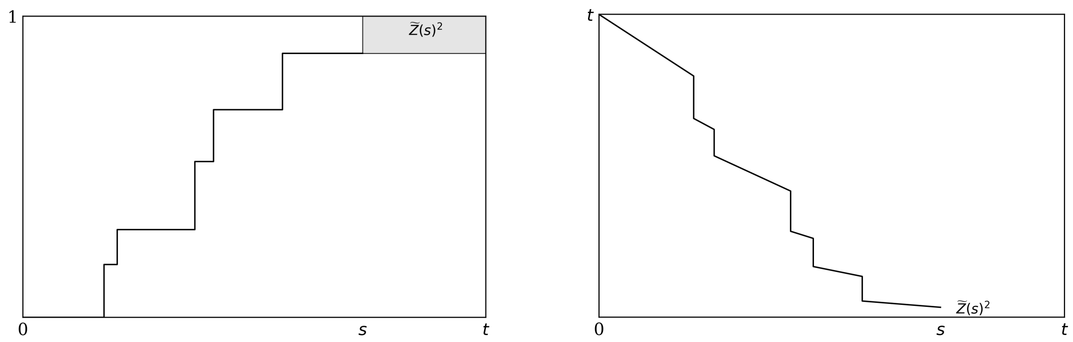
(iv) the process terminates upon reaching 0.

We denote $Z|z_0$ this process starting in position z_0. The range of $Z|z_0$ can be constructed from the set of arrivals of an inhomogeneous marked Poisson process Π with intensity (ii) and marks distributed as in (iii). The following occupancy procedure is similar to many familiar parking, packing, and scheduling models in applied probability. With each occurrence z of Π marked y relate interval $(z-y,z]$. Now, moving right-to-left from z_0 create a non-overlapping configuration by leaving the rightmost $(z_1-y_1,z_1]$ in its position and removing all other intervals that overlap this one, then proceed this way to the left of z_1-y_1 until reaching 0. The process $Z|z_0$ crosses each $(z_j-y_j,z_j]$ by jump, and drifts through the rest of $[0,z_0]$. A location $z \in (0,z_0)$ is called a *jump point* if $z \in \{z_j,\ j \geq 1\}$, a *gap point* if $z \in \cup_j(z_j-y_j,z_j]$, and a *drift point* otherwise. For the corresponding path of $Z|z_0$, there is a unique way to introduce the time variable in agreement with rule (i). Specifically, the time when $Z|z_0$ reaches z is equal to the Lebesque measure of the set of drift points within $[z, z_0]$. The path is naturally decomposed in *cycles*, each comprised of a drift interval and a jump interval in the right-to-left succession. The rightmost cycle is $(z_1 - y_1, z_1] \cup (z_1, z_0]$, and the lefmost cycle has only a drift interval.

To connect to the increasing subsequence problem fix horizon t and let $\boldsymbol{Y}$ be the running maximum process under some Markovian strategy. Let

$$\widetilde{Z}(s) := ((t-s)p(\boldsymbol{Y}(s)))^{1/(d+1)}, \quad s \in [0,t],$$

which is a drift-jump process decreasing from $t^{1/(d+1)}$ to 0, with negative jumps $\Delta\widetilde{Z}(s) = \widetilde{Z}(s) - \widetilde{Z}(s-)$ at times of selection. Figure 4.1 illustrates the correspondence for the special case $d = 1$.



(a) The running maximum and $\widetilde{Z}(s)^2$ (b) A path of $\widetilde{Z}(s)^2$

Figure 4.1: Transformation of Y to $\widetilde{Z}$

We wish to replace the observation time s by an intrinsic time parameter associated with drift. To that end, first note that the decay of $\widetilde{Z}$ due to the drift is a strictly increasing continuous process

$$\sigma(s) := t^{1/(d+1)} - \widetilde{Z}(s) + \sum_{s' \le s} \Delta\widetilde{Z}(s').$$

For $\sigma^{\leftarrow}$ the inverse function to σ, define the time-changed process

$$Z(q) := \widetilde{Z}(\sigma^{\leftarrow}(q)), \quad q \le \sigma(t). \tag{4.16}$$

Process $\boldsymbol{Y}$ over horizon $t = z^{d+1}$ has the same number of jumps as $Z|z$. This reduces the optimal selection problem with horizon t to choosing a control function θ with the objective to maximise the expected number of jumps of $Z|t^{1/(d+1)}$.

Denote $N_\theta(z)$ the number of jumps of the process $Z|z$ steered by given function θ ($0 < \theta(z) \leq z$), and let $u_\theta(z) := \mathbb{E}N_\theta(z)$. With probability $(d+1)^{d+1}/(d-1)!\,\lambda(z)\mathrm{d}z$ the process moves from a small vicinity of z to $z - y$, with y sampled from the density in (iii), in which case the expected number of jumps is equal to $u_\theta(z-y)+1$. Otherwise, the process drifts through to $z - \mathrm{d}z$. This decomposition readily yields equation

$$u'_\theta(z) = \frac{(d+1)^{d+1}}{(d-1)!}\int_0^{\theta(z)} (u_\theta(z-y) + 1 - u_\theta(z))\,\mu(z,y)\mathrm{d}y, \quad u_\theta(0) = 0, \qquad (4.17)$$

which is a special case of (4.11) derived earlier in the context of the running maximum $\boldsymbol{Y}$.

In purely analytic terms, for any fixed z, maximising $u_\theta(z)$ over admissible θ is the problem of calculus of variations. The solution is $\theta = \theta^*$, defined implicitly by equations (4.8) and (4.10).

We shall assume throughout that θ is bounded and differentiable. That the optimal θ^* is bounded can be seen at this stage of our analysis from (4.10) and (4.15).

The asymptotic comparison method based on Lemma 3 works for (4.17) smoothly. In particular, for

$$\theta_0(z) := 1/\alpha_1^* \wedge z,$$

we obtain the same expansion as (4.15). Complementing this technique, we will adopt some ideas from the potential theory for Markov processes.

The decreasing sequence of jump points of $Z|z_0$ is an embedded Markov chain with terminal state 0. Let $U_\theta(z_0, \cdot)$ be the occupation measure on $[0, z_0]$ counting the expected number of jump points, in particular $U_\theta(z, [0,z]) = u_\theta(z)$. Denote $p(z_0, z)$, for $0 \leq z \leq z_0$ the probability that z is a drift point, in particular $p(z_0, z_0) = p(z_0, 0) = 1$. There is a jump point within dz only if z does not belong to a gap, hence the occupation measure has a density which factorises as

$$U_\theta(z_0, dz) = \frac{(d+1)^{d+1}}{(d-1)!}\lambda(z)\, p(z_0, z)\ \mathrm{d}z, \quad 0 \le z \le z_0.$$

Lemma 14. *There exists a pointwise limit* $p(z) := \lim_{z_0 \to \infty} p(z_0, z)$*, which satisfies*

$$|p(z_0, z) - p(z)| < ae^{-\alpha(z_0 - z)}, \quad 0 < z < z_0,$$

with some positive constants a *and* α*.*

Proof. The proof is by coupling. Choose constant $\overline{\theta}$ big enough to have $\sup \theta(z) < \overline{\theta}$. Fix $z < z_0 < z_1$ with $z > 2\overline{\theta}$ (the latter assumption does not affect the result). Consider two independent processes Z_0 and Z_1 with $Z_0 \stackrel{d}{=} Z|z_0,\ Z_1 \stackrel{d}{=} Z|z_1$. Define Z' by running the process Z_1 until it hits a drift point ξ of Z_0, then from this point on switch over to running Z_0. Such a point ξ exists since both processes have a gap adjacent to 0. By the strong Markov property, Z' has the same distribution as Z_1. If the coupling occurs at some $\xi \in [z, z_0]$, the point z is of the same type (drift or jump) for both Z' and Z_0.

The coupling does not occur within $[z, z_0]$ only if Z_0 and Z_1 have no common drift points within these bounds. Given that $y > z$ is a drift point, the probability that the drift interval covering y extends to the left over $y - \overline{\theta}$ is at least π, for some constant $\pi > 0$. This follows since the length of drift interval dominates stochastically an exponential random variable with rate $\sup \lambda(z)\, (d+1)^{d+1}/(d-1)! < \infty$. In particular, the rightmost drift interval, adjacent to z_0, is shorter than $\overline{\theta}$ with probability at most $1-\pi$, in which case the rightmost cycle is shorter than $2\overline{\theta}$. Given ξ is not in the first cycle, the probability that ξ is not in the second is again at most $1 - \pi$, in which case also the second cycle is shorter than $2\overline{\theta}$. Continuing so forth we see that $\xi \notin [z, z_0]$ with probability at most $(1 - \pi)^k$ for $k = \lfloor (z_0 - z)/(2\overline{\theta}) \rfloor$. This readily implies an exponential bound $|p(z_0, z) - p(z_1, z)| < ae^{-\alpha(z_0 - z)}$, uniformly in $z_1 > z_0$. Sending $z_0 \to \infty$ we see that

$p(z_0, z)$ is a Cauchy sequence, whence the claim. □

In the terminology of random sets, $p(z_0, \cdot)$ is the coverage function (see [47], p. 23) for the range of $Z|z_0$. As $z_0 \to \infty$, the range converges weakly to a random set $\mathcal{Z} \subset [0, \infty)$, comprised of infinitely many intervals separated by gaps. Indeed, let $A(z_0, z) \leq z$ be the maximal point of the range of $Z|z_0$ within $[0, z]$ for $z \leq z_0$. The coupling argument in the lemma also shows that $A(z_0, z)$ has a weak limit $A(z)$, which is sufficient to justify convergence of the range intersected with $[0, z]$, due to the Markov property. By Sheffé's lemma $U_\theta(z_0, \cdot)$ converges weakly to some U_θ, which is the occupation measure for the point process of left endpoints of intervals making up $\mathcal{Z}$.

4.1.4 The reward processes

Suppose each jump point of $Z|z$ is weighted by some location-dependent reward r. Let $w_{\theta,r}(z)$ be the total expected reward accumulated by $Z|z$ controlled by θ. Now, in addition to (4.11), we also have an integral representation of $w_{\theta,r}(z)$ as the average over the occupation measure,

$$w_{\theta,r}(z) = \int_0^z r(y)\, U_\theta(z, \mathrm{d}y) = \frac{(d+1)^{d+1}}{(d-1)!} \int_0^z r(y)\lambda(y)\, p_\theta(z, y)\mathrm{d}y. \tag{4.18}$$

Lemma 15. *For an integrable function r, the solution to (4.11) has a finite limit*

$$\rho_{\theta,r} := \lim_{z\to\infty} w_{\theta,r}(z) = \frac{(d+1)^{d+1}}{(d-1)!} \int_0^\infty r(y)\lambda(y)p_\theta(y)\ \mathrm{d}y. \tag{4.19}$$

If $|r(z)| = O(z^{-\beta})$ as $z \to \infty$ for some $\beta > 1$ then $|w_{\theta,r}(z) - \rho_{\theta,r}| = O(z^{-\beta+1})$.

Proof. Since $p(z_0, z)\lambda(z) < \overline{\theta}$ the existence of the limit follows from (4.18), (4.19) and Lemma 14 by the dominated convergence. The convergence rate is estimated by splitting the difference as

$$\begin{aligned}\rho_{\theta,r} - w_{\theta,r}(z) &= \frac{(d+1)^{d+1}}{(d-1)!}\int_0^{z/2} r(y)\lambda(y)(p_\theta(z,y) - p_\theta(y))\,\mathrm{d}y \\ &+ \frac{(d+1)^{d+1}}{(d-1)!}\int_{z/2}^{\infty} r(y)\lambda(y)p_\theta(y)\,\mathrm{d}y,\end{aligned}$$

where the second integral is of the order $O(z^{-\beta+1})$ while the first is of the lesser order $O(e^{-\alpha z/2})$ by Lemma 14. □

4.1.5 Convergence of the $O(1)$-term

We are ready to derive finer asymptotics. Let

$$r(z) := \frac{(d+1)^{d+1}}{\lambda(z)(d-1)!}\int_0^{\theta} (u(z-y) - u(z) + 1)\,\frac{\partial \mu(z,y)}{\partial z}\mathrm{d}y.$$

Differentiating (4.8) and keeping an account of (4.10), we obtain

$$u''(z) = \frac{(d+1)^{d+1}}{(d-1)!}\int_0^{\theta^*(z)} (u'(z-y) + r(z) - u'(z))\,\mu(z,y)\mathrm{d}y, \quad u'(0) = 0. \tag{4.20}$$

Since $\theta^*(z) = z$ for small z, this has a simple pole at 0, but the singularity is compensated in (4.18), so Lemma 15 and (4.15) ensure that

$$u'(z) = \alpha_1^* + O(z^{-1}). \tag{4.21}$$

With (4.21) at hand, expanding the solution to (4.10) we get

$$\theta^*(z) = \frac{1}{\alpha_1^*} + O(z^{-1}).$$

Replacing θ^* by $1/\alpha_1^*$ in (4.8) incurs remainder of smaller order $O(z^{-2})$ because $\theta^*(z)$ is the stationary point of the integral viewed as a function of the upper bound. Recalling that u_2 (with $\alpha_2 = \alpha_2^*$) satisfies $u_2'(z) = \mathcal{I}u(z) + O(z^{-2})$, for the difference $w = u - u_2$ we obtain equation (4.11) with $r(z) = O(z^{-2})$, hence $u(z) - u_2(z)$ by Lemma 15 approaches a finite limit at rate $O(z^{-1})$ as $z \to \infty$. This proves an expansion

$$u(z) = \alpha_1^* z + \alpha_2^* \log z + c^* + O(z^{-1}), \quad z \to \infty \tag{4.22}$$

with some constant c^*.

Our methods are not geared to identify c^*, because the initial value $u(0) = 0$ was used nowhere, but changing it to $u(0) = b$ (which is resorting to a selection problem with terminal reward b) will result in adding b to c^*. Nevertheless, with some more effort it is possible to go beyond $O(1)$. Let us first estimate the local variation of u'.

Lemma 16. *For fixed* $\overline{h} > 0$, *as* $z \to \infty$

$$\sup_{0 \le h \le \overline{h}} |u'(z+h) - u'(z)| = O(z^{-2}).$$

Proof. Using the integral representation (4.18) of u' with $r(z) = O(z^{-2})$, write

$$u'(z+h) - u'(z) = \int_z^{z+h} r(y)\, p(z+h, y)\, \lambda(y)\, \mathrm{d}y + \int_0^z |p(z+h, y) - p(z, y)|\, \lambda(y)\, r(y)\, \mathrm{d}y.$$

The first integral is obviously $O(z^{-2})$ uniformly in $h \le \overline{h}$. By Lemma 14, the second is estimated as

$$c \int_0^z e^{-\alpha(z-y)} (y^2 + 1)^{-1} \mathrm{d}y = O(z^{-2})$$

using Laplace's method. □

The lemma applied to the right-hand side of (4.20) gives $u''(z) = O(z^{-2})$. In (4.9) we replace θ^* by $1/\alpha_1^*$, expand $u(z-y) - u(z) = -yu'(z) + O(z^{-2})$ and integrate to obtain with some algebra

$$u'(z) = \alpha_1^* + \frac{\alpha_2^*}{z} + O(z^{-2}).$$

Expanding similarly in (4.10) we get a finer formula for the optimal control function

$$\theta^*(z) = \frac{1}{\alpha_1^*} - \frac{\alpha_2^*}{\alpha_1^{*2} z} + O(z^{-2}), \quad z \to \infty, \tag{4.23}$$

in accord with (4.14). Since $u_3'(z) = \mathcal{I}u_3(z) + O(z^{-3})$ the difference $w = u - u_3$ satisfies (4.11) with $r(z) = O(z^{-3})$; hence, invoking Lemma 15, we obtain $u(z) - u_3(z) = \widehat{c} + O(z^{-2})$ for some constant $\widehat{c}$. This must agree with (4.22), therefore $\widehat{c} = c^*$. Thus we have shown that the following result holds true.

Theorem 7. *For the optimal process, the control function θ^* satisfies* (4.23)*, and the expected number of jumps has the expansion*

$$u(z) = \alpha_1^* z + \alpha_2^* \log z + c^* + \frac{\alpha_3^*}{z} + O(z^{-2}), \quad z \to \infty. \tag{4.24}$$

To appreciate the effect of the second term in (4.23) it is helpful to consider control functions of the kind

$$\theta(z) \sim \frac{1}{\alpha_1^*} + \frac{\gamma}{z}, \quad z \to \infty. \tag{4.25}$$

Theorem 8. *For control functions of the form* (4.25)*,*

$$\begin{aligned} u_\theta(z) &= \alpha_1^* z + \alpha_2^* \log z + c_1 \\ &+ \left(d\alpha_1^*\alpha_2^*\gamma + \frac{d\alpha_1^{*3}\gamma^2}{2} + \frac{(3d^3 + 13d^2 + 4d + 4)(d+1)!^{1/(d+1)}}{24(d+2)^2(d+3)(d+1)} \right) \frac{1}{z} + O(z^{-2}), \end{aligned} \tag{4.26}$$

where c_1 is constant ($u_\theta(0)$ contributes linearly to c_1).

Proof. The explicit calculation of the expansion (4.26) can be found in Appendix A.2. □

Constant c_1 in (4.26) does not exceed c^* in (4.24), but the relation between the z^{-1}-terms can be the opposite.

4.1.6 The variance expansion

For $N_\theta(z)$, the number of jumps of $Z|z$ driven by θ, let $w(z) = \mathbb{E}(N_\theta(z))^2$ be the second moment. This function satisfies

$$w'(z) = \frac{(d+1)^{d+1}}{(d-1)!} \int_0^{\theta(z)} \left(w(z-y) - w(z) + (1 + 2u(z-y))\right) \mu(z,y)\mathrm{d}y, \qquad (4.27)$$

complemented with the initial condition $w(0) = 0$. By integrating the inhomogeneous term, this can be reduced to the form (4.11), with $r(z)$ of the order of z. Applying Lemma 3 we compare w with various test functions.

We shall consider first the case of optimal $\theta = \theta^*$. It is an easy exercise to see that $w(z) \sim (\alpha_1^* z)^2$, hence the leading term in the integrand is $-2\alpha_1^{*2} yz + 2\alpha_1^* z$, which vanishes at $y = 1/\alpha_1^*$. For this reason the $O(z^{-2})$ remainder in (4.23) will contribute to the solution only $O(1)$, and not $O(\log z)$ as one might expect. Using this fact and (4.23), it is possible to match the sides of the equation by selecting coefficients of the test function

$$\widehat{w}(z) = 2z^2 + a_1 z \log z + a_2 z + a_3 (\log z)^2 + a_4 \log z,$$

achieving that the difference $w(z) - \widehat{w}(z)$ satisfies an equation of the type (4.11) with $r(z) = O(z^{-2}\log z)$. Then applying Lemma 15, $w(z) - \widehat{w}(z) \sim c_1 + z^{-1}\log z$. The calculations presented in Appendix A.3 culminated in

$$w(z) \sim (\alpha_1^* z)^2 + 2\alpha_1^*\alpha_2^* z\log z + (\alpha_4^* + 2\alpha_1^* c^*)\, z + (\alpha_2^* \log z)^2 + \left(\alpha_5^* - \frac{c^* d}{d+2}\right) \log z + c_2,$$

where

$$\alpha_4^* = \frac{2}{(d+2)(d+1)!^{1/(d+1)}}, \quad \alpha_5^* = -\frac{d^3 - 3d - 1}{3(d+2)^2(d+3)}.$$

From this and (4.22), for $\mathrm{Var}(N_{\theta^*}(z)) = w(z) - u^2(z)$ we obtain

$$\mathrm{Var}(N_{\theta^*}(z)) = \alpha_4^* z + \alpha_5^* \log z + c_3 + O\left(z^{-1}\log z\right), \quad z \to \infty.$$

with $c_3 := c_2 - (c^*)^2 - 2\alpha_1^*\alpha_3^*$. In fact, the value of c^* in (4.22) impacts c_2 but not c_3, because the latter is invariant under shifting $u(0)$.

For the general control functions, the variance is very sensitive to the behaviour of θ. The convergence $\theta(z) \to 1/\alpha_1^*$ alone does not even ensure that $O(z)$ is the right *order* for $\mathrm{Var}(N_\theta(z))$. In Appendix A.4 we compute the variance expansion for control functions satisfying (4.25)

$$\mathrm{Var}(N_{\overline{\theta}}(z)) \sim \alpha_4^* z + \left(\frac{-2 - 7d - 25d^2 + 7d^3 + 3d^4}{12(d+1)(d+2)(d+3)} - \frac{2d(d+1)\gamma}{(d+1)!^{2/(d+1)}(d+2)} \right) \log z. \tag{4.28}$$

4.1.7 Central limit theorem for the number of jumps

If the control function $\theta(z)$ approaches a constant for large z, the process Z afar from 0 is almost homogeneous. This suggests approximating the path of Z by a decreasing renewal process with two types of decrements corresponding to drift intervals and gaps.

In this section we denote $N(z)$ the number of jumps of $Z|z$ with some control function satisfying

$$\theta(z) = \frac{1}{\alpha_1^*} + O\left(z^{-1}\right), \ \text{ hence } \lambda(z) = \int_0^{\theta(z)} \mu(z,y)\mathrm{d}y = \frac{\theta(z)^d}{d} + O(z^{-1}), \quad z \to \infty. \tag{4.29}$$

Denote J_z the size of the generic gap having the right endpoint z, with density

$$\mathbb{P}(J_z \in \mathrm{d}y) = \frac{\mu(z,y)}{\lambda(z)}, \quad 0 \le y \le \theta(z),$$

and let D_z be the size of the generic drift interval with survival function

$$\mathbb{P}(D_z \ge y) = \exp\left(- \int_{z-y}^{z} \frac{(d+1)^{d+1}}{(d-1)!} \lambda(s)\ \mathrm{d}s \right), \quad 0 \le y \le z. \tag{4.30}$$

The size of the generic cycle with the right endpoint z can be written as

$$D_z + J_{z-D_z},$$

where D_z and the family of variables $J.$ are independent, and we set $J_0 = 0$.

For large z, the expected values of J_z and D_z are about equal, suggesting that about a half of $[0, z]$ is covered by drift and another half is skipped by jumps. This resembles the behaviour of the stationary selection process [10] in the Poisson setting, where the balance is kept on two scales.

It is useful to see how the mean sizes of gaps and drift intervals depend on $\theta = \theta(z)$:

$$\mathbb{E}J_z \sim \frac{\theta}{d+1} - \frac{d\,\theta^2}{2(d+1)(d+2)z}, \quad \mathbb{E}D_z \sim \frac{d\,(d-1)!}{(d+1)^{d+1}\theta^d} + \frac{d^2\,(d-1)!}{2(d+1)^{d+1}\theta^{d-1}z}.$$

For θ as in (4.29), the mean size of a cycle is

$$\mathbb{E}(J_z + D_z) = \frac{1}{\alpha_1^*} - \frac{\alpha_2^*}{\alpha_1^{*2} z} + O(z^{-2}),$$

regardless of the $O(z^{-1})$ term in (4.29). This expansion explains why the second term in (4.24) is $O(\log z)$ (but falls short of explaining the coefficient α_2^*), and why the suboptimal strategy in Theorem 7 is $O(1)$ from the optimum.

From the convergence of parameters (4.29), it is clear that as $z \to \infty$

$$D_z \xrightarrow{d} \frac{(d+1)!^{1/(d+1)}}{(d+1)^2}\,E, \quad J_z \xrightarrow{d} \frac{(d+1)!^{1/(d+1)}}{d+1}\,B,$$

and, observing the joint convergence of $(D_z, J_{z-\cdot})$, also that

$$D_z + J_{z-D_z} \xrightarrow{d} \frac{(d+1)!^{1/(d+1)}}{(d+1)^2}\,E + \frac{(d+1)!^{1/(d+1)}}{d+1}\,B, \tag{4.31}$$

where $E \stackrel{d}{=} \text{Exponential}(1)$ and $B \stackrel{d}{=} \text{Beta}(d, 1)$ are independent.

The weak convergence (4.31) of cycle sizes suggests that the behaviour of $N(z)$ for large z can be deduced from that of a renewal process with the generic step

$$H := \frac{(d+1)!^{1/(d+1)}}{(d+1)^2} E + \frac{(d+1)!^{1/(d+1)}}{d+1} B$$

which has moments

$$\mu := \mathbb{E}H = \frac{(d+1)!^{1/(d+1)}}{d+1}, \quad \sigma^2 := \mathrm{Var}(H) = \frac{2(d+1)!^{2/(d+1)}}{(d+1)^3(d+2)}.$$

Specifically, for the renewal process $R(z) := \max\{n : H_1 + \cdots + H_n \leq z\}$, with H_j's being i.i.d. replicas of H, we have the familiar CLT

$$\frac{R(z) - z\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{z}} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1),$$

and one can expect that the same limit holds for $N(z)$. This line should be pursued with care, because local discrepancies may accumulate on the large scale and bias centring or even the type of the limit distribution.

Our search of the literature on nonlinear renewal theory to cover the situation of interest showed that the most relevant work is due to Cutsem and Ycart [27]. Their setting of lattice processes is easy to modify, but the argument in [27] has a gap and, in fact, the main result fails without additional assumptions (see a remark below). In the approach taken here, we amend some details of their method of stochastic comparison. To that end, with initial state $z \to \infty$, we focus on the cycles that lie within some range $[\underline{z}, z]$, where the truncation parameter $\underline{z}$ is properly chosen to warrant approximation of the whole process.

Notation. *For two random variables X and Y, the stochastic dominance relation $X <_{st.} Y$ means $\mathbb{P}(X \geq a) \leq \mathbb{P}(Y \geq a)$ for all a.*

Replacing the variable rate in (4.30) by constant yields the bounds

$$\left((1+c/z_0)^{-1}\frac{(d+1)!^{1/(d+1)}}{(d+1)^2}E\right)\wedge(z-z_0) \quad <_{st.} \quad D_z\wedge(z-z_0)$$
$$<_{st.} \quad (1-c/z_0)^{-1}\frac{(d+1)!^{1/(d+1)}}{(d+1)^2}E.$$

Furthermore, observe that the survival function of J_z is

$$\mathbb{P}(J_z > x) = \frac{1}{\lambda(z)}\int_x^{\theta(z)} \mu(z,y)\mathrm{d}y, \quad 0 \le x \le \theta(z). \tag{4.32}$$

Let $I(x)$ be the integral in (4.32) represented as a function of the upper limit. Then we have

$$\frac{\partial\mathbb{P}(J_z > x)}{\partial\theta} = \frac{I'(\theta)\lambda(z)+I(\theta)\lambda'(z)}{(\lambda(z))^2}, \quad 0 \le x \le \theta(z).$$

Calculating the derivatives with respect to $\theta(z)$ yields

$$I'(\theta) = \mu(z,\theta).$$

Furthermore, for large enough z we have $I'(\theta) = \mu(z,\theta) > 0$; thus, $\partial\mathbb{P}(J_z > x)/\partial\theta > 0$. Whence, from (4.29),

$$(1+c/z_0)^{-1}\frac{(d+1)!^{1/(d+1)}}{d+1}B <_{st.} J_z <_{st.} (1-c/z_0)^{-1}\frac{(d+1)!^{1/(d+1)}}{d+1}B, \quad z \ge \underline{z}.$$

From these estimates follow stochastic bounds on the cycle size

$$((1+c/\underline{z})^{-1}H)\wedge(z-\underline{z}) <_{\rm st} (D_z+J_{z-D_z})\wedge(z-\underline{z}) <_{\rm st} (1-c/\underline{z})^{-1}H\,, \quad z \ge \underline{z}. \tag{4.33}$$

Setting the bounds (4.33) in terms of multiples of the same random variable H is convenient in combination with the obvious scaling property: for $k > 0$, $R(k\,\cdot)$ is the renewal process with the generic step kH. Let $N(z,\underline{z})$ be the number of cycles of $Z|z$, which fit completely within $[\underline{z}, z]$. As in [27], from (4.33) we conclude that

$$R\left((z-\underline{z})(1-c/\underline{z})\right) <_{\rm st} N(z,\underline{z}) <_{\rm st} R\left((z-\underline{z})(1+c/\underline{z})\right), \quad z \geq \underline{z}. \tag{4.34}$$

Letting $z \to \infty$ then $\underline{z} \to \infty$, and appealing to $R(z)/z \to \mu^{-1}$ a.s., (4.34) implies a weak law of large numbers for $N(z)$,

$$\frac{N(z)}{z} \stackrel{\rm d}{\to} \frac{1}{\mu}, \quad z \to \infty. \tag{4.35}$$

We aim next to show the CLT for $N(z)$. To that end, we choose $\underline{z} = \omega\sqrt{z}$, where $\omega > 0$ is a large parameter. Start with splitting

$$N(z) - z\mu^{-1} = (N(z,\underline{z}) - (z-\underline{z})\mu^{-1}) + (N(z) - N(z,\underline{z}) - \underline{z}\mu^{-1}),$$

where $N(z) - N(z,\underline{z})$ counts the cycles that start in $[0,\underline{z}]$; this component is annihilated by the scaling, since by (4.35)

$$\frac{N(z) - N(z,\underline{z}) - \underline{z}\mu^{-1}}{\sqrt{\underline{z}}} \stackrel{\rm d}{\to} 0,$$

and the same is true with $\sqrt{\underline{z}}$ replaced by bigger $\sqrt{z}$. For the leading contribution due to $N(z,\underline{z})$ we obtain using dominance (4.34) and the CLT for $R(z)$

$$\mathbb{P}\left(\frac{N(z,\underline{z}) - (z-\underline{z})\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{z}} \leq x\right) \geq \mathbb{P}\left(\frac{R((z-\underline{z})(1+c/\underline{z})) - (z-\underline{z})\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{z}} \leq x\right) =$$
$$\mathbb{P}\left(\frac{R((z-\underline{z})(1+c/\underline{z})) - (z-\underline{z})(1+c/\underline{z})\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{z}} + \frac{(z-\underline{z})c}{\omega z\mu^{-1/2}\sigma} \leq x\right)$$
$$\to 1 - \Phi\left(x - \frac{c}{\omega\sigma\mu^{-1}}\right),$$

as $z \to \infty$. Letting $\omega \to \infty$

$$\limsup_{z\to\infty} \mathbb{P}\left(\frac{N(z) - z\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{z}} \leq x\right)) = \limsup_{z\to\infty} \mathbb{P}\left(\frac{N(z,z_0) - z\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{z}} \leq x\right) \geq 1 - \Phi(x).$$

The opposite inequality is derived similarly.

Theorem 9. *For a control satisfying $\theta(z) \sim 1/\alpha_1^*$, the following central limit theorem is satisfied*

$$\frac{N_\theta(z) - \alpha_1^* z}{(\alpha_4^* z)^{1/2}} \stackrel{d}{\to} \mathcal{N}(0,1), \quad z \to \infty.$$

Remark. The renewal-type approximation for decreasing Markov chains on $\mathbb{N}$ using stochastic comparison appeared in [27]. However, their Theorem 4.1 on the normal limit for the absorption time fails without additional assumptions on the quality of convergence of the step distribution. For instance, if the decrement in position $z > 8$ assumes values 1 and 2 with probabilities $1/2 \pm 1/\log z$, the mean absorption time is asymptotic to $2z/3$, with the remainder being strictly of the order $z/\log z$, therefore not annihilated by the $\sqrt{z}$ scaling. The error in [27] appears on the bottom of p. 996, where the truncation parameter (m, a counterpart of our $\underline{z}$) is assumed independent of the initial state. Recently Alsmeyer and Marynych [2], also concerned with the lattice setting, suggested conditions on the rate of convergence of decrements in some probability metrics to ensure the normal approximation of the absorption time.

Remark. It is of interest to look at the properties of the random set $\mathcal{Z}$ which, intuitively, describes an infinite selection process. This limit object can be interpreted in the spirit of the boundary theory of Markov processes: the state space $[0, \infty)$ has a one-point compactification - the entrance Martin boundary - approached as the initial state of $Z|z$ tends to ∞. Applying the coupling argument as in Lemma 14 one can show that, at a large distance from the origin, $\mathcal{Z}$ behaves similarly to a stationary alternating renewal process, with uniformly distributed gaps and exponential drift intervals. The coverage probability and the occupation measure satisfy $p(z) \to 1/2$ and $U([0, z]) \sim \alpha_1^* z$, $z \to \infty$. Korshunov [40] studied increasing Markov processes on reals which at a distance from the origin behave similarly to renewal processes, but reverting the direction of time, required to adapt this work in our setting, does not seem straightforward.

4.1.8 Summary of the results

Translating back into the terms of the original problem, we gather and state the main results of this chapter. Let $L(t)$ be the length of the increasing subsequence chosen by the optimal selection policy from a sequence of d-dimensional items over a horizon $[0, t]$.

Theorem 10. *The following asymptotic results hold as $t \to \infty$:*

(i) *The expected length of the optimally chosen subsequence satisfies*

$$\mathbb{E}L(t) = \gamma_1^* t^{1/(d+1)} + \gamma_2^* \log t + c^* + \frac{\gamma_3^*}{t^{1/(d+1)}} + O(t^{-2/(d+1)}),$$

where

$$\gamma_1^* = \frac{(d+1)}{(d+1)!^{1/(d+1)}}, \ \gamma_2^* = -\frac{d}{2(d+2)(d+1)}, \ \gamma_3^* = \frac{(d^2+d+1)(d+1)!^{1/(d+1)}}{6(d+3)(d+2)^2(d+1)}.$$

(ii) *The variance of the length has the asymptotic expansion*

$$\mathrm{Var}(L(t)) \sim \gamma_4^* t^{1/(d+1)} + \gamma_5^* \log t + c_3,$$

where

$$\gamma_4^* = \frac{2}{(d+2)(d+1)!^{1/(d+1)}}, \ \gamma_5^* = -\frac{(d^3-3d-1)}{3(d+1)(d+2)^2(d+3)}.$$

(iii) *Then the following convergence in distribution holds*

$$\frac{L(t) - \gamma_1^* t^{1/(d+1)}}{(\gamma_4^* t^{1/(d+1)})^{1/2}} \xrightarrow{d} \mathcal{N}(0, 1).$$

Interestingly, as was also highlighted in [10], $\gamma_1 * \sim e$, as $d \to \infty$. This coincides with the asymptotics of the length of the *longest* increasing subsequence inside the sequence of d-dimensional items [13], indicating that the advantage of a prophet over the decision-maker uncovering the observations one-by-one diminishes with the number of dimensions d.

4.2 The one-dimensional setting

Samuels and Steele's [55] one-dimensional problem was the primary focus in the literature. In this section, we provide an overview of the latest developments in this classical problem and specialise our multidimensional results for the case $d = 1$.

The value function $v(t) := \mathbb{E}L(t)$ in the one-dimensional setup satisfies an integro-differential optimality equation

$$v'(t) = \int_0^1 (v(t(1-y)) - v(t) + 1)_+ \mathrm{d}y, \quad v(0) = 0,$$

which does not seem to admit a closed-form solution. Samuels and Steele [55] found the leading asymptotics $v(t) \sim \sqrt{2t}$, where the order was identified by Hammersley's subadditivity method. Bruss and Delbaen [19] combined a thorough analysis of the optimality equation with martingale methods to derive much tighter estimates

$$\sqrt{2t} - \log(1 + \sqrt{2t}) + \widetilde{c} < v(t) < \sqrt{2t}, \tag{4.36}$$

(with explicit $\widetilde{c}$) and to show that similar bounds hold for the variance $\mathrm{Var}\, L(t)$. In another paper Bruss and Delbaen [20] extended this technique to obtain a functional limit theorem for fluctuations of the shape of selected subsequence, showing in particular that the distribution of $\sqrt{3}\ (L(t) - \sqrt{2t})/(2t)^{1/4}$ converges to normal. This result and a substantial refinement of (4.36) follow readily from the special case $d = 1$ of Theorem 10. Moreover, unlike [19, 20], we do not rely on the concavity of the value function $v(t)$, but rather use tools well-suited to the analysis of a wider class of near-optimal strategies, including a continuous-time analogue of the adaptive strategy from [5].

4.2.1 The suboptimal selection strategies

To solve the optimisation problem, it is sufficient to consider a relatively small class of strategies defined recursively by means of an acceptance window $\psi(t,s,y)$ satisfying $0 \le \psi(t,s,y) \le 1-y$ for $0 \le s \le t < \infty$ and for $y \in [0,1]$.

Definition 14. *A* threshold strategy *is an online selection policy that accepts an observation* $(s,x) \in [0,t] \times [0,1]$ *if and only if*

$$0 < x - y \le \psi(t,s,y),$$

where y *is the last (hence the highest) mark selected before time* s*, with the convention that* $y = 0$ *if no selections have been made. We call function* $\psi(t,s,y)$ *a* threshold function*.*

For instance, the *greedy* strategy has the largest possible acceptance window

$$\overline{\psi}(t,s,y) = 1 - y.$$

The strategy selects the sequence of records [22], which has the expected length given by the exponential integral function

$$\mathrm{Ein}(t) = \int_0^t \frac{1-e^{-s}}{s}\,\mathrm{d}s \sim \log t, \quad t \to \infty.$$

The greedy strategy is optimal for $t \le 1.345\ldots$, when the expected number of records is not bigger than 1.

Bruss and Delbaen [19] also studied a class of suboptimal policies, which they called 'graph rules'. One of the examples of a graph rule is the policy selecting all records under the main diagonal $s = t$. This policy achieves the $\sqrt{t}$-order performance but falls short of the optimal with the value function satisfying $v_g(t) \sim \sqrt{\pi t/2}$, $t \to \infty$.

Next by the complexity is the family of *stationary* strategies, which have acceptance window of the form $\psi(t,s,y) = \delta(t) \wedge (1-y)$, depending neither on the time of observation nor on the running maximum Y, as long as Y does not overshoot $1-\delta(t)$.

In the one-dimensional setting, the analogue of the stationary strategy driven by the acceptance region $\widetilde{\boldsymbol{\Sigma}}$ is the stationary strategy with $\widetilde{\psi}(t,s,y) = (1-y) \wedge \sqrt{2/t}$. This strategy maintains a balance between increasing on the marks and time scales so that the running maximum Y fluctuates about the linear function s/t, and both resources are exhausted almost simultaneously. Let $L_{\widetilde{\psi}}(t)$ be the length of increasing sequence chosen by this stationary strategy. We may represent $L_{\widetilde{\psi}}(t)$ as a minimum of two independent renewal processes: $R_0(t)$ with a generic step $H_0 \stackrel{d}{=} \text{Uniform}[0, \sqrt{2/t}]$ and $R_1(t)$ with a generic step $H_1 \stackrel{d}{=} \text{Exponential}(\sqrt{2/t})$. By a well-known central limit theorem for renewal processes, we have, as $t \to \infty$,

$$\frac{R_0(t) - \sqrt{2t}}{(2t)^{1/4}} \stackrel{d}{\to} \xi_1, \quad \text{and} \quad \frac{\sqrt{3}(R_1(t) - \sqrt{2t})}{(2t)^{1/4}} \stackrel{d}{\to} \xi_2,$$

where ξ_1 and ξ_2 are independent standard normal variables. This leads to the following result.

Lemma 17. *The length of an increasing subsequence selected by the stationary policy with the acceptance window $\widetilde{\psi}(t)$ satisfies the following distributional convergence*

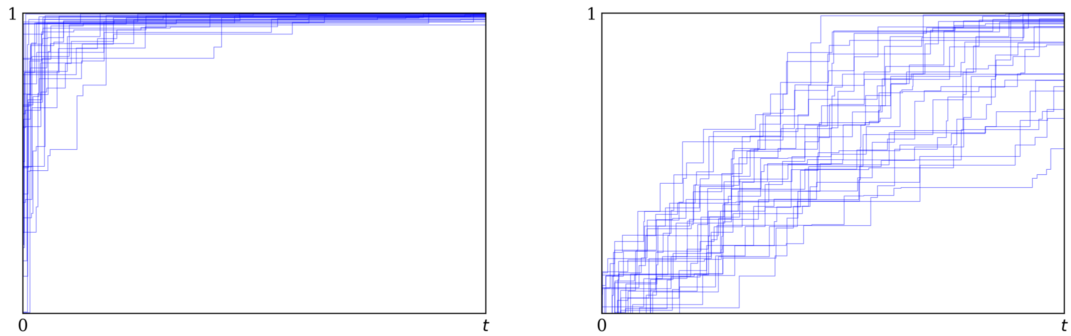
$$\sqrt{3}\,\frac{L_{\widetilde{\psi}}(t) - \sqrt{2t}}{(2t)^{1/4}} \stackrel{d}{\to} \eta, \quad t \to \infty, \tag{4.37}$$

with $\eta = \xi_1 \wedge (\xi_2/\sqrt{3})$.

Nadarajah and Kotz [48] provided exact formulae for moments of $Z \stackrel{d}{=} \min\{Z_1, Z_2\}$, where $Z_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), Z_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ are independent normal random variables

$$\begin{aligned}
\mathbb{E}Z &= \mu_1 \Phi\left(\frac{\mu_2 - \mu_1}{\theta}\right) + \mu_2 \Phi\left(\frac{\mu_1 - \mu_2}{\theta}\right) - \theta\phi\left(\frac{\mu_2 - \mu_1}{\theta}\right) \\
\mathbb{E}Z^2 &= (\sigma_1^2 + \mu_1^2)\Phi\left(\frac{\mu_2 - \mu_1}{\theta}\right) + (\sigma_2^2 + \mu_2^2)\Phi\left(\frac{\mu_1 - \mu_2}{\theta}\right) - (\mu_1 + \mu_2)\theta\phi\left(\frac{\mu_2 - \mu_1}{\theta}\right),
\end{aligned}$$

where $\theta = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$ and ρ is the correlation coefficient of Z_1 and Z_2. Specialising the formulae above yields $\mathbb{E}\eta = -\sqrt{2/\pi}$ and $\mathrm{Var}(\eta) = 2 - 2/\pi$. For comparison, the simulated paths of the running maximum process of the greedy policy and a stationary policy driven by $\widetilde{\psi}$ are demonstrated on Figure 4.2.



(a) Running maximum process of the greedy strategy

(b) Running maximum process of the stationary strategy with $\widetilde{\psi}(t, s, y) = (1 - y) \wedge \sqrt{2/t}$

Figure 4.2: Running maximum process realisations for $t = 10^2$

4.2.2 Summary of the results in one-dimensional setting

One of the convenient properties of the one-dimensional setting is the representation of the optimal policy as self-similar.

Definition 15. *A* self-similar *selection policy is a threshold policy with the control function of the form*

$$\psi(t, s, y) = (1 - y)\, \varphi((t - s)(1 - y)) \tag{4.38}$$

for some $\varphi : [0, \infty) \to [0, 1]$.

With self-similarity the connection between the running maximum Y and the transformed process $Z(\cdot)$ is more apparent. Identifying the drift rate and jump distribution reveals that (4.16) in one-dimensional case is the process $Z|\sqrt{t}$, with θ found by matching the jump rates as

$$4\lambda(z) = 2z\varphi(z^2).$$

This connection opens up the possibility to extend the results of Theorem 10 to a certain class of suboptimal self-similar selection policies. Let $L_\varphi(t)$ be the length of an increasing subsequence selected by a self-similar strategy with the acceptance window of the form (4.38).

Theorem 11. (i) *The optimal strategy has the acceptance window of the form* (4.38) *with*

$$\varphi^*(t) = \sqrt{\frac{2}{t}} - \frac{1}{3t} + O(t^{-3/2}), \quad t \to \infty,$$

and outputs an increasing subsequence with expected length

$$\mathbb{E}L(t) = \sqrt{2t} - \frac{1}{12}\log t + \widetilde{c^*} + \frac{\sqrt{2}}{144\sqrt{t}} + O(t^{-1}), \quad t \to \infty, \tag{4.39}$$

and variance

$$\mathrm{Var}(L(t)) \sim \frac{\sqrt{2t}}{3} + \frac{1}{72}\log t + c_4 + O(t^{-1/2}\log t). \tag{4.40}$$

(ii) *The strategy with* $\varphi(t) := \sqrt{2/t} \wedge 1$ *outputs an increasing subsequence with the expected length*

$$\mathbb{E}L_\varphi(t) = \sqrt{2t} - \frac{1}{12}\log t + c_5 + \frac{\sqrt{2}}{72\sqrt{t}} + O(t^{-1}), \quad t \to \infty,$$

and variance

$$\mathrm{Var}(L_\varphi(t)) \sim \frac{\sqrt{2t}}{3} + \frac{1}{24}\log t + c_6 + O(t^{-1/2}\log t), \quad t \to \infty.$$

(iii) *If* $\varphi(t) \sim \sqrt{2/t} + O(t^{-1})$, *then a central limit theorem holds:*

$$\sqrt{3}\,\frac{L_\varphi(t) - \sqrt{2t}}{(2t)^{1/4}} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1), \quad t \to \infty.$$

The selection strategy in (iii) is the analogue of Arlotto et al.'s [5] $O(\log n)$-optimal policy in the discrete-time problem, which was studied in detail in Section 3.1.3. Arlotto

t	$\mathbb{E}L_\varphi(t)$	$\sqrt{2t}$	$\sqrt{2t} - \log t/12$
10	3.63	4.472	4.28
100	13.012	14.1421	13.7584
1000	43.369	44.721	44.146
2000	62.074	63.246	62.612
3000	76.125	77.46	76.792
4000	87.992	89.443	88.752
5000	98.546	100	99.29
6000	108.099	109.545	108.896
7000	117.017	118.322	117.584
8000	125.141	126.491	125.742
9000	132.737	134.164	133.405
10000	139.855	141.421	140.654

Table 4-A: MC simulation to approximate $\mathbb{E}L_\varphi(t)$

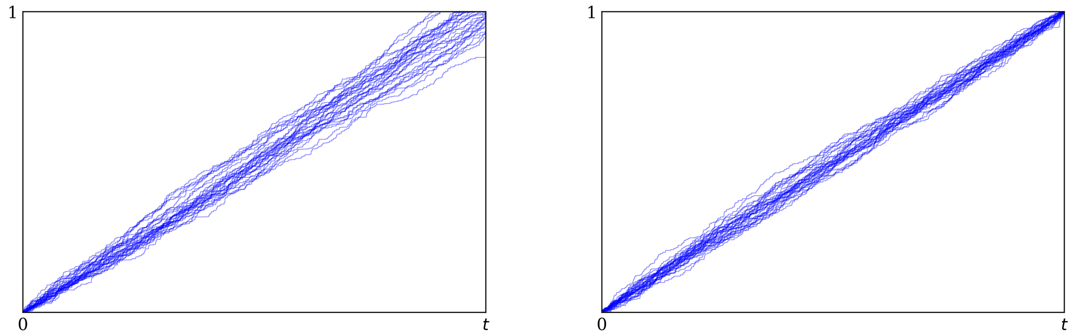
et al. approximated the value functions v_n numerically for $n = 1, \dots, 10^5$ and used the outcome as a basis for the conjecture that their policy is within $O(1)$ off optimality. We followed a similar path in the continuous-time setting. To obtain the approximations of $\mathbb{E}L_\varphi(t)$ presented in Table 4-A, we performed a Monte Carlo simulation of the selection process with 10^6 simulations for every given horizon. The approximation of the value function converges to the two-term expansion of $L_\varphi(t)$ very quickly. However, chances to capture the constant c_5 via numerical simulation are slim, as one needs an unreasonably large horizon t to start filtering out the log-term contribution.

Both the stationary strategy, driven by $\widetilde{\psi}(t, s, y) = (1-y) \wedge \sqrt{2/t}$, and the self-similar strategy in (ii) have running maximum processes fluctuating around the main diagonal. However, the running maximum paths of the stationary strategy are more dispersed, which is demonstrated on Figure 4.3.

The instance of part (c) for the optimal strategy was proved in [20]; this can be compared with the distributional limit (4.37) for the stationary strategy.

Bruss and Delbaen [20] used concavity of v to prove the bounds

$$\frac{v(t)}{3} \le \mathrm{Var}(L(t)) \le \frac{v(t)}{3} + \frac{1}{(\beta - \sqrt{2\beta})\, 6\sqrt{2}} \log \frac{t}{\beta} + 2,$$



(a) Running maximum process of the stationary strategy with $\widetilde{\psi}(t,s,y) = (1-y) \wedge \sqrt{2/t}$

(b) Running maximum process of the self-similar strategy with $\varphi(t) := \sqrt{2/t} \wedge 1$

Figure 4.3: Running maximum process realisations for $t = 10^5$

(for t no too small), where $v(\beta) = 2$. For large t, the logarithmic term in the lower bound has coefficient $-1/36$ (as is seen from (a)) and in the upper bound at least 0.55 (as can be shown by estimating β). These bounds can be compared with the coefficient $1/72$ in part (a).

4.2.3 Connection to the discrete-time problem

Arlotto et al. [4] stressed that the deep relation between the discrete-time and poissonised sequential decision models is yet to be understood. The setting with Poisson arrivals can be related to the fixed-n problem by allowing the length of the observed sequence to be used in decision strategies. However, despite the apparent similarity, translating results from one model to the other is not automatic since the information flows are very different.

Arlotto et al. [4] proved the *information* bound

$$v(n) \le v_n, \quad n \in \mathbb{N},$$

by applying an optimal policy from the poissonised problem in the discrete-time setting and using the concavity of v_n. Independently, Samuels and Steele [55] and Baryshnikov and Gnedin [10] derived the asymptotic equivalence

$$v_n \sim v(n), \quad n \to \infty,$$

by doing the reverse (in fact, Baryshnikov and Gnedin [10] proved it for the multidimensional increasing subsequence). However, extending their analysis to include the lower-order terms of expansion seems to be a challenging task.

In this dissertation we treated the two problems separately. Hence, combining the results of Theorems 1 and 11(i) confirms a strong connection between the two problems.

Corollary 5. *The discrete-time and continuous-time increasing subsequence selection problems have value functions satisfying*

$$\sup_{n\to\infty} |v_n - v(n)| < \infty.$$

4.3 The selection under a sum constraint

The stochastic knapsack problem discussed in Section 3.3 has a continuous-time analogue. Let non-negative observations $X_i \stackrel{d}{=} \text{Beta}(\alpha, 1)$ with $F(x) = Ax^\alpha,\ A, \alpha > 0$ arrive with a unit-rate homogeneous Poisson process. The objective now is to pack as many items as possible into a one-dimensional knapsack of capacity C over the time horizon t.

To preserve the convenient self-similarity property of the problem, we assume that $X_i \in [0, C]$ leading to $A = C^{-\alpha}$. The self-similarity is crucial for representing the problem in terms of piecewise deterministic Markov process and the consequent renewal approximation. However, a refined optimal value function expansion could be obtained for a much wider class of distributions — similarly to the discrete-time setup in Section 3.3 — albeit without going beyond the $O(1)$-term. The restricted distribution assumption also gives way to the equivalent results for the class of self-similar suboptimal selection policies.

In Bruss and Delbaen [19], Theorem 4.1 they showed that the poissonised bin-packing problem with standard uniformly distributed observations and a unit capacity is equivalent to the poissonised longest increasing subsequence selection by demonstrating the equivalence of the optimality equations of the two problems. The equivalence breaks with non-uniform observations in the bin-packing setup.

We need to slightly adjust the definition of the online policy for the bin-packing case before deriving the optimality equation.

Definition 16. *Let $(X_1, T_1), (X_2, T_2), \ldots$ be the atoms of a unit-rate Poisson point process, where $T_1 < T_2 < \ldots$ are the arrival times. An* online selection policy *in the continuous-time bin-packing problem is a collection of stopping times $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots)$ satisfying*

(i) *each τ_i is adapted to $(X_1, T_1), (X_2, T_2), \ldots$,*

(ii) *each τ_i assumes values in $\{T_j\}$,*

(ii) $\tau_1 < \tau_2 < \ldots$,

(iii) $X_{\tau_1} + \ldots + X_{\tau_j} \le C$.

4.3.1 The optimality equation

Let $N(t) = N(t, C)$ be the number of items packed by the optimal policy over the horizon t. The optimal value function $v(t)$, then, is $v(t) = v(t, C) := \mathbb{E}N(t, C)$. Decomposing at the first arrival over a small time interval h, we have

$$\begin{aligned} v(t,C) &= (1-h)\, v(t-h, C) \\ &+ h \int_0^C \max\{v(t-h, C-x) + 1, v(t-h, C)\}\, \mathrm{d}F(x) + o(h). \end{aligned}$$

The first term comes from the probability $1 - h$ of no arrivals, and the integral term, which evaluates the expected reward conditional on an arrival, is dictated by the dynamic programming principle. Finally, the probabilities of more than one arrival sum up to $o(h)$. Rearranging the equation above yields

$$\frac{v(t,C)-v(t-h,C)}{h}=\int_0^C (v(t-h,C-x)-v(t-h,C)+1)_+\mathrm{d}F(x)+\frac{o(h)}{h}.$$

Taking the limit as $h\to 0$ on both sides leads to a partial integro-differential equation

$$\frac{\partial v(t,C)}{\partial t}=\int_0^C (v(t,C-x)-v(t,C)+1)_+\mathrm{d}F(x). \tag{4.41}$$

Now, with the remaining capacity of $C-x$, the admissible observations arrive with a thinned Poisson process of rate $F(C-x)=(1-x/C)^\alpha$. Using a time scale transformation $t:=t(1-x/C)^\alpha$ results in a bin-packing process over observations arriving with a unit-rate Poisson process and the original remaining capacity C; thus, we have the following optimal value function equivalence $v(t,C-x)=v(t(1-x/C)^\alpha,C)=v(t(1-x/C)^\alpha)$. Substituting this into (4.41) yields the optimality equation

$$v'(t)=\int_0^C ((v\left(t(1-x/C)^\alpha\right)-v(t)+1)_+\,\mathrm{d}F(x),\quad v(0)=0.$$

Performing a substitution $x\to Cx$ to map the integration range onto the unit interval yields

$$v'(t)=\int_0^1 (v(t(1-x)^\alpha)-v(t)+1)_+\,\alpha x^{\alpha-1}\mathrm{d}x,\quad v(0)=0. \tag{4.42}$$

4.3.2 Asymptotic expansion of the value function

In this section we apply the comparison method to obtain the asymptotic estimate of the solution to (4.42). First, let us transform (4.42) by introducing a function $u(z)$ such that $u(z):=v(z^{\alpha+1})$

$$u'(z)=\alpha(\alpha+1)z^\alpha\int_0^1 (u(z(1-x)^{\alpha/(\alpha+1)})+1-u(z))_+x^{\alpha-1}\mathrm{d}x,\quad u(0)=0.$$

Substituting $z(1-x)^{\alpha/(\alpha+1)} =: z - y$, we obtain a convolution-type equation

$$u'(z) = \int_0^z (u(z-y) - u(z) + 1)_+ \nu(z,y)\mathrm{d}y, \quad u(0) = 0, \tag{4.43}$$

where

$$\nu(z,y) = (\alpha+1)^2 z^{\alpha-1} \left(1 - \frac{y}{z}\right)^{1/\alpha} \left(1 - \left(1 - \frac{y}{z}\right)^{(\alpha+1)/\alpha}\right)^{\alpha-1}.$$

Define an integral operator $\mathcal{I}$ as

$$\mathcal{I}u(z) := \int_0^z (u(z-y) - u(z) + 1)_+ \nu(z,y)\mathrm{d}y.$$

Choosing suitable test functions, we apply the comparison method based on Lemma 3 to approximate the solution to (4.43). We pick the first proxy function of the form $u_1(z) := \beta_1 z,\ \beta_1 \in \mathbb{R}_+$, for which $u_1'(z) = \beta_1$. To estimate $\mathcal{I}u_1(z)$, note that that $\psi_1(z) = 1/\beta_1$ solves $u_1(z-y) - u_1(z) + 1 = 0$.

$$\mathcal{I}u_1(z) = \int_0^{\psi_1(z)} (u(z-y) - u(z) + 1)\, \nu(z,y)\mathrm{d}y = \left(\frac{\alpha+1}{\alpha\beta_1}\right)^{\alpha} + O(z^{-1}), \quad z \to \infty.$$

By virtue of Lemma 3, we have the main asymptotic

$$u(z) \sim \beta_1^* z, \quad z \to \infty.$$

where

$$\beta_1^* = \left(\frac{\alpha+1}{\alpha}\right)^{\alpha/(\alpha+1)}.$$

We pick the second test function to be of the form $u_2(z) = \beta_1^* z + \beta_2 \log(z+1),\ \beta_2 \in \mathbb{R}$. From

$$u_2'(z) \sim \beta_1^* + \frac{\beta_2}{z}, \quad \mathcal{I}u_2(z) \sim \beta_1^* - \left(\frac{\alpha+1}{2(\alpha+2)} + \beta_2\right)\frac{1}{z}, \quad z \to \infty,$$

we obtain $u(z) \sim \beta_1^* z + \beta_2^* \log z,\ z \to \infty$, where

$$\beta_2^* = -\frac{1}{2(\alpha+2)}.$$

Finally, with the proxy function of the form $u_3(z) = \beta_1^* z + \beta_2^* \log(z+1) + \beta_3/(z+1),\ \beta_3 \in \mathbb{R}$, we have $u_3'(z) \sim \beta_1^* + \beta_2^*/z - (\beta_2^* + \beta_3)/z^2,\ z \to \infty$, and, for $\psi_2(z)$ that solves

$$u_3(z-y) - u_3(z) + 1 = 0$$

we have $\psi_2(z) \sim \beta_1^* - \beta_2^*/(\beta_1^{*2}(z+1)),\ z \to \infty$. With this in mind, expanding $\mathcal{I}u_3(z)$ yields

$$\mathcal{I}u_3(z) \sim \beta_1^* + \frac{\beta_2^*}{z} + \left(\alpha\beta_2^* + \left(\frac{\alpha+1}{\alpha}\right)^{-\alpha/(\alpha+1)} \frac{(2 - 7\alpha + 9\alpha^2 + \alpha^3 + \alpha^4)}{12\alpha(\alpha+2)^2(\alpha+3)} + \alpha\beta_3\right)\frac{1}{z^2}, \quad z \to \infty.$$

Since $u_3'(z)$ matches with $\mathcal{I}u_3(z)$ at

$$\beta_3 = \beta_3^* := -\left(\frac{\alpha+1}{\alpha}\right)^{-\alpha/(\alpha+1)} \frac{(2 - 9\alpha + \alpha^3)}{12\alpha(\alpha+2)^2(\alpha+3)},$$

by virtue of Lemma 3 we have

$$u(z) = \beta_1^* z + \beta_2^* \log z + O(1), \quad z \to \infty.$$

Reproducing the argumentation presented in Sections 4.1.3, 4.1.4, and 4.1.5, one can prove the convergence of $O(1)$-term and obtain an expansion that goes beyond the constant. That is,

$$u(z) = \beta_1^* z + \beta_2^* \log z + c_1^* + \frac{\beta_3^*}{z} + O(z^{-2}), \quad z \to \infty. \tag{4.44}$$

4.3.3 The variance expansion

In terms of $u(\cdot)$ and z, let $w(z) := \mathbb{E}(N(z))^2$; then $w(z)$ satisfies

$$u'(z) = \int_0^{\psi^*(z)} (u(z-y) - u(z) + 1)\, \nu(z,y)\mathrm{d}y, \tag{4.45}$$

with $w(0) = 0$. From (4.44), the optimal control $\psi(z)$ expands as

$$\psi^*(z) = \frac{1}{\beta_1^*} - \frac{\beta_2^*}{z\beta_1^{*2}} + O(z^{-2}), \quad z \to \infty. \tag{4.46}$$

Equation (4.45) is in the class of equations covered by Lemma 3. Therefore, we may obtain the expansion of $w(z)$ via the comparison method. Skipping over the detailed computation (which be found in Appendix A.5), we obtain, as $z \to \infty$,

$$w(z) \sim \beta_1^{*2} z^2 + 2\beta_1^*\beta_2^* z \log z + (\beta_4^* + 2\beta_1^* c_1^*)\, z + (\beta_2^* \log z)^2 + \left(\beta_5^* + \frac{c_1^*}{\alpha+2}\right) \log z,$$

where

$$\beta_4^* = \frac{2\beta_1^*}{(\alpha+1)(\alpha+2)}, \quad \beta_5* = \frac{1}{(\alpha+2)^2(\alpha+3)}.$$

as $z \to \infty$. Recalling Lemma 15, we can prove that the remainder term in the expansion of $w(z)$ converges to some constant c_7 and the order of the next term is $O(z^{-1}\log z)$. Substracting $u(z)^2$ yields

$$\mathrm{Var}N(z) \sim \beta_4^* z + \beta_5^* \log z + c_7 - c_1^{*2} - 2\beta_3^*\beta_1^* + O(z^{-1}\log z), \quad z \to \infty.$$

The variance $\mathrm{Var}N(z)$ is invariant to the change in the initial condition $u(0) = 0$. Hence, the constant $c_2^* := c_7 - c_1^{*2} - 2\beta_3^*\beta_1^*$ is independent of c_1^* (although the value of c_1^* indeed affects c_7).

4.3.4 The renewal approximation

Finally, we sketch a proof of a central limit theorem for $N(z)$ based on the renewal-type approximation. We omit many details of the derivation here; these are described carefully in Section 4.1.7. Based on (4.43) and (4.46), we approximate the number of jumps $N(z)$ with the number of renewals of a process with a generic step $H := D_z + J_z$, where

$$D_z \stackrel{d}{=} \frac{E}{\beta_1^*(\alpha+1)}, \quad J_z \stackrel{d}{=} \frac{1}{\beta_1^*} B,$$

with $E \stackrel{d}{=}$ Exponential(1) and $B \stackrel{d}{=}$ Beta$(\alpha, 1)$. H has the following moments

$$\mu = \mathbb{E}H = \frac{1}{\beta_1^*}, \quad \text{and } \sigma^2 = \text{Var}H = \frac{2}{\beta_1^{*2}(\alpha+1)(\alpha+2)}.$$

Hence, for a counting function of the renewal process with a generic step H, we have a central limit theorem

$$\frac{R(z) - z\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{z}} \xrightarrow{d} \mathcal{N}(0,1), \quad z \to \infty, \tag{4.47}$$

Analogous to Section 4.1.7, carefully dealing with the accumulating discrepancies, we obtain the distributional convergence

$$\frac{N(z) - \beta_1^* z}{\sqrt{\beta_4^* z}} \xrightarrow{d} \mathcal{N}(0,1), \quad z \to \infty.$$

4.3.5 Summary of the results

Translating back to the original t-horizon setting, we collect all the results obtained in this section. Let $N(t)$ be the number of items optimally packed into the knapsack of capacity C.

Theorem 12. *As $t \to \infty$, the optimal packing policy is self-similar with the acceptance window of the form $\varphi^*(t, y) = (C-y)^\alpha \delta^*(t(C-y)^\alpha)$, where*

$$\delta^*(t) = \frac{\alpha+1}{\beta_1^* \alpha\, t^{1/(\alpha+1)}} - \left(\frac{\alpha+1}{2\beta_1^{*2}\alpha^2} + \frac{\beta_2^*(\alpha+1)}{\beta_1^{*2}\alpha}\right)\frac{1}{t^{2/(\alpha+1)}} + O(t^{-3/(\alpha+1)}), \tag{4.48}$$

the number of packed items $N(t)$ has the mean satisfying

$$v(t) = \mathbb{E}N(t) = \beta_1^* t^{1/(\alpha+1)} + \frac{\beta_2^*}{\alpha+1}\log t + c_1^* + \frac{\beta_3^*}{t^{1/(\alpha+1)}} + O(t^{-2/(\alpha+1)}),$$

and the variance satisfying

$$\text{Var}N(t) = \beta_4^* t^{1/(\alpha+1)} + \frac{\beta_5^*}{\alpha+1}\log t + c_2^* + O(t^{-1/(\alpha+1)}\log t).$$

Moreover, the following normal convergence holds

$$\frac{N(z) - \beta_1^* z}{\sqrt{\beta_4^* z}} \xrightarrow{d} \mathcal{N}(0, 1).$$

Specialising Theorem 6 in the discrete-time bin-packing to the distribution $F(x) = (x/C)^\alpha$ with the support $[0, C]$, we see that the two problems are asymptotically similar, i.e. their value functions satisfy $|v_t(C) - v(t)| = O(1)$, $t \to \infty$.

4.4 The interval parking

Let $\mathcal{D} = \{(x, y) \in [0, 1]^2 : x \leq y\}$ and let $\mathcal{P}$ be a homogeneous Poisson point process on $[0, \infty) \times \mathcal{D}$ with 2×Lebesgue measure as intensity. Let (t_1, x_1, y_1), $(t_2, x_2, y_2), \ldots$ be the atoms of $\mathcal{P}$ labelled by increasing the time component $t_1 < t_2 < \cdots$. The points (x_i, y_i) are i.i.d. uniformly in $\mathcal{D}$. We think of (x_i, y_i) as an interval observed at time t_i, when we have to decide whether to park it into $[0, 1]$, or skip without an option to retrieve it. Restricting the time horizon to $[0, t]$ (we denote a restricted process by $\mathcal{P}_t$), our goal is to park as many intervals with the constraint that every consecutive selected interval should lie completely right-hand side of the last selected interval, e.g. with the last selected interval $[x, y]$, the next interval that we can choose must belong to $[y, 1]$.

This is known in the literature as Rényi's parking problem [52]. In his paper he derived the asymptotic 'mean filling density', which is known as Rényi's Parking Constant. He also derived an asymptotic expansion of the function measuring the 'filled' part of the interval. The latter was later improved by Dvoretzky and Robbins [31]. Many other papers considered a discrete version of the problem, where the intervals are chosen by selecting two integer points on a one-dimensional lattice [30, 42, 51, 57, 64]. However, in this section we are more interested in the number of parked intervals rather than the filling measure.

Introduce the counting function $N(t)$ that keeps track of the number of intervals parked by the optimal selection policy; then a function $v(t) : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as $v(t) := \mathbb{E}N(t)$.

We now derive the optimality equation for $v(t)$. Consider the setup with a horizon $t+h$, $h > 0$. Due to the properties of the Poisson process, with probability $1 - h + o(h)$ we expect no arrivals until instant h. On the other hand, a decision has to be made on the observation (x, y) that arrives with probability h; thus,

$$v(t+h) = (1-h)\,v(t) + 2h \int_{x=0}^{1} \int_{y=x}^{1} \max\{v(t), v(t(1-y)^2) + 1\}\,\mathrm{d}x\mathrm{d}y + o(h).$$

Rearranging, dropping the terms of smaller order, and taking limits on both sides as $h \to 0$, we obtain

$$v'(t) = 2 \int_{x=0}^{1} \int_{y=x}^{1} (v(t(1-y)^2) + 1 - v(t))_+\,\mathrm{d}x\mathrm{d}y.$$

Changing the order of integration and integrating with respect to x leads to

$$v'(t) = 2 \int_{0}^{1} (v(t(1-y)^2) + 1 - v(t))_+\,y\mathrm{d}y, \tag{4.49}$$

which should be accompanied by the initial condition $v(0) = 0$. Finally, employing substitutions $v(t^3) := u(z)$ and $z(1-y)^{2/3} := z - y$, we transform (4.49) into a convolution-type equation

$$u'(z) = 9 \int_{0}^{1} (u(z-y) - u(z) + 1)_+ \eta(z, y)\mathrm{d}y, \quad u(0) = 0, \tag{4.50}$$

where

$$\eta(z, y) = z \left(1 - \left(1 - \frac{y}{z}\right)^{3/2}\right) \left(1 - \frac{y}{z}\right)^{1/2}.$$

4.4.1 Asymptotic expansion of the optimal value function

To obtain the asymptotic expansion of the solution to (4.50), we employ the asymptotic comparison method once again. Introduce an operator $\mathcal{I}$ acting on $C^1(\mathbb{R}^+)$ as follows

$$\mathcal{I}f(z) := 9\int_0^1 (u(z-y) - u(z) + 1)_+ \eta(z,y)\mathrm{d}y$$

Specialising Lemma 3 yields the following result.

Corollary 6. *Suppose $g \in C^1(\mathbb{R}_+)$. If the function satisfies $g'(z) > \mathcal{I}g(z)$, at least for all sufficiently large z, then $\sup_{z\geq 0}(u(z) - g(z)) < \infty$. Likewise, if $g'(z) < \mathcal{I}g(z)$ for all sufficiently large z, then $\inf_{z\geq 0}(u(z) - g(z)) > -\infty$.*

Let us try first the test functions of the form $u_0(z) = \alpha_0 z$, where α_0 is a positive parameter. Firstly, let us inspect the asymptotics of $\mathcal{I}u_0(z)$

$$\mathcal{I}u_0(t) = 9\int_0^1 (u_0(z-y) - u_0(z) + 1)_+ \eta(z,y)\mathrm{d}y. \tag{4.51}$$

Setting $\delta_0(z)$ to be the solution to

$$u_0(z-y) + 1 - u_0(z) = 0;$$

rewrite (4.51) as

$$\mathcal{I}u_0(t) = 9\int_0^{\delta_0(z)} (u_0(z-y) - u_0(z) + 1)\,\eta(z,y)\mathrm{d}y. \tag{4.52}$$

From the definition of $\delta_0(z)$ it follows that $\delta_0(z) = 1/\alpha_0$; hence, plugging this into (4.52) yields

$$\mathcal{I}u_0(z) = \frac{9}{4\alpha_0^2} + O(z^{-1}), \quad z \to \infty. \tag{4.53}$$

Mathching (4.53) with $u_0'(z) = \alpha_0$ leads to

$$u_0'(z) > \mathcal{I}u_0(z), \quad \alpha_0 > \left(\frac{3}{2}\right)^{2/3},$$
$$u_0'(z) < \mathcal{I}u_0(z), \quad \alpha_0 < \left(\frac{3}{2}\right)^{2/3}.$$

Thus, an application of Corollary 6 yields the main term of the asymptotic expansion

$$u(z) \sim \left(\frac{3}{2}\right)^{2/3} z, \quad z \to \infty. \tag{4.54}$$

To refine (4.54) let us try proxy functions of the form

$$u_1(z) = \left(\frac{3}{2}\right)^{2/3} z + \alpha_1 \log(z+1), \quad \alpha_1 \in \mathbb{R}.$$

On the one hand, we have

$$\mathcal{I}u_1(z) = 9\int_0^{\delta_1(z)} (u_1(z-y) + 1 - u_1(z))\, \eta(z,y) \mathrm{d}y, \tag{4.55}$$

where $\delta_1(z)$ solves

$$u_1(z-y) + 1 - u_1(z) = 0.$$

It is not hard to see that $\delta_1(z) = (2/3)^{2/3} + O(z^{-1})$, $z \to \infty$; hence, plugging the principal term of $\delta_1(z)$ into (4.55) results in

$$\mathcal{I}u_1(z) = \left(\frac{3}{2}\right)^{2/3} - \frac{3 + 16\,\alpha_1}{8z} + O(z^{-2}), \quad z \to \infty. \tag{4.56}$$

To match (4.56) with $u_1'(z) \sim (3/2)^{2/3} + \alpha_1/z$, $z \to \infty$, one needs to choose $\alpha_1 = -1/8$; therefore,

$$u(z) \sim \left(\frac{3}{2}\right)^{2/3} z - \frac{\log z}{8}, \quad z \to \infty.$$

Finally, let us examine approximating functions of the type

$$u_2(z) = \left(\frac{3}{2}\right)^{2/3} z - \frac{\log(z+1)}{8} + \frac{\alpha_2}{(z+1)}, \quad \alpha_2 \in \mathbb{R},$$

with

$$u_2'(z) = \left(\frac{3}{2}\right)^{2/3} - \frac{1}{8z} + \left(\frac{1}{8} - \alpha_2\right)\frac{1}{z^2} + O(z^{-3}), \quad z \to \infty. \tag{4.57}$$

To work out the expansion of $\mathcal{I}u_2(z)$ up to the required order, we need the two-term expansion of $\delta_2(z)$; namely,

$$\delta_2(z) = \left(\frac{2}{3}\right)^{2/3} + \left(\frac{2}{3}\right)^{1/3} \frac{1}{12(z+1)} + O(z^{-2}), \quad z \to \infty. \tag{4.58}$$

Using (4.58) yields an expansion

$$\mathcal{I}u_2(z) \sim \left(\frac{3}{2}\right)^{2/3} - \frac{1}{8z} + \left(-\frac{1}{4} - \left(\frac{2}{3}\right)^{2/3}\frac{1}{80} + 2\alpha_2\right)\frac{1}{z^2}, \quad z \to \infty. \tag{4.59}$$

Comparing (4.57) to (4.59) and using Lemma 3 yields

$$\begin{aligned} &\limsup_{t\to\infty}[u(t) - u_2(t)] < \infty, \quad \text{when } \alpha_2 < \frac{1}{8} + \frac{1}{240}\left(\frac{2}{3}\right)^{2/3}, \text{ and} \\ &\liminf_{t\to\infty}[u(t) - u_2(t)] > \infty, \quad \text{when } \alpha_2 > \frac{1}{8} + \frac{1}{240}\left(\frac{2}{3}\right)^{2/3}. \end{aligned}$$

With the third term of $u_2(z)$ being bounded, the following asymptotic expansion holds.

$$u(z) = \left(\frac{3}{2}\right)^{2/3} z - \frac{\log z}{8} + O(1), \quad z \to \infty.$$

Applying the analysis from Sections 4.1.3, 4.1.4, and 4.1.5, one can prove the convergence of the $O(1)$ term to a constant. Subsequently, employing an analogue of Lemma 16, the following refined result can be obtained.

Theorem 13. *As $z \to \infty$, the optimal control function satisfies*

$$\delta^*(z) = \left(\frac{2}{3}\right)^{2/3} + \left(\frac{2}{3}\right)^{1/3} \frac{1}{12z} + O(z^{-2}) \tag{4.60}$$

and the optimal value function satisfies

$$u(z) = \left(\frac{3}{2}\right)^{2/3} z - \frac{1}{8}\log z + c_3^* + \left(\frac{2}{3}\right)^{2/3} \frac{1}{240z} + O(z^{-2}).$$

4.4.2 The variance expansion

In terms of $u(\cdot)$ and z, let $w(z) := \mathbb{E}N(z)^2$; then $w(z)$ satisfies

$$w'(z) = 9\int_0^\delta \left(w(z-y) - w(z) + (1 + 2u(z-y))\right)\eta(z,y)\mathrm{d}y, \tag{4.61}$$

with $w(0) = 0$. We apply the comparison method based on Lemma 3 to obtain the expansion of $w(z)$. Skipping over the detailed computation, which is presented in Appendix A.6, we obtain

$$w(z) \sim \frac{3}{2}\left(\frac{3}{2}\right)^{1/3} z^2 - \frac{1}{4}\left(\frac{3}{2}\right)^{2/3} z\log z + \left(\frac{1}{6} + 2c_3^*\right)\left(\frac{3}{2}\right)^{2/3} z + \frac{(\log z)^2}{64} + \left(\frac{1}{240} - \frac{c_3^*}{4}\right)\log z,$$

as $z \to \infty$. Substracting $u(z)^2$ yields

$$\mathrm{Var}N(z) \sim \left(\frac{3}{2}\right)^{2/3} \frac{z}{6} + \frac{\log z}{240} + c_4^*, \quad z \to \infty.$$

4.4.3 The renewal approximation

Analogously to Section 4.1.7, we investigate the renewal-type behaviour in the left-to-right packing problem. From (4.60), as z gets large, the control $\delta(z)$ approaches a constant. This suggest, approximating the number jumps $N(z)$ with the number of renewals of a process with a generic step $H := D_z + J_z$, where

$$D_z \stackrel{d}{=} \frac{2^{2/3}}{3^{5/3}} E, \quad J_z \stackrel{d}{=} \left(\frac{2}{3}\right)^{2/3} B,$$

with $E \stackrel{d}{=}$ Exponential(1) and $B \stackrel{d}{=}$ Beta$(2,1)$. The moments of H are as follows

$$\mu = \left(\frac{2}{3}\right)^{2/3} \quad \text{and} \quad \sigma^2 = \frac{1}{9}\left(\frac{2}{3}\right)^{1/3}.$$

Hence, introducing a counting function $R(z)$,

$$R(z) := \max\{n : H_1 + \ldots + H_n \le z\},$$

with H_j being i.i.d. replicas of H, we have a central limit theorem

$$\frac{R(z) - z\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{z}} \stackrel{d}{\to} \mathcal{N}(0,1), \quad z \to \infty. \tag{4.62}$$

Analogous to Section 4.1.7, carefully dealing with the accumulating discrepancies, we obtain the distributional convergence

$$\left(\frac{2}{3}\right)^{1/3} \frac{N(z) - (3/2)^{2/3} z}{\sqrt{6z}} \stackrel{d}{\to} \mathcal{N}(0,1), \quad z \to \infty.$$

4.4.4 Summary of the results

Translating back to the original t-horizon setting, we collect all the results obtained in this section. Let $N(t)$ be the number of intervals optimally packed in an online fashion.

Theorem 14. *As $t \to \infty$, the number of packed intervals has the mean satisfying*

$$\mathbb{E}N(t) = \left(\frac{3}{2}\right)^{2/3} t^{1/3} - \frac{\log t}{24} + c_3^* + \left(\frac{2}{3}\right)^{2/3} \frac{1}{240\, t^{1/3}} + O(t^{-2/3}),$$

and the variance satisfying

$$\mathrm{Var}N(t) = \left(\frac{3}{2}\right)^{2/3} \frac{t^{1/3}}{6} + \frac{\log t}{720} + c_4^* + O(t^{-1/3}\log t).$$

Moreover, the following normal convergence holds

$$\left(\frac{2}{3}\right)^{1/3} \frac{N(t) - (3/2)^{2/3}z}{t^{1/6}\sqrt{6}} \xrightarrow{d} \mathcal{N}(0,1).$$

4.4.5 The nested system of intervals

In this section we briefly touch upon a closely related problem of nesting as many random intervals as possible. As in the left-to-right packing, the random intervals arrive with unit-rate Poisson process. However, this time, if the last selected interval is $[x, y]$, the next observed interval that we can choose must lie within $[x, y]$. Decomposing at the first arrival, one can derive the optimality equation for the maximal expected number of embedded intervals, $v(t)$

$$v'(t) = 2\int_0^1 \int_x^1 (v(t(y-x)^2) + 1 - v(t))_+ \,\mathrm{d}x\mathrm{d}y, \quad v(0) = 0.$$

Now, reduce the right-hand side to a one-dimensional integral by substituting $y - x = 1 - \xi$

$$v'(t) = 2\int_0^1 (v(t(1-\xi)^2) + 1 - v(t))_+ \,\xi\mathrm{d}\xi, \quad v(0) = 0. \tag{4.63}$$

The equation (4.63) is identical to (4.49); hence, $v(t)$ must possess the properties outlined in Theorem 14.

The equivalence of the online left-to-right packing and embedding problems seems surprising at first glance. However, the explanation lies in the exchangeability of spacings generated by uniform order statistics — the equivalence breaks for non-uniformly sampled intervals.

Chapter 5

Diffusion approximations in the longest increasing subsequence problem

In this chapter we study several stochastic processes that arise naturally in the process of the longest increasing subsequence selection in continuous time. As we are interested in the time evolution of the last selection and the number of selections processes, it is convenient to extend the underlying framework slightly by considering a homogeneous Poisson random measure Π with intensity ν in the halfplane $\mathbb{R}_+ \times \mathbb{R}$, along with the filtration induced by restricting Π to $[0, t] \times \mathbb{R}$ for $t \in [0, 1]$.

Recall Definition 14 of the threshold strategy in the continuous-time longest increasing subsequence selection. For a given control ψ, define $X(t)$ and $L(t)$ to be, respectively, the last mark selected and the number of marks selected within the time interval $[0, t]$. The process $X = (X(t),\ t \in [0, 1])$, which we call the *running maximum*, is a time-inhomogeneous Markov process, jumping from the generic state x at rate $\psi(t, x)$ to another state uniformly distributed on $[x, x + \psi(t, x)]$. The *length process* $L = (L(t),\ t \in [0, 1])$ just counts the jumps of X; hence the bivariate process (X, L) is also Markovian.

Moreover, the conditional distribution of $((X(t), L(t)),\ t \geq s)$ depends on the pre-s history only through $X(s)$.

In the offline problem, some work was completed on the size of transversal fluctuations about the diagonal $x = t$ in $[0,1]^2$. Johansson [38] proved a measure concentration result asserting that, with probability approaching 1, every longest increasing subsequence (which is not unique) lies in a diagonal strip of the width of the order $\nu^{-1/6+\epsilon}$. Duvergne, Nica and Virág [28] recently proved the existence and gave some description of the functional limit, which is not Gaussian. But for smaller exponent $-1/2 < \alpha < -1/6$, Joseph and Peled [29] showed that if the increasing sequence is restricted to lie within the strip of width $\nu^{-\alpha}$, the expected maximum length remains to be asymptotic to $2\sqrt{\nu}$, while the limit distribution of the length switches to normal.

To extend the parallels and gain further insight into the optimal selection, we introduce the notation for the scaled and centred versions of running maximum and number of selection processes $L_\nu(t)$ and $X_\nu(t)$:

$$\widetilde{X}_\nu(t) := \nu^{1/4}(X_\nu(t) - t), \qquad \widetilde{L}_\nu(t) = \nu^{1/4}\left(\frac{L_\nu(t)}{\sqrt{2\nu}} - t\right), \quad t \in [0,1]. \tag{5.1}$$

To compare, in the offline problem by similar centring the critical transversal and longitudinal scaling factors appear to be $\nu^{1/6}$ and $\nu^{1/3}$, respectively. The central result of this chapter (Theorem 15) is a functional limit theorem which entails that the process $(\widetilde{X}_\nu, \widetilde{L}_\nu)$ converges weakly to a simple two-dimensional Gaussian diffusion. In particular, $\widetilde{X}_\nu$ approaches a Brownian bridge. The limit of $\widetilde{L}_\nu$ is a non-Markovian process with the covariance function

$$(s,t) \mapsto \frac{2s(2-t) - (2-s-t)\log(1-s)}{6\sqrt{2}}, \qquad 0 \leq s \leq t \leq 1,$$

which corresponds to a correlated sum of a Brownian motion and a Brownian bridge.

The question about functional limits for $L_\nu(t)$ and $X_\nu(t)$ has been initiated by Bruss and Delbaen [20]. They employed the Doob-Meyer decomposition to compensate the processes, and in an analytic tour de force showed that the scaled martingales jointly converge to a correlated Brownian motion in two dimensions. However, the compensation keeps out of sight a drift component absorbing much of the fluctuations immanent to the selection process, let alone that the compensators themselves are nonlinear integral transforms of X_ν. To break the vicious circle one needs to obtain the limit of X_ν under complete control over the centring. Curiously, in the forerunning paper Bruss and Delbaen mentioned that P.A. Meyer had suggested to them to scrutinise the generator of the Markov process (X_ν, L_ν) (see [19], Remark 2.4).

Looking at the generator of (5.1) we shall recognise the limit process without difficulty. But in order to justify the weak convergence in the Skorokhod space on the closed interval $[0, 1]$ we will need to circumvent a difficulty caused by pole singularities of the control function and the drift coefficient at the right endpoint. We shall also discuss related processes and derive tight uniform bounds on the expected values of X_ν and L_ν, thus embedding the moment expansions from Theorem 11 in the functional context.

Notation. *We sometimes omit dependence on the intensity parameter ν wherever there is no ambiguity. Notation X and L will be context-dependent, typically standing for processes associated with a near-optimal online selection strategy, while $\widetilde{X}$ and $\widetilde{L}$ will denote the normalised versions with scaling and centring as in* (5.1)*.*

5.1 Selection strategies

Intuitively, the bigger ψ, the faster X and L increase. To enable comparisons of selection processes with different controls it is very convenient to couple them by means of an additive representation through another Poisson random measure Π^*, thought of as a reserve of positive increments. The underlying properties of the planar Poisson process are translation invariance and spatial independence: Π restricted to the shifted quadrant $(t, x) + \mathbb{R}_+^2$ is independent of $\Pi_{|[0,t]\times\mathbb{R}}$ and has the same distribution as the translation of $\Pi_{|\mathbb{R}^2_+}$ by vector (t,x). So, letting Π^* to be a distributional copy of Π, a solution to the system of stochastic differential equations

$$\mathrm{d}X(t) = \int_0^{\psi(t,X(t))} x\,\Pi^*(\mathrm{d}t\mathrm{d}x), \quad \mathrm{d}L(t) = \int_0^{\psi(t,X(t))} \Pi^*(\mathrm{d}t\mathrm{d}x) \tag{5.2}$$

with initial values $X(0) = x_0$ and $L(0) = 0$ will have the same distribution as (X, L).

By virtue of the additive realisation through Π^*, the online increasing subsequence problem is transformed into an online knapsack packing problem [26]. Here, the generic item of some size x observed at time t (an atom of Π^*) can be either packed or dismissed. The objective translates as maximisation of the expected number of items added within the unit time horizon to a knapsack of unit capacity. Note that for the increasing subsequence problem the (continuous) distribution of marks does not matter, while the knapsack problem is not distribution-invariant.

Lemma 18. *For $i = 1, 2$ let X_i be selection processes driven by controls ψ_i. By coupling via* (5.2)*, each time a process with smaller acceptance window jumps, the other process also has a jump of the same size.*

Proof. Straight from (5.2),

$$\mathrm{d}(X_1 - X_2) \quad = \quad \mathrm{sgn}(\psi_1 - \psi_2) \int_{\psi_1 \wedge \psi_2}^{\psi_1 \vee \psi_2} x\,\Pi^*(\mathrm{d}t\mathrm{d}x),$$

where for shorthand $\psi_i = \psi_i(t, X_i(t))$. □

Conditionally on $(X(s), L(s)) = (x, \ell)$, the process $(X(s+\cdot) - x, L(s+\cdot) - \ell))$ has the same distribution as $(X^{(s,x)}, L^{(s,x)})$, which similarly to (5.2) is given by

$$dX^{(s,x)}(u) = \int_0^{\psi(s+u, x+X^{(s,x)}(u))} y\Pi^*(\mathrm{d}u\mathrm{d}y), \quad \mathrm{d}L^{(s,x)}(u) = \int_0^{\psi(s+u, x+X^{(s,x)}(u))} \Pi^*(\mathrm{d}u\mathrm{d}y).$$

Averaging, we obtain formulas for the predictable *compensators* of X and L

$$C_X(t) := \frac{\nu}{2}\int_0^t \psi^2(s, X(s))\mathrm{d}s, \quad C_L(t) := \nu\int_0^t \psi(s, X(s))\mathrm{d}s, \tag{5.3}$$

so $X - C_X, L - C_L$ are zero-mean martingales.

With every control we may further relate a zero-mean martingale

$$M(t) := L(t) + \mathbb{E}\{L(1) - L(t)|X(t)\} - \mathbb{E}L(1) \tag{5.4}$$

with terminal value $L(1) - \mathbb{E}L(1)$. If ψ does not depend on x, L has independent increments and $M(t) = L(t) - \mathbb{E}L(t)$.

By the setup of the problem, the running maximum must satisfy $X(1) \leq 1$. In terms of the control function this translates to the following condition.

Definition 17. *A control function $\psi(t, x)$ is called* feasible *if*

$$0 < \psi(t, x) \leq 1 - x \quad \text{for} \quad (t, x) \in [0, 1]^2.$$

In the sequel, if not stated otherwise we set $X(0) = 0$ and only consider feasible controls.

5.1.1 Principal convergence of the moments

This section follows closely the arguments found in [20], pp. 291-292.

Let

$$p(t) := \mathbb{E}X(t) = \mathbb{E}C_X(t), \quad q(t) := \frac{\mathbb{E}L(t)}{\sqrt{2\nu}} = \frac{\mathbb{E}C_L(t)}{\sqrt{2\nu}}.$$

Some general relations between the moments follow straight from formulas for the compensators (5.3). For shorthand, write $\psi = \psi(s, X(s))$. We have

$$0 \leq \mathbb{E} \int_0^t \left(1 \pm \sqrt{\nu/2}\,\psi\right)^2 \mathrm{d}s = t \pm 2q(t) + p(t),$$

where the right-hand side in increasing in t. It follows,

$$p(t) - t \geq 2\,(q(t) - t). \tag{5.5}$$

Using the Cauchy-Schwarz inequality

$$\begin{aligned} (p(t)-t)^2 &= \left(\mathbb{E}\int_0^t \left(1 - \frac{\nu}{2}\psi^2\right) \mathrm{d}s\right)^2 \\ &\leq \mathbb{E}\int_0^t \left(1 + \sqrt{\nu/2}\,\psi\right)^2 \mathrm{d}s \; \mathbb{E}\int_0^t \left(1 - \sqrt{\nu/2}\,\psi\right)^2 \mathrm{d}s \\ &= (t + 2q(t) + p(t))(t - 2q(t) + p(t)). \end{aligned} \tag{5.6}$$

Similarly

$$(q(t)-t)^2 = \left(\mathbb{E}\int_0^t 1 \cdot \left(1 - \sqrt{\nu/2}\,\psi\right) \mathrm{d}s\right)^2 \leq t\,(t - 2q(t) + p(t)) \tag{5.7}$$

The above relations did not use the feasibility constraint. For feasible control we have $p(1) < 1$, hence from (5.5) also $q(1) < 1$. Since all factors in the right-hand sides of (5.6), (5.7) are increasing, replacing them by their maximal values at $t = 1$ we obtain

$$(p(t)-t)^2 < 8(1 - q(1)), \quad (q(t)-t)^2 < 2(1 - q(1)). \tag{5.8}$$

We say that a strategy $\psi = \psi_\nu$ is *asymptotically optimal in the principal term* if $q(1) \to 1$, as $\nu \to \infty$, i.e. $\mathbb{E}L_\nu(1) \sim \sqrt{2\nu}$; in that case (5.8) imply the uniform convergence of the moments

$$\sup_{t\in[0,1]} |p(t) - t| \to 0, \quad \sup_{t\in[0,1]} |q(t) - t| \to 0.$$

It follows from (4.39) that under the optimal strategy

$$1 - q(1) \sim \frac{\log \nu}{12\sqrt{2\nu}}, \quad \nu \to \infty. \tag{5.9}$$

This relation can be called a *two-term asymptotic optimality.* Whenever this holds, the general bounds (5.8) imply that both $\sup_{t\in[0,1]} |p(t) - t|$ and $\sup_{t\in[0,1]} |q(t) - t|$ can be estimated as $O(\sqrt{\log \nu}/\nu^{1/4})$. A refinement of the convergence rate will be obtained in Section 5.6.

5.1.2 The greedy strategy

The *greedy* strategy, with control $\psi(t, x) = 1 - x$, outputs the sequence of consecutive records. The strategy is optimal for $\nu < 1.34...$ Statistical properties of records from the Poisson process is a much-studied subject [22]. It is well known that, as ν increases, the distribution of $L(1)$ approaches normal with mean and variance both asymptotic to $\log \nu$. Normalisation (5.1) is not appropriate here as most of the records concentrate near the north-west corner of the unit square (see Figure 4.2a for the simulated paths of the running maximum corresponding to the greedy selection strategy).

5.1.3 The stationary strategy

We call the strategy with control $\psi(t, x) = \sqrt{2/\nu}$ *stationary.* Although not feasible, the stationary strategy is an important benchmark. Clearly, L is a Poisson counting process with intensity $\mathbb{E}L(1) = \sqrt{2\nu}$. Taking general constant control $\psi(t, x) = \sqrt{c/\nu}$ with some $c > 0$ will yield a strategy outputting the mean length $\sqrt{\{c \wedge (2/c)\}\nu}$, which is maximal for $c = 2$. In fact, a much stronger optimality property holds: the stationary strategy achieves the maximum expected length over the class of strategies that satisfy the *mean-value* constraint $\mathbb{E}X(1) \leq 1$. This gives the mean-constraint upper bound on $\mathbb{E}L(1)$ derived in Section 4.2.1 because each feasible strategy meets the mean-value constraint.

It is seen from (5.2) that X is a compound Poisson process

$$X(t) = \sqrt{\frac{2}{\nu}} \sum_{i=0}^{L(t)} U_i \,,$$

where $U_1, U_2, \ldots$ are independent of L, uniformly distributed on $[0,1]$. Straightforward calculation of moments using Wald's identities yields

$$\mathbb{E}X(t) = t, \quad \mathrm{Var}X(t) = \frac{2^{3/2}t}{3\sqrt{\nu}}, \quad \mathrm{Cov}(X(t), L(t)) = t.$$

Since (X, L) has independent increments, a functional limit in the Skorohod topology on $D[0,1]$ follows easily from the multidimensional invariance principle:

$$(\widetilde{X}, \widetilde{L}) \Rightarrow (W_1, W_2), \quad \text{as } \nu \to \infty,$$

where $\Rightarrow$ denotes weak convergence, and the limit process $\boldsymbol{W} := (W_1, W_2)$ is a two-dimensional Brownian motion with zero drift and covariance matrix

$$\mathbb{E}\{\boldsymbol{W}(t)^T \boldsymbol{W}(t)\} = t\,\boldsymbol{\Sigma}, \quad \text{where } \boldsymbol{\Sigma} := \begin{pmatrix} \frac{2\sqrt{2}}{3} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{5.10}$$

So, marginally, W_1 and W_2 are centred Brownian motions with diffusion coefficients and correlation, respectively,

$$\sigma_1 := \frac{2^{3/4}}{\sqrt{3}}, \quad \sigma_2 := \frac{1}{2^{1/4}}, \quad \rho := \frac{\sqrt{3}}{2}. \tag{5.11}$$

Notably,

$$\rho = \frac{\sigma_2}{\sigma_1},$$

which implies that the process $\boldsymbol{W}$ satisfies the identity

$$2W_2 - W_1 \stackrel{d}{=} W_1, \tag{5.12}$$

which has a pre-limit analogue

$$2\widetilde{L} - \widetilde{X} \stackrel{d}{=} \widetilde{X}. \tag{5.13}$$

Identity (5.13) can be explained by the symmetry of the uniform distribution, $U_i \stackrel{d}{=} 1 - U_i$, which allows us to write

$$X(t) \stackrel{d}{=} \sqrt{\frac{2}{\nu}} \sum_{i=0}^{L(t)} (1 - U_i) = \sqrt{\frac{2}{\nu}}\, L(t) - X(t)\,.$$

Martingale (5.4) just coincides with naturally centred L.

The correlated Brownian motion has appeared in Bruss and Delbaen [20] (Theorem 4.1), as the limit of (X, L) centred by their compensators C_X and C_L under the optimal (feasible) strategy. This connection confirms that the key to the fluctuation problem is understanding the nature of the drift component.

5.1.4 A feasible version of the stationary strategy

The strategy driven by $\psi(t, x) = \sqrt{2/\nu} \wedge (1 - x)$ is a counterpart of that introduced by Samuels and Steele in the discrete-time setting [55]. This is a minor modification of the stationary strategy to meet the feasibility condition. Define the hitting time

$$\tau := \inf\{t \in [0, 1] : X(t) \geq 1 - \sqrt{2/\nu}\}$$

with the convention $\inf \varnothing = 1$. The strategy acts as the stationary before τ, and if $\tau < 1$ proceeds with a greedy selection, so, in essence, the selection process is frozen at time τ. Using elementary renewal theory arguments, we find asymptotics of the moments

$$\mathbb{E}L(1) \sim \sqrt{2\nu} - \frac{2^{3/4}}{\sqrt{3\pi}}\,\nu^{1/4}, \quad \mathrm{Var}L(1) \sim \frac{2^{3/2}}{3}\left(1 - \frac{1}{\pi}\right)\sqrt{\nu}.$$

Hence the strategy is asymptotically optimal in the principal term.

Furthermore, by Lemma 17, $\widetilde{L}(1)$ converges in distribution to $2^{-1/4}\{(\xi_1/\sqrt{3}) \wedge \xi_2\}$, where ξ_1, ξ_2 are independent $\mathcal{N}(0, 1)$. The normalised terminal value $\widetilde{X}(1)$ is nonpositive, and converges in distribution to $-2^{3/4}3^{-1/2}(\eta)_+$, where η is another standard normal variable and $(\cdot)_+$ denotes the positive part. By symmetry of the normal distribution, the hitting time τ assumes value 1 with probability approaching $\mathbb{P}(V_1(1) < 0) = 1/2$, and otherwise $1-\tau$ is of the order $\nu^{-1/4}$. Comparing with the stationary strategy, one can see that there is an optimality gap of order $\nu^{1/4}$, which occurs due to a premature freeze of selection in the event $\tau < 1$.

Note that the moments of terminal values satisfy $1-p(1) \sim c_1\nu^{-1/4}, 1-q(1) \sim c_2\nu^{-1/4}$ with some $c_1, c_2 > 0$, while (5.8) overestimates the first as $1-p(1) = O(\nu^{-1/8})$.

In terms of the normalised running maximum, τ is the time when $\widetilde{X}$ hits the straight line connecting points $(0, \nu^{1/4})$ and $(1, 0)$. Since $\tau \to 1$ in probability, $(\widetilde{X}, \widetilde{L})$ has the same functional limit as under the stationary strategy on every interval $[0, 1-h]$, for $h \in (0,1)$. Extending the functional limit to the closed $[0,1]$ leads to a discontinuity at $t = 1$. To capture the jump, it is enough to modify the correlated Brownian motion $\boldsymbol{W}$ by replacing the terminal value $(W_1(1), W_2(1))$ with

$$(W_1(1) - (W_1(1))_+, W_2(1) - (2/\sqrt{3})(W_1)_+).$$

5.1.5 Self-similar asymptotically optimal strategies

Recall Definition 15 of a self-similar selection policy which has the control ψ of the form

$$\psi(t,x) := (1-x)\,\delta(\nu(1-t)(1-x)), \quad (t,x) \in [0,1]^2 \tag{5.14}$$

for some function $\delta : \mathbb{R}_+ \to [0,1]$. Note that such a strategy is feasible and $\psi_\nu(0,0) = \delta(\nu)$. The rationale behind this definition is the following. Assuming x to be the running maximum at time t, the remaining part of the chain should be selected from the north-east rectangle spanned on (t,x) and $(1,1)$, and by the optimality principle the subsequence selected from the rectangle should have maximal expected length. Mapping the rectangle onto $[0,1]^2$ it is readily seen that the subproblem is an independent replica of the original problem of optimal selection from the unit square with intensity parameter $\nu(1-x)(1-t)$. The martingale (5.4) assumes the form

$$M(t) = L(t) + F(\nu(1-t)(1-X(t))) - F(\nu), \tag{5.15}$$

where the value function F (analogue of $v(t)$ from Section 4.1), for given control, depends on one variable

$$F(\nu) := \mathbb{E}L_\nu(1), \quad F(0) = 0.$$

Assumption. From this point on we assume that *the selection strategy is self-similar as defined by* (5.14)*, with function* δ *having asymptotics*

$$\delta(\nu) = \sqrt{2/\nu} + O\left(\nu^{-1}\right), \quad \nu \to \infty. \tag{5.16}$$

This assumption is central and deserves comments. Whenever $\nu(1-x)(1-t)$ is large, (5.16) implies asymptotics of the control

$$\psi(t,x) \sim \sqrt{\frac{2(1-x)}{\nu(1-t)}}, \tag{5.17}$$

which shows that near the diagonal $x = t$ the acceptance window is about the same as for the stationary strategy. Away from the diagonal, the acceptance window is close to that for the stationary strategy adjusted to the rectangle north-east of (t,x).

It is known from Theorem 11 that the optimal strategy satisfies the asymptotic expansion

$$\delta^*(\nu) \sim \sqrt{2/\nu} - (3\nu)^{-1} + O(\nu^{-3/2}).$$

A minor adjustment of Theorem 11 shows that if we assume, more generally, the relation $\delta(\nu) \sim \sqrt{2/\nu} + \beta/\nu$ with some parameter $\beta \in \mathbb{R}$, then asymptotic expansions of the moments (4.39), (4.40) are still valid, with only constant terms depending on β. Using a sandwich argument based on Lemma 18, one can further show that under the assumption (5.16) expansions of the moments hold but with constant terms being replaced by some $O(1)$ remainders. In particular, condition (5.16) ensures the two-term asymptotic optimality (5.9), equivalent to the asymptotic expansion of the value function,

$$F(\nu) = \sqrt{2\nu} - \frac{1}{12}\log(\nu+1) + O(1). \tag{5.18}$$

We stress that the logarithmic term here (as well as in the counterpart of the variance formula (4.40)) is not affected by the remainder in (5.16), rather appears due to the self-similar adjustment of (a feasible version of) the stationary strategy, as incorporated in (5.17). The impact of the second term in (5.16) on moments of the running maximum will be scrutinised in Section 5.6.

Approximation (5.17) is not useful when t or x is too close to 1 so that $\nu(1-t)(1-x)$ varies within $O(1)$. To embrace the full range of the variables, for the sequel we choose $\beta > 1$ large enough to meet the bounds

$$\left|\psi(t,x) - \sqrt{\frac{2(1-x)}{\nu(1-t)}}\right| \;<\; \frac{\beta}{\nu(1-t)}, \quad \text{for } (t,x) \in [0,1) \times [0,1). \tag{5.19}$$

This will be employed along with the bound

$$\psi(t,x) \;<\; \frac{1}{\nu(1-t)}, \quad \text{for } 1 - x < \frac{1}{\nu(1-t)} \tag{5.20}$$

which follows by feasibility.

5.2 The generators

The selection process in Section 5.1.4 demonstrates one type of possible pathology, caused by large overshooting the diagonal at times close to $t = 1$. Nevertheless, under (5.16) it is not even evident that $(\widetilde{X}, \widetilde{L})$ has a sensible limit in $D[0,1]$. A significant technical difficulty in showing the convergence is the singularity of (5.17) at $t = 1$. This will be handled in two steps. First, we bound the time variable away from $t = 1$ and show the convergence of the generators on a sufficiently big space of test functions. Then we will apply domination arguments to bound fluctuations near the right endpoint, thus justifying convergence on the full $[0,1]$.

The processes we consider are not time-homogeneous; therefore, by computing generators we include the time variable in the state vector. From (5.2), the generator of the jump process (X, L) is

$$\mathcal{L}_\nu f(t,x,\ell) = f_t(t,x,\ell) + \nu \int_0^{\psi(t,x)} \{f(t,x+u,\ell+1) - f(t,x,\ell)\} \mathrm{d}u.$$

For the processes centred by t we should include $-f_x - f_\ell$ in the generator. Then, with the change of variables

$$x \to x\nu^{-1/4} + t, \;\; \ell \to (\ell\nu^{-1/4} + t)\sqrt{2\nu}, \;\;\; \widetilde{\psi}(t,x) := \nu^{1/4}\psi(t, x\nu^{-1/4} + t)$$

we arrive at the generator of $(\widetilde{X}, \widetilde{L})$

$$\widetilde{\mathcal{L}}_\nu f = f_t - \nu^{1/4}(f_x + f_\ell) + \nu^{3/4} \int_0^{\widetilde{\psi}(t,x)} \{f(t,x+u,\ell+v) - f(t,x,\ell)\} \mathrm{d}u, \tag{5.21}$$

where we abbreviate $f = f(t,x,\ell)$ etc., and

$$v := (4\nu)^{-1/4} \tag{5.22}$$

We extend $\widetilde{\mathcal{L}}_\nu f$ by 0 outside the reachable range of $(\widetilde{X}, \widetilde{L})$. Note that the range of $\widetilde{X}(t)$ lies within the bounds

$$-t\nu^{1/4} \le x \le (1-t)\nu^{1/4}.$$

We fix $h \in (0,1)$ and focus on $t \in [0, 1-h]$, so achieving uniformly in this range

$$\widetilde{\psi}(t,x) = O(\nu^{-1/4}), \tag{5.23}$$

and for $k \geq 1$

$$\widetilde{\psi}^k(t,x) = \left(2 - \frac{2x}{\nu^{1/4}(1-t)}\right)^{k/2} \nu^{-k/4} + O(\nu^{-(k+2)/4}), \text{ for } x \leq (1-t)\nu^{1/4} - \frac{1}{\nu^{3/4}(1-t)} \tag{5.24}$$

as dictated by the bounds (5.19), (5.20).

Now let $\mathcal{D}$ be the space of vanishing at infinity functions $f \in C_0^3([0,1] \times \mathbb{R}^2)$ which satisfy a rapid decrease property

$$\sup |x^k f_\bullet(t,x,\ell)| < \infty,$$

where $f_\bullet$ is any derivative of f of the first or second order and $k > 0$. Set

$$D^>_{h,\nu} := \{(t,x,\ell) : t \in [0,1-h],\ |x| > \nu^{1/16}\},\ D^<_{h,\nu} := \{(t,x,\ell) : t \in [0,1-h],\ |x| \leq \nu^{1/16}\}.$$

We shall be using that for $f \in \mathcal{D}$

$$\lim_{\nu \to \infty} \sup_{D^>_{h,\nu}} |\nu^k f_\bullet(x)| = 0. \tag{5.25}$$

The integrand in (5.21) expands as

$$f(t, x+u, \ell+v) - f(t,x,\ell) = f_x u + f_\ell v + \frac{1}{2} f_{xx}\, u^2 + f_{x\ell}\, uv + \frac{1}{2} f_{\ell\ell}\, v^2 + R,$$

where the remainder can be estimated as

$$|R| \leq c \sum_{i=0}^{3} u^i v^{3-i},$$

with constant c chosen bigger than the maximum absolute value of any third derivative of f. Hence for the integrated remainder we have a uniform estimate

$$\nu^{3/4} \left| \int_0^{\widetilde{\psi}} R du \right| \leq \nu^{3/4} c \sum_{i=1}^{4} \widetilde{\psi}^i v^{4-i} = O(\nu^{-1/4}),$$

using (5.23), (5.22).

Integrating the Taylor polynomial yields

$$\widetilde{\mathcal{L}}_\nu f = f_t - \nu^{1/4}(f_x + f_\ell) + \nu^{3/4}\left\{\frac{1}{2} f_x\,\widetilde{\psi}^2 + f_\ell\, v\widetilde{\psi} + \frac{1}{6}\, f_{xx}\widetilde{\psi}^3 + \frac{1}{2} f_{x\ell}\,\widetilde{\psi}^2 v + \frac{1}{2} f_{\ell\ell} v^2 \widetilde{\psi}\right\} + \widetilde{R}(\nu),$$

where $\widetilde{R}(\nu) = O(\nu^{-1/4})$. Applying (5.25)

$$\lim_{\nu\to\infty} \sup_{D^{>}_{h,\nu}} |\widetilde{\mathcal{L}}_\nu f(t,x,\ell)| = 0. \tag{5.26}$$

Thus we focus on the range $D^{<}_{h,\nu}$, where (5.19) and (5.24) can be employed. From (5.19)

$$-\nu^{-1/4} f_x + \nu^{3/4}\frac{1}{2} f_x \widetilde{\psi}^2 = -\frac{x}{1-t} f_x + O(\nu^{-1/4}).$$

Observing that in this range $|x\nu^{-1/4}| \le \nu^{-3/16}$ for $k > 0$ we expand as

$$\widetilde{\psi}^k(t,x,\ell) = 2^{k/2}\nu^{-k/4} - \frac{2^{k/2-1}x}{1-t}\nu^{-(k+1)/4} + O(\nu^{-(k+1)/4-1/8}),$$

with the remainder estimate being uniform over $D^{<}_{h,\nu}$. The remaining calculations is a careful book-keeping using this formula and that the derivatives are uniformly bounded:

$$\begin{aligned}
-\nu^{1/4} f_\ell + \nu^{3/4} f_\ell v \widetilde{\psi} &= -\frac{x}{2(1-t)} f_\ell + O(\nu^{-1/8}),\\
\nu^{3/4}\frac{1}{6} f_{xx}\widetilde{\psi}^3 &= \frac{\sqrt{2}}{3} f_{xx} + O(\nu^{-3/16}),\\
\nu^{3/4}\frac{1}{2} f_{x\ell}\widetilde{\psi}^2 v &= \frac{1}{\sqrt{2}} f_{x\ell} + O(\nu^{-3/16}),\\
\nu^{3/4}\frac{1}{2} f_{\ell\ell} v^2 \widetilde{\psi} &= \frac{1}{2\sqrt{2}} f_{\ell\ell} + O(\nu^{-3/16}).
\end{aligned}$$

Define an operator

$$\widetilde{\mathcal{L}} f := f_t - \frac{x}{1-t} f_x - \frac{x}{2(1-t)} f_\ell + \frac{{\sigma_1}^2}{2} f_{xx} + \frac{{\sigma_2}^2}{2} f_{\ell\ell} + \sigma_1\sigma_2\rho f_{x\ell},$$

with σ_1, σ_2, and ρ given by (5.11).

Lemma 19. *For $f \in \mathcal{D}$ and $h \in (0,1)$*

$$\lim_{\nu\to\infty} \sup_{(t,x,\ell)\in[0,1-h]\times\mathbb{R}^2} |\widetilde{\mathcal{L}}_\nu f(t,x,\ell) - \widetilde{\mathcal{L}} f(t,x,\ell)| = 0$$

Proof. The supremum over $D^{>}_{h,\nu}$ goes to zero since by (5.25) the analogue of (5.26) holds true for $\widetilde{\mathcal{L}}$. The supremum over $D^{<}_{h,\nu}$ goes to zero by the above expansions.

□

Operator $\widetilde{\mathcal{L}}$ is the generator of a Gaussian diffusion process which satisfies the stochastic differential equations

$$\mathrm{d}Y_1(t) = -\frac{Y_1(t)}{1-t}\,\mathrm{d}t + \mathrm{d}W_1(t), \tag{5.27}$$

$$\mathrm{d}Y_2(t) = -\frac{Y_1(t)}{2(1-t)}\,\mathrm{d}t + \mathrm{d}W_2(t), \tag{5.28}$$

with zero initial value, where $\boldsymbol{W} = (W_1, W_2)$ is the two-dimensional Brownian motion with covariance $\boldsymbol{\Sigma}$ introduced in (5.10).

From the equation for the first component (5.27), it is seen that Y_1 is a Brownian bridge

$$Y_1(t) = (1-t)\int_0^t \frac{\mathrm{d}W_1(s)}{1-s}, \tag{5.29}$$

with the covariance function $\mathrm{Cov}(Y_1(s), Y_1(t)) = \sigma_1 s(1-t)$, $0 \le s \le t \le 1$. In particular, $Y_1(1) = 0$. We shall discuss the second component later on.

The space $\mathcal{D}$ is dense in a larger space $C_0^3([0,1-h]\times\mathbb{R}^2)$. Since the differentiability properties of functions are preserved under averaging over normally distributed translations, $\mathcal{D}$ is invariant under the semigroup of $\boldsymbol{Y}$. Thus by Watanabe's theorem (see [39], Proposition 17.9) $\mathcal{D}$ is a core of operator $\widetilde{\mathcal{L}}$. The above Lemma 19 and Theorem 17.25 from [39] now imply weak convergence

$$(\widetilde{X}_\nu, \widetilde{L}_\nu) \Rightarrow (Y_1, Y_2) \quad \text{in} \quad D[0,1-h] \tag{5.30}$$

for every $h \in (0,1)$. A closer inspection of the above approximation errors suggests that the quality of convergence deteriorates as $h \to 0$.

We encountered the Brownian motion $\boldsymbol{W}$ in connection with the free-endpoint stationary strategy in Sections 5.1.3 and 5.1.4. Now we see that the variable control (5.17) causes a drift that, in the $\nu \to \infty$ limit, forces the running maximum to timely arrive at the north-east corner of the square.

5.3 Convergence to diffusion: end of the proof

The martingale problem for $\widetilde{\mathcal{L}}$ is well-posed on the complete interval, and the SDE (5.27) has a unique strong solution. This suggests extending convergence (5.30) to the full $[0,1]$. To that end, we need to monitor the behaviour of $\widetilde{\mathcal{L}}_\nu f$ for t close to 1. Estimates in Bruss and Delbaen ([20], p. 294) show that $\widetilde{X}_\nu(1) \to 0$ in probability, which agrees neatly with the Brownian bridge limit, but this still does not exclude giant fluctuations of the pre-limit process near $t = 1$.

A similar kind of difficulty appears by the martingale approach to the classic problem of convergence of the empirical distribution function [36, 37]. The proof found in Jacod and Shiryaev (see [37], p.561) handles the nuisance by exploiting the time reversibility of the Brownian bridge. Our argument will rely on the self-similarity.

Since (5.30) entails the convergence of finite-dimensional distributions for times $t < 1$ and ensures that the modulus of continuity behaves correctly over $[0, 1-h]$, to justify tightness of $\widetilde{X}_\nu$'s, and hence their convergence on $[0,1]$, it will be enough to show that

$$\lim_{h\to 0} \limsup_{\nu} \mathbb{P}\left(\sup_{t\in[1-h,1]} |\widetilde{X}_\nu(t)| > h^{1/4} \right) = 0. \tag{5.31}$$

Define $\xi_{\nu,h}$ by setting

$$\widetilde{X}_\nu(1-h) = \sigma_1 \sqrt{h(1-h)}\, \xi_{\nu,h}.$$

Since $\widetilde{X}_\nu(1-h) \xrightarrow{d} Y_1(1-h)$ the distribution of $\xi_{\nu,h}$ is close to $\mathcal{N}(0,1)$ for large ν.

By self-similarity of the selection strategy, $((X_\nu(t) - t),\ t \in [1-h,1])$ has the same distribution as $(h^{-1}(X_{\nu h^2}(t) - t),\ t \in [0,1])$ with the initial value $X_{\nu h^2}(0) = \nu^{-1/4}\sigma_1\sqrt{(1-h)/h}\,\xi_{\nu,h}$, as is seen by zooming in the corner square north-east of the point $(1-h, 1-h)$ with factor h^{-1}. Changing variable $\nu h^2 \to \nu$, (5.31) translates as a compact containment condition

$$\lim_{h\to 0}\limsup_{\nu}\mathbb{P}\left(\sup_{t\in[0,1]}|\widetilde{X}_\nu(t)| > h^{-1/4}\right) = 0 \tag{5.32}$$

under the initial value $\widetilde{X}_\nu(0) = \sqrt{1-h}\,\xi_{\nu,h}$.

To verify (5.32) we shall squeeze the running maximum X between $X^\downarrow$ and $X^\uparrow$ whose normalised versions satisfy the compact containment condition. We force the majorant and the minorant to live on the opposite sides of the diagonal. Both have independent, almost stationary increments so that functional limits can be readily identified. For simplicity we will assume $X_\nu(0) = 0$. The general case with $X_\nu(0)$ of the order $\nu^{-1/4}$ can be handled by the same method.

5.3.1 A majorant

Define process $X^\uparrow = X^\uparrow_\nu$ as the solution to

$$\mathrm{d}X^\uparrow(t) = \int_0^{\psi^\uparrow(t)} x\Pi^*(\mathrm{d}t\mathrm{d}x) + \mathbb{1}(X^\uparrow(t) = t)\mathrm{d}t,$$

$X^\uparrow(0) = K\nu^{-1/2}$ for some big enough $K > 0$, with control

$$\psi^\uparrow(t) := \sqrt{\frac{2}{\nu}} + \frac{\beta}{\nu(1-t)}\,\mathbb{1}(t \le 1 - K\nu^{-1/2})$$

not depending on x. The process never drops below the line $x = K\nu^{-1/2} + t$, and whenever the line is hit the path drifts along it for some time. By the construction, above the diagonal the process $X^\uparrow$ increases faster than X, and is, in fact, a majorant.

Lemma 20. *By coupling via* (5.2), $X^\uparrow \ge X$ *a.s.*

Proof. By the virtue of (5.19), (5.20) and definition of $\psi^{\uparrow}$ we have $\psi^{\uparrow}(t,x) > \psi(t,x)$ for $x > t,\ t \le 1 - K\nu^{-1/2}$. Hence by Lemma 18, $\mathrm{d}\{X^{\uparrow}(t) - X(t)\} > 0$ conditional on $X^{\uparrow}(t) > X(t) > t$ at time $t < 1 - K\nu^{-1/2}$.

Initially $X^{\uparrow}(0) > X(0)$, and $X^{\uparrow}(t) > 1 > X(t)$ for $t > 1 - K\nu^{-1/2}$. Hence the only way the paths can cross is that X overjumps $X^{\uparrow}$ from some position $x < t \le X^{\uparrow}(t)$ at some time $t \le 1 - K\nu^{-1/2}$. The latter possibility is excluded because

$$\begin{aligned}\psi(t,x) &< \sqrt{\frac{2}{\nu}}\left(1 + \frac{t-x}{1-t}\right) + \frac{\beta}{\nu(1-t)} < \sqrt{\frac{2}{\nu}}\left(1 + \frac{t-x}{2(1-t)}\right) + \frac{\beta}{\nu(1-t)} \\ &\le \sqrt{\frac{2}{\nu}} + \frac{t-x}{K\sqrt{2}} + \frac{\beta}{K\sqrt{\nu}} < t - x + \frac{K}{\sqrt{\nu}}\end{aligned}$$

for K chosen big enough. □

Let

$$S(t) := \int_0^t \int_0^{\psi^{\uparrow}(t)} x\Pi^*(\mathrm{d}s\mathrm{d}x) - t.$$

This is a process with independent increments, which we can split into two independent components

$$S(t) = \left(\int_0^t \int_0^{\sqrt{2/\nu}} x\Pi^*(\mathrm{d}s\mathrm{d}x) - t\right) + \int_0^t \int_{\sqrt{2/\nu}}^{\psi^{\uparrow}(t)} x\Pi^*(\mathrm{d}s\mathrm{d}x).$$

The mean value of the second part is estimated as

$$\frac{2\nu}{\sqrt{\nu}} \int_0^{1-K/\sqrt{\nu}} \frac{\beta}{\nu(1-t)}\mathrm{d}t = O\left(\frac{\log \nu}{\sqrt{\nu}}\right),$$

and the first is a compensated compound Poisson process. Thus $\nu^{1/4} S \Rightarrow W_1$ as $\nu \to \infty$.

Processes akin to $(X^{\uparrow}(t) - t,\ t \in [0,1])$ are common in applied probability [7, 14]. In particular, by the interpretation as the content of a single-server M/G/1 queue, the positive increments present jobs that arrive by Poisson process and are measured in terms of the demand on the service time. The downward drift occurs due to the unit processing rate when the server is busy. Borrowing a useful identity,

$$X^{\uparrow}(t) - t = S(t) - \inf_{u \in [0,t]} S(u),$$

we conclude on the weak convergence $(\nu^{1/4}(X^{\uparrow}(t) - t),\ t \in [0,1]) \Rightarrow |W_1|$ to a reflected Brownian motion.

5.3.2 A minorant

This time we define $X^{\downarrow}$ by (5.2) with control

$$\psi^{\downarrow}(t,x) = \begin{cases} \left(\sqrt{\frac{2}{\nu}} - \frac{\beta}{\nu(1-t)}\right) \wedge (t-x), & \text{for} \quad 0 \le t \le 1 - K/\sqrt{\nu}, \\ 0, \quad \text{for} \quad 1 - K/\sqrt{\nu} < t \le 1. \end{cases}$$

where K is sufficiently large. We can regard this as a suboptimal strategy that never selects marks $x > t$. Starting at state 0, the running maximum process stays below the diagonal throughout, and gets frozen at $t = 1 - K/\sqrt{\nu}$. A counterpart of Lemma 20, $X^{\downarrow} < X$ a.s., is readily checked.

Switching general $\beta > 0$ to $\beta = 0$ impacts $\mathbb{E}X^{\downarrow}(t)$ by $O(\nu^{-1/2}\log\nu)$ uniformly in $t \in [0,1]$. Indeed, the jumps are bounded by $2/\sqrt{\nu}$, and the expected number of jumps increases by $O(\log\nu)$.

Assuming $\beta = 0$, the process $(X^{\downarrow}(t) - t,\ t \in [0, 1 - K\nu^{-1/2}])$ is a compensated compound Poisson process on the negative half-line, with reflection at 0. We have therefore

$$(\nu^{1/4}(X^{\downarrow}(t) - t),\ t \in [0,1]) \Rightarrow -|W_1|.$$

The rest of this section is dedicated to showing the convergence of the generator acting on the functions $f \in \mathcal{D}$ with $f_x(t,0) = 0$ to the generator of a reflected Brownian motion.

Computing the generator of the scaled process $(\nu^{1/4}(X^{\downarrow}(t)-t), t \in [0,1])$ yields

$$\mathcal{L}^{\downarrow}_{\nu} f(t,x) = f_t(t,x) - \nu^{-1/4} f_x(t,x) + \nu^{3/4} \int_0^{\psi^{\downarrow}(t,x)} \{f(t,x+u) - f(t,x)\} \mathrm{d}u,$$

where $\psi^{\downarrow}(t,x) = \nu^{1/4}\psi^{\downarrow}(t, x\nu^{-1/4}+t)$. To ensure the desired convergence, we choose the space $\mathcal{D}^{\downarrow}$

$$\mathcal{D}^{\downarrow} := \{f \in C_0^3([0,1]\times\mathbb{R}) : \sup|x^k f_{\bullet}(t,x)| < \infty,\ f_x(t,0) = 0\},$$

where $f_x(t,0) = 0$ is a familiar condition for the class of functions that are acted upon by the generator of a reflected Brownian motion. By Taylor's theorem

$$\mathcal{L}^{\downarrow}_{\nu} f(t,x) = f_t(t,x) + \frac{\sigma_1^2}{2} f_{xx}(t,x) + O(\nu^{-1/4}), \tag{5.33}$$

uniformly in $x \le \nu^{-1/4}\sqrt{2} - \beta\nu^{-3/4}(1-t)^{-1}$ and $0 \le t \le 1 - K/\sqrt{\nu}$.

Now, for $x > \nu^{-1/4}\sqrt{2} - \beta\nu^{-3/4}(1-t)^{-1}$ and $0 \le t \le 1 - K/\sqrt{\nu}$,

$$\mathcal{L}^{\downarrow}_{\nu} f(t,x) = f_t(t,x) - \nu^{-1/4} f_x(t,x) + \nu^{3/4} \int_0^{-x} \{f(t,x+u) - f(t,x)\} \mathrm{d}u. \tag{5.34}$$

Finally, when $1 - K/\sqrt{\nu} < t \le 1$,

$$\mathcal{L}^{\downarrow}_{\nu} f(t,x) = f_t(t,x) - \nu^{-1/4} f_x(t,x). \tag{5.35}$$

Define an operator

$$\mathcal{L}^{\downarrow} f := f_t + \frac{\sigma_1^2}{2} f_{xx}(t,x).$$

Then, from (5.33), (5.34), and (5.35) for functions $f \in \mathcal{D}^{\downarrow}$

$$\lim_{\nu\to\infty} \sup_{(t,x)\in[0,1]\times\mathbb{R}} |\mathcal{L}^{\downarrow}_{\nu} f(t,x) - \mathcal{L}^{\downarrow} f(t,x)| = 0.$$

Analogously to the proof of (5.30), we convince ourselves that $\mathcal{D}^{\downarrow}$ is a core of the operator

$\mathcal{L}^{\downarrow}$ by observing that averaging over half-normally distributed translations retains the differentibaility properties of functions.

5.3.3 The length process near termination

Having established the weak convergence of $\widetilde{X}$, we wish to estimate fluctuations of $\widetilde{L}$ near $t = 1$. To that end, we aim to verify that

$$\lim_{h\to 0}\limsup_{\nu}\mathbb{P}\left(\sup_{t\in[1-h,1]}|\widetilde{L}(t)-\widetilde{L}(1-h)|>\epsilon\right)=0. \tag{5.36}$$

Write $s = 1 - h$ and split the difference in (5.36) in three parts

$$\widetilde{L}(t)-\widetilde{L}(s)=\nu^{1/4}P_1(t)-\nu^{1/4}P_2(t)+\nu^{1/4}P_3(t),$$

where

$$\begin{aligned}
P_1(t) &:= (2\nu)^{-1/2}\{M(t)-M(s)\},\\
P_2(t) &:= (2\nu)^{-1/2}F(\nu(1-t)(1-X(t)))-(1-t),\\
P_3(t) &:= (2\nu)^{-1/2}F(\nu(1-s)(1-X(s)))-(1-s).
\end{aligned}$$

From (5.18),

$$\lim_{\nu\to\infty}\sup_{z\in[0,1]}\nu^{1/4}|(2\nu)^{-1}F(\nu z)-z|=0.$$

Using this, definition of $\widetilde{X}$ and that $|1-\sqrt{1-z}|\le|z|$ for $z<1$ we obtain

$$\begin{aligned}
&|P_2(t)|\le|\sqrt{(1-t)(1-X(t))}-(1-t)|+\\
&\{(2\nu)^{-1/2}F(\nu(1-t)(1-X(t)))-\sqrt{(1-t)(1-X(t))}\}\le\\
&\left|(1-t)\left(\sqrt{1-\frac{\widetilde{X}(t)}{\nu^{1/4}(1-t)}}-1\right)\right|+\sup_{z\in[0,1]}|(2\nu)^{-1}F(\nu z)-z|\le\\
&\nu^{-1/4}|\widetilde{X}(t)|+\sup_{z\in[0,1]}|(2\nu)^{-1}F(\nu z)-z|=\nu^{-1/4}|\widetilde{X}(t)|+o(\nu^{-1/4}),
\end{aligned}$$

so from (5.32)

$$\lim_{h\to 0}\limsup_{\nu}\mathbb{P}\left(\sup_{t\in[1-h,1]}\nu^{1/4}|P_2(t)|>\varepsilon/3\right)=0. \tag{5.37}$$

This relation also holds for P_3.

For the first part, apply Doob's maximal inequality

$$\mathbb{P}\left(\sup_{t\in[1-h,1]}\nu^{1/4}|P_1(t)|>\varepsilon/3\right)\leq\frac{9}{2\varepsilon^2\sqrt{\nu}}\,\mathrm{Var}\{M(1)-M(1-h)\}. \tag{5.38}$$

In terms of the quadratic variation (see [16], Chapter 2)

$$\mathrm{Var}\{M(1)-M(s)\}=\mathbb{E}\int_s^1\nu\,\psi(t,X(t))\,\varphi(t,X(t))\mathrm{d}t,$$

where

$$\varphi(t,x)=\mathbb{E}\{1+F(\nu(1-t)(1-x-U\psi(t,x)))-F(\nu(1-t)(1-x))\}^2$$

(with U uniform on $[0,1]$) is the mean-square size of the generic jump of M. Under the optimal strategy $0\leq\varphi(t,x)\leq 1$ (finer estimates are in [20], Section 4), and from (5.18) and (5.16) we have a uniform bound $|\varphi(t,x)|<c$. Whence

$$\mathrm{Var}\{M(1)-M(1-h)\}<c\,\mathbb{E}\int_{1-h}^1\nu\,\psi(t,X(t))\mathrm{d}t=c\,\mathbb{E}\left\{L(1)-L(1-h)\right\}<c\sqrt{2\nu h},$$

the probability in (5.38) is estimated as $O(\sqrt{h})$, and (5.36) follows from this and (5.37).

5.4 The functional central limit theorem

By the domination argument, tightness of $(\widetilde{X}_\nu,\widetilde{L}_\nu)$ follows on the whole $[0,1]$, and we arrive at our main result.

Theorem 15. *The normalised running maximum and the length process* (5.1) *driven by a control satisfying* (5.14) *and* (5.16) *(in particular, under the optimal online selection* *strategy) converge weakly in the Skorokhod space* $D[0,1]$*,*

$$(\widetilde{X}_\nu, \widetilde{L}_\nu) \Rightarrow (Y_1, Y_2), \quad \text{as} \quad \nu \to \infty,$$

where the limit bivariate process is a Gaussian diffusion defined by the equations (5.27), (5.28) *with zero initial conditions.*

We observed already that Y_1 is the Brownian bridge (5.29) and from (5.28)

$$Y_2(t) = \frac{Y_1(t)}{2} - \frac{W_1(t)}{2} + W_2(t),$$

so splitting the martingale part in independent components, we get, explicitly,

$$Y_2(t) = \int_0^t \frac{(1-t)}{2(1-s)} \mathrm{d}W_1(s) + \frac{1}{4}\, W_1(t) + \left(W_2(t) - \frac{3}{4}\, W_1(t) \right), \tag{5.39}$$

which is a sum of a Brownian motion, derived Brownian bridge, and another independent Brownian motion.

To find the covariance structure, it is convenient to resort to matrix calculations. We may write the solution to (5.27), (5.28) as

$$\boldsymbol{Y}(t)^T = e^{a(t)} \int_0^t e^{-a(u)} \mathrm{d}\boldsymbol{W}(u)^T,$$

where

$$a(t) := A \int_0^t \frac{1}{1-u} \mathrm{d}u = A \log(1-t), \qquad A := \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{pmatrix},$$

which by the Itó isometry yields

$$\mathbb{E}\{\boldsymbol{Y}(s)^T \boldsymbol{Y}(t)\} = \int_0^t e^{a(s)-a(u)} \boldsymbol{\Sigma} e^{(a(t)-a(u))^T} \mathrm{d}u, \qquad 0 \le s \le t \le 1.$$

Since A is an idempotent matrix, the exponents are readily calculated as

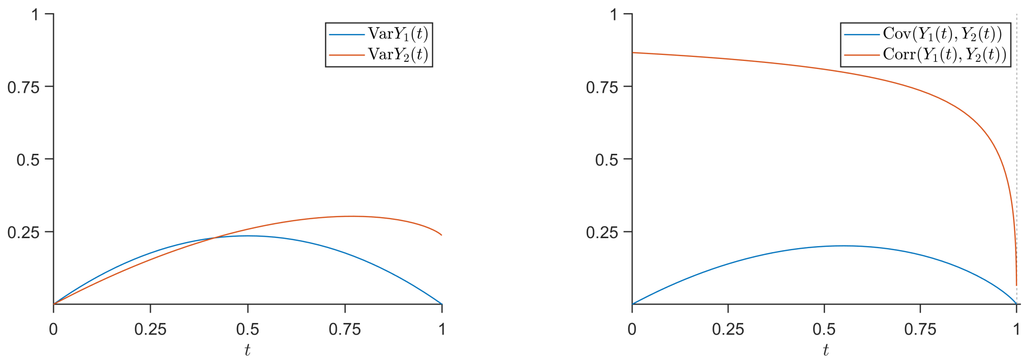
$$
\begin{aligned}
e^{a(t)} &= \sum_{i=0}^{\infty} \frac{A^i (\log(1-t))^i}{i!} = I + A \sum_{i=1}^{\infty} \frac{(\log(1-t))^i}{i!} = I - tA = \begin{pmatrix} 1-t & 0 \\ -\frac{t}{2} & 1 \end{pmatrix}, \\
e^{-a(t)} &= 1 + \frac{t}{1-t} A = \begin{pmatrix} \frac{1}{1-t} & 0 \\ \frac{t}{2(1-t)} & 1 \end{pmatrix}.
\end{aligned}
$$

With a minor help of `Mathematica` we arrive at the cross-covariance matrix

$$
\mathbb{E}\{\boldsymbol{Y}(s)^T \boldsymbol{Y}(t)\} = \begin{pmatrix} \frac{2\sqrt{2}\, s(1-t)}{3} & \frac{2s(1-t)-(1-s)\log(1-s)}{3\sqrt{2}} \\ \frac{(1-t)(2s-\log(1-s))}{3\sqrt{2}} & \frac{2s(2-t)-(2-s-t)\log(1-s))}{6\sqrt{2}} \end{pmatrix},
$$

where $0 \le s \le t \le 1$.

The following graphs illustrate the covariance structure of $\boldsymbol{Y}(t)$.



(a) Variance of the component processes

(b) Covariance and correlation of the component processes

Figure 5.1: Covariance structure of $\boldsymbol{Y}(t)$

The limit length process Y_2 is *not* Markovian since its covariance function does not satisfy the factorisation criterion (see [44], p. 148). The sum of two first terms in (5.39) is non-Markovian too.

5.5 The derived processes

From Theorem 15 follow functional limits for normalised compensators and martingale (5.4)

$$\begin{aligned} \nu^{1/4}(C_X(t) - t,\ t \in [0,1]) &\Rightarrow Y_1 - W_1, \\ 2\nu^{1/4}\left(\frac{C_L(t)}{\sqrt{2\nu}} - t,\ t \in [0,1]\right) &\Rightarrow Y_1 - W_1, \\ \sqrt{2}\nu^{-1/4}M &\Rightarrow 2W_2 - W_1 \stackrel{d}{=} W_1, \end{aligned}$$

with account of (5.12). A counterpart of (5.13) becomes

$$\widetilde{L} - 2\widetilde{X} \Rightarrow W_1.$$

Notably, the limit distributions for $t = 1$ are all the same $\mathcal{N}(\sigma_1^2, 0)$.

For a normalised square-root process

$$\widetilde{Z}(t) := \nu^{1/4}\left(\frac{Z(t)}{\sqrt{2\nu}} - (1-t)\right)$$

we have $\widetilde{Z} \Rightarrow -Y_1/2$. In Section 4.1.7 we showed that the range of Z at big distance from 0 can be split in almost independent renewal cycles with distribution close to that of $(E/2 + U)/\sqrt{2}$, where E and U are independent standard exponential and uniform variables.

From these limit relations the result of [20] on the joint convergence of normalised compensated X and L to $\boldsymbol{W}$ easily follows. Bruss and Delbaen also proved the Brownian limit for the martingale M, which by virtue of $M(1) = L(1) - F(\nu)$ led them to the central limit theorem for the total length $L(1)$.

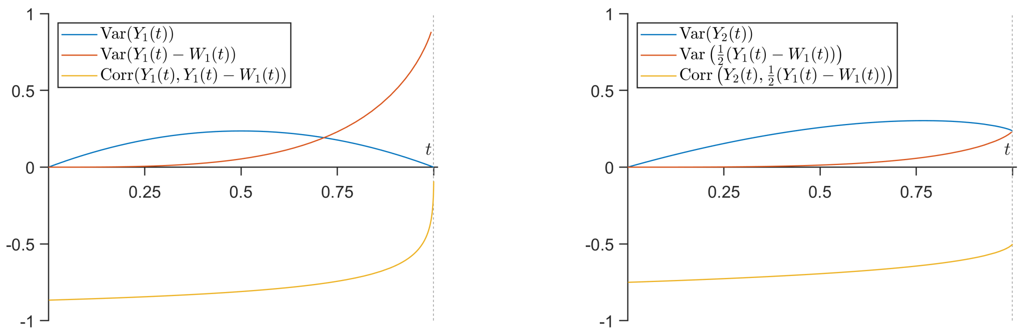
It is of interest to look at the distributions of the pairs $(X(t), C_X(t))$ and $(L(t), C_L(t))$ to capture dependence between the processes and their compensators. In the $\nu \to \infty$ limit these approach the bivariate normal distributions of $(Y_1(t), Y_1(t) - W_1(t))$ and

$(Y_2(t), \frac{1}{2}(Y_1(t) - W_1(t)))$, respectively. Calculation of the covariance matrices is straightforward from our previous findings complemented by the formula

$$\mathrm{Cov}(Y_1(t), W_1(t)) = -\sigma_1^2(1-t)\log(1-t)$$

obtained by the Itó isometry. For instance

$$\mathrm{Var}\left\{Y_1(t) - W_1(t)\right\} = \frac{4\sqrt{2}}{3}\left(t - \frac{t^2}{2} + (1-t)\log\left(1-t\right)\right).$$



(a) Covariance structure of $Y_1(t)$ and its compensator

(b) Covariance structure of $Y_2(t)$ and its compensator

Figure 5.2: Covariance structure of $\boldsymbol{Y}(t)$ and its compensator

5.6 Convergence of the moments

The weak convergence shown in Theorem 15 combined with the convergence of moments for the majorant and minorant processes imply by virtue of Pratt's lemma the expansion

$$\mathbb{E}(X(t) - t)^k = \nu^{-k/4}\,\mathbb{E}\,Y_1^k(t) + o(\nu^{-k/4}), \quad k \in \mathbb{N},$$

along with a similar expansion for the t-centred moment functions of $L/\sqrt{2\nu}$. For $k = 1$ the leading term vanishes, hence the convergence rate should be higher, as is evidenced in the instance $t = 1$ by (5.9). The logarithmic factor in (5.9) results from the optimality gap, hence it is of interest to inspect how the gap emerges in the course of selection.

We choose smallest possible constants $\beta_- > 0$, $\beta_+ \geq 0$ to squeeze the control function in the bounds

$$\sqrt{\frac{2(1-x)}{\nu(1-t)}} - \frac{\beta_-}{\nu(1-t)} \leq \psi(t,x) \leq \sqrt{\frac{2(1-x)}{\nu(1-t)}} + \frac{\beta_+}{\nu(1-t)}, \quad t,x \in [0,1). \tag{5.40}$$

The condition (5.19) thus holds with $\beta \geq \max(\beta_-, \beta_+, 1)$. To motivate introducing two parameters we note that for the optimal strategy (5.40) holds with $\beta_+ = 0$ ([20], Equation (3.5)), and that there is some asymmetry in the upper and lower estimates below.

The following auxiliary result is a special case of Grönwall's inequality:

Lemma 21. *Suppose function f with $f(0) = 0$ satisfies the integral inequality*

$$f(t) \leq -\int_0^t f(s)\left(\frac{1}{1-s} + \frac{a}{(1-s)^2}\right)\mathrm{d}s + \int_0^t g(s)\mathrm{d}s, \quad t \in [0,1),$$

$a \in \mathbb{R}$. *Then*

$$f(t) \leq (1-t)e^{-\frac{a}{1-t}} \int_0^t \frac{e^{\frac{a}{1-s}}\, g(s)}{1-s}\mathrm{d}s, \quad t \in [0,1). \tag{5.41}$$

Proof. The linear operator defined by the right-hand side of (5.41) gives a solution to the associated integral equation. The assertion follows by observing that nonnegative g is mapped to nonnegative f. □

With this result in our toolbox, we are ready to derive improved bounds on $p(t) = \mathbb{E}X(t)$ and $q(t) = \mathbb{E}L(t)/\sqrt{2\nu}$.

5.6.1 Bounds on $p(t)$

The upper bound in (5.40) implies

$$\frac{\nu}{2}\psi^2(s,x) - 1 \leq -\frac{x-s}{1-s} + \frac{\beta_+\sqrt{2}}{\sqrt{\nu}(1-s)}\sqrt{\frac{1-x}{1-s}} + \frac{\beta_+^2}{2\nu(1-s)^2}. \tag{5.42}$$

Using the elementary inequality $\sqrt{1-z} \le 1 - z/2$ for $z \le 1$ we obtain

$$\mathbb{E}\sqrt{\frac{1-X(t)}{1-t}} \le 1 - \frac{\mathbb{E}X(t)-t}{2(1-t)} = 1 - \frac{p(t)-t}{2(1-t)}.$$

Integrating in (5.42) yields

$$p(t) - t \le -\int_0^t (p(s)-s)\left(\frac{1}{1-s} + \frac{b}{(1-s)^2}\right) \mathrm{d}s + \int_0^t g(s)\mathrm{d}s,$$

where

$$g(t) = \frac{2b}{(1-t)} + \frac{b^2}{(1-t)^2}, \qquad b = \frac{\beta_+}{\sqrt{2\nu}}.$$

Applying Lemma 21 with $f(t) = p(t) - t$ and $a = b$ we obtain $p(t) - t \le G(b,t)$, where

$$G(b,t) := 1 + b - t - (1+b)(1-t)\exp\left(-\frac{bt}{1-t}\right).$$

For small $b > 0$, this is a concave function, with $G(b,t) - 2tb$ changing sign from $+$ to $-$ at some point approaching $2/3$ as $b \to 0$. The asymptotic expansion

$$G(b,t) \sim 2tb + \frac{(2t-3t^2)b^2}{2(1-t)}, \quad b \to 0,$$

holds uniformly, at least for t bounded away from 1; therefore there is an upper bound $G(b,t) < 2bt + c_+ b^2$, where the constant should be chosen to satisfy

$$c_+ > \max_{t\in[0,1]} \frac{2t-3t^2}{2(1-t)} = 2 - \sqrt{3}.$$

It follows that

$$p(t) - t \le \frac{\beta_+\sqrt{2}\ t}{\sqrt{\nu}} + \frac{c_+\beta_+^2}{2\nu}, \tag{5.43}$$

uniformly in $t \in [0,1]$ for sufficiently big ν.

To estimate in the opposite direction, we have from the lemma $p(t) - t \ge G(b,t)$, this time with negative parameter

$$b = -\frac{\beta_-}{\sqrt{2\nu}}.$$

Changing the variable to $T = (1-t)^{-1}$ simplifies analysis, and it is readily checked that

$$T \mapsto T\left(G\left(b, 1 - T^{-1}\right) - 2b\left(1 - T^{-1}\right) - b^2 T \right)$$

is a concave function, positive in the range $1 \le T < T_0$, where T_0 is such that $-b\,T_0$ approaches, as $|b| \to 0$, a limit value $1.7933\ldots$ (the positive root of $1 + x + x^2 = e^x$), which we replace by smaller $\sqrt{2}$. Thus

$$p(t) - t \ge -\frac{\beta_- \sqrt{2}\, t}{\sqrt{\nu}} - \frac{\beta_-^2}{2\nu(1-t)}, \quad \text{for} \quad t \le 1 - \frac{\beta_-}{2\sqrt{\nu}}, \tag{5.44}$$

provided ν is sufficiently large. But then by monotonicity from (5.44) it follows that

$$p(t) \ge p\left(1 - \frac{\beta_-}{2\sqrt{\nu}}\right) \ge 1 - \frac{\left(\sqrt{2} + \frac{3}{2}\right)\beta_-}{\sqrt{\nu}}, \quad \text{for} \quad t > 1 - \frac{\beta_-}{2\sqrt{\nu}},$$

hence in this range of t

$$p(t) - t \quad \ge \quad p(t) - 1 > -\frac{\left(\sqrt{2} + \frac{3}{2}\right)\beta_-}{\sqrt{\nu}} \tag{5.45}$$

(also note that the trivial upper bound $p(t) - t < 1 - t < \frac{\beta_-}{2\sqrt{\nu}}$ might improve upon (5.43) in this range).

Bounding the second term in (5.44) by its maximum, and combining with (5.45) into single inequality we obtain an estimate with simpler constant $3 > \sqrt{2} + 3/2$

$$p(t) - t \ge -\frac{3\beta_-}{\sqrt{\nu}}, \quad t \in [0,1]. \tag{5.46}$$

Similarly, the second term in (5.43) can be absorbed into the first with a larger constant.

With the full range $t \in [0,1]$ covered, we have shown that

$$\sup_{t\in[0,1]} |p(t) - t| = O\left(\frac{1}{\sqrt{\nu}}\right).$$

5.6.2 Bounds on $q(t)$

We turn to $q(t) = \mathbb{E}L(t)/\sqrt{2\nu}$. For the upper estimate we use (5.5) to obtain an integral inequality

$$q(t) = \mathbb{E}\int_0^t \sqrt{\nu/2}\ \psi(s, X(s))\mathrm{d}s \le \mathbb{E}\int_0^t \left(\sqrt{\frac{1-X(s)}{1-s}} + \frac{\beta_+}{\sqrt{2\nu}(1-s)}\right)\mathrm{d}s \le$$
$$\int_0^t \left(1 - \frac{p(s)-s}{2(1-s)} + \frac{\beta_+}{\sqrt{2\nu}(1-s)}\right)\mathrm{d}s \le \int_0^t \left(1 - \frac{q(s)-s}{(1-s)} + \frac{\beta_+}{\sqrt{2\nu}(1-s)}\right)\mathrm{d}s,$$

then apply Lemma 21 with $a = 0$ to get

$$q(t) - t \le \frac{\beta_+\, t}{\sqrt{2\nu}}. \tag{5.47}$$

The estimate approaches zero faster than in (5.9), but there is no disagreement since $q(1) < 1$. Note that applying (5.5) and (5.43) straight incurs a second term.

For the optimal strategy, (5.40) holds with $\beta_+ = 0$, thus in this case $p(t) - t \le 0$ and $q(t) - t \le 0$.

Obtaining the lower bound is more challenging. Under the optimal strategy, the value function F in (5.15) is concave [19], but under our more general assumptions on ψ this need not be the case. However, by virtue of (5.18) we may replace F by the concave function

$$\widehat{F}(\nu) := \sqrt{2\nu} - \frac{1}{12}\log(\nu+1), \tag{5.48}$$

to obtain an expansion

$$\mathbb{E}L(t) = \widehat{F}(\nu) - \mathbb{E}\{\widehat{F}(\nu(1-X(t))(1-t))\} + O(1), \tag{5.49}$$

where the absolute value of the remainder is bounded uniformly in t and ν by some constant K only depending on β_- and β_+.

By monotonicity and concavity of $\widehat{F}$, using Jensen inequality and (5.46) we estimate

$$\mathbb{E}\{\widehat{F}(\nu(1-X(t))(1-t))\} \le \widehat{F}\left(\nu(1-t)(1-p(t)\right) =$$
$$\widehat{F}\left(\nu(1-t)^2\left(1-\frac{p(t)-t}{1-t}\right)\right) \le$$
$$\widehat{F}\left(\nu(1-t)^2\left(1+\frac{3\beta_-}{\sqrt{\nu}(1-t)}\right)\right) =$$
$$\sqrt{2\nu}(1-t)\sqrt{1+\frac{3\beta_-}{\sqrt{\nu}(1-t)}} - \frac{1}{12}\log\left\{\nu(1-t)^2\left(1+\frac{3\beta_-}{\sqrt{\nu}(1-t)}\right)+1\right\} <$$
$$\sqrt{2\nu}(1-t)\left(1+\frac{3\beta_-}{2\sqrt{\nu}(1-t)}\right) - \frac{1}{12}\log\left(\nu(1-t)^2)\right) <$$
$$\sqrt{2\nu}(1-t) + \frac{3\beta_-}{\sqrt{2}} - \frac{1}{12}\log\nu - \frac{1}{6}\log(1-t).$$

Substituting this along with (5.48) into (5.49) we see that, for large enough ν,

$$\mathbb{E}L(t) \ge \sqrt{2\nu}\,t + \frac{1}{6}\log(1-t) - \left(\frac{3\beta_-}{\sqrt{2}} + K\right), \qquad t \in [0,1]. \tag{5.50}$$

The logarithmic term makes (5.50) useless for t too close to 1. However, cutting the range at, say $t_0 := 1 - 1/\sqrt{\nu}$, we can just employ the monotonicity to squeeze the expected length as

$$F(\nu) \ge \mathbb{E}L(t) \ge \mathbb{E}L(t_0) \ge \sqrt{2\nu} + \frac{1}{12}\log\nu - \left(\sqrt{2} + \frac{3\beta_-}{\sqrt{2}} + K\right), \quad t \ge t_0.$$

For a better overview, we re-write (5.47) as

$$\mathbb{E}L(t) \le \sqrt{2\nu}\,t + \beta_+\, t. \tag{5.51}$$

Comparing (5.50) with (5.51) it is seen that, uniformly in $t \in [0, 1-h]$, the mean selected length $\mathbb{E}L(t)$ is within $O(1)$ from $\sqrt{2\nu}\,t$, the latter being the exact mean length under the (unfeasible) stationary strategy.

With some more work we could show an upper bound with two leading terms as in (5.50) and a remainder uniformly bounded over $t < 1 - \nu^{-1/4+\epsilon}$.

5.7 Diffusion approximation in the bin-packing problem

In this section we sketch the potential proof of the functional central limit theorem in the continuous-time self-similar bin-packing problem from Section 4.3. Suppose that i.i.d. positive marks arrive by a Poisson process of intensity ν on $[0, 1]$, and that the marks are sampled from a density satisfying $F(x) = \mathrm{Beta}(\alpha, 1)$ with the support $[0, C]$. The stochastic optimisation task is to maximise the expected number of online selections under the constraint that their total sum does not exceed given $C > 0$. From equation (4.42), it is clear that one can equivalently consider packing into a knapsack of unit capacity.

Once again we restrict our attention to a relatively small class of distributions as our functional convergence proof relies heavily on the self-similarity of the optimal selection policy and the refined asymptotic expansions of the optimal value function obtained in Section 4.3.

Let $S_\nu(t), N_\nu(t)$, $t \in [0, 1]$ denote the running sum and the number of packed items at time t under the optimal self-similar selection policy driven by a control function $\varphi(t, s) = (1 - s)\,\delta(\nu(1 - t)(1 - s)^\alpha)$, respectively. Here we study all control functions satisfying

$$\delta(\nu) = \frac{\alpha + 1}{\beta_1^* \alpha \nu^{1/(\alpha+1)}} + O(\nu^{-2/(\alpha+1)}), \quad \nu \to \infty,$$

which includes the optimal threshold function (4.48).

We aim to prove the functional convergence of the following normalised processes

$$\widetilde{S}_\nu(t) := \nu^{1/(2(\alpha+1))} \left(S_\nu(t) - t\right), \quad \widetilde{N}_\nu(t) := \nu^{1/(2(\alpha+1))} \left(\frac{N_\nu(t)}{\beta_1^* \nu^{1/(\alpha+1)}} - t \right).$$

For the jump process $(S,N)_{t\in[0,1]}$, we have the generator

$$\mathcal{L}_\nu f(t,s,n) = f_t(t,s,n) + \nu \int_0^{\varphi(t,s)} \{f(t,s+y,n+1) - f(t,s,n)\} \alpha y^{\alpha-1} \mathrm{d}u.$$

With the change of variables

$$s \to s\nu^{-1/(2(\alpha+1))} + t, \quad n \to (n\nu^{-1/(2(\alpha+1))} + t)\beta_1^* \nu^{1/(\alpha+1)}$$

and

$$\widetilde{\varphi}(t,s) := \nu^{1/(2(\alpha+1))}\varphi(t, s\nu^{-1/(2(\alpha+1))} + t), \quad v := \frac{1}{\beta_1^* \nu^{1/(2(\alpha+1))}},$$

we have the infinitesimal generator of $(t, \widetilde{Z}_\nu(t), \widetilde{N}_\nu(t))$

$$\begin{aligned} \mathcal{L}_\nu f(t,s,n) &= f_t - \nu^{1/(2(\alpha+1))}(f_s + f_n) \\ &+ \nu^{1-\alpha/(2(\alpha+1))} \int_0^{\widetilde{\varphi}(t,s)} \left(f(t,s+y,n+v) - f(t,s,n)\right) \alpha y^{\alpha-1} \mathrm{d}y. \end{aligned}$$

A fairly long computation replicating the methods of Section 5.2 yields the asymptotics, as $\nu \to \infty$,

$$\mathcal{L}_\nu f(t,s,n) \sim f_t - \frac{s}{1-t} f_s - \frac{\alpha s}{(\alpha+1)(1-t)} f_n + \frac{{\sigma_3}^2}{2} f_{ss} + \frac{{\sigma_4}^2}{2} f_{nn} + \rho_0 \sigma_3 \sigma_4 f_{sn},$$

where

$${\sigma_3}^2 = \frac{(\alpha+1)^{(\alpha+2)/(\alpha+1)}}{\alpha^{1/(\alpha+1)}(\alpha+2)}, \quad {\sigma_4}^2 = \left(\frac{\alpha+1}{\alpha}\right)^{(\alpha-2)/(\alpha+1)}, \quad \rho_0 = \frac{\alpha^{(3\alpha-1)/(2(\alpha+1))}\sqrt{\alpha+2}}{(\alpha+1)^{2\alpha/(\alpha+1)}}.$$

Using this it should be possible to show the functional convergence

$$(\widetilde{S}_\nu, \widetilde{N}_\nu) \Rightarrow (Y_3, Y_4), \quad \text{as } \nu \to \infty,$$

in $D[0,1-h]$ for every $h \in (0,1)$, where the limit process $(Y_3(t), Y_3(t))$ is a Gaussian diffusion satisfying the SDE

$$\mathrm{d}Y_3(t) = -\frac{Y_3(t)}{1-t}\,\mathrm{d}t + \mathrm{d}W_3(t), \quad \mathrm{d}Y_4(t) = -\frac{\alpha Y_3(t)}{c(\alpha+1)(1-t)}\,\mathrm{d}t + \mathrm{d}W_4(t)$$

with zero initial conditions. Here, (W_3, W_4) is a two-dimensional centred Brownian motion with the covariance matrix $\mathbf{\Sigma_0} = \begin{pmatrix} {\sigma_3}^2 & \rho_0\sigma_3\sigma_4 \\ \rho_0\sigma_3\sigma_4 & {\sigma_4}^2 \end{pmatrix}$.

Chapter 6

Conclusion and outlook

In this manuscript we studied the infamous problem of choosing the longest increasing subsequence in an online fashion and touched upon some of the related problems. We significantly improved the existing asymptotic expansions of the mean and variance of the length of the optimally chosen subsequence, derived important limit theorems for the underlying stochastic processes, and answered long-standing questions about the form and statistical properties of a certain class of suboptimal policies.

In the first part of Chapter 3 we worked with the original discrete-time version of the problem introduced by Samuels and Steele [55]. To refine the asymptotic expansion of the mean length, in Chapter 2 we developed a method of approximating solutions to the difference equations satisfying specific monotonicity criteria. This 'asymptotic comparison' method, as we called it, allowed us to methodically obtain finer asymptotics of the solution to the optimality equation by bounding it from above and below with suitable test functions. In fact, variations of this method apply to a broad class of difference and differential equations, and its applications repeatedly appear throughout the whole dissertation. In particular, we used the asymptotic comparison to approximate the mean number of selections made by the suboptimal policy proposed by Arlotto et al. [5]. The resulting expansion confirmed the conjecture of Arlotto et al. that their suboptimal policy is within a constant off optimality.

The second part of Chapter 3 is dedicated to what may be considered a dual counterpart of the longest increasing subsequence selection, the quickest selection of the increasing subsequence. This relatively new online optimisation problem was first introduced by Arlotto et al. [3], who derived the principal asymptotics of the optimal solution. Adapting our asymptotic comparison method to the optimality recursion of the quickest selection, we obtained an expansion of the optimal solution, which is optimal up to the order of the term resulting from the shift in the initial condition. Moreover, we explicitly constructed a stationary and a variable-threshold selection policies that match the optimal performance up to one and two terms of asymptotic expansion, respectively. The results fit nicely within the framework developed over the years around the online selection problems. In particular, the choice of threshold functions for our stationary policy flows naturally from the commonly known mean constraint upper bound proof [3]. In the last sections of Chapter 3, we refine the existing expansion of the mean number of packed items in the sequential bin-packing problem initialised by Coffman et al. [26], further demonstrating the power of the asymptotic comparison method.

In Chapter 4 we turned our attention to a natural modification of the discrete-time problem: the poissonised, or the continuous-time variant of the longest increasing subsequence problem. This was first introduced as a tool to study the discrete-time problem by Samuels and Steele [55], and later studied in detail by Bruss and Delbaen [19]. Baryshnikov and Gnedin [10] generalised the continuous-time setting by considering a problem where the selection is commenced from a sample of d-dimensional vectors $\boldsymbol{X_i} \in \mathbb{R}^d$, and the selected chain must increase in all dimensions. Working in this extended setting, we significantly refined the asymptotic expansions of the mean and variance and proved the central limit theorem for the optimal length of the increasing subsequence. The main novelty of our approach is transforming the process $L(t)$ into a piecewise deterministic Markov process $Z(\cdot)$. Adapting the comparison method to the continuous-time setting, we significantly refined the asymptotics of the mean optimal length, which allowed us to prove the strong similarity between the discrete and the continuous-time cases. And, with further effort, we were able to refine the expansion beyond the $O(1)$-term, essentially closing down the study of the asymptotic analysis of $\mathbb{E}L(t)$. Specialising our results to the case $d = 1$ allowed us to significantly improve Bruss and Delbaen's estimates of the $\text{Var}L(t)$ and the optimal threshold function $\varphi(t)$. Finally, correcting an inaccuracy in the proof of Cutsem and Ycart's [27] renewal-type approximation method, we applied it to obtain a novel proof of the central limit theorem on $L(t)$. This was derived earlier by Bruss and Delbaen [20] for the one-dimensional case following a different argument. The process $Z(\cdot)$ is sandwiched between two renewal processes with increments being a sum of scaled Beta$(d, 1)$ and Exponential(1) random variables. In addition, in the original problem, we showed that the same limit theorem holds for all variable-threshold strategies satisfying $\varphi(t) = \sqrt{2/t} + O(t^{-1}),\ t \to \infty$. Although the results obtained in Baryshnikov and Gnedin's multidimensional setting are directly applicable to the one-dimensional setup, the multidimensional framework required a delicate treatment of several important differences. For example, in contrast to the original problem, the acceptance window here is a d-dimensional shape, which we approximated with a standard simplex in the asymptotic case.

In Sections 4.3 and 4.4, respectively, we demonstrated how the asymptotic comparison and renewal approximation methods could be applied to the continuous-time bin-packing and the online interval parking problems. As before, we obtained fairly complete asymptotic expansions of the mean and the variance of the number of selections and derived the central limit theorems.

The study in Chapter 5 is inspired by Bruss and Delbaen's investigation [20] of the fundamental random processes arising during the selection of the longest increasing subsequence problem. They utilised the Doob-Meyer decomposition and the martingale functional central limit theorem to prove the convergence of suitably scaled length and running maximum processes $L(t)$ and $X(t)$. However, the compensators in their approach are nonlinear transforms of $X(t)$, which makes the result less informative than desired. We, on the other hand, worked explicitly with the generator of a suitably scaled version of the three-dimensional process $(t, X(t), L(t))$. We proved the generator convergence, firstly on $(0, 1-h)$, and then, working our way around the singularities close to 1, proved the weak convergence on the whole of $[0, 1]$. The resulting limit process is a two-dimensional Gaussian diffusion consisting of a Brownian bridge and an Ornstein-Uhlenbeck-like process, which corresponds to a sum of correlated Brownian motion and a Brownian bridge. We explicitly calculated the cross-covariance matrix of $(X(t), L(t))$ and showed that the weak convergence holds for any variable-threshold selection strategy satisfying $\varphi(t) = \sqrt{2/t} + O(t^{-1}),\ t \to \infty$. The study is concluded with the derivation of mean bounds on $L(t)$ and $X(t)$, which we obtained by directly working with their respective generators.

As well as answering several fundamental questions, our work opened up possibilities for further research. The power of the asymptotic comparison method was demonstrated repeatedly, and there are numerous opportunities to adapt and apply it to other stochastic sequential selection problems. The renewal-type approximation can be generalised to a wider class of processes, which satisfy the more general form of equation (4.9), i.e. with a reward function $r(z) \neq 1$. Finally, the stochastic comparison method we utilised to prove the functional convergence of $(X(t), L(t))$ can be adapted to similar online selection problems. As an example, we sketched a proof of the weak limit in the continuous-time bin-packing problem in Section 5.7. It is also reasonable to make the conjecture that similar limit processes can be derived in the discrete-time longest increasing subsequence problem.

Appendix A

Computation of the asymptotic expansions

A.1 Asymptotic expansion of $\beta_k^{(\bar{h})}$ in Section 3.2.6

Let the first test function be of the form $y_k^{(0)} := d_0 k^2$, $d_0 \in \mathbb{R}_+$. Then,

$$y_{k+1}^{(0)} = d_0 k^2 + 2 d_0 k + 1,$$

and

$$\begin{aligned} a_k y_k^{(0)} + b_k &= d_0 k^2 \left(\frac{k}{2} + 1 \right) \log \left(1 + \frac{2}{k} \right) + \frac{k}{2} + 1 \\ &= d_0 k^2 \left(\frac{k}{2} + 1 \right) \left(\frac{2}{k} - \frac{2}{k^2} \right) + \frac{k}{2} + O(1) \\ &= d_0 k^2 + \left(\frac{1}{2} + d_0 \right) k + O(1), \quad k \to \infty. \end{aligned}$$

The expressions above match at $d_0 = 1/2$; hence, by Lemma 10, we have

$$\beta_k^{(\bar{h})} \sim \frac{k^2}{2}, \quad k \to \infty.$$

Now, for the test sequence of the type $y_k^{(1)} := k^2/2 + d_1 k \log k$, $d_1 \in \mathbb{R}$, using Taylor expansion of log we calculate

$$\begin{aligned} y_{k+1}^{(1)} &= \frac{k^2}{2} + k + 1 + d_1(k+1)\log k + 1 \\ &= \frac{k^2}{2} + d_1 k \log k + k + d_1 \log k + \left(\frac{1}{2} + d_1\right) + O(k^{-1}), \quad k \to \infty. \end{aligned}$$

and

$$a_k y_k^{(1)} + b_k = \frac{k^2}{2} + d_1 k \log k + k + d1 \log k + \frac{2}{3} + O(k^{-1}\log k), \quad k \to \infty.$$

Therefore, Lemma 10 implies

$$\beta_k^{(\bar{h})} \sim \frac{k^2}{2} + \frac{k \log k}{6}, \quad k \to \infty.$$

To bound the remainder of the expansion, we need one more approximation. To that end, consider functions of the form $y_k^{(2)} = k^2/2 + k \log k/6 + d_2 \log k$, $d_2 \in \mathbb{R}$. By computing the expansions as $k \to \infty$

$$y_{k+1}^{(2)} = \frac{k^2}{2} + \frac{k \log k}{6} + k + (1 + d_2)\log k + \frac{2}{3} + O(k^{-1}),$$

and

$$a_k y_k^{(2)} + b_k = \frac{k^2}{2} + \frac{k \log k}{6} + k + \left(\frac{1}{6} + d_2\right)\log k + \frac{2}{3} + \left(-\frac{1}{9} + d_2\right)\frac{\log k}{k} + O(k^{-1}),$$

one can see they match for $d_2 = -1/9$. However, the extra term is of order $o(k)$; thus another application of Lemma 10 yields the final expansion

$$\beta_k^{(\bar{h})} = \frac{k^2}{2} + \frac{k \log k}{6} + O(k), \quad k \to \infty.$$

A.2 Asymptotic expansion of $u_\theta(z)$ in Section 4.1.5

Since the $O(z^{-1})$-term of $\overline{\theta}$ contributes only $O(z^{-1})$ to $u_\theta(z)$, we may proceed with choosing an approximating function of the form

$$u_\theta(z) = \alpha_1^* z + \alpha_2^* \log(z+1) + c_3 + \frac{\widehat{\alpha}}{z+1},\ \widehat{\alpha} \in \mathbb{R}$$

that satisfies

$$u_\theta'(z) = \alpha_1^* + \frac{\alpha_2^*}{z} - (\alpha_2^* + \widehat{\alpha})\,\frac{1}{z^2} + O(z^{-3}), \quad z \to \infty.$$

Computing $\mathcal{I}u_\theta(z)$ as $z \to \infty$ yields

$$\begin{aligned}\mathcal{I}u_\theta(z) &= \alpha_1^* + \frac{\alpha_2^*}{z} + \Big(-\frac{d(d+1)^2\alpha_2\gamma}{(d+1)!^{1/(d+1)}} - \frac{d(d+1)^2\alpha_1^{*2}\gamma^2}{2(d+1)!^{1/(d+1)}} + d\alpha_2^* + d\widehat{\alpha} \\ &+ \frac{(3d^3+13d^2+4d+4)(d+1)!^{1/(d+1)}}{24(d+2)^2(d+3)}\Big)\frac{1}{z^2} + O(z^{-3}).\end{aligned}$$

The expansions above match for

$$\widehat{\alpha} = d\alpha_1^*\alpha_2^*\gamma + \frac{d\alpha_1^{*3}\gamma^2}{2} + \frac{(3d^3+13d^2+4d+4)(d+1)!^{1/(d+1)}}{24(d+2)^2(d+3)(d+1)} - \alpha_2^*,$$

which, in conjunction with Lemma 15, yields the expansion (4.26).

A.3 Asymptotic expansion of $w(z)$ in Section 4.1.6

Let $\mathcal{G}$ be the operator on the right-hand side of (4.27)

$$\mathcal{G}w(z) = \frac{(d+1)^{d+1}}{(d-1)!}\int_0^{\theta(z)} \left(w(z-y) - w(z) + (1 + 2u(z-y))\right)\mu(z,y)\mathrm{d}y,$$

where θ is the optimal control satisfying

$$\theta(z) = \frac{1}{\alpha_1^*} - \frac{\alpha_2^*}{\alpha_1^{*2} z} + O(z^{-2}), \quad z \to \infty,$$

and u is the optimal value function satisfying

$$u(z) = \alpha_1^* z + \alpha_2^* \log z + c_1 + \frac{\alpha_3^*}{z} + O(z^{-2}), \quad z \to \infty.$$

Our first test function is of the form $w_0(z) = \alpha_1^* z^2 + \zeta_1 z \log(z+1)$, $\zeta_1 \in \mathbb{R}$ with

$$w_0'(z) = 2{\alpha_1^*}^2 z + \alpha_4 \log z + O(1), \quad z \to \infty.$$

Computing $\mathcal{G}w_0(z)$ yields

$$\mathcal{G}w_0(z) = 2{\alpha_1^*}^2 z + (2(d+1)\alpha_1^*\alpha_2^* - d\zeta_1)\log z + O(z^{-1}\log z), \quad z \to \infty.$$

Matching the coefficients of $O(\log z)$-terms leads to the refinement

$$w(z) \sim {\alpha_1^*}^2 z^2 + 2\alpha_1^*\alpha_2^* z \log z, \quad z \to \infty.$$

Next approximation is of the form

$$w_1(z) := {\alpha_1^*}^2 z^2 + 2\alpha_1^*\alpha_2^* z \log(z+1) + \zeta_2 z, \; \alpha_5 \in \mathbb{R}.$$

For this test function, we have

$$w_1'(z) = 2{\alpha_1^*}^2 z + 2\alpha_1^*\alpha_2^* \log z + (\zeta_2 + 2\alpha_1^*\alpha_2^*) + O(z^{-3}), \quad z \to \infty.$$

Working out the expansion of $\mathcal{G}w_1(z)$ for large z yields

$$\mathcal{G}w_1(z) \sim 2{\alpha_1^*}^2 z + 2\alpha_1^*\alpha_2^* \log z + \left(\frac{(d+1)(2-d)}{(d+2)(d+1)!^{1/(d+1)}} + \frac{2(d+1)^2 c_1}{(d+1)!^{1/(d+1)}} - d\zeta_2 \right).$$

Matching the coefficients of two expansions yields the consecutive refinement

$$\begin{aligned} w(z) &\sim {\alpha_1^*}^2 z^2 + 2\alpha_1^*\alpha_2^* z \log z + \left(\frac{2}{(d+2)(d+1)!^{1/(d+1)}} + \frac{2(d+1)}{(d+1)!^{1/(d+1)}} c_1 \right) z, \\ &= \alpha_1^* z^2 + 2\alpha_1^*\alpha_2^* z \log z + (\alpha_4^* + 2\alpha_1^* c_1)\, z, \quad z \to \infty. \end{aligned}$$

We choose the third test function of the form

$$w_2(z) := \alpha_1^{*2} z^2 + 2\alpha_1^* \alpha_2^* z \log(z+1) + (\alpha_4^* + 2\alpha_1^* c_1)\, z + \zeta_3 (\log(z+1))^2.$$

To refine the asymptotic expansion of $w(z)$, the following expansions need to be computed for $z \to \infty$

$$\begin{aligned} w_2'(z) &= 2\alpha_1^{*2} z + 2\alpha_1^* \alpha_2^* \log z + 2\alpha_1^* c_1 + \alpha_4^* + 2\alpha_1^* \alpha_2^* + \frac{2\zeta_3 \log z}{z} + O(z^{-2}), \\ \mathcal{G} w_2(z) &= 2\alpha_1^{*2} z + 2\alpha_1^* \alpha_2^* \log z + 2\alpha_1^* c_1 + \alpha_4^* + 2\alpha_1^* \alpha_2^* + \left(\frac{d^2(d+1)}{2(d+2)^2} - 2d\zeta_3 \right) \frac{\log z}{z} \\ &+ O(z^{-1}). \end{aligned}$$

The above leads to the expansion

$$\begin{aligned} w(z) &\sim \alpha_1^{*2} z^2 + 2\alpha_1^* \alpha_2^* z \log z + (\alpha_4^* + 2\alpha_1^* c_1)\, z + \frac{d^2}{4(d+2)^2} (\log z)^2, \quad z \to \infty \\ &= \alpha_1^{*2} z^2 + 2\alpha_1^* \alpha_2^* z \log z + (\alpha_4^* + 2\alpha_1^* c_1)\, z + (\alpha_2^* \log z)^2, \quad z \to \infty. \end{aligned}$$

Finally, we pick the last approximating function to be

$$w_3(z) := \alpha_1^{*2} z^2 + 2\alpha_1^* \alpha_2^* z \log(z+1) + (\alpha_4^* + 2\alpha_1^* c_1)\, z + (\alpha_2^* \log(z+1))^2 + \zeta_4 \log(z+1),$$

which satisfies

$$w_3'(z) = 2\alpha_1^{*2} z + 2\alpha_1^* \alpha_2^* \log z + (2\alpha_1^* c_1 + \alpha_4^* + 2\alpha_1^* \alpha_2^*) + \frac{2\alpha_2^{*2} \log z}{z} + \frac{\zeta_4}{z} + O(z^{-3}), \quad z \to \infty.$$

Expanding $\mathcal{G} w_3(z)$ as $z \to \infty$ yields

$$\begin{aligned} \mathcal{G} w_3(z) &= 2\alpha_1^{*2} z + 2\alpha_1^* \alpha_2^* \log z + (2\alpha_1^* c_1 + \alpha_4^* + 2\alpha_1^* \alpha_2^*) + \frac{2\alpha_2^{*2} \log z}{z} \\ &- \left(\frac{(d^3 - 3d - 1)(d+1)}{3(d+2)^2(d+3)} + \frac{d(d+1)c_1}{d+2} + d\zeta_4 \right) \frac{1}{z} + O(z^{-2} \log z). \end{aligned}$$

Matching the $O(z^{-1})$-term coefficients yields the final refinement, as $z \to \infty$,

$$\begin{aligned} w(z) &\sim \alpha_1^{*2} z^2 + 2\alpha_1^* \alpha_2^* z \log z + (a_2^* + 2\alpha_1^* c_1)\, z + \alpha_2^{*2} (\log z)^2 \\ &- \left(\frac{d^3 - 3d - 1}{3(d+2)^2(d+3)} + \frac{c_1 d}{d+2} \right) \log z \\ &= \alpha_1^* z^2 + 2\alpha_1^* \alpha_2^* z \log z + (a_2^* + 2\alpha_1^* c_1)\, z + \alpha_2^{*2} (\log z)^2 + \left(\alpha_5^* - \frac{c_1 d}{d+2} \right) \log z. \end{aligned}$$

A.4 Asymptotic expansion of $w_\theta(z)$ in Section 4.1.6

Let $\mathcal{G}$ be the operator on the right-hand side of (4.27)

$$\mathcal{G} w_\theta(z) = \frac{(d+1)^{d+1}}{(d-1)!} \int_0^\theta \left[w_\theta(z-y) + (1 + 2u_\theta(z-y)) - w_\theta(z) \right] (1 - y/z)\, \mu(z,y) \mathrm{d}y,$$

where θ is a control satisfying

$$\theta(z) \sim \frac{1}{\alpha_1^*} + \frac{\gamma}{z}, \quad z \to \infty.$$

The value function corresponding to θ satisfies the expansion (4.26), as $z \to \infty$,

$$\begin{aligned} u_\theta(z) &= \alpha_1^* z + \alpha_2^* \log z + c_2 \\ &+ \left(d\alpha_1^* \alpha_2^* \gamma + \frac{d\alpha_1^{*3} \gamma^2}{2} + \frac{(3d^3 + 13d^2 + 4d + 4)(d+1)!^{1/(d+1)}}{24(d+2)^2(d+3)(d+1)} \right) \frac{1}{z} + O(z^{-2}). \end{aligned}$$

Since the $O(z^{-1})$-term of θ contributes to the $O(\log z)$-term of $\widehat{w}_\theta$, this time we only need one test function of the form

$$\widehat{w}_\theta(z) = \alpha_1^{*2} z^2 + 2\alpha_1^* \alpha_2^* z \log(z+1) + (\alpha_4^* + 2\alpha_1^* c_1)\, z + \alpha_2^{*2} (\log(z+1))^2 + b_0 \log(z+1),$$

with a parameter $b_0 \in \mathbb{R}$. The test function $\widehat{w}_\theta$ satisfies, as $z \to \infty$,

$$\widehat{w}_\theta'(z) = 2\alpha_1^{*2} z + 2\alpha_1^* \alpha_2^* \log z + (2\alpha_1^* c_1 + \alpha_4^* + 2\alpha_1^* \alpha_2^*) + \frac{2\alpha_2^{*2} \log z}{z} + \frac{b_0}{z} + O(z^{-2} \log z).$$

Computing $\mathcal{G}\widehat{w}_\theta(z)$ yields

$$\begin{aligned}\mathcal{G}\widehat{w}_\theta(z) &= 2\alpha_1^{*2} z + 2\alpha_1^*\alpha_2^* \log z + (2\alpha_1^* c_1 + \alpha_4^* + 2\alpha_1^*\alpha_2^*) + \frac{2\alpha_2^{*2}\log z}{z} \\ &+ \Big(-\frac{-2-7d-25d^2+7d^3+3d^4}{12(d+2)(d+3)} - \frac{c_2 d(d+1)}{d+2} - \frac{2d(d+1)^2\gamma}{(d+1)!^{2/(d+1)}(d+2)} \\ &- db_0\Big)\frac{1}{z} + O(z^{-2}\log z), \quad z\to\infty.\end{aligned}$$

Combining the two expansions leads to the refined asymptotics of w_θ, as $z\to\infty$,

$$\begin{aligned}w_\theta(z) &\sim \alpha_1^{*2} z^2 + 2\alpha_1^*\alpha_2^* z\log z + (\alpha_4^* + 2\alpha_1^* c_1)\, z + \alpha_2^{*2}(\log z)^2 \\ &+ \left(\frac{-2-7d-25d^2+7d^3+3d^4}{12(d+1)(d+2)(d+3)} - \frac{c_2 d}{d+2} - \frac{2d(d+1)\gamma}{(d+1)!^{2/(d+1)}(d+2)}\right)\log z.\end{aligned}$$

The above, together with the expansion of $u_\theta(z)$, yields (4.28).

A.5 Asymtptic expansion of $w(z)$ in Section 4.3.3

Let the functional $\mathcal{I}f(z)$ act as

$$\mathcal{I}f(z) := (\alpha+1)^2 \int_0^{\psi(z)} (w(z-y) - w(z) + 1 + 2u(z-y))\ \nu(z,y)\mathrm{d}y,$$

We compare $w(z)$ with various test functions.

It is an easy exercise to see that $w(z) \sim (\beta_1^* z)^2$. Let us start with $w_1(z) := (\beta_1^* z)^2 + \omega_1 z\log z,\ \omega_1 \in \mathbb{R}$. Matching $w_1'(z) = 2\beta_1^* z + \omega_1 \log z + O(1),\ z\to\infty$ with

$$\mathcal{I}w_1(z) = 2\beta_1^* z + \frac{1}{\alpha+2}\left(-\beta_1^*(\alpha+1) - \alpha\omega_1(\alpha+2)\right)\log z + O(1), \quad z\to\infty,$$

yields $w(z) \sim (\beta_1^* z)^2 + 2\beta_1^*\beta_2^* z\log z,\ z\to\infty$.

Now, consider $w_2(z) = (\beta_1^* z)^2 + 2\beta_1^*\beta_2^* z\log z + \omega_2 z$ with $w_2'(z) = 2\beta_1^* z + 2\beta_1^*\beta_2^*\log z +$

$2\beta_1^*\beta_2^* + \omega_2$. For such function, we have

$$\begin{aligned}\mathcal{I}w_2(z) &= 2\beta_1^* z + 2\beta_1^*\beta_2^* \log z + \frac{2c_1^*(\alpha+1)^{(2\alpha+1)/(\alpha+1)}}{\alpha^{\alpha/(\alpha+1)}} + \frac{(\alpha+1)^{\alpha/(\alpha+1)}}{\alpha^{\alpha/(\alpha+1)}(\alpha+2)} \\ &- \alpha\omega_2 + O(z^{-1}\log z), \quad z \to \infty.\end{aligned}$$

Hence, choosing

$$\omega_2 = 2\beta_1^* c_1^* + \frac{2\beta_1^*}{(\alpha+1)(\alpha+2)},$$

we obtain the next refinement

$$w(z) \sim (\beta_1^* z)^2 + 2\beta_1^*\beta_2^* z \log z + (2\beta_1^* c_1^* + \beta_4^*)\, z, \quad z \to \infty,$$

where

$$\beta_4^* = \frac{2\beta_1^*}{(\alpha+1)(\alpha+2)}.$$

We choose the next approximating function to be of the form $w_3(z) := (\beta_1^* z)^2 + 2\beta_1^*\beta_2^* z \log z + (2\beta_1^* c_1^* + \beta_4^*) z + \omega_3 (\log z)^2,\ \omega_3 \in \mathbb{R}$. First, compute the derivative

$$w_3'(z) = 2\beta_1^* z + 2\beta_1^*\beta_2^* \log z + 2\beta_1^*\beta_2^* + 2\beta_1^* c_1^* + \beta_4^* + 2\omega_3 \frac{\log z}{z}.$$

Second, expanding the functional $\mathcal{I}w_3(z)$ yields, as $z \to \infty$,

$$\mathcal{I}w_3(z) = 2\beta_1^* z + 2\beta_1^*\beta_2^* \log z + 2\beta_1^*\beta_2^* + 2\beta_1^* c_1^* + \beta_4^* + \left(\frac{\alpha+1}{2(\alpha+2)^2} - 2\alpha\omega_3\right)\frac{\log z}{z} + O(z^{-1}).$$

The two expressions match at $\omega_3 = \beta_2^{*2}$; thus, we arrive at a better estimate

$$w(z) \sim (\beta_1^* z)^2 + 2\beta_1^*\beta_2^* z \log z + (2\beta_1^* c_1^* + \beta_4^*)\, z + (\beta_2^* \log z)^2, \quad z \to \infty.$$

For the last approximation, consider function $w_4(z) := (\beta_1^* z)^2 + 2\beta_1^*\beta_2^* z \log z + (2\beta_1^* c_1^* + \beta_4^*)\, z +$

$(\beta_2^* \log z)^2 + \omega_4 \log z,\ \omega_4 \in \mathbb{R}$ with

$$w_4'(z) = 2\beta_1^* z + 2\beta_1^*\beta_2^* \log z + 2\beta_1^*\beta_2^* + 2\beta_1^* c_1^* + \beta_4^* + \frac{2\beta_2^{*2} \log z}{z} + \frac{\omega_4}{z}.$$

Expanding $\mathcal{I}w_4(z)$ yields

$$\begin{aligned} \mathcal{I}w_4(z) &= 2\beta_1^* z + 2\beta_1^*\beta_2^* \log z + 2\beta_1^*\beta_2^* + 2\beta_1^* c_1^* + \beta_4^* + \frac{2\beta_2^{*2} \log z}{z} \\ &+ \left(\frac{(\alpha+1)c_1^*}{\alpha+2} + \frac{\alpha+1}{(\alpha+2)^2(\alpha+3)} - \alpha\omega_4 \right) \frac{1}{z} + O(z^{-2}\log z), \quad z \to \infty. \end{aligned}$$

Therefore, matching $w_4'(z)$ with $\mathcal{I}w_4(z)$ for z large, we obtain, as $z \to \infty$,

$$w(z) \sim (\beta_1^* z)^2 + 2\beta_1^*\beta_2^* z \log z + (2\beta_1^* c_1^* + \beta_4^*)\, z + (\beta_2^* \log z)^2 + \left(\frac{c_1^*}{\alpha+2} + \beta_5^* \right) \log z,$$

where

$$\beta_5^* = \frac{1}{(\alpha+2)^2(\alpha+3)}.$$

A.6 Asymptotic expansion of $w(z)$ in Section 4.4.2

Let $\mathcal{G}$ be the integral operator

$$\mathcal{G}w(z) = 9 \int_0^{\delta(z)} \left(w(z-y) - w(z) + (1 + 2u(z-y)) \right) \eta(z,y) \mathrm{d}y,$$

where $\delta(z)$ is the optimal control satisfying

$$\delta(z) = \left(\frac{2}{3}\right)^{2/3} + \left(\frac{2}{3}\right)^{1/3} \frac{1}{12z} + O(z^{-2}), \quad z \to \infty,$$

and $u(z)$ is the optimal value function satisfying

$$u(z) = \left(\frac{3}{2}\right)^{2/3} z - \frac{\log z}{8} + c_3^* + \left(\frac{2}{3}\right)^{2/3} \frac{1}{240z} + O(z^{-2}), \quad z \to \infty.$$

Our first test function is of the form $w_0(z) = \zeta_0 z^2,\ \zeta_0 \in \mathbb{R}_+$ with

$$w_0'(z) = 2\zeta_0 z.$$

Computing $\mathcal{G}w_0(z)$ yields

$$\mathcal{G}w_0(z) = \left(9\left(\frac{3}{2}\right)^{1/3} - 4\zeta_0\right) z + O(1), \quad z \to \infty.$$

Matching the coefficients of z-terms yields

$$w(z) \sim \left(\frac{3}{2}\right)^{4/3} z^2, \quad z \to \infty.$$

Choose the next proxy function to be of the form

$$w_1(z) = (3/2)^{4/3} z^2 + \zeta_1 z \log(z+1), \quad \zeta_1 \in \mathbb{R}.$$

Matching

$$w_1'(z) \sim 2\left(\frac{3}{2}\right)^{4/3} z + \zeta_1 \log z, \quad z \to \infty,$$

with

$$\mathcal{G}w_1(z) = 2\left(\frac{3}{2}\right)^{4/3} z + \left(-\frac{2^{1/3}3^{5/3}}{8} - 2\zeta_1\right) \log z$$

Matching the coefficients of $O(\log z)$-terms leads to the refinement

$$w(z) \sim \left(\frac{3}{2}\right)^{4/3} z^2 - \frac{2^{1/3}3^{2/3}}{8} z \log z, \quad z \to \infty.$$

Next approximation is of the form

$$w_2(z) := \left(\frac{3}{2}\right)^{4/3} z^2 - \frac{2^{1/3}3^{2/3}}{8} z \log(z+1) + \zeta_2 z,\ \zeta_2 \in \mathbb{R}.$$

For this test function, we have

$$w_2'(z) = 2\left(\frac{3}{2}\right)^{4/3} z - \frac{2^{1/3}3^{2/3}}{8}\log z + \left(-\frac{2^{1/3}3^{2/3}}{8} + \zeta_2\right) + O(z^{-3}), \quad z \to \infty.$$

Working out the expansion of $\mathcal{G}w_2(z)$ for large z yields

$$\mathcal{G}w_2(z) \sim 2\left(\frac{3}{2}\right)^{4/3} z - \frac{2^{1/3}3^{2/3}}{8}\log z + \left(\frac{2^{1/3}3^{2/3}}{8} + 2^{1/3}3^{5/3}c_3^* - 2\zeta_2\right).$$

Matching the coefficients of two expansions yields the consecutive refinement

$$w(z) \sim \left(\frac{3}{2}\right)^{4/3} z^2 - \frac{2^{1/3}3^{2/3}}{8} z\log z + \left(\frac{3}{2}\right)^{2/3}\left(\frac{1}{6} + 2c_3^*\right) z, \quad z \to \infty.$$

We choose the third test function of the form

$$w_3(z) := \left(\frac{3}{2}\right)^{4/3} z^2 - \frac{2^{1/3}3^{2/3}}{8} z\log z + \left(\frac{3}{2}\right)^{2/3}\left(\frac{1}{6} + 2c_3^*\right) z + \zeta_3(\log(z+1))^2.$$

To refine the asymptotic expansion of $w(z)$, the following expansions need to be computed for $z \to \infty$

$$\begin{aligned}
w_3'(z) &= 2\left(\frac{3}{2}\right)^{4/3} z - \frac{2^{1/3}3^{2/3}}{8}\log z + \left(\frac{3}{2}\right)^{2/3}\left(\frac{1}{6} + 2c_3^*\right) + \frac{2\zeta_3 \log z}{z} + O(z^{-2}), \\
\mathcal{G}w_3(z) &= 2\left(\frac{3}{2}\right)^{4/3} z - \frac{2^{1/3}3^{2/3}}{8}\log z + \left(\frac{3}{2}\right)^{2/3}\left(\frac{1}{6} + 2c_3^*\right) + \left(\frac{3}{32} - 4\zeta_3\right)\frac{\log z}{z} \\
&+ O(z^{-1}).
\end{aligned}$$

The above leads to the expansion

$$w(z) \sim \left(\frac{3}{2}\right)^{4/3} z^2 - \frac{2^{1/3}3^{2/3}}{8} z\log z + \left(\frac{3}{2}\right)^{2/3}\left(\frac{1}{6} + 2c_3^*\right) z + \frac{(\log z)^2}{64}, \quad z \to \infty.$$

Finally, we pick the last approximating function to be

$$\begin{aligned}
w_4(z) &:= \left(\frac{3}{2}\right)^{4/3} z^2 - \frac{2^{1/3}3^{2/3}}{8} z\log(z+1) + \left(\frac{3}{2}\right)^{2/3}\left(\frac{1}{6} + 2c_3^*\right) z + \frac{(\log(z+1))^2}{64} \\
&+ \zeta_4 \log(z+1),
\end{aligned}$$

which satisfies

$$w_4'(z) = 2\left(\frac{3}{2}\right)^{4/3} z - \frac{2^{1/3}3^{2/3}}{8}\log z + \left(\frac{3}{2}\right)^{2/3}\left(\frac{1}{6} + 2c_3^*\right) + \frac{\log z}{32z} + \frac{\zeta_4}{z} + O(z^{-2}), \quad z \to \infty.$$

Expanding $\mathcal{G}w_4(z)$ as $z \to \infty$ yields

$$\begin{aligned}\mathcal{G}w_3(z) &= 2\left(\frac{3}{2}\right)^{4/3} z - \frac{2^{1/3}3^{2/3}}{8}\log z + \left(\frac{3}{2}\right)^{2/3}\left(\frac{1}{6} + 2c_3^*\right) + \frac{\log z}{32z} \\ &+ \left(\frac{1}{80} - \frac{3c_3^*}{4} - 2\zeta_4\right)\frac{1}{z} + O(z^{-2}).\end{aligned}$$

Matching the $O(z^{-1})$-term coefficients yields the final refinement, as $z \to \infty$,

$$\begin{aligned}w(z) &\sim \left(\frac{3}{2}\right)^{4/3} z^2 - \frac{2^{1/3}3^{2/3}}{8} z\log z + \left(\frac{3}{2}\right)^{2/3}\left(\frac{1}{6} + 2c_3^*\right) z \\ &+ \frac{(\log z)^2}{64} + \left(\frac{1}{240} - \frac{c_3^*}{4}\right)\log z.\end{aligned}$$

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