DIRECT AND INVERSE PROBLEMS FOR TIME-FRACTIONAL PSEUDO-PARABOLIC EQUATIONS

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Abstract. The purpose of this paper is to establish the solvability results to direct and inverse problems for time-fractional pseudo-parabolic equations with the self-adjoint operators. We are especially interested in proving existence and uniqueness of the solutions in the abstract setting of Hilbert spaces.

Contents

1. Introduction

The problems of determination of temperature at interior points of a region when the initial and boundary conditions along with diffusion source term are specified are known as direct diffusion conduction problems. In many physical problems, determination of coefficients or the right-hand side (the source term, in case of the diffusion equation) in a differential equation from some available information is required; these problems are known as inverse problems.

Inverse source problems for the diffusion, sub-diffusion and for other types of equations are well studied. There are numerous works published only in recent years in this area (for example, see [AKT19, HLIK19, KY19, RZh18, SSB19]). However, inverse problems for pseudo-parabolic equations and for their fractional analogues have been studied relatively little (see [KJ18, LT11a, LT11b, LV19, Run80, RTT19]).

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The inverse problem of determining the coefficient and the right hand side of a pseudo-parabolic equation from local over defined states has important applications in various fields of applied science and engineering. The study of inverse problems for pseudo-parabolic equations began in the 1980s by Rundell (see [Run80]).

Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{L}$, $\mathcal{M}$ be operators with the corresponding discrete spectra $\{\lambda_\xi\}_{\xi\in\mathcal{I}}$, $\{\mu_\xi\}_{\xi\in\mathcal{I}}$ on $\mathcal{H}$, where $\mathcal{I}$ is a countable set.

In this paper we consider solvability of an inverse source problem for the following pseudo-parabolic equation

$$\mathcal{D}_t^\alpha[u(t)+\mathcal{L}u(t)]+\mathcal{M}u(t)=f(t) \text{ in } \mathcal{H}, \tag{1.1}$$

for $0<t<T<\infty$, with initial data

$$u(0)=\varphi\in\mathcal{H}, \tag{1.2}$$

and final condition

$$u(T)=\psi\in\mathcal{H}. \tag{1.3}$$

Here $\mathcal{D}_t^\alpha$ is the Caputo fractional derivative of order $0<\alpha\leq 1$.

In the particular case $\alpha=1$, the equation (1.1) coincides with the classical pseudo-parabolic equation with some differential operators $\mathcal{L}$ and $\mathcal{M}$. The energy for the isotropic material can be modeled by a pseudo-parabolic equation [CG68]. Some wave processes [BBM72], filtration of the two-phase flow in porous media with the dynamic capillary pressure [BGPV97] are also modeled by pseudo-parabolic equations. Time-fractional pseudo-parabolic equation (1.1) occurs in the study of flows of the Oldroyd-B fluid, one of the most important classes for dilute solutions of polymers [FFKV09, TL05].

In this paper, we consider direct and inverse problems for the time-fractional pseudo-parabolic equation with different abstract operators. We seek generalized solutions to these problems in a form of series expansions using the elements of non-harmonic analysis (see [RT16, RTT19]) and we also examine the convergence of the obtained series solutions. The main results on well-posedness of direct and inverse problems are formulated in three theorems.

We will be making the following assumption:

Assumption 1.1. *We assume that the operators $\mathcal{L}$ and $\mathcal{M}$ are diagonalisable (can be written in the infinite dimensional matrix form) with respect to some basis $\{e_\xi\}_{\xi\in\mathcal{I}}$ of the separable Hilbert space $\mathcal{H}$ with the eigenvalues $\lambda_\xi\in\mathbb{R}:\lambda_\xi\geq c_L>0$ and $\mu_\xi\in\mathbb{R}:\mu_\xi\geq c_M>0$ for all $\xi\in\mathcal{I}$, respectively. Here c_L and c_M are some constants, $\mathcal{I}$ is some countable set.*

We will be sometimes also making the following assumption with $\mathcal{I}=\mathbb{N}^k$ or $\mathcal{I}=\mathbb{Z}^k$ for some k:

Assumption 1.2. *In further calculus for our analysis we will also require that $\lambda_\xi\to\infty$ and $\mu_\xi\to\infty$ as $|\xi|\to\infty$. Moreover, we will assume that $|\lambda_\xi|=O(|\mu_\xi|^\kappa)$ as $|\xi|\to\infty$ for some $\kappa>0$.*

1.1. **Preliminaries.** Now, for the formulation of problems we need to define fractional differentiation operators.

Definition 1.3. The Riemann-Liouville fractional integral I^α of order $0<\alpha<1$ for an integrable function f is defined by

$$I^\alpha[f](t)=\frac{1}{\Gamma(\alpha)}\int_c^t (t-s)^{\alpha-1}f(s)ds,\ t\in[c,d],$$

where Γ denotes the Euler gamma function.

The Caputo fractional derivative of order $0<\alpha<1$ of a differentiable function f is defined by

$$\mathcal{D}_t^\alpha[f](t)=I^{1-\alpha}[f'(t)]=\frac{1}{\Gamma(1-\alpha)}\int_c^t \frac{f'(s)}{(t-s)^\alpha}ds,\ t\in[c,d].$$

For more information see [KST06].

In what follows, we will widely use the properties of the Mittag-Leffler type function (see [LG99])

$$E_{\alpha,\beta}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m+\beta)}.$$

In [Sim14] the following estimate for the Mittag-Leffler function is proved, when $0<\alpha<1$ (not true for $\alpha\geq 1$)

$$\frac{1}{1+\Gamma(1-\alpha)z}\leq E_{\alpha,1}(-z)\leq\frac{1}{1+\Gamma(1+\alpha)^{-1}z},\ z>0. \tag{1.4}$$

Thus, it follows that

$$0<E_{\alpha,1}(-z)<1,\ z>0. \tag{1.5}$$

2. Direct problem

We consider the pseudo-parabolic equation

$$\mathcal{D}_t^\alpha[u(t)+\mathcal{L}u(t)]+\mathcal{M}u(t)=f(t), \tag{2.1}$$

for $0<t<T<\infty$, with the Cauchy condition

$$u(0)=\varphi\in\mathcal{H}. \tag{2.2}$$

Definition 2.1. Let X be a separable Hilbert space.

- The space $C^\alpha([0,T];X)$, $0<\alpha\leq 1$ is the space of all continuous functions $g:[0,T]\to X$ with also continuous $\mathcal{D}_t^\alpha g:[0,T]\to X$, such that
$$\|g\|_{C^\alpha([0,T];X)}=\|g\|_{C([0,T];X)}+\|\mathcal{D}_t^\alpha g\|_{C([0,T];X)}<\infty.$$
- The space $W^\alpha([0,T];X)$, $0<\alpha\leq 1$ is the space of all L^2-integrable functions $g:[0,T]\to X$ with L^2-integrable $\mathcal{D}_t^\alpha g:[0,T]\to X$, such that
$$\|g\|_{W^\alpha([0,T];X)}=\|g\|_{L^2([0,T];X)}+\|\mathcal{D}_t^\alpha g\|_{L^2([0,T];X)}<\infty.$$

A generalised solution of the direct problem (2.1)-(2.2) is the function $u\in L^2([0,T];\mathcal{H}^1_{\mathcal{L}})\cap L^2([0,T];\mathcal{H}^1_{\mathcal{M}})\cap W^\alpha([0,T];\mathcal{H}^1_{\mathcal{L}})$. Here we define $\mathcal{H}^{l,m}_{\mathcal{L},\mathcal{M}}$ as

$$\mathcal{H}^{l,m}_{\mathcal{L},\mathcal{M}}:=\{u\in\mathcal{H}:\ \mathcal{L}^l\mathcal{M}^m u\in\mathcal{H}\}, \tag{2.3}$$

for any $l,m\in\mathbb{R}$. In view of this we can define $\mathcal{H}^l_{\mathcal{L}}$, $\mathcal{H}^m_{\mathcal{M}}$ correspondingly

$$\mathcal{H}^l_{\mathcal{L}}:=\{u\in\mathcal{H}:\ \mathcal{L}^l u\in\mathcal{H}\},$$

$$\mathcal{H}^m_{\mathcal{M}} := \{u \in \mathcal{H} : \mathcal{M}^m u \in \mathcal{H}\},$$

for any $l, m \in \mathbb{R}$.

2.1. **Case I:** $1/2 < \alpha < 1$. For Problem (2.1)-(2.2), the following theorem holds true.

Theorem 2.2. *Let $1/2 < \alpha < 1$. Suppose that Assumption 1.1 holds. Let $\varphi \in \mathcal{H}^1_{\mathcal{L}} \cap \mathcal{H}^1_{\mathcal{M}}$ and $f \in L^2([0,T];\mathcal{H}) \cap L^2([0,T],\mathcal{H}^{-1,1}_{\mathcal{L},\mathcal{M}})$. Then there exists a unique solution $u(t)$ of Problem (2.1)-(2.2) such that $u \in L^2([0,T];\mathcal{H}^1_{\mathcal{M}}) \cap W^\alpha([0,T];\mathcal{H}^1_{\mathcal{L}})$. This solution can be written in the form*

$$\begin{aligned} u(t) &= \sum_{\xi\in\mathcal{I}} \varphi_\xi E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi} t^\alpha\right) e_\xi \\ &+ \sum_{\xi\in\mathcal{I}} \left[\frac{1}{1+\lambda_\xi}\int_0^t s^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\mu_\xi}{1+\lambda_\xi} s^\alpha\right) f_\xi(t-s)ds\right] e_\xi, \end{aligned}$$

where $\varphi_\xi = (\varphi, e_\xi)_{\mathcal{H}}$, $f_\xi(t) = (f(t), e_\xi)_{\mathcal{H}}$.

Proof. Let us first prove the existence result. Since the system of eigenfunctions e_ξ is a basis in $\mathcal{H}$, we seek the function $u(t)$ in the form

$$u(t) = \sum_{\xi\in\mathcal{I}} u_\xi(t) e_\xi, \tag{2.4}$$

where $u_\xi(t)$ are unknown functions. Substituting (2.4) into Equations (2.1)-(2.2) and taking into account Assumption 1.1, we obtain the following equations corresponding to the function $u_\xi(t)$:

$$\mathcal{D}^\alpha_t u_\xi(t) + \frac{\mu_\xi}{1+\lambda_\xi} u_\xi(t) = \frac{f_\xi(t)}{1+\lambda_\xi}, \tag{2.5}$$

$$u_\xi(0) = \varphi_\xi,\ \xi \in \mathcal{I}. \tag{2.6}$$

Here $f_\xi(t)$ is the coefficient function of the expansion of $f(t)$, i.e.

$$f(t) = \sum_{\xi\in\mathcal{I}} f_\xi(t) e_\xi,$$

with

$$f_\xi(t) = (f(t), e_\xi)_{\mathcal{H}},$$

and φ_ξ is the coefficient of the expansion of φ, i.e.

$$\varphi = \sum_{\xi\in\mathcal{I}} \varphi_\xi e_\xi,$$

with

$$\varphi_\xi = (\varphi, e_\xi)_{\mathcal{H}}. \tag{2.7}$$

According to [LG99], the solutions of the equation (2.5) satisfying the initial condition (2.6) can be represented in the form

$$u_\xi(t) = \varphi_\xi E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi} t^\alpha\right) - \frac{1}{\mu_\xi}\int_0^t \frac{d}{ds}\left(E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi} s^\alpha\right)\right) f_\xi(t-s)ds. \tag{2.8}$$

Putting (2.8) into (2.4), we obtain a solution of Problem (2.1)-(2.2), i.e.

$$\begin{aligned} u(t) &= \sum_{\xi\in\mathcal{I}} \varphi_\xi E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right)e_\xi \\ &- \sum_{\xi\in\mathcal{I}}\left[\frac{1}{\mu_\xi}\int_0^t \frac{d}{ds}\left(E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right)\right)f_\xi(t-s)ds\right]e_\xi. \end{aligned} \tag{2.9}$$

Here under the integral we have the derivative of the Mittag-Leffler function, which is a non-positive function, i.e.

$$\frac{d}{ds}\left(E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right)\right) = -\frac{\mu_\xi}{1+\lambda_\xi}s^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right)\leq 0,$$

in view of the fact that

$$\begin{aligned} E'_{\alpha,1}(z) &= \frac{d}{dz}\left(\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}\right) = \sum_{k=1}^{\infty}\frac{kz^{k-1}}{\Gamma(\alpha k+1)} \\ &= \frac{1}{\alpha}\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m+\alpha)} = \frac{1}{\alpha}E_{\alpha,\alpha}(z),\ z\in\mathbb{R}. \end{aligned}$$

From this we have

$$\begin{aligned} u(t) &= \sum_{\xi\in\mathcal{I}} \varphi_\xi E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right)e_\xi \\ &+ \sum_{\xi\in\mathcal{I}}\left[\int_0^t s^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right)\frac{f_\xi(t-s)}{1+\lambda_\xi}ds\right]e_\xi. \end{aligned} \tag{2.10}$$

Now, we prove the convergence of the obtained infinite series corresponding to the functions $u(t)$, $\mathcal{D}_t^\alpha u(t)$, $\mathcal{M}u(t)$ and $\mathcal{D}_t^\alpha\mathcal{L}u(t)$.

Before we get the convergence, let us calculate $\mathcal{M}u(t)$, $\mathcal{D}_t^\alpha u(t)$ and $\mathcal{D}_t^\alpha\mathcal{L}u(t)$. By using Assumption 1.1 in (2.7), we have

$$\begin{aligned} \lambda_\xi\varphi_\xi &= \lambda_\xi(\varphi,e_\xi)_\mathcal{H} = (\varphi,\mathcal{L}e_\xi)_\mathcal{H} = (\mathcal{L}\varphi,e_\xi)_\mathcal{H}; \\ \mu_\xi\varphi_\xi &= \mu_\xi(\varphi,e_\xi)_\mathcal{H} = (\varphi,\mathcal{M}e_\xi)_\mathcal{H} = (\mathcal{M}\varphi,e_\xi)_\mathcal{H}. \end{aligned} \tag{2.11}$$

Applying the operator $\mathcal{L}$ to (2.9), and taking into account formulas (2.11), we get

$$
\begin{aligned}
\mathcal{L}u(t) &= \sum_{\xi\in\mathcal{I}} \varphi_\xi E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right)\mathcal{L}e_\xi \\
&+ \sum_{\xi\in\mathcal{I}} \left[\int_0^t s^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right)\frac{f_\xi(t-s)}{1+\lambda_\xi}ds\right]\mathcal{L}e_\xi \\
&= \sum_{\xi\in\mathcal{I}} \lambda_\xi\varphi_\xi E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right)e_\xi \\
&+ \sum_{\xi\in\mathcal{I}} \frac{\lambda_\xi}{1+\lambda_\xi}\left[\int_0^t s^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right)f_\xi(t-s)ds\right]e_\xi \\
&= \sum_{\xi\in\mathcal{I}} (\mathcal{L}\varphi, e_\xi)_{\mathcal{H}} E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right)e_\xi \\
&+ \sum_{\xi\in\mathcal{I}} \frac{\lambda_\xi}{1+\lambda_\xi}\left[\int_0^t s^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right)f_\xi(t-s)ds\right]e_\xi.
\end{aligned}
\tag{2.12}
$$

Analogously, we have

$$
\begin{aligned}
\mathcal{M}u(t) &= \sum_{\xi\in\mathcal{I}} (\mathcal{M}\varphi, e_\xi)_{\mathcal{H}} E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right)e_\xi \\
&+ \sum_{\xi\in\mathcal{I}} \frac{\mu_\xi}{1+\lambda_\xi}\left[\int_0^t s^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right)f_\xi(t-s)ds\right]e_\xi.
\end{aligned}
\tag{2.13}
$$

Applying the operator $\mathcal{D}_t^\alpha$ to (2.4), we have

$$
\mathcal{D}_t^\alpha u(t) = \sum_{\xi\in\mathcal{I}} \mathcal{D}_t^\alpha u_\xi(t)e_\xi. \tag{2.14}
$$

By using (2.5), we find $\mathcal{D}_t^\alpha u_\xi(t)$

$$
\mathcal{D}_t^\alpha u_\xi(t) = \frac{f_\xi(t)}{1+\lambda_\xi} - \frac{\mu_\xi}{1+\lambda_\xi}u_\xi(t). \tag{2.15}
$$

Putting (2.10) into (2.15), we get

$$
\begin{aligned}
\mathcal{D}_t^\alpha u_\xi(t) &= \frac{f_\xi(t)}{1+\lambda_\xi} - \frac{\mu_\xi}{1+\lambda_\xi}\varphi_\xi E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right) \\
&+ \frac{\mu_\xi}{(1+\lambda_\xi)^2}\left[\int_0^t s^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right)f_\xi(t-s)ds\right].
\end{aligned}
\tag{2.16}
$$

Substituting (2.16) into (2.14), we obtain

$$
\begin{aligned}
\mathcal{D}_t^\alpha u(t) &= \sum_{\xi\in\mathcal{I}} \frac{f_\xi(t)}{1+\lambda_\xi}e_\xi - \sum_{\xi\in\mathcal{I}} \frac{\mu_\xi}{1+\lambda_\xi}\varphi_\xi E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right)e_\xi \\
&+ \sum_{\xi\in\mathcal{I}} \frac{\mu_\xi}{(1+\lambda_\xi)^2}\left[\int_0^t s^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right)f_\xi(t-s)ds\right]e_\xi.
\end{aligned}
\tag{2.17}
$$

Applying the operator $\mathcal{L}$ to (2.17) and taking into account formulas (2.11), we have

$$\mathcal{D}_t^\alpha \mathcal{L}u(t) = \sum_{\xi\in\mathcal{I}} \frac{\lambda_\xi}{1+\lambda_\xi} f_\xi(t) e_\xi - \sum_{\xi\in\mathcal{I}} \frac{\mu_\xi}{1+\lambda_\xi} (\mathcal{L}\varphi, e_\xi)_{\mathcal{H}} E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi} t^\alpha\right) e_\xi$$
$$+ \sum_{\xi\in\mathcal{I}} \frac{\mu_\xi \lambda_\xi}{(1+\lambda_\xi)^2} \left[\int_0^t s^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\mu_\xi}{1+\lambda_\xi} s^\alpha\right) f_\xi(t-s) ds\right] e_\xi e_\xi. \tag{2.18}$$

Now let us estimate $\mathcal{H}$-norms

$$\|u(t)\|^2_{\mathcal{H}} \le \sum_{\xi\in\mathcal{I}} \left| E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi} t^\alpha\right)\right|^2 |\varphi_\xi|^2$$
$$+ \sum_{\xi\in\mathcal{I}} \frac{1}{(1+\lambda_\xi)^2} \left|\int_0^t s^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\mu_\xi}{1+\lambda_\xi} s^\alpha\right) f_\xi(t-s) ds\right|^2$$
$$\le C \sum_{\xi\in\mathcal{I}} \left(\frac{1}{1+\frac{\mu_\xi}{1+\lambda_\xi} t^\alpha}\right)^2 |\varphi_\xi|^2$$
$$+ C \sum_{\xi\in\mathcal{I}} \frac{1}{(1+\lambda_\xi)^2} \left(\int_0^t \frac{s^{\alpha-1}}{1+\frac{\mu_\xi}{1+\lambda_\xi} s^\alpha} |f_\xi(t-s)| ds\right)^2 \tag{2.19}$$
$$\le C \sum_{\xi\in\mathcal{I}} |\varphi_\xi|^2 + C \sum_{\xi\in\mathcal{I}} \frac{1}{(1+\lambda_\xi)^2} \int_0^t \left(\frac{s^{\alpha-1}}{1+\frac{\mu_\xi}{1+\lambda_\xi} s^\alpha}\right)^2 ds \int_0^t |f_\xi(s)|^2 ds$$
$$\le C\|\varphi\|^2_{\mathcal{H}} + C \sum_{\xi\in\mathcal{I}} \frac{1}{(1+\lambda_\xi)^2} \int_0^t \left(\frac{1}{s^{1-\alpha}+\frac{\mu_\xi}{1+\lambda_\xi} s}\right)^2 ds \int_0^t |f_\xi(s)|^2 ds$$
$$\le C\|\varphi\|^2_{\mathcal{H}} + C \sum_{\xi\in\mathcal{I}} \frac{1}{(1+\lambda_\xi)^2} \int_0^t \frac{1}{s^{2-2\alpha}} ds \int_0^t |f_\xi(s)|^2 ds$$

Due to the assumption $\alpha > 1/2$, finally, we get

$$\|u\|^2_{L^2(0,T;\mathcal{H})} \le C(\|\varphi\|^2_{\mathcal{H}} + \|f\|^2_{L^2([0,T];\mathcal{H}_{\mathcal{L}}^{-1})}).$$

Here we take into account that

$$\sum_{\xi\in\mathcal{I}} \frac{1}{(1+\lambda_\xi)^2} \int_0^t |f_\xi(s)|^2 ds = \int_0^t \sum_{\xi\in\mathcal{I}} \left|\frac{f_\xi(s)}{1+\lambda_\xi}\right|^2 ds$$
$$= \sum_{\xi\in\mathcal{I}} \int_0^t \|f(s)\|^2_{\mathcal{H}_{\mathcal{L}}^{-1}} ds \le C\|f\|^2_{L^2([0,T];\mathcal{H}_{\mathcal{L}}^{-1})}, \tag{2.20}$$

for some constant $C > 0$.

Finally, by using (1.5) and arguing as in (2.19), from (2.9)–(2.18) we get the following estimates

$$\|\mathcal{L}u\|^2_{L^2([0,T],\mathcal{H})} \leq C(\|\mathcal{L}\varphi\|^2_{\mathcal{H}} + \|f\|^2_{L^2([0,T],\mathcal{H})}),$$

$$\|\mathcal{M}u\|^2_{L^2([0,T],\mathcal{H})} \leq C(\|\mathcal{M}\varphi\|^2_{\mathcal{H}} + \|f\|^2_{L^2([0,T],\mathcal{H}^{-1,1}_{\mathcal{L},\mathcal{M}})}),$$

$$\|\mathcal{D}^\alpha_t u\|^2_{L^2([0,T],\mathcal{H})} \leq C(\|\varphi\|^2_{\mathcal{H}^{-1,1}_{\mathcal{L},\mathcal{M}}} + \|f\|^2_{L^2([0,T],\mathcal{H}^{-1}_{\mathcal{L}})} + \|f\|^2_{L^2([0,T],\mathcal{H}^{-2,1}_{\mathcal{L},\mathcal{M}})}),$$

and

$$\|\mathcal{D}^\alpha_t \mathcal{L}u\|^2_{L^2([0,T],\mathcal{H})} \leq C(\|\varphi\|^2_{\mathcal{H}^1_{\mathcal{M}}} + \|f\|^2_{L^2([0,T],\mathcal{H})} + \|f\|^2_{L^2([0,T],\mathcal{H}^{-1,1}_{\mathcal{L},\mathcal{M}})}),$$

respectively. Here in all our estimates in the spaces $\mathcal{H}^{l,m}_{\mathcal{L},\mathcal{M}}$ for some $l, m \in \mathbb{R}$ we play with the argument as in (2.20). Thus, we finish the proof of the existence result.

Proof of the uniqueness of the solution. Let $w(t)$ and $v(t)$ be two solutions of Problem (2.1)-(2.2), i.e.

$$\mathcal{D}^\alpha_t w(t) + \mathcal{D}^\alpha_t \mathcal{L}w(t) + \mathcal{M}w(t) = f(t),$$
$$w(0) = \varphi,$$
$$\mathcal{D}^\alpha_t v(t) + \mathcal{D}^\alpha_t \mathcal{L}v(t) + \mathcal{M}v(t) = f(t),$$
$$v(0) = \varphi.$$

By subtracting these equations from each other, and denoting $u(t) = w(t) - v(t)$, we obtain

$$\mathcal{D}^\alpha_t u(t) + \mathcal{D}^\alpha_t \mathcal{L}u(t) + \mathcal{M}u(t) = 0, \tag{2.21}$$

$$u(0) = 0. \tag{2.22}$$

We also have

$$u_\xi(t) = (u(t), e_\xi)_{\mathcal{H}},\ \xi \in \mathcal{I}. \tag{2.23}$$

Applying the operator $\mathcal{D}^\alpha_t$ to (2.23), we have

$$\mathcal{D}^\alpha_t u_\xi(t) = (\mathcal{D}^\alpha_t u(t), e_\xi)_{\mathcal{H}},\ \xi \in \mathcal{I}. \tag{2.24}$$

From (2.21)–(2.22), we have

$$\mathcal{D}^\alpha_t u_\xi(t) + \frac{\mu_\xi}{1+\lambda_\xi} u_\xi(t) = 0, \tag{2.25}$$

$$u_\xi(0) = 0. \tag{2.26}$$

By the formula (2.8), when $\varphi_\xi = 0,\ f_\xi(t) = 0$, the solution of the problem (2.25)–(2.26) is $u_\xi(t) \equiv 0$.

Further, by the basis property of the system $\{e_\xi\}_{\xi\in\mathcal{I}}$ in $\mathcal{H}$, we obtain $u(t) \equiv 0$. The uniqueness of the solution of Problem (2.1)–(2.2) is proved. □

2.2. **Case II:** $0 < \alpha < 1$. Here we deal with the case when $0 < \alpha < 1$. But for this we will require more conditions on source term.

Theorem 2.3. *Let $0 < \alpha < 1$. Suppose that Assumption 1.1 holds. Let $\varphi \in \mathcal{H}^1_{\mathcal{L}} \cap \mathcal{H}^1_{\mathcal{M}}$ and $f \in W^1([0,T];\mathcal{H})$. Then there exists a unique solution $u(t)$ of Problem (2.1)-(2.2) such that $u \in L^2([0,T];\mathcal{H}^1_{\mathcal{M}}) \cap W^\alpha([0,T];\mathcal{H}^1_{\mathcal{L}})$. This solution can be written in the form*

$$
\begin{aligned}
u(t) = & \sum_{\xi\in\mathcal{I}} \varphi_\xi E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right) e_\xi \\
& + \sum_{\xi\in\mathcal{I}} \frac{f_\xi(t)}{\mu_\xi} e_\xi - \sum_{\xi\in\mathcal{I}} \frac{f_\xi(0)}{\mu_\xi} E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right) e_\xi \\
& - \sum_{\xi\in\mathcal{I}} \left[\int_0^t \frac{f'_\xi(t-s)}{\mu_\xi} E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right) ds\right] e_\xi,
\end{aligned}
$$

where $\varphi_\xi = (\varphi, e_\xi)_{\mathcal{H}}$, $f_\xi(t) = (f(t), e_\xi)_{\mathcal{H}}$.

Proof. By repeating the arguments of Theorem 2.2, we start from the formula (2.8). For the last term of the equation (2.8), we have

$$
\begin{aligned}
& -\frac{1}{\mu_\xi}\int_0^t \frac{d}{ds}\left(E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right)\right) f_\xi(t-s)ds \\
& = -\frac{1}{\mu_\xi}E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right) f_\xi(t-s)\Big|_0^t \\
& \quad -\frac{1}{\mu_\xi}\int_0^t E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right) f'_\xi(t-s)ds \\
& =\frac{1}{\mu_\xi}f_\xi(t) - \frac{1}{\mu_\xi}E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right) f_\xi(0) \\
& \quad -\frac{1}{\mu_\xi}\int_0^t E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right) f'_\xi(t-s)ds.
\end{aligned}
\tag{2.27}
$$

Thus, for the solution of the Cauchy problem

$$
\mathcal{D}^\alpha_t u_\xi(t) + \frac{\mu_\xi}{1+\lambda_\xi}u_\xi(t) = \frac{f_\xi(t)}{1+\lambda_\xi}, \quad u_\xi(0) = \varphi_\xi,
$$

we have

$$
\begin{aligned}
u_\xi(t) = & \varphi_\xi E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right) + \frac{1}{\mu_\xi}f_\xi(t) \\
& -\frac{1}{\mu_\xi}E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right) f_\xi(0) \\
& -\frac{1}{\mu_\xi}\int_0^t E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right) f'_\xi(t-s)ds,
\end{aligned}
\tag{2.28}
$$

for all $\xi \in \mathcal{I}$.

Putting (2.28) into (2.4), we obtain the solution of Problem (2.1)-(2.2) in the following form

$$
\begin{aligned}
u(t) = &\sum_{\xi\in\mathcal{I}} \varphi_\xi E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right) e_\xi \\
&+ \sum_{\xi\in\mathcal{I}} \frac{f_\xi(t)}{\mu_\xi} e_\xi - \sum_{\xi\in\mathcal{I}} \frac{f_\xi(0)}{\mu_\xi} E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right) e_\xi \\
&- \sum_{\xi\in\mathcal{I}} \left[\int_0^t \frac{f'_\xi(t-s)}{\mu_\xi} E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right) ds\right] e_\xi.
\end{aligned} \tag{2.29}
$$

To prove the convergence of the obtained infinite series corresponding to the functions $\mathcal{L}u(t)$, $\mathcal{M}u(t)$, $\mathcal{D}_t^\alpha u(t)$ and $\mathcal{D}_t^\alpha \mathcal{L}u(t)$, first, we need to calculate them.

Applying the operator $\mathcal{L}$ to (2.9), and taking into account formulas (2.11), we get

$$
\begin{aligned}
\mathcal{L}u(t) = &\sum_{\xi\in\mathcal{I}} (\mathcal{L}\varphi, e_\xi)_{\mathcal{H}} E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right) e_\xi \\
&+ \sum_{\xi\in\mathcal{I}} \frac{\lambda_\xi}{\mu_\xi} f_\xi(t) e_\xi - \sum_{\xi\in\mathcal{I}} \frac{\lambda_\xi}{\mu_\xi} f_\xi(0) E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right) e_\xi \\
&- \sum_{\xi\in\mathcal{I}} \frac{\lambda_\xi}{\mu_\xi} \left[\int_0^t f'_\xi(t-s) E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right) ds\right] e_\xi.
\end{aligned} \tag{2.30}
$$

Analogously, we have

$$
\begin{aligned}
\mathcal{M}u(t) = &\sum_{\xi\in\mathcal{I}} (\mathcal{M}\varphi, e_\xi)_{\mathcal{H}} E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right) e_\xi \\
&+ \sum_{\xi\in\mathcal{I}} f_\xi(t) e_\xi - \sum_{\xi\in\mathcal{I}} f_\xi(0) E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right) e_\xi \\
&- \sum_{\xi\in\mathcal{I}} \left[\int_0^t f'_\xi(t-s) E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right) ds\right] e_\xi.
\end{aligned} \tag{2.31}
$$

Applying the operator $\mathcal{D}_t^\alpha$ to (2.4), we have

$$
\mathcal{D}_t^\alpha u(t) = \sum_{\xi\in\mathcal{I}} \mathcal{D}_t^\alpha u_\xi(t) e_\xi. \tag{2.32}
$$

By using (2.5), we find $\mathcal{D}_t^\alpha u_\xi(t)$

$$
\mathcal{D}_t^\alpha u_\xi(t) = \frac{f_\xi(t)}{1+\lambda_\xi} - \frac{\mu_\xi}{1+\lambda_\xi} u_\xi(t). \tag{2.33}
$$

Putting (2.8) into (2.33), we get

$$
\begin{aligned}
\mathcal{D}_t^\alpha u_\xi(t) = &-\frac{\mu_\xi}{1+\lambda_\xi}\varphi_\xi E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right) \\
&+\frac{1}{1+\lambda_\xi}E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right)f_\xi(0) \\
&+\frac{1}{1+\lambda_\xi}\int_0^t E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right)f'_\xi(t-s)ds.
\end{aligned}
\tag{2.34}
$$

Substituting (2.34) into (2.32), we obtain

$$
\begin{aligned}
\mathcal{D}_t^\alpha u(t) = &-\sum_{\xi\in\mathcal{I}}\frac{\mu_\xi}{1+\lambda_\xi}\varphi_\xi E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right)e_\xi \\
&+\sum_{\xi\in\mathcal{I}}\frac{1}{1+\lambda_\xi}E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right)f_\xi(0)e_\xi \\
&+\sum_{\xi\in\mathcal{I}}\left[\frac{1}{1+\lambda_\xi}\int_0^t E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right)f'_\xi(t-s)ds\right]e_\xi.
\end{aligned}
\tag{2.35}
$$

Applying the operator $\mathcal{L}$ to (2.35) and taking into account formulas (2.11), we have

$$
\begin{aligned}
\mathcal{D}_t^\alpha \mathcal{L}u(t) = &-\sum_{\xi\in\mathcal{I}}\frac{\mu_\xi}{1+\lambda_\xi}(\mathcal{L}\varphi, e_\xi)_{\mathcal{H}} E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right)e_\xi \\
&+\sum_{\xi\in\mathcal{I}}\frac{\lambda_\xi}{1+\lambda_\xi}E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right)f_\xi(0)e_\xi \\
&+\sum_{\xi\in\mathcal{I}}\left[\frac{\lambda_\xi}{1+\lambda_\xi}\int_0^t E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right)f'_\xi(t-s)ds\right]e_\xi.
\end{aligned}
\tag{2.36}
$$

Now, let us estimate $\mathcal{H}$-norms

$$\begin{aligned}
\|u(t)\|_{\mathcal{H}}^2 \le& \sum_{\xi\in\mathcal{I}} \left|E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right)\right|^2 |\varphi_\xi|^2 \\
&+ \sum_{\xi\in\mathcal{I}} \left|\frac{f_\xi(t)}{\mu_\xi}\right|^2 + \sum_{\xi\in\mathcal{I}} \left|\frac{f_\xi(0)}{\mu_\xi}\right|^2 \left|E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha\right)\right|^2 \\
&+ \sum_{\xi\in\mathcal{I}} \left|\int_0^t \frac{f'_\xi(t-s)}{\mu_\xi} E_{\alpha,1}\left(-\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha\right)ds\right|^2 \\
\le& C\sum_{\xi\in\mathcal{I}} \left(\frac{1}{1+\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha}\right)^2 |\varphi_\xi|^2 \\
&+ \sum_{\xi\in\mathcal{I}} \left|\frac{f_\xi(t)}{\mu_\xi}\right|^2 + C\sum_{\xi\in\mathcal{I}} \left(\frac{1}{1+\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha}\right)^2 \left|\frac{f_\xi(0)}{\mu_\xi}\right|^2 \\
&+ C\sum_{\xi\in\mathcal{I}} \frac{1}{\mu_\xi^2}\left(\int_0^t \frac{1}{1+\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha}|f'_\xi(t-s)|ds\right)^2 \\
\le& C\sum_{\xi\in\mathcal{I}} |\varphi_\xi|^2 + C\sum_{\xi\in\mathcal{I}} \left|\frac{f_\xi(t)}{\mu_\xi}\right|^2 \\
&+ C\sum_{\xi\in\mathcal{I}} \frac{1}{\mu_\xi^2}\int_0^T \left(\frac{1}{1+\frac{\mu_\xi}{1+\lambda_\xi}s^\alpha}\right)^2 ds \int_0^T |f'_\xi(s)|^2 ds \\
\le& C\|\varphi\|_{\mathcal{H}}^2 + C\|f(t)\|_{\mathcal{H}_{\mathcal{M}}^{-1}}^2 + C\|f\|_{W^1(0,T;\mathcal{H}_{\mathcal{M}}^{-1})}^2.
\end{aligned} \tag{2.37}$$

Finally, we obtain

$$\|u\|_{L^2(0,T;\mathcal{H})}^2 \le C(\|\varphi\|_{\mathcal{H}}^2 + \|f\|_{W^1(0,T;\mathcal{H}_{\mathcal{M}}^{-1})}^2).$$

By using (1.5) and arguing as in (2.37), from (2.29)–(2.36) we get the following estimates

$$\|\mathcal{L}u\|_{L^2([0,T],\mathcal{H})}^2 \le C\|\mathcal{L}\varphi\|_{\mathcal{H}}^2 + C\|f\|_{W^1([0,T],\mathcal{H}_{\mathcal{L},\mathcal{M}}^{1,-1})}^2,$$

$$\|\mathcal{M}u\|_{L^2([0,T],\mathcal{H})}^2 \le C\|\mathcal{M}\varphi\|_{\mathcal{H}}^2 + C\|f\|_{W^1([0,T],\mathcal{H})}^2,$$

$$\|\mathcal{D}_t^\alpha u\|_{L^2([0,T],\mathcal{H})}^2 \le C(\|\varphi\|_{\mathcal{H}_{\mathcal{L},\mathcal{M}}^{-1,1}}^2 + \|f\|_{W^1([0,T],\mathcal{H}_{\mathcal{L}}^{-1})}^2),$$

and

$$\|\mathcal{D}_t^\alpha \mathcal{L}u\|_{L^2([0,T],\mathcal{H})}^2 \le C(\|\varphi\|_{\mathcal{H}_{\mathcal{M}}^1}^2 + \|f\|_{W^1([0,T],\mathcal{H})}^2),$$

respectively. It proves the existence result.

The proof of the uniqueness of the solution of Theorem 2.3 is the same as in the case of Theorem 2.2. ☐

3. Inverse problem

This section is concerned with an inverse problem for the pseudo-parabolic equation (1.1). We obtain existence and uniqueness results for this problem, by using the $\mathcal{L}$–Fourier method.

Problem 3.1. Find a pair of functions $(u(t), f)$ satisfying the inverse problem (1.1)-(1.3).

Let us define $\gamma := \max\{0, \kappa - 1\}$, where κ is from Assumption 1.2. A generalised solution of the inverse problem (1.1)–(1.3) is the pair of functions $(u(t), f)$, where $u \in C^{\alpha}([0,T]; \mathcal{H}^{1+\gamma}_{\mathcal{L}} \bigcap \mathcal{H}^{1+\gamma}_{\mathcal{M}})$, and $f \in \mathcal{H}$.

For Problem (1.1)–(1.3) the following statement holds true.

Theorem 3.2. *Suppose that Assumptions 1.1 and 1.2 hold. Let $\varphi, \psi \in \mathcal{H}^{1+\gamma}_{\mathcal{L}} \bigcap \mathcal{H}^{1+\gamma}_{\mathcal{M}}$. Then the generalised solution of the inverse problem (1.1)–(1.3) exists, is unique, and can be written in the form*

$$u(t) = \varphi + \sum_{\xi \in \mathcal{I}} \frac{[(\psi, e_\xi)_{\mathcal{H}} - (\varphi, e_\xi)_{\mathcal{H}}] \left(1 - E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi} t^\alpha)\right) e_\xi}{\left(1 - E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi} T^\alpha)\right)},$$

$$f = \mathcal{M}\varphi + \sum_{\xi \in \mathcal{I}} \frac{[(\mathcal{M}\psi, e_\xi)_{\mathcal{H}} - (\mathcal{M}\varphi, e_\xi)_{\mathcal{H}}] \, e_\xi}{1 - E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi} T^\alpha)}.$$

Proof. Existence. Since the system $\{e_\xi\}_{\xi \in \mathcal{I}}$ is a basis in the space $\mathcal{H}$, we expand the functions $u(t)$ and f as follows:

$$u(t) = \sum_{\xi \in \mathcal{I}} u_\xi(t) e_\xi, \tag{3.1}$$

and

$$f = \sum_{\xi \in \mathcal{I}} f_\xi e_\xi, \tag{3.2}$$

where $u_\xi(t)$ and f_ξ are

$$u_\xi(t) = (u(t), e_\xi)_{\mathcal{H}}, \ \xi \in \mathcal{I},$$
$$f_\xi = (f, e_\xi)_{\mathcal{H}}, \ \xi \in \mathcal{I}.$$

Substituting (3.1) and (3.2) into the equations (1.1)–(1.3) and using the relations

$$\mathcal{L} e_\xi = \lambda_\xi e_\xi, \ \ \mathcal{M} e_\xi = \mu_\xi e_\xi,$$

we get the following problem for the functions $u_\xi(t)$ and for the constants $f_\xi, \ \xi \in \mathcal{I}$:

$$\mathcal{D}^\alpha_t u_\xi(t) + \frac{\mu_\xi}{1+\lambda_\xi} u_\xi(t) = \frac{f_\xi}{1+\lambda_\xi}, \tag{3.3}$$

$$u_\xi(0) = \varphi_\xi, \tag{3.4}$$

$$u_\xi(T) = \psi_\xi, \tag{3.5}$$

for $t \in [0,T]$ and for any $\xi \in \mathcal{I}$. Where φ_ξ, ψ_ξ are the coefficients of the expansions of φ, ψ, i.e.

$$\varphi = \sum_{\xi\in\mathcal{I}} \varphi_\xi e_\xi, \quad \psi = \sum_{\xi\in\mathcal{I}} \psi_\xi e_\xi, \tag{3.6}$$

given by

$$\varphi_\xi = (\varphi, e_\xi)_\mathcal{H}, \quad \psi_\xi = (\psi, e_\xi)_\mathcal{H}. \tag{3.7}$$

We seek a general solution of Problem (3.3)–(3.5) in the following form

$$u_\xi(t) = \frac{f_\xi}{\mu_\xi} + C_\xi E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi} t^\alpha), \tag{3.8}$$

where the constants C_ξ, f_ξ are unknown. By using the conditions (1.2)–(1.3) we can find them.

We first find C_ξ:

$$u_\xi(0) = \frac{f_\xi}{\mu_\xi} + C_\xi = \varphi_\xi,$$

$$u_\xi(T) = \frac{f_\xi}{\mu_\xi} + C_\xi E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi} T^\alpha) = \psi_\xi,$$

$$\varphi_\xi - C_\xi + C_\xi E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi} T^\alpha) = \psi_\xi.$$

Then

$$C_\xi = \frac{\varphi_\xi - \psi_\xi}{1 - E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi} T^\alpha)}.$$

Then f_ξ is represented as

$$f_\xi = \mu_\xi \varphi_\xi - \mu_\xi C_\xi.$$

Substituting f_ξ, $u_\xi(t)$ into the expansions (3.1) and (3.2), we find

$$u(t) = \varphi + \sum_{\xi\in\mathcal{I}} C_\xi \left(E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi} t^\alpha) - 1 \right) e_\xi,$$

$$f = \sum_{\xi\in\mathcal{I}} \mu_\xi \varphi_\xi e_\xi - \sum_{\xi\in\mathcal{I}} \mu_\xi C_\xi e_\xi.$$

By using the self-adjointness of the operator $\mathcal{M}$,

$$(\mathcal{M}\varphi, e_\xi)_\mathcal{H} = (\varphi, \mathcal{M}e_\xi)_\mathcal{H},$$

and using $\mathcal{M}e_\xi = \mu_\xi e_\xi$, we obtain

$$(\varphi, e_\xi)_\mathcal{H} = \frac{(\mathcal{M}\varphi, e_\xi)_\mathcal{H}}{\mu_\xi}, \quad (\psi, e_\xi)_\mathcal{H} = \frac{(\mathcal{M}\psi, e_\xi)_\mathcal{H}}{\mu_\xi}.$$

Substituting these identities into the formula of C_ξ, we get that

$$C_\xi = \frac{(\mathcal{M}\varphi, e_\xi)_\mathcal{H} - (\mathcal{M}\psi, e_\xi)_\mathcal{H}}{\mu_\xi \left(1 - E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi} T^\alpha)\right)}.$$

Then, formally, one obtains

$$u(t) = \varphi + \sum_{\xi\in\mathcal{I}} \frac{\left[(\psi, e_\xi)_{\mathcal{H}} - (\varphi, e_\xi)_{\mathcal{H}}\right] \left(1 - E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha)\right) e_\xi}{\left(1 - E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}T^\alpha)\right)}, \tag{3.9}$$

$$f = \mathcal{M}\varphi + \sum_{\xi\in\mathcal{I}} \frac{\left[(\mathcal{M}\psi, e_\xi)_{\mathcal{H}} - (\mathcal{M}\varphi, e_\xi)_{\mathcal{H}}\right] e_\xi}{1 - E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}T^\alpha)}. \tag{3.10}$$

Since $T > T_0 \geq 0$, $T_0 = const$, for denominators of (3.9) and (3.10), the following estimate holds true by (1.4),

$$\begin{aligned} 1 - E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}T^\alpha) &\geq 1 - \frac{1}{1 + \frac{\mu_\xi}{1+\lambda_\xi}T^\alpha\Gamma(1+\alpha)^{-1}} \\ &= \frac{\frac{\mu_\xi}{1+\lambda_\xi}T^\alpha\Gamma(1+\alpha)^{-1}}{1 + \frac{\mu_\xi}{1+\lambda_\xi}T^\alpha\Gamma(1+\alpha)^{-1}} \\ &= \frac{\Gamma(1+\alpha)^{-1}}{\frac{1+\lambda_\xi}{\mu_\xi T^\alpha} + \Gamma(1+\alpha)^{-1}} \\ &\geq M > 0. \end{aligned} \tag{3.11}$$

Here, by Assumption 1.2 we have $|\lambda_\xi| = O(|\mu_\xi|^\kappa)$ as $|\xi| \to \infty$ for some $\kappa > 0$. In the case if $\kappa \leq 1$ the estimate (3.11) makes a sense. Now, suppose that $\kappa > 1$. Then, we have

$$|\mu_\xi|^{\kappa-1}(1 - E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}T^\alpha)) \geq |\mu_\xi|^{\kappa-1}\frac{\Gamma(1+\alpha)^{-1}}{\frac{1+\lambda_\xi}{\mu_\xi T^\alpha} + \Gamma(1+\alpha)^{-1}} \geq M > 0. \tag{3.12}$$

According to [LG99], we have

$$\mathcal{D}_t^\alpha\left(E_{\alpha,1}(-\lambda t^\alpha)\right) = -\lambda E_{\alpha,1}(-\lambda t^\alpha). \tag{3.13}$$

Now, we prove the convergence of the obtained infinite series corresponding to the functions $u(t)$, $\mathcal{D}_t^\alpha u(t)$, $\mathcal{M}u(t)$, $\mathcal{D}_t^\alpha \mathcal{L}u(t)$, and f.

Before we get the convergence, let us calculate $\mathcal{D}_t^\alpha u(t)$, $\mathcal{M}u(t)$ and $\mathcal{D}_t^\alpha\mathcal{L}u(t)$. Applying the operator $\mathcal{D}_t^\alpha$ to (3.9), and using (3.13), we have

$$\begin{aligned} \mathcal{D}_t^\alpha u(t) &= \sum_{\xi\in\mathcal{I}} \frac{\left[(\mathcal{M}\psi, e_\xi)_{\mathcal{H}} - (\mathcal{M}\varphi, e_\xi)_{\mathcal{H}}\right] \mathcal{D}_t^\alpha\left(1 - E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha)\right) e_\xi}{\mu_\xi\left(1 - E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}T^\alpha)\right)} \\ &= \sum_{\xi\in\mathcal{I}} \frac{\left[(\mathcal{M}\psi, e_\xi)_{\mathcal{H}} - (\mathcal{M}\varphi, e_\xi)_{\mathcal{H}}\right] E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha)\, e_\xi}{(1+\lambda_\xi)\left(1 - E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}T^\alpha)\right)}. \end{aligned} \tag{3.14}$$

Applying the operators $\mathcal{L}$ and $\mathcal{M}$ to (3.9) and taking into account (2.11), we have

$$\mathcal{L}u(t) = \mathcal{L}\varphi + \sum_{\xi\in\mathcal{I}} \frac{\left[(\mathcal{L}\psi, e_\xi)_{\mathcal{H}} - (\mathcal{L}\varphi, e_\xi)_{\mathcal{H}}\right] \left(1 - E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha)\right) e_\xi}{1 - E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}T^\alpha)},$$

$$\mathcal{M}u(t)=\mathcal{M}\varphi+\sum_{\xi\in\mathcal{I}}\frac{[(\mathcal{M}\psi,e_\xi)_{\mathcal{H}}-(\mathcal{M}\varphi,e_\xi)_{\mathcal{H}}]\left(1-E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha)\right)e_\xi}{1-E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}T^\alpha)}, \tag{3.15}$$

respectively.

Now, applying the operator $\mathcal{D}_t^\alpha$ to the first equality in (3.15), and taking into account formulas (2.11), we have

$$\mathcal{D}_t^\alpha\mathcal{L}u(t)=\sum_{\xi\in\mathcal{I}}\frac{\mu_\xi[(\mathcal{L}\psi,e_\xi)_{\mathcal{H}}-(\mathcal{L}\varphi,e_\xi)_{\mathcal{H}}]\,E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha)\,e_\xi}{(1+\lambda_\xi)\left(1-E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}T^\alpha)\right)}. \tag{3.16}$$

Let us recall that $\gamma=\max\{0,\kappa-1\}$. By using the formulas (3.9)–(3.16), and taking into account estimates (1.5), we get the following estimates

$$\begin{aligned}
\|u\|^2_{C([0,T],\mathcal{H})}&\leq C(\|\varphi\|^2_{\mathcal{H}}+\|\varphi\|^2_{\mathcal{H}^\gamma_{\mathcal{M}}}+\|\psi\|^2_{\mathcal{H}^\gamma_{\mathcal{M}}}),\\
\|\mathcal{M}u\|^2_{C([0,T],\mathcal{H})}&\leq C(\|\varphi\|^2_{\mathcal{H}^1_{\mathcal{M}}}+\|\varphi\|^2_{\mathcal{H}^{1+\gamma}_{\mathcal{M}}}+\|\psi\|^2_{\mathcal{H}^{1+\gamma}_{\mathcal{M}}}),\\
\|\mathcal{D}_t^\alpha u\|^2_{C([0,T],\mathcal{H})}&\leq C(\|\varphi\|^2_{\mathcal{H}^{-1,1+\gamma}_{\mathcal{L},\mathcal{M}}}+\|\psi\|^2_{\mathcal{H}^{-1,1+\gamma}_{\mathcal{L},\mathcal{M}}}),\\
\|\mathcal{D}_t^\alpha\mathcal{L}u\|^2_{C([0,T],\mathcal{H})}&\leq C(\|\varphi\|^2_{\mathcal{H}^{1+\gamma}_{\mathcal{M}}}+\|\psi\|^2_{\mathcal{H}^{1+\gamma}_{\mathcal{M}}}).
\end{aligned} \tag{3.17}$$

For clarity, we only show the first estimate. By taking the $\mathcal{H}$-norm from the both sides of the representation (3.9), we obtain

$$\|u(t)\|^2_{\mathcal{H}}\leq\|\varphi\|^2_{\mathcal{H}}+\sum_{\xi\in\mathcal{I}}\left[|\varphi_\xi|^2+|\psi_\xi|^2\right]\left|\frac{\left(1-E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}t^\alpha)\right)}{\left(1-E_{\alpha,1}(-\frac{\mu_\xi}{1+\lambda_\xi}T^\alpha)\right)}\right|^2. \tag{3.18}$$

Now, by using the estimates (1.5), (3.11) and (3.12), we get

$$\begin{aligned}
\|u(t)\|^2_{\mathcal{H}}&\leq\|\varphi\|^2_{\mathcal{H}}+C_1\sum_{\xi\in\mathcal{I}}|\mu_\xi|^{2\gamma}\left[|\varphi_\xi|^2+|\psi_\xi|^2\right]\\
&\leq C(\|\varphi\|^2_{\mathcal{H}}+\|\varphi\|^2_{\mathcal{H}^\gamma_{\mathcal{M}}}+\|\psi\|^2_{\mathcal{H}^\gamma_{\mathcal{M}}}),
\end{aligned} \tag{3.19}$$

for some constants $C_1>0$ and $C>0$. Thus, we finish the proof of (3.17).

Similarly, for the source term f, one obtains the estimate

$$\|f\|^2_{\mathcal{H}}\leq C(\|\varphi\|^2_{\mathcal{H}^1_{\mathcal{M}}}+\|\varphi\|^2_{\mathcal{H}^{1+\gamma}_{\mathcal{M}}}+\|\psi\|^2_{\mathcal{H}^{1+\gamma}_{\mathcal{M}}}).$$

Existence of the solution of Problem (1.1)-(1.3) is proved.

Proof of the uniqueness result. Let us suppose that $\{u_1(t),f_1\}$ and $\{u_2(t),f_2\}$ are solution of the Problem (1.1)-(1.3). Let $u(t)=u_1(t)-u_2(t)$ and $f=f_1-f_2$. Then $u(t)$ and f satisfy

$$\mathcal{D}_t^\alpha[u(t)+\mathcal{L}u(t)]+\mathcal{M}u(t)=f, \tag{3.20}$$

$$u(0)=0, \tag{3.21}$$

$$u(T)=0. \tag{3.22}$$

We also have

$$u_\xi(t) = (u(t), e_\xi)_{\mathcal{H}}, \ \ \xi \in \mathcal{I}, \tag{3.23}$$

and

$$f_\xi = (f, e_\xi)_{\mathcal{H}}, \ \ \xi \in \mathcal{I}. \tag{3.24}$$

Applying the operator $\mathcal{D}_t^\alpha$ to (3.23), we have

$$(1+\lambda_\xi)\mathcal{D}_t^\alpha u_\xi(t) = (\mathcal{D}_t^\alpha[u(t)+\mathcal{L}u(t)], e_\xi)_{\mathcal{H}} = (-\mathcal{M}u(t)+f, e_\xi)_{\mathcal{H}} = -\mu_\xi u_\xi(t) + f_\xi.$$

Thus, we get the problem with homogeneous conditions. The general solution of this equation has the form (3.8). Using the homogeneous conditions $u_\xi(0) = 0$ and $u_\xi(T) = 0$ we obtain

$$f_\xi = 0 \ \text{ and } \ u_\xi(t) \equiv 0.$$

Further, by the completeness of the system $\{e_\xi\}_{\xi\in\mathcal{I}}$ in $\mathcal{H}$, we obtain $f \equiv 0, u(t) \equiv 0$. Uniqueness of the solution of Problem (1.1)-(1.3) is proved. □

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