EQUIVARIANT K-THEORY CLASSES OF MATRIX ORBIT CLOSURES

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Abstract. The group $G = \mathrm{GL}_r(k) \times (k^\times)^n$ acts on $\mathbf{A}^{r\times n}$, the space of r-by-n matrices: $\mathrm{GL}_r(k)$ acts by row operations and $(k^\times)^n$ scales columns. A matrix orbit closure is the Zariski closure of a point orbit for this action. We prove that the class of such an orbit closure in G-equivariant K-theory of $\mathbf{A}^{r\times n}$ is determined by the matroid of a generic point. We present two formulas for this class. The key to the proof is to show that matrix orbit closures have rational singularities.

1. Introduction

Let r and n be integers, $r \leq n$, and $\mathbf{A}^{r\times n}$ the affine space of r-by-n matrices with entries in an algebraically closed field k of characteristic zero. We consider the left action of $\mathrm{GL}_r = \mathrm{GL}_r(k)$ on $\mathbf{A}^{r\times n}$ by row operations, and the right action of $T^n = (k^\times)^n$ by scaling columns. Let $v \in \mathbf{A}^{r\times n}$ be a matrix, and consider $X_v^\circ = \mathrm{GL}_r v T^n$, which is the orbit of $\mathrm{GL}_r \times T^n$ through v. We call the Zariski closure $X_v = \overline{X_v^\circ}$ a *matrix orbit closure*, and it is our primary object of interest. Matrix orbit closures were studied in [BF17, BF18] and generalizations of them were studied in [Li18, LPST20]. A primary focus of these papers is how knowledge of which collections of columns of v form linearly independent sets, data known as the matroid of v, affects both the geometry and the algebraic invariants of the matrix orbit closure.

Write $G = \mathrm{GL}_r \times T^n$. We consider the Grothendieck group of G-equivariant coherent sheaves on $\mathbf{A}^{r\times n}$, denoted $K_0^G(\mathbf{A}^{r\times n})$. Since $\mathbf{A}^{r\times n}$ is an affine space, this group can be identified with the representation ring of G. As such, the class of a coherent sheaf on $\mathbf{A}^{r\times n}$ can be written as a Laurent polynomial with integer coefficients in variables $u_1, \ldots, u_r$, $t_1, \ldots, t_n$, which generate this representation ring (see Example 2.1). We view the class of X_v in $K_0^G(\mathbf{A}^{r\times n})$ as a proxy for how complicated X_v is. Essentially the same class is also studied in the guise of the multigraded Hilbert series of X_v; either invariant is readily extracted from the other.

Our main goal is to prove the following result.

Theorem A. *Let $v \in \mathbf{A}^{r\times n}$ be any matrix.*

(1) *The class of X_v in $K_0^G(\mathbf{A}^{r\times n})$ can be determined from the matroid of v alone.*
(2) *Assume that v is a rank r matrix, and denote its matroid by M. Then, the sum*

$$\mathcal{K}(M) = \sum_{w\in S_n} \prod_{j\notin B(w)} \prod_{i\in[r]} (1-u_it_j) \cdot \prod_{i=1}^{n-1} \frac{1}{1-t_{w_{i+1}}/t_{w_i}},$$

a priori *a rational function, is a polynomial in $u_1,\dots,u_r$ and $t_1,\dots,t_n$, and it represents the class of X_v in $K_0^G(\mathbf{A}^{r\times n})$. Here $[r]=\{1,2,\dots,r\}$, S_n is the symmetric group on $[n]$, and for $w=(w_1,\dots,w_n)\in S_n$, $B(w)$ is the lexicographically first basis of M in the list w.*

There are two main motivators for this result. The first comes from [Spe09, FS12], where for any matroid M of rank r on n elements a class $y^{T^n}(M)$ is defined in the T^n-equivariant K-theory of the Grassmannian $Gr(r,n)$. This class is defined piecewise, using equivariant localization; Theorem A provides an explicit lift of $y^{T^n}(M)$ to a single polynomial expression. A similar result on equivariant Chow classes was recently obtained by Lee, Patel, Spink and Tseng [LPST20, Theorem 1]. In Proposition 9.10 we derive [LPST20, Theorem 1] from Theorem A.

The second motivation for Theorem A comes from the following problem: Given a matroid M, when is it possible to partition the ground set of M into independent sets of prescribed sizes? When M is realized by $v\in\mathbf{A}^{r\times n}$ the answer is contained in the Schur polynomial expansion of the coefficient of $t_1t_2\dots t_n$ in the multigraded Hilbert series of X_v (see [BF18, Section 8.2]). The anecdotal matroid invariance of this Schur polynomial expansion motivated the authors to connect the tensor modules in Section 7 to equivariant K-classes and ultimately conjecture the matroid invariance of the equivariant K-class of X_v in [BF18, Conjecture 5.1].

The key to proving Theorem A is the following result.

Theorem B. *Let $v\in\mathbf{A}^{r\times n}$ be a rank r matrix. Then X_v has rational singularities.*

This means, roughly, that the cohomological behavior of the structure sheaf of X_v does not change on desingularization. As a consequence of Theorem B we show the following result.

Theorem C. *Let Y be a T orbit closure in $Gr(r,n)$. Let $\mathcal{S}$ be the tautological bundle on $Gr(r,n)$ and $\mathbf{S}^\lambda$ a Schur functor where λ is a partition with at most*

r parts. Then, for all $m \geq 1$,

$$H^m(Y, \mathbf{S}^\lambda(\mathcal{S}^*)) = 0.$$

We view this as a variant of the Borel-Weil-Bott theorem for torus orbit closures in $Gr(r,n)$.

There are significant parallels between Theorems A, B and C and results occurring in Schubert calculus. The structure of the equivariant K-theory and Chow ring of the Grassmannian is governed by the geometry of the Schubert varieties, which are orbit closures of a Borel subgroup $B \subset \mathrm{GL}_n$. Taking $\mathrm{GL}_r \times B$ orbit closures in $\mathbf{A}^{r\times n}$ yields the *matrix Schubert varieties*. The equivariant K-classes of matrix Schubert varieties provide canonical representatives for the equivariant K-classes of Schubert varieties. Happily, the choice furnished by the matrix analogue coincides exactly with the well-studied (double) Grothendieck polynomials [KM05, Theorem A]. Our Theorem A presents a similar result for torus orbit closures Y in Grassmannians, where now the polynomial representing the K-class depends only on the matroid stratum of a point in the big orbit of Y.

The methods of [KM05] are those of Gröbner degenerations and avoid rational singularities entirely. In our case X_v does have a nice Gröbner degeneration [Li18, Section 4], but a key feature of this degeneration is that it purposefully breaks the symmetry under $\mathrm{GL}_r \subset G$, making it ill-suited for our purposes. An interesting open problem is to prove Theorem A via the combinatorics of Gröbner degenerations. An important step in this process was completed in [LPST20] where the identification of the G-equivariant Chow class of X_v (which ignores phenomena of positive codimension within X_v) was completed using degenerative techniques. Thus, while our Theorem A subsumes the statement of [LPST20, Theorem 1], their independent proof may ultimately be of considerable value.

Given a T^n-invariant subvariety Y of $Gr(r,n)$, one may define the matrix analogue of Y to be the closure in $\mathbf{A}^{r\times n}$ of those matrices whose row span is a point of Y. Generally, one should not expect to be able to say anything nontrivial about the equivariant K-class of the matrix analogue of Y without some additional hypothesis. Rational singularities of Y and its matrix analogue appears to be the right condition in the study of matrix orbit closures. This stands in contrast to the situation occuring in the equivariant Chow ring, where [BF17, Theorem 3.5] provides the connection between the class of Y and its matrix analogue under minimal hypotheses.

Matrix Schubert varieties are shown to have rational singularities in [KR87, Theorem 2], and this result is based on the usual Schubert varieties having rational singularities, a result shown using Frobenius splittings in [Ram85, Theorem 4] (see also [Bri01, Bri03]).

In our Theorem B we deduce rational singularities of X_v by studying the quotient of X_v by the n-torus T^n, which is a subvariety of $(\mathbf{P}^{r-1})^n$. We show that this quotient has rational singularities by applying results of Brion [Bri01, Bri03] and Li [Li18] on multiplicity-free varieties. An interesting open question is whether X_v is compatibly Frobenius split in $\mathbf{A}^{r\times n}$. The Borel-Weil-Bott style result of Theorem C is a close relative of Theorem B. Its analogue in Schubert calculus was studied extensively in [Kem76, Theorem 1]. A problem for future study is the generalization Theorem C to Coxeter matroids in flag varieties of other types. Recently, [BEST21, Section 10] initiated the study of a systematic procedure to equate certain equivariant K-theoretic Euler characteristics with equivariant integrals in Chow theory. We wonder whether combining Theorem C with this method provides any insight in the study of $H^0(Y, \mathbf{S}^\lambda(\mathcal{S}^*))$ (where Y is a torus orbit closure in $Gr(r, n)$).

After proving Theorems A, B and C we investigate K-theoretic positivity in the sense of [AGM11]. The main result here is Proposition 9.1, which says that, when working over $\mathbf{C}$, the equivariant K-class of X_v expands "positively" in terms of double Grothendieck polynomials. Once again, this result hinges on the rational singularities of X_v. We use this result as motivation for a series of progressively weaker conjectures on positivity properties of both K-classes and Chow classes of X_v. It is our hope that these conjectures provide fertile ground for future work in this area.

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2. Background on equivariant K-theory

Let k be an algebraically closed field. A variety will be an integral scheme of finite type over k. Let X be a variety with a G-action, where G is a linear algebraic group. General references for the material discussed below are [CG97, Chapter 5] and [Mer05].

We let $K_0^G(X)$ denote the Grothendieck group of G-equivariant coherent sheaves over X. We let $K_G^0(X)$ denote the Grothendieck group of G-equivariant vector bundles over X. There is a natural group homomorphism, $K_G^0(X) \to K_0^G(X)$. Using the tensor product of vector bundles, $K_G^0(X)$ is a ring, and $K_0^G(X)$ is a module over $K_G^0(X)$. If X is smooth then this map $K_G^0(X) \to K_0^G(X)$ is an isomorphism, as every equivariant coherent sheaf can be resolved by equivariant vector bundles.

If $f : Z \to X$ is a G-equivariant proper map then there is a pushforward $f_* : K_0^G(Z) \to K_0^G(X)$ defined by $f_*[\mathcal{F}] = \sum_i (-1)^i [R^i f_* \mathcal{F}]$.

Let $R(G)$ denote the representation ring of G. Then $K_0^G(X)$ is a module over $R(G)$. If X is an affine space we can identify $K_0^G(X) = K_G^0(X) = R(G)$.

It is important to emphasize that in our work, G will always be a general linear group, a torus, or a product thereof. In this case $R(G)$ is easy to describe. Let T be a maximal torus of G and let W be the Weyl group of G. The representation ring of T is a Laurent polynomial ring $\mathbf{Z}[\mathrm{Hom}(T, k^\times)]$. Then $R(G)$ is the ring of W-invariants of $R(T)$.

Let $A = k[x_1, \ldots, x_m]$ and let $\mathbf{A} = \mathrm{Spec}(A)$ be an affine space carrying a G action. An equivariant coherent sheaf on $\mathbf{A}$ is described by an equivariant coherent A-module. We describe how to compute the class of such a sheaf in $K_0^G(\mathbf{A}) = R(G) \subset R(T)$. Let T be the maximal torus of G and d its dimension, so that $R(T) = \mathbf{Z}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$. The ring $k[x_1, \ldots, x_m]$ is graded by the character group of T, which is $\mathrm{Hom}(T, k^\times) = \mathbf{Z}^d$. We assume that the grading is positive, in that the degrees of the variables x_i lie in a common open half-space of $\mathbf{Q}^d \supset \mathbf{Z}^d$.

Let M be a finitely generated, G-equivariant module over $k[x_1, \ldots, x_m]$. Then M is a $\mathbf{Z}^d$-multigraded $k[x_1, \ldots, x_m]$-module. The multigraded Hilbert series of M is

$$\mathrm{Hilb}(M) = \sum_{\mathbf{a}\in\mathbf{Z}^d} \dim_k(M_{\mathbf{a}})\, t_1^{a_1} t_2^{a_2} \dots t_d^{a_d} \in \mathbf{Z}[[t_1^{\pm 1}, \dots, t_d^{\pm 1}]].$$

Using an equivariant resolution, one can write the Hilbert series as a rational function $\mathcal{K}(M)/\prod_{i=1}^m (1 - t^{\deg(x_i)})$, as explained in [MS05, Chapter 8], where $\mathcal{K}(M) \in R(G) \subset R(T)$ is referred to as the K-polynomial of M. The K-polynomial $\mathcal{K}(M)$ represents the class of the sheaf associated to M in $K_0^G(\mathbf{A})$. When M is the coordinate ring of a closed subvariety $Z \subset \mathbf{A}$, we abuse notation and write $[Z]$ or $\mathcal{K}(Z)$ for the class of the sheaf associated to M in $K_0^G(\mathbf{A})$.

Example 2.1. Let $G = \mathrm{GL}_r \times T^n$ and let $X = \mathbf{A}^{r\times n}$. Then, the maximal torus of G is $T^r \times T^n$, where T^r is the diagonal maximal torus in GL_r. We have

$$R(T^r \times T^n) = \mathbf{Z}[u_1^{\pm 1}, \dots, u_r^{\pm 1}, t_1^{\pm 1}, \dots, t_n^{\pm 1}],$$

and,

$$R(G) = \mathbf{Z}[u_1^{\pm 1}, \dots, u_r^{\pm 1}, t_1^{\pm 1}, \dots, t_n^{\pm 1}]^{S_r},$$

where S_r is the symmetric group on $[r]$, which permutes the u variables.

Writing $A = k[x_{ij} : 1 \le i \le r, 1 \le j \le n]$ and $\mathbf{A}^{r\times n}$ for $\mathrm{Spec}(A)$, we have that $\mathbf{A}^{r\times n}$ has a G-action as described in the introduction. We take the sign convention that the character of the $(T^r \times T^n)$-action on the one-dimensional k-vector space spanned by x_{ij} is $u_i t_j$. Now, if M is a $(T^r \times T^n)$-equivariant A-module we can write

$$\mathrm{Hilb}(M) = \mathcal{K}(M) \prod_{j\in[n]} \prod_{i\in[r]} \frac{1}{1 - u_i t_j}.$$

In the future we will write T for T^n unless confusion may arise.

3. Background on rational singularities

Recall that we work over an algebraically closed field of characteristic zero. A proper birational morphism $f : Z \to Y$ of varieties, where Z is smooth, is called a resolution of singularities. It is called a rational resolution of singularities if

(i) Y is normal, i.e., $\mathcal{O}_Y \to f_*\mathcal{O}_Z$ is an isomorphism, and
(ii) $R^m f_* \mathcal{O}_Z = 0$ for $m > 0$.

We say that Y has rational singularities if there exists a rational resolution of singularities $f : Z \to Y$. We refer the reader to Kollár and Mori [KM98, Section 5.1] for more on these singularities. The following well known results will be needed.

Proposition 3.1. *If one resolution of singularities of Y is rational, then every resolution is.*

Proposition 3.2. *Let $f : Z \to Y$ be a proper birational morphism, where both Z and Y have rational singularities. Then $f_*\mathcal{O}_Z = \mathcal{O}_Y$ and $R^m f_*\mathcal{O}_Z = 0$ for $m > 0$.*

Proposition 3.3. *Assume that $Z \to Y$ is a locally trivial fiber bundle with smooth fiber and normal base. Then Z has rational singularities if and only if Y does.*

By locally trivial fiber bundle we mean the following: There is an open cover $\{U_i\}$ of Y and a smooth variety P so that for $g : Z \to Y$ we have $g^{-1}(U_i) = U_i \times P$.

Proof. The question is local on Y so we may assume that $Z = Y \times P$ where P is smooth. The map $P \to \operatorname{Spec} k$ is faithfully flat, and hence the base change $g : Y \times P \to Y$ is faithfully flat too. Let $f : \widetilde{Y} \to Y$ be a resolution of singularities. We also have natural maps $g' : \widetilde{Y} \times P \to \widetilde{Y}$ and $f' : \widetilde{Y} \times P \to Y \times P$ and these fit together into a base change diagram,

$$\begin{tikzcd} \widetilde{Y} \times P \arrow[r, "g'"] \arrow[d, "f'"] & \widetilde{Y} \arrow[d, "f"] \\ Y \times P \arrow[r, "g"] & Y. \end{tikzcd}$$

Note that f' is a resolution of singularities, since P is assumed smooth. By flat base change,

$$g^* R^i f_* \mathcal{O}_{\widetilde{Y}} = R^i f'_* g'^* \mathcal{O}_{\widetilde{Y}} = R^i f'_* \mathcal{O}_{\widetilde{Y} \times P}.$$

If Y has rational singularities then the left side above is zero for $i > 0$ and hence $R^i f'_* \mathcal{O}_{\widetilde{Y} \times P}$ for $i > 0$. When $i = 0$ then we obtain $g^* \mathcal{O}_Y = \mathcal{O}_{Y \times P}$.

If $Z = Y \times P$ has rational singularities then $R^i f'_* \mathcal{O}_{\widetilde{Y} \times P} = 0$ for $i > 0$ and hence $g^* R^i f_* \mathcal{O}_{\widetilde{Y}} = 0$. Because g^* is faithful, $R^i f_* \mathcal{O}_{\widetilde{Y}} = 0$ for $i > 0$. Since Y is assumed normal we have $f_* \mathcal{O}_{\widetilde{Y}} = \mathcal{O}_Y$, so we are done. □

4. Rational singularities of matrix orbit closures

We now begin our study of matrix orbit closures. Matroids enter the story in this section. For those unfamiliar with matroids, a reference with viewpoint similar to our own is [Kat16]; a general text such as [Oxl11] may also be helpful.

The goal of this section is to prove the following theorem.

Theorem 4.1. *Let $v \in \mathbf{A}^{r\times n}$ be a rank r matrix. Then X_v has rational singularities.*

To prove this, we immediately reduce to the case that v has no zero columns. Let $p : \mathbf{A}^{r\times n} \dashrightarrow (\mathbf{P}^{r-1})^n$ denote the natural rational map. Then $V = p(X_v)$ is the GL_r orbit closure of $p(v)$ in $(\mathbf{P}^{r-1})^n$. The class of V in the Chow ring of $(\mathbf{P}^{r-1})^n$ can be described using a special case of work of Li [Li18, Theorem 1.1]. To describe Li's result, recall that the Chow ring of $(\mathbf{P}^{r-1})^n$ is isomorphic to $\mathbf{Z}[t_1, \dots, t_n]/(t_1^r, \dots, t_n^r)$. Here, the class of t_i represents the class of a hyperplane in the ith factor.

For a matroid M with rank function rk_M, define the set

$$S(M) = \{s \in \mathbf{N}^n : \sum_{i\in I} s_i < r\,\mathrm{rk}_M(I) \text{ for all } I \subseteq [n], \sum_{i=1}^n s_i = r^2 - 1\}.$$

The elements of $S(M)$ are all of the lattice points of the Minkowski difference $rP(M) - \mathrm{conv}\{e_1, \dots, e_n\}$, where $P(M)$ is the basis polytope of M [LPST20, Theorem 5.3].

Theorem 4.2 ([Li18, Theorem 1.1])**.** *Let M denote the matroid of v. The class of V in the Chow ring $A^*((\mathbf{P}^{r-1})^n)$ is*

$$\sum_{s\in S(M)} \prod_{i=1}^n t_i^{r-1-s_i}.$$

The variety $(\mathbf{P}^{r-1})^n$ is a flag variety, and as such its Chow ring has a privileged generating set as a $\mathbf{Z}$-module consisting of the classes of Schubert varieties. In the case of $(\mathbf{P}^{r-1})^n$ the Schubert varieties are products of linear subspaces in each $\mathbf{P}^{r-1}$, whose Chow classes are exactly the monomials in $t_1, \dots, t_n$. It follows from Theorem 4.2 that when the class of V is expressed in this Schubert basis, the coefficients involved are either 0 or 1. Thus, V is *multiplicity free* in the sense of Brion [Bri03], whose main theorem on such varieties is this.

Theorem 4.3 ([Bri03, Theorem 1])**.** *Let $\mathcal{F}\ell$ be a flag variety of a semisimple algebraic group. Let $V \subset \mathcal{F}\ell$ be a subvariety whose Chow class is a linear combination of classes of Schubert varieties, where all coefficients involved are 0 or 1. Then,*

(1) *V is arithmetically normal and Cohen-Macaulay in the projective embedding given by an ample line bundle on $\mathcal{F}\ell$;*
(2) *For any globally generated line bundle $\mathcal{L}$ on $\mathcal{F}\ell$, the restriction map $H^0(\mathcal{F}\ell, \mathcal{L}) \to H^0(V, \mathcal{L})$ is surjective. All higher cohomology groups $H^m(V, \mathcal{L})$, $m \geq 1$, vanish. If $\mathcal{L}$ is ample then $H^m(V, \mathcal{L}^{-1}) = 0$ for $m < \dim(V)$;*

(3) *V has rational singularities.*

Proof. Only the third item is not part of the statement of [Bri03, Theorem 0.1]. For this item write $\mathcal{F}\ell = G/Q$, where Q is a parabolic subgroup. One constructs the variety

$$Y = \{g \in G : g^{-1}Q/Q \in V\}.$$

By the argument of [Bri01, Theorem 5] one deduces that Y has rational singularities. One must assume that Y does not contain a G orbit to apply this argument (*cf.* [Bri03, Remark 3.3] where a small error is noted), which we may because if Y did contain a G orbit then $Y = G$, which is smooth and thus has rational singularities. Since Y has rational singularities and the natural map $Y \to V$ is a locally trivial fiber bundle with smooth connected fiber Q and normal base V (see [Bri03, Lemma 1.1]), we conclude by Proposition 3.3 that V has rational singularities. ☐

Consider now the line bundle $\mathcal{L}_i$ on $(\mathbf{P}^{r-1})^n$ whose fiber over $(\ell_1, \dots, \ell_n)$ is the line $\ell_i \subset \mathbf{A}^r$. Note that $\mathcal{L}_i^{-1}$ is globally generated for all i and the line bundle $\mathcal{L}_1^{-1-m_1} \otimes \cdots \otimes \mathcal{L}_n^{-1-m_n}$ is ample over $(\mathbf{P}^{r-1})^n$, provided that each m_i is non-negative. We construct the vector bundle

$$\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n,$$

which is a subbundle of the trivial bundle with fiber $(\mathbf{A}^r)^n = \mathbf{A}^{r\times n}$. Let $A = k[x_{ij} : 1 \le i \le r, 1 \le j \le n]$ be the coordinate ring of $\mathbf{A}^{r\times n}$. We can identify $H^0((\mathbf{P}^{r-1})^n, \mathrm{Sym}(\mathcal{E}^*))$ with A.

Proposition 4.4. *For $V = p(X_v)$, the natural map $A \to H^0(V, \mathrm{Sym}(\mathcal{E}^*))$ is surjective.*

Proof. Decompose $\mathrm{Sym}(\mathcal{E}^*)$ as $\bigoplus_{(a_1,\dots,a_n)\in\mathbf{N}^n} \mathcal{L}_1^{-a_1} \otimes \cdots \otimes \mathcal{L}_n^{-a_n}$. Since $\mathcal{L}^{-a_1} \otimes \cdots \otimes \mathcal{L}^{-a_n}$ is globally generated on $(\mathbf{P}^{r-1})^n$ we may apply Theorem 4.3(2), and the result follows. ☐

Proposition 4.5. *Let $k[X_v]$ denote the coordinate ring of X_v. Then there is a G-equivariant isomorphism of A-modules, $H^0(V, \mathrm{Sym}(\mathcal{E}^*)) \approx k[X_v]$.*

Proof. Recall that A is multigraded by $\mathrm{Hom}(T, k^\times) = \mathbf{Z}^r \oplus \mathbf{Z}^n$, where the degree of x_{ij} is (e_i, e_j). The prime ideal of X_v is homogeneous for this grading, i.e. is generated by homogeneous elements, since X_v is G- and hence T-invariant. The identification $A = H^0((\mathbf{P}^{r-1})^n, \mathrm{Sym}(\mathcal{E}^*))$ is G-equivariant, and the restriction map $A \to H^0(V, \mathrm{Sym}(\mathcal{E}^*))$ is too. The kernel of the latter is generated by those homogeneous polynomials in A whose restriction to the orbit $G \cdot v$ is zero; thus the kernel is the prime ideal of X_v, which we have just seen is homogeneous. By Proposition 4.4 we obtain the desired result. ☐

Our proof of rational singularities now follows quite quickly from the following result of Kempf and Ramanathan.

Theorem 4.6 (Kempf, Ramanathan [KR87, Theorem 1])**.** *Let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be globally generated line bundles on a complete variety X with rational singularities. If*

(1) $H^i(X, \mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_n^{m_n}) = 0$ *for all* $i > 0$ *and all* $m_1, \ldots, m_n \in \mathbf{N}^n$ *and*
(2) $H^i(X, \mathcal{L}_1^{-1-m_1} \otimes \cdots \otimes \mathcal{L}_n^{-1-m_n}) = 0$ *for all* $i < \dim(X)$ *and* $m_1, \ldots, m_n \in \mathbf{N}^n$,

then the spectrum of the ring of sections $\bigoplus_{(m_1,\ldots,m_n)\in\mathbf{N}^n} H^0(X, \mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_n^{m_n})$ *has rational singularities.*

Proof of Theorem 4.1. Theorem 4.6 applies to X_v by Theorem 4.3 and Proposition 4.5. □

5. Borel-Weil-Bott theorem for torus orbits in Grassmannians

In this section we use a variant of the Gel′fand-Macpherson correspondence to obtain a variant of the Borel-Weil-Bott theorem for T orbit closures in $Gr(r, n)$.

The group $G = \mathrm{GL}_r \times T$ acts on $Gr(r, n) = \mathrm{GL}_n//P$, where $T = T^n$ is the maximal torus of GL_n and GL_r acts trivially. Thus, a T orbit closure in $Gr(r, n)$ is the same thing as a G orbit closure.

Let $\mathcal{S}$ be the rank r tautological bundle over the Grassmannian $Gr(r, n)$. Its fiber over a subspace is precisely that subspace. Recall that the Borel-Weil-Bott theorem for the Grassmannian describes the cohomology groups of various Schur functors applied to $\mathcal{S}^*$. It says, in a weakened form, that the higher cohomology groups of such bundles vanish and gives a formula for their global sections. Our variant of this result is below.

Theorem 5.1. *Let Y be a T orbit closure in $Gr(r, n)$. Let $\mathcal{S}$ be the tautological bundle over $Gr(r, n)$, and $\mathbf{S}^\lambda$ be a Schur functor where λ is a partition with at most r parts. Then for all $m \geq 1$,*

$$H^m(Y, \mathbf{S}^\lambda(\mathcal{S}^*)) = 0.$$

The result will follow by applying Weyman's geometric method [Wey03, Chapter 5] (*cf.* [KR87, p. 355, Condition I′]), knowing in advance that X_v has rational singularities.

In this section we assume that v has rank r. This means its row span represents a point in the Grassmannian $Gr(r, n)$.

Let Y be the torus orbit closure through the row space of v; Y is a normal toric variety. Let Z be the total space of the vector bundle $\mathcal{S}^{\oplus r}$ restricted to Y, which is a subbundle of the trivial bundle with fiber $(\mathbf{A}^n)^r = \mathbf{A}^{r\times n}$. We put a GL_r action on $\mathcal{S}^{\oplus r}$ using the left action on fibers; since this commutes with the natural T action we see that $\mathcal{S}^{\oplus r}$ is a G-equivariant vector bundle over Y. We have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "s'"] & \mathbf{A}^{r\times n} \times Y \arrow[r, "\pi"] \arrow[d, "s"] & Y \\ X_v \arrow[r, "i'"] & \mathbf{A}^{r\times n} & \end{tikzcd}$$

Here the vertical arrows are projections to the first factor, π is projection to the second factor and other maps are inclusions. All these maps are G-equivariant.

Proposition 5.2. *The higher direct images $R^m s_*\mathcal{O}_Z$, $m \geq 1$, vanish and $s_*\mathcal{O}_Z$ is isomorphic to $k[X_v]$ as a G-equivariant A-module.*

Proof. The map $s' : Z \to X_v$ is a partial desingularization, in that it is proper and a birational isomorphism: The inverse is the map $u \mapsto (u, \mathrm{rowSpan}(u))$, defined over the set of full rank matrices in X_v. Since Y is a toric variety it has rational singularities and thus Z has rational singularities too, by Proposition 3.3. It follows from Proposition 3.2 that $R^m s'_*\mathcal{O}_Z = 0$ for $m \geq 1$ and that $s'_*\mathcal{O}_Z = \mathcal{O}_{X_v}$.

The result follows by viewing these as statements about G-equivariant A-modules (i.e., applying i'_*). ☐

Proof of Theorem 5.1. By [Wey03, Theorem 5.1.2(b)] we may identify $R^m s_*\mathcal{O}_Z$ with $H^m(Y, \mathrm{Sym}((\mathcal{S}^{\oplus r})^*))$, which is zero for $m \geq 1$. Now use the Cauchy formula [Wey03, Theorem 2.3.2] to write

$$\mathrm{Sym}((\mathcal{S}^{\oplus r})^*) = \bigoplus_{\lambda} \mathbf{S}^{\lambda}(k^r) \otimes \mathbf{S}^{\lambda}(\mathcal{S}^*),$$

the sum over partitions λ with at most r parts. The result follows from the additivity of global sections over direct sums and the linear independence of the Schur functors. ☐

Remark 5.3. Weyman's geometric method can be used to construct a free resolution of $k[X_v]$. Define

$$F_i = \bigoplus_{j\geq 0} H^j(Y, \bigwedge^{i+j}(\mathcal{Q}^*)^{\oplus r}) \otimes_k A(-i-j),$$

where $\mathcal{Q} = k^n/\mathcal{S}$ is the tautological quotient bundle on the Grassmannian. [Wey03, Theorem 5.1.3] asserts that there are minimal differentials $d_i : F_i \to F_{i-1}$ so that $F_\bullet$ is a finite free resolution of $k[X_v]$ as an A-module. By the dual Cauchy theorem, the cohomologies of $\bigwedge^m(\mathcal{Q}^*)^{\oplus r}$ can be expressed in terms of Schur functors applied to $\mathcal{Q}^*$. While it can be shown that $H^i(Y, \mathbf{S}^\lambda(\mathcal{Q})) = 0$ for $i > 0$, similar to the argument above, the higher cohomology groups of $\mathbf{S}^\lambda(\mathcal{Q}^*)$ (and dually, $\mathbf{S}^\lambda(\mathcal{S})$) are currently unknown. This is the subject of future work.

6. Matroid invariance of the equivariant K class of X_v

To prove the first part of the theorem in the introduction, the matroid invariance of the class of X_v for *any* matrix v, we consider the case when v has rank $r \leq n$. By [BF18, Proposition 6.6], it is enough to prove the theorem in this special case. We thus assume without loss of generality that v has rank r for the remainder of our article.

In this section we give a formula for the equivariant K class of X_v in terms of the class of Y, the T orbit closure in $Gr(r,n)$ through the row space of v. That this formula solves the problem of the matroid invariance of the class of X_v follows from a result of Speyer.

Theorem 6.1 (Speyer [Spe09, proof of Prop. 12.5])**.** *The class of a torus orbit closure Y in $K_0^T(Gr(r,n))$ depends only of the matroid of a point in the big orbit of Y.*

As mentioned above, the class of Y can be expressed as a linear combination of classes of Schubert varieties. These have lifts to $\mathbf{A}^{r\times n}$, known as *matrix Schubert varieties* [KM05], which we will use in an analogous expression for the class of X_v. Let Ω_λ be a Schubert variety in $Gr(r,n)$, and let X_λ° denote the locus of r-by-n matrices in $\mathbf{A}^{r\times n}$ whose row space lies in Ω_λ. The closure of X_λ° in $\mathbf{A}^{r\times n}$, denoted X_λ, is a matrix Schubert variety.

Theorem 6.2. *Let v be a rank r matrix.*

(1) *The class of the structure sheaf of X_v in $K_0^G(\mathbf{A}^{r\times n})$ can be determined from the matroid of v alone.*
(2) *The class of X_v can be written as a $\mathbf{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$-linear combination of the classes of matrix Schubert varieties, X_λ, where $\lambda_1 \leq n - r$.*

Proof of Theorem 6.2(1). Let $Y \subset Gr(r,n)$ be the torus orbit closure through the row span of v. Recall that $G = \mathrm{GL}_r \times T$. Then G acts on the Grassmannian $Gr(r,n)$, where the GL_r-factor acts trivially. So there is a natural isomorphism of $R(G) = R(\mathrm{GL}_r) \otimes R(T)$-modules,

$$R(\mathrm{GL}_r) \otimes K_0^T(Gr(r,n)) \approx K_0^G(Gr(r,n)).$$

Under this isomorphism $1 \otimes [\mathcal{O}_Y]$ maps to $[\mathcal{O}_Y]$.

Now, we use [Mer05, Corollary 12] to see that the projection map $\pi : \mathbf{A}^{r\times n} \times Gr(r,n) \to Gr(r,n)$ induces a pullback isomorphism

$$\pi^* : K_0^G(Gr(r,n)) \to K_0^G(\mathbf{A}^{r\times n} \times Gr(r,n)).$$

The pullback of $\mathcal{O}_Y$ is $\mathcal{O}_{\mathbf{A}^{r\times n}\times Y}$. We now multiply by the class of the vector bundle $\mathcal{S}^{\oplus r}$, which is say, the class of its locally free sheaf of sections. The result is the class of $\mathcal{O}_Z$, which is the class of the sheaf of sections of the restriction of $\mathcal{S}^{\oplus r}$ to Y. Now we apply s_* to $[\mathcal{O}_Z]$, which by Proposition 5.2, gives $[\mathcal{O}_{X_v}]$. In summary,

$$[\mathcal{O}_{X_v}] = s_*([\mathcal{S}^{\oplus r}] \cdot \pi^*[\mathcal{O}_Y]),$$

and since the right side is determined by the matroid of v, by Theorem 6.1, so is the left. □

Proof of Theorem 6.2(2). In $K_0^G(Gr(r,n))$ write $[\mathcal{O}_Y] = \sum_\lambda c_\lambda[\Omega_\lambda]$, where the sum is over partitions λ with $\lambda_1 \le n-r$ and $c_\lambda \in \mathbf{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ (since GL_r acts trivially on Y). A matrix Schubert variety X_λ has rational singularities [KM05, Theorem 2.4.3] and is (partially) resolved by the total space of the vector bundle $\mathcal{S}^{\oplus r}$ restricted to Ω_λ, just as we have done for X_v in part (1). Thus,

$$\begin{aligned}
[\mathcal{O}_{X_v}] &= s_*([\mathcal{S}^{\oplus r}] \cdot \pi^*[\mathcal{O}_Y]) \\
&= s_*\left([\mathcal{S}^{\oplus r}] \cdot \pi^* \sum_\lambda c_\lambda[\Omega_\lambda]\right) \\
&= \sum_\lambda c_\lambda s_*\left([\mathcal{S}^{\oplus r}] \cdot \pi^*[\Omega_\lambda]\right) = \sum_\lambda c_\lambda [\mathcal{O}_{X_\lambda}].
\end{aligned}$$ □

We note the following immediate corollary of the proof.

Corollary 6.3. *In the notation of the previous results, if we uniquely expand $[\mathcal{O}_Y]$ in the Schubert basis,*

$$[\mathcal{O}_Y] = \sum_\lambda c_\lambda[\Omega_\lambda] \in K_0^T(Gr(r,n)),$$

then, for the same coefficients $\{c_\lambda\}$ we have,

$$[\mathcal{O}_{X_v}] = \sum_\lambda c_\lambda[X_\lambda] \in K_0^G(\mathbf{A}^{r\times n}).$$

7. Consequences of matroid invariance

In this section we state the consequences of the matroid invariance of $[X_v]$, which we studied further in [BF18]. Let $v_1, \dots, v_n$ denote the columns of the matrix v. Write $G(v)$ for the span in $(k^r)^{\otimes n}$ of the tensors

$$(gv_1) \otimes (gv_2) \otimes \cdots \otimes (gv_n), \quad g \in \mathrm{GL}_r.$$

Clearly $G(v)$ is a representation of GL_r.

Corollary 7.1. *The class of $G(v)$ in $R(\mathrm{GL}_r)$ is determined by the matroid of v.*

Let $\mathfrak{S}(v)$ denote the span in $(k^r)^{\otimes n}$ of the tensors

$$v_{w_1} \otimes v_{w_2} \otimes \cdots \otimes v_{w_n}, \quad w \in S_n$$

Clearly $\mathfrak{S}(v)$ is a representation of S_n.

Corollary 7.2. *The class of $\mathfrak{S}(v)$ in $R(S_n)$ is determined by the matroid of v.*

Describing the irreducible decomposition of these representations was the motivation for studying X_v and its class in K-theory. The first corollary follows because the character of $G(v)$ is the coefficient of $t_1 t_2 \dots t_n$ in $\mathrm{Hilb}(k[X_v])$. The second corollary follows from the first by Schur-Weyl duality. For the proofs of these statements and further information, we refer the reader to [BF18].

8. Explicit formula for the K-class of X_v

In this section we compute a formula for the class of X_v in $K_0^G(\mathbf{A}^{r \times n})$ using equivariant localization. This class is represented as the numerator of the Hilbert series $\mathrm{Hilb}(k[X_v])$, as explained in Section 2.

Let $\mathcal{E}$ be a finite dimensional T-equivariant vector bundle over Y. We use the same symbol $\mathcal{E}$ for its sheaf of sections. The equivariant Euler characteristic of $\mathcal{E}$ is $\chi_T(\mathcal{E}) = \sum_i (-1)^i [H^i(Y, \mathcal{E})]$, where $[-]$ means to compute the character as a representation of T.

Since we have shown the G-equivariant identification $H^0(Y, \mathrm{Sym}((\mathcal{S}^*)^{\oplus r})) = k[X_v]$, and the higher cohomology groups of $\mathrm{Sym}((\mathcal{S}^*)^{\oplus r})$ on Y vanish, we can compute the Hilbert series of $k[X_v]$ by computing the T-equivariant Euler characteristic of $\mathrm{Sym}((\mathcal{S}^*)^{\oplus r})$.

To state the formula for $\chi_T(\mathcal{E})$ we let $T_B Y$ be the Zariski tangent space of Y at a fixed point B, and let $C_B \subset T_B Y$ be the tangent cone. The localization formula given in the corollary to [AGP20, Theorem 1.5] states that there is an equality in $\mathbf{Q}(u_1, \dots, u_r, t_1, \dots, t_n)$, the field of fractions of $R(T)$, to wit

$$\chi_T(\mathcal{E}) = \sum_{B \in Y^T} [\mathcal{E}_B] \cdot \mathrm{Hilb}(C_B),$$

where the sum is over T-fixed points B of Y. This formula finds its motivation in the Atiyah-Bott-Berline-Vergne localization formula for smooth varieties. (Alternately, one could pass to a desingularization of Y and compute the Euler characteristic of the pullback of $\mathrm{Sym}((\mathcal{S}^*)^{\oplus r})$ using the aforementioned formula.) Above, $\mathrm{Hilb}(C_B) \in \mathbf{Q}(u_1, \dots, u_r, t_1, \dots, t_n)$ is the Hilbert series of $C_B \subset T_B$ and $\mathcal{E}_B$ is the fiber of $\mathcal{E}$ over B. The formula applies in our case since Y has finitely many T-fixed points and the trivial character does not appear in $[T_B Y]$. The Hilbert series $\mathrm{Hilb}(C_B)$ are referred to as "equivariant multiplicities" in [AGP20] (see Proposition 6.3 in *loc. cit.*).

We now compute $\chi_T(\mathrm{Sym}^m((\mathcal{S}^*)^{\oplus r}))$ using this formula. The T-fixed points of Y correspond to the bases B of the matroid M of v. Over a fixed point B, we can compute the character of $\mathrm{Sym}((\mathcal{S}^*)^{\oplus r})$ to be

$$\prod_{j \in B} \prod_{i \in [r]} \frac{1}{1 - u_i t_j}$$

The equivariant multiplicities were computed in [BF17, Lemma 5.2] as

$$\mathrm{Hilb}(C_B) = \sum_{(w_1, \dots, w_n)} \prod_{i=1}^{n-1} \frac{1}{1 - t_{w_{i+1}}/t_{w_i}}, \tag{1}$$

where the sum is over those permutations $w \in S_n$ whose lexicographically first basis is B. Putting all this together and condensing the summation over fixed points and permutations gives

$$\mathrm{Hilb}(k[X_v]) = \chi_T(\mathrm{Sym}((\mathcal{S}^*)^{\oplus r}) = \sum_{w \in S_n} \prod_{j \in B(w)} \prod_{i \in [r]} \frac{1}{1 - u_i t_j} \prod_{i=1}^{n-1} \frac{1}{1 - t_{w_{i+1}}/t_{w_i}},$$

where $B(w)$ denotes the lexicographically first basis of the matroid M occurring in the list $w = (w_1, \dots, w_n)$. Finally, to obtain the K-theory class $\mathcal{K}(k[X_v])$ we multiply by $\prod_{j \in [n]} \prod_{i \in [r]} (1 - u_i t_j)$.

Theorem 8.1. *The class of X_v in $K_0^G(\mathbf{A}^{r \times n})$ is given by the formula*

$$\mathcal{K}(k[X_v]) = \sum_{w \in S_n} \prod_{j \notin B(w)} \prod_{i \in [r]} (1 - u_i t_j) \cdot \prod_{i=1}^{n-1} \frac{1}{1 - t_{w_{i+1}}/t_{w_i}}.$$

Note that the left side is *a priori* a Laurent polynomial in $u_1, \dots, u_r$ and $t_1, \dots, t_n$. We have thus shown the claims of the second part of the theorem from the introduction.

Example 8.2. Let v represent the matroid M of rank $r = 2$ on $n = 4$ elements with bases

$$\{\{1,2\},\{1,3\},\{2,3\},\{1,4\},\{2,4\}\}.$$

We first evaluate the equivariant multiplicities. One way to do this is afforded by the proof of [BF17, Lemma 5.2]: they are Hilbert series of affine toric varieties, and therefore lattice point enumerators of cones. The cones in question are the tangent cones to $P(M)$. For this M, the toric variety Y is smooth except at the fixed point $\operatorname{span}\{e_1, e_2\}$, so the tangent cones other than $C_{\{1,2\}}$ are unimodular simplicial cones. We get

$$\operatorname{Hilb}(C_{\{1,3\}}) = \frac{1}{(1-t_2/t_3)(1-t_4/t_3)(1-t_2/t_1)}$$

and

$$\operatorname{Hilb}(C_{\{1,2\}}) = \frac{1-t_3t_4/t_1t_2}{(1-t_3/t_2)(1-t_4/t_2)(1-t_3/t_1)(1-t_4/t_1)},$$

and the other three equivariant multiplicities are images of $\operatorname{Hilb}(C_{\{1,3\}})$ under transposing 1 and 2 and/or transposing 3 and 4.

The reader may check that the same rational functions are obtained from (1). To facilitate this we describe the permutations w that achieve $B = B(w)$ for each basis B. If $\{w_1, w_2\}$ is a basis of M, then $B(w) = \{w_1, w_2\}$; this gives four permutations for each basis. The $4! - 5 \cdot 4 = 4$ permutations unaccounted for are 3412, 3421, 4312, and 4321, whose lexicographically first bases are $\{1,3\}$, $\{2,3\}$, $\{1,4\}$, and $\{2,4\}$ respectively.

Using these equivariant multiplicities, the sum in Theorem 8.1 works out to

$$\mathcal{K}(k[X_v]) = 1 - u_1u_2t_3t_4. \tag{2}$$

Elements 3 and 4 are parallel, so M is a parallel extension of the uniform matroid $U_{2,3}$, and therefore the above class can also be computed from Theorem 9.3 or Proposition 9.5 of [BF18]. The procedure described in [BF18, Theorem 9.3] is to apply a Demazure divided difference operator δ_3 to the class of the matroid $U_{2,3} \oplus U_{0,1}$, which is $(1-u_1t_4)(1-u_2t_4)$ since the associated matrix orbit closure is a linear subspace of $\mathbf{A}^{2\times 4}$. Application of the divided difference gives

$$\begin{aligned}\delta_3((1-u_1t_4)(1-u_2t_4)) &= \frac{(1-u_1t_4)(1-u_2t_4) - t_4/t_3(1-u_1t_3)(1-u_2t_3)}{1-t_4/t_3}\\ &= 1 - u_1u_2t_3t_4.\end{aligned}$$

We check the agreement of this class with Theorem 4.2, first deriving the Chow class featuring in that theorem from (2). The class of X_v in $K_0^T(\mathbf{A}^{r\times n})$, forgetting the GL_r-action, can be computed by evaluating the u_i at 1. The resulting class is $1 - t_3t_4$. The class of $V = p(X_v)$ in

$$K_0((\mathbf{P}^{r-1})^n) = \mathbf{Z}[t_1, \ldots, t_n]/((1-t_1)^r, \ldots, (1-t_n)^r)$$

is represented by this same polynomial. Finally, [MS05, Section 8.5] asserts that the class of V in the Chow ring $A^*((\mathbf{P}^{r-1})^n)$ is the sum of terms of lowest degree after substituting $1 - t_i$ for t_i ($i \in [n]$) in the last K-class. The substitution yields

$$1 - (1-t_3)(1-t_4) = t_3 + t_4 - t_3t_4$$

so the Chow class of V is $t_3 + t_4$.

For the other side of the comparison we must compute the set $S(M)$. As singletons have rank 1 in M it follows that $S(M) \subseteq \{0,1\}^4$, and of the remaining inequalities defining $S(M)$ the only one not automatically satisfied once $\sum_{i=1}^4 s_i = 2^2 - 1 = 3$ is $s_3 + s_4 < 2$. Therefore

$$S(M) = \{(1,1,0,1), (1,1,1,0)\},$$

from which Theorem 4.2 also produces the Chow class $t_3 + t_4$.

We now address the polynomiality of the rational function

$$\mathcal{K}(M) = \sum_{w \in S_n} \prod_{j \notin B(w)} \prod_{i \in [r]} (1 - u_i t_j) \cdot \prod_{i=1}^{n-1} \frac{1}{1 - t_{w_{i+1}}/t_{w_i}}.$$

for arbitrary matroids M, which we will call the K-class of M.

Theorem 8.3. *For any matroid M of rank r on $[n]$, $\mathcal{K}(M)$ is a polynomial in $u_1, \ldots, u_r, t_1, \ldots, t_n$.*

Proof. For a fixed permutation $w \in S_n$ let $B \mapsto [B = B(w)]$ be the indicator function of $B(w)$, and write $D(w)$ for $\prod_{i=1}^{n-1} \frac{1}{1 - t_{w_{i+1}}/t_{w_i}}$. Then,

$$\begin{aligned} \mathcal{K}(M) &= \sum_{w \in S_n} \prod_{j \notin B(w)} \prod_{i \in [r]} (1 - u_i t_j) \cdot D(w) \\ &= \sum_{B \in \binom{[n]}{r}} \sum_{w \in S_n} \prod_{j \notin B} \prod_{i \in [r]} (1 - u_i t_j) \cdot D(w) \cdot [B = B(w)]. \end{aligned}$$

If $B = \{w_{i_1}, \ldots, w_{i_r}\}$ with $i_1 < \cdots < i_r$, then $B = B(w)$ if and only if $\mathrm{rk}_M(\{w_1, \ldots, w_i\}) = \max\{j : i_j \leq i\}$ for all $1 \leq i \leq n$. Therefore, viewed as a function of M, the function $[B = B(w)]$ is valuative on matroid polytope subdivisions by [DF10, Proposition 5.3]. In the expansion of $\mathcal{K}(M)$ above the only factors depending on M are the expressions $[B = B(w)]$, so $\mathcal{K}(M)$

is a $\mathbf{Q}(u_1, \ldots, u_r, t_1, \ldots, t_n)$-linear combination of valuative functions, and is therefore valuative itself.

The dual to the abelian group of valuative functions is spanned by Schubert matroids for all orderings of the ground set $[n]$ [DF10, Theorem 5.4] (see also the discussion after that work's Theorem 6.3). That is, if M is an arbitrary matroid, then there exist Schubert matroids $M_1, \ldots, M_k$ and integers $a_1, \ldots, a_k$ such that $\mathcal{K}(M) = \sum_{i=1}^{k} a_i \mathcal{K}(M_i)$. All Schubert matroids are representable over any infinite field, so each $\mathcal{K}(M_i)$ is a polynomial because it is the K-theory class of a sheaf. Thus $\mathcal{K}(M)$ is a polynomial too. □

Remark 8.4. The polynomiality of the formula for $\mathcal{K}(k[X_v])$ can be deduced directly from the cohomological variant of this result proved in [LPST20, Theorem 1] (Hunter Spink, private communication). From this one can conclude that the right side above defines an honest element of $K_0^G(\mathbf{A}^{r\times n})$ (as opposed to an element of its field of fractions) and further, using equivariant localization, that it is a lift of $[\mathcal{O}_Y] \in K_0^T(Gr(r, n))$. However, without rational singularities of X_v, or some other means, one cannot conclude that this natural lift of $[\mathcal{O}_Y]$ is the class of X_v.

9. K-theoretic positivity

In this section we consider the Schubert expansion of the class of matrix orbit closures in $\mathbf{A}^{r\times n}$. By Corollary 6.3, this is equivalent to the Schubert expansion of a torus orbit closure Y in the Grassmannian.

For matroids realizable over $\mathbf{C}$, we deduce K-theoretic positivity of the class of X_v via results of Anderson, Griffeth and Miller [AGM11]. We conjecture a strengthening of this property for all matroids and pose a series of progressively weaker conjectures concerning different types of positivity.

9.1. Positivity for matroids realizable over C.

Let M be a matroid realized be a complex r-by-n matrix v. Write $\mathcal{K}(M) = \mathcal{K}(k[X_v])$ for the polynoimal of Section 8.

There are two types of positivity that $\mathcal{K}(k[X_v])$ could exhibit, being a simultaneous lift of the class of the T^n-equivariant subvariety $Y \subset Gr(r, n)$ as well as the class of the T^r-equivariant subvariety $p(X_v) \subset (\mathbf{P}^{r-1})^n$. Our primary concern will be the positivity obtained from the Grassmannian, although in this subsection we will consider both types of positivity.

In order to state the first result we need to be explicit about which (matrix) Schubert varieties we use to express our classes. For a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r)$, X_λ° is the set of matrices $m \in \mathbf{A}^{r\times n}$ whose rank increases occur in columns $n-r+1-\lambda_1 < n-r+2-\lambda_{r-1} < \cdots < n-\lambda_r$, when the matrix is read columnwise left-to-right. The matrix Schubert variety is $X_\lambda = \overline{X_\lambda^\circ}$. Thus X_λ is a point when $\lambda = (0, \dots, 0)$ and $X_\lambda = \mathbf{A}^{r\times n}$ when $\lambda = (n-r, \dots, n-r)$. In this way X_λ has codimension $|\lambda| = \lambda_1 + \cdots + \lambda_r$. The class of X_λ in $K_0^G(\mathbf{A}^{r\times n})$ is given by the double Grothendieck polynomial $\mathfrak{G}_\lambda(u,t)$ [KM05, Theorem A]; explicit formulas for these polynomials can be found in [KM05, KMY09].

Proposition 9.1. *Let $v \in \mathbf{A}^{r\times n}$ be a rank r matrix realizing a matroid M with e connected components. Write $\mathcal{K}(M)$ in terms of double Grothendieck polynomials,*

$$\mathcal{K}(M) = \sum_{\lambda:\lambda_1 \le n-r} c_\lambda(t)\mathfrak{G}_\lambda(u,t).$$

Then, the Laurent polynomials $(-1)^{r(n-r)-(n-e)-|\lambda|}c_\lambda(t) \in \mathbf{Z}[t_1^{\pm1}, \dots, t_n^{\pm1}]$ are polynomials in $t_2/t_1 - 1, t_3/t_4 - 1, \dots, t_n/t_{n-1} - 1$ with positive integer coefficients.

Recall that a matroid M is connected if it is indecomposable with respect to direct sum (simplicial complex join). Every matroid can be expressed uniquely as a direct sum of connected matroids. When M is realized by a matrix $v \in \mathbf{A}^{r\times n}$, then M is a direct sum of e connected matroids if and only if the codimension of X_v is equal to $r(n-r) - (n-e)$. The effect of direct sum on K-classes is investigated in [BF18, Section 7].

Proof. By Corollary 6.3, it suffices to prove the class of the toris orbit closure $Y \subset Gr(r,n)$ through $\pi(v)$ has the analogous property in $K_0^T(Gr(r,n))$. This follows from [AGM11, Corollary 5.1] since Y has rational singularities and is of codimension $r(n-r) - (n-e)$. □

Example 9.2. Let $v \in \mathbf{A}^{2\times 4}$ be a generic matrix, whose matroid is therefore the uniform matroid $U_{2,4}$. We have

$$\mathcal{K}(k[X_v]) = 1 - u_1^2u_2^2t_1t_2t_3t_4,$$

which can be seen from either Theorem 8.1, [BF18, Proposition 5.2] or the fact that X_v is a hypersurface in $\mathbf{A}^{2\times 4}$. We expand the above polynomial in terms of the classes of the matrix Schubert varieties and obtain

$$\begin{aligned} 1 - u_1^2u_2^2t_1t_2t_3t_4 = t_1^{-1}t_4[\mathcal{O}_{X_{(2,1)}}] - t_1^{-1}t_4[\mathcal{O}_{X_{(2,0)}}] - t_1^{-1}t_4[\mathcal{O}_{X_{(1,1)}}] + \\ (t_1^{-1}t_4 + t_1^{-1}t_2^{-1}t_3t_4)[\mathcal{O}_{X_{(1,0)}}] - (t_1^{-1}t_2^{-1}t_3t_4 - 1)[\mathcal{O}_{X_{(0,0)}}]. \end{aligned}$$

Writing $\beta_i = t_{i+1}/t_i - 1$, $1 \le i \le 3$, we see that

$$t_1^{-1}t_4 = (\beta_1+1)(\beta_2+1)(\beta_3+1)$$

and

$$t_1^{-1}t_2^{-1}t_3t_4 = (\beta_1+1)(\beta_2+1)^2(\beta_3+1)$$

have positive coefficients in the β_i, and the right hand side of the latter has 1 as a term so that $t_1^{-1}t_2^{-1}t_3t_4 - 1$ is positive in the β_i as well. These computations verify the result of Proposition 9.1.

Proposition 9.3. *Let $v \in \mathbf{A}^{r\times n}$ be a realization of a loopless matroid M having e connected components. We may write*

$$\mathcal{K}(M) = \sum_{\alpha} d_\alpha \prod_{j=1}^{n}\prod_{i=1}^{\alpha_j}(1-u_it_j),$$

where the sum ranges over compositions $\alpha = (\alpha_1,\ldots,\alpha_n)$ with $0 \le \alpha_i \le r-1$, and the coefficients d_α are Laurent polynomials in $u_1,\ldots,u_r$. Then, $(-1)^{r(n-r)-(n-e)-\sum_i \alpha_i} d_\alpha$ is a polynomial in $u_2/u_1-1, u_3/u_2-1, \ldots, u_r/u_{r-1}-1$ with positive integer coefficients.

Proof. As discussed in Section 4, $(\mathbf{P}^{r-1})^n$ can be regarded as a flag variety. Hence, $K_0^{T^r}((\mathbf{P}^{r-1})^n)$ has a $\mathbf{Z}[u_1^{\pm1},\ldots,u_r^{\pm1}]$-module basis given by the classes of Schubert varieties, which now are products of linear spaces indexed by compositions α of length n whose parts α_i satisfy $0 \le \alpha_i \le r-1$. The Schubert variety Σ_α of α consists of those points $(p_1,\ldots,p_n)$ where $(p_i)_j = 0$ for $0 \le j \le \alpha_i$, and has codimension $\sum_i \alpha_i$. The corresponding affine analogue X_α of Σ_α clearly has rational singularities since it is a linear subspace of $\mathbf{A}^{r\times n}$. The K-class of $X_\alpha \subset \mathbf{A}^{r\times n}$ is readily computed to be $\prod_{j=1}^{n}\prod_{i=1}^{\alpha_j}(1-u_it_j)$. When we work in $K_0^{T^r}((\mathbf{P}^{r-1})^n)$, we have

$$[\mathcal{O}_{p(X_v)}] = \sum_\alpha d_\alpha[\mathcal{O}_{\Sigma_\alpha}]$$

to which we apply [AGM11, Corollary 5.1]. Since the codimension of $p(X_v)$ is $r(n-r)-(n-e)$, we conclude that $(-1)^{r(n-r)-(n-e)-\sum_i\alpha_i}d_\alpha$ is a polynomial in u_2/u_1-1, u_3/u_2-1, $\ldots u_r/u_{r-1}-1$ with positive integer coefficients. Our goal is to lift this result to $\mathcal{K}(k[X_v])$, following the proof of Theorem 6.2.

Let q_i be the projection of $\mathbf{A}^{r\times n}\times(\mathbf{P}^{r-1})^n$ to its ith factor, and let $\mathcal{E} \subset \mathbf{A}^{r\times n}$ be the vector bundle of Section 4. As before if $X \subset \mathbf{A}^{r\times n}$ has rational singularities (and does not lay in the space of matrices with a zero column) then $q_{1*}\left([\mathcal{E}]\cdot q_2^*[\mathcal{O}_{p(X)}]\right) = [\mathcal{O}_X]$. Then, as all the varieties below have rational singularities,

$$\begin{aligned}[\mathcal{O}_{X_v}] = q_{1*}\left([\mathcal{E}]\cdot q_2^*[\mathcal{O}_{p(X_v)}]\right) &= q_{1*}\left([\mathcal{E}]\cdot q_2^*\sum_\alpha d_\alpha[\mathcal{O}_{\Sigma_\alpha}]\right)\\ &= \sum_\alpha d_\alpha q_{1*}\left([\mathcal{E}]\cdot q_2^*[\mathcal{O}_{\Sigma_\alpha}]\right) = \sum_\alpha d_\alpha[\mathcal{O}_{X_\alpha}].\end{aligned}$$

The result follows. □

Example 9.4. We continue the above example with a generic 2-by-4 matrix v and expand

$$\mathcal{K}(k[X_v]) = 1 - u_1^2u_2^2t_1t_2t_3t_4,$$

as described in Proposition 9.3. We have

$$1 - u_1^2u_2^2t_1t_2t_3t_4 = 1 - (u_2/u_1)^2 \sum_{\alpha}(-1)^{|\alpha|-1} \prod_{j:\alpha_j>0}(1 - u_1t_j)$$

where the sum is over compositions α, with $\alpha_i \leq 1$ for all i. The coefficient of $1 = [\mathcal{O}_{X_{(0,0,0,0)}}]$ on the right hand side is $-((u_2/u_1)^2 - 1)$, and the remaining coefficients are greater by unity, $\pm(u_1/u_2)^2$. Since

$$(u_2/u_1)^2 - 1 = (u_2/u_1 - 1)^2 + 2(u_2/u_1 - 1),$$

we have verified the proposition.

9.2. **Positivity conjectures for all matroids.** We begin by conjecturing a strengthening of Proposition 9.1 for all matroids.

Conjecture 9.5. *Let M be a matroid of rank r on n elements with e connected components. Write $\mathcal{K}(M)$ in terms of double Grothendieck polynomials,*

$$\mathcal{K}(M) = \sum_{\lambda:\lambda_1\leq n-r} c_\lambda(t)\mathfrak{S}_\lambda(u,t).$$

Then, the Laurent polynomials $(-1)^{r(n-r)-(n-e)-|\lambda|}c_\lambda(t) \in \mathbf{Z}[t_1^{\pm1},\ldots,t_n^{\pm1}]$ can be expressed (non-uniquely) as a square-free polynomial in the quantities $t_j/t_i - 1$, $1 \leq i < j \leq n$, with positive integer coefficients.

This property has been observed to hold for K-theoretic Littlewood-Richardson coefficients [PY17, Corollary 1.5], although no geometric reason is available at this time.

Example 9.6. We continue Example 9.2, where $M = U_{2,4}$. It suffices to see that $t_1^{-1}t_2^{-1}t_3t_4$ can be a written as a square-free polynomial in $t_j/t_i - 1$, $1 \leq i < j \leq 4$, with non-negative integer coefficients, since all other coefficients evidently already have the desired property. We have, for example,

$$t_1^{-1}t_2^{-1}t_3t_4 = (t_4/t_2 - 1)(t_3/t_1 - 1) + (t_3/t_1 - 1) + (t_4/t_2 - 1) + 1,$$

so the conjecture holds in this case.

Of course, Conjecture 9.5 implies the following slightly weaker version, with the same conclusion as Proposition 9.1.

Conjecture 9.7. *Let M be a matroid of rank r on n elements with e connected components. Write $\mathcal{K}(M)$ in terms of double Grothendieck polynomials,*

$$\mathcal{K}(M) = \sum_{\lambda:\lambda_1 \le n-r} c_\lambda(t)\mathfrak{G}_\lambda(u,t).$$

Then, the Laurent polynomials $(-1)^{r(n-r)-(n-e)-|\lambda|}c_\lambda(t) \in \mathbf{Z}[t_1^{\pm 1},\ldots,t_n^{\pm 1}]$ can be expressed as a polynomial in the quantities $t_{i+1}/t_i - 1$, $1 \le i < n$, with positive integer coefficients.

The existence of the decomposition in the conjecture follows from the argument in Theorem 8.3. Indeed, $\mathcal{K}(M)$ is a valuative matroid invariant and hence it suffices to prove the existence of the decomposition for matroids realizable over $\mathbf{C}$, which we did in Proposition 9.1. In general, when writing $\mathcal{K}(M)$ as a $\mathbf{Z}[t_1,\ldots,t_n]$-linear combination of K-classes of matroids realizable over $\mathbf{C}$ one loses control on the signs of coefficients involved, so valuativity does not immediately reduce the conjecture to the realizable case.

We now state the non-equivariant analogue of these results. Recall that the matrix Schubert variety $X_\lambda \subset \mathbf{A}^{r\times n}$ has its class in $K_0^{\mathrm{GL}_r}(\mathbf{A}^{r\times n})$ equal to the Grothendieck polynomial $\mathfrak{G}_\lambda(u)$. This polynomial represents the class of the Schubert variety $\Omega_\lambda \subset Gr(r,n)$ in $K_0(Gr(r,n))$. These polynomials satisfy $\mathfrak{G}_\lambda(u) = \mathfrak{G}_\lambda(u,(1,\ldots,1))$, since the restriction $K_0^G(\mathbf{A}^{r\times n}) \to K_0^{\mathrm{GL}_r}(\mathbf{A}^{r\times n})$ is obtained by setting $t_1 = \cdots = t_n = 1$. Indeed, evaluating all $t_i = 1$ in the above conjectures yields the following.

Conjecture 9.8. *Let M be a matroid of rank r on n elements with e connected components. Write $\mathcal{K}(M)|_{t_1=\cdots=t_n=1}$ in terms of Grothendieck polynomials,*

$$\mathcal{K}(M)|_{t_1=\cdots=t_n=1} = \sum_{\lambda:\lambda_1\le n-r} c_\lambda \mathfrak{G}_\lambda(u).$$

Then, the constants $(-1)^{r(n-r)-(n-e)-|\lambda|}c_\lambda$ are non-negative integers.

We now explain how these conjectures imply analogous results on G-equivariant Chow classes. Let $A_G^\bullet(\mathbf{A}^{r\times n})$ be the G-equivariant Chow ring of $\mathbf{A}^{r\times n}$. Since $\mathbf{A}^{r\times n}$ is an affine space, one may write

$$A_G^\bullet(\mathbf{A}^{r\times n}) = \mathbf{Z}[u_1,\ldots,u_r,t_1,\ldots,t_n]^{S_r}$$

and the class of the equivariant coherent sheaf associated to an R-module N is its multidegree $\mathcal{C}(N)$ [KMS06, Proposition 1.9]. There is a general procedure to extract $\mathcal{C}(N)$ from the K-polynomial $\mathcal{K}(N)$ which we now describe.

Proposition 9.9. *Let N be a G-equivariant coherent R-module whose class in $K_0^G(\mathbf{A}^{r\times n})$ is represented by its K-polynomial $\mathcal{K}(N)$. To obtain the class of $\mathcal{C}(N)$ in the equivariant Chow ring $A_G^\bullet(\mathbf{A}^{r\times n})$, replace each u_i with $1-u_i$,* *each t_j with $1-t_j$, expand the resulting formal power series and then gather the lowest non-zero term, which will be of degree* $\operatorname{codim}(N)$*.*

Proof. The G-equivariant K and Chow classes of N are, by definition, equal to the T-equivariant K and Chow classes of N where T is the maximal torus of G. (The G-equivariant K and Chow rings are equal to the subring of Weyl group invariants in T-equivariant K and Chow rings.) The result now follows from Proposition 1.9 and the definition of the multidegree in [KMS06]. See also [MS05, Section 8.5]. □

The G-equivariant Chow class of X_v was studied in [BF17, LPST20]. For each matroid M of rank r on ground set $\{1,2,\ldots,n\}$, define a rational function by the formula

$$\mathcal{C}(M) = \sum_{w\in S_n}\prod_{j\notin B(w)}\prod_{i\in[r]}(u_i+t_j)\cdot\prod_{i=1}^{n-1}\frac{1}{t_{w_{i+1}}-t_{w_i}}.$$

Since each summand in the definition of $\mathcal{K}(M)$ is homogeneous, $\mathcal{C}(M)$ is obtained from $\mathcal{K}(M)$ by applying the substitutions $u_i\mapsto 1-u_i$ and $t_j\mapsto 1-t_j$ and extracting the terms of smallest possible degree after simplifying. By Proposition 9.9, we have the following result.

Proposition 9.10. *If $v\in\mathbf{A}^{r\times n}$ is a realization of a rank r matroid M on n elements then $\mathcal{C}(M)$ equals the class of X_v in the G-equivariant Chow ring of $\mathbf{A}^{r\times n}$.*

We remark that this is exactly the statement of [LPST20, Theorem 1], which obtains the result through entirely different means.

The G-equivariant Chow class of the matrix Schubert variety X_λ is equal to the double Schur polynomial $s_\lambda(u,t)$, which is a special case of a double Schubert polynomial for Grassmannian permutations [KM05, Theorem A] (see also [KMY09]). It is equal to the lowest degree term of $\mathfrak{S}_\lambda(1-u,1-t)$ after simplification. The following conjecture is the result of applying Proposition 9.9 to Conjecture 9.5.

Conjecture 9.11. *Let M be a matroid of rank r on n elements with e connected components. Write $\mathcal{C}(M)$ in terms of double Schur polynomials,*

$$\mathcal{C}(M) = \sum_{\lambda:\lambda_1\le n-r} d_\lambda(t)s_\lambda(u,t).$$

Then, the polynomial $(-1)^{r(n-r)-(n-e)-|\lambda|}d_\lambda(t)\in\mathbf{Z}[t_1,\ldots,t_n]$ can be expressed (non-uniquely) as a square-free polynomial in the quantities t_i-t_j, $1\le i<j\le n$ with non-negative integer coefficients.

The following weaker result holds for all matroids M realizable over $\mathbf{C}$ by Proposition 9.1, or by [Gra01, Theorem 3.2].

Conjecture 9.12. *Let M be a matroid of rank r on n elements with e connected components. Write $\mathcal{C}(M)$ in terms of double Schur polynomials,*

$$\mathcal{C}(M) = \sum_{\lambda:\lambda_1 \leq n-r} d_\lambda(t) s_\lambda(u,t).$$

Then, the polynomial $(-1)^{r(n-r)-(n-e)-|\lambda|} d_\lambda(t) \in \mathbf{Z}[t_1,\ldots,t_n]$ can be expressed as a polynoimal in the quantities $t_i - t_{i+1}$, $1 \leq i < j \leq n$ with non-negative integer coefficients.

We finally present the weakest but most accessible version of our conjectures by passing from the G-equivariant Chow ring of $\mathbf{A}^{r\times n}$ to the GL_r-equivariant Chow ring. It follows from the above conjecture by $t_1 = \cdots = t_n = 0$.

Conjecture 9.13. *Let M be a matroid of rank r on n elements with e connected components. Write $\mathcal{C}(M)|_{t_1=\cdots=t_n=0}$ in terms of Schur polynomials,*

$$\mathcal{C}(M)|_{t_1=\cdots=t_n=0} = \sum_{\lambda:\lambda_1\leq n-r, |\lambda| = r(n-r)-(n-e)} d_\lambda s_\lambda(u).$$

Then, the constants d_λ are non-negative integers.

Example 9.14. If $M = U_{2,4}$ then $\mathcal{K}(M) = 1 - u_1^2 u_2^2 t_1 t_2 t_3 t_4$. It follows that $\mathcal{C}(M) = 2u_1 + 2u_2 + t_1 + t_2 + t_3 + t_4$. We write

$$\mathcal{C}(M) = 2s_{(1,0)}(u,t) - (t_1 + t_2 - t_3 - t_4) s_{(0,0)}(u,t).$$

Since $-(t_1 + t_2 - t_3 - t_4) = (-1)((t_1 - t_3) + (t_2 - t_4))$, $\mathcal{C}(M)$ satisfies Conjecture 9.11. Setting $t_1 = \cdots = t_4 = 0$ we see at once that Conjecture 9.13 is satisfied as well.

Remark 9.15. The coefficients that appear in Conjecture 9.13 are related to the Schur classes of tautological bundles of matroids studied in [BEST21]. When M is connected they are degrees of such Schur classes.

Let Y be the torus orbit closure of a point in $Gr(r,n)$ with matroid M. Write $y^T(M)$ for the class of Y in $A_T^\bullet(Gr(r,n))$, which is represented by the multidegree $\mathcal{C}(M)$ in the appropriate quotient of $A_G^\bullet(\mathbf{A}^{r\times n})$. Let $y(M)$ denote the image of $y^T(M)$ in the ordinary (non-equivariant) Chow ring of $Gr(r,n)$. Assume M is connected, i.e., Y has dimension $n-1$.

The permutohedral toric variety X_{A_n} is a toric resolution of singularities of Y. For a class $\xi \in A^\bullet(Gr(r,n))$, the push-pull formula for this resolution says that

$$\deg_{Gr(r,n)}(y(M)\,\xi) = \deg_{X_{A_n}}(\xi_M)$$

where equivariant localization yields the following description of ξ_M. Let $\xi^T \in A^\bullet_T(Gr(r,n))$ be an equivariant lift of ξ. Given an r-subset B of $[n]$, let $\xi^T_B \in \mathbf{Z}[t_1,\dots,t_n]$ denote the localization of ξ^T at the T fixed point of $Gr(r,n)$ indexed by B. For a matroid M of rank r on $[n]$, the collection of polynomials $\xi^T_M = (\xi^T_{B(w)})_{w\in S_n}$ defines a class in the T-equivariant Chow ring of X_{A_n}, and ξ_M is the corresponding ordinary class. This is the Chow-theoretic analogue of, and is implied by, [BEST21, Proposition 3.13]. The argument in [BEST21, Lemma 10.9] extends this to non-realizable matroids.

We may uniquely write

$$\mathcal{C}(M)|_{t_1=\cdots=t_n=0} = \sum_{\lambda:\lambda_1\le n-r, |\lambda|=r(n-r)-(n-1)} d_\lambda s_\lambda(u).$$

Then Corollary 6.3 implies that

$$y(M) = \sum_{\lambda:\lambda_1\le n-r, |\lambda|=r(n-r)-(n-1)} d_\lambda[\Omega_\lambda] \in A^\bullet(Gr(r,n))$$

for the same constants d_λ. For an (equivariant) vector bundle $\mathcal{E}$ let $s_\lambda(\mathcal{E})$ denote the Schur polynomial evaluated at its (equivariant) Chern roots. There is an equality of non-equivariant classes $[\Omega_\lambda] = s_\lambda(\mathcal{S}^\vee)$ in $A^\bullet(Gr(r,n))$. Let λ^* denote the partition corresponding to the complement of λ in a r-by-$(n-r)$ box, so that $[\Omega_\lambda][\Omega_{\mu^*}] = \delta_{\lambda\mu}$. Assume μ is a partition of $n-1$. Pairing $y(M)$ with $[\Omega_\mu]$ and taking degree gives

$$\begin{aligned} d_{\mu^*} = \deg_{Gr(r,n)}(y(M)[\Omega_\mu]) &= \deg_{Gr(r,n)}(y(M)s_\mu(\mathcal{S}^\vee)) \\ &= \deg_{X_{A_{n-1}}}(s_\mu(\mathcal{S}^\vee)_M) = \deg_{X_{A_{n-1}}}(s_\mu(\mathcal{S}|_M^\vee)), \end{aligned}$$

where $\mathcal{S}|_M$ is the tautological sub-bundle of the matroid M from [BEST21].

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