A noncooperative foundation of the competitive divisions for bads*

Marco Mariotti[a]

[a]*School of Economics & Finance, Queen Mary University of London*

Mile End Road, London E1 4NS, UK

E-mail: m.mariotti@qmul.ac.uk

Quan Wen[b,†]

[b]*Department of Economics, University of Washington, Box 353330*

Seattle, WA 98195-3330, USA

E-mail: wenq2@uw.edu

First version: February 2020
Revised version: September 2020
Final version: January 2021

Abstract

Many economic situations involve the division of bads. We study a noncooperative game model for this type of division problem. The game resembles a standard multilateral bargaining model, but in our case, perpetual disagreement is not a feasible outcome. The driving feature of the model is that a player that makes an unacceptable proposal (causing breakdown with some probability) is made to internalize all the costs in case of breakdown. We show that as the probability of exogenous breakdown goes to zero, this game implements some *competitive divisions* in Markov perfect equilibria: the limit of any convergent sequence of equilibrium outcomes is a competitive division, but a competitive division may not be a limit of the equilibrium outcomes.

JEL Classification: C71; C72, C73; C78

Keywords: Competitive division; Mixed manna; Markov perfect equilibrium

*We thank Kalyan Chatterjee, Hülya Eraslan, Jean-Jacques Herings, Maria Montero, Akira Okada, Karl Shell, William Thomson, John Weymark, an associate editor, and two anonymous referees for their comments and suggestions. Usual disclaimer applies.

†Corresponding author.

1 Introduction

The competitive equilibrium with equal income of Varian (1974), also simply known as the *competitive division*, is a prominent solution concept to division problems for private goods, which satisfies several desirable theoretical and practical properties. Gale (1960), Eisenberg (1961), and others show that, under a certain class of utility functions, the competitive division of goods can be simply obtained by solving a maximization problem of the Nash product function of the individuals' utilities. This result, which is often called the Gale-Eisenberg theorem, has been recently extended by Bogomolnaia, Moulin, Sandomirskiy and Yanovskaya (2017, 2019) to division problems of "mixed manna", which contain both goods and bads.[1] They show that for a "positive" problem where the amount of the goods dominates the amount of the bads (in the sense that individuals' utility profile when no one receives anything is Pareto dominated by some feasible utility profiles), there is a unique competitive division utility profile that maximizes the Nash product of individuals' utilities over the set of feasible utility profiles. This result provides a striking link between a "physical" concept and a "welfarist" concept of fair division. For a "negative" problem, where the utility profile when no one receives anything is not Pareto dominated by any feasible utility profile, Bogomolnaia et al. (2017) show that any competitive division is a critical point of the Nash product of individuals' disutilities over the Pareto frontier of feasible utility profiles. As a consequence, while the link between physical and welfarist concepts is preserved in a negative division problem, there could be multiple competitive divisions with distinct utility profiles.

In this paper, we are interested in providing a non-cooperative game foundation for the competitive division, thus providing a non-cooperative implementation mechanism for this concept. For a positive problem, the Gale-Eisenberg theorem implies that the competitive division utility profile is identical to the Nash (1950) bargaining solution, and there is a large

[1] We refer readers to their papers for many real-life division problems of mixed manna, such as dissolving partnership, bankruptcy, and cost sharing.

body of non-cooperative bargaining literature that implements the Nash bargaining solution, such as Rubinstein (1982) and some others that we will discuss later. However, the same cannot be said for the competitive divisions of a negative problem. Using the characterization by Bogomolnaia et al. (2017) of the competitive divisions for negative problems, we introduce a non-cooperative bargaining model to solve a negative problem. We show that a feasible utility profile is a competitive utility profile if it is the limit of any convergent sequence of Markov perfect equilibrium payoffs as bargaining friction vanishes.

The non-cooperative bargaining model that we study in this paper resembles the "alternating offer" bargaining games that require unanimous agreement with an exogenous probability of breakdown, as in Binmore et al. (1986). There are (potentially) infinitely many periods, and in each period before a settlement is reached one player makes a proposal in the set of feasible utility profiles, corresponding to a feasible division of the mixed manna. There is no free disposal and all the manna must be allocated to the players. If the proposal is accepted unanimously by the other players, then the problem is solved and the accepted proposal is implemented. Otherwise, the bargaining game may break down with some exogenous probability. If the bargaining does not break down, another player will propose in the following period.

The peculiarity of our model, compared to standard bilateral/multilateral bargaining models, lies in the outcome in the case of exogenous breakdown after any standing offer is rejected. Unlike in a positive problem, there are no "disagreement payoffs" that result from perpetual disagreement, since not allocating the mixed manna is not a feasible option in a negative problem. After bargaining exogenously breaks down, the proposing player will take all the mixed manna and each responding player does not receive anything. How to allocate the mixed manna after breakdown depends on the identity of the proposer who made the latest rejected proposal. Note that, in a negative problem, receiving all the mixed manna is the worst possible outcome for any player. Therefore, this player receives maximum punishment for making a "wrong" offer.

Even with a convex set of feasible utility profiles, our bargaining mechanism for a negative problem has multiple subgame perfect equilibrium outcomes in general. We show that as the probability of breakdown vanishes, the limit of a convergence sequence of Markov perfect equilibrium payoff vectors is a competitive utility profile, but not vice versa. In this sense, our bargaining mechanism provides a non-cooperative foundation for some competitive divisions in negative problems.

This paper mainly contributes to the literature on how to support or implement various cooperative solutions non-cooperatively. Rubinstein (1982), Binmore et al. (1986) provide several bilateral bargaining models where the unique subgame perfect equilibrium converges to the Nash (1950) bargaining solution. Miyagawa (2002) introduces a four-stage game model that implements a large class of two-person bargaining solutions in subgame perfect equilibrium. Early studies, such as Chae and Yang (1994), show that multilateral versions of the alternating-offer or random-proposer bargaining models typically have multiple subgame perfect equilibrium outcomes. However, there is a unique subgame perfect equilibrium outcome in the multilateral bargaining model of Krishna and Serrano (1996) and the multi-agent bilateral bargaining models of Suh and Wen (2006), and they all converge to the Nash bargaining solution or the competitive division of the corresponding positive division or bargaining problem due to the Gale-Eisenberg theorem. Britz et al. (2014) study an action-dependent multilateral bargaining protocol where its stationary subgame perfect equilibrium outcome also converges to the Nash bargaining solution or the competitive division in a positive problem. More recently, Brügemann et al. (2019) examine a Rolodex game to support the Shapley values of a general cooperative game. The competitive division and the Nash rationing solution are just two of many solutions for bankruptcy and taxation problems, see Thomson (2015) on a recent update on this vast literatures. Researchers also seek how to support other cooperative solutions non-cooperatively, such as Aumann and Maschler (1985) and Ashlagi et al.(2012), or how to solve a division problem non-cooperatively, such as Stiglitz (2019), O'Neil (2009) and García-Jurado et al, (2006).

The rest of this paper is organized as follows. Section 2 presents the competitive divisions of bads and the main result of Bogomolnaia et al.(2017). In Section 3, we introduce a non-cooperative bargaining model with probabilistic breakdown to solve a negative division problem and establish the existence of a Markov perfect equilibrium in this mechanism. Our main result is presented in Section 4: a feasible utility profile is a competitive profile if it is the limit of a convergent sequence of Markov perfect equilibrium payoffs as the probability of breakdown goes to zero. We demonstrate that a competitive profile may not be the limit of Markov perfect equilibrium payoffs of our game model. Section 5 offers some concluding remarks. All proofs are provided in the Appendix.

2 Preliminaries

A set of $n \geq 2$ players, denoted by $N = \{1, \ldots, n\}$, divide m commodities $\omega \in \mathbb{R}^m_+$ among themselves.[2] An allocation, denoted by $(z_1, \ldots, z_n)$, is feasible if $z_1 + \cdots + z_n = \omega$, where $z_i \geq 0$ denotes player i's consumption bundle. For all $i \in N$, player i's utility function $u_i(\cdot)$ is assumed to be strictly monotonically decreasing, continuous, concave, and homogenous of degree 1. As we will focus on negative problems, where all the commodities are bads to all the players, $u_i(z_i) \leq u_i(0)$ for all $i \in N$ and for all $z_i \in \mathbb{R}^m_+$.

A feasible allocation $(z_1, \ldots, z_n)$ is a *competitive division* if there exists a negative price vector $p \ll 0$ such that for all $i \in N$,[3]

$$z_i \in d_i(p) = \arg\max u_i(y_i) \quad \text{subject to } y_i \in B(p) = \{y \geq 0 \mid p \cdot y \leq -1\}, \tag{1}$$

$$z_i \in \arg\min p \cdot y \quad \text{subject to } y \in d_i(p). \tag{2}$$

Condition (1) is the usual demand property, while (2) requires that every player $i \in N$ spends as little as possible in her demand set at the competitive division bundle z_i. Note that the budget set $B(p) = \{y \geq 0 \mid p \cdot y \leq \beta\}$ here is given in a normalized form with $p \ll 0$

[2] Our model description focuses on a negative problem, where the utility profile when no one receives anything is not Pareto dominated by any feasible utility profile, such as when all the commodities are bad to all the players. The dimensionality of the commodity space is not particularly important. We refer readers to Bogomolnaia et al (2017) for a more complete and detailed description of a general division problem.

[3] Here $p = (p_1, \ldots, p_m) \ll 0$ means that $p_k < 0$ for all $k = 1, \ldots, m$.

and $\beta = -1$ for a negative problem, and with $p \gg 0$ and $\beta = 1$ for a positive problem. In a negative problem where all commodities are bads, every player prefers consuming less to more and the value of a consumption bundle is a negative number due to $p \ll 0$. So every player maximizes her utility on a budget set that is different from the standard budget set for goods.

A competitive division satisfies several desirable properties, such as efficiency and non-envy, even in the case when all the commodities are bads to all the players. Bogomolnaia et al.(2017) provide a constructive method to show the existence of competitive division in a general problem. Their characterization of a competitive division is given in terms of players' utilities corresponding to a competitive division allocation, called a competitive utility profile or simply a competitive profile. Accordingly, let

$$S = \big\{(u_1(z_1), \ldots, u_n(z_n)) \in \mathbb{R}^n_- : z_1 + \cdots + z_n = \omega\big\} \tag{3}$$

denote the set of all feasible utility profiles. Due to the concavity and monotonicity of players' utility functions and $\omega > 0$, set S is a convex subset of $\mathbb{R}^n_-$ and $u(0) \notin S$. Since all competitive allocations are Pareto efficient, we can focus on the set of feasible and Pareto efficient utility profiles, denoted by

$$P(S) = \{s \in S : \; s \ll s' \Rightarrow s' \notin S\}.$$

Unlike a positive division problem, a negative problem could be multiple competitive allocations for bads, and hence multiple competitive utility profiles. By convention, we often denote a utility profile as $s = (s_i, s_{-i})$ for all $i \in N$.

We now restate the following key characterization of Bogomolnaia et al.(Theorem 1, 2017) for the competitive utility profiles in a negative problem for later reference. Given an arbitrary smooth function $g(\cdot)$ and a closed convex set C, $x \in C$ is called a *critical point* of function $g(\cdot)$ in set C if the upper contour set of function $g(\cdot)$ at x has a supporting hyperplane that also supports set C. Note that any local maximum or a local minimum of function $g(\cdot)$ on the boundary of set C must be a critical point of function $g(\cdot)$ in set C, but not vice versa.

Lemma 1 *A feasible allocation z is a competitive allocation if and only if its corresponding utility profile $s = (u_1(z_1), \dots, u_n(z_n)) \in P(S)$ such that $s \ll u(0)$ and s is a critical point of the Nash product function $\prod_{i \in N} |x_i - u_i(0)|$ in set S.*

Lemma 1 implies that any local maximum or local minimum of the Nash product function $\prod_{i \in N} |x_i - u_i(0)|$ over the set of Pareto efficient utility profiles $P(S)$ must be a competitive utility profile. However, a later example demonstrates that a competitive utility profile may be neither a local maximum nor a local minimum of the Nash product function on $P(S)$. At any local maximum or local minimum of the Nash product function on $P(S)$, the corresponding contour set of the Nash product function $\prod_{i \in N} |x_i - u_i(0)|$ is tangent to $P(S)$ either from above or from below locally around the competitive utility profile, but these two sets cannot cross each other. Lemma 1 inspires the non-cooperative bargaining model that we will study next to solve a negative division problem non-cooperatively, where the limit of any convergent sequence of Markov perfect equilibrium payoffs is a competitive utility profile.

3 A Non-Cooperative Bargaining Model to Divide Bads

In this section, we introduce and study a non-cooperative bargaining game where the players take turns to propose how to divide the bads ω among themselves; or, equivalently, to achieve a feasible utility profile in S. There are potentially infinitely many periods, denoted by $t = 1, 2, \dots$. In period $t = 1$, player 1 proposes a feasible allocation with utility profile, denoted by $s^1 \in S$. Then all the other $n-1$ players simultaneously decide whether to accept or reject player 1's proposal $s^1 \in S$. If player 1's proposal s^1 is accepted unanimously, then player 1's proposal s^1 will be implemented immediately. Otherwise, the game will either continue, with probability $\rho \in (0, 1)$, to period 2 where player 2 will propose, or end with probability $1 - \rho$, in which case player 1 will receive all the bads ω and every other player will receive 0. In period $t = kn + i$ for some non-negative integer k and $i \in N$, player i proposes a feasible allocation with utility profile $s^i \in S$, and the other $n-1$ players $N \backslash \{i\}$ simultaneously decide whether to accept player i's proposal s^i. Again, if player i's proposal $s^i \in S$ is accepted unanimously by all the other $n-1$ players, then s^i will be implemented immediately in period t. Otherwise, i.e., after player i's proposal is rejected by some of the other players, there are two possible continuations similar to those after period 1. With probability $\rho \in (0,1)$, the game will continue to the following period $t+1$ where player $i+1$ (or player 1 if $i = n$) will make a new proposal $s^{t+1} \in S$, and with probability $1-\rho$ the game will be terminated exogenously. In the latter case, every responding player $j \neq i$ will receive nothing with payoff $u_j(0)$, while the proposing player i will receive all the bads ω with payoff $u_i(\omega) < 0$. In other words, in this game the proposing player bears all the responsibility if he makes a "wrong" proposal that is rejected. Considering that $u_i(\omega)$ is player i's worst "possible" payoff when player i takes all the bads ω, we can focus on the utility profile $s \in S$ such that $s_i \in [u_i(\omega), u_i(0)]$ for all $i \in N$. Note that any utility profile after the game is terminated exogenously is always feasible and efficient because only one player will receive all the bads ω. More specifically, after player i's proposal is rejected and the game is terminated, the utility profile is

$$(u_1(0), \ldots, u_i(\omega), \ldots, u_n(0)) \in P(S).$$

The model described so far is a well-defined non-cooperative game of complete information. Histories and strategies are defined in the usual fashion. For example, a history at the beginning of a period consists of all past rejected proposals, the identities of those who rejected those proposals, etc. The identity of the proposing player in a period is deterministic; it does not depend on the history, but what proposal the proposing player makes and how the other responding players respond to a proposal could all depend on the entire history prior to the current period. After any finite history where no settlement has been reached and the game has not been terminated exogenously, a strategy profile specifies the proposing player' proposal and the responding players' responses to all the possible proposals. Players' payoffs in such a dynamic game are assumed to be their expected payoffs, because of the possibility of probabilistic termination. For example, from an outcome with no agreement in the first period and an agreement $s^* \in S$ in the second period if the bargaining is not terminated in the first period, player 1's expected payoff is given by $(1-\rho)\, u_1(\omega) + \rho s_1^*$ and every other player $j \neq 1$ has an expected payoff of $(1-\rho)\, u_j(0) + \rho s_j^*$. There is no time discount between periods, but having a time discount will not change our analysis and result much. Completely specifying all players' expected payoffs from an arbitrary outcome can be not only extremely involved but also unnecessary for our main analysis.

As in most non-cooperative multilateral bargaining models with either convex or non-convex sets of possible agreements, our bargaining game also has multiple subgame perfect equilibria in general. As we have learned from the non-cooperative bargaining literature, subgame perfect equilibria in these models can be involved with complicated strategies, including possible delays in reaching an agreement/settlement.[4] Given our main objective in this paper, we will focus on a simple class of Markov strategy profiles where all players' behavioral strategies in any period depend on the identity of the current proposing player only. Viewing this game as an implementation mechanism, this can be considered as a design feature. Accordingly, a Markov strategy profile can be described by n proposals $\{s^1, \ldots, s^n\} \subset S$, where player i will always propose $s^i \in S$ whenever player i is the proposing player, and accept any player j's proposal if and only if his utility in player j's proposal is not less than s_i^j for all $j \neq i$. Consequently, player i's proposal s^i will be accepted by all the other players $N \backslash \{i\}$. Note that these n equilibrium proposals depend on the value of $\rho \in (0,1)$. A subgame perfect equilibrium is a *Markov perfect equilibrium* (MPE) if the equilibrium strategy profile is a Markov strategy profile as defined.

We now derive the necessary and sufficient conditions for a Markov strategy profile $\{s^1, \ldots, s^n\} \subset S$ to be a subgame perfect equilibrium and establish the existence of a MPE.

[4] Without time discount, delay in reaching an agreement does not lead to any waste in allocating the estate. However, delay is still Pareto inefficient due to players' risk aversion.

For exposition purposes, denote $f_i(\cdot) : \times_{j\neq i} [u_j(\omega), u_j(0)] \to [u_i(\omega), u_i(0)]$ for all $i \in N$ as

$$f_i(s_{-i}) = \max s_i \quad \text{subject to}$$
$$(s_i, s_{-i}) \in \{s \in \times_{j\in N} [u_j(\omega), u_j(0)] : s \leq s' \text{ for some } s' \in S\}.$$

When the other players receive s_{-i}, player i could obtain at most $f_i(s_{-i})$. Observe that $f_i(s_{-i}) = u_i(0)$ when $s_j \leq u_j(z_j)$ for $j \neq i$ and $\sum_{j\neq i} z_j \geq \omega$, and utility profile $(f_i(s_{-i}), s_{-i})$ is Pareto efficient, i.e., $(f_i(s_{-i}), s_{-i}) \in P(S)$, when $s_j = u_j(z_j)$ for $j \neq i$ and $\sum_{j\neq i} z_j \leq \omega$. Because S is closed and convex, every $f_i(\cdot)$ is well-defined, weakly monotonically decreasing, concave, and continuous for all $s_{-i} \in \times_{j\neq i} [u_j(\omega), u_j(0)]$.

The following proposition provides the necessary and sufficient conditions for a MPE:

Proposition 1 *The Markov strategy profile with proposals $\{s^1, \ldots, s^n\} \subset S$ constitutes a subgame perfect equilibrium, i.e., a MPE, if and only if for all i and $j \in N$,*

$$s^i_i = f_i(s^i_{-i}) \quad \textit{and} \quad s^i_j = (1 - \rho^{\langle j-i\rangle})u_j(0) + \rho^{\langle j-i\rangle} s^j_j, \tag{4}$$

$$\textit{where} \quad \langle j - i\rangle = \begin{cases} j - i, & \textit{if } j - i \geq 0, \\ n + j - i, & \textit{otherwise.} \end{cases} \tag{5}$$

Note that $\langle j - i\rangle$ defined by (5) is the number of periods that player j will propose after player i's proposal is rejected. The second equation in (4), $s^i_j = (1 - \rho^{\langle j-i\rangle})u_j(0) + \rho^{\langle j-i\rangle} s^j_j$, gives player j's expected continuation payoff if player j rejects player i's proposal. Also note that, as an important implication of the $f_i(\cdot)$ function, equation (4) implies that either $s^i \in P(S)$ or $s^i_i = u_i(0)$. In such a MPE, the proposing player i will propose utility profile s^i such that every other player j receives his expected continuation payoff and player i claims the remainder $s^i_i = f_i(s^i_{-i})$, as stated in (4). In the proof of Proposition 1, we will show that in any period where player i proposes, player i will indeed propose s^i and every player $j \in N\backslash\{i\}$ will accept any player i's proposal if and only if player j's utility in player i's proposal is not less than s^i_j.

Our next proposition establishes the existence of a solution (a fixed point) to equation system (4), and hence the existence of a MPE in our non-cooperative bargaining game. In addition, Proposition 2 shows that in any MPE, every player $i \in N$ must receive a payoff that is between $u_i(\omega)$ and $u_i(0)$ for all $\rho \in (0,1)$.

Proposition 2 *Equation system (4) admits at least one solution. Moreover, at any solution* $\{s^1, \ldots, s^n\}$ *to (4), we have* $u_i(\omega) < s_i^i < u_i(0)$ *and* $s^i \in P(S)$ *for all* $i \in N$ *and for all* $\rho \in (0,1)$.

Although Proposition 2 establishes the existence of a MPE as a fixed point to (4), the uniqueness of a MPE is not guaranteed. As matter of a fact, there could be multiple MPE outcomes as our later examples demonstrate.

4 Competitive Utility Profiles as Limits of MPE Payoffs

In this section, we show that the limit of any convergent sequence of MPE payoffs in our non-cooperative game is a competitive utility profile as the probability of exogenous breakdown goes to zero. For expositional simplicity and without loss of generality, we can normalize a problem so that $u_i(0) = 0$ for all $i \in N$. For a normalized problem, Lemma 1 states that $s \in P(S)$ is a competitive profile if and only if $s \ll 0$ and s is a critical point of the Nash product function $\prod_{i \in N} |x_i|$ in set S. Therefore, the most relevant part of the utility profiles to our analysis is $P(S) \cap \mathbb{R}^n_-$.

Recall that a Markov strategy profile can be represented by the n players' MPE proposals $\{s^1, \ldots, s^n\}$. Proposition 1 asserts that these n proposals form a MPE if and only if

$$\left(s^1, \ldots, s^n\right) = \begin{pmatrix} s_1^1 & \rho^{n-1} s_1^1 & \cdots & \rho s_1^1 \\ \rho s_2^2 & s_2^2 & & \rho^2 s_2^2 \\ \vdots & & \ddots & \vdots \\ \rho^{n-1} s_n^n & \rho^{n-2} s_n^n & \cdots & s_n^n \end{pmatrix}. \tag{6}$$

Because of (6), we can simply describe a MPE strategy profile by n players' "demands" $\{s_1^1, \ldots, s_n^n\}$ in a MPE, because these demands uniquely determine all the other utilities in the n equilibrium proposals.

Our next proposition asserts that as the probability of continuation goes to one, the limit of any convergent sequence of MPE payoffs is a competitive utility profile in the corresponding division problem.

Proposition 3 *Suppose that MPE payoff vector $s^1 \in P(S)$, as player 1 proposes at the beginning of the game, converges to $s^* \in P(S)$ as $\rho \to 1$. Then s^* is a competitive utility profile.*

Similar to the proof that the unique subgame perfect equilibrium in the alternating-offer bargaining game of Rubinstein (1982) converges to the Nash bargaining solution, the proof of Proposition 3 relies on the fact that all the equilibrium proposals of the same MPE have the same Nash product. As $\rho \to 1$, all these MPE proposals have the same limit, which must be a local maximum or local minimum, and hence a critical point, of the Nash product function over set $P(S)$. By Lemma 1, the limit of these MPE proposals must be a competitive utility profile.

Recall Lemma 1 and the properties of a critical point, a feasible profile s^* is a competitive utility profile if it is a local solution to

$$\max / \min \prod_{j \in N} |s_j| \qquad \text{subject to} \quad s \in P(S).$$

This implies that $P(S)$ intersects that the contour set $\left\{ s \in \mathbb{R}^n_- : \prod_{j \in N} |s_j| = \alpha \right\}$ when α is sufficiently close to $\prod_{j \in N} |s^*_j|$ from below for a local maximum or from above for a local minimum. Next, we will use these properties to show that any local maximum or local minimum of the Nash product function on $P(S)$ can be the limit of MPE payoffs when there are two players. Let $B_\varepsilon(s) = \{s' : \|s - s'\| < \varepsilon\}$ denote the ε-open ball centered at s.

Proposition 4 *Consider a competitive profile s^* of a 2-player division problem with feasible utility profiles S. If there exist $\bar{\varepsilon} > 0$ such that*

$$s^*_1 s^*_2 \geq (\textit{or} \ \leq) \ s_1 s_2 \qquad \textit{for all } s \in B_{\bar{\varepsilon}}(s^*) \cap P(S), \tag{7}$$

then, for all $\varepsilon < \bar{\varepsilon}$, there is a MPE with payoff in $B_\varepsilon(s^*) \cap P(S)$ for some $\rho \in (0, 1)$.

The proof of Proposition 4 is constructive by utilizing Lemma 1. Such a competitive utility profile, either a local maximum or a local maximum of the Nash profile function, can be approximated by a MPE payoff vector in the sense that a smaller $\varepsilon > 0$ requires a higher probability of continuation $\rho \in (0, 1)$.

To illustrate Propositions 3 and 4, we reconsider an example with two bads of Bogomolnaia et al.(2017, page 1856). The set of feasible utility profiles is the convex hull of

$$(-4, 0),\ (-1, -1),\ (-3, -2), \text{ and } (0, -3).$$

In this example, there are three competitive utility profiles, local minimum $\left(-2, -\frac{2}{3}\right)$, local maximum $(-1, -1)$, and local minimum $\left(-\frac{3}{4}, -\frac{3}{2}\right)$, as shown in the following Figure 1:

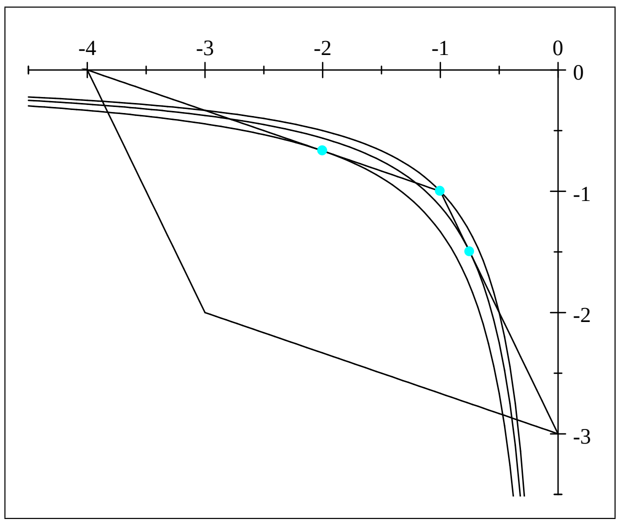

Fig. 1. Feasible utility profiles and three competitive profiles.

When ρ is sufficiently close to 1, there are three MPE outcomes in our non-cooperative bargaining game due to Proposition 1. One MPE is determined by two players' demands $s_1^1 = -\frac{4}{\rho+1}$ and $s_2^2 = -\frac{4}{3\rho+3}$. The corresponding MPE payoff $\left(-\frac{4}{\rho+1}, -\frac{4\rho}{3\rho+3}\right)$ (when player 1 proposes) converges to the competitive profile $\left(-2, -\frac{2}{3}\right)$ as $\rho \to 1$. Players' demands in the second MPE are $s_1^1 = -\frac{9\rho-4}{6\rho^2-1}$ and $s_2^2 = -\frac{8\rho-3}{6\rho^2-1}$. The corresponding MPE payoff $\left(-\frac{9\rho-4}{6\rho^2-1}, -\frac{8\rho^2-3\rho}{6\rho^2-1}\right)$ converges to the competitive utility profile $(-1, -1)$ as $\rho \to 1$. Lastly, players' demands in the third MPE are $s_1^1 = -\frac{3}{2\rho+2}$ and $s_2^2 = -\frac{3}{\rho+1}$, and the corresponding MPE payoff $\left(-\frac{3}{2\rho+2}, -\frac{3\rho}{\rho+1}\right)$ converges to the competitive utility profile $\left(-\frac{3}{4}, -\frac{3}{2}\right)$ as $\rho \to 1$.

Notice that second MPE always exists for all $\rho \in (0,1)$, while the first and third MPEs exist only when ρ is sufficiently close to 1.

This example demonstrates two interesting facts: First, when a problem has multiple competitive utility profiles, our bargaining model will have multiple MPE outcomes when ρ is sufficiently close to 1. Second, as the probability of continuation ρ goes to one, the limit of any convergent sequence of MPE outcomes of the bargaining model is a competitive utility profile of the underlying problem.

On the other hand, however, a competitive utility profile may not be approximated by any MPE outcome. To conclude this section, we present such an example.[5] There are two players $\{1,2\}$ and two commodities $\{a,b\}$ with one unit each. Two players' utility functions are given by, respectively,

$$u_1(a_1,b_1) = -2a_1 - 2b_1 \quad \text{and} \quad u_2(a_2,b_2) = -\frac{7}{3}a_2 - \frac{2}{3}b_2.$$

Consequently, the set of feasible utility profile S is the convex hull of

$$(-4,0),\ (-2,-\frac{7}{3}),\ \left(-2,-\frac{2}{3}\right),\ \text{and}\ (0,-3),$$

as illustrated in Figure 2.

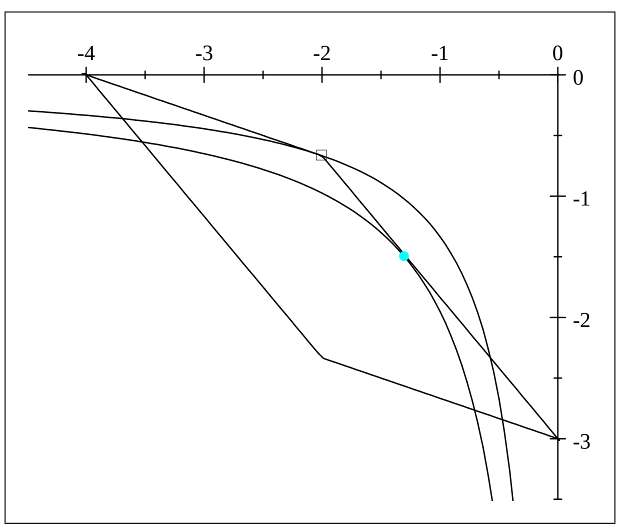

Fig. 2. Two competitive profiles but a unique limit of MPE utility profile.

[5]We would like to thank a referee for helping us to perfect this example.

One can verify that there are two competitive allocations

$$\begin{pmatrix} \frac{9}{14} & 0 \\ \frac{5}{14} & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with competitive prices $\left(-\frac{14}{9}, -\frac{4}{9}\right)$ and $(-1, -1)$, respectively. The corresponding competitive utility profiles are $\left(-\frac{9}{7}, -\frac{3}{2}\right)$ as marked by a point and $\left(-2, -\frac{2}{3}\right)$ as marked by a box in Figure 2. Note that competitive profile $\left(-\frac{9}{7}, -\frac{3}{2}\right)$ is a local minimum of the Nash product function on the Pareto frontier, but competitive profile $\left(-2, -\frac{2}{3}\right)$ is neither a local maximum nor a local minimum of the Nash product function on the Pareto frontier of set S.

The bargaining model based on this negative problem has a unique MPE for all $\rho \in (0, 1)$ based the necessary and sufficient conditions of Proposition 1. Depending on the value of $\rho \in (0, 1)$, the equilibrium proposals may be on the two linear segments of the Pareto frontier of S when ρ is less than $\frac{2}{7}$, or on the same linear segment when ρ is greater $\frac{2}{7}$. More specifically, for $\rho \in (0, \frac{2}{7}]$, the two players' demands in this unique MPE are $-\frac{18\rho-8}{7\rho^2-2}$ and $-\frac{28\rho-18}{21\rho^2-6}$, respectively. For $\rho \in (\frac{2}{7}, 1)$, however, the two players' equilibrium demands are $-\frac{18}{7\rho+7}$ and $-\frac{3}{\rho+1}$, respectively. As $\rho \to 1$, this unique MPE utility profile converges to the competitive profile $\left(-\frac{9}{7}, -\frac{3}{2}\right)$. This MPE profile skips the competitive utility profile $\left(-2, -\frac{2}{3}\right)$ as ρ passes $\frac{2}{7}$. Again, this example demonstrates that the converse statement of Proposition 3 is not true.

5 Concluding Remarks

Classical literature (Gale, 1960; Eisenberg, 1961) has shown a striking correspondence between a "physical" solution concept to division problems for goods, the competitive division, and a welfarist solution concept, the maximization of the Nash product in utility space. Bogomolnaia et al. (2017, 2019) have recently extended the analysis to the case of "bads" (mixed manna). While in such a case, the link between physical and welfarist concept is retained, it is somewhat more complicated due to the presence of multiplicities. This case poses special challenges for the non-cooperative foundation, or implementation, of the competitive divisions when all commodities are bads, which is the focus of this paper.

We proposed a model that resembles the classical alternating-offer bilateral bargaining game with exogenous probability of breakdown. In this context, however, the outcome "perpetual disagreement" is not a feasible outcome: the bads must be allocated. One key novelty we introduced is that a player who makes a proposal that leads to breakdown will have to internalize all costs for the failure to reach an agreement. More specifically, the proposer will be assigned all the bads in the case his proposal is not accepted unanimously by the other players. Therefore, every player needs to balance the risk of a large punishment for not satisfying everybody and the desire for a higher share when making a proposal. This game implements some competitive divisions in the sense that, with a vanishing breakdown probability, all (Markov perfect) equilibrium payoffs approximate competitive profiles.

It is worthwhile to notice that we proved Proposition 4 by construction for the case of $n = 2$ only. Our constructive approach is not applicable to generalize Proposition 4 to the problems with more than two players. With more than two players, even for a competitive utility profile that locally maximizes or minimizes the Nash product on the Pareto frontier of feasible profiles, there are infinite efficient profiles in any neighborhood of the competitive profile that have the same Nash product.

Appendix

Proof of Proposition 1. In the bargaining game where the players rotate in making proposals, $\langle j - i \rangle$ defined by (5) is the number of periods that player j will propose after player i's proposal is rejected. According to (4) and the strategy profile, s^i_j is player j's expected continuation payoff if player i's proposal s^i is rejected in the current period. Observe that after any of player i's proposal (not necessarily player i's equilibrium proposal s^i) is rejected, player j's expected continuation payoff is equal to

$$\begin{aligned}(1-\rho)u_j(0)+\rho s_j^{i+1} &= (1-\rho)u_j(0)+\rho\left[(1-\rho)u_j(0)+\rho s_j^{i+2}\right]\\ &= \cdots \ = (1-\rho^{\langle j-i\rangle})u_j(0)+\rho^{\langle j-i\rangle}s_j^j,\end{aligned}$$

because if the bargaining is ever terminated before player j proposes, which will happen with probability $1-\rho^{\langle j-i\rangle}$, player j will always receive $u_j(0)$. With probability $\rho^{\langle j-i\rangle}$, player j will proposal s^j that will be accepted by the other players $\langle j-i\rangle$ periods later. Therefore, it is sequentially rational for player j to accept any of player i's proposal if and only if player j's payoff in the proposal is not less than $s_j^i=(1-\rho^{\langle j-i\rangle})u_j(0)+\rho^{\langle j-i\rangle}s_j^j$ as stated in (4).

Now we show that whenever player i proposes, player i will propose $s^i\in P(S)$ as defined in (4) rather than demand more than s_i^i for himself in his proposal. Suppose that player i deviates from this Markov strategy profile by demanding more than s_i^i. Because $s^i\in P(S)$, player i will has to offer less to some of the other players than those in s^i, and such a proposal will be rejected by these players. According to the bargaining model, after player i's proposal is rejected, his expected continuation payoff will be

$$(1-\rho)u_i(\omega)+\rho s_i^{i+1} \tag{A.1}$$

because with probability $1-\rho$, the game will be terminated after which player i will receive $u_i(\omega)$, and with probability ρ, player $i+1$ will propose in the following period from which player i will receive s_i^{i+1}. On the other hand, if player i proposes s^i as prescribed by the Markov strategy profile, player i will receive

$$s_i^i=f_i(s_{-i}^i)=f_i\left((1-\rho)u_{-i}(0)+\rho s_{-i}^{i+1}\right)\geq(1-\rho)f_i\left(u_{-i}(0)\right)+\rho f_i\left(s_{-i}^{i+1}\right), \tag{A.2}$$

due to the concavity of $f_i(\cdot)$. Notice that $f_i\left(u_{-i}(0)\right)=u_i(\omega)$ and

$$s^{i+1}=(s_i^{i+1},s_{-i}^{i+1})\in P(S) \quad\Longrightarrow\quad s_i^{i+1}\leq f_i\left(s_{-i}^{i+1}\right).$$

(A.1) and (A.2) together imply that it is sequentially rational for player i to propose exactly s^i, as given in (4), to the other players when player i proposes. We can then conclude that (4) provides both the necessary and sufficient conditions for a MPE. $\square$

Proof of Proposition 2. For all $i \in N$, first observe that continuous function $f_i(\cdot)$ maps from $\times_{j\neq i}[u_j(\omega), u_j(0)]$ into $[u_i(\omega), u_i(0)]$. We can rewrite condition (4) as

$$s_i^i = f_i\left(\left\{(1-\rho^{\langle j-i\rangle})u_j(0) + \rho^{\langle j-i\rangle}s_j^j\right\}_{j\neq i}\right) \quad \text{for all } i \in N. \tag{A.3}$$

Notice that s_i^i does not appear on the right hand side of (A.3). Rewrite all these n functions as $F(\cdot) : \times_{i\in N}[u_i(\omega), u_i(0)] \rightarrow \times_{i\in N}[u_i(\omega), u_i(0)]$, where

$$F\left(s_1^1, \ldots, s_n^n\right) \equiv \begin{pmatrix} f_1\left(\left\{(1-\rho^{\langle j-1\rangle})u_j(0) + \rho^{\langle j-1\rangle}s_j^j\right\}_{j\neq 1}\right) \\ f_2\left(\left\{(1-\rho^{\langle j-2\rangle})u_j(0) + \rho^{\langle j-2\rangle}s_j^j\right\}_{j\neq 2}\right) \\ \vdots \\ f_n\left(\left\{(1-\rho^{\langle j-n\rangle})u_j(0) + \rho^{\langle j-n\rangle}s_j^j\right\}_{j\neq n}\right) \end{pmatrix}$$

is a continuous function that maps from convex and compact set $\times_{i\in N}[u_i(\omega), u_i(0)] \subset \mathbb{R}^n$ into itself. By the Brower's fixed point theorem, $F(\cdot)$ has, at least, a fixed point

$$\left(\hat{s}_1^1, \ldots, \hat{s}_n^n\right)^T = F\left(\hat{s}_1^1, \ldots, \hat{s}_n^n\right) \in \times_{i\in N}[u_i(\omega), u_i(0)],$$

where $(\hat{s}_1^1, \ldots, \hat{s}_n^n)^T$ denotes the transpose vector of $(\hat{s}_1^1, \ldots, \hat{s}_n^n)$ in order to match function $F(\cdot)$. Proposition 1 then implies that there is a MPE where player i will always propose $\hat{s}^i$, where

$$\begin{aligned} \hat{s}_i^i &= f_i\left((1-\rho^{\langle 1-i\rangle})u_1(0) + \rho^{\langle 1-i\rangle}\hat{s}_1^1 \ldots, \hat{s}_i^i, \ldots (1-\rho^{\langle n-1\rangle})u_n(0) + \rho^{\langle n-i\rangle}\hat{s}_n^n\right), \\ \hat{s}_i^j &= (1-\rho^{\langle j-i\rangle})u_j(0) + \rho^{\langle j-i\rangle}\hat{s}_j^j \qquad\qquad \text{for } j \neq i, \end{aligned}$$

and player j will always accept player i's proposal if and only if player j's payoff in the proposal is not less than $\hat{s}_j^i = (1-\rho^{\langle 1-i\rangle})u_j(0)+\rho^{\langle j-i\rangle}\hat{s}_j^j$. Note that any fixed point $(s_1^1, \ldots, s_n^n)$ of $F(\cdot)$ is not a feasible utility profile but it provides n feasible equilibrium proposals in the same MPE by (4) and (5).

For the second part of this proposition, we prove by contradiction that $u_i(\omega) < s_i^i < u_i(0)$ for all $i \in N$. Note that $u(0) = (u_1(0), \ldots, u_n(0)) \notin S$ implies that point $u(0)$ can never be a solution to (4). Without loss of generality, suppose that $\hat{s}_1^1 = u_1(0)$ and $\hat{s}_2^2 < u_2(0)$. Then $\hat{s}_1^j = u_1(0)$ for all $j \in N$ by (4). Notice that

$$\begin{aligned}\hat{s}^1 &= (u_1(0), (1-\rho)u_2(0) + \rho\hat{s}_2^2, \ldots, (1-\rho^{n-1})u_n(0) + \rho^{n-1}\hat{s}_n^n) \\ &\geq (u_1(0), \hat{s}_2^2, \ldots, (1-\rho^{n-2})u_n(0) + \rho^{n-2}\hat{s}_n^n) = \hat{s}^2.\end{aligned}$$

The last inequality implies that if player 2 proposes $\hat{s}^1 \in S$ instead of $\hat{s}^2 \in S$, player 2's proposal $\hat{s}^1 \in S$ will be accepted by all other players and player 2 will receive $(1-\rho)u_2(0)+\rho\hat{s}_2^2$. Because $\hat{s}_2^2 < u_2(0)$, we have that for all $\rho \in (0,1)$,

$$(1-\rho)u_2(0) + \rho\hat{s}_2^2 > \hat{s}_2^2,$$

which contradicts to the fact that $\{\hat{s}^1, \ldots, \hat{s}^n\}$ is a set of MPE proposals. To conclude, we must have that for all $i \in N$ and for all $\rho \in (0,1)$, $\hat{s}_i^i < u_i(0)$ which implies that $\hat{s}^i \in P(S)$, and hence $\hat{s}_i^i > u_i(\omega)$ due to $\hat{s}^i \in P(S)$ and $\hat{s}_j^i < u_i(0)$. $\square$

Proof of Proposition 3. For $\rho \in (0,1)$, let $\{s^1, \ldots, s^n\}$ be a set of MPE proposals that support the MPE payoff $s^1 \in P(S)$, where player 1's proposal s^1 will be accepted in period 1. First note that these n MPE proposals are linearly independent because the determinant of (6) is

$$\begin{vmatrix} s_1^1 & \rho^{n-1}s_1^1 & \cdots & \rho s_1^1 \\ \rho s_2^2 & s_2^2 & & \rho^2 s_2^2 \\ \vdots & & \ddots & \vdots \\ \rho^{n-1}s_n^n & \rho^{n-2}s_n^n & \cdots & s_n^n \end{vmatrix}$$

$$= \prod_{i=1}^{n} s_i^i \cdot \begin{vmatrix} 1 & \rho^{n-1} & \cdots & \rho \\ \rho & 1 & & \rho^2 \\ \vdots & & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \cdots & 1 \end{vmatrix}$$

$$= (1-\rho^n)^{n-1}\prod_{i=1}^{n} s_i^i \cdot \begin{vmatrix} 1-\rho^n & 0 & \cdots & 0 \\ \rho & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \cdots & 1 \end{vmatrix} \text{(multiply the last row by } \rho^k \text{ and subtract it from the } k\text{-th row)}$$

$$= (1-\rho^n)^n \prod_{i=1}^{n} s_i^i \neq 0,$$

due to the second part of Proposition 2, i.e., $s_i^i < 0$ for all $i \in N$. Because $\{s^1, \ldots, s^n\}$ are linearly independent, they span a unique hyperplane in $\mathbb{R}^n$. For a given $\rho \in (0,1)$, denote the normalized norm vector of this unique hyperplane by $w(\rho) \in \Delta^n$. Since the unit simplex Δ^n is compact, without loss of generality, assume that $w(\rho) \to w^* \in \Delta^n$ as $\rho \to 1$. By (6), we have that for all $i \in N$, $s^i \to s^*$ as $\rho \to 1$. Because $s^i \in P(S)$, we have $s^* \in P(S)$ and the hyperplane $w^* \cdot s = w^* \cdot s^*$ supports set S at s^*. One the other hand, note that these n MPE proposals have the same Nash product: For all $i \in N$, we have

$$\prod_{j=1}^{n} \left|s_j^i\right| = \rho^{1+2+\cdots+(n-1)} \left|s_1^1 \cdots s_n^n\right|.$$

Hence, hyperplane $w^* \cdot s = w^* \cdot s^*$ also supports contour set $\left\{ s \in \mathbb{R}_-^n : \prod_{j=1}^n |s_j| = \prod_{j=1}^n \left|s_j^*\right| \right\}$ at s^*, from which we can conclude that s^* is a competitive profile due to Lemma 1.

Observe that given $\rho \in (0,1)$, these n MPE proposals are uniquely determined by the n diagonal elements of the $n \times n$ matrix of (6). Accordingly, we can denote a MPE strategy profile by the n players' utility "demands" in their equilibrium proposals for simplicity. □

Proof of Proposition 4. Suppose that a competitive utility profile $s^* \in P(S)$ maximizes the Nash product $|s_1 s_2|$ within $B_{\bar{\varepsilon}}(s^*) \cap S$ as stated in (7). Then for all $\varepsilon < \bar{\varepsilon}$, s^* must also maximize the Nash product $|s_1 s_2|$ within $B_\varepsilon(s^*) \cap P(S)$. In what follows, we show that $B_\varepsilon(s^*) \cap P(S)$ must contain, at least, two distinct Pareto efficient utility profiles that have the same Nash product; say $\{s^1, s^2\} \subset B_\varepsilon(s^*) \cap P(S)$ such that $s_1^1 s_2^1 = s_1^2 s_2^2$.

For the existence of s^1 and s^2, we need to consider two cases. First, if s^* is not a unique local maximum point to the Nash product, then we can choose two different maximum points as s^1 and s^2 (in fact, one of them can be s^*) of the Nash product in $B_\varepsilon(s^*) \cap P(S)$. Hence, s^1 and s^2 must have the same Nash product. Second, if s^* is the unique local maximum point, then we must have s^1 and s^2 such that $s_1^1 < s_1^* < s_1^2$ and $s_1^1 s_2^1 = s_1^2 s_2^2 < s_1^* s_2^*$. More specifically, first arbitrarily choose $\hat{s}^1$ and $\hat{s}^2$ from $B_\varepsilon(s^*) \cap P(S)$ such that $\hat{s}_1^1 < s_1^* < \hat{s}_1^2$, we know $s_1^* s_2^* > \max\{\hat{s}_1^1 \hat{s}_2^1, \hat{s}_1^2 \hat{s}_2^2\}$ because s^* is the unique local maximum point of the Nash product by (7). With loss of generality, suppose $\hat{s}_1^1 \hat{s}_2^1 > \hat{s}_1^2 \hat{s}_2^2$. Note that $s_1 f_2(s_1)$ is a continuous function of $s_1 \in [s_1^*, \hat{s}_1^2]$ and

$$s_1^* s_2^* = s_1^* f_2(s_1^*) > \hat{s}_1^1 \hat{s}_2^1 > \hat{s}_1^2 \hat{s}_2^2 = \hat{s}_1^2 f_2(\hat{s}_1^2),$$

by the Intermediate Value Theorem, there exists $s_1^2 \in (s_1^*, \hat{s}_1^2)$ such that $s_1^2 f_2(s_1^2) = \hat{s}_1^1 \hat{s}_2^1$. Now choose $s^1 = \hat{s}^1$ and $s^2 = (s_1^2, f_2(s_1^2)) \in B_\varepsilon(s^*) \cap P(S)$. Then, s^1 and s^2 have the same Nash product by construction.

To complete this proof, we show that the so-selected feasible and efficient profiles s^1 and $s^2 \in B_\varepsilon(s^*) \cap P(S)$ form a set of MPE proposals in our game for some $\rho \in (0, 1)$. Because $s_1^1 < s_1^2 < 0$, define $\rho = s_1^2 / s_1^1 \in (0, 1)$, and hence we have $s_1^2 = \rho s_1^1$. Recall that s^1 and s^2 have the same Nash product,

$$s_1^1 s_2^1 = s_1^2 s_2^2 = \rho s_1^1 s_2^2 \quad \Rightarrow \quad s_2^1 = \rho s_2^2.$$

Proposition 1, or equivalently (6), implies that s^1 and s^2 form a set of MPE proposals in our game with $\rho = s_1^2 / s_1^1$. Note that as ε is closer to 0, s_1^2 and s_1^1 are closer to each other, and so ρ is closer to 1. By construction, both s^1 and s^2 are in $B_\varepsilon(s^*) \cap P(S)$ for $\varepsilon \in (0, \bar{\varepsilon})$.

The proof for the case when s^* is a local minimum point of (7) is similar and hence is omitted here. □

References

Ashlagi, I., Karagözoğlu, E., Klaus, B., 2012. A non-cooperative support for equal division in estate division problems. Mathematical Social Sciences 63, 228-233.

Aumann, R. J., Maschler, M., 1985. Game theoretic analysis of a bankruptcy problem from the Talmud. Journal of Economic Theory 36, 195-213.

Binmore, K., Rubinstein, A., Wolinsky, A., 1986. The Nash bargaining solution in economic modelling. Rand Journal of Economics 17, 176-188.

Bogomolnaia, A., Moulin, H., Sandomirskiy, F., Yanovskaia, E., 2019. Dividing bads under additive utilities. Social Choice and Welfare 52, 395-417.

Bogomolnaia, A., Moulin, H., Sandomirskiy, F., Yanovskaya, E., 2017. Competitive division of a mixed manna. Econometrica 85, 1847-1871.

Britz, V., Herings, P. J.-J., Predtetchinski A., 2014. On the convergence to the Nash bargaining solution for action-dependent bargaining protocols. Games and Economic Behavior 86, 178-183.

Brügemann, B., Gautier, P., Menzio, G., 2019. Intra firm bargaining and Shapley values. Review of Economic Studies 86, 564-592.

Chae, S., Yang, J.-A., 1994. An N-person pure bargaining game. Journal of Economic Theory 62, 86-102.

Eisenberg, E., 1961. Aggregation of utility functions. Management Science 7, 337–350.

Gale, D., 1960. Linear Economic Models. McGraw Hill.

García-Jurado, I., González-Díaz, J., Villar, A., 2006. A non-cooperative approach to bankruptcy problems. Spanish Economic Review 8, 189-197.

Herrero, M.J., 1989. The Nash program: non-convex bargaining problems. Journal of Economic Theory 49, 266-277.

Krishna, V., Serrano, R., 1996. Multilateral bargaining. Review of Economic Studies 63, 61-80.

Miyagawa, E., 2002. Subgame-perfect implementation of bargaining solutions. Games and Economic Behavior 41, 292-308.

Nash, J.F., 1950. The bargaining problem. Econometrica 18, 155-162.

O'Neil, B., 2009. Bargaining with a claims structure: possible solution to the Talmudic division problem. Working Paper, UCLA.

Osborne, M., Rubinstein, A., 1990. Bargaining and Markets, Academic Press, San Diego.

Rubinstein, A., 1982. Perfect equilibrium in a bargaining model. Econometrica 50, 97-109.

Stiglitz, J.E., 2019. An Agenda for reforming economic theory. Frontiers of Economics in China 14(2), 149-167.

Suh, S.-C., Wen, Q., 2006. Multi-agent bilateral bargaining and the Nash bargaining solution. Journal of Mathematical Economics 42, 61-73.

Thomson, W., 2015. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update. Mathematical Social Sciences 74, 41-59.

Varian, H.R., 1974. Equity, envy and efficiency. Journal of Economic Theory 9, 63-91.