

# EIGENFUNCTION EXPANSIONS OF ULTRADIFFERENTIABLE FUNCTIONS AND ULTRADISTRIBUTIONS. III. HILBERT SPACES AND UNIVERSALITY

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ABSTRACT. In this paper we analyse the structure of the spaces of smooth type functions, generated by elements of arbitrary Hilbert spaces, as a continuation of the research in our papers [4] and [5]. We prove that these spaces are perfect sequence spaces. As a consequence we describe the tensor structure of sequential mappings on the spaces of smooth type functions and characterise their adjoint mappings. As an application we prove the universality of the spaces of smooth type functions on compact manifolds without boundary.

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## 1. INTRODUCTION

The present paper is a continuation of our papers [4] and [5]. In [5], we analysed the structure of the spaces of coefficients of eigenfunction expansions of functions in Komatsu classes on compact manifolds. We also described the tensor structure of sequential mappings on spaces of Fourier coefficients and characterised their adjoint mappings. In particular, these classes include spaces of analytic and Gevrey functions, as well as spaces of ultradistributions, dual spaces of distributions and ultradistributions, in both Roumieu and Beurling settings. In another work, [4], we have characterised Komatsu spaces of ultradifferentiable functions and ultradistributions on compact manifolds in terms of the eigenfunction expansions related to

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positive elliptic operators. Here we note that, using properties of the elliptic operators and the Plancherel formula one can get such type of characterisation of smooth functions in terms of their Fourier coefficients. For example, if  $E$  is a positive elliptic pseudo-differential operator on a compact manifold  $X$  without boundary and  $\lambda_j$  denotes its eigenvalues in the ascending order, then smooth functions on  $X$  can be characterised in terms of their Fourier coefficients:

$$f \in C^\infty(X) \Leftrightarrow \forall N \exists C_N : |\widehat{f}(j, l)| \leq C_N \lambda_j^{-N} \text{ for all } j \geq 1, 1 \leq l \leq d_j, \quad (1.1)$$

where  $\widehat{f}(j, l) = (f, e_j^l)_{L^2}$  with  $e_j^l$  being the  $l^{\text{th}}$  eigenfunction corresponding to the eigenvalue  $\lambda_j$  (of multiplicity  $d_j$ ). Such characterisations for analytic functions were obtained by Seeley in [25], with a subsequent extension to Gevrey and, more generally, to Komatsu classes, in [4]. The results obtained in [5] do not include the cases of smooth functions on compact manifolds. We will extend the results in [5] to the spaces of smooth functions. Moreover in this work, we aim at discussing an abstract analysis of the spaces of smooth type functions generated by basis elements of an arbitrary Hilbert space. Considering an abstract point of view has an advantage that the results will cover the analysis of smooth functions on different spaces like compact Lie groups and manifolds. In particular, we introduce a notion of smooth functions generated by elements of a Hilbert space  $\mathcal{H}$  forming a basis. We will show that the appearing spaces of coefficients with respect to expansions in eigenfunctions of positive self-adjoint operators are perfect spaces in the sense of the theory of sequence spaces (see, e.g., Köthe [11]). Consequently, we obtain tensor representations for linear mappings between spaces of smooth type functions. Such discrete representations in a given basis are useful in different areas of time-frequency analysis, in partial differential equations, and in numerical investigations.

Using the obtained representations we establish the universality properties of the appearing spaces. In [29], L. Waelbroeck proved the so-called *universality* of the space of Schwartz distributions  $\mathcal{E}(V)$  with compact support on a  $C^\infty$ -manifold  $V$ , with the  $\delta$ -mapping  $\delta : V \rightarrow \widehat{\mathcal{E}(V)}$ , that is, any vector valued  $C^\infty$ -mapping  $f : V \rightarrow E$ , from  $V$  to a sequence space  $E$ , factors through  $\delta : V \rightarrow \widehat{\mathcal{E}(V)}$  by a unique linear morphism  $\tilde{f} : \widehat{\mathcal{E}(V)} \rightarrow E$  as  $f = \tilde{f} \circ \delta$ . The universality of the spaces of Gevrey functions on the torus has been established in [27]. As an application of our tensor representations, we prove the universality of the spaces of smooth functions on compact manifolds.

Our analysis is based on the global Fourier analysis on arbitrary Hilbert spaces using techniques similar to compact manifold which was consistently developed in [6], with a number of subsequent applications, for example to the spectral properties of operators [7], or to the wave equations for the Landau Hamiltonian [21]. The corresponding version of the Fourier analysis is based on expansions with respect to orthogonal systems of eigenfunctions of a self-adjoint operator. The non self-adjoint version has been developed in [20], with a subsequent extension in [22].

The paper is organised as follows. In Section 2 we will briefly recall the constructions leading to the global Fourier analysis on arbitrary Hilbert spaces and define the smooth type function spaces. In Section 3 we very briefly recall the relevant definitions from the theory of sequence spaces. In Section 4 we present the main results of this

paper and their proofs. In Section 5 we prove the universality results for the smooth functions on compact manifold.

## 2. FOURIER ANALYSIS ON HILBERT SPACES

Let  $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$  be a separable Hilbert space and denote by

$$\mathcal{U} := \{e_{jl} : e_{jl} \in \mathcal{H}, 1 \leq l \leq d_j, d_j \in \mathbb{N}\}_{j \in \mathbb{N}}$$

a collection of elements of  $\mathcal{H}$ . We assume that  $\mathcal{U}$  is a basis of the space  $\mathcal{H}$  with the property

$$(e_{jl}, e_{mn})_{\mathcal{H}} = \delta_{jm} \delta_{ln}, \quad j, m \in \mathbb{N} \text{ and } 1 \leq l \leq d_j, 1 \leq n \leq d_m,$$

where  $\delta_{jm}$  is the Kronecker delta, equal to 1 for  $j = m$ , and to zero otherwise. Also let us fix a sequence of positive numbers  $\Lambda := \{\lambda_j\}_{j \in \mathbb{N}}$  such that  $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ , and the series

$$\sum_{j=1}^{\infty} d_j \lambda_j^{-s_0} < \infty \tag{2.1}$$

converges for some  $s_0 > 0$ . For example, in a compact  $C^\infty$  manifold  $X$  of dimension  $n$  without boundary and with a fixed measure we have

$$\sum_{j=1}^{\infty} d_j (1 + \lambda_j)^{-q} < \infty \text{ if and only if } q > \frac{n}{\nu},$$

where  $0 < \lambda_1 < \lambda_2 < \dots$  are eigenvalues of a positive elliptic pseudo-differential operator  $E$  of an integer order  $\nu$ , with  $H_j \subset L^2(X)$  the corresponding eigenspace and

$$d_j := \dim H_j, \quad H_0 := \ker E, \quad \lambda_0 := 0, \quad d_0 := \dim H_0.$$

We associate to the pair  $\{\mathcal{U}, \Lambda\}$  a linear self-adjoint operator  $E : \mathcal{H} \rightarrow \mathcal{H}$  such that

$$E^s f = \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} \lambda_j^s (f, e_{jl}) e_{jl} \tag{2.2}$$

for  $s \in \mathbb{R}$  and those  $f \in \mathcal{H}$  for which the series converges in  $\mathcal{H}$ . Then  $E$  is densely defined since

$$E^s e_{jl} = \lambda_j^s e_{jl}, \quad 1 \leq l \leq d_j, \quad j \in \mathbb{N},$$

and  $\mathcal{U}$  is a basis of  $\mathcal{H}$ . Also we write  $H_j = \text{span}\{e_{jl}\}_{1 \leq l \leq d_j}$ , and so  $\dim H_j = d_j$ . Then we have

$$\mathcal{H} = \bigoplus_{j \in \mathbb{N}} H_j.$$

The Fourier transform for  $f \in \mathcal{H}$  is defined as

$$\widehat{f}(j, l) := (f, e_{jl})_{\mathcal{H}}, \quad j \in \mathbb{N}, \quad 1 \leq l \leq d_j.$$

We next define the following notions:

The spaces of smooth type functions are defined by

$$H_E^\infty := \bigcap_{s \in \mathbb{R}} H_E^s,$$

where

$$H_E^s := \left\{ \phi \in \mathcal{H} : \sum_{j=1}^{\infty} \lambda_j^{2s} \|\widehat{\phi}(j)\|_{\text{HS}}^2 < \infty \right\}.$$

There exists a linear pairing

$$(f, g)_{H_E^s} = (E^s f, E^{-s} g)_{\mathcal{H}},$$

for  $f \in H_E^s$  and  $g \in H_E^{-s}$ . It is easy to see from this that every continuous linear functional on  $H_E^s$  is of the form  $f \rightarrow (f, g)_{H_E^s}$  for some  $g \in H_E^{-s}$ , that is  $(H_E^s)' = H_E^{-s}$ . Then we denote the space of distributions as  $(H_E^\infty)' = \bigcup_{s \in \mathbb{R}} H_E^{-s}$ .

### 3. SEQUENCE SPACES AND SEQUENTIAL LINEAR MAPPINGS

We briefly recall that a sequence space  $V$  is a linear subspace of

$$\mathbb{C}^{\mathbb{Z}} = \{a = (a_j) | a_j \in \mathbb{C}, j \in \mathbb{Z}\}.$$

The dual  $\widehat{V}$  ( $\alpha$ -dual in the terminology of G. Köthe [11]) is a sequence space defined by

$$\widehat{V} = \{a \in \mathbb{C}^{\mathbb{Z}} : \sum_{j \in \mathbb{Z}} |f_j| |a_j| < \infty \text{ for all } f \in V\}.$$

A sequence space  $V$  is called *perfect* if  $\widehat{\widehat{V}} = V$ . A sequence space is called *normal* if  $f = (f_j)_{j \in \mathbb{N}} \in V$  implies  $|f| = (|f_j|)_{j \in \mathbb{N}} \in V$ . A dual space  $\widehat{V}$  is normal so that any perfect space is normal.

A pairing  $\langle \cdot, \cdot \rangle_V$  on  $V$  is a bilinear function on  $V \times \widehat{V}$  defined by

$$\langle f, g \rangle_V = \sum_{j \in \mathbb{Z}} f_j g_j \in \mathbb{C},$$

which converges absolutely by the definition of  $\widehat{V}$ .

**Definition 3.1.**  $\phi : V \rightarrow \mathbb{C}$  is called a sequential linear functional if there exists some  $a \in \widehat{V}$  such that  $\phi(f) = \langle f, a \rangle_V$  for all  $f \in V$ . We abuse the notation by also writing  $a : V \rightarrow \mathbb{C}$  for this mapping.

**Definition 3.2.** A mapping  $\phi : V \rightarrow W$  between two sequence spaces is called a sequential linear mapping if

- (1)  $\phi$  is algebraically linear,
- (2) for any  $g \in \widehat{W}$ , the composed mapping  $g \circ \phi : V \rightarrow \mathbb{C}$  is in  $\widehat{V}$ .

### 4. TENSOR REPRESENTATIONS AND THE ADJOINTNESS

In this section we discuss  $\alpha$ -duals of the spaces, tensor representations for mappings between these spaces and their  $\alpha$ -duals, and obtain the corresponding adjointness theorem.

#### 4.1. Duals and $\alpha$ -duals.

In this section we first prove that the  $\alpha$ -dual,  $\widehat{H}_E^s$  of the space  $H_E^s$ , where

$$\widehat{H}_E^s = \left\{ v = (v_j)_{j \in \mathbb{N}}, v_j \in \mathbb{C}^{d_j}; \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} |\widehat{\phi}(j, l)| |v_{jl}| < \infty, \text{ for all } \phi \in H_E^s \right\},$$

coincides with the space  $H_E^{-s}$ .

**Remark 4.1.** Here we observe that,

$$w \in \widehat{H}_E^s \implies \sum_{j=1}^{\infty} \lambda_j^{-s-s_0/2} \|w_j\|_{\text{HS}} < \infty.$$

Indeed, let  $w \in \widehat{H}_E^s$ . Define  $\widehat{\phi}(j, l) = \lambda_j^{-(s+s_0/2)}$  for  $j \in \mathbb{N}$  and  $1 \leq l \leq d_j$ .

Then  $\|\widehat{\phi}(j)\|_{\text{HS}} = d_j^{1/2} \lambda_j^{-s-s_0/2}$ . It follows from (2.1) and the definition of  $H_E^s$  that  $\phi \in H_E^s$ .

From the definition of the  $\alpha$ -dual and using the inequality,

$$\|w_j\|_{\text{HS}} \leq \|w_j\|_{l^1},$$

we can then conclude that,

$$\begin{aligned} \sum_{j=1}^{\infty} \lambda_j^{-s-s_0/2} \|w_j\|_{\text{HS}} &\leq \sum_{j=1}^{\infty} \lambda_j^{-s-s_0/2} \|w_j\|_{l^1} \\ &= \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} |\widehat{\phi}(j, l)| |w_{jl}| < \infty. \end{aligned} \quad (4.1)$$

Our first result is the identification of the topological dual with the  $\alpha$ -dual.

**Theorem 4.2.**  $\widehat{H}_E^s = H_E^{-s}$ .

*Proof.* First we will show  $H_E^{-s} \subseteq \widehat{H}_E^s$ .

Let  $u \in H_E^{-s}$ .

Then from the definition we have  $\sum_{j=1}^{\infty} \lambda_j^{-2s} \|\widehat{u}(j)\|_{\text{HS}}^2 < \infty$ . We denote for  $j \in \mathbb{N}$ ,

$u_{jl} = \widehat{u}(j, l) \in \mathbb{C}^{d_j}$ ,  $l = 1, 2, \dots, d_j$ . Using the Cauchy-Schwartz inequality, for any  $\phi \in H_E^s$  we get

$$\begin{aligned} &\sum_{j=1}^{\infty} \sum_{l=1}^{d_j} |\widehat{\phi}(j, l)| |u_{jl}| \\ &\leq \left( \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} \lambda_j^{2s} |\widehat{\phi}(j, l)|^2 \right)^{1/2} \left( \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} \lambda_j^{-2s} |u_{jl}|^2 \right)^{1/2} < \infty, \text{ (by assumption)}. \end{aligned}$$

This implies that  $u \in \widehat{H}_E^s$ . We thus obtain  $H_E^{-s} \subseteq \widehat{H}_E^s$ .

Next we will show that  $\widehat{H}_E^s \subseteq H_E^{-s}$ .

By duality we know that  $(H_E^s)' = H_E^{-s}$ . So it will be enough if we can prove  $\widehat{H}_E^s \subseteq (H_E^s)'$ .

Let  $w \in \widehat{H}_E^s$ . Then for  $\phi \in H_E^s$  we define

$$w(\phi) = \sum_{j=1}^{\infty} \lambda_j^{-s_0/2} \widehat{\phi}(j) \cdot w_j = \sum_{j=1}^{\infty} \lambda_j^{-s_0/2} \sum_{l=1}^{d_j} \widehat{\phi}(j, l) w_{jl}.$$

Then using Remark 4.1 we have

$$\begin{aligned} |w(\phi)| &\leq \sum_{j=1}^{\infty} \lambda_j^{-s_0/2} \|\widehat{\phi}(j)\|_{\text{HS}} \|w_j\|_{\text{HS}} \\ &\leq C \sum_{j=1}^{\infty} \lambda_j^{-s_0/2} \lambda_j^{-s} \|w_j\|_{\text{HS}} < \infty, \end{aligned} \tag{4.2}$$

since  $\phi \in H_E^s$ . So  $w(\phi)$  is well defined.

We next check that  $w$  is continuous. Let  $\phi_m \rightarrow \phi$  as  $m \rightarrow \infty$  in  $H_E^s$ . This means,  $\|\phi_m - \phi\|_{H_E^s} \rightarrow 0$  as  $m \rightarrow \infty$ , which implies  $\lambda_j^s \|\widehat{\phi}_m(j) - \widehat{\phi}(j)\|_{\text{HS}} \rightarrow 0$  as  $m \rightarrow \infty$ , for  $j \in \mathbb{N}$ . So we have

$$\|\widehat{\phi}_m(j) - \widehat{\phi}(j)\|_{\text{HS}} \leq C_m \lambda_j^{-s},$$

where  $C_m \rightarrow 0$  as  $m \rightarrow \infty$ . Then

$$\begin{aligned} |w(\phi_m - \phi)| &\leq \sum_{j=1}^{\infty} \lambda_j^{-s_0/2} \|\widehat{\phi}_m(j) - \widehat{\phi}(j)\|_{\text{HS}} \|w_j\|_{\text{HS}} \\ &\leq C_m \sum_{j=1}^{\infty} \lambda_j^{-(s+s_0/2)} \|w_j\|_{\text{HS}} \rightarrow 0, \end{aligned}$$

as  $m \rightarrow \infty$ . Hence  $w$  is continuous. This gives  $w \in (H_E^s)' = H_E^{-s}$ . So we have  $\widehat{H}_E^s \subseteq H_E^{-s}$ , that implies  $\widehat{H}_E^s = H_E^{-s}$ .  $\square$

From this we can have the following corollary.

**Corollary 4.3.**  $v \in \widehat{H}_E^s \iff \sum_{j=1}^{\infty} \lambda_j^{-2s} \|v_j\|_{\text{HS}}^2 < \infty$  and  $v_j \in \mathbb{C}^{d_j}$ , we denote  $v_{jl} = \widehat{v}_{jl} = (v, e_{jl})_{\mathcal{H}}$ .

**Remark 4.4.** From the definition of  $H_E^\infty$ ,

$$v \in H_E^\infty \iff \sum_{j=1}^{\infty} \lambda_j^{2s} \|v_j\|_{\text{HS}}^2 < \infty, \quad \forall s \in \mathbb{R},$$

where  $v_j = \widehat{v}(j) \in \mathbb{C}^{d_j}$ .

We next define the  $\alpha$ -dual of the space  $\widehat{H}_E^\infty$ ,

$$\widehat{H}_E^\infty = \left\{ v = (v_j), v_j \in \mathbb{C}^{d_j} : \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} |v_{jl}| |\widehat{\phi}(j, l)| < \infty, \text{ for all } \phi \in H_E^\infty \right\}.$$

Also observe that  $\widehat{H}_E^\infty = \widehat{\bigcap_{s \in \mathbb{R}} H_E^s} = \bigcup_{s \in \mathbb{R}} \widehat{H}_E^s = \bigcup_{s \in \mathbb{R}} H_E^{-s}$ . From this we can state the following lemma and the proof will follow from our above observation.

**Lemma 4.5.**  $v \in \widehat{H}_E^\infty \iff$  for some  $s \in \mathbb{R}$  we have  $\sum_{j=1}^{\infty} \lambda_j^{-2s} \|v_j\|_{\text{HS}}^2 < \infty$ .

Next we proceed to prove that  $H_E^\infty$  is a perfect space. But before that let us prove the following lemma.

**Lemma 4.6.** We have  $w \in [\widehat{H}_E^\infty]^\wedge$  if and only if  $\sum_{j=1}^{\infty} \lambda_j^{2s} \|w_j\|_{\text{HS}}^2 < \infty$  for all  $s \in \mathbb{R}$ .

*Proof.* Let  $w \in [\widehat{H}_E^\infty]^\wedge$ . Let  $s \in \mathbb{R}$ , we define

$$v_{jl} = \widehat{v}(j, l) = \lambda_j^s, \quad 1 \leq l \leq d_j.$$

Then  $\|v_j\|_{\text{HS}}^2 = d_j \lambda_j^{2s}$  and so  $v \in H_E^{-s - \frac{s_0}{2}}$  because in view of (2.1) we have

$$\sum_{j=1}^{\infty} \lambda_j^{-2s} \lambda_j^{-s_0} \|v_j\|_{\text{HS}}^2 = \sum_{j=1}^{\infty} d_j \lambda_j^{-2s} \lambda_j^{-s_0} \lambda_j^{2s} = \sum_{j=1}^{\infty} d_j \lambda_j^{-s_0} < \infty. \quad (4.3)$$

Then  $v \in \widehat{H}_E^\infty = \bigcup_{s \in \mathbb{R}} H_E^{-s}$  which gives  $\sum_{j=1}^{\infty} \sum_{l=1}^{d_j} |w_{jl}| |v_{jl}| < \infty$ . Now we observe first that

$$\begin{aligned} \sum_{j=1}^{\infty} \lambda_j^{2s} \|w_j\|_{\text{HS}}^2 &\leq \sum_{j=1}^{\infty} d_j \lambda_j^{2s} \|w_j\|_{\text{HS}}^2 \\ &= \sum_{j=1}^{\infty} \|v_j\|_{\text{HS}}^2 \|w_j\|_{\text{HS}}^2, \end{aligned} \quad (4.4)$$

since  $d_j \geq 1$ .

We want to show that  $\sum_{j=1}^{\infty} \|v_j\|_{\text{HS}}^2 \|w_j\|_{\text{HS}}^2 < \infty$ . To prove this we will use the following identity:

$$\sum_{i=1}^n a_i^2 \sum_{l=1}^n b_l^2 = \sum_{i=1}^n a_i^2 b_i^2 + \sum_{i=1}^n a_i^2 \left( \sum_{l=1}^n b_l^2 - b_i^2 \right).$$

From this we get

$$\sum_{i=1}^{d_j} |w_{ji}|^2 \sum_{l=1}^{d_j} |v_{jl}|^2 = \sum_{i=1}^{d_j} |v_{ji}|^2 |w_{ji}|^2 + \sum_{i=1}^{d_j} |w_{ji}|^2 \left( \sum_{l=1}^{d_j} |v_{jl}|^2 - |v_{ji}|^2 \right).$$

We consider the second term of the above inequality, that is,

$$\sum_{i=1}^{d_j} |w_{ji}|^2 \left( \sum_{l=1}^{d_j} |v_{jl}|^2 - |v_{ji}|^2 \right) \leq \sum_{i=1}^{d_j} |w_{ji}|^2 \sum_{l=1}^{d_j} |v_{jl}|^2 \leq \sum_{i=1}^{d_j} |w_{ji}|^2 (C\lambda_j^{2s+s_0}),$$

since  $v \in H_E^{-s-s_0/2}$ . Then we get

$$\begin{aligned} \sum_{j=1}^{\infty} \|v_j\|_{\text{HS}}^2 \|w_j\|_{\text{HS}}^2 &\leq \sum_{j=1}^{\infty} \sum_{i=1}^{d_j} |w_{ji}|^2 (|v_{ji}|^2 + C\lambda_j^{2s+s_0}) \\ &\leq \sum_{j=1}^{\infty} \sum_{i=1}^{d_j} |w_{ji}|^2 |u_{ji}|^2 \\ &\leq C \left( \sum_{j=1}^{\infty} \sum_{i=1}^{d_j} |w_{ji}| |u_{ji}| \right)^2, \end{aligned} \quad (4.5)$$

where  $|u_{ji}|^2 = |v_{ji}|^2 + C\lambda_j^{2s+s_0}$ . Now

$$\begin{aligned} \sum_{j=1}^{\infty} \lambda_j^{-2s-2s_0} \|u_j\|_{\text{HS}}^2 &= \sum_{j=1}^{\infty} \lambda_j^{-2s-2s_0} \|v_j\|_{\text{HS}}^2 + C \sum_{j=1}^{\infty} d_j \lambda_j^{-2s-2s_0} \lambda_j^{2s+s_0} \\ &\leq C' \sum_{j=1}^{\infty} \lambda_j^{-s_0} + C \sum_{j=1}^{\infty} d_j \lambda_j^{-s_0} < \infty, \end{aligned} \quad (4.6)$$

which implies that  $u \in H_E^{-s-s_0}$ , that is  $u \in \widehat{H}_E^\infty$ . This gives that

$$\sum_{j=1}^{\infty} \sum_{i=1}^{d_j} |w_{ji}| |u_{ji}| < \infty, \quad (4.7)$$

as  $w \in [\widehat{H}_E^\infty]^\wedge$  and  $u \in \widehat{H}_E^\infty$ . So we have from (4.4), (4.5) and (4.7) that

$$\sum_{j=1}^{\infty} \lambda_j^{2s} \|w_j\|_{\text{HS}}^2 < \infty.$$

Next we proceed to prove the opposite direction. Let

$$w \in \Sigma = \{v = (v_j)_{j \in \mathbb{N}}, v_j \in \mathbb{C}^{d_j}\}$$

be such that  $\sum_{j=1}^{\infty} \lambda_j^{2s} \|w_j\|_{\text{HS}}^2 < \infty$  for all  $s \in \mathbb{R}$ . Let  $v \in \widehat{H}_E^\infty = \bigcup_{s \in \mathbb{R}} H_E^{-s}$ . In particular, we have  $v \in H_E^{-s}$  for some  $s \in \mathbb{R}$ . We have to show

$$\sum_{j=1}^{\infty} \sum_{l=1}^{d_j} |w_{jl}| |v_{jl}| < \infty.$$



By the Cauchy-Schwartz inequality we have

$$\begin{aligned} \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} |w_{jl}| |v_{jl}| &= \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} \lambda_j^s |w_{jl}| \lambda_j^{-s} |v_{jl}| \\ &\leq \left( \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} \lambda_j^{2s} |w_{jl}|^2 \right)^{1/2} \left( \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} \lambda_j^{-2s} |v_{jl}|^2 \right)^{1/2} < \infty. \end{aligned} \quad (4.8)$$

It follows that  $w \in [\widehat{H_E^\infty}]^\wedge$ , completing the proof.  $\square$

Now using Remark 4.4 and Lemma 4.6 we can prove that the spaces  $H_E^\infty$  are perfect spaces.

**Theorem 4.7.**  $H_E^\infty$  is a perfect space.

*Proof.* From the definition we always have  $H_E^\infty \subseteq [\widehat{H_E^\infty}]^\wedge$ . We will prove the other direction. Let  $w \in [\widehat{H_E^\infty}]^\wedge$ ,  $w = (w_j)_{j \in \mathbb{N}}$  and  $w_j \in \mathbb{C}^{d_j}$ .

Define

$$\phi = \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} w_{jl} e_{jl}.$$

The series is convergent and  $\phi \in \mathcal{H}$  since

$$\begin{aligned} \|\phi\|_{\mathcal{H}} &= \left\| \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} w_{jl} e_{jl} \right\|_{\mathcal{H}} \\ &\leq \left( \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} \lambda_j^{s_0} |w_{jl}|^2 \right)^{1/2} \left( \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} \lambda_j^{-s_0} \|e_{jl}\|_{\mathcal{H}}^2 \right)^{1/2} \\ &= \left( \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} \lambda_j^{s_0} |w_{jl}|^2 \right)^{1/2} \left( \sum_{j=1}^{\infty} d_j \lambda_j^{-s_0} \right)^{1/2} < \infty, \end{aligned} \quad (4.9)$$

since  $w \in [\widehat{H_E^\infty}]^\wedge$  and using (2.1) and Lemma 4.6.

Also from the property  $(e_{jl}, e_{mn})_{\mathcal{H}} = \delta_{jm} \delta_{ln}$ , for  $j, m \in \mathbb{N}$  and  $1 \leq l \leq d_j$ ,  $1 \leq n \leq d_m$ , it is obvious that  $\widehat{\phi}(j, l) = (\phi, e_{jl})_{\mathcal{H}} = w_{jl}$ . This gives  $\|\widehat{\phi}(j)\|_{\mathcal{HS}} = \|w_j\|_{\mathcal{HS}}$ . So by Lemma 4.6,

$$\begin{aligned} w \in [\widehat{H_E^\infty}]^\wedge &\implies \sum_{j=1}^{\infty} \lambda_j^{2s} \|w_j\|_{\mathcal{HS}}^2 < \infty, \text{ for all } s \in \mathbb{R} \\ &\implies \sum_{j=1}^{\infty} \lambda_j^{2s} \|\widehat{\phi}(j)\|_{\mathcal{HS}}^2 < \infty, \end{aligned}$$

and so from Remark 4.4 we have  $w \in H_E^\infty$  which implies that  $[\widehat{H_E^\infty}]^\wedge \subseteq H_E^\infty$  holds.  $\square$

## 4.2. Adjointness.

Before proving the adjointness theorem we first prove the following lemma,

**Lemma 4.8.** *Let  $v \in H_E^\infty$  and  $w \in \widehat{H}_E^\infty$ . Then we have*

$$\sum_{j=1}^{\infty} \|v_j\|_{\text{HS}} \|w_j\|_{\text{HS}} < \infty, \quad (4.10)$$

where  $v_j = (v_{jl})_{1 \leq l \leq d_j} = \widehat{v}(j, l)$ , and  $j \in \mathbb{N}$ , and the same for  $w$ . Moreover, suppose that for all  $v \in H_E^\infty$ , (4.10) holds. Then we must have  $w \in \widehat{H}_E^\infty$ . Also, if for all  $w \in \widehat{H}_E^\infty$ , (4.10) holds, we have  $v \in H_E^\infty$ .

*Proof.* Let us assume first that

$$\sum_{j=1}^{\infty} \|v_j\|_{\text{HS}} \|w_j\|_{\text{HS}} < \infty,$$

for  $v \in H_E^\infty$  or  $w \in \widehat{H}_E^\infty$ . We observe that

$$\sum_{l=1}^{d_j} |v_{jl}| |w_{jl}| \leq \|v_j\|_{\text{HS}} \|w_j\|_{\text{HS}}.$$

And so we have

$$\sum_{j=1}^{\infty} \sum_{l=1}^{d_j} |v_{jl}| |w_{jl}| \leq \sum_{j=1}^{\infty} \|v_j\|_{\text{HS}} \|w_j\|_{\text{HS}} < \infty.$$

Now we prove the other direction. Here we will use the following inequality,

$$\sum_{i=1}^n |a_i| \sum_{l=1}^n |b_l| = \sum_{i=1}^n |a_i| |b_i| + \sum_{i=1}^n |a_i| \left( \sum_{l=1}^n |b_l| - |b_i| \right),$$

for any  $a_i, b_i \in \mathbb{R}$ , yielding

$$\begin{aligned} \|v_j\|_{\text{HS}} \|w_j\|_{\text{HS}} &\leq \sum_{i=1}^{d_j} |w_{ji}| \sum_{l=1}^{d_j} |v_{jl}| \\ &= \sum_{i=1}^{d_j} |v_{ji}| |w_{ji}| + \sum_{i=1}^{d_j} |w_{ji}| \left( \sum_{l=1}^{d_j} |v_{jl}| - |v_{ji}| \right). \end{aligned} \quad (4.11)$$

We consider the second term of the above inequality, that is,

$$\sum_{i=1}^{d_j} |w_{ji}| \left( \sum_{l=1}^{d_j} |v_{jl}| - |v_{ji}| \right) \leq \sum_{i=1}^{d_j} |w_{ji}| (C d_j \lambda_j^{-s}), \quad \forall s \in \mathbb{R},$$

since  $v \in H_E^\infty$  and so  $v \in H_E^s, \forall s \in \mathbb{R}$ . Then we get

$$\sum_{j=1}^{\infty} \|v_j\|_{\text{HS}} \|w_j\|_{\text{HS}} \leq \sum_{j=1}^{\infty} \sum_{i=1}^{d_j} |w_{ji}| (|v_{ji}| + C d_j \lambda_j^{-s}). \quad (4.12)$$

Let  $|u_{ji}| = |v_{ji}| + Cd_j\lambda_j^{-s}$ , for  $i = 1, 2, \dots, d_j$ . Then

$$\begin{aligned} \sum_{i=1}^{d_j} |u_{ji}|^2 &= \sum_{i=1}^{d_j} |v_{ji}|^2 + \sum_{i=1}^{d_j} C^2 d_j^2 \lambda_j^{-2s} + \sum_{i=1}^{d_j} 2Cd_j \lambda_j^{-s} |v_{ji}| \\ &= \|v_j\|_{\text{HS}}^2 + C^2 d_j^3 \lambda_j^{-2s} + 2Cd_j^2 \lambda_j^{-2s}. \end{aligned} \quad (4.13)$$

Then for any  $t > 0$  we have, using (4.13), that

$$\begin{aligned} \sum_{j=1}^{\infty} \lambda_j^{2t} \|u_j\|_{\text{HS}}^2 &= \sum_{j=1}^{\infty} \lambda_j^{2t} \|v_j\|_{\text{HS}}^2 + C^2 \sum_{j=1}^{\infty} \lambda_j^{2t} d_j^3 \lambda_j^{-2s} + 2C \sum_{j=1}^{\infty} \lambda_j^{2t} d_j^2 \lambda_j^{-2s} \\ &\leq \sum_{j=1}^{\infty} \lambda_j^{2t} \|v_j\|_{\text{HS}}^2 + C^2 \sum_{j=1}^{\infty} d_j \lambda_j^{2t+2s_0} \lambda_j^{-2s} \\ &\quad + 2C \sum_{j=1}^{\infty} \lambda_j^{2t+2s_0} \lambda_j^{-2s}. \end{aligned} \quad (4.14)$$

Now since  $v \in H_E^\infty$ , in particular we can have  $2s = 2t + 3s_0$ , for any  $t > 0$ , which gives, using (2.1)

$$\sum_{j=1}^{\infty} \lambda_j^{2t} \|u_j\|_{\text{HS}}^2 \leq \sum_{j=1}^{\infty} \lambda_j^{2t} \|v_j\|_{\text{HS}}^2 + C^2 \sum_{j=1}^{\infty} d_j \lambda_j^{-s_0} + 2C \sum_{j=1}^{\infty} \lambda_j^{-s_0} < \infty.$$

So  $u \in H_E^\infty$ . Then we have for  $w \in \widehat{H_E^\infty}$ ,

$$\sum_{j=1}^{\infty} \|v_j\|_{\text{HS}} \|w_j\|_{\text{HS}} \leq \sum_{j=1}^{\infty} \sum_{i=1}^{d_j} |w_{ji}| |u_{ji}| < \infty,$$

completing the proof.  $\square$

We next prove the adjointness theorem, also recalling Definition 3.1. Let  $\mathcal{H}, \mathcal{G}$  be two Hilbert spaces and  $E$  and  $F$  be the operators defined by (2.2) corresponding to the bases  $\{e_j\}_{j \in \mathbb{N}}, \{h_k\}_{k \in \mathbb{N}}$ , respectively, where  $d_j = \dim X_j$  and  $g_k = \dim Y_k$ , and  $X_j = \text{span}\{e_{jl}\}_{l=1}^{d_j}$ ,  $Y_k = \text{span}\{h_{ki}\}_{i=1}^{g_k}$ . Also  $\mathcal{H} = \bigoplus_{j \in \mathbb{N}} X_j$  and  $\mathcal{G} = \bigoplus_{k \in \mathbb{N}} Y_k$ . We denote the corresponding spaces to the operators  $E$  and  $F$  in the Hilbert space  $\mathcal{H}$  and  $\mathcal{G}$  respectively by  $H_E^\infty$  and  $G_F^\infty$ .

**Theorem 4.9.** *A linear mapping  $f : H_E^\infty \rightarrow G_F^\infty$  is sequential if and only if  $f$  is represented by an infinite tensor  $(f_{kjl})$ ,  $k, j \in \mathbb{N}$ ,  $1 \leq l \leq d_j$  and  $1 \leq i \leq g_k$  such that for any  $u \in H_E^\infty$  and  $v \in \widehat{G_F^\infty}$  we have*

$$\sum_{j=1}^{\infty} \sum_{l=1}^{d_j} |f_{kjl}| |\widehat{u}(j, l)| < \infty, \text{ for all } k \in \mathbb{N}, i = 1, 2, \dots, g_k, \quad (4.15)$$

and

$$\sum_{k=1}^{\infty} \sum_{i=1}^{g_k} |(v_k)_i| \left| \left( \sum_{j=1}^{\infty} f_{kjl} \widehat{u}(j) \right)_i \right| < \infty. \quad (4.16)$$

Furthermore, the adjoint mapping  $\widehat{f} : \widehat{G}_F^\infty \rightarrow \widehat{H}_E^\infty$  defined by the formula  $\widehat{f}(v) = v \circ f$  is also sequential, and the transposed matrix  $(f_{kj})^t$  represents  $\widehat{f}$ , with  $f$  and  $\widehat{f}$  related by  $\langle f(u), v \rangle_{G_F^\infty} = \langle u, \widehat{f}(v) \rangle_{H_E^\infty}$ .

Let us summarise the ranges for indices in the used notation as well as give more explanation to (4.16). For  $f : H_E^\infty \rightarrow G_F^\infty$  and  $u \in H_E^\infty$  we write

$$\mathbb{C}^{g_k} \ni f(u)_k = \sum_{j=1}^{\infty} f_{kj} \widehat{u}(j) = \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} f_{kjl} \widehat{u}(j, l), \quad k \in \mathbb{N}, \quad (4.17)$$

so that

$$f_{kjl} \in \mathbb{C}^{g_k}, \quad f_{kjl} \in \mathbb{C}, \quad k, j \in \mathbb{N}, \quad 1 \leq l \leq d_j, \quad 1 \leq i \leq g_k, \quad (4.18)$$

and

$$\mathbb{C} \ni (f(u)_k)_i = f(u)_{ki} = \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} f_{kjl} \widehat{u}(j, l), \quad k \in \mathbb{N}, \quad 1 \leq i \leq g_k, \quad (4.19)$$

where we view  $f_{kj}$  as a matrix,  $f_{kj} \in \mathbb{C}^{g_k \times d_j}$ , and the product of the matrices has been explained in (4.17).

**Remark 4.10.** Let us now describe how the tensor  $(f_{kjl})$ ,  $k, j \in \mathbb{N}$ ,  $1 \leq l \leq d_j$ ,  $1 \leq i \leq g_k$ , is constructed given a sequential mapping  $f : H_E^\infty \rightarrow G_F^\infty$ . For every  $k \in \mathbb{N}$  and  $1 \leq i \leq g_k$ , define the family  $v^{ki} = (v_j^{ki})_{j \in \mathbb{N}}$  such that each  $v_j^{ki} \in \mathbb{C}^{d_j}$  is defined by

$$v_j^{ki}(l) = \begin{cases} 1, & j = k, l = i, \\ 0, & \text{otherwise.} \end{cases} \quad (4.20)$$

Then  $v^{ki} \in \widehat{G}_F^\infty$ , and since  $f$  is sequential we have  $v^{ki} \circ f \in \widehat{H}_E^\infty$ , and we can write  $v^{ki} \circ f = (v^{ki} \circ f)_{j \in \mathbb{N}}$ , where  $(v^{ki} \circ f)_j \in \mathbb{C}^{d_j}$ . Then for each  $1 \leq l \leq d_j$  we set

$$f_{kjl} := (v^{ki} \circ f)_j(l), \quad (4.21)$$

the  $l^{\text{th}}$  component of the vector  $(v^{ki} \circ f)_j \in \mathbb{C}^{d_j}$ . The formula (4.21) will be shown in the proof of Theorem 4.9. In particular, since for  $\phi \in H_E^\infty$  we have  $f(\phi) \in G_F^\infty$ , it will be a consequence of (4.32) and (4.33) later on that

$$v^{ki} \circ f(\phi) = (\widehat{f(\phi)})(k, i) = \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} f_{kjl} \widehat{\phi}(j, l), \quad (4.22)$$

so that the tensor  $(f_{kjl})$  is describing the transformation of the Fourier coefficients of  $\phi$  into those of  $f(\phi)$ .

To prove Theorem 4.9 we first establish the following lemma.

**Lemma 4.11.** Let  $f : H_E^\infty \rightarrow G_F^\infty$  be a linear mapping represented by an infinite tensor  $(f_{kjl})_{k, j \in \mathbb{N}, 1 \leq l \leq d_j, 1 \leq i \leq g_k}$  satisfying (4.15) and (4.16). Then for all  $u \in H_E^\infty$  and  $v \in \widehat{G}_F^\infty$ , we have

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} |(v_k)_i| \left| \left( \sum_{1 \leq j \leq n} f_{kj} \widehat{u}(j) \right)_i \right| = \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} |(v_k)_i| \left| \left( \sum_{j=1}^{\infty} f_{kj} \widehat{u}(j) \right)_i \right|.$$

*Proof of Lemma 4.11.* Let  $u \in H_E^\infty$  and  $u \approx (\widehat{u}(j))_{j \in \mathbb{N}}$ . Define  $u^n := (\widehat{u}^{(n)}(j))_{j \in \mathbb{N}}$  by setting

$$\widehat{u}^{(n)}(j) = \begin{cases} \widehat{u}(j), & j \leq n, \\ 0, & j > n. \end{cases}$$

Then for any  $w \in \widehat{H_E^\infty}$ , we get  $\langle u - u^n, w \rangle_{H_E^\infty} \rightarrow 0$  as  $n \rightarrow \infty$ . This is true since  $\sum_{j=1}^\infty |\widehat{u}(j) \cdot w_j| < \infty$  so that

$$|\langle u - u^n, w \rangle_{H_E^\infty}| \leq \sum_{j \geq n} |\widehat{u}_j \cdot w_j| \rightarrow 0$$

as  $n \rightarrow \infty$ . Now for any  $u \in H_E^\infty$  and  $v \in \widehat{G_F^\infty}$  and from (4.15) and (4.16) we have

$$\begin{aligned} \langle f(u), v \rangle_{G_F^\infty} &= \sum_{k=1}^\infty (f(u))_k \cdot v_k = \sum_{k=1}^\infty \left( \sum_{j=1}^\infty f_{kj} \widehat{u}(j) \right) \cdot v_k \\ &= \sum_{k=1}^\infty \sum_{j=1}^\infty \sum_{\ell=1}^{d_j} \sum_{i=1}^{g_k} f_{kj\ell i} \widehat{u}(j, \ell) (v_k)_i = \sum_{j=1}^\infty \sum_{\ell=1}^{d_j} \widehat{u}(j, \ell) \sum_{k=1}^\infty \sum_{i=1}^{g_k} f_{kj\ell i} (v_k)_i \\ &= \sum_{j=1}^\infty \sum_{\ell=1}^{d_j} \widehat{u}(j, \ell) \sum_{k=1}^\infty f_{kj\ell} \cdot v_k = \sum_{j=1}^\infty \widehat{u}(j) \cdot (v \circ f)_j = \langle u, v \circ f \rangle_{H_E^\infty}, \quad (4.23) \end{aligned}$$

where

$$\mathbb{C}^{d_j} \ni (v \circ f)_j = \left\{ \sum_{k=1}^\infty f_{kj\ell} \cdot v_k \right\}_{\ell=1}^{d_j}, \quad j \in \mathbb{N},$$

and

$$v \circ f = \{(v \circ f)_j\}_{j=1}^\infty.$$

Now we have the mapping  $f : H_E^\infty \rightarrow G_F^\infty$  and  $v \in \widehat{G_F^\infty}$ , so we have  $v \circ f : H_E^\infty \rightarrow \Sigma$ , where  $\Sigma = \{v = (v_j)_{j \in \mathbb{N}}, v_j \in \mathbb{C}^{d_j}\}$ . For any  $u \in H_E^\infty$  and  $v \in \widehat{G_F^\infty}$  we have

$$\langle f(u), v \rangle_{G_F^\infty} = \sum_{j=1}^\infty \sum_{\ell=1}^{d_j} \sum_{k=1}^\infty \sum_{i=1}^{g_k} f_{kj\ell i} \widehat{u}(j, \ell) (v_k)_i = \langle u, v \circ f \rangle_{H_E^\infty}.$$

Then from (4.15) and (4.16) we have  $\langle u, (v \circ f) \rangle < \infty$ , and since  $H_E^\infty$  is perfect, we have for any  $u \approx (\widehat{u}(j))_{j \in \mathbb{N}} \in H_E^\infty$  that the series  $\sum_{j=1}^\infty |(v \circ f)_j \cdot \widehat{u}(j)|$  is convergent. So then  $v \circ f \in \widehat{H_E^\infty}$ . Then we have

$$\langle f(u) - f(u^n), v \rangle_{G_F^\infty} = \langle u - u^n, v \circ f \rangle_{H_E^\infty} \rightarrow 0$$

as  $n \rightarrow \infty$ . Therefore,

$$\langle f(u), v \rangle_{G_F^\infty} = \lim_{n \rightarrow \infty} \langle f(u^n), v \rangle_{G_F^\infty},$$

for all  $u \in H_E^\infty$  and  $v \in \widehat{G}_F^\infty$ . Hence for any  $u \in H_E^\infty$  and  $v \in \widehat{G}_F^\infty$  we have

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} v_k \cdot \left( \sum_{1 \leq j \leq n} f_{kj} \widehat{u}(j) \right) = \sum_{k=1}^{\infty} v_k \cdot \left( \sum_{j=1}^{\infty} f_{kj} \widehat{u}(j) \right),$$

that is,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} (v_k)_i \left( \sum_{1 \leq j \leq n} f_{kj} \widehat{u}(j) \right)_i = \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} (v_k)_i \left( \sum_{j=1}^{\infty} f_{kj} \widehat{u}(j) \right)_i.$$

Now we will use the fact that if  $u \in H_E^\infty$  then  $|u| \in H_E^\infty$  where  $|u| = \left( |\widehat{u}|_j \right)_{j \in \mathbb{N}}$ ,  $|\widehat{u}|_j \in \mathbb{R}^{d_j}$ , with

$$|\widehat{u}|_j := \begin{bmatrix} |\widehat{u}(j, 1)| \\ |\widehat{u}(j, 2)| \\ \vdots \\ |\widehat{u}(j, d_j)| \end{bmatrix},$$

in view of Theorem 4.7. The same is true for the dual space  $[G_F^\infty]^\wedge$ . So then this argument gives

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} |(v_k)_i| \left| \left( \sum_{1 \leq j \leq n} f_{kj} \widehat{u}(j) \right)_i \right| = \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} |(v_k)_i| \left| \left( \sum_{j=1}^{\infty} f_{kj} \widehat{u}(j) \right)_i \right|.$$

The proof is complete.  $\square$

**Remark 4.12.** *This proof does not require sequentiality and it can be used to improve the argument in [5, Theorem 4.7].*

*Proof of Theorem 4.9.* Let us assume first that the mapping  $f : H_E^\infty \rightarrow G_F^\infty$  can be represented by  $f = (f_{kjl})_{k, j \in \mathbb{N}, 1 \leq l \leq d_j, 1 \leq i \leq g_k}$ , an infinite tensor such that

$$\sum_{j=1}^{\infty} \sum_{l=1}^{d_j} |f_{kjl}| |\widehat{u}(j, l)| < \infty, \text{ for all } k \in \mathbb{N}, i = 1, 2, \dots, g_k, \quad (4.24)$$

and

$$\sum_{k=1}^{\infty} \sum_{i=1}^{g_k} |(v_k)_i| \left| \left( \sum_{j=1}^{\infty} f_{kj} \widehat{u}(j) \right)_i \right| < \infty \quad (4.25)$$

hold for all  $u \in H_E^\infty$  and  $v \in \widehat{G}_F^\infty$ .

Let  $\widehat{u}_1 = (\widehat{u}_1(p))_{p \in \mathbb{N}}$  be such that for some  $j, l$  where  $j \in \mathbb{N}, 1 \leq l \leq d_j$ , we have

$$\widehat{u}_1(p, q) = \begin{cases} 1, & p = j, q = l, \\ 0, & \text{otherwise.} \end{cases}$$

Then  $u_1 \in H_E^\infty$  so  $fu_1 = f(u_1) \in G_F^\infty$  and

$$\begin{aligned}
 (fu_1)_k &= \sum_{p=1}^{\infty} f_{kp} \widehat{u}_1(p) \\
 &= \sum_{p=1}^{\infty} \sum_{q=1}^{d_p} f_{kpq} \widehat{u}_1(p, q) \\
 &= \sum_{q=1}^{d_j} f_{kj} \widehat{u}_1(j, q) \\
 &= f_{kjl} \in \mathbb{C}^{g_k}.
 \end{aligned} \tag{4.26}$$

We now first show that

$$(\widehat{fu})(k) = \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} f_{kjl} \widehat{u}(j, l),$$

where  $f_{kjl} \in \mathbb{C}$  for each  $k, j \in \mathbb{N}$ ,  $1 \leq l \leq d_j$  and  $1 \leq i \leq g_k$ . The way in which  $f$  has been defined we have

$$(fu)_k = \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} f_{kjl} \widehat{u}(j, l), \quad f_{kjl} \in \mathbb{C}^{g_k}.$$

Also since  $u \in H_E^\infty$ , from our assumption we have  $fu \in G_F^\infty$  and  $fu \approx \left( (\widehat{fu})(j) \right)_{j \in \mathbb{N}}$ , so that  $(fu)_k \approx (\widehat{fu})(k)$ .

We can then write  $(\widehat{fu})(k) = \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} f_{kjl} \widehat{u}(j, l)$ . Since we know that  $v \in [G_F^\infty]^\wedge$  and  $fu \in G_F^\infty$ , we have

$$\sum_{k=1}^{\infty} \sum_{i=1}^{g_k} |(v_k)_i| |(\widehat{fu})(k)|_i = \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} |(v_k)_i| \left| \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} f_{kjl} \widehat{u}(j, l) \right| < \infty.$$

In particular using the definition of  $u_1$  and (4.26) we get

$$\sum_{k=1}^{\infty} \sum_{i=1}^{g_k} |(v_k)_i| \left| \sum_{p=1}^{\infty} \sum_{q=1}^{d_p} f_{kpqi} \widehat{u}_1(p, q) \right| = \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} |(v_k)_i| |f_{kjl}| < \infty, \tag{4.27}$$

for any  $j \in \mathbb{N}$  and  $1 \leq l \leq d_j$ .

Now for any  $u \in H_E^\infty$  consider

$$J = \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} \left| \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} (v_k)_i f_{kjl} \right| |\widehat{u}(j, l)|.$$

Then we consider the series

$$I_n := \sum_{1 \leq j \leq n} \sum_{l=1}^{d_j} \left| \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} (v_k)_i f_{kjl} \right| |\widehat{u}(j, l)|,$$

so that we have

$$\begin{aligned} I_n &= \sum_{1 \leq j \leq n} \sum_{l=1}^{d_j} \left| \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} (v_k)_i f_{kjl i} |\widehat{u}(j, l)| \right| \\ &= \sum_{1 \leq j \leq n} \sum_{l=1}^{d_j} \left| \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} (v_k)_i f_{kjl i} \widehat{u}(j, l) \right|. \end{aligned}$$

Let  $\epsilon = (\epsilon_i)_{1 \leq i \leq d_k}$ ,  $k \in \mathbb{N}$ , be such that  $\epsilon_i \in \mathbb{C}$  and  $|\epsilon_i| \leq C$ , for all  $i$  and such that

$$\left| \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} (v_k)_i f_{kjl i} \widehat{u}(j, l) \right| = \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} (v_k)_i f_{kjl i} \widehat{u}(j, l) \epsilon_i.$$

Then

$$\begin{aligned} I_n &= \sum_{1 \leq j \leq n} \sum_{l=1}^{d_j} \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} (v_k)_i f_{kjl i} \widehat{u}(j, l) \epsilon_i \\ &\leq C \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} |(v_k)_i| \left| \sum_{1 \leq j \leq n} \sum_{l=1}^{d_j} f_{kjl i} \widehat{u}(j, l) \epsilon_i \right|. \end{aligned} \quad (4.28)$$

It follows from Lemma 4.11 that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} |(v_k)_i| \left| \sum_{1 \leq j \leq n} \sum_{l=1}^{d_j} f_{kjl i} \widehat{u}(j, l) \epsilon_i \right| = \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} |(v_k)_i| \left| \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} f_{kjl i} \widehat{u}(j, l) \epsilon_i \right| < \infty.$$

Then

$$J = \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} \left| \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} (v_k)_i f_{kjl i} |\widehat{u}(j, l)| \right| < \infty. \quad (4.29)$$

So we proved that if  $(f_{kjl i})$  satisfies

- $\sum_{j=1}^{\infty} \sum_{l=1}^{d_j} |f_{kjl i}| |\widehat{u}(j, l)| < \infty$ ,
- $\sum_{k=1}^{\infty} \sum_{i=1}^{g_k} |(v_k)_i| \left| \left( \sum_{j=1}^{\infty} f_{kjl i} \widehat{u}(j) \right)_i \right| < \infty$ ,

then for any  $u \in H_E^\infty$  and  $v \in [G_F^\infty]^\wedge$  we have from (4.27) and (4.29), respectively, that

- (i)  $\sum_{k=1}^{\infty} \sum_{i=1}^{g_k} |(v_k)_i| |f_{kjl i}| < \infty$ ,
- (ii)  $\sum_{j=1}^{\infty} \sum_{l=1}^{d_j} \left| \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} (v_k)_i f_{kjl i} \right| |\widehat{u}(j, l)| < \infty$ .

Now recall that for  $f : H_E^\infty \rightarrow G_F^\infty$  we have

$$(f(u))_k = \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} f_{kjl i} \widehat{u}(j, l),$$



for any  $u \in H_E^\infty$ , then for any  $v \in [G_F^\infty]^\wedge$ , the composed mapping  $v \circ f : H_E^\infty \rightarrow \mathbb{C}$  is given by

$$\begin{aligned} (v \circ f)(u) &= \sum_{k=1}^{\infty} v_k \cdot (f(u))_k = \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} (v_k)_i \left( \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} f_{kjl} \widehat{u}(j, l) \right) \\ &= \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} \left( \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} (v_k)_i f_{kjl} \right) \widehat{u}(j, l). \end{aligned} \quad (4.30)$$

So by (ii) we get that

$$|(v \circ f)(u)| \leq \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} \left| \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} (v_k)_i f_{kjl} \right| |\widehat{u}(j, l)| < \infty.$$

So  $\widehat{f}(v) = (\widehat{f}(v)_{jl})_{j \in \mathbb{N}, 1 \leq l \leq d_j}$ , with  $\widehat{f}(v)_{jl} = \sum_{k=1}^{\infty} \sum_{i=1}^{g_k} (v_k)_i f_{kjl} \in \widehat{H}_E^\infty$  (from the definition of  $\widehat{H}_E^\infty$ ), that is  $f$  is sequential. And then  $\langle f(u), v \rangle_{G_F^\infty} = \langle u, \widehat{f}(v) \rangle_{H_E^\infty}$  is also true.

Now to prove the converse part we assume that  $f : H_E^\infty \rightarrow G_F^\infty$  is sequential. We have to show that  $f$  can be represented as  $f \approx (f_{kjl})_{k, j \in \mathbb{N}, 1 \leq l \leq d_j, 1 \leq i \leq g_k}$  and satisfies (4.15) and (4.16).

Define for  $k, i$  where  $k \in \mathbb{N}$  and  $1 \leq i \leq g_k$ , the sequence  $u^{ki} = (u_j^{ki})_{j \in \mathbb{N}}$  such that  $u_j^{ki} \in \mathbb{C}^{d_j}$  and  $u_j^{ki}(l) = \widehat{u}^{ki}(j, l)$ , given by

$$u_j^{ki}(l) = \widehat{u}^{ki}(j, l) = \begin{cases} 1, & j = k, l = i, \\ 0, & \text{otherwise.} \end{cases}$$

Then  $u^{ki} \in [G_F^\infty]^\wedge$ . Now since  $f$  is sequential we have  $u^{ki} \circ f \in \widehat{H}_E^\infty$  and  $u^{ki} \circ f = \left( (u^{ki} \circ f)_j \right)_{j \in \mathbb{N}}$ , where  $(u^{ki} \circ f)_j \in \mathbb{C}^{d_j}$ . We denote  $u^{ki} \circ f = (f_j^{ki})_{j \in \mathbb{N}}$ , where  $f_j^{ki} = (u^{ki} \circ f)_j$ . Then  $(f_j^{ki})_{j \in \mathbb{N}} \in \widehat{H}_E^\infty$  and  $f_j^{ki} \in \mathbb{C}^{d_j}$ .

Then for any  $\phi \approx \left( \widehat{\phi}(j) \right)_{j \in \mathbb{N}} \in H_E^\infty$  we have

$$\sum_{j=1}^{\infty} \sum_{l=1}^{d_j} |f_j^{ki}| |\widehat{\phi}(j, l)| < \infty. \quad (4.31)$$

For  $\phi \in H_E^\infty$  we can write  $f(\phi) \in G_F^\infty$ . We can also write

$$f(\phi) = \left( \widehat{f(\phi)}(p) \right)_{p \in \mathbb{N}}.$$

So

$$\begin{aligned}
 u^{ki} \circ f(\phi) &= \sum_{j=1}^{\infty} u_j^{ki} \widehat{(f(\phi))}_j \\
 &= \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} u_{jl}^{ki} \widehat{(f(\phi))}(j, l) \\
 &= \widehat{(f(\phi))}(k, i) \text{ (from the definition of } u^{ki}\text{)}. \tag{4.32}
 \end{aligned}$$

We have  $u^{ki} \circ f = (f^{ki}) \in \widehat{H}_E^{\infty}$ , so

$$\begin{aligned}
 (u^{ki} \circ f)(\phi) &= \sum_{j=1}^{\infty} f_j^{ki} \widehat{\phi}(j) \\
 &= \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} f_{jl}^{ki} \widehat{\phi}(j, l). \tag{4.33}
 \end{aligned}$$

From (4.32) and (4.33) we have  $\widehat{(f(\phi))}(k, i) = \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} f_{jl}^{ki} \widehat{\phi}(j, l)$ .

Hence  $(f(\phi))_{ki} = \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} f_{jl}^{ki} \widehat{\phi}(j, l)$ ,  $k \in \mathbb{N}$ , and  $1 \leq i \leq g_k$ , that is  $f$  is represented by the tensor  $\{(f_{jl}^{ki})\}_{k, j \in \mathbb{N}, 1 \leq i \leq g_k, 1 \leq l \leq d_j}$ .

If we denote  $f_{jl}^{ki}$  by  $f_{kjl}^{ki} = f_{kjl}^{ki}$ , we can say that  $f$  is represented by the tensor  $(f_{kjl}^{ki})_{k, j \in \mathbb{N}, 1 \leq l \leq d_j, 1 \leq i \leq g_k}$ . Also let  $v \in \widehat{G}_F^{\infty}$ . Since  $f(\phi) \in G_F^{\infty}$  for  $\phi \in H_E^{\infty}$ , then from the definition of  $\widehat{G}_F^{\infty}$  we have

$$\sum_{k=1}^{\infty} \sum_{i=1}^{g_k} |(v_k)_i| \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} |f_{kjl}^{ki} \widehat{\phi}(j, l)| < \infty.$$

This completes the proof of Theorem 4.9.  $\square$

## 5. APPLICATIONS TO UNIVERSALITY

In this section we give an application of the developed analysis to the universality problem. We start with the spaces of smooth functions, and then make some remarks how the same arguments can be extended to the Komatsu classes setting from [4].

First we recall the notations:

Let  $\mathcal{H}, \mathcal{G}$  be two Hilbert spaces and let  $E$  and  $F$  be the operators corresponding to the bases  $\{e_j\}_{j \in \mathbb{N}}, \{h_k\}_{k \in \mathbb{N}}$ , as in (2.2), where  $d_j = \dim X_j$  and  $g_k = \dim Y_k$ , and  $X_j = \text{span}\{e_{jl}\}_{l=1}^{d_j}$ ,  $Y_k = \text{span}\{h_{ki}\}_{i=1}^{g_k}$ . Also  $\mathcal{H} = \bigoplus_{j \in \mathbb{N}} X_j$  and  $\mathcal{G} = \bigoplus_{k \in \mathbb{N}} Y_k$ .

We denote the spaces of smooth type functions corresponding to the operators  $E$  and  $F$  in the Hilbert space  $\mathcal{H}$  and  $\mathcal{G}$ , respectively, by  $H_E^{\infty}$  and  $G_F^{\infty}$ .

The main application of Theorem 4.9 will be in the setting when  $X, Y$  are compact manifold without boundary, where  $\mathcal{H} = L^2(X)$  and  $H_E^\infty = C^\infty(X)$ , and  $\mathcal{G} = L^2(Y)$ ,  $G_F^\infty = C^\infty(Y)$ .

Using Theorem 4.9 we prove the universality of the spaces of the smooth type functions,  $C^\infty(X)$ , where we can write

$$f \in C^\infty(X) \iff \forall N \exists C_N : |\widehat{f}(j, k)| \leq C_N(1 + \lambda_j)^{-N} \text{ for all } j, k.$$

Further details of such spaces can be found in [6]. In particular, if  $E$  is an elliptic pseudo-differential operator of positive order, then this is just the usual space of smooth functions on  $X$ .

**Definition 5.1.** *Let  $E$  be a self-adjoint, positive operator. A mapping  $f : X \rightarrow W$  from the compact manifold  $X$  to a sequence space  $W$ , is said to be a  $H_E^\infty$ -mapping if for any  $u \in \widehat{W}$ , the composed mapping  $u \circ f : X \rightarrow \mathbb{C}$  belongs to  $H_E^\infty$ .*

Next we prove the universality of the spaces of smooth type functions.

**Theorem 5.2.** *Let  $X$  be a compact manifold.*

(i) *The delta mapping  $\delta : X \rightarrow \widehat{H_E^\infty}$  defined by*

$$\delta(x) = \delta_x,$$

and

$$\delta_x(\phi) = \langle \delta_x, \phi \rangle_{H_E^\infty} = \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} \widehat{\phi}(j, l)(\delta_x)_{jl} = \phi(x), \text{ for all } \phi \in H_E^\infty, x \in X,$$

is a  $H_E^\infty$ -mapping.

(ii) *If  $\tilde{g} : \widehat{H_E^\infty} \rightarrow G_F^\infty$  is a sequential linear mapping, then the composed mapping  $\tilde{g} \circ \delta : X \rightarrow G_F^\infty$  is a  $H_E^\infty$ -mapping.*

(iii) *For any  $H_E^\infty$ -mapping  $f : X \rightarrow G_F^\infty$ , there exists a unique sequential linear mapping  $\widehat{f} : \widehat{H_E^\infty} \rightarrow G_F^\infty$  such that  $f = \widehat{f} \circ \delta$ .*

*Proof.* (i) Recall that  $\widehat{H_E^\infty} = \bigcup_{s \in \mathbb{R}} H_E^{-s} = [H_E^\infty]'$ .

Let  $v \in [\widehat{H_E^\infty}]^\wedge = H_E^\infty$ . We define the composed mapping

$$v \circ \delta_x = \langle \delta_x, v \rangle_{H_E^\infty} = \sum_{j=1}^{\infty} \sum_{l=1}^{d_j} v_{jl}(\delta_x)_{jl} = v(x) \text{ (by definition of } \delta).$$

This is well-defined since  $v \in H_E^\infty$  and  $\delta_x \in [H_E^\infty]' = \widehat{H_E^\infty}$ . Also since  $v \in H_E^\infty$ , we see that  $v \circ \delta \in H_E^\infty$  and that implies  $\delta$  is a  $H_E^\infty$ -mapping.

(ii) Let  $u \in \widehat{G_F^\infty} = [G_F^\infty]'$ .

From the definition of  $H_E^\infty$ -mapping we have to show that  $u \circ \tilde{g} \circ \delta \in H_E^\infty$ .

Now by given condition  $\tilde{g} : [H_E^\infty]' \rightarrow G_F^\infty$ , so we have  $u \circ \tilde{g} \in [H_E^\infty]''$ .

Here we claim that  $[H_E^\infty]'' = H_E^\infty$ .

Note that  $H_E^\infty \subseteq [H_E^\infty]''$ .

Recall that,  $[H_E^\infty]' = \bigcup_{s \in \mathbb{R}} H_E^{-s}$  and so for any  $s \in \mathbb{R}$ ,

$$H_E^{-s} \subseteq [H_E^\infty]' \Rightarrow [H_E^\infty]'' \subseteq [H_E^{-s}]' \Rightarrow [H_E^\infty]'' \subseteq H_E^s.$$

Since the above is true for any  $s \in \mathbb{R}$ , we have  $[H_E^\infty]'' \subseteq \bigcap_{s \in \mathbb{R}} H_E^s = H_E^\infty$  and this gives

$$[H_E^\infty]'' = H_E^\infty.$$

Then  $u \circ \tilde{g} \in H_E^\infty$ . Using same argument as in the proof of (i) and from the definition of the mapping  $\delta$  we have  $u \circ \tilde{g} \circ \delta_x = u \circ \tilde{g}(x)$  and  $u \circ \tilde{g} \circ \delta$  belongs to  $H_E^\infty$ .

So from Definition 5.1,  $\tilde{g} \circ \delta$  is a  $H_E^\infty$ -mapping.

(iii) Existence of  $\hat{f} : \widehat{H}_E^\infty \rightarrow G_F^\infty$ .

By hypothesis,  $f : X \rightarrow G_F^\infty$  is a  $H_E^\infty$ -mapping so that for any  $v \in \widehat{G}_F^\infty$ ,  $v \circ f \in H_E^\infty$ . A sequential linear mapping  $\tilde{f} : \widehat{G}_F^\infty \rightarrow H_E^\infty$  can be defined by  $\tilde{f}(v) = v \circ f$ , and  $v(f(x)) = \langle f(x), v \rangle_{G_F^\infty}$ .

Hence by Theorem 4.9 there is an adjoint mapping, we denote it by  $\hat{f}$ , where  $\hat{f} : \widehat{H}_E^\infty \rightarrow [\widehat{G}_F^\infty]^\wedge$  is a sequential mapping. By the definition of the adjoint mapping  $\hat{f}$  we have

$$\langle u, \tilde{f}(v) \rangle_{H_E^\infty} = \langle \hat{f} \circ u, v \rangle_{G_F^\infty}, \quad u \in [H_E^\infty]^\wedge,$$

where  $\langle \cdot, \cdot \rangle$  is the bilinear function on  $H_E^\infty \times \widehat{H}_E^\infty$  defined in Section 3. The above can be written as

$$\langle u, v \circ f \rangle_{H_E^\infty} = \langle \hat{f} \circ u, v \rangle_{G_F^\infty}, \quad u \in \widehat{H}_E^\infty.$$

For  $u = \delta_x$ , this gives

$$\langle \hat{f} \circ \delta_x, v \rangle_{G_F^\infty} = \langle \delta_x, v \circ f \rangle_{H_E^\infty} = (v \circ f)(x) = v(f(x)) = \langle f(x), v \rangle_{G_F^\infty},$$

for any  $v \in \widehat{G}_F^\infty$ . This proves  $f = \hat{f} \circ \delta$ .

Uniqueness of  $\hat{f} : \widehat{H}_E^\infty \rightarrow G_F^\infty$ .

Suppose  $\hat{f} \circ \delta = f = 0$ . We have to show that  $\hat{f} = 0$  on  $\widehat{H}_E^\infty$ . Since  $\hat{f}$  is sequential, there exists  $g : \widehat{G}_F^\infty \rightarrow H_E^\infty$  such that

$$\langle \hat{f} \circ u, v \rangle_{G_F^\infty} = \langle u, g \circ v \rangle_{H_E^\infty}.$$

Take  $u = \delta_x \in \widehat{H}_E^\infty$ , then

$$\langle \hat{f} \circ \delta_x, v \rangle_{G_F^\infty} = \langle \delta_x, g \circ v \rangle_{H_E^\infty} = g(v(x)) = 0$$

for any  $v \in \widehat{G}_F^\infty$ , that is,  $g = 0$  on  $\widehat{G}_F^\infty$ . From  $\langle \hat{f} \circ u, v \rangle_{G_F^\infty} = 0$  for any  $v \in \widehat{G}_F^\infty$ , we get  $\hat{f} \circ u = 0$  for any  $u \in \widehat{H}_E^\infty$ , that is  $\hat{f} = 0$  on  $\widehat{H}_E^\infty$ .  $\square$

### 5.1. Extension to Komatsu classes.

Here we briefly outline how the analysis above can be extended to the setting of Komatsu classes from [4, 5].

**Remark 5.3.** *In another work ([5]) we studied the Komatsu classes of ultra-differentiable functions  $\Gamma_{\{M_k\}}(X)$  on a compact manifold  $X$ , where  $M_{\{k\}}$  be a sequence of positive numbers such that*

- (1)  $M_0 = 1$ ,
- (2)  $M_{k+1} \leq AH^k M_k$ ,  $k = 0, 1, 2, \dots$ ,
- (3)  $M_{2k} \leq AH^k \min_{0 \leq q \leq k} M_q M_{k-q}$ ,  $k = 0, 1, 2, \dots$ , for some  $A, H > 0$ .

*In [5] we have characterised the dual spaces of these Komatsu classes and have shown that these spaces are perfect spaces, i.e, these spaces coincide with their second dual spaces. Furthermore, in [5, Theorem 4.7] we proved the following theorem for Komatsu classes of functions on a compact manifold  $X$ :*

**Theorem 5.4** (Adjointness Theorem). *Let  $\{M_k\}$  and  $\{N_k\}$  satisfy conditions (M.0)–(M.3). A linear mapping  $f : \Gamma_{\{M_k\}}(X) \rightarrow \Gamma_{\{N_k\}}(X)$  is sequential if and only if  $f$  is represented by an infinite tensor  $(f_{kjl})$ ,  $k, j \in \mathbb{N}_0$ ,  $1 \leq l \leq d_j$  and  $1 \leq l \leq d_k$  such that for any  $u \in \Gamma_{\{M_k\}}(X)$  and  $v \in \widehat{\Gamma_{\{N_k\}}(X)}$  we have*

$$\sum_{j=0}^{\infty} \sum_{l=1}^{d_j} |f_{kjl}| |\widehat{u}(j, l)| < \infty, \text{ for all } k \in \mathbb{N}_0, i = 1, 2, \dots, d_k,$$

and

$$\sum_{k=0}^{\infty} \sum_{i=1}^{d_k} |(v_k)_i| \left| \left( \sum_{j=0}^{\infty} f_{kj} \widehat{u}(j) \right)_i \right| < \infty.$$

*Furthermore, the adjoint mapping  $\widehat{f} : \widehat{\Gamma_{\{N_k\}}(X)} \rightarrow \widehat{\Gamma_{\{M_k\}}(X)}$  defined by the formula  $\widehat{f}(v) = v \circ f$  is sequential, and the transposed matrix  $(f_{kj})^t$  represents  $\widehat{f}$ , with  $f$  and  $\widehat{f}$  related by  $\langle f(u), v \rangle = \langle u, \widehat{f}(v) \rangle$ .*

The above theorem described the tensor structure of sequential mappings on spaces of Fourier coefficients and characterised their adjoint mappings. Now in particular the considered classes include spaces of analytic and Gevrey functions (which are perfect spaces too), as well as spaces of ultradistributions, yielding tensor representations for linear mappings between these spaces on compact manifolds. Now using [5, Theorem 4.7] and the same techniques used in this paper to prove the universality of smooth functions in Theorem 5.2, on compact manifolds in Section 5, one also obtains the universality of the Gevrey classes of ultradifferentiable functions on compact groups (from [3]) and Komatsu classes of functions in compact manifolds. As the proof would be a repetition of those arguments, we omit it here.

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