From amplitudes to gravitational radiation with cubic interactions and tidal effects

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We study the effect of cubic and tidal interactions on the spectrum of gravitational waves emitted in the inspiral phase of the merger of two nonspinning objects. There are two independent parity-even cubic interaction terms, which we take to be $I_1 = R^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}$ and $G_3 = I_1 - 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}R^{\rho}{}_{\alpha}{}^{\sigma}{}_{\beta}$. The latter has vanishing pure graviton amplitudes but modifies mixed scalar/graviton amplitudes which are crucial for our study. Working in an effective field theory setup, we compute the modifications to the quadrupole moment due to I_1, G_3 and tidal interactions, from which we obtain the power of gravitational waves radiated in the process to first order in the perturbations and leading order in the post-Minkowskian expansion. The I_1 predictions are novel, and we find that our results for G_3 are related to the known quadrupole corrections arising from tidal perturbations, although the physical origin of the G_3 coupling is unrelated to the finite-size effects underlying tidal interactions. We show this by recomputing such tidal corrections and by presenting an explicit field redefinition. In the post-Newtonian expansion our results are complete at leading order, which for the gravitational-wave flux is 5PN for G_3 and tidal interactions and 6PN for I_1. Finally, we compute the corresponding modifications to the waveforms.

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I. INTRODUCTION

The first direct detection of gravitational waves and the first observation of a binary black hole merger by the LIGO/Virgo collaboration [1] has opened a new observational window potentially challenging our understanding of gravity. Anticipating improved experimental sensitivity in the future, high-precision theoretical predictions from general relativity will be required, and in the recent few years much effort went into developing new theoretical tools using traditional and novel approaches. This includes important calculations of the effective gravitational potential at second [2,3], third [4–7], fourth [8–20], fifth [21–23] and sixth [24,25] post-Newtonian order (PN),[1] as well as in the post-Minkowskian expansion [28–30] and formal developments in computing classical observables from scattering amplitudes [31–50]. A related, ambitious question is whether gravitational waves can, now or in the near future, provide feasible tests of modifications of general relativity as implied by string theory or other extensions of Einstein-Hilbert (EH) gravity. Even if the experimental precision has not been reached today, one can entertain this tantalizing possibility.

An effective field theory (EFT) framework for gravity was advocated in [51], and is ideally suited to study systematically higher-derivative corrections to the EH theory. In [52], this approach was followed to compute the corrections to the gravitational potential between compact objects and their effective mass and current quadrupoles due to perturbations quartic in the Riemann tensor, and the corresponding modifications to the waveforms were then analyzed in [53]. Modifications to the gravitational potential due to cubic interactions in the Riemann tensor were computed in [54,55] using amplitude techniques, and the deflection angle and time delay/advance of massless particles of spin 0,1 and 2 were derived in [56] for cubic and quartic perturbations in the Riemann tensor as well as for interactions of the type FFR [56]. Terms that are quadratic in the Riemann tensor do not contribute to the classical scattering of particles in four dimensions [57]. In this paper we wish to describe dissipative effects in the dynamics of binaries, that is gravitational-wave radiation, from appropriate five-point amplitudes with four massive scalars and one radiation graviton. We perform this study in the presence of cubic

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[1] For a recent review of the EFT approach to the binary problem [26] in the PN expansion see [27].

modifications to the EH action and tidal effects. Interestingly, we will see that there is an overlap between these two types of corrections, which are linked by appropriate field redefinitions [58,59] which we construct explicitly. We note however that the physical origin of these interactions is very different—for instance, I_1 and G_3 appear in the low-effective action of bosonic strings, or can be induced by integrating out massive matter [60,61].

In the presence of scalars and restricting our focus to parity-even interactions, there are two independent cubic terms: $I_1 \coloneqq R^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}$ and $I_2 \coloneqq R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}R^{\rho}{}_{\alpha}{}^{\sigma}{}_{\beta}$. A more natural combination is in fact $G_3 \coloneqq I_1 - 2I_2$, which, as is well known, is topological in six dimensions [62] and has vanishing graviton amplitudes. In [63], it was argued from studying the scattering of polarized gravitons that I_1 potentially leads to superluminal effects/causality violation in the propagation of gravitons for impact parameter $b \lesssim \alpha^{\frac{1}{4}}$. Here $\alpha \sim \Lambda^{-4}$ is the coupling constant of the I_1 interaction, and Λ is the cutoff of the theory. In that paper, α was chosen to be much larger than $G^2 \sim M_{\mathrm{Planck}}^{-4}$. This allows us to treat the gravitational scattering in a semiclassical setup, where predictions can be trusted up to $M_{\mathrm{Planck}}(> \Lambda)$. Because of these considerations, cubic terms were not considered in the analysis of [52,53] (while not conclusively excluding their potential relevance). The issue of superluminality was reinvestigated in an EFT framework in [56], where it was found that the I_1 interaction leads to a time advance in the propagation of gravitons (but not photons and scalars) when $b \lesssim \alpha^{\frac{1}{4}}$. Finally, G_3 does not lead to any time advance/delay for massless particles [56], while still correcting the gravitational potential [54,55]. An identical conclusion for the propagation of massless particles in the background of a black hole was reached in [64], both for the I_1 and G_3 interactions.[2]

In this respect, an important observation was made in [65], namely that such superluminality effects (and those observed earlier on in [66–68]) are unresolvable within the regime of validity of the EFT, and do not lead to violations of causality. In our setup such violations would indeed occur at $b \lesssim \Lambda^{-1}$, which is at the boundary of the regime of validity of our EFT, while the processes we are interested in only probe the regime where the EFT is valid. Above Λ, the only known way to restore causality is to introduce an infinite tower of massive particles [63]. In conclusion, these observations do not rule out cubic interactions for our EFT computation, although they may impose constraints on the cutoff—it needs to be such that possible effects due to the massive modes, required to ensure causality, cannot be resolved with current-day experiments. We also note that, assuming that these interactions can contribute to any classical gravitational scattering ($\Lambda < M_{\mathrm{Planck}}$), then we have $\alpha > G^2$, independently of precise estimates of the cutoff Λ.

In the following we work in an effective theory containing cubic and tidal perturbations, and compute a five-point amplitude with four massive scalars (representing the black holes) and one radiated soft graviton. From this, one can in principle extract all radiative multipole moments to this order, but for the sake of our applications we will only focus on the quadrupole moment induced by the cubic and tidal interactions, from which we then derive the corresponding changes to the power radiated by gravitational waves and to the waveforms. Our results for the quadrupole correction are exact to leading order in the perturbations and in the post-Minkowskian expansion. We also take the post-Newtonian expansion of our results, which are complete at 5PN order for the G_3 and tidal interaction corrections, and at 6PN order for the I_1 corrections. We find that the corrections due to G_3 have the same form as those generated by a particular type of tidal interaction (although the corresponding coefficients in the EFT action are independent). We also explain this result by constructing an explicit field redefinition relating the two couplings. For the PN-expanded result of the tidal corrections to the mass quadrupole we find agreement with [69–71]. The remaining tasks consist in using the corrected quadrupole moment to compute the modifications compared to EH gravity to the power emitted by the radiated gravitational waves, and the corresponding corrections to the waveforms in the stationary phase approximation (SPA).[3] Here we follow closely [53], and also present a comparison with their result obtained with perturbations that are quartic in the Riemann tensor.

The rest of the paper is organized as follows. In Sec. II we introduce the EFT we are discussing, reviewing some of the relevant results, including the corrections to the gravitational potential from cubic [54,55] and tidal interactions [74–76]. Furthermore, we point out the vanishing of all graviton amplitudes in the pure gravity plus G_3 theory, and explicitly construct a field redefinition that maps G_3 into a tidal perturbation. Section III contains the calculation of the relevant four-scalar, one soft graviton amplitude in our EFT, from which we extract the perturbations to the quadrupole moment. In Sec. IV we compute the power radiated by the gravitational waves, and finally in Sec. V the corrections to the waveforms in the SPA. In an Appendix we present some details on the modifications to the circular orbits due to the perturbations.

II. DESCRIPTION OF THE THEORY

A. The EFT action

We consider an EFT describing EH gravity with higher-derivative couplings interacting with two massive scalars.

[2] Note that for G_3 the coefficient $2d_9 + d_{10}$ in Eq. (2.24) of [64] vanishes.

[3] See e.g., [72,73] for details of this approximation.

These model spinless heavy objects, and we also include the leading tidal interactions in our description which describe finite size effects of the heavy objects. Specifically, the EFT action we consider is

$$S = S_{\mathrm{eff}} + S_{\phi_1\phi_2} + S_{\mathrm{tidal}}, \tag{2.1}$$

where

$$S_{\mathrm{eff}} = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{2}{\kappa^2}R - \frac{2}{\kappa^2}\mathcal{L}_6 - \cdots\right] \tag{2.2}$$

is the effective action for gravity, with

$$\mathcal{L}_6 = \frac{\alpha_1}{48}I_1 + \frac{\alpha_2}{24}G_3. \tag{2.3}$$

I_1 and G_3 are the parity-even cubic couplings defined as

$$I_1 \coloneqq R^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}, \qquad G_3 \coloneqq I_1 - 2I_2, \tag{2.4}$$

with

$$I_2 \coloneqq R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}R^{\rho}{}_{\alpha}{}^{\sigma}{}_{\beta}. \tag{2.5}$$

The dots in (2.2) stand for higher-derivative interactions that we will not consider here. The two scalars, with masses m_1 and m_2, couple to gravity with an action

$$S_{\phi_1\phi_2} = \int \mathrm{d}^4x\sqrt{-g}\frac{1}{2}\sum_{i=1,2}(\partial_\mu\phi_i\partial^\mu\phi_i - m_i^2\phi_i^2), \tag{2.6}$$

and in addition we include higher-derivative couplings describing tidal effects of extended heavy objects,

$$S_{\mathrm{tidal}} = \int \mathrm{d}^4x\sqrt{-g}\frac{1}{4}R_{\mu\alpha\nu\beta}R^{\rho\alpha\sigma\beta} \times \sum_{i=1,2}\left(\lambda_i\phi_i^2\delta^\mu_\rho\delta^\nu_\sigma + \frac{\eta_i}{m_i^4}\nabla^\mu\nabla^\nu\phi_i\nabla_\rho\nabla_\sigma\phi_i\right) + \cdots. \tag{2.7}$$

These tidal interactions were recently studied in [76], and the dots stand for the (Hilbert) series of higher-dimensional operators classified in [59,77], which will not play any role in this work. We now briefly discuss some properties of the interactions we consider.

B. Cubic interactions

The I_1 and G_3 interactions naturally arise in the low-energy effective description of bosonic string theory, whose terms cubic in the curvature can be obtained by making the replacement

$$\alpha_1 = \alpha_2 \to \alpha'^2 e^{-4\Phi} \tag{2.8}$$

in (2.3), where Φ is the dilaton. These interactions are also produced in the process of integrating out massive matter [60,61]. In pure gravity only one of them is independent in four dimensions [78,79], while in the presence of matter coupled to gravity they become independent. For the sake of the computation of the power radiated by the gravitational waves performed in later sections we need the correction induced by the cubic interactions to the gravitational potential. The full 2nd Post-Minkowskian (PM) computation of this quantity was performed in [54,55], and expanding their result one obtains

$$V(\vec{r},|\vec{p}|) = -\frac{Gm_1m_2}{r} + \frac{3}{8}\frac{\alpha_1G^2}{r^6}\frac{(m_1+m_2)^3}{m_1m_2}\vec{p}^2 - \frac{3}{4}\frac{\alpha_2G^2}{r^6}m_1m_2(m_1+m_2)\left(1 - \frac{m_1^2+m_2^2}{2m_1^2m_2^2}\vec{p}^2\right) + \cdots, \tag{2.9}$$

where the dots indicate higher PN corrections which we do not consider here. Note that the terms proportional to α_1 and α_2 are the result of a one-loop computation. In the PN expansion, the term proportional to α_1 (from the I_1 interaction) is suppressed by a factor of $\vec{p}^2/m_{1,2}^2$ compared to the dominant correction proportional to α_2 (from G_3).

1. Amplitudes from the G_3 interaction

It is well known that, unlike I_1, the G_3 interaction has a vanishing three-graviton amplitude and does not contribute to graviton scattering up to four particles [62,80]—and in fact to any number of gravitons. This can be understood by the fact that G_3 is topological in six dimensions [62], and therefore computing tree-level four-dimensional graviton amplitudes from dimensionally reducing the six-dimensional ones automatically gives zero. Combining this observation with unitarity techniques leads to

$$\mathcal{M}_{\mathrm{EH}+G_3}(h_1,\ldots,h_n)|_{d<6} = \mathcal{M}_{\mathrm{EH}}(h_1,\ldots,h_n)|_{d<6}, \tag{2.10}$$

for any n. Hence the G_3 interaction does not affect the perturbative dynamics in theories of pure gravity. However, if we consider a theory of gravity with matter, e.g., massive scalars mimicking black holes or neutron stars, the presence of a G_3 coupling alters their dynamics. In particular the four-point amplitude with two gravitons and two scalars becomes [54,55]

$$\mathcal{M}^{(0)}_{\mathrm{EH}+G_3}(\phi_1,\phi_2,h_3^{++},h_4^{++}) = \mathcal{M}^{(0)}_{\mathrm{EH}}(\phi_1,\phi_2,h_3^{++},h_4^{++}) + i\frac{\alpha_2}{32}\left(\frac{\kappa}{2}\right)^2[34]^4(2m^2+s). \tag{2.11}$$

The nontrivial contribution to the scattering amplitude of two massive scalars and two gravitons from the G_3 interactions modifies the classical potential in the two-body system, as shown in [54,55]. As we will show below, both G_3 and I_1 produce corrections to the quadrupole moment already at tree level. Specifically we find that the G_3 quadrupole correction is dominant in the PN expansion, which parallels the results found for the corresponding corrections to the gravitational potential quoted earlier in (2.9).

2. The G_3 interaction as a tidal effect

It is easy to show that the contact term proportional to $[34]^4(2m^2+s)$ in the amplitude (2.11) is (up to a numerical coefficient) the amplitude arising from a particular tidal interactions of the form $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}m^2\phi^2 - \nabla^\alpha R^{\mu\nu\rho\sigma}\nabla_\alpha R_{\mu\nu\rho\sigma}\phi^2$. This suggests that there should exist a four-dimensional field redefinition mapping the G_3 interaction into a tidal effect, as already noticed in [58,59].[4] In this section we construct this field redefinition explicitly.

We begin by rewriting G_3 in a more convenient form, making use of two identities in four dimensions [83]:

$$R^{\alpha\beta}{}_{[\alpha\beta}R^{\mu\nu}{}_{\mu\nu}R^{\rho}{}_{\rho]} = 0, \tag{2.12}$$

which translates into

$$R^{\alpha}{}_{\beta}R^{\beta\rho}{}_{\mu\nu}R^{\mu\nu}{}_{\alpha\rho} = \frac{1}{4}R^3 - 2RR^{\alpha}{}_{\beta}R^{\beta}{}_{\alpha} + 2R^{\alpha}{}_{\beta}R^{\mu}{}_{\nu}R^{\beta\nu}{}_{\alpha\mu} + 2R^{\alpha}{}_{\beta}R^{\beta}{}_{\mu}R^{\mu}{}_{\alpha} + \frac{1}{4}RR^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\alpha\beta}, \tag{2.13}$$

and

$$R^{\alpha\beta}{}_{[\alpha\beta}R^{\mu\nu}{}_{\mu\nu}R^{\rho\sigma}{}_{\rho\sigma]} = 0, \tag{2.14}$$

which, in combination with (2.13), leads to

$$\begin{aligned} R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}R^{\rho}{}_{\alpha}{}^{\sigma}{}_{\beta} = {}& \frac{1}{2}R^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta} - \frac{5}{8}R^3 \\ &+ \frac{9}{2}RR^{\alpha}{}_{\beta}R^{\beta}{}_{\alpha} - \frac{3}{8}RR^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\alpha\beta} \\ &- 3R^{\alpha}{}_{\beta}R^{\mu}{}_{\nu}R^{\beta\nu}{}_{\alpha\mu} - 4R^{\alpha}{}_{\beta}R^{\beta}{}_{\mu}R^{\mu}{}_{\alpha}. \end{aligned} \tag{2.15}$$

The latter identity implies that, in four dimensions, G_3 can be rewritten as

$$\begin{aligned} G_3|_{d=4} = {}& \frac{3}{4}RR^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\alpha\beta} + \frac{5}{4}R^3 - 9RR^{\alpha}{}_{\beta}R^{\beta}{}_{\alpha} \\ &- 8R^{\alpha}{}_{\beta}R^{\beta}{}_{\mu}R^{\mu}{}_{\alpha} + 6R^{\alpha}{}_{\beta}R^{\mu}{}_{\nu}R^{\beta\nu}{}_{\alpha\mu} \\ \sim{}& \frac{3}{4}RR^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\alpha\beta}, \end{aligned} \tag{2.16}$$

where in the second line we have dropped all terms involving more than one Ricci scalar/tensor. These terms can be traded, via a further field redefinition, for a contact term of the form $R_{\mu\nu\rho\sigma}\partial^\mu\phi_1\partial^\nu\phi_2\partial^\rho\phi_1\partial^\sigma\phi_2$, which only contributes to quantum corrections to the quadrupole moment. Thus

$$\begin{aligned} S_{\mathrm{eff}} &= \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{2}{\kappa^2}R - \frac{\alpha_2}{12\kappa^2}G_3\right] + S_{\phi_1,\phi_2}, \\ &= \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{2}{\kappa^2}R - \frac{\alpha_2}{16\kappa^2}R(R^{\alpha\beta\mu\nu})^2 + \cdots\right] + S_{\phi_1,\phi_2} \\ &\to \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{2}{\kappa^2}R + \frac{\alpha_2}{64}(R^{\alpha\beta\mu\nu})^2 \right. \\ &\qquad \left. \times \sum_{i=1,2}(2m_i^2\phi_i^2 - \partial_\mu\phi_i\partial^\mu\phi_i) + \mathcal{O}(\alpha_2^2)\right] + S_{\phi_1,\phi_2}, \end{aligned} \tag{2.17}$$

where in the last line we have used the field redefinition

$$g_{\alpha\beta} \to g_{\alpha\beta} - \frac{\alpha_2}{32}g_{\alpha\beta}R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\mu\nu}. \tag{2.18}$$

Finally, integrating by parts and discarding boundary contributions, we can rewrite the new interaction term in (2.17) as

$$\begin{aligned} &(R^{\alpha\beta\mu\nu})^2(2m^2\phi^2 - \partial_\mu\phi\partial^\mu\phi) \\ &\quad = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}m^2\phi^2 - \nabla^\alpha R^{\mu\nu\rho\sigma}\nabla_\alpha R_{\mu\nu\rho\sigma}\phi^2, \end{aligned} \tag{2.19}$$

where the second term does not give any classical contribution to the scattering amplitude at leading order. Hence, for the sake of computing classical contributions to amplitudes, we can replace

$$\begin{aligned} S_{\mathrm{eff}} \to {}& \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{2}{\kappa^2}R + \frac{\alpha_2}{64}(R^{\alpha\beta\mu\nu})^2\sum_{i=1,2}m_i^2\phi_i^2 + \mathcal{O}(\alpha_2^2)\right] \\ &+ S_{\phi_1,\phi_2}, \end{aligned} \tag{2.20}$$

thereby explicitly showing that the G_3 interaction can be absorbed into the first of the two tidal interactions in (2.7).

C. Tidal effects

During the inspiral phase of binary systems involving at least one extended heavy object like a neutron star, corrections due to the finite size of the object(s) increase as the distance between the objects decreases. These effects can be included systematically using a tidal expansion, i.e., a multipole expansion dominated by the mass quadrupole moment. Finite-size effects are bound to become of ever increasing importance in the light of future gravitational-wave experiments, and will likely play a key role in a deeper understanding of the internal structure of compact

[4]We also observe that black holes in four dimensions have nonvanishing Love numbers when higher-derivative interactions are considered [81,82].

objects. The computation of tidal effects has been addressed in the past by a wide variety of methods, recently including complete PM results [74–76] for the conservative dynamics.

In order to compute the modifications to the waveform coming from the tidal interactions in (2.7) we need to expand the 2PM potential in the conservative Hamiltonian computed in [74–76] up to $\mathcal{O}(\vec{p}^2)$, with the result

$$V_{\mathrm{tidal}}(\vec{r},\vec{p}) = -\frac{3}{2}\frac{G^2}{r^6}\frac{m_2^2}{m_1}\left[8\left(1-\frac{m_1^2+m_2^2}{2m_1^2m_2^2}\vec{p}^2\right)\lambda_1 + \left(1+\frac{2m_1^2+2m_2^2+5m_1m_2}{m_1^2m_2^2}\vec{p}^2\right)\eta_1\right] + 1\leftrightarrow 2+\cdots, \tag{2.21}$$

where the dots indicate higher PN terms.

III. QUADRUPOLE MOMENTS IN EFTs OF GRAVITY

In the PN framework, the conservative and dissipative dynamics of two objects of mass m_1 and m_2, coupled to the gravity effective action (2.2), is described by the following point-particle effective action [26,52]:

$$S_{\mathrm{pp}} = \int dt\left[\frac{1}{2}\mu\dot{\vec{r}}^2 - V(\vec{r},\vec{p}) + \frac{1}{2}Q^{ij}(\vec{r},\vec{p})R^{0i0j}+\cdots\right], \tag{3.1}$$

where

$$\mu \coloneqq \frac{m_1m_2}{m_1+m_2} \tag{3.2}$$

is the reduced mass, and $\vec{r}(t)$ is the relative position of the two objects. $V(\vec{r},\vec{p})$ denotes the potential, whose explicit expression to first order in α_1, α_2 [54,55], and $\lambda_{1,2}$, $\eta_{1,2}$ [74–76] is obtained by summing (2.9) and (2.21), and $Q^{ij}(\vec{r},\vec{p})$ is the quadrupole moment, to be computed below. The dots represent higher-order terms that will be irrelevant in our analysis. This action can be trusted in the inspiral phase before the objects reach relativistic velocities.

We now present the computation of the five-point amplitude $\phi_1\phi_2 \to \phi_1\phi_2 + \bar{h}(k)$ with four scalars and one radiated soft graviton $\bar{h}(k)$. Its momentum k^μ is on shell, while the momentum of the graviton exchanged between the two objects is purely spacelike (corresponding to an instantaneous interaction), and in our setup is given by $q^\mu = -p_1^\mu - p_2^\mu = (0,\vec{q})$. Furthermore, the energy of the radiated graviton is such that $k^0 \ll |\vec{q}|$, so that k^μ can be ignored for practical purposes, and the radiated graviton enters the amplitude only through its associated Riemann curvature tensor $\bar{R}_{\alpha\beta\mu\nu}$. Finally, because we are only interested in classical contributions [i.e., $\mathcal{O}(\hbar^0)$], we keep only the leading terms in $\vec{q}^2$.

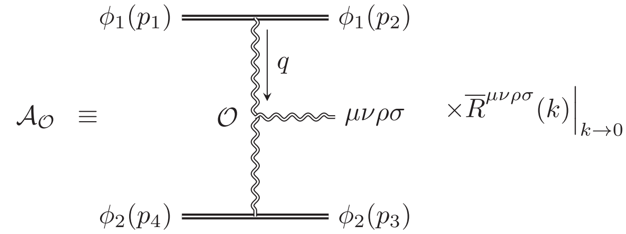

FIG. 1. The single diagram contributing to the radiation process with an insertion of the operators $\mathcal{O} = I_1, I_2$. All momenta are treated as outgoing and the radiated graviton is taken to be soft.

In the following we first compute fully relativistic scattering amplitudes and then perform the PN expansion to extract the correction to the quadrupole term in the effective action (3.1). In the center-of-mass frame, the momenta of the particles can be parametrized as

$$p_1^\mu = -\left(E_1, \vec{p}-\frac{\vec{q}}{2}\right), \qquad p_4^\mu = -\left(E_4, -\vec{p}+\frac{\vec{q}}{2}\right),$$
$$p_2^\mu = \left(E_2, \vec{p}+\frac{\vec{q}}{2}\right), \qquad p_3^\mu = \left(E_3, -\vec{p}-\frac{\vec{q}}{2}\right), \tag{3.3}$$

with $p_1^2 = p_2^2 = m_1^2$, $p_3^2 = p_4^2 = m_2^2$. Furthermore, we have

$$E_1 = E_2 = \sqrt{m_1^2+\vec{p}^2+\vec{q}^2/4},$$
$$E_3 = E_4 = \sqrt{m_2^2+\vec{p}^2+\vec{q}^2/4}, \tag{3.4}$$

where $\vec{p}\cdot\vec{q} = 0$ because of momentum conservation. In our all-outgoing convention for the external lines, the four-momenta p_1 and p_4 correspond to the incoming particles, and hence their energies are negative.

A. The amplitude with cubic interactions

Our next task is to compute the five-point amplitude $\mathcal{A}_{\mathcal{O}}$ shown in Fig. 1, with $\mathcal{O} = I_1, I_2$ (which we can then combine to obtain $\mathcal{A}_{G_3}$). We first obtain its relativistic expression, factoring out a single Riemann tensor associated with the radiated graviton, and then split the Lorentz indices into time and spatial components and isolate the terms contracted into $\bar{R}_{0i0j}$. Upon Fourier transforming to position space, these components will allow to directly read off Q_{ij} by matching to the Hamiltonian density associated to the point particle effective action (3.1). The classical relativistic results are, for I_1,

$$\mathcal{A}_{I_1} = i(\alpha_1+2\alpha_2)\left(\frac{\kappa}{2}\right)^2\frac{q^\mu q^\rho}{q^2}[m_1^2p_3^\nu p_3^\sigma + m_2^2p_1^\nu p_1^\sigma - 2(p_1\cdot p_3)p_1^\nu p_3^\sigma]\bar{R}_{\mu\nu\rho\sigma}, \tag{3.5}$$

while, for I_2,

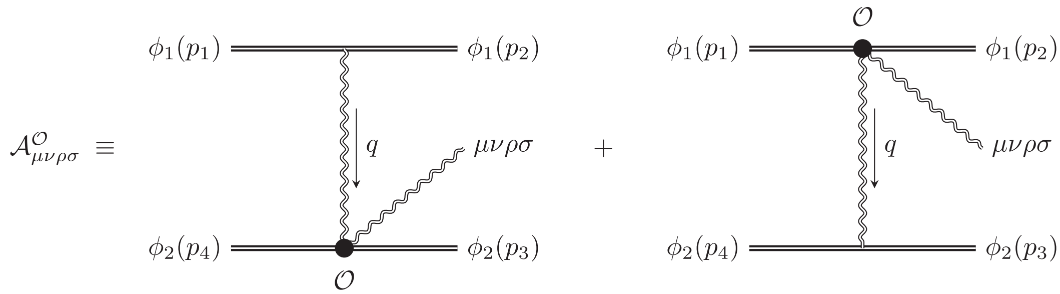

FIG. 2. The two diagrams contributing to the gravitational radiation, where $\mathcal{O}$ denotes any of the two tidal interactions in (2.7). An overall Riemann tensor of the radiated graviton is factored out, so that $\mathcal{A}^{\mathcal{O}} = \mathcal{A}^{\mathcal{O}}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma}(k \to 0)$.

$$\mathcal{A}_{I_2} = \frac{i}{2}\alpha_2\left(\frac{\kappa}{2}\right)^2 \frac{q^\mu q^\rho}{q^2}(m_1^2 p_3^\nu p_3^\sigma + m_2^2 p_1^\nu p_1^\sigma)\bar{R}_{\mu\nu\rho\sigma}. \tag{3.6}$$

Note that the result for the G_3 interaction introduced in (2.3) can be obtained as

$$\mathcal{A}_{G_3} \coloneqq (\mathcal{A}_{I_1} + \mathcal{A}_{I_2})|_{\alpha_1=0}. \tag{3.7}$$

The terms in the amplitude contributing to the quadrupole radiation are then

$$\mathcal{A}_{I_1}(q) = -i(\alpha_1 + 2\alpha_2)\left(\frac{\kappa}{2}\right)^2 (m_1^2 E_4^2 + m_2^2 E_1^2 - 2E_1^2 E_4^2 - 2\vec{p}^2 E_1 E_4)\frac{q^i q^j}{\vec{q}^2}\bar{R}_{0i0j} + \cdots, \tag{3.8}$$

and

$$\mathcal{A}_{I_2}(q) = -i\frac{\alpha_2}{2}\left(\frac{\kappa}{2}\right)^2 (m_1^2 E_4^2 + m_2^2 E_1^2)\frac{q^i q^j}{\vec{q}^2}\bar{R}_{0i0j} + \cdots, \tag{3.9}$$

where we have used that $E_3 = E_4$ in order to write the result as a function of the energies and momenta of the incoming particles p_1 and p_4. The dots stand for additional terms proportional to $\bar{R}_{0ijk}$ and $\bar{R}_{ijkl}$, which can also be extracted from our result.

B. The amplitude with tidal effects

A calculation similar to the one outlined in the previous section (see Fig. 2) leads to the fully relativistic result

$$\mathcal{A}_{\text{tidal}}(q) = i\left(\frac{\kappa}{2}\right)^2 \frac{q^\mu q^\rho}{q^2}\left\{8\lambda_1 p_4^\nu p_4^\sigma + 8\lambda_2 p_1^\nu p_1^\sigma + \frac{1}{2}[(m_1^2 + m_2^2 - t)^2 - 2m_1^2 m_2^2] \times \left(\frac{\eta_2}{m_2^4}p_4^\nu p_4^\sigma + \frac{\eta_1}{m_1^4}p_1^\nu p_1^\sigma\right)\right\}\bar{R}_{\mu\nu\rho\sigma}, \tag{3.10}$$

which, upon expanding in the spatial and time components, reads

$$\mathcal{A}_{\text{tidal}}(q) = -i\left(\frac{\kappa}{2}\right)^2\left\{8\lambda_1 E_4^2 + 8\lambda_2 E_1^2 + [2(E_1 E_4 + \vec{p}^2)^2 - m_1^2 m_2^2] \times \left(\eta_2 \frac{E_4^2}{m_2^4} + \eta_1 \frac{E_1^2}{m_1^4}\right)\right\}\frac{q^i q^j}{\vec{q}^2}\bar{R}_{0i0j} + \cdots, \tag{3.11}$$

where the ellipses stand once again for terms proportional to $\bar{R}_{0ijk}$ and $\bar{R}_{ijkl}$, which we will not need in the remainder of this paper.

C. The quadrupole corrections

Next we extract the corrections to the mass quadrupole moment Q_{ij} from (3.8), (3.9) and (3.11). To do so we simply match the appropriately normalized and Fourier-transformed $\mathcal{A}_{\mathcal{O}}$, as defined in (3.13) below, to the quadrupole contribution in (3.1).[5] To begin with, we perform the relevant Fourier transforms using

$$\int dt \int \frac{d^3 q}{(2\pi)^3}\frac{q_i q_j}{|\vec{q}|^2}e^{i\vec{q}\cdot\vec{r}}\bar{R}^{0i0j} = -\frac{3}{4\pi}\int dt \frac{1}{r^5}\left(x_i x_j - \frac{1}{3}r^2\delta_{ij}\right)\bar{R}^{0i0j}. \tag{3.12}$$

Taking into account the nonrelativistic normalization factor of $-i/4E_1E_4$, we arrive at the quadrupolelike terms

$$\tilde{\mathcal{A}}^{\text{quad}}_{\mathcal{O}}(r) \coloneqq -i\frac{\mathcal{A}^{\text{quad}}_{\mathcal{O}}(r)}{4E_1E_4}, = \frac{1}{2}C_{\mathcal{O}}(E_i, m_i, \vec{p}^2)\int dt \frac{1}{r^5}\left(x_i x_j - \frac{1}{3}r^2\delta_{ij}\right)\bar{R}^{0i0j}, \tag{3.13}$$

where $C_{\mathcal{O}}$ are coefficients depending on the energies and masses as well as $\vec{p}^2$ of the heavy particles, with

[5]For further details on the procedure see for example [26,52].

$$\begin{aligned}C_{I_1}(E_i, m_i, \vec{p}^2) &= \frac{3}{8\pi}(\alpha_1 + 2\alpha_2)\left(\frac{\kappa}{2}\right)^2 \\ &\quad\times\left(m_1^2\frac{E_4}{E_1} + m_2^2\frac{E_1}{E_4} - 2E_1E_4 - 2\vec{p}^2\right), \\ C_{I_2}(E_i, m_i, \vec{p}^2) &= \frac{3}{16\pi}\alpha_2\left(\frac{\kappa}{2}\right)^2\left(m_1^2\frac{E_4}{E_1} + m_2^2\frac{E_1}{E_4}\right), \\ C_{\mathrm{tidal}}(E_i, m_i, \vec{p}^2) &= \frac{3}{8\pi}\left(\frac{\kappa}{2}\right)^2\left\{8\lambda_1\frac{E_4}{E_1} + 8\lambda_2\frac{E_1}{E_4}\right. \\ &\quad + [2(E_1E_4 + \vec{p}^2)^2 - m_1^2m_2^2] \\ &\quad\left.\times\left(\eta_1\frac{E_1}{E_4m_1^4} + \eta_2\frac{E_4}{E_1m_2^4}\right)\right\}. \end{aligned} \tag{3.14}$$

Comparing (3.13) with the Hamiltonian density obtained from the action (3.1), we conclude that the modifications to the quadrupole moment arising from the cubic and tidal couplings are given by

$$Q^{ij}_{\mathcal{O}} = \frac{C_{\mathcal{O}}}{\mu r^5}Q^{ij}_N, \tag{3.15}$$

where we have introduced the leading-order quadrupole moment in the EH theory for a binary system with masses m_1 and m_2,

$$Q^{ij}_N = \mu\left(x^ix^j - \frac{1}{3}r^2\delta^{ij}\right), \tag{3.16}$$

with μ being the reduced mass defined in (3.2). Combining the various correction terms, we arrive at

$$\begin{aligned}Q^{ij} &= Q^{ij}_N + Q^{ij}_{I_1} + Q^{ij}_{I_2} + Q^{ij}_{\mathrm{tidal}} \\ &= \left(1 + \frac{C_{I_1}}{\mu r^5} + \frac{C_{I_2}}{\mu r^5} + \frac{C_{\mathrm{tidal}}}{\mu r^5}\right)Q^{ij}_N. \end{aligned} \tag{3.17}$$

It is interesting to write the three coefficients C_{I_1}, C_{I_2} and C_{tidal} in the PN expansion. Keeping terms up to first order in $\vec{p}^2$ one has

$$\begin{aligned}C^{\mathrm{PN}}_{I_1} &= -3G(\alpha_1 + 2\alpha_2)M\frac{\vec{p}^2}{\mu}, \\ C^{\mathrm{PN}}_{I_2} &= 3G\alpha_2m_1m_2, \\ C^{\mathrm{PN}}_{\mathrm{tidal}} &= 3G\left[8\lambda_1 + \eta_1 + \frac{1}{2M}(8(m_1 - m_2)\lambda_1\right. \\ &\quad\left. + (3m_1 + 5m_2)\eta_1)\frac{\vec{p}^2}{\mu^2}\right]\frac{m_2}{m_1} + 1 \leftrightarrow 2, \end{aligned} \tag{3.18}$$

where

$$M \coloneqq m_1 + m_2, \tag{3.19}$$

and, as usual, $\kappa^2 \coloneqq 32\pi G$. For convenience we also quote the contribution due to the G_3 interaction alone—this is given by

$$Q^{ij}_{G_3} = (Q^{ij}_{I_1} + Q^{ij}_{I_2})|_{\alpha_1=0} = 3G\alpha_2\frac{M}{r^5}\left(1 - \frac{2\vec{p}^2}{\mu^2}\right)Q^{ij}_N. \tag{3.20}$$

IV. POWER RADIATED BY THE GRAVITATIONAL WAVES

We can now compute the power radiated by the gravitational waves in the approximation of circular orbits. In the EH theory, the radius of the circular orbit is given by the well-known formula

$$r_N = \left(\frac{GM}{\Omega^2}\right)^{\frac{1}{3}}. \tag{4.1}$$

In the presence of the cubic and tidal interactions, this quantity gets modified as

$$\begin{aligned}r_\circ &= r_N + \delta r, \\ \delta r &= \Omega^3\left[-\frac{\alpha_1}{2}v + \left(\frac{\alpha_2}{2} + 8\lambda_{12}\right)\left(\frac{3}{v} + v(2\nu - 1)\right)\right. \\ &\quad\left. + \eta_{12}\left(\frac{3}{v} + v(\nu + 2)\right)\right] + \mathcal{O}(g_i^2), \end{aligned} \tag{4.2}$$

where g_i stands for any of the coupling constants of the cubic and tidal perturbations. We also introduced the symmetric mass ratio ν defined as

$$\nu \coloneqq \frac{m_1m_2}{M^2}, \tag{4.3}$$

and the parameter

$$v \coloneqq r_N\Omega = (GM\Omega)^{\frac{1}{3}}, \tag{4.4}$$

as well as the following combinations of the couplings

$$\lambda_{12} \coloneqq \mu\left(\frac{\lambda_1}{m_1^3} + \frac{\lambda_2}{m_2^3}\right), \qquad \eta_{12} \coloneqq \mu\left(\frac{\eta_1}{m_1^3} + \frac{\eta_2}{m_2^3}\right). \tag{4.5}$$

Finally, Ω denotes the angular velocity on the circular orbit, and the value δr has been computed using (4.2) and (A5), where the potentials entering (A5) are given in (2.9) and (2.21). The total energy per unit mass M of the system, to first order in the couplings, is then given by

$$\begin{aligned}E(v) &= -\frac{1}{2}\nu v^2 + \frac{9}{4}\frac{v^{12}}{(GM)^4}\nu(\alpha_2 + 16\lambda_{12} + 2\eta_{12}) \\ &\quad + \frac{11}{8}\frac{v^{14}}{(GM)^4}[-\nu\alpha_1 + \nu(2\nu - 1)(\alpha_2 + 16\lambda_{12}) \\ &\quad + 4\nu(\nu + 2)\eta_{12}]. \end{aligned} \tag{4.6}$$

The above formula is complete at leading order in all of the perturbations [that is $\mathcal{O}(v^{12})$] and at $\mathcal{O}(v^{14})$ for the α_1 correction only. The remaining $\mathcal{O}(v^{14})$ terms have been obtained from a small-velocity expansion of our 2PM result, and in order to get a complete result at that PN order one would need to include also the 3PM corrections to the potential generated by cubic and tidal interactions.[6] We have also compared the contribution to the energy from the $\eta_{1,2}$ corrections to [71], finding agreement (after mapping their coefficients $\mu_A^{(2)}$ to ours).[7]

Next, we compute the leading-order gravitational-wave flux using the quadrupole formula

$$\mathcal{F}(v) = \frac{G}{5}\langle \dddot{Q}^{ij}\,\dddot{Q}^{ij}\rangle, \tag{4.7}$$

using the result of our computation for Q^{ij} in (3.17). To first order in the couplings α_1 and α_2 the flux becomes

$$\mathcal{F}(v)=\frac{G}{5}\langle \dddot{Q}_N^{ij}\,\dddot{Q}_N^{ij}\rangle\left[1+\frac{2}{\mu r^5}(C_{I_1}^{\mathrm{PN}}+C_{I_2}^{\mathrm{PN}}+C_{\mathrm{tidal}}^{\mathrm{PN}})\right]+\mathcal{O}(\alpha_i^2), \tag{4.8}$$

where the PN-expanded coefficients $C_{\mathcal{O}}^{\mathrm{PN}}$ are explicitly given in (3.18).

Two comments are in order here. First, we note that the prefactor $\langle \dddot{Q}_N^{ij}\dddot{Q}_N^{ij}\rangle$ is evaluated on the radius $r_\circ$ of the circular orbit in the presence of the cubic and tidal interactions, as given in (4.2). Furthermore, the quantity $\vec{p}^2 \coloneqq p_r^2 + p_\phi^2/r^2$ can be obtained using the fact that $p_r = 0$ on the circular orbit while $p_\phi \coloneqq l$ is a constant, which can be determined from Hamilton's equations, with the result

$$l \coloneqq \frac{\mu r_\circ^2 \Omega}{1 + 2\mu U(r_\circ)}, \tag{4.9}$$

where $r_\circ$ is given in (4.2) and $U(r)$ is the part of the potential proportional to $\vec{p}^2$, following the conventions of the Appendix. Using these relations, $\vec{p}^2$ is reexpressed as a function of Ω, the masses and the couplings.

Factoring out the standard power radiated by the gravitational wave in EH,

$$\mathcal{F}_N(v) \coloneqq \frac{G}{5}\langle \dddot{Q}_N^{ij}\dddot{Q}_N^{ij}\rangle\Big|_{r=r_N} = \frac{32}{5}G\mu^2 r_N^4 \Omega^6 = \frac{32}{5}\frac{\nu^2 v^{10}}{G}, \tag{4.10}$$

we can rewrite the expression for the flux as

$$\begin{aligned}\mathcal{F}(v)=\frac{32}{5}\frac{\nu^2 v^{10}}{G}\Bigg[1+\frac{v^{10}}{(GM)^4}&(12\alpha_2+144\lambda_{12}+48\lambda'_{12}\\ &+18\eta_{12}+6\eta'_{12})+\frac{v^{12}}{(GM)^4}[-8\alpha_1+2(2\nu-7)\alpha_2\\ &+8(8\nu-7)\lambda_{12}+24\lambda'_{12}+(8\nu+31)\eta_{12}+9\eta'_{12}]\Bigg],\end{aligned} \tag{4.11}$$

with λ_{12} and η_{12} defined in (4.5) and

$$\lambda'_{12} \coloneqq \frac{1}{M}\left(\frac{\lambda_1}{m_1}+\frac{\lambda_2}{m_2}\right), \quad \eta'_{12} \coloneqq \frac{1}{M}\left(\frac{\eta_1}{m_1}+\frac{\eta_2}{m_2}\right). \tag{4.12}$$

Similarly to (4.6), the first line and the α_1 term in the second line of (4.11) are complete. We also note that the $\eta_{1,2}$ part of the tidal flux is in agreement with [71].

V. WAVEFORMS IN EFT OF GRAVITY

Following [53] we can also compute the correction induced by the cubic and tidal interactions to the gravitational phase in the saddle point approximation. In this approach, the waveform in the frequency domain is written as[8]

$$\tilde{h}_{\mathrm{SPA}}(f) \sim \exp\left[i\left(\psi_f(t_f) - \frac{\pi}{4}\right)\right], \tag{5.1}$$

where

$$\psi(t) \coloneqq 2\pi f t - \phi(t). \tag{5.2}$$

Here $\phi(t)$ is the orbital phase, while $\dot{\phi}(t) = \pi F(t)$ defines the instantaneous frequency $F(t)$ of the gravitational wave. t_f is defined as the time where

$$\dot{\psi}(t)|_{t=t_f} = 0, \tag{5.3}$$

implying that $F(t_f) = 2f$. In the adiabatic approximation, the work of [72,73] provides explicit formulas for $\psi_{\mathrm{SPA}}(t_f)$ and t_f:

$$\psi_{\mathrm{SPA}}(t_f)=2\pi f t_{\mathrm{ref}}-2\phi_{\mathrm{ref}}+\frac{2}{G}\int_{v_f}^{v_{\mathrm{ref}}} dv(v_f^3-v^3)\frac{E'(v)}{\mathcal{F}(v)}, \tag{5.4}$$

$$t_f = t_{\mathrm{ref}} + M\int_{v_f}^{v_{\mathrm{ref}}} dv\frac{E'(v)}{\mathcal{F}(v)}, \tag{5.5}$$

where $v_{\mathrm{ref}} = v(t_{\mathrm{ref}})$ and t_{ref} are integration constants, $v_f \coloneqq (\pi G M f)^{\frac{1}{3}}$, and $E(v)$ and $\mathcal{F}(v)$ were computed to

[6]Similar considerations apply to our results for the flux in (4.11).

[7]For further details on mapping field theory to point-particle actions see e.g., [36,84].

[8]See for example Sec. III F of [73] for a detailed derivation.

lowest order in the cubic and tidal perturbations in (4.6) and (4.8), respectively.

We can now compute the correction to $\psi_{\rm SPA}(t_f)$ due to the presence of the perturbations, expanding the ratio $E'(v)/\mathcal{F}(v)$ at consistent PN order and performing the integration in (5.4). Doing so we arrive at

$$\psi_{\rm SPA}(t_f) = \psi^{\rm EH}_{\rm SPA}(t_f) + \psi^{I_1+I_2}_{\rm SPA}(t_f) + \psi^{\rm tidal}_{\rm SPA}(t_f). \quad (5.6)$$

Here

$$\psi^{\rm EH}_{\rm SPA}(t_f) = 2\pi f t'_{\rm ref} - 2\phi'_{\rm ref} + \frac{3}{128\nu v_f^5} \quad (5.7)$$

is the EH contribution, where we have also included the reference time and phase $t'_{\rm ref}$ and $\phi'_{\rm ref}$, which have been redefined in order to absorb terms that depend on $v_{\rm ref}$; and

$$\psi^{I_1+I_2}_{\rm SPA}(t_f) = -\frac{3}{128\nu v_f^5}\left[156\frac{\alpha_2}{(GM)^4}v_f^{10} - \frac{545\alpha_1 + (665-850\nu)\alpha_2}{14(GM)^4}v_f^{12}\right],$$

$$\psi^{\rm tidal}_{\rm SPA}(t_f) = -\frac{3}{128\nu v_f^5}\left\{24\frac{v_f^{10}}{(GM)^4}(8(12\lambda_{12} + \lambda'_{12}) + 12\eta_{12} + \eta'_{12}) - \frac{10}{7}\frac{v_f^{12}}{(GM)^4}[4((91-170\nu)\lambda_{12} - 6\lambda'_{12}) - 5(17\nu+37)\eta_{12} - 9\eta'_{12}]\right\}, \quad (5.8)$$

are the new contributions due to cubic and tidal perturbations. Similarly to our comment after (4.6), we note that all the terms at leading order in velocity in (5.8) are complete, while the remaining ones would also receive further modifications from a 3PM computation of the potential and a 2PM computation of the quadrupole.

Finally, it is interesting to compare our results with those of [53]. The perturbations considered in that paper have the form

$$\mathcal{L}_8 = \beta_1\mathcal{C}^2 + \beta_2\mathcal{C}\tilde{\mathcal{C}} + \beta_3\tilde{\mathcal{C}}^2, \quad (5.9)$$

where

$$\mathcal{C} \coloneqq R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \qquad \tilde{\mathcal{C}} \coloneqq \frac{1}{2}R_{\mu\nu\alpha\beta}\epsilon^{\alpha\beta}{}_{\gamma\delta}R^{\gamma\delta\mu\nu}. \quad (5.10)$$

The modifications to $\psi_{\rm SPA}(t_f)$ due to quartic interactions as found in [53] are [reinstating powers of G in the result of that paper, and converting their d_Λ into our β_1 as defined in (5.9)],

$$\psi^{\rm quartic}_{\rm SPA}(t_f) = \psi^{\rm EH}_{\rm SPA}(t_f) + \frac{3}{128\nu v_f^5}\left[\left(\frac{234240}{11} - \frac{522240}{11}\nu\right)\frac{\beta_1}{(GM)^6}v_f^{16}\right]. \quad (5.11)$$

Note the different dependence on v_f in the correction terms in (5.8) and (5.11), which are of $\mathcal{O}(v_f^{10})$ and $\mathcal{O}(v_f^{16})$ in the leading cubic and tidal, and quartic cases, respectively, in order to constrain the coefficients of the higher-dimensional interactions. Finally, it will be interesting to perform a comparison of our result in (5.6) to experimental data, as performed in [53] for the case of quartic perturbations in the Riemann tensor. A natural extension of our work is to consider spinning particles, and study the effect of higher-derivative interactions on the Kerr geometry from an amplitudes perspective, complementing traditional approaches [85,86].

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APPENDIX: HAMILTONIANS WITH MOMENTUM-DEPENDENT POTENTIALS

Consider a momentum-dependent Hamiltonian of the form

$$H = \frac{\vec{p}^2}{2\mu}[1 + 2\mu U(r)] + V(r), \quad (A1)$$

where $\vec{p}^2 = p_r^2 + \frac{p_\phi^2}{r^2}$. From Hamilton’s equations we learn that $p_\phi \coloneqq l$ is constant, as well as $\dot{\phi} = \frac{l}{\mu r^2}[1 + 2\mu U(r)]$. The latter equation can be used to reexpress l as a function of Ω. We also have

$$\dot{r} = \frac{p_r}{\mu}[1 + 2\mu U(r)], \quad (A2)$$

and, for circular orbits, we see that $p_r = 0$ and hence $\dot{p}_r = 0$. In this case, the Hamilton equation $\dot{p}_r = -\frac{\partial H}{\partial r}$ simplifies to

$$V'(r_\circ) - \frac{l^2}{\mu r_\circ^3}[1 + 2\mu U(r_\circ)] + \frac{l^2}{r_\circ^2}U'(r_\circ) = 0, \tag{A3}$$

where $r_\circ$ is the radius of the circular orbit. We will also set $\Omega := \dot{\phi}(r = r_\circ)$, or

$$\Omega := \frac{l}{\mu r_\circ^2}[1 + 2\mu U(r_\circ)]. \tag{A4}$$

Using this to eliminate l in favor of Ω, we finally get

$$V'(r_\circ) - \frac{\mu r_\circ \Omega^2}{1 + 2\mu U(r_\circ)}\left[1 - \frac{\mu r_\circ U'(r_\circ)}{1 + 2\mu U(r_\circ)}\right] = 0. \tag{A5}$$

This equation determines $r_\circ$ as a function of Ω. In the absence of a perturbation, we have

$$\Omega_N = \frac{l}{\mu r_N^2}, \tag{A6}$$

where r_N is the radius of the circular orbit in the EH theory, given in (4.1).

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