Uniform semi-Latin squares and their pairwise-variance aberrations

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Abstract

For integers $n > 2$ and $k > 0$, an $(n \times n)/k$ *semi-Latin square* is an $n \times n$ array of k-subsets (called *blocks*) of an nk-set (of *treatments*), such that each treatment occurs once in each row and once in each column of the array. A semi-Latin square is *uniform* if every pair of blocks, not in the same row or column, intersect in the same positive number of treatments. It is known that a uniform $(n \times n)/k$ semi-Latin square is Schur optimal in the class of all $(n \times n)/k$ semi-Latin squares, and here we show that when a uniform $(n \times n)/k$ semi-Latin square exists, the Schur optimal $(n \times n)/k$ semi-Latin squares are precisely the uniform ones. We then compare uniform semi-Latin squares using the criterion of pairwise-variance (PV) aberration, introduced by J. P. Morgan for affine resolvable designs, and determine the uniform $(n \times n)/k$ semi-Latin squares with minimum PV aberration when there exist $n-1$ mutually orthogonal Latin squares of order n. These do not exist when $n = 6$, and the smallest uniform semi-Latin squares in this case have size $(6 \times 6)/10$. We present a complete classification of the uniform $(6 \times 6)/10$ semi-Latin squares, and display the one with least PV aberration. We give a construction producing a uniform $((n+1) \times (n+1))/((n-2)n)$ semi-Latin square when there exist $n-1$ mutually orthogonal Latin squares of order n, and determine the PV aberration of such a uniform semi-Latin square. Finally, we describe how certain affine resolvable designs and balanced incomplete-block designs can be constructed from uniform semi-Latin squares. From the uniform $(6 \times 6)/10$ semi-Latin squares we classified, we obtain (up to block design isomorphism) exactly 16875 affine resolvable designs for 72 treatments in 36 blocks of size 12 and 8615 balanced incomplete-block designs for 36 treatments in 84 blocks of size 6. In particular, this shows that there are at least 16875 pairwise non-isomorphic orthogonal arrays $\mathrm{OA}(72, 6, 6, 2)$.

Keywords: Design optimality; Block design; Schur optimality; Affine resolvable design; Balanced incomplete-block design; Orthogonal array

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1 Introduction

For integers $n > 2$ and $k > 0$, an $(n \times n)/k$ *semi-Latin square* is an $n \times n$ array of k-subsets (called *blocks*) of an nk-set (of *treatments*), such that each treatment occurs once in each row and once in each column of the array. Note that an $(n \times n)/1$ semi-Latin square is the same thing as a Latin square of order n. We consider two $(n \times n)/k$ semi-Latin squares to be *isomorphic* if one can be obtained from the other by applying an *isomorphism*, which is a sequence of zero or more of: permuting the rows, permuting the columns, transposing the array, and renaming the treatments. An *automorphism* of a semi-Latin square S is an isomorphism mapping S onto itself. The applications of semi-Latin squares include the design of agricultural experiments, consumer testing, and via their duals, human-machine interaction (see Bailey (1992, 2011)).

A (v, b, r, k)-*design* is a binary block design for v treatments in b blocks of size k (considered as k-subsets of the set of treatments), such that each treatment is in exactly r blocks. If we ignore the block structure of an $(n \times n)/k$ semi-Latin square S then we obtain an (nk, n^2, n, k)-design called the *underlying block design* of S. A (v, b, r, k)-design with $k < v$ and $r > 1$ is *resolvable* if its collection of blocks can be partitioned into r partitions of the treatments (called *parallel classes*), and such a resolvable design is *affine resolvable* if every pair of blocks in distinct parallel classes intersect in the same positive number μ of treatments. A (v, b, r, k)-design is a (v, b, r, k, λ)-*balanced incomplete-block design* (*BIBD*) if $1 < k < v$ and every pair of distinct treatments occur together in exactly λ blocks. Two (v, b, r, k)-designs are *isomorphic* (as block designs) if there is a bijection from the treatments of the first to those of the second which maps the list of blocks of the first onto that of the second in some order. Such a bijection is called a (block design) *isomorphism*, and an *automorphism* of a (v, b, r, k)-design is an isomorphism from that block design to itself.

An *orthogonal array* of *strength* t with N rows, r columns ($r \geq t$), and based on s symbols ($s \geq 2$), here taken to be $1, 2, \ldots s$, or an *orthogonal array* $\mathrm{OA}(N, r, s, t)$, is an $N \times r$ array whose entries are symbols, such that for every $N \times t$ subarray, each of the possible s^t t-tuples of symbols occurs as a row equally often (which must be N/s^t times). As Hedayat et al. (1999) point out, there are many trivial constructions of orthogonal arrays of strength one, so we ignore this case. Two orthogonal arrays $\mathrm{OA}(N, r, s, t)$ are *isomorphic* if one can be obtained from the other by permuting the rows, permuting the columns, and permuting the symbols separately within each column. It is known that orthogonal arrays of strength 2 and affine resolvable designs are equivalent combinatorial objects. In particular, Bailey et al. (1995) describe how to construct an equivalent $\mathrm{OA}(v, r, s, 2)$ from an affine resolvable $(v, b, r, v/s)$-design (see also Morgan (2010)), such that two affine resolvable $(v, b, r, v/s)$-designs are isomorphic if and only if their equivalent orthogonal arrays are isomorphic. We remark that affine resolvable designs were introduced by Bose (1942) (in the context of BIBDs), and orthogonal arrays were introduced by Rao (1947).

An $(n \times n)/k$ semi-Latin square S is *uniform* if every pair of blocks, not in the same row or column, intersect in the same positive number $\mu = \mu(S)$ of treatments (in which case $k = \mu(n-1)$). For example, here is a $(3 \times 3)/4$ uniform semi-Latin square with $\mu = 2$:

1 4 7 10	2 5 8 11	3 6 9 12
3 6 8 11	1 4 9 12	2 5 7 10
2 5 9 12	3 6 7 10	1 4 8 11

(1)

Uniform semi-Latin squares were introduced, constructed, and studied by Soicher (2012), where it was shown that a uniform $(n \times n)/k$ semi-Latin square is Schur optimal (defined later) in the class of all $(n \times n)/k$ semi-Latin squares.

In this paper, we further the study of uniform semi-Latin squares. We show that, if a uniform $(n \times n)/k$ semi-Latin square exists, then the Schur optimal $(n \times n)/k$ semi-Latin squares are precisely the uniform ones.

We then compare uniform $(n \times n)/k$ semi-Latin squares using the criterion of pairwise-variance (PV) aberration, introduced by Morgan (2010) for affine resolvable designs, and determine the uniform $(n \times n)/k$ semi-Latin squares with minimum PV aberration when there exist $n - 1$ mutually orthogonal Latin squares (MOLS) of order n. These do not exist when $n = 6$, and the smallest uniform semi-Latin squares in this case have size $(6 \times 6)/10$. We describe a complete classification of the uniform $(6 \times 6)/10$ semi-Latin squares, and find that, up to isomorphism, there are exactly 8615 such designs. We compare their PV aberrations, and display the one with least PV aberration.

We give a construction producing a uniform $((n+1)\times(n+1))/((n-2)n)$ semi-Latin square when there exist $n-1$ MOLS of order n, and determine the PV aberration of such a uniform semi-Latin square.

Finally, we describe how a uniform $(n\times n)/(\mu(n-1))$ semi-Latin square can be used to construct two (possibly isomorphic) affine resolvable $(\mu n^2, n^2, n, \mu n)$-designs and an $(n^2, \mu n(n+1), \mu(n+1), n, \mu)$-BIBD. From the uniform $(6\times 6)/10$ semi-Latin squares we classified, we obtain (up to block design isomorphism) exactly 16875 affine resolvable $(72, 36, 6, 12)$-designs and 8615 $(36, 84, 14, 6, 2)$-BIBDs. In particular, this shows that there are at least 16875 pairwise non-isomorphic orthogonal arrarys $\mathrm{OA}(72, 6, 6, 2)$.

2 Schur optimality

Let Δ be a (v, b, r, k)-design. The *concurrence matrix* Λ of Δ is the $v\times v$ matrix whose rows and columns are indexed by the treatments of Δ, and whose (α, β)-entry is the number of blocks containing both α and β (this entry is the *concurrence* of treatments α and β).

The *scaled information matrix* of Δ is

$$F(\Delta) := I_v - (rk)^{-1}\Lambda,$$

where I_v denotes the $v\times v$ identity matrix. The eigenvalues of $F(\Delta)$ are all real and lie in the interval $[0, 1]$. The all-1 vector is an eigenvector of $F(\Delta)$ with corresponding eigenvalue 0. The remaining eigenvalues (counting repeats) are called the *canonical efficiency factors* of Δ. It is well known that these canonical efficiency factors are all non-zero if and only if Δ is connected (that is, its treatment-block incidence graph is connected), and they are all equal to 1 if and only if $k = v$.

Now suppose that Δ has canonical efficiency factors $\delta_1 \le \cdots \le \delta_{v-1}$. We say that Δ is *Schur optimal* in a class $\mathcal{C}$ of (v, b, r, k)-designs containing Δ if for each design $\Gamma \in \mathcal{C}$, with canonical efficiency factors $\gamma_1 \le \cdots \le \gamma_{v-1}$, we have

$$\sum_{i=1}^{\ell} \delta_i \ge \sum_{i=1}^{\ell} \gamma_i,$$

for $\ell = 1, \ldots, v-1$. A Schur optimal design need not exist within a given class $\mathcal{C}$, but, when it does, that design is optimal in $\mathcal{C}$ with respect to a very wide range of statistical optimality criteria, including being Φ_p-optimal, for all $p \in (0, \infty)$, and also A-, D-, and E-optimal (see Giovagnoli and Wynn (1981); see also Bailey and Cameron (2009) or Shah and Sinha (1989) for definitions of these optimality criteria).

As recommmended by Bailey (1992), for the purposes of statistical optimality, we compare $(n\times n)/k$ semi-Latin squares as their underlying (nk, n^2, n, k)-designs. Thus, we take the *canonical efficiency factors* of a semi-Latin square to be those of its underlying block design, and to say that an $(n \times n)/k$ semi-Latin square S is *Schur optimal* means that its underlying block design is Schur optimal in the class of underlying block designs of $(n \times n)/k$ semi-Latin squares.

The *dual* of the (v, b, r, k)-design Δ is the (b, v, k, r)-design Δ' obtained by interchanging the roles of treatments and blocks, so the point-block incidence matrix of Δ' is the transpose of that of Δ. As the canonical efficiency factors of Δ' differ from those of Δ only in the number of times 1 occurs, it follows that Δ is Schur optimal in a class $\mathcal{C}$ of (v, b, r, k)-designs if and only if Δ' is Schur optimal in the class of the duals of the elements of $\mathcal{C}$ (see Bailey and Cameron (2009)). We take the treatments of the *dual* S' of an $(n \times n)/k$ semi-Latin square S to be the Cartesian product $\{1, \ldots, n\} \times \{1, \ldots, n\}$, with treatment (i, j) corresponding to the (i, j)-cell of S, and then, for each treatment α of S, the corresponding block in S' is the set of those (i, j) such that α is in the (i, j)-cell of S. In particular, S' is an (n^2, nk, k, n)-design. See Bailey (2011) for more on duals of semi-Latin squares, including applications. Also relevant are parts of Suen (1982) and Suen and Chakravarti (1986).

Bailey et al. (1995) studied and constructed affine resolvable designs and proved:

Theorem 1. *Let Δ be an affine resolvable (v, b, r, k)-design with $r > 2$, and let $s = v/k > 1$. Then:*

1. *the canonical efficiency factors of Δ are $1 - 1/r$, with multiplicity $r(s - 1)$, and 1, with multiplicity $v - 1 - r(s - 1)$;*

2. *the affine resolvable (v, b, r, k)-designs are precisely the Schur optimal designs in the class of all resolvable (v, b, r, k)-designs.*

We prove the following analogous result for uniform semi-Latin squares:

Theorem 2. *Let $n > 2$ and let S be a uniform $(n \times n)/k$ semi-Latin square. Then:*

1. *the canonical efficiency factors of S are $1 - 1/(n - 1)$, with multiplicity $(n - 1)^2$, and 1, with multiplicity $nk - 1 - (n - 1)^2$;*

2. *the uniform $(n \times n)/k$ semi-Latin squares are precisely the Schur optimal designs in the class of all $(n \times n)/k$ semi-Latin squares.*

Proof. Soicher (2012) determined the canonical efficiency factors of S and its Schur optimality. Here, we complete the proof of the theorem.

Let T be any Schur optimal $(n \times n)/k$ semi-Latin square and let T' be the dual of T. We shall show that the concurrence matrix A' of S' is equal to the concurrence matrix B' of T', showing that T is uniform.

The canonical efficiency factors of S' are $1 - 1/(n-1)$, with multiplicity $(n-1)^2$ and 1, with multiplicity $2n-2$. Now the argument in the proof of Theorem 3.4 of Soicher (2012) shows that the 0-eigenspace of A' is contained in the 0-eigenspace of B', so in particular T' has at least $2n-2$ canonical efficiency factors equal to 1. Then, since the sum of the canonical efficiency factors is the same for S' and T', the Schur optimality of both S' and T' implies that T' must have precisely $2n-2$ canonical efficiency factors equal to 1, and the remaining ones equal to $1 - 1/(n-1)$.

We now know that A' and B' have the same nk-eigenspace (spanned by the all-1 vector), as well as the same 0-eigenspace. It follows that the orthogonal complement of the direct sum of these two eigenspaces must be the eigenspace for the remaining eigenvalue of both A' and B'. Thus $A'\mathbf{e} = B'\mathbf{e}$ as $\mathbf{e}$ runs over a basis of $\mathbb{R}^{n^2}$ (consisting of common eigenvectors), and so $A' = B'$.

Now concurrence in S' and T' is block intersection size in S and T, respectively, and so S uniform implies T uniform. $\square$

Caliński (1971) emphasised the importance of block designs with just two distinct canonical efficiency factors, one of which is 1. As a result, some authors call this the C-property, and call such designs C-designs: for example, see Saha (1976) and Ceranka et al. (1986).

3 Pairwise-variance aberration

Now, given a collection $\mathcal{S}$ of Schur optimal designs in a given class $\mathcal{C}$ of (v, b, r, k)-designs, we want a criterion to choose between them. For affine resolvable designs, Morgan (2010) proposed choosing a design with minimum pairwise-variance (PV) aberration, a combinatorial criterion which translates to a statistical one when the Schur optimal (v, b, r, k)-designs under consideration are all connected and all have the same two distinct canonical efficiency factors (and no others).

Definition 1. *Let Δ be a (v, b, r, k)-design. Define*

$$\eta(\Delta) := (\eta_0(\Delta), \ldots, \eta_r(\Delta)),$$

where $\eta_i(\Delta)$ is the number of unordered pairs of distinct treatments of Δ with concurrence equal to i. If Δ is connected and has at most two distinct canonical efficiency factors, then $\eta(\Delta)$ is called the pairwise-variance (or PV) aberration of Δ. Where Δ and Γ are connected (v, b, r, k)-designs having the same two distinct canonical efficiency factors, the design Δ is considered to have smaller PV aberration than Γ if $\eta(\Delta)$ is lexicograhically less than $\eta(\Gamma)$.

Now the underlying (nk, n^2, n, k)-design of a uniform $(n \times n)/k$ semi-Latin square is connected (since $n > 2$) and has exactly two distinct canonical efficiency factors, $1 - 1/(n-1)$ and 1. We thus make the following:

Definition 2. *For S a semi-Latin square with underlying block design Δ, we define $\eta(S)$ and $\eta_i(S)$ to be $\eta(\Delta)$ and $\eta_i(\Delta)$, respectively, and if S is uniform then we call $\eta(S)$ the PV aberration of S. Where S and T are uniform $(n \times n)/k$ semi-Latin squares then S is considered to have smaller PV aberration than T if $\eta(S)$ is lexicograhically less than $\eta(T)$.*

These definitions are justified by the following result, which is a corollary of Theorem 1 of Bailey (2009).

Theorem 3. *Let Δ be a connected (v, b, r, k)-design having at most two distinct canonical efficiency factors. Then the variance of the estimator of the difference of the effects of distinct treatments α and β is a function depending only on r, k, the canonical efficiency factors, and the concurrence of α and β. Moreover, as a function of the concurrence of α and β this function is strictly decreasing.*

Thus, by minimising PV aberration in an appropriate class of designs, we are not only minimising the maximum pairwise-variance, but, for those designs in the class with the same maximum pairwise-variance, we are minimising the number of pairs of distinct treatments having that maximum pairwise-variance, and when these numbers are the same, we are minimising the number of pairs with the next largest pairwise-variance, and so on.

Morgan (2010) studied affine resolvable designs with minimum PV aberration, and placed an extensive catalogue of these online at `http://designtheory.org/database/v-r-k-ARD-MV/`.

We remark that when a (v, b, r, k, λ)-BIBD exists, with $b > v$, it may be of interest to determine the duals of (v, b, r, k, λ)-BIBDs with minimum PV aberration, and for this, the block intersection size distribution of (v, b, r, k, λ)-BIBDs would need to be studied.

We next present a general result providing designs with minimum PV aberration in certain circumstances, but first we need a definition. For s a positive integer, an *s-fold inflation* of a (v, b, r, k)-design or an $(n \times n)/k$ semi-Latin square is obtained by replacing each treatment α by s treatments $\sigma_{\alpha,1}, \ldots, \sigma_{\alpha,s}$, such that $\sigma_{\alpha,i} = \sigma_{\beta,j}$ if and only if $\alpha = \beta$ and $i = j$. In particular, an s-fold inflation of a (v, b, r, k)-design is an (sv, b, r, sk)-design and an s-fold inflation of an $(n \times n)/k$ semi-Latin square is an $(n \times n)/(sk)$ semi-Latin square.

Theorem 4. *Suppose that Δ is a (v, b, r, k)-design, with $r > 1$, such that every pair of distinct non-disjoint blocks meet in a positive constant number*

μ of treatments. Then

$$\eta_0(\Delta) \geq v(v-k-(r-1)(k-\mu))/2, \tag{2}$$

with equality holding if and only if

$$\eta(\Delta) = (v(v-k-(r-1)(k-\mu))/2, vr(k-\mu)/2, 0, \ldots, 0, v(\mu-1)/2), \tag{3}$$

which happens if and only if Δ is a μ-fold inflation of a $(v/\mu, b, r, k/\mu)$-design with the property that every pair of distinct non-disjoint blocks meet in just one treatment.

Proof. Let α be any treatment of Δ, and let $B_1, \ldots, B_r$ be the blocks containing α (in some fixed, but arbitrary, order). We have that $|B_i \cap B_j| = \mu$ for $1 \leq i < j \leq r$.

Now let d_i be the number of treatments in B_i that are not in any of $B_1, \ldots, B_{i-1}$. Then α is concurrent with exactly $d := \sum_{i=1}^{r} d_i$ distinct treatments (including α itself). Now $d_1 = k$, $d_2 = k - \mu$, and for $i = 3, \ldots, r$, $d_i \leq k-\mu$, with equality if and only if $B_1, \ldots, B_i$ meet pairwise in exactly the same μ treatments. Thus α has concurrence 0 with $v-d \geq v-k-(r-1)(k-\mu)$ treatments, with equality if and only if $B_1, \ldots, B_r$ meet pairwise in exactly the same μ treatments.

From the preceding argument, we have that the inequality (2) holds, and that if equality holds then every treatment must have concurrence 0 with $v-k-(r-1)(k-\mu)$ treatments, concurrence 1 with $r(k-\mu)$ treatments, and concurrence r with the remaining $\mu - 1$ treatments other than itself, in which case (3) holds.

Now suppose that (3) holds, and define a relation $\sim$ on the set of treatments by $\alpha \sim \beta$ if and only if α and β have concurrence r. Then $\sim$ is easily seen to be an equivalence relation, with each equivalence class having exactly μ elements. Choose equivalence class representatives $\alpha_1, \ldots, \alpha_{v/\mu}$, and form the design Γ having these representatives as treatments and blocks $B \cap \{\alpha_1, \ldots, \alpha_{v/\mu}\}$, where B runs over the blocks of Δ. Then Γ is a $(v/\mu, b, r, k/\mu)$-design such that every pair of distinct non-disjoint blocks meet in just one treatment, and Δ is a μ-fold inflation of Γ.

Now if Δ is a μ-fold inflation of a $(v/\mu, b, r, k/\mu)$-design with the property that that every pair of distinct non-disjoint blocks meet in just one treatment, then for each treatment α of Δ, the blocks of Δ containing α meet in the same μ treatments, and it follows that equality holds in (2). $\square$

We shall apply the preceding theorem to uniform semi-Latin squares. The *superposition* of an $(n \times n)/k$ semi-Latin square with an $(n \times n)/\ell$ semi-Latin square (with disjoint sets of treatments) is made by superimposing the first square upon the second, resulting in an $(n \times n)/(k + \ell)$ semi-Latin square. For example, the uniform $(3 \times 3)/4$ semi-Latin square (1) is a 2-fold inflation of a superposition of two MOLS of order 3. Indeed, it is easy to see that when $n > 2$ and there exist $n-1$ MOLS of order n that the superposition T of these $n-1$ MOLS is a uniform $(n \times n)/(n-1)$ semi-Latin square, and so, for every positive integer μ, a μ-fold inflation of T is a uniform $(n \times n)/(\mu(n-1))$ semi-Latin square (see Soicher (2012)).

Theorem 5. *Suppose that S is a uniform $(n \times n)/k$ semi-Latin square, with $\mu := \mu(S)$. Then*

$$\eta_0(S) \geq nk^2/2,$$

and the following are equivalent:

1. $\eta_0(S) = nk^2/2$;
2. $\eta(S) = (nk^2/2, n^2k(k-\mu)/2, 0, \ldots, 0, nk(\mu-1)/2)$;
3. *S is a μ-fold inflation of a superposition of $n-1$ MOLS of order n;*
4. *$n-1$ MOLS of order n exist and S has minimum PV aberration in the class of uniform $(n \times n)/k$ semi-Latin squares.*

Proof. The underlying block design of S is an (nk, n^2, n, k)-design in which every pair of distinct non-disjoint blocks meet in exactly $\mu = k/(n-1)$ treatments. Thus, by Theorem 4, we have that $\eta_0(S) \geq nk(nk - k - (n-1)(k - k/(n-1)))/2 = nk^2/2$.

Now we prove the equivalence of statements 1 to 4.

$(1 \Rightarrow 2)$ This follows from Theorem 4.

$(2 \Rightarrow 3)$ Assume that statement 2 holds. It follows from Theorem 4 that S must be a μ-fold inflation of a uniform $(n \times n)/(n-1)$ semi-Latin square T, and, by Theorem 3.3 of Soicher (2012), T is a superposition of $n-1$ MOLS of order n.

$(3 \Rightarrow 4)$ Now assume that S is a μ-fold inflation of a superposition of $n-1$ MOLS of order n. In particular, $n-1$ MOLS of order n exist. Now $\eta_0(S) = nk^2/2$, so if U is any uniform $(n \times n)/k$ semi-Latin square then $\eta_0(U) \geq \eta_0(S)$, with equality if and only if $\eta(U) = \eta(S)$. Thus S has minimum PV aberration in the class of uniform $(n \times n)/k$ semi-Latin squares.

$(4 \Rightarrow 1)$ Finally, assume that $n-1$ MOLS of order n exist, and that N is a μ-fold inflation of the superposition of these $n-1$ MOLS. Then N is a uniform $(n \times n)/k$ semi-Latin square. Now if S has minimum PV aberration in the class of uniform $(n \times n)/k$ semi-Latin squares, we must have $\eta_0(S) \leq \eta_0(N) = nk^2/2$, but since $\eta_0(S) \geq nk^2/2$, we must have equality. □

Corollary 6. *Let $n > 2$ and assume that there exist $n-1$ MOLS of order n. Then for every positive integer μ, the uniform $(n \times n)/(\mu(n-1))$ semi-Latin squares with minimum PV aberration are precisely the μ-fold inflations of the superpositions of $n-1$ MOLS of order n.*

When the integer $n > 2$ is a prime-power, a well-known construction of Bose (1938) gives $n-1$ MOLS of order n, and so a μ-fold inflation of the superposition of these $n-1$ MOLS yields a uniform (and hence Schur optimal) $(n \times n)/(\mu(n-1))$ semi-Latin square with minimum PV aberration in the class of all Schur optimal (and hence by Theorem 2 uniform) $(n \times n)/(\mu(n-1))$ semi-Latin squares.

For example, the $(5 \times 5)/12$ uniform semi-Latin squares were classified by Soicher (2013a). Up to isomorphism, there are exactly 277 such semi-Latin squares, and we calculated their PV aberrations. The least such PV aberration is

$$(360, 1350, 0, 0, 0, 60),$$

coming from a 3-fold inflation of the superposition of four MOLS of order 5. The next best PV aberration is

$$(488, 1062, 128, 64, 0, 28),$$

and the worst is

$$(720, 450, 600, 0, 0, 0).$$

This shows that when searching for designs with minimum, or near minimum, PV aberration, one cannot restrict the search to designs having concurrences differing by as little as possible. This is contrary to the usual thinking for eigenvalue-based optimality criteria, as discussed in John and Mitchell (1977) and in Section 2.5 of John and Williams (1995).

4 The uniform $(6 \times 6)/10$ semi-Latin squares

Let $n > 2$. If n is a prime power, we can use $n-1$ MOLS of order n to construct a uniform $(n \times n)/(\mu(n-1))$ semi-Latin square with minimum PV aberration, for all $\mu \geq 1$. This focuses attention on the case $n = 6$. There does not exist a uniform $(6 \times 6)/5$ semi-Latin square, since, by Theorem 3.3 of Soicher (2012), such a square would be a superposition of five MOLS of order 6, and even just two MOLS of order 6 do not exist. On the other hand, in Section 5 of Soicher (2012), uniform $(6 \times 6)/(5\mu)$ semi-Latin squares are constructed for all $\mu \geq 2$.

In this section, we describe a complete classification of the uniform $(6 \times 6)/10$ semi-Latin squares, and display the one amongst these having least PV aberration. The computations described took place on a desktop PC with 16GB RAM and an Intel(R) i7-6700 CPU running at 3.4GHz.

Consider now the Hamming graph $H(2, n)$. This graph has vertex-set $\{1, \ldots, n\} \times \{1, \ldots, n\}$, with distinct vertices (i, j) and (i', j') joined by an edge if and only if $i = i'$ or $j = j'$. We observe that a block design Δ is the dual of a uniform $(n \times n)/(\mu(n-1))$ semi-Latin square if and only if:

- the treatments of Δ are the vertices of $H(2, n)$;
- each block of Δ is a co-clique (independent set) of size n of $H(2, n)$ (and has multiplicity at most μ in Δ);
- the concurrence of distinct treatments α, β of Δ is 0 or μ, according as $\{\alpha, \beta\}$ is an edge or non-edge of $H(2, n)$.

Note that, in particular, these conditions imply that every treatment of Δ is in exactly $\mu(n-1)$ blocks.

The (graph) automorphism group $\mathrm{Aut}(H(2, n))$ of $H(2, n)$ is the wreath product $S_n \wr C_2$, which is generated by the direct product $D := S_n \times S_n$ of two copies of the symmetric group on $\{1, \ldots, n\}$ and an element τ satisfying $\tau^2 = 1$ and $\tau(g, h) = (h, g)\tau$ for all $(g, h) \in D$. If (i, j) is a vertex of $H(2, n)$ then the image of (i, j) under $(g, h) \in D$ is (i^g, j^h) and the image of (i, j) under τ is (j, i).

Now suppose that S is a uniform $(n \times n)/(\mu(n-1))$ semi-Latin square, and $\xi \in \mathrm{Aut}(H(2, n))$. Then the image of the dual S' of S under ξ is the block design whose treatments are the ξ-images of the treatments of S' and whose blocks are obtained by applying ξ to every treatment in every block of S'. This image is the dual of a uniform $(n \times n)/(\mu(n-1))$ semi-Latin square. Now let T be any uniform $(n \times n)/(\mu(n-1))$ semi-Latin square. By Theorem 3 of Soicher (2013b), we have that S and T are isomorphic as semi-Latin squares if and only if there is an element of $\mathrm{Aut}(H(2, n))$ mapping the dual S' of S to the dual T' of T. Moreover, due to the structure of both S' and T' as duals of uniform semi-Latin squares, any block design isomorphism from S' to T' must be a graph automorphism of $H(2, n)$, and so we have that S and T are isomorphic as semi-Latin squares if and only if S' and T' are isomorphic as block designs.

We classify the uniform $(6 \times 6)/10$ semi-Latin squares via backtrack searches (see, for example, Section 6.2 of Gibbons and Östergård (2007), for an introduction to using backtrack search for the enumeration of block designs). Our searches are for the block multisets of the duals of the uniform semi-Latin squares we seek to classify. We represent a block multiset as a set of (block,multiplicity)-pairs, where the blocks in these pairs are distinct and the associated multiplicity of a block gives the number of times that block occurs in the block multiset. Our backtrack searches work (block,multiplicity)-pair by (block,multiplicity)-pair.

We define a *partial solution* to be a set of (block,multiplicity)-pairs, such that the blocks are distinct co-cliques of size 6 of $H(2, 6)$, the (block) multiplicities are 1 or 2, and no non-edge of $H(2,6)$ is contained in more than two blocks (counting multiplicities). A *solution* is a partial solution for which every non-edge of $H(2,6)$ is contained in exactly two blocks (counting multiplicities). Hence, the solutions are precisely the block multisets of the duals of the uniform $(6 \times 6)/10$ semi-Latin squares.

We first programmed a partial backtrack search exploiting the automorphism group of $H(2,6)$, using the GAP system (The GAP Group, 2020) and adapting code from its DESIGN (Soicher, 2019a) and GRAPE (Soicher, 2019b) packages. This search was used to generate a sequence

$$(P_1, A_1), (P_2, A_2), \ldots, (P_t, A_t),$$

where each P_i is a partial solution and its corresponding A_i is a set of (block,multiplicity)-pairs, such that no block of A_i is a block of P_i, and the following hold:

- each isomorphism class of duals of uniform $(6 \times 6)/10$ semi-Latin squares has at least one representative whose block multiset is a solution consisting of some P_i extended by elements belonging to the corresponding A_i;
- each P_i (considered as a multiset of independent sets of $H(2,6)$) has trivial stabiliser in $\mathrm{Aut}(H(2,6))$.

For our program, it turned out that $t = 2214$, and the search ran in under four minutes.

Then, for each pair (P_i, A_i), we used a newly developed C program to perform a backtrack search to determine all the solutions which are extensions of P_i by elements from A_i. The total run time for this step was about eight and a half hours, or on average, about 14 seconds for each i.

Finally, we took all the (1340930 as it turned out) solutions found by the C program backtrack searches and determined isomorphism class representatives amongst all the duals of uniform $(6 \times 6)/10$ semi-Latin squares having those solutions as their block multisets. We did this using the DESIGN package making heavy use of the bliss program (Junttila and Kaski, 2007) via GRAPE. The run time for this step was a little over five hours.

We found that, up to isomorphism, there are exactly 8615 uniform $(6 \times 6)/10$ semi-Latin squares. These semi-Latin squares, their PV aberrations, and their duals are available from `http://www.maths.qmul.ac.uk/~lsoicher/usls/`. There is a unique (up to isomorphism) uniform $(6 \times 6)/10$ semi-Latin square M with least PV aberration, which is

$$(532, 906, 294, 30, 6, 0, 2).$$

We present this M in Figure 1. The automorphism group of the dual of M has order 12, but the automorphism group of M has order 48, since there

1 2 3 4 5 6 7 8 9 10	11 12 13 14 15 16 17 18 19 20	21 22 23 24 25 26 27 28 29 30	31 32 33 34 35 36 37 38 39 40	41 42 43 44 45 46 47 48 49 50	51 52 53 54 55 56 57 58 59 60
11 12 21 22 31 32 41 42 51 52	1 2 23 24 33 34 43 44 53 54	3 4 13 14 35 36 45 46 55 56	5 6 15 16 25 26 47 48 57 58	7 8 17 18 27 28 37 38 59 60	9 10 19 20 29 30 39 40 49 50
17 19 25 27 35 39 43 45 53 57	7 9 21 22 36 40 47 48 55 59	1 2 15 20 31 37 41 49 58 60	3 10 13 18 23 28 44 50 51 52	4 5 11 16 29 30 32 33 54 56	6 8 12 14 24 26 34 38 42 46
14 15 24 28 37 40 47 50 54 56	3 5 26 27 32 39 46 49 51 60	8 9 11 19 33 38 44 48 52 57	1 2 12 17 29 30 42 45 55 59	6 10 13 20 21 22 34 35 53 58	4 7 16 18 23 25 31 36 41 43
13 16 23 29 34 38 48 49 55 60	4 8 28 30 31 35 42 50 57 58	6 10 12 18 32 39 43 47 54 59	7 9 11 20 24 27 41 46 53 56	1 2 14 19 25 26 36 40 51 52	3 5 15 17 21 22 33 37 44 45
18 20 26 30 33 36 44 46 58 59	6 10 25 29 37 38 41 45 52 56	5 7 16 17 34 40 42 50 51 53	4 8 14 19 21 22 43 49 54 60	3 9 12 15 23 24 31 39 55 57	1 2 11 13 27 28 32 35 47 48

Figure 1: Uniform $(6 \times 6)/10$ semi-Latin square with least PV aberration

are automorphisms of M fixing every cell, but interchanging the treatments in one or both of the pairs of treatments with concurrence 6. We note that $\eta_0(M) = 532$, which is well off the lower bound of 300 given by Theorem 5.

The next best PV aberration of a uniform $(6 \times 6)/10$ semi-Latin square is

$$(532, 912, 276, 48, 0, 0, 2),$$

and the worst is

$$(600, 720, 450, 0, 0, 0, 0).$$

Using the DESIGN package, we found that no dual of a uniform $(6 \times 6)/10$ semi-Latin square is resolvable. Equivalently, no uniform $(6 \times 6)/10$ semi-Latin square is a superposition of Latin squares, which answers a question raised by Soicher (2013a).

We have checked that the results of our computations are consistent with some previous partial classifications of uniform $(6 \times 6)/10$ semi-Latin squares done by the authors using the DESIGN package. These include finding that (up to isomorphism) there are exactly 5828 uniform $(6 \times 6)/10$ semi-Latin squares whose dual has a non-trivial automorphism, exactly 7 uniform $(6 \times 6)/10$ semi-Latin squares having at least two treatments with concurrence 6, and (see Soicher (2013a)) exactly 98 uniform $(6 \times 6)/10$ semi-Latin squares having no concurrence greater than 2. We have also checked that our programs agree with Soicher (2013a) that, up to isomorphism, there are exactly 10 uniform $(5 \times 5)/8$ semi-Latin squares and exactly 277 uniform $(5 \times 5)/12$ semi-Latin squares.

5 A new construction of uniform semi-Latin squares

Suppose that $n > 2$ and there exist $n-1$ mutually orthogonal Latin squares $\Lambda_1, \ldots, \Lambda_{n-1}$ of order n with disjoint sets of symbols $\mathcal{L}_1, \ldots, \mathcal{L}_{n-1}$. We present a construction of an $((n+1) \times (n+1))/(\mu n)$ uniform semi-Latin square with $\mu = n-2$, which generalises the uniform $(6 \times 6)/15$ semi-Latin square used in the proof of Theorem 5.1 of Soicher (2012).

For $i = 1, \ldots, n-2$ and $j = 1, \ldots, n$, put $\mathcal{L}_{ij} = \{(\alpha, j) : \alpha \in \mathcal{L}_i\}$. Put $\bar{\mathcal{L}}_i = \mathcal{L}_{i1} \cup \cdots \cup \mathcal{L}_{in}$. Create an n-fold inflation $\bar{\Lambda}_i$ of Λ_i using the symbols in $\bar{\mathcal{L}}_i$.

Put $\bar{\mathcal{L}}_{n-1} = \mathcal{L}_{n-1} \times \{1, \ldots, n-2\}$, and create an $(n-2)$-fold inflation $\bar{\Lambda}_{n-1}$ of Λ_{n-1} using the symbols in $\bar{\mathcal{L}}_{n-1}$.

Superpose $\bar{\Lambda}_1, \ldots, \bar{\Lambda}_{n-1}$ to give an $(n \times n)/\ell$ semi-Latin square S, where $\ell = (n-2)n + n - 2 = (n-2)(n+1)$.

Add an extra row and an extra column to this. For $i = 1, \ldots, n$, $j = 1, \ldots, n$ and $t = 1, \ldots, n-2$, each cell (i, j) contains a unique symbol from $\mathcal{L}_{tj}$: remove this from cell (i, j) and insert it in cells $(i, n+1)$ and $(n+1, j)$. Put all the symbols in $\bar{\mathcal{L}}_{n-1}$ into cell $(n+1, n+1)$.

Now we have an array $\bar{S}$ of size $((n+1) \times (n+1))/k$ where $k = \ell - (n-2) = n(n-2) = \left|\bar{\mathcal{L}}_{n-1}\right|$. Every symbol in $\bar{\mathcal{L}}_1 \cup \cdots \cup \bar{\mathcal{L}}_{n-2} \cup \bar{\mathcal{L}}_{n-1}$ occurs precisely once in each row and once in each column, so $\bar{S}$ is a semi-Latin square.

For $1 \leq i, j \leq n$, cell (i, j) contains symbols $(\alpha, 1), \ldots, (\alpha, n-2)$ for a single symbol α in $\mathcal{L}_{n-1}$: thus it has $n-2$ symbols in common with cell $(n+1, n+1)$.

Suppose that $i \neq i'$ and $j \neq j'$ with i, i', j, j' in $\{1, \ldots, n\}$. Then there is exactly one value of t such that the original cells (i, j) and (i', j') have the same symbol in Λ_t. If $t = n-1$ then the symbols in common to the new cells (i, j) and (i', j') in $\bar{S}$ are precisely those in $\bar{\mathcal{L}}_{n-1}$: there are $n-2$ of these symbols. If $1 \leq t \leq n-2$ then the symbols in common to the new cells (i, j) and (i', j') are precisely those in $(\bar{\mathcal{L}}_t \setminus \mathcal{L}_{tj}) \setminus \mathcal{L}_{tj'}$: there are $n-2$ of these.

We have shown that new cell (i, j) in $\bar{S}$ has $n-2$ symbols in common with new cell (i', j') for all j' in $\{1, \ldots, n\} \setminus \{j\}$. This accounts for $(n-1)(n-2)$ symbols in new cell (i, j). Since it has no symbols in common with new cell (i', j), the remaining $n-2$ symbols in new cell (i, j) must be in cell $(i', n+1)$. Similarly, new cell (i, j) has $n-2$ symbols in common with cell $(n+1, j')$.

For $1 \leq i \leq n$ and $1 \leq j \leq n$, we have shown that cell $(i, n+1)$ has precisely $n-2$ symbols in common with new cell (i', j) when $i' \neq i$ and none in common with new cell (i, j). For any fixed j, this accounts for $(n-1)(n-2)$ symbols in column j. Hence the remaining $n-2$ symbols in cell $(i, n+1)$ must be in cell $(n+1, j)$.

This completes the proof that $\bar{S}$ is a uniform semi-Latin square.

Theorem 7. *Let $\bar{S}$ be the $((n+1)\times(n+1))/((n-2)n)$ uniform semi-Latin square constructed above. If $n \geq 5$ then $n-2>2$ and*

$$\begin{aligned}
\eta_{n+1}(\bar{S}) &= \frac{n(n-2)(n-3)}{2},\\
\eta_{n-2}(\bar{S}) &= \frac{n^2(n-1)(n-2)}{2},\\
\eta_{2}(\bar{S}) &= \frac{n^2(n-2)(n-3)(2n-1)}{2},\\
\eta_{1}(\bar{S}) &= \frac{n(n-1)(n-2)(n^3-4n^2+8n-2)}{2},\\
\eta_{0}(\bar{S}) &= \frac{n^2(n-2)(3n^2-9n+4)}{2},
\end{aligned}$$

and $\eta_m(\bar{S})=0$ for all other non-negative integers m.

Proof. Suppose that $1 \leq i \leq n-2$ and $\alpha \in \mathcal{L}_i$. For j and j' in $\{1,\ldots,n\}$ with $j \neq j'$, the concurrence of (α, j) and (α, j') is $n-2$. If $\beta \in \mathcal{L}_i \setminus \{\alpha\}$, then the concurrence of (α, j) and (β, j) is 1 and there is one other value of j' such that the concurrence of (α, j) and (β, j') is 1; otherwise this concurrence is 0.

Suppose that $1 \leq i' \leq n-2$ and $i' \neq i$. Then we can show that, of the n^2 elements in $\bar{\mathcal{L}}_{i'}$, $2n-1$ have concurrence 2 with (α, j), $(n-1)(n-3)$ have concurrence 1 with (α, j), and the remaining $2(n-1)$ have concurrence 0 with (α, j).

Now consider γ in $\mathcal{L}_{n-1}$. For j and j' in $\{1,\ldots,n-2\}$ with $j \neq j'$, the concurrence of (γ, j) and (γ, j') is $n+1$. If $\delta \in \mathcal{L}_{n-1} \setminus \{\gamma\}$ and $j' \in \{1,\ldots,n-2\}$ then the concurrence of (γ, j) and (δ, j') is 1. If $\alpha \in \mathcal{L}_i$, as above, then the concurrence of (γ, j) and (α, j') is 0 for one value of j' and is 1 for the remaining $n-1$ values of j'.

It follows that $\eta_{n+1}(\bar{S}) = n(n-2)(n-3)/2$, and that $\eta_{n-2}(\bar{S}) = n(n-2)n(n-1)/2$. Concurrence 2 occurs only between $\bar{\mathcal{L}}_i$ and $\bar{\mathcal{L}}_{i'}$ for $i \neq i'$, so we find that $\eta_2(\bar{S}) = (n-2)(n-3)/2 \times n^2(2n-1)$.

Concurrence 1 occurs $(n-2)n^2 \times 2(n-1)/2$ times within treatment sets $\bar{\mathcal{L}}_1, \ldots, \bar{\mathcal{L}}_{n-2}$, and $(n-2)(n-3)n^2/2 \times (n-1)(n-3)$ times between such sets. It occurs $n(n-1)/2 \times (n-2)^2$ times within $\bar{\mathcal{L}}_{n-1}$, and $n(n-2) \times (n-2) \times n(n-1)$ times between $\bar{\mathcal{L}}_{n-1}$ and other treatments. Finally, concurrence 0 occurs $(n-2)n^2 \times (n-1)(n-2)/2$ times within $\bar{\mathcal{L}}_1, \ldots, \bar{\mathcal{L}}_{n-2}$, $(n-2)(n-3)n^2/2 \times 2(n-1)$ times between these sets, and $n(n-2) \times (n-2) \times n$ times between these and $\mathcal{L}_{n-1}$. □

Let $n \geq 5$. We observe that the lower bound for η_0 given by Theorem 5 applied to $\bar{S}$ is $(n+1)n^2(n-2)^2/2$, which is $(n+1)(n-2)/(3n^2-9n+4)$

times $\eta_0(\bar{S})$. Thus, when $n+1$ is a prime power, $\bar{S}$ is far from optimal with respect to PV aberration. However, when $n+1$ is not a prime power, we do not know how far from optimal $\bar{S}$ is with respect to PV aberration.

Putting $n = 5$ in Theorem 7 shows that the uniform $(6\times 6)/15$ semi-Latin square constructed by Soicher (2012) has PV aberration

$$(1275, 1890, 675, 150, 0, 0, 15).$$

On the other hand, the least PV aberration of any uniform $(6 \times 6)/15$ semi-Latin square found by the authors so far is

$$(1260, 1943, 630, 125, 40, 0, 7),$$

and a uniform $(6 \times 6)/15$ semi-Latin square with this PV aberration is available from `http://www.maths.qmul.ac.uk/~lsoicher/usls/`.

Suppose now $n > 2$ is a prime power. Then there exist $n-1$ MOLS of order n, and so we can make a uniform $((n+1)\times(n+1))/((n-2)n)$ semi-Latin square $\bar{S}$ as above. In this case, there is also the construction of Theorem 4.3 of Soicher (2012) (equivalent to the construction of Corollary 4.1.2 of Suen (1982)) which gives a uniform semi-Latin square T of size $((n+1) \times (n+1))/((n-1)n)$ (and also one of size $((n+1) \times (n+1))/((n-1)n/2)$ when n is odd). Now let s and t be non-negative integers, not both zero. Let U be an s-fold inflation of $\bar{S}$ if $t = 0$, let U be a t-fold inflation of T if $s = 0$, and otherwise let U be the superposition of an s-fold inflation of $\bar{S}$ and a t-fold inflation of T. Then U is a uniform $((n+1) \times (n+1))/(\mu n)$ semi-Latin square with $\mu = s(n-2) + t(n-1)$. Since $n-2$ and $n-1$ are coprime, by the well-known solution of the "Frobenius coin problem" for two denominations, every integer greater than or equal to $(n-3)(n-2)$ is a non-negative integer linear combination of $n-2$ and $n-1$. Therefore, when $n > 2$ is a prime power, there exists a uniform semi-Latin square of size $((n+1) \times (n+1))/(\mu n)$ for every positive integer $\mu \geq (n-3)(n-2)$ (and for every positive integer $\mu \geq (n-3)((n-1)/2-1)$ if in addition n is odd).

6 Constructing affine resolvable designs and BIBDs from uniform semi-Latin squares

Let n be any integer greater than 2, let S be a uniform $(n \times n)/(\mu(n-1))$ semi-Latin square, with rows $R_1, \dots, R_n$, and let $\Delta(S)$ be its underlying block design. Obtain the block design $\Delta_1(S)$ from $\Delta(S)$ by adding, for each $i = 1, \dots, n$, μ new treatments $R_{i,1}, \dots, R_{i,\mu}$, each incident precisely with the blocks in row R_i. In $\Delta_1(S)$, the set of blocks in each column form a replicate, and every pair of blocks in different columns have exactly μ treatments in common. Hence $\Delta_1(S)$ is an affine resolvable $(\mu n^2, n^2, n, \mu n)$-design. The analogous construction using columns in place of rows gives another affine resolvable design $\Delta_2(S)$, which may or may not be isomorphic to $\Delta_1(S)$. However, if T is any uniform $(n \times n)/(\mu(n-1))$ semi-Latin square isomorphic to S, then the isomorphism classes of $\Delta_1(T)$ and $\Delta_2(T)$ are those of $\Delta_1(S)$ and $\Delta_2(S)$, in some order.

Each of the 8615 uniform $(6 \times 6)/10$ semi-Latin squares S classified in Section 4 gives two affine resolvable designs $\Delta_1(S)$ and $\Delta_2(S)$. We found that, up to block design isomorphism (determined by the DESIGN package using bliss via GRAPE), these give in total 16875 affine resolvable $(72, 36, 6, 12)$-designs. The only isomorphisms of affine resolvable designs resulted from the 355 uniform $(6 \times 6)/10$ semi-Latin squares T having an automorphism interchanging rows and columns, in which case $\Delta_1(T)$ is isomorphic to $\Delta_2(T)$.

An affine resolvable $(72, 36, 6, 12)$-design is equivalent to an orthogonal array $\mathrm{OA}(72, 6, 6, 2)$. Recent work finding all isomorphism classes of orthogonal arrays for certain parameter tuples, such as that by Bulutoglu and Margot (2008), Bulutoglu and Ryan (2015, 2018), Geyer et al. (2019) and Schoen et al. (2010), does not yet include the orthogonal arrays $\mathrm{OA}(72, 6, 6, 2)$, but our present work can be used to produce 16875 pairwise non-isomorphic orthogonal arrays having this parameter tuple. An arbitrary orthogonal array $\mathrm{OA}(72, 6, 6, 2)$ A is isomorphic to one of these 16875 orthogonal arrays if and only if A is isomorphic to an orthogonal array B with the property that for each $i = 1, 2, 3, 4, 5, 6$, there are two rows of B of the form (i, i, i, i, i, i). An instance of this, which extends to an $\mathrm{OA}(72, 7, 6, 2)$, is given in Example 4.1.2 of Suen (1982).

For every uniform $(n \times n)/(\mu(n-1))$ semi-Latin square S we can make a third design $\Delta_3(S)$ from $\Delta(S)$ by adding μ new treatments for each row and μ new treatments for each column, as above (so each new treatment is incident with the blocks in its corresponding row or column), and then taking the dual. This gives a block design with n^2 treatments in $\mu n(n-1) + 2\mu n$ blocks of size n, in which every pair of distinct treatments has concurrence μ. In other words, $\Delta_3(S)$ is an $(n^2, \mu n(n+1), \mu(n+1), n, \mu)$-BIBD.

When $\mu = 1$, this construction is essentially that of Bose (1938) to obtain an affine plane of order n (i.e. an $(n^2, n(n+1), n+1, n, 1)$-BIBD) from $n-1$ MOLS of order n. When $\mu \geq 2$, the BIBDs we construct have repeated blocks. However, their parameter tuples seem to have no overlap with those of the BIBDs with repeated blocks constructed by Bailey and Cameron (2007).

If S and T are isomorphic uniform $(n \times n)/k$ semi-Latin squares then $\Delta_3(S)$ and $\Delta_3(T)$ must be isomorphic, but the converse need not hold. For example, it was shown by Owens and Preece (1995) (and confirmed by Egan and Wanless (2016)) that there are exactly 15 sets of eight MOLS of order 9, up to trisotopism, that is, up to uniform permutation of the rows, uniform permutation of the columns, permuting the symbols separately within each Latin square, and optionally, simultaneously transposing the Latin squares. It follows, from Theorem 3.3 of Soicher (2012), that there are exactly 15 isomorphism classes of uniform $(9 \times 9)/8$ semi-Latin squares. However, there are only seven isomorphism classes of affine planes of order 9 (see, for example, Owens and Preece (1995)), and so there are non-isomorphic uniform $(9 \times 9)/8$ semi-Latin squares S and T such that $\Delta_3(S)$ and $\Delta_3(T)$ are isomorphic affine planes.

We have, however, verified (via the DESIGN package using bliss via GRAPE), that the 8615 $(36, 84, 14, 6, 2)$-BIBDs $\Delta_3(S)$ obtained from the 8615 uniform $(6 \times 6)/10$ semi-Latin squares S classified in Section 4 are pairwise non-isomorphic. According to Mathon and Rosa (2007), only five BIBDs with these parameters were known at that time.

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