A NOTE ON K-FUNCTIONAL, MODULUS OF SMOOTHNESS, JACKSON THEOREM AND NIKOLSKII-STECHKIN INEQUALITY ON DAMEK-RICCI SPACES

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Abstract. In this paper we study approximation theorems for L^2-space on Damek-Ricci spaces. We prove direct Jackson theorem of approximations for the modulus of smoothness defined using spherical mean operator on Damek-Ricci spaces. We also prove Nikolskii-Stechkin inequality. To prove these inequalities we use functions of bounded spectrum as a tool of approximation. Finally, as an application we prove equivalence of the K-functional and modulus of smoothness for Damek-Ricci spaces.

Contents

1. Introduction

The main purpose of this paper is to study the equivalence of the K-functional and the modulus of smoothness generated by the spherical mean operator on Damek-Ricci spaces. Damek-Ricci spaces, also known as Harmonic NA groups, are solvable (non-unimodular)

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Lie groups. It is worth mentioning that Damek-Ricci spaces contain non-compact symmetric spaces of rank one as a very small subclass and, in general, Damek-Ricci spaces are not symmetric. Damek-Ricci spaces were introduced by Eva Damek and Fulvo Ricci in [14] and the geometry of these spaces was studied by Damek [13] and Cowling-Dooley-Koranyi [6]. Fourier analysis on these spaces has been developed and studied by many authors including Anker-Damek-Yacoub [1], Astengo-Comporesi-Di Blasio [2] , Damek-Ricci [15], Di Blasio [16], Ray-Sarkar [40], Kumar-Ray-Sarkar [22]. One of the interesting features of these spaces is that the radial analysis on these spaces behaves similar to the hyperbolic spaces as observed in [1] and therefore it fits into the perfect setting of Jacobi analysis developed by Flensted-Jensen and Koornwinder [23, 19, 20].

The study of the K-functional is a classical and important topic in interpolation theory and approximation theory. Peetre the K-functional is useful for describing the interpolation spaces between two Banach spaces. First, let us recall the definition of the K-functional. For two Banach spaces A_1 and A_2, the Peetre the K-functional is given by

$$K(f, \delta, A_1, A_2) := \inf\{\|f_1\|_{A_1} + \delta \|f_2\|_{A_2} : \; f = f_1 + f_2, \, f_1 \in A_1, f_2 \in A_2\},$$

where δ is a positive parameter. Now, the Peetre interpolation space $(A_1, A_2)_{\theta,r}$ for $0 < \theta < 1$, $0 < r \leq \infty$, is defined by the norm

$$|f|_{(A_1,A_2)_{\theta,r}} := \begin{cases} \left(\int_0^\infty [\delta^{-\theta} K(f, \delta, A_1, A_2)]^r \frac{d\delta}{\delta}\right)^{\frac{1}{r}} & \text{if } 0 < r < \infty, \\ \sup_{\delta>0} \delta^{-\theta} K(f, \delta, A_1, A_2) & \text{if } r = \infty. \end{cases}$$

The characterizations of the K-functional has several applications in approximation theory [12]. In [27], Peetre started characterization of the K-functional by proving an equivalence of it with the modulus of smoothness for L^p-spaces on $\mathbb{R}^n$ which proved to be very helpful to study apporximation theory. Later, in [10] the authors showed its equivalence in terms of the rearrangement of derivatives for a pair of Sobolev spaces W_p^m and for the pair (L^p, W_p^m). In particular, a characterization of the K-functional for $(L^2(\mathbb{R}), W_2^m(\mathbb{R}))$ can be found in the classical book of Berens and Buter [4]. The characterizations of the K-functional for the pair $(L^2(X), W_2^m(X))$ were explored by several authors for different choices of X. Classically, this equivalence was proved for $X = \mathbb{R}^n$ by Peetre [27] and after that it was proved for $X = [a, b]$ by De Vore-Scherer [10], for weighted setting by Ditzian [11], for $X = \mathbb{R}^n$ with Dunkl translation by Belkina and Platonov [3], for rank one symmetric spaces [18], for Jacobi analysis in [17] and for compact symmetric spaces on [38]. In this paper our aim is to extend this characterization to more general setting of solvable (non unimodular) Lie groups. We consider the pair $(L^2(X), W_2^m(X))$ for X being the Damek-Ricci spaces. We will prove the equivalence of the K-functional and modulus of smoothness generated by spherical mean operator on Damek-Ricci space. Modulus of smoothness for Damek-Ricci space has been introduced in [22]. We prove our main result by establishing two classical results, namely, Direct Jackson theorem [26] and Nikolskii-Stechkin inequality [25] for Damek-Ricci spaces. Platonov studied Direct Jackson theorem and Nikolskii-Stechkin inequality for compact homogeneous manifolds and for noncompact symmetric spaces of rank one ([33, 35, 34, 38, 32, 39]).

2. Essentials about harmonic NA groups

For basics of harmonic NA groups and Fourier analysis on them, one can refer to seminal research papers [13, 14, 15, 16, 1, 2, 6, 40, 22, 21]. However, we give necessary definitions, notation and terminology that we shall use in this paper.

Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra, equipped with an inner product $\langle\, , \rangle$. Denote by $\mathfrak{z}$ the center of $\mathfrak{n}$ and by $\mathfrak{v}$ the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$ with respect to the inner product of $\mathfrak{n}$. We assume that dimensions of $\mathfrak{v}$ and $\mathfrak{z}$ are m and l respectively as real vector spaces. The Lie algebra $\mathfrak{n}$ is H-type algebra if for every $Z \in \mathfrak{z}$, the map $J_Z : \mathfrak{v} \to \mathfrak{v}$ defined by

$$\langle J_Z X, Y\rangle = \langle Z, [X, Y]\rangle, \quad X, Y \in \mathfrak{v},\, Z \in \mathfrak{z},$$

satisfies the condition $J_Z^2 = -\|Z\|^2 I_{\mathfrak{v}}$, where $I_{\mathfrak{v}}$ is the identity operator on $\mathfrak{v}$. It follows that for $Z \in \mathfrak{z}$ with $\|Z\| = 1$ one has $J_Z^2 = -I_{\mathfrak{v}}$; that is, J_Z induced a complex structure on $\mathfrak{v}$ and hence $m = \dim(\mathfrak{v})$ is always even. A connected and simply connected Lie group N is called H-type if its Lie algebra is of H-type. The exponential map is a diffeomorphism as N is nilpotent, we can parametrize the element of $N = \exp \mathfrak{n}$ by (X, Z), for $X \in \mathfrak{v}$ and $Z \in \mathfrak{z}$. The multiplication on N follows from the Campbell-Baker-Hausdorff formula given by

$$(X, Z)(Z', Z') = (X + X', Z + Z' + \frac{1}{2}[X, X']).$$

The group $A = \mathbb{R}_+^*$ acts on N by nonisotropic dilations as follows: $(X, Y) \mapsto (a^{\frac{1}{2}}X, aZ)$. Let $S = N \ltimes A$ be the semidirect product of N with A under the aforementioned action.

The group multiplication on S is defined by

$$(X, Z, a)(X', Z', a') = (X + a^{\frac{1}{2}} X', Z + aZ' + \frac{1}{2} a^{\frac{1}{2}} [X, X'], aa').$$

Then S is a solvable (connected and simply connected) Lie group with Lie algebra $\mathfrak{s} = \mathfrak{z} \oplus \mathfrak{v} \oplus \mathbb{R}$ and Lie bracket

$$[(X, Z, \ell), (X', Z', \ell')] = (\frac{1}{2}\ell X' - \frac{1}{2}\ell' X, \ell Z' - \ell' Z + [X, X]', 0).$$

The group S is equipped with the left-invariant Riemannian metric induced by

$$\langle (X, Z, \ell), (X', Z', \ell') \rangle = \langle X, X' \rangle + \langle Z, Z' \rangle + \ell\ell'$$

on $\mathfrak{s}$. The homogneous dimension of N is equal to $\frac{m}{2} + l$ and will be denoted by Q. At times, we also use symbol ρ for $\frac{Q}{2}$. Hence $\dim(\mathfrak{s}) = m + l + 1$, denoted by d. The associated left Haar measure on S is given by $a^{-Q-1} dX dZ da$, where dX, dZ and da are the Lebesgue measures on $\mathfrak{v}, \mathfrak{z}$ and $\mathbb{R}_+^*$ respectively. The element of A will be identified with $a_t = e^t$, $t \in \mathbb{R}$. The group S can be realized as the unit ball $B(\mathfrak{s})$ in $\mathfrak{s}$ using the Cayley transform $C : S \to B(\mathfrak{s})$ (see [1]).

To define (Helgason) Fourier transform on S we need to introduce the notion of Poisson kernel [2]. The Poisson Kernel $\mathcal{P} : S \times N \to \mathbb{R}$ is defined by $\mathcal{P}(na_t, n') = P_{a_t}(n'^{-1}n)$, where

$$P_{a_t}(n) = P_{a_t}(X, Z) = C a_t^Q \left(\left(a_t + \frac{|X|^2}{4} \right)^2 + |Z|^2 \right)^{-Q}, \quad n = (X, Z) \in N.$$

The value of C is suitably adjusted so that $\int_N P_a(n) dn = 1$ and $P_1(n) \leq 1$. The Poisson kernel satisfies several useful properties (see [22, 40, 2]), we list here a few of them. For $\lambda \in \mathbb{C}$, the complex power of the Poisson kernel is defined as

$$\mathcal{P}_\lambda(x, n) = \mathcal{P}(x, n)^{\frac{1}{2} - \frac{i\lambda}{Q}}.$$

It is known ([40, 2]) that for each fixed $x \in S$, $\mathcal{P}_\lambda(x, \cdot) \in L^p(N)$ for $1 \leq p \leq \infty$ if $\lambda = i\gamma_p \rho$, where $\gamma_p = \frac{2}{p} - 1$. A very special feature of $\mathcal{P}_\lambda(x, n)$ is that it is constant on the hypersurfaces $H_{n,a_t} = \{n\sigma(a_t n') : n' \in N\}$. Here σ is the geodesic inversion on S, that is an involutive, measure-preserving, diffeomorphism which can be explicitly given by [6]:

$$\sigma(X, Z, a_t) = \left(\left(e^t + \frac{|V|^2}{4} \right)^2 + |Z|^2 \right)^{-1} \left(\left(-\left(e^t + \frac{|X|^2}{4} \right) + J_Z \right) X, -Z, a_t \right).$$

Let Δ_S be the Laplace-Beltrami operator on S. Then for every fixed $n \in N$, $\mathcal{P}_\lambda(x,n)$ is an eigenfunction of Δ_S with eigenvalue $-(\lambda^2 + \frac{Q^2}{4})$ (see [2]). For a measurable function f on S, the (Helgason) Fourier transform is defined as

$$\widetilde{f}(\lambda, n) = \int_S f(x)\,\mathcal{P}_\lambda(x,n)dx$$

whenever the integral converge. For $f \in C_c^\infty(S)$, the following inversion formula holds ([2, Theorem 4.4]):

$$f(x) = C\int_{\mathbb{R}}\int_N \widetilde{f}(\lambda,n)\,\mathcal{P}_{-\lambda}(\lambda,n)|c(\lambda)|^{-2}\,d\lambda dn,$$

where $C = \frac{c_{m,l}}{2\pi}$. The authors also proved that the (Helgason) Fourier transform extends to an isometry from $L^2(S)$ onto the space $L^2(\mathbb{R}_+ \times N, C|c(\lambda)|^{-2}d\lambda dn)$. In fact they have the precise value of constants, we refer the reader to [2]. The following estimates for the function $|c(\lambda)|$ holds: $c_1|\lambda|^{d-1} \le |c(\lambda)|^{-2} \le (1+|\lambda|)^{d-1}$ for all $\lambda \in \mathbb{R}$ (e. g. see [40]). In [40, Theorem 4.6], the authors proved the following version of the Hausdorff-Young inequality: For $1 \le p \le 2$ we have

$$\left(\int_{\mathbb{R}}\int_N |\widetilde{f}(\lambda + i\gamma_{p'}\rho, n)|^{p'} dn\,|c(\lambda)|^{-2}d\lambda\right)^{\frac{1}{p'}} \le C_p\|f\|_p. \tag{1}$$

A function f on S is called *radial* if for all $x, y \in S$, $f(x) = f(y)$ if $\mu(x,e) = \mu(y,e)$, where μ is the metric induced by the canonical left invariant Riemannian structure on S and e is the identity element of S. Note that radial functions on S can be identified with the functions $f = f(r)$ of the geodesic distance $r = \mu(x,e) \in [0,\infty)$ to the identity. It is clear that $\mu(a_t, e) = |t|$ for $t \in \mathbb{R}$. At times, for any radial function f we use the notation $f(a_t) = f(t)$. For any function space $\mathcal{F}(S)$ on S, the subspace of radial functions will be denoted by $\mathcal{F}(S)^\#$. The elementary spherical function $\phi_\lambda(x)$ is defined by

$$\phi_\lambda(x) := \int_N \mathcal{P}_\lambda(x,n)\mathcal{P}_{-\lambda}(x,n)\,dn.$$

It follows ([1, 2]) that ϕ_λ is a radial eigenfunction of the Laplace-Beltrami operator Δ_S of S with eigenvalue $-(\lambda^2 + \frac{Q^2}{4})$ such that $\phi_\lambda(x) = \phi_{-\lambda}(x)$, $\phi_\lambda(x) = \phi_\lambda(x^{-1})$ and $\phi_\lambda(e) = 1$. It is also evident from the fact that, for every fixed $n \in N$, $\mathcal{P}_\lambda(x,n)$ is an eigenfunction of Δ_S with eigenvalue $-(\lambda^2 + \frac{Q^2}{4})$, that, for suitable function f on S, we have

$$\widetilde{\Delta_S^l f}(\lambda, n) = -(\lambda^2 + \frac{Q^2}{4})^l\widetilde{f}(\lambda,n)$$

for every natural number l (see [2, p. 416]). In [1], the authors showed that the radial part (in geodesic polar coordinates) of the Laplace-Beltrami operator Δ_S given by

$$\operatorname{rad}\Delta_S = \frac{\partial^2}{\partial t} + \{\frac{m+l}{2}\coth\frac{t}{2} + \frac{k}{2}\tanh\frac{t}{2}\}\frac{\partial}{\partial t},$$

is (by substituting $r = \frac{t}{2}$) equal to $\frac{1}{4}\mathcal{L}_{\alpha,\beta}$ with indices $\alpha = \frac{m+l+1}{2}$ and $\beta = \frac{l-1}{2}$, where $\mathcal{L}_{\alpha,\beta}$ is the Jacobi operator studied by Koornwinder [23] in detail. It is worth noting that we are in the ideal situation of Jacobi analysis with $\alpha > \beta > \frac{-1}{2}$. In fact, the Jacobi functions $\phi_\lambda^{\alpha,\beta}$ and elementary spherical functions ϕ_λ are related as ([1]): $\phi_\lambda(t) = \phi_{2\lambda}^{\alpha,\beta}(\frac{t}{2})$. As consequence of this relation, the following estimates for the elementary spherical functions hold true (see [36]).

Lemma 2.1. *The following inequalities are valid for spherical functions $\phi_\lambda(t)$ $(t, \lambda \in \mathbb{R}_+)$:*

- $|\phi_\lambda(t)| \leq 1$.
- $|1 - \phi_\lambda(t)| \leq \frac{t^2}{2}(\lambda^2 + \frac{Q^2}{4})$.
- *There exists a constant $c > 0$, depending only on λ, such that $|1 - \phi_\lambda(t)| \geq c$ for $\lambda t \geq 1$.*

Let σ_t be the normalized surface measure of the geodesic sphere of radius t. Then σ_t is a nonnegative radial measure. The spherical mean operator M_t on a suitable function space on S is defined by $M_t f := f * \sigma_t$. It can be noted that $M_t f(x) = \mathcal{R}(f^x)(t)$, where f^x denotes the right translation of function f by x and $\mathcal{R}$ is the radialization operator defined, for suitable function f, by

$$\mathcal{R}f(x) = \int_{S_\nu} f(y)\, d\sigma_\nu(y),$$

where $\nu = r(x) = \mu(C(x), 0)$, here C is the Cayley transform, and $d\sigma_\nu$ is the normalized surface measure induced by the left invariant Riemannian metric on the geodesic sphere $S_\nu = \{y \in S : \mu(y, e) = \nu\}$. It is easy to see that $\mathcal{R}f$ is a radial function and for any radial function f, $\mathcal{R}f = f$. Consequently, for a radial function f, $M_t f$ is the usual translation of f by t. In [22], the authors proved that, for a suitable function f on S, $\widetilde{M_t f}(\lambda, n) = \widetilde{f}(\lambda, n)\phi_\lambda(t)$ whenever both make sense. Also, $M_t f$ converges to f as $t \to 0$, i.e., $\mu(a_t, e) \to 0$. It is also known that M_t is a bounded operator on $L^2(S)$ with operator norm equal to $\phi_0(a_t)$. In particular, for $f \in L^2(S)$, we have $\|M_t f\|_2 \leq \phi_0(a_t)\|f\|_2$. The following Lemmata are taken from [5].

Lemma 2.2. *Let $\alpha > \frac{-1}{2}$. Then there are positive constant $c_{1,\alpha}$ and $c_{2,\alpha}$ such that*

$$c_{1,\alpha} \min\{1, (\lambda t)^2\} \leq 1 - j_\alpha(\lambda t) \leq c_{2,\alpha} \min\{1, (\lambda t)^2\},$$

where j_α is the usual Bessel function of first kind normalized by $j_\alpha(0) = 1$.

Lemma 2.3. *Let $\alpha > \frac{-1}{2}$ and $t_0 > 0$. Then, for all $\lambda \in \mathbb{R}$, there exist a constant $c_1 > 0$ such that for all $0 \leq t \leq t_0$, the function ϕ_λ satisfies*

$$|1 - \phi_\lambda(t)| \geq c_1 |1 - j_\alpha(\lambda t)|,$$

where j_α is the usual Bessel function of first kind normalized by $j_\alpha(0) = 1$.

3. Main results

In this section we present our main results. Throughout this section, we denote a Damek-Ricci space by S. We denote by $L^2(S)$ the Hilbert space of all square integrable function on S with respect to Haar measure λ on S. We begin this section by recalling the definition of Sobolev spaces on Damek-Ricci spaces.

The Sobolev space $W_2^m(S)$ on Damek-Ricci space S is defined by

$$W_2^m(S) := \{f \in L^2(S) : \Delta_S^l f \in L^2(S), \quad l = 1, 2, \ldots, m\}.$$

The space $W_2^m(S)$ can be equipped with seminorm $|f|_{W_2^m(S)} := \|\Delta_S^m f\|_2$ and with the norm $\|f\|_{W_2^m(S)} = \|f\|_2 + \|\Delta_S^m f\|_2$.

The modulus of smoothness (continuity) Ω_k is defined by using the spherical mean operator M_t as follows:

$$\Omega_k(f, \delta)_2 := \sup_{0<t\leq\delta} \|\Delta_t^k f\|_2,$$

where $\Delta_t^k f = (I - M_t)^k f$. The modulus of smoothness $\Omega_k(f, \delta)_2$ satisfies the following properties:

(i) The function $\delta \mapsto \Omega_k(f, \delta)_2$ is a decreasing function and satisfies

$$\Omega_k(f \pm g, \delta)_2 \leq \Omega_k(f, \delta)_2 + \Omega_k(g, \delta)_2$$

for all $f, g \in L^2(S)$.

(ii) $\Omega_k(f,\delta)_2 \le (\phi_0(a_t)+1)^k\|f\|_2$ and $\Omega_k(f,\delta)_2 \le (1+\phi_0(a_t))^{k-l}\Omega_l(f,\delta)_2$ for $l \le k$.

(iii) If $f \in W_2^m(S)$ then we have $\Omega_k(f,\delta)_2 \le \delta^{2k}\|\Delta_S^k f\|_2, \quad k \le m$.

The proof of (i) and (ii) follows from the definition of modulus of continuity and norm estimate for M_t on $L^2(S)$. To show (iii), we note, by Plancherel formula, that,

$$\|\Delta_t^k f\|_2^2 = \int_0^\infty \int_N |\widetilde{(\Delta_t^k f)}(\lambda,n)|^2\, |c(\lambda)|^{-2} d\lambda\, dn.$$

Since $\widetilde{(\Delta_t^k f)}(\lambda,n) = |1-\phi_\lambda(a_t)|^k \widetilde{f}(\lambda,n)$ we have, by Lemma 2.1, that

$$\begin{aligned}
\|\Delta_t^k f\|_2^2 &= \int_0^\infty \int_N |1-\phi_\lambda(a_t)|^{2k} |\widetilde{f}(\lambda,n)|\, |c(\lambda)|^{-2} d\lambda\, dn \\
&\le t^{4k} \int_0^\infty \int_N (\lambda^2+\frac{Q^2}{4})^{2k} |\widetilde{f}(\lambda,n)|\, |c(\lambda)|^{-2} d\lambda\, dn \\
&= t^{4k} \int_0^\infty \int_N |\widetilde{\Delta_S^k} f(\lambda,n)|^2\, |c(\lambda)|^{-2} d\lambda\, dn = t^{4k}\|\Delta_S^k f\|_{L^2(S)}^2.
\end{aligned}$$

3.1. **Direct Jackson theorem.** This subsection is devoted for proving the Direct Jackson theorem of approximations theory for Damek-Ricci spaces. For the approximation we will use the functions of bounded spectrum. The functions of bounded spectrum were used by Platonov [39, 32, 33, 34] to prove Jackson type direct theorem for Jacobi transform and for symmetric spaces. Such kind of functions also appear in the work of Pesenson [31] under the name of Paley-Wiener functions for studying approximation theory on homogeneous manifolds.

A function $f \in L^2(S)$ is called a *function with bounded spectrum* (or a *Paley-Wiener function*) of order $\nu > 0$ if

$$\mathcal{F}f(\lambda,n) = 0 \quad \text{for } |\lambda| > \nu.$$

Denote the space of all function on S with bounded spectrum of order ν by $\mathrm{BS}_\nu(S)$. The best approximation of a function $f \in L^2(S)$ by the functions in $\mathrm{BS}_\nu(S)$ is defined by

$$E_\nu(f) := \inf_{g \in \mathrm{BV}_\nu(S)} \|f-g\|_{L^2(S)}.$$

Lemma 3.1. *Let $\nu > 0$. For any function $f \in L^2(S)$, the function $P_\nu(f)$ defined by*

$$P_\nu(f)(x) := \mathcal{F}^{-1}(\mathcal{F}f(\lambda,n)\chi_\nu(\lambda)),$$

where χ_ν is a function defined by $\chi_\nu(\lambda) = 1$ for $|\lambda| \le \nu$ and 0 otherwise, satisfies the following properties:

(i) *For every* $f \in L^2(S)$, $P_\nu(f) \in BS_\nu(S)$.
(ii) *For every function* $f \in BS_\nu(S)$, $P_\nu(f) = f$.
(iii) *If* $f \in L^2(S)$ *then* $\|P_\nu(f)\|_{L^2(S)} \le \|f\|_{L^2(S)}$ *and* $\|f - P_\nu(f)\|_{L^2(S)} \le 4E_\nu(f)$.

Proof. (i) This is trivial to see. Indeed, by definition we have

$$\mathcal{F}P_\nu(f)(x) = \mathcal{F}f(\lambda, n)\chi_\nu(\lambda) = 0$$

for $|\lambda| > \nu$. Therefore, $P_\nu(f) \in BS_\nu(S)$.

(ii) Let $f \in BS_\nu(S)$. Then $\mathcal{F}f(\lambda, n) = 0$ for $|\lambda| > \nu$ and $\mathcal{F}P_\nu(f)(\lambda, n) = \mathcal{F}f(\lambda, n)$ for $|\lambda| \le \nu$. So, by using the inversion formula we have

$$\begin{aligned} P_\nu(f)(x) &= C \int_{\mathbb{R}} \int_N \mathcal{F}P_\nu(f)(\lambda, n)\ |c(\lambda)|^{-2} d\lambda\, dn \\ &= C \int_{|\lambda| \le \nu} \int_N \mathcal{F}f(\lambda, n)\ |c(\lambda)|^{-2} d\lambda\, dn \\ &= C \int_{\mathbb{R}} \int_N \mathcal{F}f(\lambda, n)\ |c(\lambda)|^{-2} d\lambda\, dn = f(x). \end{aligned}$$

(iii) Take $f \in L^2(S)$. By Plancherel formula, we get

$$\begin{aligned} \|P_\nu(f)\|^2_{L^2(S)} &= \int_0^\infty \int_N |\mathcal{F}P_\nu(f)(\lambda, n)|^2 \, |c(\lambda)|^{-2}\, d\lambda\, dn \\ &= \int_0^\nu \int_N |\mathcal{F}f(\lambda, n)|^2 \, |c(\lambda)|^{-2}\, d\lambda\, dn \\ &\le \int_0^\infty \int_N |\mathcal{F}f(\lambda, n)|^2 \, |c(\lambda)|^{-2}\, d\lambda\, dn = \|f\|^2_{L^2(S)}. \end{aligned}$$

Also, for proving second inequality take any $g \in BS_\nu(S)$ such that

$$\|f - g\| \le 2E_\nu(f)_2.$$

Now, by using the fact that $P_\nu(g) = g$ we get

$$\begin{aligned} \|f - P_\nu(f)\|_{L^2(S)} &= \|f - g - P_\nu(g - f)\|_{L^2(S)} \\ &\le \|f - g\|_{L^2(S)} + \|f - g\|_{L^2(S)} \le 4E_\nu(f)_2. \end{aligned}$$

□

The following two theorems are analogues of Jackson's direct theorem in classical approximation theorem for Damek-Ricci spaces.

Theorem 3.2. *If $f \in L^2(S)$ then for every $\nu > 0$ we have*

$$E_\nu(f) \leq c_k\, \Omega_k \left(f, \frac{1}{\nu}\right)_2, \quad k \in \mathbb{N}, \tag{2}$$

where c_k is a constant.

Proof. The Plancherel formula gives that

$$\begin{aligned}\|f - P_\nu(f)\|^2_{L^2(S)} &= \int_0^\infty \int_N |\mathcal{F}(f - P_\nu(f))(\lambda, n)|^2\ |c(\lambda)|^{-2}\, d\lambda\, dn \\ &= \int_0^\infty \int_N |1 - \chi_\nu(\lambda)|^2\, |\mathcal{F}(f(\lambda, n)|^2\ |c(\lambda)|^{-2}\, d\lambda\, dn \\ &= \int_{\lambda \geq \nu} \int_N |\mathcal{F}(f(\lambda, n)|^2\ |c(\lambda)|^{-2}\, d\lambda\, dn.\end{aligned}$$

By Lemma 2.1 we have $|1 - \phi_\lambda\left(\frac{1}{\nu}\right)| \geq c$ for $\lambda \geq \nu$. Therefore, by Plancherel formula, we get

$$\begin{aligned}\|f - P_\nu(f)\|^2_{L^2(S)} &\leq c^{-2k} \int_{\lambda \geq \nu} \int_N |1 - \phi_\lambda\left(1/\nu\right)|^{2k} |\mathcal{F}f(\lambda, n)|^2\, |c(\lambda)|^{-2}\, d\lambda\, dn \\ &= c^{-2k} \int_{\lambda \geq \nu} \int_N |\mathcal{F}((I - M_{1/\nu})^k f)(\lambda, n)|^2\, |c(\lambda)|^{-2}\, d\lambda\, dn \\ &\leq c^{-2k} \int_0^\infty \int_N |\mathcal{F}((I - M_{1/\nu})^k f)(\lambda, n)|^2\, |c(\lambda)|^{-2}\, d\lambda\, dn \\ &= c^{-2k} \|(I - M_{1/\nu})^k f\|^2_{L^2(S)}.\end{aligned}$$

Therefore, as $P_\nu(f) \in BS_\nu(S)$, we get

$$\begin{aligned}E_\nu(f) = \inf_{g \in \mathrm{BV}_\nu(S)} \|f - g\|_{L^2(S)} \leq \|f - P_\nu(f)\|_{L^2(S)} &\leq c^{-k} \|(I - M_{1/\nu})^k f\|_{L^2(S)} \\ &= c^{-k} \|\Delta^k_{1/\nu} f\|_{L^2(S)} \leq c_k\, \Omega_k \left(f, \frac{1}{\nu}\right)_2,\end{aligned}$$

proving (2) and hence the theorem is proved. ◻

Theorem 3.3. *Let $r \in \mathbb{N}$ and $\nu > 0$. Assume that $f, \Delta_S f, \Delta^2 f, \ldots, \Delta^r f$ are in $L^2(S)$. Then*

$$E_\nu(f) \leq c'_k\ \nu^{-2r} \Omega_k \left(\Delta^r_S f, \frac{1}{\nu}\right)_2, \quad k \in \mathbb{N}, \tag{3}$$

where c'_k is a constant.

Proof. Let $r \in \mathbb{N}$ and $t > 0$. Suppose that $f, \Delta_S f, \Delta_S^2 f, \dots, \Delta_S^r f$ are in $L^2(S)$. Then Lemma 2.1 and Plancherel formula give that

$$\begin{aligned}
\|(I - M_t)f\|_{L^2(S)}^2 &= \int_0^\infty \int_N |\mathcal{F}((I - M_t)f)(\lambda, n)|^2 |c(\lambda)|^{-2}\, d\lambda\, dn \\
&= \int_0^\infty \int_N |1 - \phi_\lambda(a_t)|^2 |\mathcal{F}f(\lambda, n)|^2 |c(\lambda)|^{-2}\, d\lambda\, dn \\
&\le \frac{t^4}{4} \int_0^\infty \int_N (\lambda^2 + \frac{Q^2}{4})^2 |\mathcal{F}f(\lambda, n)|^2 |c(\lambda)|^{-2}\, d\lambda\, dn \\
&= \frac{t^4}{4} \int_0^\infty \int_N |\mathcal{F}(\Delta_S f)(\lambda, n)|^2 |c(\lambda)|^{-2}\, d\lambda\, dn = \frac{t^4}{4} \|\Delta_S f\|_{L^2(S)}^2.
\end{aligned}$$

Therefore,

$$\|(I - M_t)f\|_{L^2(S)} \le \frac{t^2}{2} \|\Delta_S f\|_{L^2(S)}. \tag{4}$$

By proceeding similar to the proof of Theorem 3.2 we get

$$\|f - P_\nu(f)\|_{L^2(S)} \le c^{-(k+r)} \|(I - M_{1/\nu})^{k+r} f\|_{L^2(S)}. \tag{5}$$

By applying inequality (4) on the right hand side of (5) r-times we obtain that

$$\begin{aligned}
\|f - P_\nu(f)\|_{L^2(S)} &\le c^{-(k+r)} 2^{-r} \nu^{-2r} \|(I - M_{1/\nu})^k \Delta_S^r f\|_{L^2(S)} \\
&= c_k'\ \nu^{-2r} \Omega_k \left(\Delta_S^r f, \frac{1}{\nu} \right)_2,
\end{aligned}$$

where $c_k' = c^{-(k+r)} 2^{-r}$. Now, the theorem follows from the definition of $E_\nu(f)$ by noting that

$$E_\nu(f) = \inf_{g \in \mathrm{BV}_\nu(S)} \|f - g\|_{L^2(S)} \le \|f - P_\nu(f)\|_{L^2(S)} \le c_k'\ \nu^{-2r} \Omega_k \left(\Delta_S^r f, \frac{1}{\nu} \right)_2,$$

completing the proof. □

3.2. **Nikolskii-Stechkin inequality.** In this subsection, we will prove Nikolskii-Stechkin inequality [25] for Damek-Ricci spaces.

Theorem 3.4. *For any $f \in L^2(S)$ and $\nu > 0$ we have*

$$\|\Delta_S^k (P_\nu(f))\|_{L^2(S)} \le c_3\, \nu^{2k} \|\Delta_{1/\nu}^k f\|_{L^2(S)}, \quad k \in \mathbb{N}. \tag{6}$$

Proof. First note that

$$\mathcal{F}(\Delta_S^k P_\nu(f))(\lambda, n) = (-1)^k \left(\lambda^2 + \frac{Q^2}{4}\right)^k \mathcal{F}(P_\nu(f))(\lambda, n).$$

Using Plancherel formula we have

$$\begin{aligned}
\|\Delta_S^k(P_\nu(f))\|_{L^2(S)}^2 &= \int_0^\infty \int_N |\mathcal{F}(\Delta_S^k P_\nu(f))(\lambda, n)|^2 |c(\lambda)|^{-2}\, d\lambda\, dn \\
&= \int_{|\lambda|\leq \nu} \int_N \left(\lambda^2 + Q^2/4\right)^{2k} |\mathcal{F}f(\lambda, n)|^2\, |c(\lambda)|^{-2}\, d\lambda\, dn \\
&= \int_0^\infty \int_N \frac{\left(\lambda^2 + Q^2/4\right)^{2k} \chi_\nu(\lambda)}{|1-\phi_\lambda(1/\nu)|^{2k}} |1-\phi_\lambda(1/\nu)|^{2k} |\mathcal{F}f(\lambda, n)|^2\, |c(\lambda)|^{-2}\, d\lambda\, dn.
\end{aligned}$$

Now note that by Lemma 2.3 we have

$$\begin{aligned}
\sup_{\lambda\in\mathbb{R}} \frac{\left(\lambda^2 + Q^2/4\right)^{2k} \chi_\nu(\lambda)}{|1-\phi_\lambda(1/\nu)|^{2k}} &= \nu^{4k} \sup_{|\lambda|\leq\nu} \frac{\left((\lambda^2 + Q^2/4)/\nu^2\right)^{2k}}{|1-\phi_\lambda(1/\nu)|^{2k}} \\
&\leq \frac{\nu^{4k}}{c_1} \sup_{|\lambda|\leq\nu} \frac{\left((\lambda^2 + Q^2/4)/\nu^2\right)^{2k}}{|1-j_\alpha(\lambda/\nu)|^{2k}} \\
&= \frac{\nu^{4k}}{c_1} \sup_{|t|\leq 1} \frac{\left(t^2 + Q^2/4\nu^2\right)^{2k}}{|1-j_\alpha(t)|^{2k}} = \frac{C'}{c_1}\nu^{4k},
\end{aligned}$$

where $C' = \sup_{|t|\leq 1} \frac{\left(t^2+Q^2/4\nu^2\right)^{2k}}{|1-j_\alpha(t)|^{2k}}$.

Therefore, we get

$$\begin{aligned}
\|\Delta_S^k(P_\nu(f))\|_{L^2(S)}^2 &\leq \frac{C'}{c_1}\nu^{4k} \int_0^\infty \int_N |1-\phi_\lambda(1/\nu)|^{2k} |\mathcal{F}f(\lambda, n)|^2\, |c(\lambda)|^{-2}\, d\lambda\, dn \\
&= \frac{C'}{c_1}\nu^{4k} \int_0^\infty \int_N |\mathcal{F}(\Delta_{1/\nu}^k f)(\lambda, n)|^2\, |c(\lambda)|^{-2}\, d\lambda\, dn \\
&= \frac{C'}{c_1}\nu^{4k} \|\Delta_{1/\nu}^k f\|_{L^2(S)}^2.
\end{aligned}$$

Hence, $\|\Delta_S^k(P_\nu(f))\|_{L^2(S)} \leq c_3\, \nu^{2k}\, \|\Delta_{1/\nu}^k f\|_{L^2(S)}$. □

As noted in Lemma 3.1 that $P_\nu(f) = f$ for any $f \in BS_\nu(S)$, the following corollary is immediate.

Corollary 3.5. *For $\nu > 0$, $k \in \mathbb{N}$ and $f \in BV_\nu(S)$ we have the following inequality:*

$$\|\Delta_S^k f\|_{L^2(S)} \leq c_3\, \nu^{2k}\, \|\Delta_{1/\nu}^k f\|_{L^2(S)}.$$

The following corollary follows from the definition of modulus of smoothness.

Corollary 3.6. *For $\nu > 0$, $k \in \mathbb{N}$ and $f \in L^2(S)$ we have the following inequality:*

$$\|\Delta_S^k f\|_{L^2(S)} \leq c_3 \, \nu^{2k} \, \Omega_k \left(f, \frac{1}{\nu} \right)_2 .$$

3.3. **Equivalence of the K-functional and modulus of smoothness.** Our main objective will be proved here. We will prove in the following theorem that the K-functional for the pair $(L^2(S), W_2^m(S))$ and modulus of smoothness generated by spherical mean operators are equivalent. The Peetre the K-functional $K(f, \delta, L^2(S), W_2^m(S))$ for the pair $(L^2(S), W_2^m(S))$ is defined by

$$K_m(f, \delta) := \inf\{\|f - g\|_{L^2(S)} + \delta \|\Delta_S^m g\|_{L^2(S)} : \ f \in L^2(S) \ g \in W_2^m(S)\}.$$

The next theorem presents the equivalence of the K-functional $K_m(f, \delta^{2m})$ and the modulus of smoothness $\Omega_m(f, \delta)_2$ for $f \in L^2(S)$ and $\delta > 0$.

Theorem 3.7. *For $f \in L^2(S)$ and $\delta > 0$ we have*

$$\Omega_m(f, \delta)_2 \asymp K_m(f, \delta^{2m}). \tag{7}$$

In other words, there exist $c_1 > 0$, $c_2 > 0$ such that for all $f \in L^2(S)$ and $\delta > 0$ we have

$$c_1 \, \Omega_m(f, \delta)_2 \leq K_m(f, \delta^{2m}) \leq c_2 \, \Omega_m(f, \delta)_2.$$

Proof. Take $g \in W_2^m(S)$. Now by using the properties of modulus of continuity $\Omega_m(f, \delta)_2$ we get

$$\begin{aligned} \Omega_m(f, \delta)_2 &\leq \Omega_m(f - g, \delta)_2 + \Omega_m(g, \delta)_2 \\ &\leq (\phi_0(a_t) + 1)^m \|f - g\|_{L^2(S)} + \delta^{2m} \|\Delta_S^m g\|_{L^2(S)} \\ &\leq \tilde{c}(\|f - g\|_{L^2(S)} + \delta^{2m} \|\Delta_S^m g\|_{L^2(S)}), \end{aligned}$$

where $\tilde{c} = (\phi_0(a_t) + 1)^m$. By taking the infimum over all $g \in W_2^m(S)$, we obtain

$$\Omega_m(f, \delta)_2 \lesssim K_m(f, \delta^{2m}).$$

Now, to prove the other side we take $g = P_\nu(f)$ for $\nu > 0$, then, from the definition of $K_m(f, \delta^{2m})$, it follows that

$$K_m(f, \delta^{2m}) \leq \|f - P_\nu(f)\|_{L^2(S)} + \delta^{2m} \|\Delta_S^m (P_\nu(f))\|_{L^2(S)}. \tag{8}$$

Now, from Lemma 3.1 (iii), (2) and Corollary 3.6 we get that

$$K_m(f, \delta^{2m}) \leq 4E_v(f) + c_3\delta^{2m}\nu^{2m}\Omega_m\left(f, \frac{1}{\nu}\right)_2$$
$$\leq 4c_2\Omega_m\left(f, \frac{1}{\nu}\right)_2 + c_3(\delta\nu)^{2m}\Omega_m\left(f, \frac{1}{\nu}\right)_2 \leq c_4(1+(\delta\nu)^{2m})\Omega_m\left(f, \frac{1}{\nu}\right)_2.$$

By taking $\nu = \frac{1}{\delta}$ we get

$$K_m(f, \delta^{2m}) \lesssim \Omega_m\left(f, \delta\right)_2$$

proving (7). ☐

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