Communication

On the proof of recursive Vogler algorithm for multiple knife-edge diffraction

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Abstract—We consider the problem of multiple knife-edge diffraction estimation which is a fundamental task in many wireless communication applications. So far, one of the most accurate methods for this problem is the Vogler one whose recursive implementation is efficient to reduce the high computational complexity of the direct one. However, in the original report, Vogler only presented the final result of the recursive algorithm without a rigorous mathematical proof, thus making the method difficult to understand and implement in practice. To tackle this shortcoming, we first analyze the mathematical structure of the problem and then present a formal proof of the result. To gain intuition of the proof and the key steps, we provide a simplified study case of four knife-edges. The insight from our proposed analysis and proof can be used to obtain a comprehensive interpretation, initiate a practical implementation and develop new efficient algorithms with similar structure.

Index Terms—Multiple knife-edge diffraction, Vogler method, recursive algorithm, RF transmission

I. INTRODUCTION

Estimation of diffraction attenuation plays an important role in many applications of wireless communication. Some classical ones can be taken into account such as evaluating propagation loss over irregular terrains [1]–[3] or aeronautical mobile and ground station interactions [4]. More recently, diffraction estimation is also used in several emerged applications, for example, channel modeling at cmWave and mmWave bands for 5G [5] or influence of vehicles as obstacles in Vehicular Ad Hoc Networks (VANET) [6]. Generally, to obtain better accuracy, it is essential to use multiple knife-edge approximation instead of single one which is simple but inadequate.

To date, many approaches are proposed in the literature for calculating the multiple knife-edge diffraction loss. Some well-known representatives includes: (i) 'analytical' methods, such as the Vogler method and the variants of the uniform theory of diffraction (UTD) [7]–[13], to name a few; (ii) graphical-based methods, such as the Bullington method [14], the Deygout method [15], the Causebrook [16] and Giovanelli methods [17]. The first group of methods is characterized

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by accurate estimation but high computational complexity whereas the second group is represented by low computation complexity but inadequate accuracy. The Vogler method was developed based on a generalized residual series formulation [18] for electromagnetic waves over a sequence of smooth cylindrical obstacles. The obtained result of such derivation has the form of a multiple integral whose computation can be represented efficiently in terms of series representation. The Vogler method is the first of its kind allowing to predict a precise diffraction attenuation value up to ten knife-edges.

In terms of estimation accuracy, the Vogler method stands out from other methods [19], presenting the ultimate solution supported by both simulations and real-life measured experiment results [3], [19], [20]. For example, the Vogler method is also used as the ground truth in a comparative studies for fifty scenarios reported in [20]. Comparison results of multiple diffraction models for a digital broadcasting coverage prediction with different frequencies can be found in [3], [19].

However, the naive implementation of the Vogler algorithm results in high computational complexity, making it difficult to use in practice. To overcome this computational challenge, the recursive version was proposed by Vogler [21] to reduce the computational time. Unfortunately, this version came without a rigorous mathematical proof, but only the final result [21], [22], leading to difficult understanding and implementation. Thus, we aim to provide insight of the mathematical structure of this problem to gain a comprehensive interpretation. Our contributions are threefold. First, we analyze why the Vogler algorithm can be represented in a recursive manner. Second, we present a formal proof by induction. Finally, to gain intuition of the proof, we provide a case study of four knife-edges.
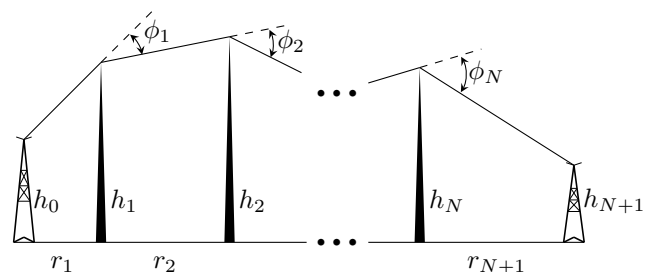
II. THE VOGLER METHOD

Fig. 1: Geometry of multiple knife edges.

In this section the background on the Vogler method will be presented. We consider the geometry of N knife-edge

diffraction in Fig. 1 where h_0 and h_{N+1} denote the transmitter and receiver heights respectively, $\{h_n\}_{n=1}^{N}$ are the knife-edge heights to a reference surface, $\{\phi_n\}_{n=1}^{N}$ are diffraction angles, and $\{r_n\}_{n=1}^{N+1}$ are $N+1$ separation distances between knife-edges. Then the diffraction attenuation, A_N, for N knife-edges is given by [22]

$$A_N = \frac{1}{2^N} C_N \exp(\sigma_N) \left(\frac{2}{\sqrt{\pi}}\right)^N \underbrace{\int_{\beta_1}^{\infty} \cdots \int_{\beta_N}^{\infty}}_{N-\text{fold}}$$
$$\exp(2f) \exp\left(-\sum_{n=1}^{N} u_n^2\right) du_1 \cdots du_N \quad (1)$$

where

$$f = \begin{cases} 0 & \text{for } N = 1 \\ \sum_{j=1}^{N-1} \gamma_j & \text{for } N \geq 2 \end{cases} \quad (2)$$

$$\gamma_j = \alpha_j (u_j - \beta_j)(u_{j+1} - \beta_{j+1}) \quad (3)$$

$$\alpha_j = \left[\frac{r_j r_{j+1}}{(r_j + r_{j+1})(r_{j+1} + r_{j+2})}\right]^{1/2}, \text{where } 1 \leq j \leq N-1 \quad (4)$$

$$\beta_j = \phi_j \left[\frac{\mathrm{i} k r_j r_{j+1}}{2(r_j + r_{j+1})}\right]^{1/2}, \text{where } 1 \leq j \leq N \quad (5)$$

$$\sigma_N = \sum_{n=1}^{N} \beta_n^2 \quad (6)$$

$$C_N = \begin{cases} 1 & \text{for } N = 1 \\ \left[\frac{\left(\sum_{n=1}^{N+1} r_n\right) \prod_{n=1}^{N} r_n}{\prod_{n=1}^{N}(r_n + r_{n+1})}\right]^{1/2} & \text{for } N \geq 2 \end{cases} \quad (7)$$

with $\mathrm{i} = \sqrt{-1}$ being an imaginary number and the wave number $\mathrm{k} = 2\pi/\lambda$.

To calculate (1), the main idea in [22] is that instead of computing the N-fold integral, we convert such task into computing N single integrals. To this end, Vogler proposed to express $\exp(2f)$ in terms of power series as:

$$\exp(2f) = \sum_{m=0}^{\infty} \frac{(2f)^m}{m!} \quad (8)$$

Then by exploiting the fact that

$$\frac{2}{\sqrt{\pi}} \int_{\beta}^{\infty} (u-\beta)^m \exp(-u^2) du = m! I(m, \beta) \quad (9)$$

where $m!$ refers to the factorial of m and $I(m, \beta)$ defines the repeated integrals of the complementary error function [23], we reach a closed-form solution for A_N. The detail calculations are omitted here and can be found in [21]. The results are summarized in the following theorem

Theorem 1. *For $N = 1$, equation (1) is reduced to the standard form of a single knife-edge diffraction:*

$$A_1 = \frac{1}{\sqrt{\pi}} \exp(\beta_1^2) \int_{\beta_1}^{\infty} \exp(-u^2) du. \quad (10)$$

For $N = 2$, we reach a closed-form solution for A_2 as follows:

$$A_2 = \frac{1}{2^2} C_2 \exp(\sigma_2) \sum_{m=0}^{\infty} I_m \quad (11)$$

where

$$I_m = 2^m \alpha_1^m m! I(m, \beta_1) I(m, \beta_2). \quad (12)$$

For $N > 2$, a general solution for A_N is given by:

$$A_N = \frac{1}{2^N} C_N \exp(\sigma_N) \sum_{m=0}^{\infty} I_m \quad (13)$$

where

$$I_m =$$
$$2^m \sum_{m_1=0}^{m} \cdots \sum_{m_{N-2}=0}^{m_{N-3}} \prod_{i=1}^{N} \frac{(m_{i-1} - m_{i+1})!}{(m_i - m_{i+1})!} \alpha_i^{m_{i-1} - m_i} I(n_i, \beta_i) \quad (14)$$

with, by using notation $m_0 = m$,

$$n_i = \begin{cases} m_0 - m_1 & i = 1 \\ m_{i-2} - m_i & 2 \leq i \leq N-1 \\ m_{N-2} - m_{N-1} & i = N \end{cases} \quad (15)$$

III. ON THE RECURSIVE SOLUTION OF THE VOGLER METHOD

Many existing studies in the literature highlight the high complexity of the Vogler method (see [3], [20], [24] and references therein). However, the technical report by Vogler [21] indeed includes a recursive version which provides a faster solution than the direct implementation. Unfortunately, there is a lack of a rigorous proof, making such version difficult for understanding and implementation. To overcome this problem, we provide here a mathematical proof of this recursive algorithm. We first show that the recursive structure of the Vogler algorithm stems from the structure of expansion of f^m. Then, we propose a general proof for the recursive algorithm (i.e., for $N > 2$).

To show that the Vogler solution can provide a recursive structure, we first present the following lemma:

Lemma 1. *Assume that f has the following form as by (2) when $N > 1$*

$$f = \sum_{i=1}^{N-1} \gamma_i, \quad (16)$$

for $N \geq 3$, we can represent f^m as given by (17) and (18) at the top of the next page where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \quad (19)$$

denotes the binomial coefficient.

Proof. See Appendix A. $\square$

The non-recursive version of the Vogler algorithm is obtained by substituting (18) and (8) into (1) and then computing the integral to achieve (14). The main drawback of this version is that, considering two successive terms, I_m and I_{m+1}, the computation of I_{m+1} requires recalculating completely the sub-calculations in I_m, thus omitting the recursive structure of (17). By the recursive structure, we mean that f^m can be represented in an abstract form as follows

$$f^m(\gamma_1, \cdots, \gamma_{N-1}) = \psi(m, \gamma_1, \psi(m_1, \gamma_2, \psi(\cdots, \\ \psi(m_{N-3}, \gamma_{N-2}, \gamma_{N-1}) \cdots))) \quad (20)$$

$$f^m = \sum_{m_1=0}^{m} \binom{m}{m_1} \gamma_1^{m-m_1} \underbrace{\left(\sum_{m_2=0}^{m_1} \binom{m_1}{m_2} \gamma_2^{m_1-m_2} \left(\cdots \underbrace{\left(\underbrace{\sum_{m_{N-2}=0}^{m_{N-3}} \binom{m_{N-3}}{m_{N-2}} \gamma_{N-2}^{m_{N-3}-m_{N-2}} \gamma_{N-1}^{m_{N-2}}}_{1^{\text{st}}\text{term}} \right)}_{(N-3)^{\text{nd}}\text{term}} \right)\right)}_{(N-2)^{\text{nd}}\text{term}} \tag{17}$$

or

$$f^m = m! \sum_{m_1=0}^{m} \sum_{m_2=0}^{m_1} \cdots \sum_{m_{N-2}=0}^{m_{N-3}} \prod_{i=1}^{N-1} \frac{\gamma_i^{m_{i-1}-m_i}}{(m_{i-1}-m_i)!} \tag{18}$$

where

$$0 \le m_{N-2} \le \cdots \le m_1 \le m, \tag{21}$$

$$\psi(m,a,b) = \sum_{i=0}^{m} \binom{m}{i} b^{m-i} a^i, \tag{22}$$

Theorem 2. *(Recursive computation of I_m [22]): Let*

$$C(N-1, m_{N-2}, m_{N-3})$$
$$= (m_{N-3})! \alpha_{N-1}^{m_{N-2}} I(m_{N-3}, \beta_{N-1}) I(m_{N-2}, \beta_N). \tag{23}$$

Then, given the following notation

$$i = m_{N-L}, j = m_{N-L-1}, k = m_{N-L-2}$$
$$2 \le L \le N-2, N \ge 4$$

and the recursive relationship

$$C(N-L,j,k)$$
$$= \sum_{i=0}^{j} \frac{(k-i)!}{(j-i)!} \alpha_{N-L}^{j-i} I(k-i, \beta_{N-L}) C(N-L+1, i, j), \tag{24}$$

the recursive computation of I_m is given by

$$I_m = 2^m \sum_{m_1=0}^{m_0} \alpha_1^{m-m_1} I(m-m_1, \beta_1) C(2, m_1, m). \tag{25}$$

Proof. See Appendix B. □

IV. CASE STUDY OF FOUR KNIFE-EDGES ($N = 4$)

To simplify the discussion and illustrate the main idea, we present here a case study of $N = 4$. We note that by substituting (8) into (1), A_N can be rewritten as

$$A_N = \frac{1}{2^N} C_N \exp(\sigma_N) \sum_{m=0}^{\infty} I_m \tag{26}$$

where

$$I_m = \left(\frac{2^m}{m!}\right) \left(\frac{2}{\sqrt{\pi}}\right)^N$$
$$\times \int_{\beta_1}^{\infty} \cdots \int_{\beta_N}^{\infty} f^m \exp\left(-\sum_{n=1}^{N} u_n^2\right) du_1 \cdots du_N, \tag{27}$$

Since we aim to obtain a recursive formula for I_m, by substituting (17) into (27), I_m for $N = 4$ is given by

$$I_m =$$
$$\left(\frac{2^m}{m!}\right) \left(\frac{2}{\sqrt{\pi}}\right)^4$$
$$\times \int_{\beta_1}^{\infty} \int_{\beta_2}^{\infty} \int_{\beta_3}^{\infty} \int_{\beta_4}^{\infty} \left(\sum_{m_1=0}^{m} \binom{m}{m_1} \gamma_1^{m-m_1} \sum_{m_2=0}^{m_1} \binom{m_1}{m_2} \gamma_2^{m_1-m_2} \gamma_3^{m_2} \right)$$
$$\times \exp\left(-\sum_{n=1}^{4} u_n^2\right) du_1 \cdots du_4. \tag{28}$$

Using (3) and the definition of binomial coefficients (19), we can expand I_m as follows

$$I_m =$$
$$\left(\frac{2^m}{m!}\right) \left(\frac{2}{\sqrt{\pi}}\right)^4 \int_{\beta_1}^{\infty} \int_{\beta_2}^{\infty} \int_{\beta_3}^{\infty} \int_{\beta_4}^{\infty}$$
$$\sum_{m_1=0}^{m} \frac{m!}{(m-m_1)! m_1!} \alpha_1^{m-m_1} (u_1-\beta_1)^{m-m_1} (u_2-\beta_2)^{m-m_1}$$
$$\times \left(\left(\sum_{m_2=0}^{m_1} \frac{m_1!}{(m_1-m_2)! m_2!} \alpha_2^{m_1-m_2} (u_2-\beta_2)^{m_1-m_2} \right.\right.$$
$$\left.\left. \alpha_3^{m_2} (u_3-\beta_3)^{m_1} (u_4-\beta_4)^{m_2} \right) \right)$$
$$\times \exp\left(-\sum_{n=1}^{4} u_n^2\right) du_1 \cdots du_4. \tag{29}$$

Using (23), we can write

$$C(3, m_2, m_1) = m_1! \alpha_3^{m_2} I(m_1, \beta_3) I(m_2, \beta_4) \tag{30}$$

which represents the first coefficient needed to compute the recursive form. Then, we can rewrite I_m as

$$I_m = \left(\frac{2^m}{m!}\right) \left(\frac{2}{\sqrt{\pi}}\right) \int_{\beta_1}^{\infty} \sum_{m_1=0}^{m} \frac{m!}{(m-m_1)!} \alpha_1^{m-m_1}$$
$$\times (u_1-\beta_1)^{m-m_1} C(2, m_1, m) \exp(-u_1^2) du_1 \tag{31}$$

where

$$C(2, m_1, m) =$$
$$\sum_{m_2=0}^{m_1} \frac{(m-m_2)!}{(m_1-m_2)!} \alpha_2^{m_1-m_2} I(m-m_2, \beta_2) C(3, m_2, m_1). \tag{32}$$

Finally, by computing the integral of (31), we achieve

$$I_m = \left(\frac{2^m}{m!}\right) \sum_{m_1=0}^{m} m! \alpha_1^{m-m_1}$$
$$\times \left[\frac{2}{\sqrt{\pi}} \int_{\beta_1}^{\infty} \frac{(u_1-\beta_1)^{m-m_1}}{(m-m_1)!} \exp\left(-u_1^2\right) du_1 \right] C(2, m_1, m)$$
$$= 2^m \sum_{m_1=0}^{m} \alpha_1^{m-m_1} I(m-m_1, \beta_1) C(2, m_1, m) \tag{33}$$

which is the exact result of the Theorem 2 for $N = 4$.

V. CONCLUSION

In this communication, we have introduced a rigorous proof for the recursive Vogler algorithm after analyzing its mathematical structure. A case study of four knife-edges is also presented in a 'closed form' to bring intuition of the key steps. Our study can provide a deep overview of the problem and clarify the algorithm derivation. Moreover, we could use the insight from the proof to develop an efficient recursive algorithm for the multiple bridged knife-edge diffraction problem [25]. The recursive Vogler algorithm can also be used to generate faster training dataset for function approximation-based methods [24], [26] proposed to bring a better trade-off in terms of the accuracy and the computational complexity.

ACKNOWLEDGMENT

APPENDIX

A. Proof of Lemma 1

We obtain the first form (17) by applying the binomial theorem $N-2$ times to f^m

$$\left(\sum_{i=1}^{N-1} \gamma_i\right)^m = \sum_{m_1=0}^{m} \binom{m}{m_1} \gamma_1^{m-m_1} \left(\sum_{i=2}^{N-1} \gamma_i\right)^{m_1}$$
$$\left(\sum_{i=2}^{N-1} \gamma_i\right)^{m_1} = \sum_{m_2=0}^{m_1} \binom{m_1}{m_2} \gamma_2^{m_1-m_2} \left(\sum_{i=3}^{N-1} \gamma_i\right)^{m_2}$$
$$\vdots$$
$$\left(\gamma_{N-3} + (\gamma_{N-2} + \gamma_{N-1})\right)^{m_{N-4}} =$$
$$\sum_{m_{N-3}=0}^{m_{N-4}} \binom{m_{N-4}}{m_{N-3}} \gamma_{N-3}^{m_{N-4}-m_{N-3}} (\gamma_{N-2} + \gamma_{N-1})^{m_{N-3}}$$
$$(\gamma_{N-1} + \gamma_{N-2})^{m_{N-3}} =$$
$$\sum_{m_{N-2}=0}^{m_{N-3}} \binom{m_{N-3}}{m_{N-2}} \gamma_{N-2}^{m_{N-3}-m_{N-2}} \gamma_{N-1}^{m_{N-2}}$$

To achieve the second form (18), we observe that, by re-arranging the summation and the coefficients, (17) can be rewritten as

$$f^m = \sum_{m_1=0}^{m} \sum_{m_2=0}^{m_1} \cdots \sum_{m_{N-2}=0}^{m_{N-3}} \prod_{i=1}^{N-1} \binom{m_{i-1}}{m_i} \gamma_i^{m_{i-1}-m_i} \tag{34}$$

where we used the notation $m_i = 0, i \geq N-1$. Note that,

$$\prod_{i=1}^{N-1} \binom{m_{i-1}}{m_i} = \prod_{i=1}^{N-1} \frac{m_{i-1}!}{m_i!} \prod_{i=1}^{N-1} \frac{1}{(m_{i-1}-m_i)!} \tag{35}$$

and, using again the notation $m_0 = m$,

$$\prod_{i=1}^{N-1} \frac{m_{i-1}!}{m_i!} = \frac{m_0!}{m_1!} \frac{m_1!}{m_2!} \cdots \frac{m_{N-2}!}{m_{N-1}!} = \frac{m_0!}{m_{N-1}!} = m! \tag{36}$$

then, we obtain the desired result

$$f^m = m! \sum_{m_1=0}^{m} \sum_{m_2=0}^{m_1} \cdots \sum_{m_{N-2}=0}^{m_{N-3}} \prod_{i=1}^{N-1} \frac{\gamma_i^{m_{i-1}-m_i}}{(m_{i-1}-m_i)!}.$$

This concludes the proof.

B. Proof of Theory 2

First, we denote the following set of integrals

$$\mathcal{I}_1 = \left(\frac{2}{\sqrt{\pi}}\right)^2 \int_{\beta_{N-1}}^{\infty} \int_{\beta_N}^{\infty} \sum_{m_{N-2}=0}^{m_{N-3}} \binom{m_{N-3}}{m_{N-2}} \gamma_{N-2}^{m_{N-3}-m_{N-2}}$$
$$\times \gamma_{N-1}^{m_{N-2}} \exp\left(-\left(u_{N-2}^2 + u_{N-1}^2\right)\right) du_{N-2} du_{N-1}, \tag{37}$$

$$\mathcal{I}_2 = \left(\frac{2}{\sqrt{\pi}}\right)^3 \int_{\beta_{N-2}}^{\infty} \int_{\beta_{N-1}}^{\infty} \int_{\beta_N}^{\infty} \sum_{m_{N-3}=0}^{m_{N-4}} \binom{m_{N-4}}{m_{N-3}} \gamma_{N-3}^{m_{N-4}-m_{N-3}}$$
$$\times (\gamma_{N-2} + \gamma_{N-1})^{m_{N-3}} \exp\left(-\sum_{i=N-2}^{N} u_i^2\right) du_{N-2} \cdots du_N, \tag{38}$$

$$\vdots$$

$$\mathcal{I}_n = \left(\frac{2}{\sqrt{\pi}}\right)^{n+1} \int_{\beta_{N-n}}^{\infty} \cdots \int_{\beta_N}^{\infty} \sum_{m_{N-n-1}=0}^{m_{N-n-2}} \binom{m_{N-n-2}}{m_{N-n-1}}$$
$$\times \gamma_{N-n-1}^{m_{N-n-2}-m_{N-n-1}} \left(\sum_{i=N-n}^{N-1} \gamma_i\right)^{N-n-1}$$
$$\times \exp\left(-\sum_{i=N-n}^{N} u_i^2\right) du_{N-n} \cdots du_N,$$

$$\vdots$$

$$\mathcal{I}_{N-2} = \left(\frac{2}{\sqrt{\pi}}\right)^{N-1} \int_{\beta_2}^{\infty} \cdots \int_{\beta_N}^{\infty} \sum_{m_1=0}^{m} \binom{m}{m_1} \gamma_1^{m-m_1}$$
$$\times \left(\sum_{i=2}^{N-1} \gamma_i\right)^{m_1} \exp\left(-\sum_{i=2}^{N} u_i^2\right) du_2 \cdots du_N. \tag{39}$$

We notice that, the following relationships hold:

$$
\mathcal{I}_n = \frac{2}{\sqrt{\pi}} \int_{\beta_{N-n}}^{\infty} \sum_{m_{N-n-1}=0}^{m_{N-n-2}} \binom{m_{N-n-2}}{m_{N-n-1}}
$$
$$
\times \gamma_{m_{N-n-1}}^{m_{N-n-2}-m_{N-n-1}} \mathcal{I}_{n-1} \exp\left(u_{N-n}^2\right) du_{N-n} \quad (40)
$$

for $2 \leq n \leq N-2$, and

$$
I_m = \left(\frac{2^m}{m!}\right) \frac{2}{\sqrt{\pi}} \int_{\beta_1}^{\infty} \mathcal{I}_{N-2} \exp\left(-u_1^2\right) du_1, \quad (41)
$$

stemming from definitions of $\mathcal{I}_n$ and I_m.

Our main idea is to find a general formula for $\mathcal{I}_n$. Then it is straightforward to obtain I_m through (41). To this end, we will provide a general formula for $\mathcal{I}_n$ and prove it by induction. In particular, our proof can be divided into three steps:

- Step 1: We compute $\mathcal{I}_1$
- Step 2: We provide a general formula of $\mathcal{I}_n$ for $2 \leq n \leq N-2$. We show that this formula is true by induction
- Step 3: We show that the results of Theorem 2 are consequences of Step 2, thus concluding the proof.

Now we provide the technical details.

Step 1: *We compute $\mathcal{I}_1$*

First, from (3) and (19), we note that

$$
\gamma_{N-2}^{m_{N-3}-m_{N-2}} \gamma_{N-1}^{m_{N-2}} =
$$
$$
\times \alpha_{N-2}^{m_{N-3}-m_{N-2}} (u_{N-2}-\beta_{N-2})^{m_{N-3}-m_{N-2}} \alpha_{N-1}^{m_{N-2}}
$$
$$
\times (u_{N-1}-\beta_{N-1})^{m_{N-3}} (u_N-\beta_N)^{m_{N-2}} \quad (42)
$$

Then by substituting (42) into (37), we obtain

$$
\mathcal{I}_1 = \sum_{m_{N-2}=0}^{m_{N-3}} \frac{(m_{N-3})!}{(m_{N-3}-m_{N-2})!} \alpha_{N-2}^{m_{N-3}-m_{N-2}}
$$
$$
\times (u_{N-2}-\beta_{N-2})^{m_{N-3}-m_{N-2}} C\left(N-1, m_{N-2}, m_{N-3}\right) \quad (43)
$$

where

$$
C\left(N-1, m_{N-2}, m_{N-3}\right)
$$
$$
= (m_{N-3})! \alpha_{N-1}^{m_{N-2}} I\left(m_{N-3}, \beta_{N-1}\right) I\left(m_{N-2}, \beta_N\right). \quad (44)
$$

Step 2: *We provide formula of $\mathcal{I}_n$ for $2 \leq n \leq N-2$ and show that this formula is true by induction.*

To this end, we introduce the following lemma

Lemma 2. *For $2 \leq n \leq N-2$, $\mathcal{I}_n$ is given by*

$$
\mathcal{I}_n = \sum_{m_{N-n-1}=0}^{m_{N-n-2}} \frac{(m_{N-n-2})!}{(m_{N-n-2}-m_{N-n-1})!} \alpha_{N-n-1}^{m_{N-n-2}-m_{N-n-1}}
$$
$$
\times (u_{N-n-1}-\beta_{N-n-1})^{m_{N-n-2}-m_{N-n-1}} \quad (45)
$$
$$
\times C\left(N-n, m_{N-n-1}, m_{N-n-2}\right) \quad (46)
$$

where

$$
C\left(N-n, m_{N-n-1}, m_{N-n-2}\right) =
$$
$$
\sum_{m_{N-n}=0}^{m_{N-n-1}} \frac{(m_{N-n-2}-m_{N-n})!}{(m_{N-n-1}-m_{N-n})!} \alpha_{N-n}^{m_{N-n-1}-m_{N-n}} \times
$$
$$
I\left(m_{N-n-2}-m_{N-n}, \beta_{N-n}\right) C\left(N-n+1, m_{N-n}, m_{N-n-1}\right). \quad (47)
$$

Proof. We show that, this formula is true for $n=2$. Then we assume that (46) holds. Subsequently we prove that it is also true for I_{n+1}. Due to (40), we can rewrite (38) as:

$$
\mathcal{I}_2 = \frac{2}{\sqrt{\pi}} \int_{\beta_{N-2}}^{\infty} \sum_{m_{N-3}=0}^{m_{N-4}} \binom{m_{N-4}}{m_{N-3}} \gamma_{N-3}^{m_{N-4}-m_{N-3}}
$$
$$
\times \mathcal{I}_1 \exp\left(-u_{N-2}^2\right) du_{N-2}. \quad (48)
$$

By substituting (43) into (48) and compute the integral to the variable u_{N-2}, we have

$$
\mathcal{I}_2 = \sum_{m_{N-3}=0}^{m_{N-4}} \frac{(m_{N-4})!}{(m_{N-4}-m_{N-3})!} \alpha_{N-3}^{m_{N-4}-m_{N-3}}
$$
$$
\times (u_{N-3}-\beta_{N-3})^{m_{N-4}-m_{N-3}} C\left(N-2, m_{N-3}, m_{N-4}\right) \quad (49)
$$

where

$$
C\left(N-2, m_{N-3}, m_{N-4}\right)
$$
$$
= \sum_{m_{N-2}=0}^{m_{N-3}} \frac{(m_{N-4}-m_{N-2})!}{(m_{N-3}-m_{N-2})!} \alpha_{N-2}^{m_{N-3}-m_{N-2}}
$$
$$
\times I\left(m_{N-4}-m_{N-2}, \beta_{N-2}\right) C\left(N-1, m_{N-2}, m_{N-3}\right). \quad (50)
$$

Now, assuming that (46) holds. We show that it is also true for I_{n+1}. In fact, because of (40), we have

$$
\mathcal{I}_{n+1} = \frac{2}{\sqrt{\pi}} \int_{\beta_{N-n-1}}^{\infty} \sum_{m_{N-n-2}=0}^{m_{N-n-3}} \binom{m_{N-n-3}}{m_{N-n-2}}
$$
$$
\times \gamma_{m_{N-n-2}}^{m_{N-n-3}-m_{N-n-2}} \mathcal{I}_n \exp\left(u_{N-n-1}\right) du_{N-n-1}. \quad (51)
$$

Substituting (46) into (51) and computing the integral to u_{N-n-1} yield

$$
\mathcal{I}_{n+1} = \sum_{m_{N-n-2}=0}^{m_{N-n-3}} \frac{(m_{N-n-3})!}{(m_{N-n-3}-m_{N-n-2})!}
$$
$$
\times \alpha_{N-n-2}^{m_{N-n-3}-m_{N-n-2}} (u_{N-n-2}-\beta_{N-n-2})^{m_{N-n-3}-m_{N-n-2}}
$$
$$
\times \sum_{m_{N-n-1}=0}^{m_{N-n-2}} \frac{(m_{N-n-3}-m_{N-n-1})!}{(m_{N-n-2}-m_{N-n-1})!} \alpha_{N-n-1}^{m_{N-n-2}-m_{N-n-1}}
$$
$$
\times C\left(N-n, m_{N-n-1}, m_{N-n-2}\right)
$$
$$
\times \left[\frac{2}{\sqrt{\pi}} \int_{\beta_{N-n-1}}^{\infty} \frac{(u_{N-n-1}-\beta_{N-n-1})^{m_{N-n-3}-m_{N-n-1}}}{(m_{N-n-3}-m_{N-n-1})!} \right.
$$
$$
\left. \times \exp\left(u_{N-n-1}\right) du_{N-n-1} \right]
$$
$$
= \sum_{m_{N-n-2}=0}^{m_{N-n-3}} \frac{(m_{N-n-3})!}{(m_{N-n-3}-m_{N-n-2})!} \alpha_{N-n-2}^{m_{N-n-3}-m_{N-n-2}}
$$
$$
\times (u_{N-n-2}-\beta_{N-n-2})^{m_{N-n-3}-m_{N-n-2}}
$$
$$
\times C\left(N-n-1, m_{N-n-2}, m_{N-n-3}\right) \quad (52)
$$

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where

$$C\left(N-n-1, m_{N-n-2}, m_{N-n-3}\right)$$

$$= \sum_{m_{N-n-1}=0}^{m_{N-n-2}} \frac{(m_{N-n-3} - m_{N-n-1})!}{(m_{N-n-2} - m_{N-n-1})!} \alpha_{N-n-1}^{m_{N-n-2}-m_{N-n-1}}$$

$$\times I\left(m_{N-n-3} - m_{N-n-1}, \beta_{N-n-1}\right)$$

$$\times C\left(N-n, m_{N-n-1}, m_{N-n-2}\right). \qquad (53)$$

Thus, the formula (46) holds by induction, concluding the proof. $\qquad \square$

Step 3: *We show that the results of Theorem 2 are consequences of Step 2, thus concluding the proof.*

By using the relationship (41) and (47) with $n = N - 2$, we obtain

$$I_m = 2^m \sum_{m_1=0}^{m} \alpha_1^{m-m_1} C\left(2, m_1, m\right)$$

$$\times \left[\frac{2}{\sqrt{\pi}} \int_{\beta_1}^{\infty} \frac{(u_1 - \beta_1)^{m-m_1}}{(m-m_1)!} \exp\left(-u_1^2\right) du \right]$$

$$= 2^m \sum_{m_1=0}^{m} \alpha_1^{m-m_1} I\left(m - m_1, \beta_1\right) C\left(2, m_1, m\right). \qquad (54)$$

Moreover, let

$$m_0 = m,$$

$$i = m_{N-L}, j = m_{N-L-1}, k = m_{N-L-2}$$

$$2 \le L \le N - 2, N \ge 4.$$

The following expression holds

$$C\left(N-L, j, k\right)$$

$$= \sum_{i=0}^{j} \frac{(k-i)!}{(j-i)!} \alpha_{N-L}^{j-i} \left(2/\sqrt{\pi}\right) I\left(k-i, \beta_{N-L}\right) C\left(N-L+1, i, j\right).$$

$$(55)$$

This is a consequence from the proof of Lemma 2 (i.e., it holds by induction) on the recursive relationship of $C\left(N-1, m_{N-2}, m_{N-3}\right), C\left(N-2, m_{N-3}, m_{N-4}\right), \cdots,$ $C\left(N-n, m_{N-n-1}, m_{N-n-2}\right)$ (see Equations (44), (50) and (47) respectively) and the identical form of terms in (17). We thus conclude the proof.

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