Approximations in L^1 with convergent Fourier series

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Abstract

For a separable finite diffuse measure space $\mathcal{M}$ and an orthonormal basis $\{\varphi_n\}$ of $L^2(\mathcal{M})$ consisting of bounded functions $\varphi_n \in L^\infty(\mathcal{M})$, we find a measurable subset $E \subset \mathcal{M}$ of arbitrarily small complement $|\mathcal{M} \setminus E| < \epsilon$, such that every measurable function $f \in L^1(\mathcal{M})$ has an approximant $g \in L^1(\mathcal{M})$ with $g = f$ on E and the Fourier series of g converges to g, and a few further properties. The subset E is universal in the sense that it does not depend on the function f to be approximated. Further in the paper this result is adapted to the case of $\mathcal{M} = G/H$ being a homogeneous space of an infinite compact second countable Hausdorff group. As a useful illustration the case of n-spheres with spherical harmonics is discussed. The construction of the subset E and approximant g is sketched briefly at the end of the paper.

Introduction

In the present paper we work with finite measure spaces $(\mathcal{M}, \Sigma, \mu)$. For efficiency of nomenclature we will write $\mathcal{M} = (\mathcal{M}, \Sigma, \mu)$ and $|A| = |A|_\mu = \mu(A)$ for every $A \in \Sigma$, where the σ-algebra Σ and the measure μ are clear from the context. Consider a separable finite measure space $(\mathcal{M}, \Sigma, \mu)$. Separability here simply means that all spaces $L^p(\mathcal{M})$ for $1 \le p < \infty$ are separable. Let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis of $L^2(\mathcal{M})$ with $\varphi_n \in L^\infty(\mathcal{M})$ for all $n \in \mathbb{N}$. For a function $f \in L^1(\mathcal{M})$ we

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denote its Fourier components by

$$Y_n(x; f) = c_n(f)\varphi_n(x), \quad c_n(f) = (f, \varphi_n)_2 = \int_{\mathcal{M}} f(x)\varphi_n^*(x)d\mu(x), \quad \forall n \in \mathbb{N},$$

where φ_n^* denotes the complex conjugate of φ_n. The possibly divergent Fourier series of f will be

$$\sum_{n=1}^{\infty} Y_n(x; f).$$

Note that already for the trigonometric system on the interval there exists an integrable function of which the Fourier series diverges in L^1 ([1], Chapter VIII, §22). We will often make use of Fourier polynomials and orthogonal series of the form

$$Q(x) = \sum_n Y_n(x) = \sum_n c_n\varphi_n(x), \quad c_n \in \mathbb{C}, \tag{1}$$

without reference to a particular function for which these may be the Fourier components. Denote by

$$\sigma(f) = \{n \in \mathbb{N} | \quad c_n(f) \neq 0\}, \quad f \in L^1(\mathcal{M}), \tag{2}$$

the spectrum of a function f.

Before stating our main theorem let us recall the notion of diffuseness for a measure space.

Definition 1 *In a measure space $(\mathcal{M}, \Sigma, \mu)$, a measurable subset $A \in \Sigma$ is called an atom if $|A| > 0$ and for every $B \in \Sigma$ with $B \subseteq A$ either $|B| = |A|$ or $|B| = 0$. The measure space $(\mathcal{M}, \Sigma, \mu)$ is called diffuse or non-atomic if it has no atoms.*

The main result of this paper is the following

Theorem 1 *Let $\mathcal{M}$ be a separable finite diffuse measure space, and let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal system in $L^2(\mathcal{M})$ consisting of bounded functions $\varphi_n \in L^\infty(\mathcal{M})$, $n \in \mathbb{N}$. For every $\epsilon, \delta > 0$ there exists a measurable subset $E \in \Sigma$ with measure $|E| > |\mathcal{M}| - \delta$ and with the following property; for each function $f \in L^1(\mathcal{M})$ with $\|f\|_1 > 0$ there exists an approximating function $g \in L^1(\mathcal{M})$ that satisfies:*

1. *$\|f - g\|_1 < \epsilon$,*
2. *$f = g$ on E,*

3. the Fourier series of g converges in $L^1(\mathcal{M})$,

4. we have

$$\sup_m \left\| \sum_{n=1}^{m} Y_n(g) \right\|_1 < 2 \min\left\{ \|f\|_1, \|g\|_1 \right\}.$$

Luzin proved that every almost everywhere finite function f on $[0,1]$ can be modified on a subset of arbitrarily small positive measure so that it becomes continuous. Further results in this direction were obtained by Menshov and others. See [15], [16], [11], [17], [19], [18], [14], [13], [4], [5], [6], [7], [8], [10], [9] for earlier results in this direction for classical orthonormal systems. Let us note that if $\mathcal{M}$ is not diffuse (i.e., it has atoms) then Statement 4 of this theorem may not hold with any coefficient on the right hand side. This is illustrated in the next

Example 1 *For every natural $N \in \mathbb{N}$, let $(\mathcal{M}, \Sigma, \mu) = (\mathbb{N}_2, 2^{\mathbb{N}_2}, P)$ be the probability space with orthonormal basis $\{\varphi_1, \varphi_2\}$ of $L^2(\mathcal{M})$, where*

$$\mathbb{N}_2 = \{1, 2\}, \quad P(\{1\}) = \frac{3}{16N^2 - 1}, \quad \varphi_1 = \left(2N, \frac{1}{2}\right).$$

Take $\delta = 1/35$ and $f = (1, 0)$. Then $|E| > 1 - \delta$ forces $E = \mathcal{M}$, and therefore $f = g$ on E implies $f = g$ on $\mathcal{M}$. Now

$$\|Y_1(g)\|_1 = \|Y_1(f)\|_1 = |c_1(f)| \|\varphi_1\|_1 = \frac{3N[(8N+3)^2 - 25]}{4(16N^2 - 1)} > N\|f\|_1 = \frac{3N}{16N^2 - 1}.$$

Theorem 1 is equivalent to the following theorem, which can be obtained by repeatedly applying Theorem 1 with fixed $f \in L^1(\mathcal{M})$ and $\epsilon_m = \frac{1}{m}$, $\delta_m = \frac{1}{m}$, $m = 1, 2, \ldots$.

Theorem 2 *Let $\mathcal{M}$ be a separable finite diffuse measure space, and let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal system in $L^2(\mathcal{M})$ consisting of bounded functions $\varphi_n \in L^\infty(\mathcal{M})$, $n \in \mathbb{N}$. There exists an increasing sequence of subsets $\{E_m\}_{m=1}^{\infty}$, $E_m \subset E_{m+1} \subset \mathcal{M}$, with $\lim_{m\to\infty} |E_m| = |\mathcal{M}|$, such that for every integrable function $f \in L^1(\mathcal{M})$ with $\|f\|_1 > 0$ there exists a sequence of approximating functions $\{g_m\}_{m=1}^{\infty}$, $g_m \in L^1(\mathcal{M})$, so that the following statements hold:*

1. $g_m \xrightarrow[m\to\infty]{} f$ in $L^1(\mathcal{M})$,

2. $f = g_m$ on E_m, $\forall m \in \mathbb{N}$,

3. the Fourier series of g_m converges in $L^1(\mathcal{M})$, $\forall m \in \mathbb{N}$, *4. we have*

$$\sup_N \left\| \sum_{n=1}^{N} Y_n(g_m) \right\|_1 < 2 \min \{ \|f\|_1, \|g_m\|_1 \}, \quad \forall m \in \mathbb{N}.$$

Remark 1 *Not for every orthonormal system $\{\varphi_n\}_{n=1}^{\infty}$ does an arbitrary integrable function $f \in L^1(\mathcal{M})$ have an orthogonal series $\sum_{n=1}^{\infty} Y_n$ of the form (1) that converges to f in $L^1(\mathcal{M})$, and if that happens then $\sum_{n=1}^{\infty} Y_n$ is necessarily the Fourier series of f, i.e., $Y_n = Y_n(f)$.*

For instance, in case of spherical harmonics this is guaranteed only in $L^2(\mathbb{S}^2)$ [2]. However, the following weaker statement is a corollary of Theorem 2 and holds true for all integrable functions.

Corollary 1 *Under the assumptions of Theorem 2, there exists an increasing sequence of subsets $\{E_m\}_{m=1}^{\infty}$, $E_m \subset E_{m+1} \subset \mathcal{M}$, such that $\lim\limits_{m\to\infty} |E_m| = |\mathcal{M}|$ with the following property. For any fixed integrable function $f \in L^1(\mathcal{M})$ with $\|f\|_1 > 0$ and for every natural number $m \in \mathbb{N}$ there is an orthogonal series $\sum_{n=1}^{\infty} Y_n^{(m)}$ of which the restriction $\sum_{n=1}^{\infty} Y_n^{(m)}|_{E_m}$ to the subset E_m converges to the restriction $f|_{E_m}$ in $L^1(E_m)$. In $L^1(\mathcal{M})$ the series $\sum_{n=1}^{\infty} Y_n^{(m)}$ converges to a function $g_m \in L^1(\mathcal{M})$. The sequence of these functions $\{g_m\}_{m=1}^{\infty}$ converges to f in $L^1(\mathcal{M})$.*

The general case

Theorem 1 is true for every finite separable diffuse measure space $\mathcal{M}$, but it will be more convenient to reduce the problem to that for a smaller class of measure spaces and then to prove the theorem for that class. First let us show that Theorem 1 is invariant under isomorphisms of measure algebras. For that purpose we will reformulate Theorem 1 in a way that makes no reference to the actual measure space $\mathcal{M}$ but only to its measure algebra $\mathcal{B}(\mathcal{M})$. We note that if we replace the set E produced by Theorem 1 by another measurable set $E' \in \Sigma$ such that the symmetric difference is null, $|E \triangle E'| = 0$, then all statements of the theorem remain valid with E' instead of E. This brings us to the following equivalent formulation of Theorem 1.

Theorem 3 *Let $\mathcal{M}$ be a finite separable diffuse measure space, and let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal system in $L^2(\mathcal{M})$ consisting of bounded functions $\varphi_n \in L^\infty(\mathcal{M})$, $n \in \mathbb{N}$. For every $\epsilon, \delta > 0$ there exists a function $\chi_E \in L^\infty(\mathcal{M})$ with $\chi_E^2 = \chi_E$ and $\|\chi_E\|_1 > |M| - \delta$, with the following properties; for each function $f \in L^1(\mathcal{M})$ with $\|f\|_1 > 0$ there exists an approximating function $g \in L^1(\mathcal{M})$ that satisfies:*

1. $\|f-g\|_1 < \epsilon$,

2. $(f-g)\chi_E = 0$,

3. *the Fourier series of* g *converges in* $L^1(\mathcal{M})$,

4. *we have*

$$\sup_m \left\| \sum_{n=1}^{m} Y_n(g) \right\|_1 < 2 \min \left\{ \|f\|_1, \|g\|_1 \right\}.$$

In this form the theorem relies only upon spaces $L^p(\mathcal{M})$, $p = 1, 2, \infty$, which can be constructed purely out of the measure algebra $\mathcal{B}(\mathcal{M})$ with no recourse to the underlying measure space $\mathcal{M}$. In particular, if two measure spaces have isomorphic measure algebras then the statements of Theorem 1 on these two spaces are equivalent.

Remark 2 *It is known in measure theory that every finite separable diffuse measure space* $\mathcal{M}$ *satisfies*

$$\mathcal{B}(\mathcal{M}) \simeq \mathcal{B}\left([0,a]\right),$$

where $a > 0$ *is a positive real number.*

Thus, without loss of generality, we can restrict ourselves to measure spaces $\mathcal{M} = [0,a]$. The next reduction comes from the following observation.

Remark 3 *If Theorem 3 is true for the finite separable measure space* $(\mathcal{M}, \mu)$ *then it is true also for* $(\mathcal{M}, \lambda\mu)$ *for every* $\lambda > 0$.

Indeed, for every $p \in [1,\infty]$ the operator $\mathrm{T}_p\, f \doteq \lambda^{-\frac{1}{p}} f$ defines an isometric isomorphism $\mathrm{T}_p : L^p(\mathcal{M},\mu) \to L^p(\mathcal{M},\lambda\mu)$. It is now straightforward to check that if the statements of Theorem 3 hold on $(\mathcal{M},\mu)$ with data $\{\varphi_n\}_{n=1}^{\infty}$, ϵ, δ, χ_E, f, g, then they hold on $(\mathcal{M},\lambda\mu)$ with data $\{\mathrm{T}_2\, \varphi_n\}_{n=1}^{\infty}$, ϵ, $\lambda\delta$, $\mathrm{T}_\infty\, \chi_E$, $\mathrm{T}_1\, f$, $\mathrm{T}_1\, g$.

Thus we established that without loss of generality we are allowed to prove the theorem just for the unit interval $\mathcal{M} = [0,1]$. In fact, in the next sections we will prove Theorem 1 on separable cylindric probability spaces, i.e., separable probability spaces of the form $\mathcal{M} = [0,1] \otimes \mathcal{N}$, where $\mathcal{N}$ is another probability space. The unit interval is trivially cylindric, $[0,1] \simeq [0,1] \otimes \{1\}$, and it may seem an unnecessary effort to prove the theorem for a cylindric space instead of $[0,1]$. But note that the result cited in Remark 2 is very abstract and the produced isomorphisms are in general far from being geometrically natural. Our proof of Theorem 1 is constructive, and the construction of the set E highly depends on the cylindric structure. If the space at hand has a natural cylindric structure then this approach gives a geometrically more sensible set E than what we would expect had we identified the cylinder with the unit interval through a wild measure algebra isomorphism.

The particular case

In this section we will prove the main theorem for the particular case where $\mathcal{M} = (\mathcal{M}, \Sigma, \mu)$ is a separable cylindric probability space

$$\mathcal{M} = [0,1] \otimes \mathcal{N}. \tag{3}$$

Here $\mathcal{N} = (\mathcal{N}, \Sigma_0, \nu)$ is any separable probability space. We will write $\mathcal{M} \ni x = (t, y) \in [0,1] \times \mathcal{N}$.

The core lemmata

First let us state a variant of Féjér's lemma.

Lemma 1 *Let $a, b \in \mathbb{R}$, $a < b$. For every $f \in L^1[a,b]$ and $g \in L^\infty(\mathbb{R})$, g being $(b-a)$-periodic,*

$$\lim_{\lambda \to +\infty} \int_a^b f(t) g(\lambda t) dt = \frac{1}{b-a} \int_a^b f(t) dt \int_a^b g(t) dt.$$

This lemma is given in [1, page 77] with $[a,b] = [-\pi, \pi]$, but the proof for arbitrary a and b follows with only trivial modifications.

We proceed to our first critical lemma.

Lemma 2 *Let $\Delta = [a,b] \times \Delta_0 \in \Sigma$ with $[a,b] \subset [0,1]$ and $\Delta_0 \in \Sigma_0$, $\quad 0 \neq \gamma \in \mathbb{R}$, $\quad \epsilon, \delta \in (0,1)$ and $N \in \mathbb{N}$ be given. Then there exists a function $g \in L^\infty(\mathcal{M})$, a measurable set $\Sigma \ni E \subset \Delta$ and a Fourier polynomial of the form*

$$Q(x) = \sum_{n=N}^{M} Y_n(x), \quad N \leq M \in \mathbb{N},$$

such that

1. *$|E| > |\Delta|(1-\delta)$,*
2. *$g(x) = \gamma$ for $x \in E$ $\quad$ and $\quad$ $g(x) = 0$ for $x \notin \Delta$,*
3. *$|\gamma||\Delta| < \|g\|_1 < 2|\gamma||\Delta|$,*

4. $\|Q-g\|_1 < \epsilon$,

5. and

$$\max_{N\le m\le M}\left\|\sum_{n=N}^{m} Y_n\right\|_1 \le \frac{|\gamma|\sqrt{|\Delta|(1+\delta)}}{\sqrt{\delta}}.$$

Proof: Set

$$\delta_* \doteq \frac{\delta}{1+\delta} \in \left(0, \frac{1}{2}\right). \tag{4}$$

Define the 1-periodic function $I:\mathbb{R}\to\mathbb{R}$ by setting

$$I(t) = 1 - \frac{1}{\delta_*}\chi_{[0,\delta_*)}(t) = \begin{cases} 1 & \text{if} \quad t\in[\delta_*,1), \\ 1-\frac{1}{\delta_*} & \text{if} \quad t\in[0,\delta_*) \end{cases}, \tag{5}$$

for $t\in[0,1)$ and continuing periodically. Then obviously

$$\int_0^1 I(t)dt = 0. \tag{6}$$

By Féjér's lemma

$$\lim_{s\to+\infty}\int_\Delta I(st)\varphi_n^*(t,y)dtdy = \lim_{s\to+\infty}\int_a^b I(st)\int_{\Delta_0}\varphi_n^*(t,y)dydt$$
$$= \lim_{s\to+\infty}\int_0^1 I(st)\left[\chi_{[a,b]}(t)\int_{\Delta_0}\varphi_n^*(t,y)dy\right]dt = \int_0^1 I(t)dt\int_\Delta \varphi_n^*(x)dx = 0. \tag{7}$$

Choose a natural number $s_0\in\mathbb{N}$ sufficiently large so that

$$s_0 > \frac{(1-\delta_*)^2}{\delta_*^2(b-a)} \quad \text{and} \quad \left|\int_\Delta I(s_0t)\varphi_n^*(t,y)dtdy\right| < \frac{\epsilon}{2N|\gamma|}, \quad 1\le n\le N. \tag{8}$$

Set

$$g(x) \doteq \gamma I(s_0t)\chi_\Delta(x), \quad E \doteq \{x\in\Delta| \quad g(x)=\gamma\}. \tag{9}$$

Then it can be seen that

$$|E| \ge |\Delta|\frac{\lfloor s_0(b-a)\rfloor(1-\delta_*)}{s_0(b-a)} > |\Delta|(1-\delta_*)\left(1-\frac{1}{s_0(b-a)}\right) > |\Delta|(1-\delta), \tag{10}$$

where the first inequality of formula (8) and then formula (4) were used in the last step. Clearly,

$g \in L^\infty(\mathcal{M})$ and thus we have proven Statements 1 and 2. Next we note using (4) that

$$\int_\Delta |I(s_0 t)| dx = \int_E dx + \int_{\Delta \setminus E} \left| 1 - \frac{1}{\delta_*} \right| dx = |E| + \frac{1}{\delta}(|\Delta| - |E|), \tag{11}$$

and then by $(1-\delta)|\Delta| < |E| < |\Delta|$ we find that

$$|\Delta| < |E| + \frac{1}{\delta}(|\Delta| - |E|) < 2|\Delta|, \tag{12}$$

which together entail

$$|\Delta| < \int_\Delta |I(s_0 t)| dx < 2|\Delta|. \tag{13}$$

Similarly,

$$\int_\Delta |I(s_0 t)|^2 dx = \int_E dx + \int_{\Delta \setminus E} \left(1 - \frac{1}{\delta_*} \right)^2 dx = |E| + \frac{1}{\delta^2}(|\Delta| - |E|) < \left(1 + \frac{1}{\delta} \right) |\Delta|. \tag{14}$$

Formulae (9) and (13) imply that

$$|\gamma||\Delta| < \|g\|_1 = \int_\mathcal{M} |g(x)| dx = |\gamma| \int_\Delta |I(s_0 t)| dx < 2|\gamma||\Delta|, \tag{15}$$

which proves Statement 3. In a similar fashion we obtain

$$\|g\|_2^2 = \int_\mathcal{M} |g(x)|^2 dx = \gamma^2 \int_\Delta |I(s_0 t)|^2 dx < \left(1 + \frac{1}{\delta} \right) \gamma^2 |\Delta|. \tag{16}$$

We have $g \in L^\infty(\mathcal{M}) \subset L^2(\mathcal{M})$, and therefore the Fourier series $\sum Y_n(g)$ converges to g in $L^2(\mathcal{M})$. Thus we can choose the natural number $M \in \mathbb{N}$ so large that

$$\left\| \sum_{n=1}^{M} Y_n(g) - g \right\|_2 < \frac{\epsilon}{2}. \tag{17}$$

Further, from formula (8) we estimate the magnitude of the first N Fourier coefficients of g as

$$|c_n(g)| = \left| \int_\mathcal{M} g(x) \varphi_n^*(x) dx \right| = |\gamma| \left| \int_\Delta I(s_0 t) \varphi_n^*(x) dx \right| < \frac{\epsilon}{2N}, \quad 1 \le n \le N. \tag{18}$$

Finally, set

$$Q(x) \doteq \sum_{n=N}^{M} Y_n(x; g), \quad \forall x \in \mathcal{M}. \tag{19}$$

In order to prove Statement 4 we write

$$\begin{aligned} \|Q - g\|_1 \leq \|Q - g\|_2 = \left\| \sum_{n=N}^{M} Y_n(g) - g \right\|_2 &\leq \left\| \sum_{n=1}^{M} Y_n(g) - g \right\|_2 + \left\| \sum_{n=1}^{N-1} Y_n(g) \right\|_2 \\ &\leq \left\| \sum_{n=1}^{M} Y_n(g) - g \right\|_2 + \sum_{n=1}^{N-1} |c_n(g)| \|\varphi_n\|_2 < \epsilon, \end{aligned} \tag{20}$$

where formulae (17) and (18) were used in the last step along with the normalization $\|\varphi_n\|_2 = 1$. Using the pairwise orthogonality of the Fourier components $Y_n(g)$ and formula (16) we can obtain the coarse estimate

$$\left\| \sum_{n=N}^{m} Y_n(g) \right\|_2^2 = \sum_{n=N}^{m} \|Y_n(g)\|_2^2 \leq \sum_{n=1}^{\infty} \|Y_n(g)\|_2^2 = \|g\|_2^2 < \left(1 + \frac{1}{\delta}\right) \gamma^2 |\Delta|, \tag{21}$$

which immediately yields

$$\left\| \sum_{n=N}^{m} Y_n(g) \right\|_1 \leq \left\| \sum_{n=N}^{m} Y_n(g) \right\|_2 < \frac{|\gamma| \sqrt{|\Delta|(1+\delta)}}{\sqrt{\delta}}, \quad m > N, \tag{22}$$

thus proving Statement 5. Note that the inequality $\|.\|_1 \leq \|.\|_2$ used above hold thanks to the convenient assumption that we are in a probability space. □

Lemma 3 *Let $f \in L^1(\mathcal{M})$, $\epsilon, \delta \in (0, 1)$, $N_0 \in \mathbb{N}$. Then $\exists E \in \Sigma$, $g \in L^\infty(\mathcal{M})$ and*

$$Q(x) = \sum_{n=N_0}^{N} Y_n(x), \quad N \in \mathbb{N},$$

such that

1. $|E| > 1 - \delta$,
2. $x \in E$ *implies* $g(x) = f(x)$,
3. $\frac{1}{3}\|f\|_1 < \|g\|_1 < 3\|f\|_1$,
4. $\|g - Q\|_1 < \epsilon$,

5. and

$$\sup_{N_0 \le m \le N} \left\| \sum_{n=N_0}^{m} Y_n \right\|_1 < 3\|f\|_1.$$

Proof: For every measurable partition of $\mathcal{N}$

$$\mathcal{N} = \bigsqcup_{i=1}^{\tilde{\nu}_0} \tilde{\Delta}_i, \quad \tilde{\Delta}_i \in \Sigma_0, \quad i \neq j \quad \Rightarrow \quad \left| \tilde{\Delta}_i \cap \tilde{\Delta}_j \right| = 0, \quad \forall i, j = 1, \ldots \tilde{\nu}_0, \quad \tilde{\nu}_0 \in \mathbb{N}, \tag{23}$$

and every partition $0 = x_0 < x_1 < \ldots < x_{\bar{\nu}_0} = 1$, $\bar{\nu}_0 \in \mathbb{N}$, of the unit interval, the product partition

$$\Delta_k = [x_{l-1}, x_l] \times \tilde{\Delta}_i, \quad k = \bar{\nu}_0 \cdot i + l - 1 = 1, \ldots, \nu_0 \doteq \bar{\nu}_0 \cdot \tilde{\nu}_0, \quad l = 1, \ldots, \bar{\nu}_0, \quad i = 1, \ldots, \tilde{\nu}_0, \tag{24}$$

is a measurable partition of $\mathcal{M}$ with the property that

$$\max_{1 \le k \le \nu_0} |\Delta_k| \le \max_{1 \le l \le \bar{\nu}_0} |x_l - x_{l-1}|. \tag{25}$$

For every product partition $\{\Delta_k\}_{k=1}^{\nu_0}$ as above and every tuple of real numbers $\{\gamma_k\}_{k=1}^{\nu_0}$ consider the step function

$$\Lambda(x) = \sum_{k=1}^{\nu_0} \gamma_k \chi_{\Delta_k}(x), \quad \forall x \in \mathcal{M}. \tag{26}$$

By the assumption of separability of $\mathcal{M}$ we know that step functions of the form (26) subordinate to product partitions are dense in all spaces $L^p(\mathcal{M})$ for $1 \le p < \infty$. Choose a product partition and a subordinate step function such that

$$\|\Lambda - f\|_1 < \min\left\{ \frac{1}{2}\epsilon, \frac{1}{3}\|f\|_1 \right\}. \tag{27}$$

Note that the numbers γ_k are not assumed to be distinct, thus we can refine the given partition without changing γ_k and the function $\Lambda(x)$. We use the property (25) to refine the product partition $\{\Delta_k\}_{k=1}^{\nu_0}$ until it satisfies

$$144\gamma_k^2 |\Delta_k| (1 + \delta) < \delta \|f\|_1^2, \quad k = 1, \ldots, \nu_0. \tag{28}$$

Now we apply Lemma 2 iteratively with

$$\Delta \leftarrow \Delta_k, \quad \gamma \leftarrow \gamma_k, \quad \epsilon \leftarrow \frac{1}{2^{\nu_0+2}} \min\{\epsilon, \|f\|_1\}, \quad \delta \leftarrow \delta, \quad N \leftarrow N_{k-1} \tag{29}$$

for $k = 1, \ldots, \nu_0$, obtaining at each k a function $g_k \in L^\infty(\mathcal{M})$, a set $\Sigma \ni E_k \subset \Delta_k$, a number $N_{k-1} \leq N_k \in \mathbb{N}$ and a Fourier polynomial

$$Q_k(x) = \sum_{n=N_{k-1}}^{N_k-1} Y_n(x) \tag{30}$$

with the following properties:

1°. $|E_k| > |\Delta_k|(1-\delta)$,

2°. $g_k(x) = \gamma_k$ for $x \in E_k$ and $g_k(x) = 0$ for $x \notin \Delta_k$,

3°. $|\gamma_k||\Delta_k| < \|g_k\|_1 < 2|\gamma_k||\Delta_k|$,

4°. $\|Q_k - g_k\|_1 < \frac{1}{2^{\nu_0+2}} \min\{\epsilon, \|f\|_1\}$,

5°. and

$$\max_{N_{k-1} \leq m < N_k} \left\| \sum_{n=N_{k-1}}^{m} Y_n \right\|_1 \leq \frac{|\gamma_k|\sqrt{|\Delta_k|(1+\delta)}}{\sqrt{\delta}}.$$

Set

$$E \doteq \bigcup_{k=1}^{\nu_0} E_k, \quad g(x) \doteq f(x) - \left[\Lambda(x) - \sum_{k=1}^{\nu_0} g_k(x)\right], \tag{31}$$

$$N \doteq N_{\nu_0} - 1, \quad Q(x) \doteq \sum_{k=1}^{\nu_0} Q_k(x) = \sum_{n=N_0}^{N} Y_n(x), \quad \forall x \in \mathcal{M}. \tag{32}$$

First we check that from (26), (1°), (2°) and (31) it follows that

$$|E| = \sum_{k=1}^{\nu_0} |E_k| > \sum_{k=1}^{\nu_0} |\Delta_k| (1-\delta) = 1 - \delta, \tag{33}$$

$$x \in E \implies x \in E_k \implies \Lambda(x) = \gamma_k = g_k(x),\, g_l(x) = 0,\, l \neq k \implies g(x) = f(x), \tag{34}$$

so that Statements 1 and 2 are proven. Next we observe using (4°), (27), (31) and (32) that

$$\|Q - g\|_1 = \left\| \sum_{k=1}^{\nu_0} [Q_k - g_k] + [f - \Lambda] \right\|_1 \leq \sum_{k=1}^{\nu_0} \|Q_k - g_k\|_1 + \|f - \Lambda\|_1 < \epsilon, \tag{35}$$

which proves Statement 4. Further, from (26), (27), (2°) and (3°) we deduce that

$$\begin{aligned}\|g\|_1 &\leq \sum_{k=1}^{\nu_0}\|g_k\|_1 + \|f-\Lambda\|_1 \leq 2\sum_{k=1}^{\nu_0}|\gamma_k||\Delta_k| + \|f-\Lambda\|_1 \\ &= 2\|\Lambda\|_1 + \|f-\Lambda\|_1 \leq 3\|f-\Lambda\|_1 + 2\|f\|_1 < 3\|f\|_1. \end{aligned} \tag{36}$$

Moreover, the same formulae also imply

$$\begin{aligned} &\|g\|_1 + \frac{1}{3}\|f\|_1 > \|g\|_1 + \|f-\Lambda\|_1 \geq \|g-f+\Lambda\|_1 \\ = \sum_{k=1}^{\nu_0^2}\|g_k\|_1 > \sum_{k=1}^{\nu_0^2}|\gamma_k||\Delta_k| &= \|\Lambda\|_1 \geq \big|\|\Lambda - f\|_1 - \|f\|_1\big| > \frac{2}{3}\|f\|_1, \end{aligned} \tag{37}$$

i.e., $\|f\|_1 < 3\|g\|_1$, thus proving Statement 3. In order to prove Statement 5 let us fix an $N_0 \leq m \leq N$. Then there is a $1 \leq k_0 \leq \nu_0$ such that $N_{k_0-1} \leq m < N_{k_0}$, and thus by (30) and (32) we have

$$\sum_{n=N_0}^{m} Y_n(x) = \sum_{k=1}^{k_0-1} Q_k(x) + \sum_{N_{k_0-1}}^{m} Y_n(x). \tag{38}$$

Finally we use this along with formulae (3°), (4°), (5°) and (28) to obtain

$$\begin{aligned}\left\|\sum_{n=N_0}^{m} Y_n(x)\right\|_1 &\leq \sum_{k=1}^{k_0-1}\|Q_k - g_k\|_1 + \sum_{k=1}^{k_0-1}\|g_k\|_1 + \left\|\sum_{k=N_{k_0-1}}^{m} Y_n\right\|_1 \\ &< \frac{1}{4}\|f\|_1 + 2\|\Lambda\|_1 + \frac{|\gamma_{k_0}|\sqrt{|\Delta_{k_0}|(1+\delta)}}{\sqrt{\delta}} < 3\|f\|_1, \end{aligned} \tag{39}$$

and this completes the proof. □

Lemma 4 *Let $\{R_k\}_{k=1}^{\infty}$ be any fixed ordering of the set of all Fourier polynomials with rational coefficients into a sequence. Then for every $f \in L^1(\mathcal{M})$ and sequence $\{b_s\}_{s=1}^{\infty}$ of positive numbers $b_s > 0$ there exists subsequence $\{R_{k_s}\}_{s=0}^{\infty}$ such that*

1. $\|R_{k_0} - f\|_1 \leq \frac{1}{2}\|f\|_1$
2. $\|R_{k_s}\|_1 < b_s$ *for* $s \geq 1$
3. $\sum_{s=0}^{\infty} R_{k_s} = f$ *in* $L^1(\mathcal{M})$.

Proof: Let us first convince ourselves that Fourier polynomials with rational coefficients are dense in $L^1(\mathcal{M})$. Indeed, by the assumption of separability, step functions are dense in $L^1(\mathcal{M})$, but they all belong also to the separable Hilbert space $L^2(\mathcal{M})$. On the other hand, Fourier polynomials are clearly dense in $L^2(\mathcal{M})$. And finally, an arbitrary Fourier polynomial can be approximated in $L^2(\mathcal{M})$ by a Fourier polynomial with rational coefficients (this amounts to approximating the Fourier coefficients by rational numbers). A three-epsilon argument together with $\|.\|_1 \leq \|.\|_2$ then yields the assertion.

Using the denseness of $\{R_k\}_{k=1}^{\infty}$ let us choose a natural number $k_0 \in \mathbb{N}$ such that

$$\|R_{k_0} - f\|_1 \leq \frac{1}{2} \min\{\|f\|_1, b_1\}. \tag{40}$$

Then we can choose further natural numbers $k_s \in \mathbb{N}$ iteratively as follows. For every $s \in \mathbb{N}$, again by using the denseness argument, choose a number k_s so that $k_s > k_{s-1}$ and

$$\left\| f - \sum_{r=0}^{s} R_{k_r} \right\|_1 = \left\| R_{k_s} - \left(f - \sum_{r=0}^{s-1} R_{k_r} \right) \right\|_1 < \frac{1}{2} \min\left\{ b_s, b_{s+1}, \frac{1}{s} \right\}, \quad \forall s \in \mathbb{N}. \tag{41}$$

Statements 1 and 3 are clearly satisfied. For Statement 2 we have

$$\|R_{k_s}\|_1 = \left\| R_{k_s} - \left(f - \sum_{r=0}^{s-1} R_{k_r} \right) + \left(f - \sum_{r=0}^{s-1} R_{k_r} \right) \right\|_1$$
$$\leq \left\| R_{k_s} - \left(f - \sum_{r=0}^{s-1} R_{k_r} \right) \right\|_1 + \left\| f - \sum_{r=0}^{s-1} R_{k_r} \right\|_1 < \frac{1}{2} b_s + \frac{1}{2} b_s = b_s, \quad \forall s \in \mathbb{N}. \tag{42}$$

Lemma is proven. □

The main theorem

Here we will prove Theorem 1 for the particular case of $(\mathcal{M}, \Sigma, \mu)$ being a separable cylindric probability space as in (3).

Proof: Recall that $\epsilon, \delta > 0$ and $f \in L^1(\mathcal{M})$ with $\|f\|_1 > 0$ are given. Denote

$$\epsilon_0 \doteq \min\{\epsilon, \|f\|_1\}. \tag{43}$$

Let $\{R_k\}_{k=1}^{\infty}$ be any ordering of the set of all Fourier polynomials with rational coefficients into a sequence. Iteratively applying Lemma 3 with

$$f \leftarrow R_k, \quad \epsilon \leftarrow \frac{\epsilon_0}{2^{k+7}}, \quad \delta \leftarrow \frac{\delta}{2^k}, \quad N_0 \leftarrow N_{k-1} \tag{44}$$

for $k = 1, 2, \ldots$ we obtain for each $k \in \mathbb{N}$ a subset $\tilde{E}_k \in \Sigma$, a function $\tilde{g}_k \in L^\infty(\mathcal{M})$, a number $N_{k-1} \leq N_k \in \mathbb{N}$ (set $N_0 = 0$) and a Fourier polynomial

$$\tilde{Q}_k(x) = \sum_{n=N_{k-1}}^{N_k-1} \tilde{Y}_n(x) \tag{45}$$

with the following properties:

1†. $|\tilde{E}_k| > 1 - \frac{\delta}{2^k}$,

2†. $x \in \tilde{E}_k$ implies $\tilde{g}_k(x) = R_k(x)$,

3†. $\frac{1}{3}\|R_k\|_1 < \|\tilde{g}_k\|_1 < 3\|R_k\|_1$,

4†. $\|\tilde{g}_k - \tilde{Q}_k\|_1 < \epsilon_0 2^{-k-7}$,

5†. and

$$\sup_{N_{k-1} \leq m < N_k} \left\| \sum_{n=N_{k-1}}^{m} \tilde{Y}_n \right\|_1 < 3\|R_k\|_1.$$

Define the desired set E as

$$E \doteq \bigcap_{k=1}^{\infty} \tilde{E}_k. \tag{46}$$

Observe from (1†) that

$$|E| = 1 - |\mathcal{M} \setminus E| \geq 1 - \sum_{s=1}^{\infty} |\mathcal{M} \setminus \tilde{E}_k| > 1 - \sum_{k=1}^{\infty} \frac{\delta}{2^k} = 1 - \delta. \tag{47}$$

Note that E is universal, i.e., independent of f.

Let $\{R_{k_s}\}_{s=0}^{\infty}$ be the subsequence of Fourier polynomials provided by Lemma 4 applied with

$$f \leftarrow f, \quad b_s \leftarrow \frac{\epsilon_0}{2^{s+6}}. \tag{48}$$

It satisfies

1°. $\|R_{k_0} - f\|_1 \leq \frac{1}{2}\|f\|_1$,

2°. $\|R_{k_s}\|_1 < \epsilon_0 2^{-s-6}$ for $s \geq 1$,

3°. $\sum_{s=0}^{\infty} R_{k_s} = f$ in $L^1(\mathcal{M})$.

We want to use mathematical induction in order to define a sequence of natural numbers $1 < \nu_1 < \nu_2 < \ldots$ and a sequence of functions $\{g_s\}_{s=1}^{\infty}$, $g_s \in L^1(\mathcal{M})$, such that for all $s \in \mathbb{N}$ we have

1^*. $x \in \tilde{E}_{\nu_s}$ implies $g_s(x) = R_{k_s}(x)$,

2^*. $\|g_s\|_1 < \epsilon_0 2^{-s-2}$,

3^*.

$$\left\| \sum_{j=1}^{s} [\tilde{Q}_{\nu_j} - g_j] \right\|_1 < \frac{\epsilon_0}{2^{s+6}},$$

4^*.

$$\max_{N_{\nu_s - 1} \le m < N_{\nu_s}} \left\| \sum_{n=N_{\nu_s - 1}}^{m} \tilde{Y}_n \right\|_1 < \frac{\epsilon_0}{2^s}.$$

Assume that for some $s \in \mathbb{N}$, the choice of $1 < \nu_1 < \nu_2 < \ldots < \nu_{s-1}$ and $g_1, g_2, \ldots, g_{s-1}$ satisfying (3^*) has been already made (for $s = 1$ this is trivially correct). Remember that by $\sigma(h)$ we have denoted the $\{\varphi_n\}$-spectrum of a function $h \in L^1(\mathcal{M})$, i.e., the support of its Fourier series. Using the denseness of $\{R_k\}_{k=1}^{\infty}$ (see Lemma 4) choose a natural number $\nu_s \in \mathbb{N}$ such that $N_{\nu_1 - 1} > \max \sigma(R_{k_0})$ and $\nu_s > \nu_{s-1}$ for $s > 1$, and

$$\left\| R_{\nu_s} - \left(R_{k_s} - \sum_{j=1}^{s-1} [\tilde{Q}_{\nu_j} - g_j] \right) \right\|_1 < \frac{\epsilon_0}{2^{s+7}}. \tag{49}$$

Then by (2°) and (3^*) we have for all $s \in \mathbb{N}$ that

$$\left\| R_{k_s} - \sum_{j=1}^{s-1} [\tilde{Q}_{\nu_j} - g_j] \right\|_1 \le \|R_{k_s}\|_1 + \left\| \sum_{j=1}^{s-1} [\tilde{Q}_{\nu_j} - g_j] \right\|_1 < \frac{3\,\epsilon_0}{2^{s+6}}, \tag{50}$$

which combined with (49) implies

$$\|R_{\nu_s}\|_1 \le \left\| R_{\nu_s} - \left(R_{k_s} - \sum_{j=1}^{s-1} [\tilde{Q}_{\nu_j} - g_j] \right) \right\|_1 + \left\| R_{k_s} - \sum_{j=1}^{s-1} [\tilde{Q}_{\nu_j} - g_j] \right\|_1 < \frac{7\,\epsilon_0}{2^{s+7}}. \tag{51}$$

Set

$$g_s(x) \doteq R_{k_s}(x) + \tilde{g}_{\nu_s}(x) - R_{\nu_s}(x). \tag{52}$$

Condition (1^*) is easily satisfied thanks to ($2^\dagger$) with $k = \nu_s$. For condition (2^*) we write

$$\|g_s\|_1 = \|R_{k_s} + \tilde{g}_{\nu_s} - R_{\nu_s}\|_1$$
$$\leq \left\| R_{\nu_s} - R_{k_s} + \sum_{j=1}^{s-1} [\tilde{Q}_{\nu_j} - g_j] \right\|_1 + \left\| \sum_{j=1}^{s-1} [\tilde{Q}_{\nu_j} - g_j] \right\|_1 + \|\tilde{g}_{\nu_s}\|_1 < \frac{\epsilon_0}{2^{s+2}}, \tag{53}$$

where we used (49), ($3^\dagger$), ($3^*|_{s-1}$) and (51) in the last step. To show that condition (3^*) is satisfied we observe that

$$\left\| \sum_{j=1}^{s} [\tilde{Q}_{\nu_j} - g_j] \right\|_1 = \left\| \tilde{Q}_{\nu_s} - g_s + \sum_{j=1}^{s-1} [\tilde{Q}_{\nu_j} - g_j] \right\|_1 = \left\| \tilde{Q}_{\nu_s} - R_{k_s} - \tilde{g}_{\nu_s} + R_{\nu_s} + \sum_{j=1}^{s-1} [\tilde{Q}_{\nu_j} - g_j] \right\|_1$$
$$\leq \left\| \tilde{Q}_{\nu_s} - \tilde{g}_{\nu_s} \right\|_1 + \left\| R_{\nu_s} - \left(R_{k_s} - \sum_{j=1}^{s-1} [\tilde{Q}_{\nu_j} - g_j] \right) \right\|_1 < \frac{\epsilon_0}{2^{s+6}}, \tag{54}$$

where ($4^\dagger$), (49) and $\nu_s > s$ were used in the second inequality. Finally we satisfy condition (4^*) using ($5^\dagger$) and (51),

$$\max_{N_{\nu_s-1} \leq m < N_{\nu_s}} \left\| \sum_{n=N_{\nu_s-1}}^{m} \tilde{Y}_n \right\|_1 < 3\|R_{\nu_s}\|_1 < \frac{\epsilon_0}{2^s}. \tag{55}$$

The iteration is thus complete, and by mathematical induction we construct the sequences $\{\nu_s\}_{s=1}^\infty$ and $\{g_s\}_{s=1}^\infty$ satisfying conditions (1^*) through (4^*) for all $s \in \mathbb{N}$. Define

$$g(x) \doteq R_{k_0}(x) + \sum_{s=1}^{\infty} g_s(x), \quad \forall x \in \mathcal{M}. \tag{56}$$

From (53) it follows that

$$\sum_{s=1}^{\infty} \|g_s\|_1 < \frac{13\,\epsilon_0}{64} < \infty, \tag{57}$$

thus $g \in L^1(\mathcal{M})$. The construction is now complete, and it remains to verify the statements of the theorem.

To prove Statement 2 of the theorem we note that $x \in E$ means $x \in \tilde{E}_{\nu_s}$, and hence by ($2^\dagger$)

$g_s(x) = R_{k_s}(x)$ for all $s \in \mathbb{N}$. It then follows from (3°) that

$$g(x) = R_{k_0}(x) + \sum_{s=1}^{\infty} g_s(x) = \sum_{s=0}^{\infty} R_{k_s}(x) = f(x), \quad \forall x \in E. \tag{58}$$

Let $\{Y_n\}_{n=1}^{\infty}$ be the series of $Y_n = c_n \varphi_n$ such that

$$\sum_{n=1}^{N_{\nu_s}-1} Y_n = R_{k_0} + \sum_{j=1}^{s} \tilde{Q}_{\nu_j}, \quad \forall s \in \mathbb{N}. \tag{59}$$

Let $m \in \mathbb{N}$, and let $r \in \mathbb{N}$ be the largest natural number such that $N_{\nu_{r-1}} \leq m$ (if $m < N_{\nu_1 - 1}$ set $r = 1$). Set $m_* \doteq \min\{m, N_{\nu_r} - 1\}$. Then by (56), (3^*), (4^*) and (53) we get

$$\begin{aligned}\left\|\sum_{n=1}^{m} Y_n - g\right\|_1 &= \left\|\sum_{n=1}^{m_*} Y_n - g\right\|_1 = \left\|\sum_{j=1}^{r-1} \tilde{Q}_{\nu_j} + \sum_{n=N_{\nu_{r-1}}}^{m_*} \tilde{Y}_n - \sum_{j=1}^{r-1} g_j - \sum_{j=r}^{\infty} g_j\right\|_1 \\ &\leq \left\|\sum_{j=1}^{r-1} \left[\tilde{Q}_{\nu_j} - g_j\right]\right\|_1 + \left\|\sum_{n=N_{\nu_{r-1}}}^{m_*} \tilde{Y}_n\right\|_1 + \left\|\sum_{j=r}^{\infty} g_j\right\|_1 < \frac{23\,\epsilon_0}{2^{r+4}}.\end{aligned} \tag{60}$$

Now as $m \to \infty$ obviously $r \to \infty$ as well, thus making the above expression vanish, which proves that $\sum Y_n$ is the Fourier series of g, i.e., $Y_n = Y_n(g)$, and it converges to g as required in Statement 3. Further, from (2°), (3°), (56) and (57) we have that

$$\|f - g\|_1 = \left\|\sum_{s=1}^{\infty} R_{k_s} - \sum_{s=1}^{\infty} g_s\right\|_1 \leq \sum_{s=1}^{\infty} \|R_{k_s}\|_1 + \sum_{s=1}^{\infty} \|g_s\|_1 < \frac{7\,\epsilon_0}{32}, \tag{61}$$

which in view of (43) proves Statement 1. Finally, using (60) and (61) we establish that

$$\left\|\sum_{n=1}^{m} Y_n\right\|_1 \leq \left\|\sum_{n=1}^{m} Y_n - g\right\|_1 + \|f - g\|_1 + \|f\|_1 < \frac{15\,\epsilon_0}{16} + \|f\|_1 < 2\|f\|_1, \tag{62}$$

but also

$$\left\|\sum_{n=1}^{m} Y_n\right\|_1 \leq \left\|\sum_{n=1}^{m} Y_n - g\right\|_1 + \|g\|_1 < \frac{23}{32}\|f\|_1 + \|g\|_1. \tag{63}$$

Note that by (61)

$$\|f\|_1 \leq \|g\|_1 + \|f - g\|_1 < \|g\|_1 + \frac{7}{32}\|f\|_1, \tag{64}$$

and therefore $25\|f\|_1 < 32\|g\|_1$. This together with (63) yields

$$\left\|\sum_{n=1}^{m} Y_n\right\|_1 < \frac{23}{25}\|g\|_1 + \|g\|_1 < 2\|g\|_1, \tag{65}$$

which establishes Statement 4. The proof of the theorem is accomplished. □

Compact groups

Let G be a compact Hausdorff topological group, Σ the Borel σ-algebra and $d\mu(x) = dx$ the normalized Haar measure. Then μ is diffuse if and only if G is infinite, which we will assume here. Now by [12, Theorem 28.2] we have that $\dim L^2(G) = \mathrm{w}(G)$, therefore (G, Σ, μ) is separable if and only if G is second countable. This will also be assumed in what follows. This implies in particular, through Peter-Weyl theorem, that the dual $\hat{G}$ is countable. For a detailed exposition of harmonic analysis on compact groups consult, e.g., [20] or [3].

For every irreducible unitary representation $\rho \in \hat{G}$ (or rather $[\rho] \in \hat{G}$), let $\mathcal{H}_\rho$ be the representation Hilbert space of dimension $\dim \mathcal{H}_\rho \doteq d_\rho \in \mathbb{N}$. Choose an arbitrary orthonormal basis $\{e_i^\rho\}_{i=1}^{d_\rho}$ of $\mathcal{H}_\rho$, and denote

$$\varphi_{\rho,i,j}(x) \doteq \sqrt{d_\rho}(\rho(x)e_j^\rho, e_i^\rho), \quad \forall x \in G, \quad i, j = 1, \ldots, d_\rho, \quad \forall \rho \in \hat{G}. \tag{66}$$

By the Peter-Weyl theorem $\{\varphi_{\rho,i,j}\}$ is an orthonormal basis in $L^2(G)$. Moreover, since $\rho(x)$ is unitary for all $x \in G$, we get

$$|\varphi_{\rho,i,j}(x)| = \sqrt{d_\rho}\,|(\rho(x)e_j^\rho, e_i^\rho)| \leq \sqrt{d_\rho}\,\|\rho(x)e_j^\rho\|\|e_i^\rho\| \leq \sqrt{d_\rho}, \tag{67}$$

so that $\varphi_{\rho,i,j} \in L^\infty(G)$. Therefore, if we put an arbitrary (total) order on $\hat{G}$ then Theorem 1 is directly applicable to (G, Σ, μ) with $\{\varphi_{\rho,i,j}\}$.

But the arbitrary choice of the bases $\{e_i^\rho\}_{i=1}^{d_\rho}$ is artificial from the viewpoint of the group G. The more natural construction is the operator valued Fourier transform,

$$\hat{f}(\rho) = \int_G f(x)\rho^*(x)dx, \quad \forall f \in L^1(G), \tag{68}$$

and the corresponding block Fourier series

$$\sum_{\rho\in\hat{G}} d_\rho \operatorname{tr}\left[\hat{f}(\rho)\rho(x)\right] = \sum_{\rho\in\hat{G}}\sum_{i,j=1}^{d_\rho} Y_{\rho,i,j}(x,f),$$
$$Y_{\rho,i,j}(x,f) = c_{\rho,i,j}(f)\varphi_{\rho,i,j}(x), \quad c_{\rho,i,j}(f) = (f,\varphi_{\rho,i,j})_2. \tag{69}$$

More generally, if we work on a homogeneous space $\mathcal{M} \simeq G/H$ of a compact group G as above with (closed) isotropy subgroup $H \subset G$, then we define multiplicities

$$d_\rho^H = \operatorname{mult}(\mathbf{1}, \rho\big|_H), \quad \forall \rho \in \hat{G} \tag{70}$$

(this reduces to $d_\rho^H = d_\rho$ if $H = \{\mathbf{1}\}$ as before). Moreover, we restrict to

$$\widehat{G/H} = \left\{\rho \in \hat{G}\middle| \quad d_\rho^H > 0\right\}. \tag{71}$$

A point $x \in G/H$ is a coset $x = \mathrm{x}H$, $\mathrm{x} \in G$. If dh is the normalized Haar measure on H then there is a unique normalized left G-invariant measure μ on G/H (the pullback of $d\mathrm{x}$ through the quotient map) such that

$$\int_G f(\mathrm{x})d\mathrm{x} = \int_{G/H}\left(\int_H f(\mathrm{x}h)dh\right)d\mu(x), \quad \forall f \in C(G). \tag{72}$$

Denote

$$\mathbb{P}_H \doteq \int_H \rho(h)dh, \quad \rho(x) \doteq \rho(\mathrm{x})\mathbb{P}_H, \quad \forall \rho \in \widehat{G/H}, \quad \forall x = \mathrm{x}H \in G/H. \tag{73}$$

Note that $d_\rho^H = \dim \mathbb{P}_H\mathcal{H}_\rho$. Let $\{e_i^\rho\}_{i=1}^{d_\rho}$ be an orthonormal basis in $\mathcal{H}_\rho$ as before, and choose an orthonormal basis $\{e_\alpha^\rho\}_{\alpha=1}^{d_\rho^H}$ in $\mathbb{P}_H\mathcal{H}_\rho$. Now the Fourier transform of a function $f \in L^1(G/H)$ becomes

$$\hat{f}(\rho) = \int_{G/H} f(x)\rho^*(x)d\mu(x), \quad \forall \rho \in \widehat{G/H}, \tag{74}$$

and the corresponding block Fourier series is

$$\sum_{\rho\in\widehat{G/H}} d_\rho \operatorname{tr}\left[\hat{f}(\rho)\rho(x)\right] = \sum_{\rho\in\widehat{G/H}}\sum_{i=1}^{d_\rho}\sum_{\alpha=1}^{d_\rho^H} Y_{\rho,i,\alpha}(x;f), \tag{75}$$

$$Y_{\rho,i,\alpha}(x;f) = c_{\rho,i,\alpha}(f)\varphi_{\rho,i,\alpha}(x), \quad c_{\rho,i,\alpha}(f) = (f,\varphi_{\rho,i,\alpha})_2, \quad \varphi_{\rho,i,\alpha}(x) = \sqrt{d_\rho}(\rho(x)e_\alpha^\rho, e_i^\rho). \tag{76}$$

If the homogeneous space G/H is infinite then the invariant measure μ is diffuse. Applying Theorem 1 to $(G/H, \Sigma_H, \mu)$ (Σ_H is the Borel σ-algebra on G/H) and the system $\{\varphi_{\rho,i,\alpha}\}$ with $\widehat{G/H}$ ordered arbitrarily, we obtain the following modification.

Theorem 4 *Let $\mathcal{M} = G/H$ be an infinite homogeneous space of a compact second countable Hausdorff group G with closed isotropy subgroup $H \subset G$. For every $\epsilon, \delta > 0$ there exists a measurable subset $E \in \Sigma_H$ with measure $|E| > 1 - \delta$ and with the following property; for each function $f \in L^1(G/H)$ with $\|f\|_1 > 0$ there exists an approximating function $g \in L^1(G/H)$ that satisfies:*

1. *$\|f - g\|_1 < \epsilon$,*
2. *$f = g$ on E,*
3. *the block Fourier series (75) of g converges in $L^1(G/H)$,*
4. *we have*

$$\sup_{\rho \in \widehat{G/H}} \left\| \sum_{\varrho \leq \rho} d_\varrho \operatorname{tr}\left[\hat{g}(\varrho)\varrho(x)\right] \right\|_1 < 2 \min\left\{\|f\|_1, \|g\|_1\right\}.$$

Note that E depends on the chosen order in $\widehat{G/H}$.

As discussed before, the proof of the above theorem becomes more constructive and transparent if we have a natural cylindric structure in G/H. Then the proof based on the cylindric structure becomes directly applicable (without intermediate measurable transformations), and each step in the proof retains its original interpretation in terms of cylindrical coordinates. To this avail, below we will establish a natural Borel almost isomorphism of measure spaces between G/H and the cylindric space $K \times K\backslash G/H$ for certain infinite closed subgroups $K \subset G$, which will provide an obvious identification of all spaces $L^p(G/H)$ and $L^p(K \times K\backslash G/H)$. Note that since K is infinite and compact, the probability space (K, dk) is isomorphic to the unit interval.

Definition 2 *A measurable map between two measure spaces $\varphi : (\Omega_1, \mu_1) \to (\Omega_2, \mu_2)$ is an almost isomorphism of measure spaces if there exist full measure subspaces $X \subset \Omega_1$ and $Y \subset \Omega_2$, $|\Omega_1 \setminus X| = |\Omega_2 \setminus Y| = 0$, such that the restriction of φ is an isomorphism of measure spaces $\varphi|_X : (X, \mu_1) \to (Y, \mu_2)$.*

If $K \subset G$ is a closed subgroup of G then denote by $G/H^{(K)} \subset G/H$ the set of those points $x \in G/H$ with a non-trivial stabilizer within K,

$$G/H^{(K)} = \{x \in G/H| \quad \exists\, \mathbf{1} \neq k \in K \quad \text{s.t.} \quad kx = x\}. \tag{77}$$

Let us now fix an infinite closed subgroup $K \subset G$, and let $\mathrm{q} : G/H \to K\backslash G/H$ be the natural quotient map. Then the pullback measure $\nu = \mu \circ \mathrm{q}^{-1} = \mathrm{q}^* \mu$ is the natural probability measure on $K\backslash G/H$. Provided that the subset $G/H^{(K)}$ in G/H is μ-null, we obtain a natural product structure in the following way.

Proposition 1 *If* $\left|G/H^{(K)}\right|_\mu = 0$ *then there exists a Borel almost isomorphism* $\varphi : K \times K\backslash G/H \to G/H$ *such that*

$$\varphi(k'k, Kx) = k'\varphi(k, Kx), \quad \mathrm{q}\,(\varphi(k, Kx)) = Kx, \quad \forall k, k' \in K, \quad \forall Kx \in K\backslash G/H. \tag{78}$$

Proof: Both G/H and $K\backslash G/H$ are compact Hausdorff second countable, hence metrizable by Urysohn's metrization theorem. The canonical quotient map $\mathrm{q} : G/H \to K\backslash G/H$ is a continuous surjection between compact metrizable spaces. By Federer-Morse theorem there exists a Borel subset $Z \subset G/H$ such that the restriction $\mathrm{q}\,|_Z : Z \to K\backslash G/H$ is a Borel isomorphism. Let $W \doteq Z \setminus G/H^{(K)}$ and $X = \mathrm{q}(W)$, so that $\mathrm{q}\,|_W : W \to X$ is a Borel isomorphism. Define $\varphi : K \times X \to G/H$ by setting

$$\varphi(k, Kx) \doteq k \cdot \mathrm{q}\,|_W^{-1}(Kx), \quad \forall k \in K, \quad \forall Kx \in X. \tag{79}$$

φ is Borel bi-measurable, since it is the composition of bi-measurable maps $(k, x) \mapsto k \cdot x$ and $\mathrm{q}\,|_W^{-1}$. The properties (78) are easily implied by the definition of φ. The map φ is also injective. Indeed, if $\varphi(k_1, Kx_1) = \varphi(k_2, Kx_2)$ then

$$\mathrm{q}(\varphi(k_1, Kx_1)) = Kx_1 = \mathrm{q}(\varphi(k_2, Kx_2)) = Kx_2, \tag{80}$$

and

$$\varphi(k_1, Kx_1) = k_1\varphi(\mathbf{1}, Kx_1) = \varphi(k_2, Kx_2) = k_2\varphi(\mathbf{1}, Kx_1), \tag{81}$$

so that $k_2^{-1}k_1\varphi(\mathbf{1}, Kx_1) = \varphi(\mathbf{1}, Kx_1)$. If $k_1 \neq k_2$ then $\varphi(\mathbf{1}, Kx_1)$ has a non-trivial stabilizer, that is, $\varphi(\mathbf{1}, Kx_1) \in G/H^{(K)}$. But $\varphi(\mathbf{1}, Kx_1) = \mathrm{q}\,|_W^{-1}(Kx_1) \in W$ and $W \cap G/H^{(K)} = \emptyset$, which is a contradiction. Thus $k_1 = k_2$ and the injectivity is proven. Denoting $Y \doteq \varphi(X) \subset G/H$ we see that $\varphi : X \to Y$ is a Borel isomorphism.

For every $f \in C(G/H)$, by measure disintegration theorem, we have

$$\int_{G/H} f(x)d\mu(x) = \int_{K\backslash G/H} d\nu(Kx) \int_K f(kx)dk. \tag{82}$$

Let χ_X and χ_Y be the indicator functions of the subsets X and Y, respectively. Since $K \cdot Y \subset Y$ we have that $\chi_Y(x) = \chi_X(Kx)$. It follows that

$$\int_Y f(x)d\mu(x) = \int_{G/H} f(x)\chi_Y(x)d\mu(x) = \int_{K\backslash G/H} d\nu(Kx) \int_K f(kx)\chi_Y(kx)dk = \int_{K\backslash G/H} \chi_X(Kx)d\nu(Kx) \int_K f(kx)dk = \int_X d\nu(Kx) \int_K f(\varphi(k, Kx))dk, \tag{83}$$

which shows that $\varphi : (X, dk \otimes \nu) \to (Y, \mu)$ is a measure space isomorphism.

Finally, let us note that $K \cdot Z = G/H$. Indeed, for every $x \in G/H$ we have that $z = \mathrm{q}\,|_Z^{-1}(\mathrm{q}(x)) \in Z$, and since $\mathrm{q}(x) = \mathrm{q}(z)$ we have that $\exists k \in K$ such that $kz = x$. On the other hand, it is easy to see that the subset $G/H^{(K)} \subset G/H$ is left K-invariant, for if $x \in G/H^{(K)}$ with $k_x \in K$ such that $k_x x = x$ then for every $k \in K$ it follows that $k_y y = y$, where $y = kx$ and $k_y = k_0 k_x k_0^{-1}$, which means that $y \in G/H^{(K)}$. Therefore

$$|G/H \setminus Y|_\mu = |K \cdot Z \setminus K \cdot W|_\mu = |K \cdot (Z \setminus W)|_\mu \leq \left|K \cdot G/H^{(K)}\right|_\mu = \left|G/H^{(K)}\right|_\mu = 0, \tag{84}$$

so that $|Y|_\mu = |X|_{dk\otimes\nu} = 1$. This completes the proof. □

Spheres

As an instructive illustration of the above constructions we will consider spheres $\mathbb{S}^d$, $2 \leq d \in \mathbb{N}$, with their Euclidean (Lebesgue) probability measures (surface area normalized to one). For $d = 2$ the Statements 2 and 3 of Theorem 1 were obtained in [4].

The sphere $\mathbb{S}^d$ can be considered as the homogeneous space G/H with $G = \mathrm{SO}(d+1)$ and $H = \mathrm{SO}(d)$. Harmonic analysis in these homogeneous spaces is a classical subject widely available in the literature (see e.g. [21]). The dual space $\widehat{G/H}$ consists of irreducible representations by harmonic polynomials of fixed degree $\rho \in \mathbb{N}_0$, and it is conveniently ordered according to that degree, $\rho \in \widehat{G/H} \simeq \mathbb{N}_0$. The dimension of the representation ρ is

$$d_\rho = \dim \mathcal{H}_\rho = \binom{d+\rho}{d} - \binom{d+\rho-2}{d}, \quad \forall \rho \in \mathbb{N}_0, \tag{85}$$

whereas the multiplicities are all $d_\rho^H = 1$. We choose standard spherical coordinates $x = (\theta_1, \ldots, \theta_{d-1}, \phi)$, where $\theta_j \in [0, \pi]$, $j = 1, \ldots, d-1$, and $\phi \in [0, 2\pi)$. The orthonormal system $\{\varphi_{\rho,i,\alpha}\}$ in this case consists of spherical harmonics

$$\varphi_{\rho,i,\alpha}(x) = Y_\rho^i(\theta_1, \ldots, \theta_{d-1}, \phi), \quad \forall \rho \in \mathbb{N}_0, \quad i = 1, \ldots, d_\rho. \tag{86}$$

The block Fourier series of a function $f \in L^1(\mathbb{S}^d)$ is

$$\sum_{\rho=1}^{\infty} \sum_{i=1}^{d_\rho} \hat{f}(\rho; i) Y_\rho^i(\theta_1, \ldots, \theta_{d-1}, \phi), \tag{87}$$

$$\hat{f}(\rho; i) = \int_0^{2\pi} \int_0^{\pi} \ldots \int_0^{\pi} f(\theta_1, \ldots, \theta_{d-1}, \phi) \bar{Y}_\rho^i(\theta_1, \ldots, \theta_{d-1}, \phi) d\mu(\theta_1, \ldots, \theta_{d-1}, \phi), \tag{88}$$

$$d\mu(\theta_1, \ldots, \theta_{d-1}, \phi) = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{\frac{d+1}{2}}} \sin^{d-1}(\theta_1) d\theta_1 \ldots \sin(\theta_{d-1}) d\theta_{d-1} d\phi. \tag{89}$$

A natural cylindric structure is obtained by choosing $K = \mathrm{SO}(2)$, the circle subgroup responsible for rotation in the longitudinal variable ϕ. The subset $G/H^{(K)}$ here contains only the two poles - $\theta_j = 0$, $j = 1, \ldots, d-1$, and $\theta_j = \pi$, $j = 1, \ldots, d-1$, respectively. Thus indeed $\left|G/H^{(K)}\right|_\mu = 0$, and Proposition 1 applies. The section $Z \subset G/H$ appearing in the proof of Proposition 1 can be chosen to correspond to the meridian $\phi = 0$ in $\mathbb{S}$, which is Borel isomorphic to $\mathrm{SO}(2)\backslash\mathrm{SO}(d+1)/\mathrm{SO}(d)$ through the quotient map q. In this way we have the almost isomorphism

$$\varphi : \mathrm{SO}(2) \times \mathrm{SO}(2)\backslash\mathrm{SO}(d+1)/\mathrm{SO}(d) \to \mathbb{S}^d \tag{90}$$

given by $\varphi(\phi, (\theta_1, \ldots, \theta_{d-1})) = (\theta_1, \ldots, \theta_{d-1}, \phi)$, i.e., simply by separation of the variable ϕ. As a final step we parameterize $\mathrm{SO}(2) = \mathbb{S}^1$ by $t = \phi/2\pi$ to obtain an almost isomorphism

$$[0, 1] \times \mathrm{SO}(2)\backslash\mathrm{SO}(d+1)/\mathrm{SO}(d) \to \mathbb{S}^d. \tag{91}$$

This is the cylindric structure used implicitly in [4].

Construction of E and g

The proof of the main theorem above is constructive, although the construction of the set E and of the approximating function g may be hard to follow due to the complexity of the proof. In this last section we will very briefly sketch that construction step by step.

- Choose an arbitrary ordering $\{R_k\}_{k=1}^{\infty}$ of all Fourier polynomials with rational coefficients.
- For every $k \in \mathbb{N}$, choose a partition $\{\Delta_l(k)\}_{l=1}^{\nu_0(k)}$ of the cylindric measure space $\mathcal{M} = [0,1] \times \mathcal{N}$ of the form $\Delta_l(k) = [a_l(k), b_l(k)] \times \tilde{\Delta}_l(k)$ such that the measures $|\Delta_l(k)|$ are small enough, as well as a subordinate real step function $\Lambda(k) = \sum_{l=1}^{\nu_0(k)} \gamma_l(k)\chi_{\Delta_l(k)}$ (χ_X is the indicator function of the subset X), such that $\|\Lambda(k) - R_k\|_1$ is sufficiently small.
- For every $k \in \mathbb{N}$, choose a number $\delta_*(k) \in (0, \frac{1}{2})$ so that $\{\delta_*(k)\}_{k=1}^{\infty}$ decays sufficiently rapidly. Define the periodic step function

  $$I(t) = 1 - \frac{1}{\delta_*(k)}\chi_{[0,\delta_*(k))}(t \mod 1),$$

  and the measurable function $\hat{g}_l^k \in L^{\infty}(\mathcal{M})$ by

  $$\hat{g}_l^k(x) = \gamma_l(k) I(s_0(k)t)\chi_{\Delta_l(k)}(x), \quad \forall x \in \mathcal{M},$$

  where the positive number $s_0(k)$ is sufficiently large. Define the measurable subsets $\hat{E}_l(k) \subset \Delta_l(k)$ by

  $$\hat{E}_l(k) = \left\{x \in \Delta_l(k) \middle| \quad \hat{g}_l^k(x) = \gamma_l(k)\right\}.$$

  Define inductively the natural numbers $\hat{N}_l(k)$, $l = 0, \ldots, \nu_0(k)$ and Fourier polynomials $\hat{Q}_l^k$, $l = 1, \ldots, \nu_0(k)$, by setting $\hat{N}_0(1) = 1$, $\hat{N}_0(k) = \hat{N}_{\nu_0(k)}(k-1)$ for $k > 1$, and $\hat{Q}_l^k = \sum_{n=\hat{N}_{l-1}(k)}^{\hat{N}_l(k)-1} Y_n(\hat{g}_l^k)$ (here $\{Y_n(g)\}_{n=1}^{\infty}$ is the Fourier series of the function $g \in L^2(\mathcal{M})$), so that the quantities

  $$\left\| \sum_{n=1}^{\hat{N}_l(k)-1} Y_n(\hat{g}_l^k) - \hat{g}_l^k \right\|_2$$

  are sufficiently small.
- For every $k \in \mathbb{N}$, define the natural numbers $N_k = \hat{N}_{\nu_0(k)} - 1$, measurable subsets $\tilde{E}_k =$

$\bigcup_{l=1}^{\nu_0(k)} \hat{E}_l(k)$, and measurable functions $\tilde{g}_k \in L^\infty(\mathcal{M})$ by

$$\tilde{g}_k = R_k - \Lambda(k) + \sum_{l=1}^{\nu_0(k)} \hat{g}_l^k,$$

as well as Fourier polynomials

$$\tilde{Q}_k = \sum_{l=1}^{\nu_0(k)} \hat{Q}_l^k = \sum_{n=N_{k-1}}^{N_k - 1} \tilde{Y}_n.$$

- Set
$$E = \bigcap_{k=1}^{\infty} \tilde{E}_k.$$

- Choose by Lemma 4 a subsequence $\{R_{k_s}\}_{s=0}^\infty$ such that $\|R_{k_s}\|_1$ decay sufficiently rapidly, and $\sum_{s=0}^\infty R_{k_s} = f$ in $L^1(\mathcal{M})$.

- Define inductively the sequence of natural numbers $\{\nu_s\}_{s=1}^\infty$, $\nu_s > \nu_{s-1}$ for $s > 1$, and measurable functions $g_s \in L^\infty(\mathcal{M})$ by choosing ν_1 so that $N_{\nu_1 - 1} > \max \sigma(R_{k_0})$ and
$$\left\| R_{\nu_s} - R_{k_s} + \sum_{j=1}^{s-1} \left[\tilde{Q}_{\nu_j} - g_j \right] \right\|_1$$
is sufficiently small, and setting $g_s = R_{ks} + \tilde{g}_{\nu_s} - R_{\nu_s}$.

- Finally, set
$$g = R_{k_0} + \sum_{s=1}^{\infty} g_s.$$

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