An exotic presentation of Q_{28}

W.H. MANNAN
TOMASZ POPIEL

We introduce a new family of presentations for the quaternion groups and show that for the quaternion group of order 28, one of these presentations has non-standard second homotopy group.

57M05, 57M20; 20C05, 16S34, 20C10, 55P15, 55Q91, 55N25

1 Introduction

Since the work of Johnson [16, 17] and Beyl and Waller [3, 4] in the early 2000's, the hunt has been on to find out if a finite balanced presentation of a quaternion group Q_{4n} can have non-standard second homotopy group. This has largely been fuelled by the connection to Wall's famous $D(2)$ problem [17]. However until the present work, for each quaternion group Q_{4n}, all known presentations have had second homotopy group IQ_{4n}^*, the dual of the augmentation ideal, and it was conjectured that anything else would be impossible.

We show that such a presentation is in fact possible. That is, the purpose of the present work is to introduce a new family of presentations for Q_{4n}, and to show that in the case $n = 7$, one (at least) of these presentations has a non-standard second homotopy group:

Theorem A. *We have a presentation for the quaternion group Q_{28}:*

$$\mathcal{P}' = \langle x, y \,|\, y^2 = x^7, \quad y^{-1}xyx^2 = x^3y^{-1}x^2y \rangle,$$

which has a non-standard second homotopy group. That is, if $X_{\mathcal{P}'}$ is the Cayley complex associated to $\mathcal{P}'$ and $X_{\mathcal{P}}$ is the Cayley complex associated to the standard presentation:

$$\mathcal{P} = \langle x, y \,|\, y^2 = x^7, \quad xyx = y \rangle,$$

then $\pi_2(X_{\mathcal{P}'}) \ncong \pi_2(X_{\mathcal{P}})$ as modules over $\mathbb{Z}[Q_{28}]$.

Note that whilst $\pi_2(X_{\mathcal{P}})$ is known to be generated by a single element over $\mathbb{Z}[Q_{28}]$, we will show that $\pi_2(X_{\mathcal{P}'})$ is not. Thus when we say that $\pi_2(X_{\mathcal{P}'}) \ncong \pi_2(X_{\mathcal{P}})$, we mean

Published: XX Xxxember 20XX DOI: 10.2140/agt.20XX.XX.1001

that this holds with respect to all identifications of the groups presented by $\mathcal{P}$ and $\mathcal{P}'$. Therefore $X_{\mathcal{P}}$ and $X_{\mathcal{P}'}$ are not homotopy equivalent.

In fact we will show that $X_{\mathcal{P}'}$ has the same second homotopy group as a 3-complex constructed by Beyl and Waller [3], sometimes referred to in the community as Nancy's Toy. They state [3, p.908] that such an $X_{\mathcal{P}'}$ if it exists will not be homotopy equivalent to the spine of a closed 3–manifold. We thank J. Nicholson for pointing out that our $\mathcal{P}'$ thus resolves the question of whether all finite balanced presentations of closed 3–manifold fundamental groups are homotopy equivalent to such spines.

Another application is given by [5, Proposition 5.5], where our construction is used to present non-homotopy equivalent manifolds in dimensions 4 and above, which become diffeomorphic under stabilisation by taking the connected sum with a product of spheres.

Given a presentation $\mathcal{Q}$ for a group G, let $X_{\mathcal{Q}}$ denote its Cayley complex. By the second homotopy group of $\mathcal{Q}$ we refer to the $\mathbb{Z}[G]$ module with underlying abelian group $\pi_2(X_{\mathcal{Q}})$ with natural (right) G-action arising intuitively from stretching elements of $\pi_2(X_{\mathcal{Q}})$ back along loops in $G = \pi_1(X_{\mathcal{Q}})$.

It is non-trivial to construct finite presentations $\mathcal{Q}, \mathcal{Q}'$ of the same group G, with the same deficiency (number of generators minus number of relators) but with different second homotopy groups. In particular, the Hurewicz homomorphism identifies $\pi_2(X_{\mathcal{Q}}) \cong H_2(\widetilde{X_{\mathcal{Q}}})$, and Schanuel's lemma then implies that:

$$\pi_2(X_{\mathcal{Q}}) \oplus F \cong \pi_2(X_{\mathcal{Q}'}) \oplus F,$$

for some free finitely generated module F over $\mathbb{Z}[G]$.

In other words, we require non-cancellation of free modules over $\mathbb{Z}[G]$. Note that in the case of finite groups, we have cancellation over $\mathbb{Q}[G]$ for all finitely generated modules. Thus distinguishing $\pi_2(X_{\mathcal{Q}})$ from $\pi_2(X_{\mathcal{Q}'})$ requires subtle number theoretic considerations.

None the less it has been achieved [12, § 1.7]. For the trefoil group, Lustig, building on the work of Dunwoody and Berridge produced infinitely many presentations with the same deficiency but pairwise distinct second homotopy groups [1, 7, 20]. For finite groups, homotopically distinct presentations with the same deficiency were found by Metzler for ${C_5}^3$ [28, 18, p.105]. Linnell [19, Corollary 1.4(iii) and (iv)] clarified the situation for second homotopy groups: for p a prime satisfying $p \equiv 1 \mod 4$ there are precisely two homotopy types of presentation for ${C_p}^3$, but they have isomorphic second homotopy groups. On the other hand for p a prime satisfying $p \equiv 1 \mod 6$, we have three homotopically distinct presentations of ${C_p}^4$ and they all have non-isomorphic second homotopy groups (with respect to any identification of the presented groups).

The case of quaternion groups has been the subject of much analysis [16, 17, 3, 4], largely because of its relation to Wall's $D(2)$ problem. In 1965 Wall showed that for $n > 2$, if a finite cell complex is cohomologically n dimensional (in the sense of having no non-trivial cohomology in dimensions above n with respect to any coefficient bundle), then it is in fact homotopy equivalent to an actual n dimensional cell complex [37]. Subsequently it was shown by Swan and Stallings that the only cohomologically 1 dimensional finite cell complexes are disjoint unions of wedges of circles [34, 35]. However decades later the case $n = 2$ remains a major open problem, known as Wall's $D(2)$ problem.

A $D(2)$–complex is a finite (connected) 3 dimensional cell complex Y, with no cohomology above dimension 2. To solve the problem one would need to produce a $D(2)$–complex which was not homotopy equivalent to a finite 2-complex (or show that this cannot be done). Note that without loss of generality such a 2-complex is (the Cayley complex of) a finite presentation of $\pi_1(Y)$. The existence of such a space Y with a particular fundamental group is equivalent to there being an algebraic 2–complex over the group which is not geometrically realisable [15, 17, 22, 23]. Using this it has been show that such a space Y cannot have certain fundamental groups, such as cyclic groups, products of the form $C_\infty \times C_n$ [8] or dihedral groups [14, 17, 21, 33, 26, 11].

On the other hand $D(2)$–complexes have been produced and conjectured to not be homotopy equivalent to any finite presentation of their fundamental group. Broadly these spaces fall into two categories:

(1) Those where it is conjectured that there is no finite presentation of their fundamental group with the same Euler characteristic.

(2) Those where there are finite presentations of their fundamental group with the same Euler characteristic, but it is conjectured that none of them have the same second homotopy module.

A third less explored option would be to prohibit a finite presentation based on k-invariants (see [17, Chapter 6]), rather than Euler characteristic or second homotopy group.

Many spaces falling into the first category have been proposed [6, 10, 27]. To actually verify that there is no presentation with sufficiently low Euler characteristic will require a fundamentally novel obstruction. Ideas from geometric group theory and algebraic geometry [25] have been mooted.

A quintessential example of a space that fell into the first category had fundamental group a free product of several $C_p \times C_p$ as p ranged over distinct primes. However it was shown that presentations of these groups with sufficiently low Euler characteristics did indeed exist [13].

As has been mentioned, fundamental groups of spaces falling into the second category require a certain failure in the cancellation of free modules. Although not necessary, the most prominent examples proposed with finite fundamental group are those where cancellation fails even within the stable class of free modules. From the Swan–Jacobinski Theorem [17, § 15] we know that such groups must necessarily have a binary polyhedral group as a quotient (see [29] for more detailed analysis of which groups this failure of cancellation occurs over).

This makes it natural to look at the binary polyhedral groups themselves. Swan showed that the binary polyhedral groups where cancellation fails in the stable class of free modules are precisely Q_{4n} for $n \geq 6$ [36, Theorem I].

Based on this work, spaces were constructed which fell into the second category with fundamental group Q_{2^k} with $k \geq 5$ [16] and fundamental group Q_{28} [3, 4]. That is, their second homotopy group was not IQ_{4n}^*, and it was conjectured that no finite presentation of Q_{4n} would have a second homotopy group other than IQ_{4n}^*.

We prove this conjecture false, by displaying a finite presentation with a second homotopy group different to IQ_{4n}^*. In fact, based on our result Nicholson has shown that there are no solutions to Wall's $D(2)$ problem with fundamental group Q_{28} [30, Theorem 7.7]. That is any $D(2)$–complex with fundamental group Q_{28} having minimal Euler characteristic is either homotopy equivalent to $\mathcal{P}$ or $\mathcal{P}'$. It is worth noting that this means that finite presentations have now been found which defy the relevant conjectures for quintessential examples of spaces which fell into both the first and second category. It is worth then considering the possibility that finite presentations can always be found, homotopy equivalent to a given $D(2)$–complex.

Note that a finite presentation of a group (possibly not the fundamental group of the 3–complex) may always be found, so that applying Quillen's plus construction results in a space homotopy equivalent to the 3–complex [24]. On the other hand a famous result of Bestvina and Brady yields a similar situation where there is no finite presentation of the group at all [2].

Broadly, the prevailing opinion is that an example from category 1 or 2, will succeed in not being homotopy equivalent to a finite presentation. The present work is not sufficient to alter that prevailing opinion, but it does draw attention to the possibility.

Acknowledgments We acknowledge that the present work is built on the foundations laid by F.E.A. Johnson, F. Rudolf Beyl, and the late Nancy Waller, who is greatly missed. We would also like to thank the National Science Foundation for award DMS-0918418 which allowed the first author to meet two of these key players in the field. Also Johnny Nicholson has made several valuable observations since the first draft of this work was produced, in addition to extending the work in various directions in his own articles. Finally we thank the referee for diligent scrutiny and helpful suggestions.

2 The standard presentation

Let Q_{4n} denote the quaternion group with standard presentation:

$$\mathcal{P} = \langle x, y \,|\, y^2 = x^n,\ y = xyx \rangle.$$

Let $X_{\mathcal{P}}$ denote the Cayley complex of this presentation (where relations $a = b$ are interpreted as relators $a^{-1}b$). The edges in $X_{\mathcal{P}}$ corresponding to x, y may be lifted to edges in $\widetilde{X_{\mathcal{P}}}$, represented by generators $e_{\mathbf{1}}, e_{\mathbf{2}} \in C_1(\widetilde{X_{\mathcal{P}}})$ respectively. Similarly the two disks in $X_{\mathcal{P}}$ corresponding to the two relations in $\mathcal{P}$ may be lifted to disks in $\widetilde{X_{\mathcal{P}}}$, represented by generators $E_{\mathbf{1}}, E_{\mathbf{2}} \in C_2(\widetilde{X_{\mathcal{P}}})$ respectively.

Then $\pi_2(X_{\mathcal{P}})$ is a (right) module over $\mathbb{Z}[\pi_1(X_{\mathcal{P}})] = \mathbb{Z}[Q_{4n}]$. Further $\pi_2(X_{\mathcal{P}})$ may be identified via the Hurewicz isomorphism (as modules over $\mathbb{Z}[Q_{4n}]$), with the kernel of the boundary map:

$$\partial_2\colon C_2(\widetilde{X_{\mathcal{P}}}) \to C_1(\widetilde{X_{\mathcal{P}}}).$$

We may describe the boundary map ∂_2 explicitly as follows:

$$\begin{aligned}
\partial_2\colon E_{\mathbf{1}} &\mapsto e_{\mathbf{1}}\partial_x(y^{-2}x^n) + e_{\mathbf{2}}\partial_y(y^{-2}x^n) &&= e_{\mathbf{1}}\sigma_x - e_{\mathbf{2}}(1+y),\\
\partial_2\colon E_{\mathbf{2}} &\mapsto e_{\mathbf{1}}\partial_x(y^{-1}xyx) + e_{\mathbf{2}}\partial_y(y^{-1}xyx) &&= e_{\mathbf{1}}(1+yx) + e_{\mathbf{2}}(x-1).
\end{aligned}$$

Here ∂_x, ∂_y denote the free Fox derivative with respect to x, y respectively [9] and σ_x denotes the group ring element $1 + x + x^2 + x^3 + \cdots + x^{n-1}$.

For proofs of the following see for example [3, Lemma 4.2] or [16]. The module $\pi_2(X_{\mathcal{P}}) = \ker(\partial_2)$ is generated by:

$$u = E_{\mathbf{1}}(x-1) + E_{\mathbf{2}}(1-yx). \tag{1}$$

Further, the annihilator of u is precisely $\Sigma_G\mathbb{Z}[Q_{4n}]$, where Σ_G denotes the sum of all group elements in Q_{4n}. Letting IG^* denote the module $\mathbb{Z}[Q_{4n}]/\Sigma_G\mathbb{Z}[Q_{4n}]$, we may conclude that

$$\pi_2(X_{\mathcal{P}}) = u\mathbb{Z}[Q_{4n}] \cong IG^*. \tag{2}$$

3 The new presentations

We now describe a new family of presentations $\mathcal{E}_{n,r}$, where the parameter r is an integer:

$$\mathcal{E}_{n,r} = \langle x, y \mid y^2 = x^n, \quad y^{-1}xyx^{r-1} = x^r y^{-1} x^2 y \rangle.$$

Clearly Q_{4n} is a quotient of the group presented by $\mathcal{E}_{n,r}$, for any $r \in \mathbb{Z}$, as both relations hold for the standard generators $x, y \in Q_{4n}$. In particular:

$$y^{-1}xyx^{r-1} = x^{-1}x^{r-1} = x^r x^{-2} = x^r y^{-1} x^2 y \tag{3}$$

However $\mathcal{E}_{n,r}$ need not be a presentation for Q_{4n}. If we specialize to $r = 3$ though, it is a presentation for Q_{4n}, as we shall see.

Lemma 3.1 *Let $a, b \in G$ for some group G satisfy:*

$$ab^2 = b^3a^2, \tag{4}$$

$$ba^2 = a^3b^2. \tag{5}$$

Then $ba = 1$.

Proof Multiplying (4) through by a^2 on the left we get:

$$a^2b^3a^2 = a^3b^2 = ba^2,$$

from (5). Thus $a^2b^2 = 1$ so $b^2a^2 = 1$ and (5) reduces to $b^{-1} = a$. □

Lemma 3.2 *The presentation $\mathcal{E}_{n,3}$ presents Q_{4n} for all $n \geq 2$.*

Proof In the light of (3) we know that any relation satisfied by x, y in $\mathcal{E}_{n,3}$, is also satisfied in $\mathcal{P}$. It remains to show that in the group presented by $\mathcal{E}_{n,3}$, the following identity holds:

$$y = xyx$$

Let $a = y^{-1}xy, b = x$. From the second relation in $\mathcal{E}_{n,3}$ we have that $ab^2 = b^3a^2$. As $y^2 = x^n$ we know that y^2 is central and conjugating the second relation in $\mathcal{E}_{n,3}$ by y, we get $ba^2 = a^3b^2$. Thus Lemma 3.1 tells us that $ba = 1$. That is $xy^{-1}xy = 1$ and $xyx = y$. □

Let $\mathcal{P}'$ denote $\mathcal{E}_{7,3}$. The remainder of this article will be devoted to showing that $\pi_2(X_{\mathcal{P}'}) \ncong IG^*$, in the case $n = 7$. However we briefly pause to consider other possible presentations $\mathcal{E}_{n,r}$ for Q_{4n}. Computations in Magma suggest that $\mathcal{E}_{n,r}$ is frequently a presentation of Q_{4n}. This has been the case for every value of n, r that we have tried where either $r \not\equiv 2 \mod 3$, or $3 \nmid n$. We provide one further result in that direction.

Lemma 3.3 *Let $a, b \in G$ for some group G satisfy:*

$$\begin{aligned} ab &= b^2a^2, \\ ba &= a^2b^2, \\ a^n &= b^n, \end{aligned}$$

where $3 \nmid n$. Then $ba = 1$.

Proof We have $a^3b = a^2(ab) = a^2b^2a^2 = (ba)a^2 = ba^3$. Thus a^3 is central and so is a^n. As $3, n$ are coprime we have that a is central. Thus $ba = 1$ follows from either of the first two equations. □

Lemma 3.4 *The presentation $\mathcal{E}_{n,2}$ presents Q_{4n} for all $n \geq 2$ with $3 \nmid n$.*

Proof Again we need only show that:

$$y = xyx$$

holds in the group with presentation $\mathcal{E}_{n,2}$. Again let $a = y^{-1}xy, b = x$. From the second relation in $\mathcal{E}_{n,2}$ we have that $ab = b^2a^2$. As $y^2 = x^n$ we know that y^2 is central and conjugating the second relation in $\mathcal{E}_{n,2}$ by y, we get $ba = a^2b^2$. Clearly $a^n = b^n$, so Lemma 3.3 tells us that $ba = 1$. That is $xy^{-1}xy = 1$ and $xyx = y$. □

4 Computing $\pi_2(X_{\mathcal{P}'})$

From now on we fix $n = 7$ and we wish to show that $\pi_2(X_{\mathcal{P}'}) \ncong \pi_2(X_{\mathcal{P}})$. In this section we will describe $\pi_2(X_{\mathcal{P}'})$ as a submodule of IG^* with explicit generators. Then in §5 we will decompose $\pi_2(X_{\mathcal{P}'})$ via Milnor squares, to show that it is indeed not $\pi_2(X_{\mathcal{P}})$.

First note that multiplying both sides of a relation by the same generator on the same side does not alter the homotopy type of the associated Cayley complex.

Thus replacing the second relation in $\mathcal{P}'$ with any of the following, results in homotopy equivalent Cayley complexes:

$$\begin{aligned}
y^{-1}xyx^2 &= x^3y^{-1}x^2y, && \text{Original relation}\\
x^{-3}(y^{-1}xyx)x^3x^{-2} &= y^{-1}x^2y, && \text{Multiplying on left by } x^{-3}\\
x^{-3}(y^{-1}xyx)x^3 &= y^{-1}x^2yx^2, && \text{Multiplying on right by } x^2\\
x^{-3}(y^{-1}xyx)x^3 &= (y^{-1}xyx)(x^{-1}(y^{-1}xyx)x). && \text{Rebracketing}
\end{aligned}$$

Now let R denote the word $(y^{-1}xyx)$. The last relation then becomes $x^{-3}Rx^3 = R(x^{-1}Rx)$.

Then $\pi_2(X_{\mathcal{P}'})$ may be identified with the kernel of the boundary map ∂_2' associated to the presentation:

$$\langle x, y \,|\, y^2 = x^7,\ R(x^{-1}Rx) = x^{-3}Rx^3 \rangle.$$

Let $F_{\mathbf{1}}, F_{\mathbf{2}}$ denote the generators corresponding to these two relations.

For a general group presentation containing relators $R_1, \cdots, R_n$, integers $s_1, \cdots, s_n$, words $w_1, \cdots, w_n$ in the generators, and a generator t:

$$\partial_t\left((w_1^{-1}R_1^{s_1}w_1)\cdots(w_n^{-1}R_n^{s_n}w_n)\right) = (\partial_t R_1)s_1w_1 + \cdots + (\partial_t R_n)s_nw_n$$

Thus we have:

$$\begin{aligned}
\partial_x\left((x^{-1}R^{-1}x)R^{-1}(x^{-3}Rx^3)\right) &= (\partial_x R)(x^3 - x - 1),\\
\partial_y\left((x^{-1}R^{-1}x)R^{-1}(x^{-3}Rx^3)\right) &= (\partial_y R)(x^3 - x - 1).
\end{aligned}$$

We may describe ∂_2' explicitly:

$$\begin{aligned}
&\partial_2' \colon F_{\mathbf{1}} \mapsto \partial_2 E_{\mathbf{1}},\\
&\partial_2' \colon F_{\mathbf{2}} \mapsto \partial_2 E_{\mathbf{2}}(x^3 - x - 1).
\end{aligned}$$

Thus given any $F_{\mathbf{1}}a + F_{\mathbf{2}}b \in \ker(\partial_2')$, for $a, b \in \mathbb{Z}[Q_{28}]$, we have:

$$E_{\mathbf{1}}a + E_{\mathbf{2}}(x^3 - x - 1)b = u\gamma, \tag{6}$$

for some unique $\gamma \in IG^*$.

We will show that the right annihilator of $x^3 - x - 1$ is $\{0\}$, so in fact γ determines a, b.

Lemma 4.1 *In the ring $\mathbb{Z}[x]/(x^{14} - 1)$, the ideal generated by $x^3 - x - 1$ contains 4.*

Proof Dividing $x^{14}-1$ by x^3-x-1 leaves a remainder of

$$\alpha_1 = 12x^2+16x+8.$$

Let

$$\begin{aligned}\alpha_2 &= \alpha_1 x-(x^3-x-1)12 = 16x^2+20x+12,\\ \alpha_3 &= \alpha_2 x-(x^3-x-1)16 = 20x^2+28x+16.\end{aligned}$$

Thus $\alpha_1,\alpha_2,\alpha_3$ are divisible by x^3-x-1 in the ring $\mathbb{Z}[x]/(x^{14}-1)$. Finally note $\alpha_3+\alpha_2-\alpha_1 3=4$. □

Thus we have an element $p\in\mathbb{Z}[x]/(x^{14}-1)\subset\mathbb{Z}[Q_{28}]$ satisfying $p(x^3-x-1)=4$. If $(x^3-x-1)b=0$ for some $b\in\mathbb{Z}[Q_{28}]$ then $p(x^3-x-1)b=0$ so $4b=0$ and $b=0$. Thus we can conclude that the right annihilator of x^3-x-1 is indeed $\{0\}$.

Lemma 4.2 *We have*

$$\pi_2(X_{\mathcal{P}'})\cong\{\gamma\in IG^*|\ \exists b\in\mathbb{Z}[Q_{28}]|\ (1-yx)\gamma=(x^3-x-1)b\}.$$

In other words, we have that $\pi_2(X_{\mathcal{P}'})$ is the kernel of the homomorphism

$$IG^*\to\mathbb{Z}[Q_{28}]/(x^3-x-1)\mathbb{Z}[Q_{28}],$$

mapping $1\mapsto 1-yx$.

Proof We have identified $\pi_2(X_{\mathcal{P}'})$ with the kernel of ∂_2', which consists of elements F_1a+F_2b, with $a,b\in\mathbb{Z}[Q_{28}]$ satisfying (6), for some $\gamma\in IG^*$. From (2) we know that if this condition is satisfied for some γ, then it is unique.

Conversely given γ satisfying (6), for some $a,b\in\mathbb{Z}[Q_{28}]$, we know that

$$a=(x-1)\gamma, \tag{7}$$

$$(x^3-x-1)b=(1-yx)\gamma. \tag{8}$$

As the right annihilator of (x^3-x-1) is $\{0\}$, we know that there is a unique $F_1a+F_2b\in\pi_2(X_{\mathcal{P}'})$, for which a,b satisfy (6).

Thus $\pi_2(X_{\mathcal{P}'})$ may be identified with the set of $\gamma\in IG^*$, satisfying (8), for some $b\in\mathbb{Z}[Q_{28}]$. □

We next seek to better understand the module:

$$M=\mathbb{Z}[Q_{28}]/(x^3-x-1)\mathbb{Z}[Q_{28}].$$

From Lemma 4.1 we know that any element of M may be written in the form $a_0+a_1x+a_2x^2+(a_3+a_4x+a_5x^2)y$, with the $a_i \in \{0,1,2,3\}$. Let $A = \mathbb{Z}_4[x]/(x^3-x-1)$. Note that in A, we have $x(x^2-1) = 1$, so x is invertible.

Lemma 4.3 *We have a well defined $\mathbb{Z}[Q_{28}]$ module $A \oplus A$ with $\mathbb{Z}[Q_{28}]$ action given by*:

$$\begin{aligned}(a,b)y &= (bx^7, a)\\(a,b)x &= (ax, bx^{-1})\end{aligned}$$

for all $a, b \in A$.

Proof For any $\mathbb{Z}[x]/(x^{14}-1)$ module A', the above defines a $\mathbb{Z}[Q_{28}]$ action on $A' \oplus A'$ as direct application of x^7, y^2, xyx, y demonstrates that the given action respects the identities $x^7 = y^2, xyx = y$. It thus suffices to show that x^{14} acts trivially on A. We may verify this immediately by recalling from the proof of Lemma 4.1 that:

$$x^{14} - 1 = (x^3 - x - 1)q + 4(3x^2 + 4x + 2)$$

for some polynomial q in x with integer coefficients. □

Lemma 4.4 *We have an isomorphism of $\mathbb{Z}[Q_{28}]$ modules $M \cong A \oplus A$.*

Proof The homomorphism $A \oplus A \to M$ mapping $(a,b) \mapsto a + by$ has inverse $M \to A \oplus A$, mapping $1 \mapsto (1,0)$. □

Lemma 4.2 identifies $\pi_2(X_{\mathcal{P}'})$ with the kernel of the map $\psi\colon IG^* \to M$, mapping $1 \mapsto 1 - yx$. Let

$$\begin{aligned}\phi_1 &= x^6 + x^5 - x^4 - 3x^3 - x^2 + x + 1,\\\phi_2 &= 2 + 2x - x^3 + x^3y.\end{aligned}$$

Lemma 4.5 *We have $4, \phi_1, \phi_2 \in \pi_2(X_{\mathcal{P}'})$.*

Proof Clearly $4 \in \ker\,\psi$. Also $(x^3 - x - 1)(x^{-3} - x^{-1} - 1)$ commutes with y and is divisible by $x^3 - x - 1$, so it too lies in $\ker\,\psi$. In particular, $\phi_1 = -(x^3 - x - 1)(x^{-3} - x^{-1} - 1)x^3$ lies in $\ker\,\psi$.

Finally we note that:

$$(1 - yx)\phi_2 = (x^3 - x - 1)(-2 + (x^4 + x^2 + x - 1)x^{-4}y).$$

□

Our goal in this section is to show that these three elements generate $\pi_2(X_{\mathcal{P}'})$. To that end we must understand the map $\psi\colon IG^* \to M \cong A \oplus A$. Firstly, we note the following holds in A:

Lemma 4.6 *In A we have*:

$$
\begin{array}{lll}
x^3 = x+1, & x^7 = 2x^2+2x+1, & x^{11} = x^2+3x, \\
x^4 = x^2+x, & x^8 = 2x^2+3x+2, & x^{12} = 3x^2+x+1, \\
x^5 = x^2+x+1, & x^9 = 3x^2+2, & x^{13} = x^2+3. \\
x^6 = x^2+2x+1, & x^{10} = x+3, &
\end{array}
$$

Proof To deduce each identity from the preceding one, we need only note that if $x^i = ax^2+bx+c$ in A, then $x^{i+1} = bx^2+(a+c)x+a$ in A. □

Lemma 4.7 *We have*:

$$
\begin{array}{rcl}
\psi(1) & = & (1, 3x^2+1), \\
\psi(x) & = & (x, x^2+3x+3), \\
\psi(x^2) & = & (x^2, 3x^2+x), \\
\psi(x^3) & = & (x+1, 3x+1), \\
\psi(x^4) & = & (x^2+x, x^2+2), \\
\psi(x^5) & = & (x^2+x+1, 2x^2+x+2).
\end{array}
$$

Proof We note that $(1-yx)x^i \in M$ corresponds to the element $(x^i, -x^{13-i}) \in A \oplus A$. Lemma 4.6 then gives the above expressions. □

Lemma 4.8 *The elements $4, \phi_1, \phi_2 \in \pi_2(X_{\mathcal{P}'})$ generate $\pi_2(X_{\mathcal{P}'})$ as a right module.*

Proof From any element of $\pi_2(X_{\mathcal{P}'})$, one may subtract appropriate multiples of ϕ_1, ϕ_2, in order to be left with an element $\alpha \in \pi_2(X_{\mathcal{P}'})$ of the form:

$$\alpha = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5,$$

with the $a_i \in \mathbb{Z}$. It will suffice to show that $4|a_0, a_1, a_2, a_3, a_4, a_5$. We have $\psi(\alpha) = 0$ which by Lemma 4.7 is equivalent to:

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 1 & 3 & 3 \\ 1 & 0 & 0 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 & 3 & 1 \\ 1 & 1 & 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 2 & 1 & 2 \end{pmatrix} = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0),$$

working modulo 4. To deduce that the $a_i \equiv 0 \mod 4$, it suffices to show that the above matrix is invertible, as a matrix over $\mathbb{Z}_4$. This follows from elementary row or column reduction over $\mathbb{Z}_4$. $\square$

5 Milnor square decompositions

Lemma 4.8 gives us an explicit generating set for $\pi_2(X_{\mathcal{P}'})$ as a submodule of IG^*. In order to show that this is not isomorphic to $\pi_2(X_{\mathcal{P}})$, we will decompose this submodule via a series of Milnor squares (see for example [3, Section 2]).

Firstly, let S denote the ring $\mathbb{Z}[Q_{28}]/\mathbb{Z}[Q_{28}](1+y^2)$. Then

$$\begin{aligned} \pi_2(X_{\mathcal{P}}) \otimes_{\mathbb{Z}[Q_{28}]} S &\cong IG^*/IG^*(1+y^2) \\ &\cong \mathbb{Z}[Q_{28}]/(\Sigma_G, 1+y^2)\mathbb{Z}[Q_{28}] \cong S, \end{aligned}$$

as $\Sigma_G = (1+y^2)(1+x+x^2+x^3+x^4+x^5+x^6)(1+y)$.

Then if N denotes the right S module $\pi_2(X_{\mathcal{P}'}) \otimes_{\mathbb{Z}[Q_{28}]} S$ we get:

Lemma 5.1 *If N is not a rank one free module over S, then*

$$\pi_2(X_{\mathcal{P}'}) \ncong \pi_2(X_{\mathcal{P}})$$

as $\mathbb{Z}[Q_{28}]$ modules.

Note that if N has $\mathbb{Z}$ torsion, then it cannot be a rank one free S module and we would have that $\pi_2(X_{\mathcal{P}'}) \ncong \pi_2(X_{\mathcal{P}})$ as desired. Hence for the remainder we only need to consider the case where N is $\mathbb{Z}$ torsion free.

Lemma 5.2 *The module N is isomorphic to the right ideal of S generated by $\{4, \phi_1, \phi_2\}$.*

Proof The right ideal of S generated by $\{4, \phi_1, \phi_2\}$ is isomorphic to:

$$\pi_2(X_{\mathcal{P}'})/(\pi_2(X_{\mathcal{P}'}) \cap IG^*(1+y^2)).$$

Thus we must show that if $\beta \in \pi_2(X_{\mathcal{P}'})$ and $\beta \in IG^*(1+y^2)$, then $\beta \in \pi_2(X_{\mathcal{P}'})(1+y^2)$. We know that if $\beta \in \pi_2(X_{\mathcal{P}'})$ and $\beta \in IG^*(1+y^2)$, then $4\beta \in \pi_2(X_{\mathcal{P}'})(1+y^2)$. Thus 4β represents 0 in N. As we have that N is $\mathbb{Z}$ torsion free as an assumption, we can conclude that β also represents 0 in N. Thus $\beta \in \pi_2(X_{\mathcal{P}'})(1+y^2)$. □

From now on N will denote the right ideal $(4, \phi_1, \phi_2)S$. Let

$$\sigma_{-x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6.$$

We have a Milnor square decomposition of the ring S [3, §2, II]:

(9)
$$\begin{tikzcd} S \arrow[r] \arrow[d] & S/\sigma_{-x} \arrow[d] \\ S/(x+1) \arrow[r] & S/(x+1,\, \sigma_{-x}) \end{tikzcd}$$

where the arrows all denote the natural projections. We have natural identifications:

$$\begin{aligned} S/(x+1) &= \mathbb{Z}[y]/(1+y^2), \\ S/(x+1,\, \sigma_{-x}) &= \mathbb{Z}_7[y]/(1+y^2). \end{aligned}$$

Let $\Lambda = S/\sigma_{-x}$. Note that $\mathbb{Z}[x]/\sigma_{-x}$ is the cyclotomic ring of degree 7, which embeds in $\mathbb{C} \subset \mathbb{H}$. This embedding may be extended to embed Λ in $\mathbb{H}$. In particular Λ contains no zero divisors. Similarly, the Gaussian integers $\mathbb{Z}[y]/(1+y^2)$ embed in $\mathbb{C}$ and contain no zero divisors. The ring $\mathbb{Z}_7[y]/(1+y^2)$ is just the field of order 49. We may rewrite (9):

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \Lambda \arrow[d] \\ \mathbb{Z}[y]/(1+y^2) \arrow[r] & \mathbb{Z}_7[y]/(1+y^2) \end{tikzcd}$$

We have a commutative square of modules over the corresponding rings:

(10)
$$\begin{tikzcd} N \arrow[r, "p_1"] \arrow[d, "q_1"] & N \otimes \Lambda \arrow[d, "q_2"] \\ N \otimes \mathbb{Z}[y]/(1+y^2) \arrow[r, "p_2"] & N \otimes \mathbb{Z}_7[y]/(1+y^2) \end{tikzcd}$$

where again the maps p_1, p_2, q_1, q_2 are the natural projections, and each $\otimes$ is over S.

Lemma 5.3 *We may rewrite the square (10) as:*

$$\begin{tikzcd} N \arrow[r, "p_1"] \arrow[d, "q_1"] & N/S\sigma_{-x} \arrow[d, "q_2"] \\ \mathbb{Z}[y]/(1+y^2) \arrow[r, "p_2"] & \mathbb{Z}_7[y]/(1+y^2) \end{tikzcd} \tag{11}$$

where p_1 is the natural projection, p_2 is reduction modulo 7, and q_1, q_2 are restrictions of the ring homomorphisms:

$$S \to \mathbb{Z}[y]/(1+y^2), \qquad\qquad \Lambda \to \mathbb{Z}_7[y]/(1+y^2),$$

respectively, both mapping $x \mapsto -1, y \mapsto y$.

Proof In S we have $\sigma_{-x} = \phi_1\sigma_{-x}$, so we have $N\sigma_{-x} = S\sigma_{-x}$. Thus

$$N \otimes \Lambda \cong N/N\sigma_{-x} \cong N/S\sigma_{-x},$$

and the map p_1 is the natural projection.

We have:

$$\phi_1 = 1 + (x^5 - x^3 - 2x^2 + x)(x+1). \tag{12}$$

Given $w \in N \cap S(x+1)$ we have $w = a(x+1) + by(x+1)$ for some polynomial expressions a, b in x over $\mathbb{Z}$. Note that $y\phi_1 = \phi_1 x^{-6}y$. Thus we have

$$\begin{aligned} w\phi_1 &= (a+by)\phi_1(x+1) \\ &= (a\phi_1 + b\phi_1 x^{-6}y)(x+1) \\ &= \phi_1(a + bx^{-6}y)(x+1). \end{aligned}$$

Thus multiplying (12) on the left by w and rearranging gives:

$$w = \phi_1(a + bx^{-6}y)(x+1) - w(x^5 - x^3 - 2x^2 + x)(x+1).$$

In particular $w \in N(x+1)$. Thus we have $N \cap S(x+1) = N(x+1)$. We conclude:

$$\begin{aligned} N \otimes \mathbb{Z}[y]/(1+y^2) &\cong N/N(x+1) \\ &\cong N/(N \cap S(x+1)) \subseteq \mathbb{Z}[y]/(1+y^2), \end{aligned}$$

and q_1 is the desired restriction. However from (12) we know $q_1(\phi_1) = 1$, so in fact we may identify $N \otimes \mathbb{Z}[y]/(1+y^2) \cong \mathbb{Z}[y]/(1+y^2)$.

Finally note that $q_1(\sigma_{-x}) = 7$, so p_2 is reduction modulo 7, and the square commutes, so q_2 must also be the desired restriction. □

We have $N/S\sigma_{-x} \subset S/S\sigma_{-x} \cong \Lambda$. Thus we may identify $N/S\sigma_{-x}$ with the right ideal, $I = (4, \phi_1, \phi_2)\Lambda$.

Lemma 5.4 *The right ideal I is freely generated over Λ by the element $1 + yx$.*

Proof In Λ we have:

$$\phi_1 = \phi_1 + \sigma_{-x} = 2(x^3 - 1)^2.$$

However we also know that:

$$(x^3 - 1)(-x^3 + x^2 - x) = -x^6 + x^5 - x^4 + x^3 - x^2 + x = 1,$$

so $x^3 - 1$ is a unit and $2 \in I$.

Thus $-x^3 + x^3y = \phi_2 - 2(1 + x) \in I$. Multiplying (on the right) by x^4 gives us that $1 + yx \in I$.

We next show that $(1+yx)$ divides (on the left) $4, \phi_1, \phi_2$. Note first that $(1+yx)(1-yx) = 2$, so $2 \in (1 + yx)\Lambda$. Thus $4 \in (1 + yx)\Lambda$ and $\phi_1 = 2(x^3 - 1)^2 \in (1 + yx)\Lambda$.

Finally note that:

$$\phi_2 = 2(1 + x) - (1 + yx)x^3 \in (1 + yx)\Lambda.$$

We conclude that $1 + yx$ generates the ideal I. Further, as Λ contains no zero divisors, we know that $1 + yx$ must generate I freely. □

Suppose now that N is free of rank one. Then it must be freely generated by some element $v \in N$. Then $p_1(v), q_1(v)$ must freely generate I, $\mathbb{Z}[y]/(1 + y^2)$ respectively. That is:

$$\begin{aligned} p_1(v) &= (1 + yx)\mu_1, \\ q_1(v) &= \mu_2, \end{aligned}$$

for units $\mu_1 \in \Lambda^*, \mu_2 \in (\mathbb{Z}[y]/(1 + y^2))^*$. By commutativity of (11) we have:

$$p_2(\mu_2) = q_2((1 + yx)\mu_1). \tag{13}$$

Let

$$\begin{aligned} \hat{p_2}\colon (\mathbb{Z}[y]/(1 + y^2))^* &\to (\mathbb{Z}_7[y]/(1 + y^2))^*, \\ \hat{q_2}\colon \Lambda^* &\to (\mathbb{Z}_7[y]/(1 + y^2))^*, \end{aligned}$$

denote the induced maps on units by the natural projections. Then from (13) we get:

$$\hat{p_2}(\mu_2) = (1 - y)\hat{q_2}(\mu_1). \tag{14}$$

Lemma 5.5 *Let H denote the subgroup of the abelian group*

$$(\mathbb{Z}_7[y]/(1+y^2))^*,$$

generated by the images of $\hat{p}_2, \hat{q}_2$. Then H is generated by $3, y$ and has cosets $H, (1+2y)H, (-3+4y)H, (1+4y)H$.

Proof See proof of [3, Theorem 3.2] or proof of [36, Lemma 10.13]. □

Lemma 5.6 *The module N is not free.*

Proof If N were free then by (14) we would have $1-y \in H$. However $-3y(1-y) = (-3+4y)$, so $1-y \in (-3+4y)H$. □

Combining lemmas 3.2, 5.1, 5.6 we deduce:

Theorem A. *We have a presentation for the quaternion group Q_{28}:*

$$\mathcal{P}' = \langle x, y \,|\, y^2 = x^7, \quad y^{-1}xyx^2 = x^3y^{-1}x^2y \rangle,$$

which has a non-standard second homotopy group. That is, if $X_{\mathcal{P}'}$ is the Cayley complex associated to $\mathcal{P}'$ and $X_{\mathcal{P}}$ is the Cayley complex associated to the standard presentation:

$$\mathcal{P} = \langle x, y \,|\, y^2 = x^7, \quad xyx = y \rangle,$$

then $\pi_2(X_{\mathcal{P}'}) \ncong \pi_2(X_{\mathcal{P}})$ as modules over $\mathbb{Z}[Q_{28}]$.

The fact that our procedure resulted in the coset $(-3+4y)H$ actually tells us (see proof of [3, Theorem 3.2]) that our presentation $\mathcal{P}'$ has the same second homotopy group as the algebraic 2–complex constructed in [3]: the so called Nancy's Toy [12, § 1.9.4]. This is no surprise given that N is non-free, as from [36, pp. 110–111] we know that $(-3+4y)H$ is the only coset corresponding to a non-free stably free module and N had to be stably free, as the Hurewicz isomorphism theorem and Schanuel's lemma combine to imply $\pi_2(X_{\mathcal{P}'})$ and $\pi_2(X_{\mathcal{P}})$ are stably equivalent.

6 The $D(2)$-property for Q_{4n}

Thus we have shown that it is possible for a finite balanced presentation of Q_{28} to have a non-standard second homotopy group. Let Y be a $D(2)$–complex, with $\pi_1(Y) = Q_{4n}$

for some $n \geq 2$. In particular we have shown that Y having a non-standard second homotopy group is not sufficient for it to solve Wall's $D(2)$ problem.

One might ask if every such Y of minimal Euler characteristic is homotopy equivalent to $\mathcal{E}_{n,r}$ for some r. Nicholson has answered this question in the negative. In the discussion proceeding [32, Theorem B] he notes that whilst the number of homotopically distinct presentations in our family grows linearly in n, the number of minimal Y as above grows exponentially.

Nonetheless he does show that our presentations are enough to verify the $D(2)$ property for Q_{28} [30, Theorem 7.7].

It remains possible that:

Conjecture 6.1 *Every $D(2)$–complex Y with $\pi_1(Y) = Q_{4n}$ and $\chi(Y) = 1$, is homotopy equivalent to a presentation of Q_{4n} of the form:*

$$\mathcal{Q} = \langle x, y \,|\, y^2 = x^n, \quad \mathrm{Eq}(y^{-1}xy, x) \rangle,$$

where $\mathrm{Eq}(a,b)$ is an equation implied by $ab = 1$, equating words in a, b.

As a starting point for proving this, we would require a more systematic way of computing generators of $\pi_2(\widetilde{X_\mathcal{Q}})$. We are grateful to the referee for suggesting the following approach:

Let $G_\mathbf{1}, G_\mathbf{2}$ be generators of $C_2(\widetilde{X_\mathcal{Q}})$ corresponding to the relations of $\mathcal{Q}$, respectively. Let $\partial_2'' \colon C_2(\widetilde{X_\mathcal{Q}}) \to C_1(\widetilde{X_\mathcal{Q}})$ be the boundary map. Recall from section 2 the generators $E_\mathbf{1}, E_\mathbf{2}$ for $C_2(\widetilde{X_\mathcal{P}})$, where $\mathcal{P}$ was the standard presentation for Q_{4n}. As the second relation of $\mathcal{Q}$ is implied by the second relation of $\mathcal{P}$ and $\mathcal{Q}$ presents Q_{4n} we have:

$$\begin{aligned} \partial_2(E_\mathbf{1}) &= \partial_2''(G_\mathbf{1}) \\ \partial_2''(G_\mathbf{2}) &= \partial_2(E_\mathbf{2})\lambda, \\ \partial_2(E_\mathbf{2}) &= \partial_2''(G_\mathbf{1})\mu_1 + \partial_2''(G_\mathbf{2})\mu_2, \end{aligned}$$

for some $\lambda, \mu_1, \mu_2 \in \mathbb{Z}[Q_{4n}]$.

Thus we have a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \pi_2(\widetilde{X_\mathcal{Q}}) \arrow[r, hook] \arrow[d, shift right] & C_2(\widetilde{X_\mathcal{Q}}) \arrow[r, "\partial_2''"] \arrow[d, shift right, "f"'] & C_1(\widetilde{X_\mathcal{Q}}) \arrow[d, equal] \\ 0 \arrow[r] & \pi_2(\widetilde{X_\mathcal{P}}) \arrow[r, hook] \arrow[u, shift right] & C_2(\widetilde{X_\mathcal{P}}) \arrow[r, "\partial_2"] \arrow[u, shift right, "g"'] & C_1(\widetilde{X_\mathcal{P}}) \end{tikzcd}$$

where f, g are given by:

$$f\colon \begin{array}{l} G_{\mathbf{1}} \mapsto E_{\mathbf{1}}, \\ G_{\mathbf{2}} \mapsto E_{\mathbf{2}}\lambda, \end{array} \qquad\qquad g\colon \begin{array}{l} E_{\mathbf{1}} \mapsto G_{\mathbf{1}}, \\ E_{\mathbf{2}} \mapsto G_{\mathbf{1}}\mu_1 + G_{\mathbf{2}}\mu_2. \end{array}$$

The following may then be useful for computing $\pi_2(\widetilde{X_Q})$ in various cases, as a starting point to prove conjecture 6.1.

Lemma 6.2 *Regarding $\pi_2(\widetilde{X_Q})$ as a submodule of $C_2(\widetilde{X_Q})$ in the natural way, it is generated by the elements:*

$$G_{\mathbf{1}}((x-1)+\mu_1(1-yx)) + G_{\mathbf{2}}\mu_2(1-yx), \qquad -G_{\mathbf{1}}\mu_1\lambda + G_{\mathbf{2}}(1-\mu_2\lambda).$$

Proof Suppose $G_{\mathbf{1}}\alpha_1 + G_{\mathbf{2}}\alpha_2 \in \pi_2(\widetilde{X_Q})$. We have:

$$G_{\mathbf{1}}\alpha_1 + G_{\mathbf{2}}\alpha_2 = gf(G_{\mathbf{1}}\alpha_1 + G_{\mathbf{2}}\alpha_2) + ((1-gf)G_2)\alpha_2.$$

Here $f(G_{\mathbf{1}}\alpha_1 + G_{\mathbf{2}}\alpha_2) \in \pi_2(\widetilde{X_{\mathcal{P}}})$ so

$$f(G_{\mathbf{1}}\alpha_1 + G_{\mathbf{2}}\alpha_2) = (E_{\mathbf{1}}(x-1) + E_{\mathbf{2}}(1-yx))\gamma,$$

for some $\gamma \in \mathbb{Z}[Q_{4n}]$, by (1).

Thus $\pi_2(\widetilde{X_Q})$ is generated by $g(E_{\mathbf{1}}(x-1) + E_{\mathbf{2}}(1-yx)), (1-gf)G_2$ over $\mathbb{Z}[Q_{4n}]$. We conclude by noting:

$$\begin{aligned} g(E_{\mathbf{1}}(x-1) + E_{\mathbf{2}}(1-yx)) &= G_{\mathbf{1}}((x-1)+\mu_1(1-yx)) + G_{\mathbf{2}}\mu_2(1-yx), \\ (1-gf)(G_2) &= -G_{\mathbf{1}}\mu_1\lambda + G_{\mathbf{2}}(1-\mu_2\lambda). \end{aligned}$$

□

However at present Q_{28} remains the only quaternionic group with chain homotopically distinct minimal algebraic 2-complexes for which the $D(2)$ property has been verified. Recent progress for Q_{24} (see discussion proceeding [31, Lemma 8.3]) shows that the two non-standard algebraic 2–complexes would be realised by a single exotic presentation (with different identifications of the presented group with Q_{24}). However even in this case it is not known if such a presentation exists.

References

[1] **P. H. Berridge, M. J. Dunwoody** ; *Non-free projective modules for torsion-free groups* : J. London Math. Soc. 19 (1979), Issue 2, pp. 433–436

[2] **M.Bestvina, N. Brady** ; *Morse theory and finiteness properties of groups* : Inventiones mathematicae 129 (1997), Issue 3 : pp. 445–470

[3] **F.R. Beyl, N. Waller** ; *A stably free nonfree module and its relevance for homotopy classification, case Q_{28}* : Algebr. Geom. Topol. 5 (2005), pp. 899–910

[4] **F.R. Beyl, N. Waller** ; *Examples of exotic free 2–complexes and stably free non–free modules for quaternion groups* : Algebr. Geom. Topol. 8 (2008), pp. 1–17

[5] **Imre Bokor, Diarmuid Crowley, Stefan Friedl, Fabian Hebestreit, Daniel Kasprowski, Markus Land, and J. Nicholson** ; *Connected sum decompositions of high-dimensional manifolds* : arXiv: 1909.02628

[6] **M.R. Bridson, M. Tweedale** ; *Deficiency and abelianized deficiency of some virtually free groups* : Math. Proc. Cambridge Philos. Soc. 143 (2007), pp. 257–264

[7] **M. J. Dunwoody** ; *The homotopy type of a two-dimensional complex* : Bull. London Math. Soc. 8 (1976), pp. 282–285

[8] **T. Edwards** ; *Generalised Swan modules and the D(2)–problem* : Algebr. Geom. Topol. 6 (2006), pp. 71–89

[9] **R.H. Fox** ; *Free differential calculus. V. The Alexander matrices re-examined* : Ann. of Math. .2/ 71 (1960), pp. 408–422

[10] **K. Gruenberg, P. Linnell** ; *Generation gaps and abelianized defects of free products* : J. Group Theory 11 (2008), Issue 5, pp. 587–608

[11] **Ian Hambleton** ; *Two remarks on Wall's D2 problem* : Math. Proc. Camb. Phil. Soc. 167 (2019), Issue 2, pp. 361–368

[12] **J. Harlander, F.R.Beyl** ; *A Survey of recent progress on some problems in 2-dimensional topology* : Advances in two-dimensional homotopy and combinatorial group theory : LMS 446 (2018), pp. 1–26

[13] **C. Hog–Angeloni** ; *Beiträge zum (einfachen) Homotopietyp bei freien Produkten und anderen gruppentheoretischen Konstruktionen* : PhD thesis, Johann Wolfgang Goethe-Universität Frankfurt am Main (1988)

[14] **F.E.A. Johnson** ; *Explicit homotopy equivalences in dimension two* : Math. Proc. Camb. Phil. Soc. 133 (2002), pp. 411–430

[15] **F.E.A. Johnson** ; *Stable modules and Wall's D(2)-problem* : Comment. Math. Helv. 78 (2003), Issue 1, pp. 18–44

[16] **F.E.A. Johnson** ; *Minimal 2-complexes and the D(2) problem* : Proc. AMS 132 (2003), pp. 579–586

[17] **F.E.A. Johnson** ; *Stable Modules and the D(2)–Problem* : LMS 301 (2003)

[18] **M.P. Latiolais** ; *Homotopy and homology classification of 2–complexes* : Two-dimensional homotopy and combinatorial group theory : LMS 197 (1993), pp. 97–124

[19] **P. A. Linnell** ; *Minimal free resolutions and (G, n)–complexes for finite abelian groups* : Proc. London Math. Soc. s3–66 (1993), Issue 2, pp. 303–326.

[20] **M. Lustig** ; *Infinitely many pairwise homotopy inequivalent 2–complexes K_i with fixed π_1 and Euler characteristic*, J. Pure Appl. Algebra 88 (1993), no. 1–3, pp. 173–175

[21] **W.H. Mannan** ; *The D(2) property for D_8* : Algebr. Geom. Topol. 7 (2007), pp. 517–528

[22] **W.H. Mannan** ; *Homotopy types of truncated projective resolutions* : Homology, Homotopy and Applications Vol. 9 (2007), No. 2, pp. 445–449

[23] **W.H. Mannan** ; *Realizing algebraic 2-complexes by cell complexes* : Math. Proc. Camb. Phil. Soc. Vol. 146 (2009), Issue 03, pp. 671–673

[24] **W.H. Mannan** ; *Quillen's Plus Construction and the D(2) problem* : Algebr. Geom. Topol. 9 (2009), pp. 1399–1411

[25] **W.H. Mannan** ; *A commutative version of the group ring* : J. of Algebra 379 (2013), pp. 113–143

[26] **W.H. Mannan, S. O'Shea** ; *Minimal algebraic complexes over D_{4n}* : Algebr. Geom. Topol. 13 (2013), pp. 3287–3304

[27] **W.H. Mannan** ; *Explicit generators for the relation module in the example of Gruenberg–Linnell* : Math. Proc. Camb. Phil. Soc. 161 (2016), pp. 199–202

[28] **W. Metzler** ; *Über den Homotopietyp zweidimensionaler CW–Komplexe und Elementartransformationen bei Darstellungen von Gruppen durch Erzeugende und definierende Relationen* : J. für die reine und angewandte Mathematik 285 (1976), pp. 7–23

[29] **J. Nicholson** ; *A cancellation theorem for modules over integral group rings* : Math. Proc. Camb. Phil. Soc. (to appear), arXiv:1807.00307

[30] **J. Nicholson** ; *On CW-complexes over groups with periodic cohomology* : arXiv: 1905.12018

[31] **J. Nicholson** ; *Projective modules and homotopy classification* : arXiv: 2004.04252

[32] **J. Nicholson** ; *Cancellation for (G,n)–complexes and the swan finiteness obstruction* : arXiv: 2005.01664

[33] **S. O'Shea** ; *The D(2)–problem for dihedral groups of order 4n* : Algebr. Geom. Topol. 12 (2012), pp. 2287–2297

[34] **J.R. Stallings** ; *On torsion-free groups with infinitely many ends* : Ann. of Math. Vol. 88 (1968), No. 2, pp. 312–334

[35] **R.G. Swan** ; *Groups of cohomological dimension one* : J. Algebra 12 (1969), pp. 585–610

[36] **R. G. Swan** ; *Projective modules over binary polyhedral groups* : Journal für die reine und angewandte Mathematik 342 (1983), pp. 66–172

[37] **C. T. C. Wall** ; *Finiteness conditions for CW–complexes* : Ann. of Math. 81, (1965), pp. 56–69.

Queen Mary University of London, School of Mathematical Sciences, Mile End Road, London E1 4NS.

No affiliation

wajid@mannan.info, tomasz.popiel@uwa.edu.au

https://www.qmul.ac.uk/maths/profiles/mannanw.html