

# Optimal escape from metastable states driven by non-Gaussian noise

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We investigate escape processes from metastable states that are driven by non-Gaussian noise. Using a path integral approach, we define a weak-noise scaling limit that identifies optimal escape paths as minima of a stochastic action, while retaining the infinite hierarchy of noise cumulants. This enables us to investigate the effect of different noise amplitude distributions. We find generically a reduced effective potential barrier but also fundamental differences, particularly for the limit when the non-Gaussian noise pulses are relatively slow. Here we identify a class of amplitude distributions that can induce a single-jump escape from the potential well. Our results highlight that higher-order noise cumulants crucially influence escape behaviour even in the weak-noise limit.

Escape from a metastable state underlies a large variety of phenomena in physics, chemistry, and biology [1] and has accordingly received an extraordinary amount of attention since Kramers' seminal work [2]. Recently, the problem has seen renewed interest in continuous systems driven by non-Gaussian noise. Poissonian shot noise (PSN) has been investigated in the context of full counting statistics [3–10], where the escape rate in e.g. Josephson junctions can be related to the current statistics. In the limit of weak noise intensity  $D$ , the Kramers rate assumes the exponential form  $r \cong e^{-S/D}$ , generalizing the classical Arrhenius result. The key challenge is to determine the action  $S$  in the presence of the infinite hierarchy of noise cumulants that describes non-Gaussian noise. Previous studies have truncated this hierarchy, typically after the cubic term [3–8, 11]. Analytical results retaining the full hierarchy are available only for particular amplitude distributions, such as one-sided PSN with exponentially distributed [12–17] or constant amplitudes [9, 10, 18]. On the other hand, the exponential scaling in  $1/D$  breaks down for escape driven by Lévy-stable noise (LSN), as has been established both in simulations [19–26] and analytical work [27–29]. Here, we present a unified framework where exact results for the Kramers rate are obtained for a large class of non-Gaussian noise types. Our approach characterizes in particular the dynamics of the escape process and explains the difference in the scalings of PSN and LSN driven escape.

We consider the time evolution of a single degree of freedom  $q$  under the influence of a conservative force, with potential  $V$ , and noise  $\xi$ ,  $\dot{q}(t) = -V'(q) + \xi(t)$ . We specify the noise  $\xi$  very generally as the derivative of a process with stationary increments. It can then be described via the characteristic functional

$$G_\xi[g] = \langle e^{i \int_0^t ds \xi(s)g(s)} \rangle = e^{\int_0^t ds \psi(g(s))}, \quad (1)$$

where the characteristic function  $\psi$  has the form [30]

$$\psi(g) = ig - Dg^2/2 + \lambda\phi(g) \quad (2)$$

$$\phi(g) = \int dA (e^{igA} - igA - 1) \rho(A). \quad (3)$$

The first two terms in Eq. (2) represent a constant drift  $a$  and Gaussian noise of intensity  $D$ , capturing the effect of continuous fluctuations. The term  $\phi$  is the contribution of discontinuous jumps. This becomes evident when  $\rho$  is interpreted as the density of independently and identically distributed jump amplitudes  $A_j$  and  $\lambda$  as the rate parameter of a Poisson process generating jump times  $t_j$ .  $G_\xi$  is then just the characteristic function of Gaussian white noise with drift and superimposed zero-mean PSN  $z(s) - \langle z(s) \rangle$ , where  $z(s) = \sum_{j=1}^n A_j \delta(s - t_j)$  and  $n$  is Poisson distributed with mean  $\lambda t$ . If  $\rho(A)$  is not normalizable, e.g. because of a power law divergence  $\rho(A) \propto |A|^{-\alpha-1}$  for small  $A$  [31] with  $0 < \alpha < 2$ , it cannot strictly be interpreted as an amplitude distribution. Nevertheless,  $\lambda\rho(A)dA$  still gives the rate of jumps with size in the range  $[A, A + dA]$ .

For our weak-noise limit we need to have a scale for noise amplitudes, which we achieve by writing  $\rho(A) = \rho_0(A/A_0)/A_0$  in terms of a base distribution  $\rho_0(x)$ . For  $\phi$  this scaling translates into  $\phi(g) = \phi_0(A_0g)$ , where  $\phi_0$  is given by Eq. (3) with  $\rho$  replaced by  $\rho_0$ . Table I lists the amplitude distributions  $\rho_0(x)$  we consider, normalized so that  $\langle x^2 \rangle \equiv \int dx \rho_0(x)x^2 = 1$ . We also include the case where the cumulants of the non-Gaussian noise are truncated after some low order by expanding  $\phi(g)$ . In the following we focus on symmetric noise with zero drift ( $a = 0$ ). A straightforward calculation from  $G_\xi$  then shows that the noise covariance is  $\text{Cov}(\xi(t), \xi(t')) = (D + D_\phi)\delta(t - t')$ , where  $D_\phi = \lambda\langle A^2 \rangle = \lambda A_0^2$ .

The starting point for our approach is a path integral for the propagator, i.e. the probability of arriving at  $q_b$  at time  $t$  when starting from  $q_a$ . We adopt an Itô convention and express the propagator as an inverse functional Fourier transform, following [32]:

$$p(q_b, t|q_a) = \int_{(q_a, 0)}^{(q_b, t)} \mathcal{D}[q] \int \mathcal{D}\left[\frac{g}{2\pi}\right] e^{-S[q, g]}. \quad (4)$$

The action functional is given by  $S[q, g] = \int_0^t ds \mathcal{L}(q(s), g(s))$  with the underlying Lagrangian  $\mathcal{L}(q, g) = ig(\dot{q} + V'(q)) + Dg^2/2 - \lambda\phi(g)$ . Now introduce

Distribution	$\rho_0(x)$	$\bar{\phi}_0(k) = \phi_0(-ik) = \langle e^{kx} - kx - 1 \rangle$
Gaussian	$e^{-x^2/2}/\sqrt{2\pi}$	$\bar{\phi}_{0,G}(k) = e^{k^2/2} - 1$
Constant modulus	$[\delta(x-1) + \delta(x+1)]/2$	$\bar{\phi}_{0,\text{const}}(k) = \cosh(k) - 1$
Gamma ( $\alpha < 2$ )	$ x ^{-1-\alpha} e^{- x }/[2\Gamma(2-\alpha)]$	$\bar{\phi}_{0,\alpha}(k) = [(1-k)^\alpha + (1+k)^\alpha - 2]/[2\alpha(\alpha-1)]$
Exponential ( $\alpha = -1$ )	$e^{- x }/4$	$\bar{\phi}_{0,\text{exp}}(k) = k^2/[2(1-k^2)]$
Truncated $\phi$		$\bar{\phi}_{0,\text{trunc}}(u) = \frac{1}{2}k^2 + bk^4$

TABLE I: Symmetric noise amplitude base distributions  $\rho_0$  studied. For Gamma noise,  $\rho_0(x)$  can be normalized to  $\int dx \rho_0(x) = 1$  for  $\alpha < 0$ , with the exponential a special case, but not for  $\alpha > 0$ . At generic amplitude scale  $A_0$  one has  $\bar{\phi}(k) = \bar{\phi}_0(A_0 k)$ .

the scaling  $g \rightarrow \tilde{g}/D$ , so that  $\mathcal{L} = \tilde{\mathcal{L}}/D$  with

$$\tilde{\mathcal{L}}(q, \tilde{g}) = i\tilde{g}(\dot{q} + V'(q)) + \tilde{g}^2/2 - \lambda D\phi(\tilde{g}/D). \quad (5)$$

In the absence of the  $\phi$ -component ( $\lambda = 0$ ),  $\tilde{\mathcal{L}}$  is independent of  $D$  and we have the familiar scaling form of the Gaussian action. This means that taking  $D \rightarrow 0$  yields the usual weak-noise limit of the path integral. For  $\lambda \neq 0$  we need to apply a suitable scaling for  $\phi$ . Expanding the exponential in Eq. (3) and using  $\phi(g) = \phi_0(gA_0)$  yields

$$\phi\left(\frac{\tilde{g}}{D}\right) = \frac{A_0^2 \langle x^2 \rangle (i\tilde{g})^2}{D^2 2!} + \frac{A_0^3 \langle x^3 \rangle (i\tilde{g})^3}{D^3 3!} + \dots \quad (6)$$

We now consider the scaling  $\lambda \rightarrow \tilde{\lambda}/D^\mu$ ,  $A_0 \rightarrow \tilde{A}_0 D^\nu$ , which makes the  $O(\tilde{g}^n)$  term of  $\lambda D\phi$  scale as  $D^{1-\mu+n(\nu-1)}$ . The exponents  $\mu, \nu$  define different scaling regimes as  $D \rightarrow 0$ , as shown in the inset of Fig. 1. In regime I, all orders ( $n \geq 2$ ) in  $\tilde{g}$  diverge as  $D \rightarrow 0$ . In regime II, there are always some higher orders that diverge as  $D \rightarrow 0$ , while in regime III all orders scale to zero as  $D \rightarrow 0$  so that one effectively recovers the case  $\lambda = 0$ . For the particular combination  $\nu = \frac{1}{2}(\mu + 1)$  with  $\mu > 1$  (red line) only the  $\tilde{g}^2$  term remains in Eq. (5) as  $D \rightarrow 0$ . The non-Gaussian noise strength  $D_\phi \propto D \rightarrow 0$  here, so this is a valid weak noise-limit but one that reduces to effective Gaussian noise. Only for  $\mu = \nu = 1$  do all orders in  $\tilde{g}$  remain in Eq. (5) as  $D \rightarrow 0$ . This is the scaling we adopt: it represents a genuine weak-noise limit of our generic noise, since  $D_\phi \propto D \rightarrow 0$  while the infinite hierarchy of noise cumulants is retained.

After scaling, we obtain for the path integral

$$p(q_b, t|q_a) = \int_{(q_a, 0)}^{(q_b, t)} \mathcal{D}[q] \int \mathcal{D}\left[\frac{\tilde{g}}{2\pi D}\right] e^{-S[q, \tilde{g}]/D}, \quad (7)$$

with Lagrangian

$$\tilde{\mathcal{L}}(q, \tilde{g}) = i\tilde{g}(\dot{q} + V'(q)) + \tilde{g}^2/2 - \tilde{\lambda}\bar{\phi}(\tilde{g}) \quad (8)$$

and  $\bar{\phi}(\tilde{g}) = \phi_0(\tilde{g}\tilde{A}_0)$ . We now drop the tildes as only rescaled variables are used below. As  $D \rightarrow 0$  we can calculate  $p(q_b, t|q_a)$  within a saddle-point approach as  $p(q_b, t|q_a) \cong e^{-S[q^*, g^*]/D}$  where  $\cong$  indicates the asymptotic behaviour for small  $D$ . The optimal paths  $q^*, g^*$  are determined from the condition  $\delta S = 0$ , which leads to the coupled Euler-Lagrange (EL) equations

$$\dot{q} = -V'(q) + ig - i\lambda\phi'(g) \quad (9)$$

$$\dot{g} = V''(q)g, \quad (10)$$

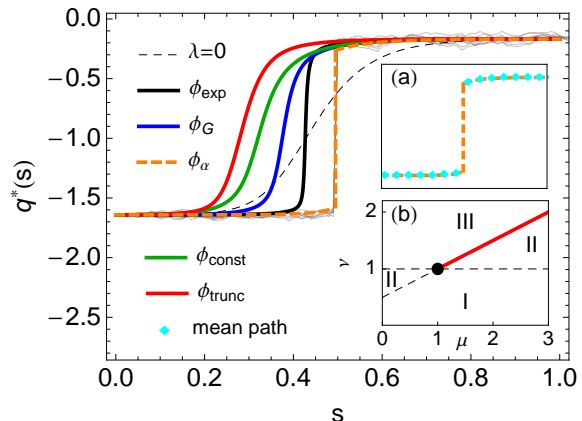


FIG. 1: (Colors online) Optimal excitation paths for the different  $\phi_0$  (see Table I) in the potential  $V(x) = x^4 - 6x^2 - 2x + 5$ , where  $q_a = -1.64$ ,  $q_b = -0.17$ . Parameter values:  $\lambda = 1$ ,  $A_0 = 1$ ,  $\alpha = 1.2$ ,  $b = 1/2$  (corresponding to fourth order truncation of  $\phi_{\text{exp}}$ ). Thin lines: paths directly sampled from the weight (4) with  $g$  integrated out, for  $D = 0.01$ . (a) Mean sample path compared with theory for  $\phi_\alpha$ . (b) Different scaling regimes of non-Gaussian noise given by the scaling exponents  $\mu, \nu$ .

The boundary conditions are  $q(0) = q_a$ ,  $q(t) = q_b$ , and accordingly we write the optimal action as  $S(q_b, t; q_a)$ . Comparing Eq. (9) with Eq. (8) shows that it can be expressed as a functional of  $g^*(t)$  only. On physical grounds we require real solutions for  $q^*$  and hence purely imaginary  $g^*$ , so we define  $k(s) = ig^*(s)$  and get for the action

$$S(q_b, t; q_a) = \int_0^t ds \left[ \frac{1}{2}k^2 + \lambda(k\bar{\phi}'(k) - \bar{\phi}(k)) \right] \quad (11)$$

where  $\bar{\phi}(k) = \phi(g) = \phi(-ik)$ . Evaluating  $\phi(g)$  for imaginary arguments in this way requires a continuation into the complex plane and so a nonzero radius of convergence for  $\phi(g)$ , a condition that is met if  $\rho(A)$  has tails that are no heavier than exponential. One could formulate the solution of (9,10) in terms of a Hamiltonian  $\mathcal{H}$  with the conjugate momentum  $\partial\mathcal{L}/\partial\dot{q} = ig$ :  $\mathcal{L} = ig\dot{q} - \mathcal{H}(q, ig)$ . The EL-equations are then just Hamilton's equations describing motion with a conserved  $\mathcal{H}$ .

Let us now consider specifically escape from a metastable state  $q_a$ , located at the minimum of the metastable basin of  $V$ , to the top of the potential barrier at  $q_b (> q_a)$ . For Gaussian noise, the path integral solu-

tion of this problem is essentially analogous to the quantum mechanical tunneling problem treated in a semiclassical approximation [33] (see also [34–36] for a discussion in the statistical mechanics context). The theory of large deviations gives the dominant scaling of the escape rate  $r$  in the weak-noise limit as [37, 38]

$$r \cong e^{-S(q_b; q_a)/D}, \quad S(q_b; q_a) = \inf_{t \geq 0} S(q_b, t; q_a). \quad (12)$$

The optimal path that provides the lowest (infimum in Eq. (12)) action is achieved for  $t \rightarrow \infty$  [33]. For  $\lambda = 0$ ,  $S(q_b; q_a)$  can be determined analytically as twice the height of the energy barrier,  $\Delta V = V(q_b) - V(q_a)$ . Since  $D = 2T$  for thermal noise, Eq. (12) thus recovers the Arrhenius result [1]. In this case, the optimal escape or “excitation” path is the time-reverse of a deterministic relaxation path from  $q_b$  to  $q_a$  [34–36].

For  $\lambda \neq 0$ , deterministic relaxations with  $g^* = 0$  still solve the EL equations and have zero action, but their time-reversal no longer gives the excitation paths. This is clear from the predictions for different amplitude distributions in Fig. 1, which we have confirmed by direct path sampling. The optimal escape paths have the characteristic instanton shape: for large  $t$  the system spends most of its time close to  $q_a$  and  $q_b$  while the actual barrier transition is sharply localized in time. The key observation is that the instanton shape varies with  $\phi$ , while the deterministic relaxation and its time reverse are entirely independent of  $\phi$  and  $\lambda$ . Moreover, we observe that the optimal action  $S(q_b; q_a)$  is reduced compared to the Gaussian limit of the noise for a range of small  $\lambda$  values: the non-Gaussian noise makes escape faster. Differences between amplitude distributions become pronounced especially in the limit  $\lambda \rightarrow 0$ : the Gaussian case is approached continuously with the low-order truncated  $\phi$  and for constant and Gaussian distributed noise amplitudes, though the approach is extremely slow for the latter (Fig. 2). For exponential and Gamma noise, the action is discontinuous at  $\lambda = 0$ : as  $\lambda \rightarrow 0$  it converges to a value considerably smaller than  $2\Delta V$ . Puzzlingly, for  $\alpha > 0$  the small  $\lambda$  regime appears inaccessible, with  $q^*$  becoming complex below some threshold.

In order to understand these surprising observations, we proceed analytically and integrate out  $g$  directly from Eq. (7). In the weak noise-limit this can again be done by saddle point integration and gives an action for  $q$  alone. (Technically we discretize into small time intervals  $dt$  and take  $dt \rightarrow 0$  after  $D \rightarrow 0$ .) Defining  $\beta(k) = k^2/2 + \lambda\tilde{\phi}(k)$ , the resulting action is  $S[q] = \int ds \pi(\dot{q}(s) + V'(q(s)))$ . Here  $\pi(\cdot)$  is the Legendre transform of  $\beta(\cdot)$ , i.e.  $\pi(f) = \max[kf - \beta(k)]$  with the maximum taken over the range of  $k$  where  $\tilde{\phi}(k)$  remains non-singular. Note that the function  $\pi$  is not equivalent to the Hamiltonian  $\mathcal{H}$  since we have integrated out the momenta  $ig$ . In fact, since  $\dot{q} + V'(q) = \xi$  from the original equation of motion, the action  $S[q] = \int ds \pi(\xi(s))$  gives the weight of any trajectory

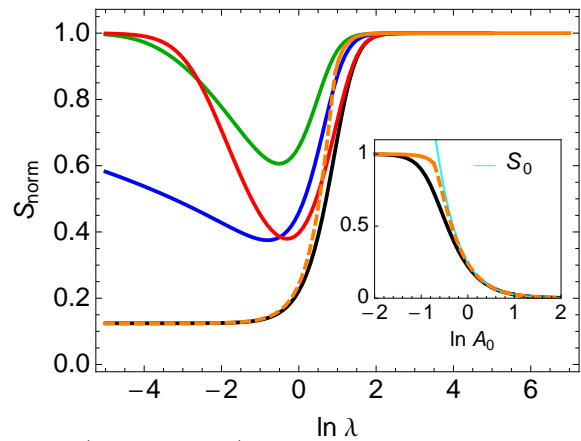


FIG. 2: (Colors online) The normalized action  $S_{\text{norm}} = S(q_b; q_a)/(2\Delta V/(1 + \lambda A_0^2))$  with  $A_0 = 1$  obtained for the same potential as in Fig. 1. The reference value  $2\Delta V/(1 + \lambda A_0^2)$  is the action for the Gaussian limit of  $\phi$ , i.e. truncated after the quadratic term. The  $\phi_\alpha$  curve is dashed where optimal paths cannot be found from the EL equations because they involve jumps. Inset:  $S_{\text{norm}}$  as a function of  $\ln A_0$  for  $\phi_{\text{exp}}$  and  $\phi_\alpha$ , at fixed  $D_\phi = \lambda A_0^2 = 1$ . For large  $A_0$  (small  $\lambda$ ) the curves converge to  $S_0 = \int dq \min(2V'(q), 1/A_0)$ .

of the noise (averaged again over small  $dt$ ) in the large deviation limit  $D \rightarrow 0$ . The function  $\pi(\xi)$  thus generalizes the simple quadratic  $\xi^2/2$  appearing in the Wiener measure  $\exp[-\int ds \xi(s)^2/(2D)]$  for Gaussian noise.

One can now think of  $q(t)$  as a path in the  $(q, v)$ -plane, with  $v = \dot{q}$ . Then the action reads  $S = \int dq \pi(v + V'(q))/|v|$  and for each  $q$  we can find  $v = \dot{q}$  simply as the minimum of  $\pi(v + V'(q))/|v|$ . We do not need to enforce the total time constraint  $t = \int dq/|v|$  as we want  $t \rightarrow \infty$  and the integral automatically diverges at both ends for paths between stationary points of  $V$ . The trivial global minimum is  $v = -V'(q)$ , which describes deterministic relaxation. For an excitation from  $q_a$  to  $q_b > q_a$  we need  $v > 0$ , on the other hand. The condition for a minimum of  $\pi(v + V'(q))/v$  with respect to  $v$  for excitation paths can be cast in the form

$$V'(q) = \beta(k)/k, \quad (13)$$

$$v = \beta'(k) - V'(q) \quad (14)$$

using basic properties of Legendre transforms. Here,  $k$  has to be found from Eq. (13) and then gives  $v$  using Eq. (14). This implicitly defines a function  $v = \dot{q} = \Xi(V'(q))$  and hence characterizes the shape of the excitation path. Moreover, we obtain the action simply as

$$S = \int_{q_a}^{q_b} dq k(q), \quad (15)$$

where  $k(q)$  is the solution of Eq. (13). Eqs. (13–15) reproduce existing results for special cases. In the Gaussian case ( $\lambda = 0$ ) one has  $\beta(k) = k^2/2$ ; thus  $k(q) = 2V'(q)$  and  $v = V'(q)$ , the expected time reverse of

the relaxation path. For escape driven by one-sided PSN without a Gaussian component, as investigated in Refs. [12, 13, 16, 18], we have  $\beta(k) = \lambda A_0^2 k^2 / [2(1 - A_0 k)]$  for exponential and  $\beta(k) = \lambda (e^{A_0 k} - A_0 k - 1)$  for constant amplitudes. This yields  $k(q) = 2V'(q) / (\lambda A_0^2 + 2A_0 V'(q))$  [12, 13, 16] for the exponential case and  $k(q)$  as the solution of  $k = \ln(1 + k(A_0 + V'(q)/\lambda)) / A_0$  [18] for the constant case, respectively.

To use Eq. (13) to understand the observations in Fig. 2 we distinguish three cases: (a)  $\bar{\phi}$  has no singularities for real  $k$ , which includes our scenarios  $\phi_{\text{trunc}}$ ,  $\phi_G$ ,  $\phi_{\text{const}}$ ; (b)  $\bar{\phi}$  has singularities for real  $k$  beyond which it is not defined, and diverges as these singularities are approached; this happens for  $\phi_{\text{exp}}$  or more generally  $\phi_\alpha$  with  $\alpha < 0$ , with singularities at  $k_\pm = \pm 1/A_0$ ; (c)  $\bar{\phi}$  has singularities but remains bounded on approaching them, as for our  $\phi_\alpha$  with  $\alpha > 0$ . Now write Eq. (13) as

$$\lambda = \frac{k}{\bar{\phi}(k)} (V'(q) - k/2) \equiv R(k) \quad (16)$$

The RHS  $R(k)$  diverges as  $1/k$  for  $k \rightarrow 0$  and decreases as  $k$  increases. In case (a) defined above, it then hits zero at  $k = k_G = 2V'(q)$ . The limit  $\lambda \rightarrow 0$  thus gives back purely Gaussian behaviour ( $\lambda = 0$ ) as we found. For case (b) the same is true if  $V'(q)$  is small enough for  $k_G$  to lie in the allowed range  $[k_-, k_+]$ . For larger  $V'(q)$ ,  $R(k)$  goes to zero already as  $k \rightarrow k_+$  because  $\bar{\phi}(k)$  diverges, so that in the limit  $\lambda \rightarrow 0$  we get the non-Gaussian solution  $k = k_+ = 1/A_0$ . As then also  $\phi'(k)$  diverges, the corresponding trajectory  $q^*(s)$  must have a section with infinite slope, i.e. a jump. For  $\lambda \rightarrow 0$ ,  $k(q)$  has the value  $k_G = 2V'(q)$  or  $k_+ = 1/A_0$  depending on which is smaller, giving for the action (15):  $S \rightarrow S_0 = \int dq \min(2V'(q), 1/A_0)$ . This is clearly lower than the Gaussian limit  $S = 2\Delta V$ . Therefore, non-Gaussian noise of infinitesimal rate  $\lambda$  discontinuously lowers the effective barrier for escape. This effect occurs for large enough noise amplitudes, specifically  $2A_0 V'_{\text{max}} > 1$  where  $V'_{\text{max}} = \max V'(q)$  for  $q \in [q_a, q_b]$ . We emphasize that truncation of the cumulant hierarchy at any finite order, which gives polynomial  $\bar{\phi}$ , always brings us back to case (a) and cannot reproduce even qualitatively the discontinuous behaviour for  $\lambda \rightarrow 0$ .

In case (c), small  $\lambda$  requires  $k \rightarrow k_G$  due to the boundedness of  $\bar{\phi}$ . However, if  $k_+ < k_G$ , then this point is outside the allowed range of  $k$ , and so for small enough  $\lambda$  there is no solution for  $k$  in Eq. (13), in agreement with our observation for Gamma noise with  $\alpha > 0$  in Fig. 2. Small  $\lambda$  here means  $\lambda < \frac{k_+}{\bar{\phi}(k_+)} (V'(q) - k_+/2)$ , or equivalently  $V'(q) > V'_0 \equiv \beta(k_+)/k_+$ . In the range of  $q$  where this is true, one can now check that  $\pi(v + V'(q))/v$  is monotonically decreasing for  $v > 0$ , reaching the limit  $k_+$  for  $v \rightarrow \infty$ : the optimal velocity is infinite,  $\Xi(V'(q)) = \infty$ , and there must be a jump in the optimal path. To the action this then contributes  $k_+ \int dq$  where

the integral covers the  $q$ -range of the jump. Our conclusion is therefore that for case (c) the optimal paths will have jumps even for nonzero noise rates  $\lambda$ , provided that  $V'_{\text{max}} > V'_0$ ; for Gamma noise with  $\alpha > 0$  one has  $V'_0 = \{1 + \lambda A_0^2 (2^\alpha - 2) / [\alpha(\alpha - 1)]\} / (2A_0)$ , which as in case (b) scales with  $1/A_0$  if we keep  $D_\phi = \lambda A_0^2$  constant. The presence of a jump in certain parameter regimes indicates that there is competition between escape via a sequence of small amplitude noise steps, and waiting for an atypically large ( $O(1)$ ) noise pulse to kick the system close to the barrier. The latter is preferred when the noise amplitudes have an exponential tail with large enough scale  $A_0$ , and when non-Gaussian noise pulses arrive at a sufficiently small rate to make successful accumulation of many small pulses less likely.

One can push the analysis further, e.g. to show that in the limit  $\lambda \rightarrow 0$  at constant  $A_0$  the function  $\Xi(\cdot)$  and hence the excitation paths become identical for Gamma and exponential amplitude distributions, i.e. independent of the exponent  $\alpha$ . The limiting shape is  $\Xi(V'(q)) = V'(q)$  for  $V'(q) < 1/(2A_0)$ , and  $\Xi(V'(q)) = \infty$  otherwise; thus the optimal excitation paths consist of initial and final segments of time-reversed relaxations, connected by a jump, and the resulting action is just  $S_0$ . For large  $A_0$  one then has  $S_0 \approx (q_b - q_a)/A_0$  and the excitation path becomes a jump directly from  $q_a$  to  $q_b$ . In the limit  $A_0 \rightarrow \infty$  our Gamma noise retrieves LSN where  $\rho(A) \propto |A|^{-1-\alpha}$ . Our results show that the effective barrier governing the escape rate then vanishes, and hence clarify why the exponential scaling with  $1/D$  must break down in this case as found both numerically [19–26] and analytically [27–29]. Note that our arguments for all cases (a,b,c) above imply that the solution of Eq. (16) always obeys  $k < k_G = 2V'(q)$ , and hence the action (15) is *lowered* by the non-Gaussian noise,  $S < 2\Delta V$ . Fig. 2 suggests that even  $S < 2\Delta V / (1 + \lambda A_0^2)$ , i.e. the effective barrier is smaller than for the equivalent Gaussian noise.

In summary, we have defined a weak noise scaling limit that allows the effects of non-Gaussian noise to be fully accounted for. We have identified a class of amplitude distributions that lead to optimal escape paths containing jumps; these encompass the entire path for large noise amplitudes. We have shown that even very rare non-Gaussian noise ( $\lambda \rightarrow 0$ ) can discontinuously lower the effective barrier for escape, an effect that cannot be captured by approximations that truncate the hierarchy of noise cumulants. It would be highly interesting to observe such non-Gaussian escape events in an experimental system, e.g. in Josephson junctions [39, 40]. We remark that our path integral approach can be extended to give the prefactor in Eq. (12), by including the fluctuation determinant and regularizing multi-instanton contributions [34, 35, 41]. Application of our approach to systems with more than one degree of freedom could also answer open questions such as the effect of non-Gaussian noise on the selection of transition configurations [42].

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