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Global rigidity of 2-dimensional linearly constrained frameworks

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A linearly constrained framework in $\mathbb{R}^d$ is a point configuration together with a system of constraints which fixes the distances between some pairs of points and additionally restricts some of the points to lie in given affine subspaces. It is globally rigid if the configuration is uniquely defined by the constraint system. We show that a generic linearly constrained framework in $\mathbb{R}^2$ is globally rigid if and only if it is redundantly rigid and 'balanced'. For unbalanced generic frameworks, we determine the precise number of solutions to the constraint system whenever the rigidity matroid of the framework is connected. We obtain a stress matrix sufficient condition and a Hendrickson type necessary condition for a generic linearly constrained framework to be globally rigid in $\mathbb{R}^d$.

1 Introduction

A (bar-joint) framework (G,p) in $\mathbb{R}^d$ is the combination of a finite, simple graph $G=(V,E)$ and a realisation $p:V\to\mathbb{R}^d$. The framework (G,p) is rigid if every edge-length preserving continuous motion of the vertices arises as a congruence of $\mathbb{R}^d$. Moreover (G,p) is globally rigid if every framework (G,q) with the same edge lengths as (G,p) arises from a congruence of $\mathbb{R}^d$.

In general it is an NP-hard problem to determine the global rigidity of a given framework [22]. The problem becomes more tractable, however, if we consider generic frameworks i.e. frameworks in which the set of coordinates of the points is algebraically independent over $\mathbb{Q}$. Hendrickson [11] obtained two necessary conditions for a generic framework (G,p) in $\mathbb{R}^d$ to be globally rigid: the graph G should be $(d+1)$-connected, and the framework (G,p) should be redundantly rigid i.e. it remains rigid after deleting any edge. While Hendrickson's conditions are insufficient to imply generic global rigidity when $d\geq 3$ [4, 19], they are sufficient when $d=1,2$. In particular, we have the following theorem of Jackson and Jordán [12] when $d=2$.

Theorem 1.1. A generic framework (G,p) in $\mathbb{R}^2$ is globally rigid if and only if G is either a complete graph on at most three vertices or G is 3-connected and redundantly rigid. □

Theorem 1.1 implies that global rigidity is a *generic property* of 2-dimensional bar-joint frameworks, in the sense that it depends only on the underlying graph and not the particular generic realisation. Gortler, Healy and Thurston [10] extended this result to show that global rigidity is a generic property in all dimensions.

A linearly constrained framework is a bar-joint framework in which certain vertices are constrained to lie in given affine subspaces, in addition to the usual distance constraints between pairs of vertices. Linearly constrained frameworks are motivated by numerous practical applications, notably in mechanical engineering and biophysics, see for example [9, 24]. Streinu and Theran [23] give a characterisation for generic rigidity of linearly constrained frameworks in $\mathbb{R}^2$. Together with Cruickshank [8], we recently obtained an analogous characterisation for generic rigidity of linearly constrained frameworks in $\mathbb{R}^d$ as long as the dimensions of the affine subspaces at each vertex are sufficiently small (compared to d). In this article we consider global rigidity for linearly constrained frameworks. Global rigidity of bar-joint frameworks has its own suite of practical applications, for example in sensor network localisation [14], and we expect our extension to have similar uses.

Throughout this paper we will consider graphs whose only possible multiple edges are multiple loops. We call such a graph $G=(V,E,L)$ a *looped simple graph* where E denotes the set of (non-loop) edges and L the set of

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loops. A *d-dimensional linearly constrained framework* is a triple (G, p, q) where $G = (V, E, L)$ is a looped simple graph, $p : V \to \mathbb{R}^d$ and $q : L \to \mathbb{R}^d$. For $v_i \in V$ and $e_j \in L$ we put $p(v_i) = p_i$ and $q(e_j) = q_j$. The framework (G, p, q) is *generic* if the set of coordinates of $\{p, q\}$ is algebraically independent over $\mathbb{Q}$ i.e. the transcendence degree of $\mathbb{Q}(p, q)$ over $\mathbb{Q}$ is $d(|V| + |L|)$.

Two d-dimensional linearly constrained frameworks (G, p, q) and $(G, \tilde{p}, q)$ are *equivalent* if

$$\begin{aligned} \|p_i - p_j\|^2 &= \|\tilde{p}_i - \tilde{p}_j\|^2 \text{ for all } v_iv_j \in E, \text{ and} \\ p_i \cdot q_j &= \tilde{p}_i \cdot q_j \text{ for all incident pairs } v_i \in V \text{ and } e_j \in L. \end{aligned}$$

We say that (G, p, q) is *globally rigid* if its only equivalent framework is itself.

We give an illustration of rigidity and global rigidity in $\mathbb{R}^2$ in Figure 1. First note that a loop at a vertex constrains that vertex to lie on a specific line. Every realisation of the graph H as a generic linearly constrained framework will be globally rigid as having two different line constraints at each vertex fixes the position of the vertices in $\mathbb{R}^2$. Every generic realisation (G, p) of the graph G is rigid by Theorem 2.1 below, but is not globally rigid, since we can obtain an equivalent realisation by reflecting the vertex v_2 in the line through $p(v_1)$ which is perpendicular to the line constraint at v_2.

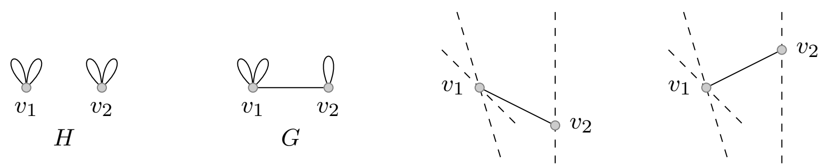

Fig. 1. Every realisation of the graph H as a generic linearly constrained framework in $\mathbb{R}^2$ is globally rigid. Every generic realisation of the graph G is rigid but not globally rigid. Two distinct equivalent realisations of G are given on the right of the figure.

Our main results, Theorems 8.4 and 9.1, characterise global rigidity for generic linearly constrained frameworks in $\mathbb{R}^2$ and determine the precise number of frameworks which are equivalent to a given generic framework whenever the underlying rigidity matroid of the given framework is connected. We also obtain results, Theorems 4.5 and 3.2, which give a stress matrix sufficient condition and a Hendrickson type necessary condition for generic global rigidity in $\mathbb{R}^d$. Theorem 8.4 implies that global rigidity is a generic property of 2-dimensional linearly constrained frameworks. We do not know whether this remains true in all dimensions but we do show, in Lemma 4.6 below, that the sufficient condition for generic global rigidity given in Theorem 4.5 is a generic property in all dimensions. One way to verify that global rigidity is a generic property in all dimensions would be to extend the above mentioned result of Gortler, Healy and Thurston [10] to show that the stress matrix condition in Theorem 4.5 is also a necessary condition for the global rigidity of d-dimensional generic linearly constrained frameworks when they are connected and have at least two vertices.

A more detailed description of the content of the paper is as follows. In Section 2 we provide a brief background on infinitesimal rigidity for linearly constrained frameworks. In Section 3 we give necessary conditions for generic global rigidity analogous to Hendrickson's conditions. We obtain an algebraic sufficient condition for the generic global rigidity of a linearly constrained framework in Section 4. This sufficient condition in terms of the rank of an appropriate stress matrix extends a key result of Connelly [5] for bar-joint frameworks. We focus on characterising generic global rigidity in $\mathbb{R}^2$ in the remainder of the paper. We obtain structural results on rigid circuits in the generic 2-dimensional linearly constrained rigidity matroid in Section 5. These are used in Sections 6 and 7 to obtain a recursive construction for the family of looped simple graphs which are balanced and generically redundantly rigid. The recursive construction is then used in Section 8 to characterise generic global rigidity. We determine the precise number of frameworks which are equivalent to a given generic framework whenever the underlying rigidity matroid of the given framework is connected in Section 9. We close by considering the analogous problem of characterising global rigidity for 'spherically constrained' frameworks, i.e. frameworks in which the vertices are constrained to lie on a sphere rather than a hyperplane, in Section 10.

2 Infinitesimal rigidity

An *infinitesimal motion* of a linearly constrained framework (G, p, q) is a map $\dot{p} : V \to \mathbb{R}^d$ satisfying the system of linear equations:

$$\begin{aligned} (p_i - p_j) \cdot (\dot{p}_i - \dot{p}_j) &= 0 \text{ for all } v_iv_j \in E \\ q_j \cdot \dot{p}_i &= 0 \text{ for all incident pairs } v_i \in V \text{ and } e_j \in L. \end{aligned}$$

The second constraint implies that, for each vertex v_i, its *infinitesimal velocity* $\dot{p}(v_i)$ is constrained to lie in the intersection of the hyperplanes with normals q_j for every loop e_j incident to v_i.

The *rigidity matrix* $R(G, p, q)$ of the framework is the matrix of coefficients of this system of equations for the unknowns $\dot{p}$. Thus $R(G, p, q)$ is a $(|E| + |L|) \times d|V|$ matrix, in which: the row indexed by an edge $v_iv_j \in E$ has $p(u) - p(v)$ and $p(v) - p(u)$ in the d columns indexed by v_i and v_j, respectively and zeros elsewhere; the row indexed by a loop $e_j = v_iv_i \in L$ has q_j in the d columns indexed by v_i and zeros elsewhere.

The framework (G, p, q) is *infinitesimally rigid* if its only infinitesimal motion is $\dot{p} = 0$, or equivalently if $\operatorname{rank} R(G, p, q) = d|V|$. We say that the graph G is *rigid* in $\mathbb{R}^d$ if $\operatorname{rank} R(G, p, q) = d|V|$ for some realisation (G, p, q) in $\mathbb{R}^d$, or equivalently if $\operatorname{rank} R(G, p, q) = d|V|$ for all *generic* realisations (G, p, q).

Streinu and Theran [23] characterised the looped simple graphs G which are rigid in $\mathbb{R}^2$. Given a looped simple graph $G = (V, E, L)$ and $F \subseteq E \cup L$, let V_F denote the set of vertices incident to F.

Theorem 2.1. Let H be a looped simple graph. Then H is rigid in $\mathbb{R}^2$ if and only if H has a spanning subgraph $G = (V, E, L)$ such that $|E| + |L| = 2|V|$, $|F| \leq 2|V_F|$ for all $F \subseteq E \cup L$ and $|F| \leq 2|V_F| - 3$ for all $\emptyset \neq F \subseteq E$. □

3 Necessary conditions for global rigidity

We say that a looped graph $G = (V, E, L)$ is *redundantly rigid* if $G - e$ is rigid for any $e \in E \cup L$, and that G is *d-balanced* if, for all $X \subset V$ with $|X| = d$, each connected component of $G - X$ has at least one loop. We will show that the properties of being redundantly rigid and d-balanced are necessary conditions for a connected generic linearly constrained framework with at least two vertices to be globally rigid in $\mathbb{R}^d$. (Note that a framework with one vertex is globally rigid if and only if it is rigid, and that a disconnected framework is globally rigid if and only if each of its connected components is globally rigid.)

Given a linearly constrained framework (G, p, q) in $\mathbb{R}^d$ we define its *configuration space* $C(G, p, q)$ to be the set

$$C(G, p, q) = \{\hat{p} \in \mathbb{R}^{d|V|} \, : \, (G, \hat{p}, q) \text{ is equivalent to } (G, p, q)\}.$$

In order to establish that globally rigid linearly constrained frameworks are also redundantly rigid an important step is to prove that the configuration space is compact. Since it is easy to see that the configuration space is closed this will follow from the following lemma.

Lemma 3.1. Let (G, p, q) be a generic linearly constrained framework in $\mathbb{R}^d$. Then $C(G, p, q)$ is bounded if and only if each connected component of G contains at least d loops. □

Proof. Let H be a connected component of G. Suppose H does not contain d loops. Let W be the subspace of $\mathbb{R}^d$ spanned by the vectors $q(f)$ for $f \in L(H)$. Choose $0 \neq t \in W^\perp$ and define (G, p', q) by putting $p'(v) = p(v) + t$ for all $v \in V(H)$, and $p'(v) = p(v)$ for all $v \in V(G) \setminus V(H)$. Then (G, p', q) is equivalent to (G, p, q) and since we can choose t to be arbitrarily large, $C(G, p, q)$ is not bounded.

Suppose H contains d loops $f_1, f_2, \ldots, f_d$. Let $B := \sum_{xy \in E(H)} |p(x) - p(y)|$ and choose $v \in V(H)$. Then the fact that H is connected implies that $|p(v) \cdot q(f_i)| \leq B$ for all $1 \leq i \leq d$. Since q is generic we have $\mathbf{e_1} = (1, 0, \ldots, 0) = \sum_{i=1}^d \alpha_i q(f_i)$ for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_d$. Hence

$$|p(v) \cdot \mathbf{e_1}| = \left| \sum_{i=1}^d \alpha_i p(v) \cdot q(f_i) \right| \leq B \sum_{i=1}^d |\alpha_i|.$$

A similar argument shows that $|p(v) \cdot \mathbf{e_j}|$ is bounded for all vectors $\mathbf{e_j}$ in the standard basis for $\mathbb{R}^d$. Since v is arbitrary, $C(H, p|_H, q|_H)$ is bounded.

We may apply the same argument to each connected component of G to deduce that $C(G, p, q)$ is bounded. ■

Theorem 3.2. Suppose (G, p, q) is a generic globally rigid linearly constrained framework in $\mathbb{R}^d$. Then each connected component of G is either a single vertex with at least d loops or is d-balanced and redundantly rigid in $\mathbb{R}^d$. □

Proof. Since (G,p,q) is globally rigid if and only if each of its connected components are globally rigid, we may assume that G is connected and is rigid in $\mathbb{R}^d$. It is easy to see that the theorem holds when G has one vertex so we may assume that $|V| \geq 2$.

We first prove that G is d-balanced. Let $X \subseteq V$ with $|X| = d$. Suppose some connected component G_1 of $G - X$, is incident with no loops. Then we can obtain an equivalent but noncongruent realisation from (G,p,q) by reflecting G_1 in the hyperplane spanned by the points $p(v)$, $v \in X$. This contradicts the global rigidity of (G,p,q). Hence G is d-balanced.

We next show that G is redundantly rigid in $\mathbb{R}^d$. Suppose for a contradiction that there exists an edge $e \in E \cup L$ such that $(G-e,p,q)$ is not rigid. A key step in obtaining a contradiction is to show that the configuration space $C(G-e,p,q)$ is bounded. By Lemma 3.1, it will suffice to show that each connected component H of $G-e$ has at least d loops. This follows from the fact that G is d-balanced when $d=1$ so we may assume that $d \geq 2$.

Suppose that $|V(H)| = n \geq 2$. Since G is rigid, the rank of $R(H, p|_H, q|_H)$ is at least $dn-1$. On the other hand, the rank of the submatrix of $R(H, p|_H, q|_H)$ consisting of the rows indexed by the (non-loop) edges of H is at most $dn - \binom{d+1}{2}$ when $n \geq d$ and $\binom{n}{2}$ when $n < d$. This implies that the number of loops in H is at least $(dn-1) - dn + \binom{d+1}{2} \geq d$ when $n \geq d$, and at least $(dn-1) - \binom{n}{2} \geq d$ when $n < d$, since $d \geq 2$.

It remains to consider the case when $|V(H)| = 1$, say $V(H) = \{v\}$. Suppose that $|L(H)| < d$. The facts that G is rigid and connected imply that H has exactly $(d-1)$ loops and that $e = uv$ for some $u \neq v$. Let ℓ be the line through $p(v)$ which is perpendicular to $q(f)$ for all loops incident to v and let P be the point on ℓ which is closest to $p(u)$. Let $p'(x) = p(x)$ for all $x \in V - v$ and $p'(v) = 2P - p(v)$. Then (G,p',q) is equivalent to (G,p,q) and $p' \neq p$. This contradicts the fact that (G,p,q) is globally rigid. Hence $|L(H)| \geq d$.

Since every component of $G-e$ has at least d loops, $C(G-e,p,q)$ is bounded by Lemma 3.1. We can now use a similar argument to that given in [17] to deduce that the component $\mathcal{C}$ of $C(G-e,p,q)$ that contains p is diffeomorphic to a circle, and that there exists a $p' \in \mathcal{C} - p$ with (G,p',q) equivalent to (G,p,q). ∎

We will show in Section 8 that the necessary conditions for generic global rigidity given in Theorem 3.2 are also sufficient when $d = 2$.

4 Equilibrium stresses

We will obtain an algebraic sufficient condition for a generic linearly constrained framework in $\mathbb{R}^d$ to be globally rigid, and show that the property that this condition holds is preserved by the graph '1-extension operation'. These results are key to our characterisation of global rigidity for 2-dimensional generic frameworks.

An *equilibrium stress* for a linearly constrained framework (G,p,q) in $\mathbb{R}^d$ is a pair (ω,λ), where $\omega : E \to \mathbb{R}$, $\lambda : L \to \mathbb{R}$ and (ω,λ) belongs to the cokernel of $R(G,p,q)$. Thus (ω,λ) is an equilibrium stress for (G,p) in $\mathbb{R}^d$ if and only if, for all $v_i \in V$,

$$\sum_{v_j \in V} \omega_{ij}(p(v_i) - p(v_j)) + \sum_{e_j \in L} \lambda_{i,j} q(e_j) = 0. \qquad (1)$$

where ω_{ij} is taken to be equal to $\omega(e)$ if $e = v_iv_j \in E$ and to be equal to 0 if $v_iv_j \notin E$, and λ_{ij} is equal to $\lambda(e_j)$ if e_j is a loop at v_i and is equal to 0 otherwise. An example is presented in Figure 2.

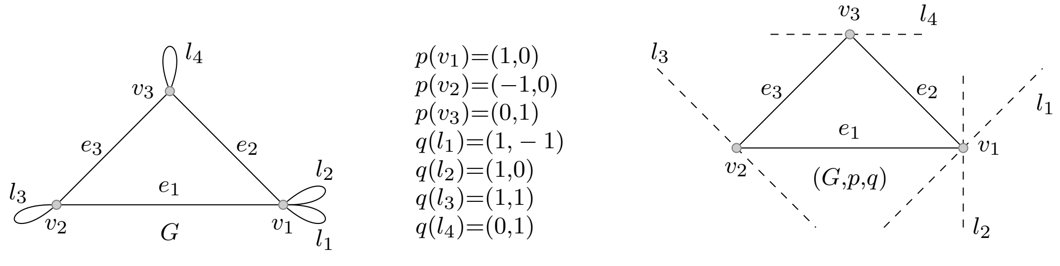

Fig. 2. A looped simple graph G and the corresponding realisation as a linearly constrained framework in $\mathbb{R}^2$. The vectors $\omega = (0,1,1)$ and $\lambda = (-1,0,1,-2)$ give an equilibrium stress for this framework.

We can write (1) in matrix form

$$\Pi(p)\Omega(\omega) + \Pi(q)\Lambda(\lambda)^T = 0. \qquad (2)$$

where:

- the *stress matrix* $\Omega(\omega)$ is a $|V| \times |V|$-matrix in which the off diagonal entry in row v_i and column v_j is $-\omega_{ij}$, and the diagonal entry in row v_i is $\sum_{v_j \in V} \omega_{ij}$;
- the *linear constraint matrix* $\Lambda(\lambda)$ is a $|V| \times |L|$-matrix in which the entry in row v_i and column e_j is λ_{ij};
- for any set $S = \{s_1, s_2, \dots, s_t\}$ and map $f : S \to \mathbb{R}^d$, the *coordinate matrix* $\Pi(f)$ is the $d \times t$ matrix in which the i'th column is $f(s_i)$.

Lemma 4.1. Suppose (ω, λ) is an equilibrium stress for a d-dimensional linearly constrained framework (G, p, q). Then $\operatorname{rank} \Omega(\omega) \leq |V| - 1$. In addition, if (G, p) is generic and G has at least two vertices and at most $d - 1$ loops, then $\operatorname{rank} \Omega(\omega) \leq |V| - 2$. □

Proof. The first assertion follows from the fact that the vector $(1, 1, \dots, 1)$ belongs to the cokernel of $\Omega(\omega)$. To prove the second assertion, we suppose that (G, p) is generic and G has at least two vertices and at most $d - 1$ loops. Choose a nonzero vector $q_0 \in \mathbb{R}^d$ such that q_0 is orthogonal to $q(f)$ for all loops f of G. Then the vector $(q_0 \cdot p(v_1), q_0 \cdot p(v_2), \dots, q_0 \cdot p(v_n))$ belongs to the cokernel of $\Omega(\omega)$, since $q_0 \Pi(p) \Omega(\omega) = 0$ by Equation (2), and is linearly independent from $(1, 1, \dots, 1)$ since (G, p, q) is generic and $|V| \geq 2$. Hence $\operatorname{rank} \Omega(\omega) \leq |V| - 2$. ■

We say that (ω, λ) is a *full rank equilibrium stress* for (G, p, q) if $\operatorname{rank} \Omega(\omega) = |V| - 1$. For example, the framework (G, p, q) drawn in Figure 2 has a stress matrix

$$\Omega(\omega) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

with respect to the given equilibrium stress $(\omega, \lambda) = (0, 1, 1, -1, 0, 1, -2)$ and we have $\operatorname{rank} \Omega(\omega) = 2$. Thus (ω, λ) is a full rank equilibrium stress for the framework (G, p, q) drawn in Figure 2. We will show in Theorem 4.5 below that the property of having a full rank equilibrium stress is a sufficient condition for a generic linearly constrained framework to be globally rigid in $\mathbb{R}^d$.

Lemma 4.2. Let (G, p, q) and (G, p', q) be frameworks and (ω, λ) be a full rank equilibrium stress for both (G, p, q) and (G, p', q). Then, for some fixed $t \in \mathbb{R}^d$, we have $p(v_i) = t + p'(v_i)$ for all $v_i \in V(G)$. □

Proof. Since $\operatorname{rank} \Omega(\omega) = |V(G)| - 1$, the cokernel of $\Omega(\omega)$ is spanned by $(1, 1, \dots, 1)$. Since (ω, λ) is an equilibrium stress for both (G, p, q) and (G, p', q), we may use (2) to deduce that $(\Pi(p) - \Pi(p'))\Omega(\omega) = 0$. This implies that each row of $\Pi(p - p')$ belongs to $\operatorname{coker} \Omega(\omega)$ and hence is a scalar multiple of $(1, 1, \dots, 1)$. This gives $p(v_i) = t + p'(v_i)$ for all $v_i \in V$, where $t = (t_1, t_2, \dots, t_d)$ and $t_i(1, 1, \dots, 1)$ is the i'th row of $\Pi(p - p')$. ■

Lemma 4.3. Let (G, p, q) be a generic linearly constrained framework in $\mathbb{R}^d$ and (G, p', q) be an equivalent framework. Suppose that G has at least d loops and that $p(v) = p'(v) + t$ for some fixed $t \in \mathbb{R}^d$, for all $v \in V$. Then $p = p'$. □

Proof. Choose distinct loops $e_i \in L$ for $1 \leq i \leq d$ and let v_i be the vertex incident to e_i. Let P, P' and Q be the $d \times d$ matrices whose i'th columns are $p(v_i)$, $p'(v_i)$ and $q(e_i)$, respectively and let A be the $d \times d$ diagonal matrix with the coordinates of t on the diagonal. Then $(P - P')Q^T = 0$ since (G, p, q) and (G, p', q) are equivalent. Since $P - P' = AJ$ where J is the $d \times d$ matrix all of whose entries are 1, we have $AJQ^T = 0$. Since (G, p, q) is generic, Q is nonsingular and hence $AJ = 0$. This implies that $A = 0$ and hence $t = 0$. ■

Proposition 4.4 (Connelly [5]). Suppose that $f : \mathbb{R}^a \to \mathbb{R}^b$ is a function, where each coordinate is a polynomial with coefficients in some finite extension $\mathbb{K}$ of $\mathbb{Q}$, $p \in \mathbb{R}^a$ is generic over $\mathbb{K}$ and $f(p) = f(\hat{p})$, for some $\hat{p} \in \mathbb{R}^a$. Then there are (open) neighbourhoods N_p of p and $N_{\hat{p}}$ of $\hat{p}$ in $\mathbb{R}^a$ and a diffeomorphism $g : N_{\hat{p}} \to N_p$ such that for all $x \in N_q$, $f(g(x)) = f(x)$, and $g(\hat{p}) = p$. □

Theorem 4.5. Suppose (G, p, q) is a generic linearly constrained framework in $\mathbb{R}^d$ with at least two vertices, and (ω, λ) is a full rank equilibrium stress for (G, p, q). Then (G, p, q) is globally rigid. □

Proof. Let $(G, \hat{p}, q)$ be equivalent to (G, p, q). Let $F : \mathbb{R}^{d|V|} \to \mathbb{R}^{|E|+|L|}$ be the rigidity map defined by $F(p) = (F_E(p), F_L(p))$ where $F_E : \mathbb{R}^{d|V|} \to \mathbb{R}^{|E|}$ is the usual rigidity map defined by $F_E(p) = (\dots, \|p(u) - p(v)\|^2, \dots)_{e=uv \in E}$, and $F_L : \mathbb{R}^{d|V|} \to \mathbb{R}^{|L|}$ is defined by $F_L(p) = (\dots, p(v) \cdot q(f), \dots)_{f=vv \in L}$.

Proposition 4.4 implies that there exist neighbourhoods N_p of (p, q) and $N_{\hat{p}}$ of $(\hat{p}, q)$ and a diffeomorphism $g : N_{\hat{p}} \to N_p$ such that $g(\hat{p}) = p$ and for all $\bar{p} \in N_{\hat{p}}$ we have $F(g(\hat{p})) = F(\hat{p})$. By differentiating at $\hat{p}$, observing that the differential of F is (up to scaling) the rigidity matrix $R(G, p, q)$ and using the fact that (ω, λ) is an equilibrium stress for (G, p, q), we have $(\omega, \lambda)R(G, \hat{p}, q) = (\omega, \lambda)R(G, p, q)D = 0 \cdot D = 0$ where D is the Jacobian of g at $\hat{p}$. Hence (ω, λ) is an equilibrium stress for $(G, \hat{p}, q)$. We can now use Lemmas 4.1, 4.2 and 4.3 to deduce that $p = \hat{p}$. ■

We will prove a partial converse to Theorem 4.5 in Section 8 by showing that every *connected, 2-dimensional,* globally rigid generic framework with at least two vertices has a full rank equilibrium stress. Note that the converse to Theorem 4.5 is false for disconnected frameworks since a framework is globally rigid if and only if each of its connected components is globally rigid, whereas no disconnected framework can have a full rank equilibrium stress (since we may apply Lemma 4.1 to each connected component).

Our next result shows that we can apply Theorem 4.5 whenever we can find a full rank equilibrium stress in an arbitrary infinitesimally rigid framework.

Lemma 4.6. Suppose $G = (V, E, L)$ can be realised in $\mathbb{R}^d$ as an infinitesimally rigid linearly constrained framework with a full rank equilibrium stress. Then every generic realisation of G in $\mathbb{R}^d$ is infinitesimally rigid and has a full rank equilibrium stress which is nonzero on all elements of $E \cup L$. □

Proof. Let $|V| = n$, $|E \cup L| = m$, and let (G, p, q) be a realisation of G in $\mathbb{R}^d$. Since the entries in the rigidity matrix $R(G, p, q)$ are polynomials in p and q, the rank of $R(G, p, q)$ will be maximised whenever (G, p', q') is generic. Hence (G, p, q) will be infinitesimally rigid whenever (G, p, q) is generic.

We adapt the proof technique of Connelly and Whiteley [6, Theorem 5] to prove the second part of the theorem. Since the entries in $R(G, p, q)$ are polynomials in p and q, and the space of equilibrium stresses of (G, p, q) is the cokernel of $R(G, p, q)$, each equilibrium stress of (G, p, q) can be expressed as a pair of rational functions $(\omega(p, q, t), \lambda(p, q, t))$ of p, q and t, where t is a vector of $m - dn$ indeterminates. This implies that the entries in the corresponding stress matrix $\Omega(\omega(p, q, t))$ will also be rational functions of p, q and t. Hence the rank of $\Omega(\omega(p, q, t))$ will be maximised whenever p, q, t is algebraically independent over $\mathbb{Q}$. In particular, for any generic $p, q \in \mathbb{R}^{dn}$, we can choose $t \in \mathbb{R}^{m-dn}$ such that $\operatorname{rank} \Omega(\omega(p, q, t)) = dn - 1$ and hence $(\omega(p, q, t), \lambda(p, q, t))$ will be a full rank equilibrium stress for (G, p, q).

Now suppose that (G, p, q) is generic and that (ω, λ) is a full rank stress for (G, p, q) chosen such that the total number of edges $e \in E \cup L$ with $\omega_e = 0$ and loops $\ell \in L$ with $\lambda_\ell = 0$ is as small as possible. Suppose for a contradiction that, for some $f \in E \cup L$, we have $\omega_f = 0$ if $f \in E$ and $\lambda_f = 0$ if $f \in L$. Then $(\omega|_{E-f}, \lambda|_{L-f})$ is a full rank equilibrium stress for $(G - f, p, q)$. By Theorem 4.5, $(G - f, p, q|_{L-f})$ is globally rigid. In particular $(G - f, p, q|_{L-f})$ is rigid, and hence, since $(G - f, p, q|_{L-f})$ is generic, it is infinitesimally rigid. This implies that the row of $R(G, p, q)$ indexed by f is contained in a minimal linearly dependent set of rows, and gives us an equilibrium stress $(\hat{\omega}, \hat{\lambda})$ for (G, p, q) with $\hat{\omega}_f \neq 0$ when $f \in E$, and $\hat{\lambda}_f \neq 0$ when $f \in L$. Then $(\omega', \lambda') = (\omega, \lambda) + c(\hat{\omega}, \hat{\lambda})$ is an equilibrium stress for (G, p), for any $c \in \mathbb{R}$. We can now choose a sufficiently small $c > 0$ so that $\operatorname{rank} \Omega(\omega') = n - 1$, $\omega'_e \neq 0$ for all $e \in E$ for which $\omega_e \neq 0$, and $\lambda'_\ell \neq 0$ for all $\ell \in L$ for which $\lambda_\ell \neq 0$. This contradicts the choice of (ω, λ). ■

Theorem 4.5, Lemma 4.6 and the fact that the framework (G, p, q) drawn in Figure 2 is infinitesimally rigid and has a full rank equilibrium stress imply that every generic realisation of G as a linearly constrained framework in $\mathbb{R}^2$ is globally rigid. A similar argument will allow us to show that the '1-extension operation' preserves the property of having a full rank equilibrium stress.

Let $G = (V, E, L)$ be a looped simple graph. The *d-dimensional* 1*-extension operation* forms a new looped simple graph from G by deleting an edge or loop $e \in E \cup L$ and adding a new vertex v and $d + 1$ new edges or loops incident to v, with the provisos that each end vertex of e is incident to exactly one new edge, and, if $e \in L$, then there is at least one new loop incident to v. See Figure 3 for an illustration of the types of 1-extension we will use.

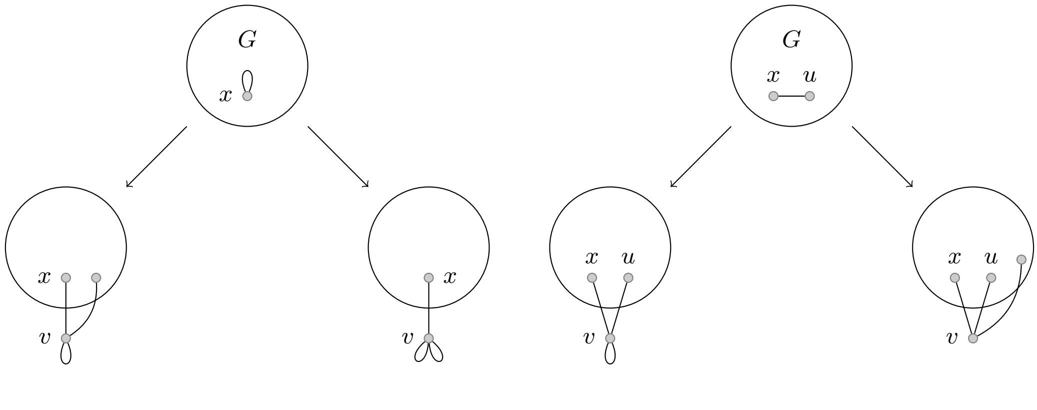

Fig. 3. Possible 2-dimensional 1-extensions on a loop (on the left) and on an edge (on the right) of a graph G.

Theorem 4.7. Let H be a looped simple graph and G be obtained from H by either adding an edge or a loop, or by applying the d-dimensional 1-extension operation. Suppose that H can be realised in $\mathbb{R}^d$ as an infinitesimally rigid linearly constrained framework with a full rank equilibrium stress. Then G can be realised in $\mathbb{R}^d$ as an infinitesimally rigid linearly constrained framework with a full rank equilibrium stress. □

Proof. Let (H,p,q) be a generic realisation of H as a d-dimensional linearly constrained framework. By Lemma 4.6, (H,p,q) has a full rank equilibrium stress (ω,λ) which is nonzero on all elements of $E \cup L$.

Suppose G is obtained from H by adding an edge or loop f. Let (G,p,q') be the realisation of G obtained from (H,p,q) by assigning an arbitrary value to $q'(f)$ if f is a loop. Then (G,p,q') is infinitesimally rigid and the equilibrium stress (ω',λ') for (G,p,q') obtained from (ω,λ) by setting (ω',λ') to be zero on f has full rank. Hence we may suppose that G is a 1-extension of H.

There are two cases to consider depending on whether the 1-extension deletes an edge or a loop. In the former case we can proceed by the standard collinear triangle technique given in [5] (if a new edge is a loop then the loop is assigned stress 0). Hence we present the latter case.

Let $V=\{v_1,v_2,\ldots,v_n\}$. Suppose that the 1-extension deletes $f_1 \in L$, where f_1 is a loop at v_1 and adds a new vertex v_0 with neighbours $v_1,v_2,\ldots,v_{k_1}$ and loops $f_0^1,\ldots,f_0^{k_2}$ at v_0 (where $k_1+k_2=d+1$). Let (G,p',q') be defined by putting $p'(v)=p(v)$ for all $v\in V$, $q'(f)=q(f)$ for all $f\in L-f_1$, $p'(v_0)=p(v_1)+q(f_1)$ and $q'(f_0^1)=q(f_1)$ and choosing $\{q'(f_0^1),\ldots,q'(f_0^{k_2})\}$ to be algebraically independent.

We first show that the framework $(G+f_1-v_0v_1,p',q')$ is infinitesimally rigid. Its rigidity matrix R can be constructed from $R(H,p,q)$ by adding d new columns indexed by v_0, and d new rows indexed by $v_0v_2,\ldots v_0v_{k_1}, f_0^1,\ldots,f_0^{k_2}$, respectively. Since (p,q) is generic and $\{q'(f_0^1),\ldots,q'(f_0^{k_2})\}$ is algebraically independent, the $d\times d$ submatrix M of R with rows indexed by $v_0v_2,\ldots v_0v_{k_1}, f_0^1,\ldots,f_0^{k_2}$ and columns indexed by v_0 is nonsingular. The fact that the new columns contain zeros everywhere except in the new rows now gives $\operatorname{rank} R = \operatorname{rank} R(H,p,q)+d$. By the choice of p',q', the rows in R corresponding to v_0v_1, f_1, f_0^1 are a minimal linearly dependent set. Thus

$$\operatorname{rank} R(G,p',q') = \operatorname{rank} R = \operatorname{rank} R(H,p,q)+d.$$

The fact that (H,p,q) is infinitesimally rigid now implies that (G,p',q') is infinitesimally rigid.

Let (ω',λ') be the stress for (G,p',q') defined by putting $\omega'_e=\omega_e$ for all $e\in E$, $\omega'_{v_0v_i}=0$ for all $i>1$, $\omega'_{v_0v_1}=-\lambda(f_1)$, $\lambda'(f)=\lambda(f)$ for all $f\in L$, $\lambda'(f_0^1)=\lambda(f_1)$ and $\lambda'(f_0^i)=0$ for all $i>1$. It is straightforward to verify that (ω',λ') is an equilibrium stress for (G,p',q'). We let ω_{ij} denote the stress $\omega'_e=\omega_e$ on $e=v_iv_j\in E$ and put $\lambda_1=\lambda(f_1)$. We have

$$\Omega(\omega')=\begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & 0 & \ldots & 0\\ \lambda_1 & \sum_{j\geq 1}\omega_{1j}-\lambda_1 & -\omega_{12} & -\omega_{13} & \ldots & -\omega_{1n}\\ 0 & -\omega_{12} & \sum_{j\geq 1}\omega_{2j} & -\omega_{23} & \ldots & -\omega_{2n}\\ 0 & -\omega_{13} & -\omega_{23} & \sum_{j\geq 1}\omega_{3j} & \ldots & -\omega_{3n}\\ \vdots & \vdots & \vdots & & & \vdots \end{bmatrix}.$$

By adding row 1 to row 2 and then column 1 to column 2 this reduces to

$$\begin{bmatrix} -\lambda_1 & 0 & 0 & 0 & \ldots & 0\\ 0 & \sum_j \omega_{1j} & -\omega_{12} & -\omega_{13} & \ldots & -\omega_{1n}\\ 0 & -\omega_{12} & \sum_j\omega_{2j} & -\omega_{23} & \ldots & -\omega_{2n}\\ \vdots & \vdots & \vdots & \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 0 & \ldots & 0\\ 0 & & & \\ \vdots & & \Omega(\omega) & \\ 0 & & & \end{bmatrix}.$$

Since $-\lambda_1\neq 0$, we have $\operatorname{rank}\Omega(\omega')=\operatorname{rank}\Omega(\omega)+1$. Hence (ω',λ') is a full rank equilibrium stress for (G,p',q'). ■

5 Circuits in the 2-dimensional linearly constrained rigidity matroid

We will focus on 2-dimensional linearly constrained frameworks in the following sections, and will suppress specific reference to the dimension. In particular we will refer to the 2-dimensional 1-extension operation as a 1-*extension*, to the property of being 2-balanced as *balanced*, and say that a graph is *rigid* to mean it is rigid in $\mathbb{R}^2$.

This section will contain a combinatorial analysis of the simplest redundantly rigid graphs - these correspond to rigid circuits in the linearly constrained generic rigidity matroid. Our results are analogous to those obtained in [1] for bar-joint frameworks and [15] for direction-length frameworks.

Let $G = (V, E, L)$ be a looped graph. For $X \subset V$, we define a matroid $\mathcal{M}_{lc}(G)$ on $E \cup L$ by the conditions of Theorem 2.1: a set $F \subset E \cup L$ is independent if $|F'| \leq 2|V_{F'}|$ for all $F' \subseteq F$, and $|F'| \leq 2|V_{F'}| - 3$ for all $\emptyset \neq F' \subseteq F \cap E$. We will refer to subgraphs of G whose edge-set is a circuit in $\mathcal{M}_{lc}(G)$ as $\mathcal{M}_{lc}$*-circuits*, see Figure 4. The definition of independence in $\mathcal{M}_{lc}(G)$ gives rise to the following characterisation of its circuits.

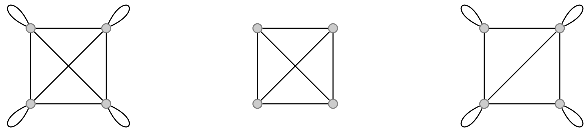

Fig. 4. A graph G on the left and two $\mathcal{M}_{lc}$-circuits of G in the middle and on the right. It can be verified by Lemma 5.1 that the subgraph in the middle is a flexible $\mathcal{M}_{lc}$-circuit and the subgraph on the right is a rigid $\mathcal{M}_{lc}$-circuit.

For $X \subseteq V$, let $i_E(X)$ and $i_L(X)$ denote the number of elements of E and L respectively in the subgraph induced by X in G and put $i_{E\cup L}(X) = i_E(X) + i_L(X)$.

Lemma 5.1. Let $G = (V, E, L)$ be a looped simple graph. Then G is an $\mathcal{M}_{lc}$-circuit if and only if, either
(a) $|E| + |L| = 2|V| + 1$, $i_E(X) \leq 2|X| - 3$ for all $X \subset V$ with $|X| \geq 2$ and $i_{E\cup L}(X) \leq 2|X|$ for all $X \subsetneq V$, or
(b) $L = \emptyset$, $|E| = 2|V| - 2$ and $i_E(X) \leq 2|X| - 3$ for all $X \subsetneq V$ with $|X| \geq 2$. □

We will refer to the circuits described in (a) as *rigid* $\mathcal{M}_{lc}$*-circuits* and to those in (b) as *flexible* $\mathcal{M}_{lc}$*-circuits*. The smallest rigid $\mathcal{M}_{lc}$-circuits are the looped graphs $K_1^{[3]}$ and $K_2^{[2]}$. The flexible $\mathcal{M}_{lc}$-circuits are precisely the circuits of the 2-dimensional generic bar-joint rigidity matroid. A recursive construction for these circuits is given in [1]. We will use the results of this section to obtain a recursive construction for rigid $\mathcal{M}_{lc}$-circuits in Section 6.

Let $d_G^\dagger(v)$ denote the number of edges or loops that are incident with a vertex v in a looped simple graph $G = (V, E, L)$, $d_G(v) = d_G^\dagger(v) + i_L(v)$ denote the *degree* of v in G, and $V_G^3 = \{v \in V : d_G^\dagger(v) = 3\}$ denote the set of *nodes* of G. We will suppress the subscript in $d_G^\dagger$, d_G and V_G^3 when it is obvious which graph we are referring to. For disjoint $X, Y \subseteq V$, let $G[X]$ denote the subgraph of G induced by X and $d(X, Y)$ denote the number of edges of G between X and Y.

Lemma 5.2. Let $G = (V, E, L)$ be a rigid $\mathcal{M}_{lc}$-circuit. Then $V^3 \neq \emptyset$ and the graph H obtained by deleting all loops from $G[V^3]$ is a forest. □

Proof. Since G is a rigid $\mathcal{M}_{lc}$-circuit, $d^\dagger(v) \geq 3$ for all $v \in V$,

$$4|V| + 2 = 2|E| + 2|L| = \sum_{v\in V} d(v) = \sum_{v\in V} d^\dagger(v) + |L|,$$

and $|L| \geq 3$. This gives $\sum_{v\in V} d^\dagger(v) \leq 4|V| - 1$ so G has at least one node.

Suppose $C = (S, F)$ is an induced cycle in H. Let $T = V \setminus S$. Then we have

$$\begin{aligned} i_{E\cup L}(T) &= 2|V| + 1 - i_{E\cup L}(S) - d(S, T) \\ &= 2|V| + 1 - |S| - i_L(S) - d(S, T) \\ &= 2|T| + 1, \end{aligned}$$

where the second equality follows from the fact that C is a cycle, and the last equality follows from the fact that a vertex in S contributes 1 to exactly one of $i_L(S)$ and $d(S, T)$, so $i_L(S) + d(S, T) = |S|$. This contradicts the facts that G is a rigid $\mathcal{M}_{lc}$-circuit and T is a proper subset of V. ■

Let $G = (V, E, L)$ be a rigid $\mathcal{M}_{lc}$-circuit. We say that $X \subseteq V$ is *mixed critical* if $i_{E\cup L}(X) = 2|X|$ and is *pure critical* if $i_E(X) = 2|X| - 3$. We say that X is *critical* if it is either mixed or pure critical. Note that if $uv \in E$ then $X = \{u, v\}$ is pure critical.

Lemma 5.3. Let X be a mixed critical set in a rigid $\mathcal{M}_{lc}$-circuit $G = (V, E, L)$ and $Y = V \setminus X$.
(a) Then $|V^3 \cap Y| \geq 1$, with strict inequality when $|Y| \geq 2$.
(b) If $d(X, Y) \geq 3$ then Y contains a vertex of degree three in G. □

Proof. (a) If $|Y|=1$ then the unique vertex in Y must be a node of G. Hence we may assume that $|Y|\geq 2$.

Suppose $|Y\cap V^3|\leq 1$. Note that for a vertex $v\in Y$, we have $d(v)=d^\dagger(v)+i_L(v)$, so

$$\sum_{v\in Y} d(v)=\sum_{v\in Y} d^\dagger(v)+i_L(Y)\geq 4|Y|-1+i_L(Y).$$

We also have

$$2i_{E\cup L}(Y)=\sum_{v\in Y} d_{G[Y]}(v)=\sum_{v\in Y} d(v)-d(X,Y).$$

We can combine these two (in)equalities to obtain

$$2i_E(Y)+i_L(Y)+d(X,Y)\geq 4|Y|-1. \tag{3}$$

Since $|Y|\geq 2$ we have $i_E(Y)\leq 2|Y|-3$ and Equation (3) now gives

$$i_{E\cup L}(Y)+d(X,Y)=i_E(Y)+i_L(Y)+d(X,Y)\geq 4|Y|-1-i_E(Y)\geq 2|Y|+2.$$

The fact that X is mixed critical now gives

$$|E|+|L|=i_{E\cup L}(X)+i_{E\cup L}(Y)+d(X,Y)\geq 2|X|+2|Y|+2=2|V|+2,$$

a contradiction.

(b) We have $\sum_{v\in Y} d(v)=2i_{E\cup L}(Y)+d(X,Y)$ and

$$i_{E\cup L}(Y)=|E|-i_{E\cup L}(X)-d(X,Y)=2|V\setminus X|-d(X,Y)+1.$$

This implies that

$$\sum_{v\in Y} d(v)=4|V\setminus X|-2d(X,Y)+d(X,Y)+2=4|Y|-d(X,Y)+2.$$

If $d(X,Y)\geq 3$ then $\sum_{v\in Y} d(v)<4|Y|$ and hence Y contains a vertex of degree less than four. ■

We next consider two versions of the 1-extension operation for a looped simple graph $G=(V,E,L)$. The 1-*extension operation at an edge* $uw\in E$ deletes uw and adds a new vertex v, new edges uv and wv and either a new edge vx for some $x\in V\setminus\{u,w\}$ or a new loop at v. The 1-*extension operation at a loop* $uu\in L$ deletes uu, adds a new vertex v, a new edge uv and a new loop at v, and either a new edge wv, for some $w\in V-u$ or a second new loop at v. It is straightforward to show that both versions of the 1-extension operation transform a rigid $\mathcal{M}_{lc}$-circuit into another rigid $\mathcal{M}_{lc}$-circuit using Lemma 5.1(a).

We refer to the inverse operation to each of the above 1-extension operations as a *1-reduction to an edge* or *loop*, respectively. When G is a rigid $\mathcal{M}_{lc}$-circuit and $v\in V$, we say that these reduction operations are *admissible* if they result in a smaller rigid $\mathcal{M}_{lc}$-circuit, and that v is *admissible* if there is an admissible 1-reduction at v.

The remainder of this section will be devoted to obtaining a structural characterisation of 'non-admissibility'. This will be used in the next section to show that every balanced rigid $\mathcal{M}_{lc}$-circuit on at least two vertices contains an admissible vertex.

For a vertex v in a looped simple graph $G=(V,E,L)$, the *neighbour set* of v is the set $N(v)$ of all vertices in $V-v$ which are adjacent to v.

Lemma 5.4. Let $G=(V,E,L)$ be a rigid $\mathcal{M}_{lc}$-circuit and v be a node in G.
(a) Suppose $N(v)=\{u,w,z\}$. Then the 1-reduction at v which adds uw is non-admissible if and only if there exists a critical set X in G with $u,w\in X$ and $v,z\notin X$.
(b) Suppose $N(v)=\{u,w\}$. Then:
(i) the 1-reduction at v which adds uw is non-admissible if and only if there exists a pure critical set X with $u,w\in X$ and $v\notin X$;
(ii) the 1-reduction at v which adds uu is non-admissible if and only if there exists a mixed critical set X with $u\in X$ and $w,v\notin X$.
(c) If $N(v)=\{u\}$, then the 1-reduction at v which adds uu is admissible. □

Proof. It is straightforward to show that the existence of each of the critical sets described in (a) and (b) implies non-admissibility.

For the converse, we first suppose that the 1-reduction described in case (a) is non-admissible. Then the graph resulting from this 1-reduction is not a rigid $\mathcal{M}_{lc}$-circuit. This implies that there exists either an $X \subseteq V - v$ with $i_E(X) \geq 2|X| - 3$ or a $Y \subsetneq V - v$ with $i_{E \cup L}(Y) \geq 2|Y|$. These subsets are pure critical and mixed critical, respectively, in G. If the first alternative holds then $z \notin X$, since otherwise $|X| \geq 3$, $i_E(X + v) = i_E(X) + 3 = 2|X| - 3 + 3 = 2|X + v| - 2$ and we would contradict the fact that G is a rigid $\mathcal{M}_{lc}$-circuit. Similarly, if the second alternative holds and $z \in Y$ then $i_{E \cup L}(Y + v) = i_{E \cup L}(Y) + 3 = 2|Y| + 3 = 2|Y + v| + 1$, again contradicting the fact that G is a rigid $\mathcal{M}_{lc}$-circuit.

The arguments in cases (b) and (c) are similar. ■

Lemma 5.5. Let G be a rigid $\mathcal{M}_{lc}$-circuit.
(a) If $X, Y \subset V$ are pure critical with $|X \cap Y| \geq 2$, then $X \cup Y$ and $X \cap Y$ are pure critical and $d(X \setminus Y, Y \setminus X) = 0$.
(b) If $X, Y \subset V$ are mixed critical and $|X \cup Y| \leq |V| - 1$, then $X \cup Y$ and $X \cap Y$ are mixed critical and $d(X \setminus Y, Y \setminus X) = 0$.
(c) If $X \subset V$ is mixed critical, $Y \subset V$ is pure critical, $|X \cap Y| \geq 2$ and $|X \cup Y| \leq |V| - 1$, then $X \cup Y$ is mixed critical, $X \cap Y$ is pure critical and $i_L(Y \setminus X) = 0 = d(X \setminus Y, Y \setminus X)$. □

Proof. We prove (c), parts (a) and (b) can be proved similarly. We have

$$\begin{aligned} 2|X| + 2|Y| - 3 &= i_{E \cup L}(X) + i_E(Y) \\ &= i_{E \cup L}(X \cup Y) + i_E(X \cap Y) - i_L(Y \setminus X) - d(X \setminus Y, Y \setminus X) \\ &\leq 2|X \cup Y| + 2|X \cap Y| - 3 - i_L(Y \setminus X) - d(X \setminus Y, Y \setminus X) \\ &= 2|X| + 2|Y| - 3 - i_L(Y \setminus X) - d(X \setminus Y, Y \setminus X) \end{aligned}$$

Hence $i_L(Y \setminus X) = d(X \setminus Y, Y \setminus X) = 0$ and equality holds throughout the above displayed calculation. In particular, $i_{E \cup L}(X \cup Y) = 2|X \cup Y|$, $i_E(X \cap Y) = 2|X \cap Y| - 3$. ■

Lemma 5.6. Let G be a rigid $\mathcal{M}_{lc}$-circuit and v be a node of G with three distinct neighbours u, w, t. Suppose that X, Y are mixed critical sets in G satisfying $\{u, w\} \subseteq X \subseteq V \setminus \{v, t\}$ and $\{w, t\} \subseteq Y \subseteq V \setminus \{v, u\}$. Suppose further that Z is a (mixed or pure) critical set with $\{u, t\} \subseteq Z \subseteq V \setminus \{v, w\}$. Let $W^* = (V - v) \setminus W$ for each $W \in \{X, Y, Z\}$. Then:
(a) $X \cup Y = X \cup Z = Y \cup Z = V - v$;
(b) $d(X^*, Y^*) = d(Y^*, Z^*) = d(X^*, Z^*) = 0$;
(c) either $\{X^*, Y^*, Z^*, X \cap Y \cap Z\}$ is a partition of $V - v$, or $X \cap Y \cap Z = \emptyset$ and $\{X^*, Y^*, Z^*\}$ is a partition of $V - v$;
(d) if Z is pure critical then $i_L(X^*) = 0 = i_L(Y^*)$. □

Proof. Since X, Y are mixed critical, $X \cup Y$ is mixed critical and $d(X \setminus Y, Y \setminus X) = 0$ by Lemma 5.5(b). The first assertion gives $i_{E \cup L}(X \cup Y \cup \{v\}) = 2|X \cup Y \cup \{v\}| + 1$. Since G is an $\mathcal{M}_{lc}$-circuit, this imples that $X \cup Y = V - v$. We now have $X^* = Y \setminus X$ and $Y^* = X \setminus Y$. When Z is mixed critical, a similar argument for X, Z and Y, Z tells us that (a) and (b) hold. Part (c) follows immediately from (a). Hence we may assume that Z is pure critical.

Since Z is pure critical, $G[Z]$ is connected and hence there is a path P in $G[Z]$ from u to t. If $X \cap Z = \{u\}$ then P would contain no vertices of $X - u$. The existence of such a path P would contradict the fact that $u \in X \setminus Y$, $t \in Y \setminus X$ and $d(X \setminus Y, Y \setminus X) = 0$. Hence $|X \cap Z| \geq 2$ and we can use Lemma 5.5(c) to deduce that $X \cup Z$ is mixed critical and $i_L(X^*) = 0 = d(X \setminus Z, Z \setminus X)$. A similar argument as in the previous paragraph now gives $X \cup Z = V - v$. We can now use symmetry to deduce that $Y \cup Z = V - v$ and $i_L(Y^*) = 0 = d(Y \setminus Z, Z \setminus Y)$. This gives (a), (b), (c) and (d) in the case when Z is pure critical. ■

We call a triple (X, Y, Z) of three sets satisfying the hypotheses of Lemma 5.6 a *strong flower on* v when Z is mixed critical and a *weak flower on* v when Z is pure critical. Note that it is possible for (X, Y, Z) to be both a strong and weak flower on v, see Figure 5.

Lemma 5.7. Let G be a rigid $\mathcal{M}_{lc}$-circuit and v be a node of G with three distinct neighbours r, s, t. Suppose that Y, Z are pure critical sets satisfying $\{s, t\} \subseteq Y \subseteq V \setminus \{v, r\}$ and $\{r, s\} \subseteq Z \subseteq V \setminus \{v, t\}$, and X is a critical set with $\{r, t\} \subseteq X \subseteq V \setminus \{v, s\}$. Then X is mixed critical, $X \cap Y = \{t\}$, $X \cap Z = \{r\}$, $Y \cap Z = \{s\}$, $G - v = G[X] \cup G[Y] \cup G[Z]$, and the component of $G - \{r, t\}$ which contains $\{v, s\}$ has no loops. □

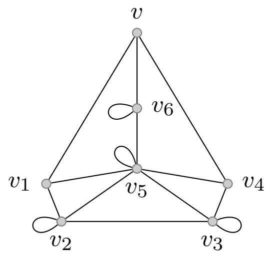

Fig. 5. Let $X = \{v_1, v_2, v_3, v_5, v_6\}$, $Y = \{v_2, v_3, v_4, v_5, v_6\}$, and $Z = \{v_1, v_2, v_3, v_4, v_5\}$. Then Z is pure critical and mixed critical so (X, Y, Z) is both a strong and a weak flower on v.

Proof. If $|Y \cap Z| \geq 2$ then $Y \cup Z$ would be pure critical by Lemma 5.5(a) and we would have $i_E(Y \cup Z \cup \{v\}) = 2|Y \cup Z \cup \{v\}| - 2$. This would contradict the fact that G is a rigid $\mathcal{M}_{lc}$-circuit. Hence $Y \cap Z = \{s\}$. Since Y is pure critical, $G[Y]$ is connected and hence there exists a path P in G from s to t which avoids $Z - s$.

Suppose X is pure critical. Then a similar argument to the above gives $X \cap Z = \{r\}$ and $Y \cap X = \{t\}$, $X \cup Y \cup Z$ is pure critical and $i_E(X \cup Y \cup Z \cup \{v\}) = 2|X \cup Y \cup Z \cup \{v\}| - 2$. This would again contradict the fact that G is a rigid $\mathcal{M}_{lc}$-circuit. Hence X is mixed critical.

Suppose $|X \cap Z| > 1$. Then $X \cup Z$ is mixed critical and $d(X \setminus Z, Z \setminus X) = 0$ by Lemma 5.5(c). This gives $i_{E \cup L}(X \cup Z \cup \{v\}) = 2|X \cup Z \cup \{v\}| + 1$ so $X \cup Z = V - v$. The path P now implies that $d(X \setminus Z, Z \setminus X) > 0$, a contradiction.

Hence we have $X \cap Z = \{r\}$ and, by symmetry, $X \cap Y = \{t\}$. This gives

$$\begin{aligned} i_{E \cup L}(X \cup Y \cup Z \cup \{v\}) &\geq i_{E \cup L}(X) + i_E(Y) + i_E(Z) + 3 \\ &= 2|X| + (2|Y| - 3) + (2|Z| - 3) + 3 \\ &= 2|X \cup Y \cup Z \cup \{v\}| + 1. \end{aligned}$$

Since G is a rigid $\mathcal{M}_{lc}$-circuit, we must have $X \cup Y \cup Z \cup \{v\} = V$ and $i_{E \cup L}(X) + i_E(Y) + i_E(Z) + 3 = |E| + |L|$. This implies that all loops in G are contained in $G[X]$ and that the component of $G - \{r, t\}$ which contains $\{v, s\}$ has no loops. Hence G is not balanced. ■

We call a triple (X, Y, Z) of three sets satisfying the hypotheses of Lemma 5.7 for G an *unbalanced flower on* v. Note that (X, Y, Z) cannot be both an unbalanced flower and a strong or weak flower since, in the former, $X \cap Y \cap Z = \emptyset$ and $G[Y], G[Z]$ are connected, while, for every strong or weak flower with $X \cap Y \cap Z = \emptyset$, each of $G[X], G[Y], G[Z]$ are disconnected.

Lemma 5.8. Let G be a rigid $\mathcal{M}_{lc}$-circuit and v be a non-admissible node of G with three distinct neighbours. Then at least one of the following holds:
(a) there exists a strong flower on v in G;
(b) there exists a weak flower on v in G;
(c) there exists an unbalanced flower on v in G. □

Proof. This follows immediately from Lemmas 5.4, 5.6 and 5.7. ■

Our final result of this section is a decomposition lemma for *unbalanced* rigid $\mathcal{M}_{lc}$-circuits i.e. rigid $\mathcal{M}_{lc}$-circuits which are not balanced. It uses the following graph operation.

Given three looped simple graphs $G = (V, E, L)$, $G_1 = (V_1, E_1, L_1)$ and $G_2 = (V_2, E_2, L_2)$, we say that G *is the* 2*-sum of* G_1 *and* G_2 *along an edge* uv if $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \{u, v\}$, $E = (E_1 \cup E_2) - uv$, $E_1 \cap E_2 = \{uv\}$, $L = L_1 \cup L_2$ and $L_1 \cap L_2 = \emptyset$. Figure 6 gives an example of an unbalanced rigid $\mathcal{M}_{lc}$-circuit which is the 2-sum of a rigid and a flexible $\mathcal{M}_{lc}$-circuit. Lemma 5.9 below shows that every unbalanced rigid $\mathcal{M}_{lc}$-circuit can be obtained in this way.

Lemma 5.9. Let $G = (V, E, L)$ be a looped simple graph and $u, v \in V$. Then G is a rigid $\mathcal{M}_{lc}$-circuit such that $G - \{u, v\}$ has a loopless connected component if and only if G is the 2-sum of a rigid $\mathcal{M}_{lc}$-circuit and a flexible $\mathcal{M}_{lc}$-circuit along a common edge uv. □

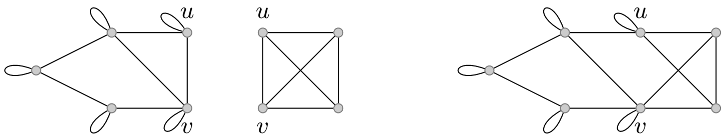

Fig. 6. An unbalanced rigid $\mathcal{M}_{lc}$-circuit on the right (removing u and v results in a loopless component) obtained from a rigid $\mathcal{M}_{lc}$-circuit and a flexible $\mathcal{M}_{lc}$-circuit by a 2-sum operation along the edge uv on the left.

Proof. First suppose that G is a 2-sum of a rigid $\mathcal{M}_{lc}$-circuit $G_1 = (V_1, E_1, L_1)$ and a flexible $\mathcal{M}_{lc}$-circuit $G_2 = (V_2, E_2)$ along the edge $e = uv \in E_1 \cap E_2$. It is straightforward to check that G satisfies the conditions given in Lemma 5.1(a). Hence G is a rigid $\mathcal{M}_{lc}$-circuit. It is unbalanced since $G_2 - \{u, v\}$ is a connected component of $G - \{u, v\}$ which contains no loops.

Now suppose G is a rigid $\mathcal{M}_{lc}$-circuit such that $G - \{u, v\}$ has a component H with no loops. Let $H_1 = (V_1, E_1, L_1)$ be the subgraph of G induced by $V \setminus V(H)$, and $H_2 = (V_2, E_2)$ be the simple subgraph of G obtained from $G - (V_1 \setminus \{u, v\})$ by deleting any loops at u or v. We have

$$2|V| + 1 = |E| + |L| \leq i_{E_1 \cup L_1}(V_1) + i_{E_2}(V_2) \leq 2|V_1| + 2|V_2| - 3 = 2|V| + 1$$

Thus equality must occur throughout. This implies that $uv \notin E$ (by equality in the first inequality), $|E_1| + |L_1| = 2|V_1|$ and $|E_2| = 2|V_2| - 3$ (by equality in the second inequality). Let G_1 and G_2 be the graphs obtained from H_1 and H_2, respectively, by adding the edge uv. It is straightforward to check that G_1 and G_2 satisify the conditions of Lemma 5.1 (a) and (b), respectively. Hence G_1 is a rigid $\mathcal{M}_{lc}$-circuit, G_2 is a flexible $\mathcal{M}_{lc}$-circuit, and G is the 2-sum of G_1 and G_2 along uv. ■

6 $\mathcal{M}_{lc}$-connected graphs

Our long term aim is to obtain a recursive construction for balanced redundantly rigid graphs. To accomplish this we first consider the closely related family of '$\mathcal{M}_{lc}$-connected graphs'. It is easy to see that a looped simple graph $G = (V, E, L)$ is redundantly rigid if and only if it is rigid and every element of $E \cup L$ belongs to an $\mathcal{M}_{lc}$-circuit. The graph G is $\mathcal{M}_{lc}$*-connected* if every pair of elements of $E \cup L$ belong to a common $\mathcal{M}_{lc}$-circuit in G.

We will show that any balanced $\mathcal{M}_{lc}$-connected graph other than $K_1^{[3]}$ can be reduced to a smaller $\mathcal{M}_{lc}$-connected graph using the operation of edge/loop deletion or 1-reduction. Our proof uses the concept of an 'ear decomposition' of a matroid and follows a similar strategy to that used in [3, 12].

Recall that a matroid $M = (E, r)$ is *connected* if every pair of elements of M is contained in a common circuit. Given a non-empty sequence of circuits $C_1, C_2, \ldots, C_m$ in M, let $D_i = C_1 \cup C_2 \cup \ldots C_i$ for all $1 \leq i \leq m$, and put $\tilde{C}_i = C_i \setminus D_{i-1}$. The sequence $C_1, C_2, \ldots, C_m$ is a *partial ear decomposition* of M if, for all $2 \leq i \leq m$,

(E1) $C_i \cap D_{i-1} \neq \emptyset$,
(E2) $C_i \setminus D_{i-1} \neq \emptyset$, and
(E3) no circuit C'_i satisfying (E1) and (E2) has $C'_i \setminus D_{i-1} \subset C_i \setminus D_{i-1}$.

A partial ear decomposition $C_1, C_2, \ldots, C_m$ is an *ear decomposition* of M if $D_m = E$.

Lemma 6.1 ([7])**.** Let $M = (E, r)$ be a matroid with $|E| \geq 2$. Then:

(i) M is connected if and only if M has an ear decompostion.
(ii) If M is connected then every partial ear decomposition is extendable to an ear decomposition of M.
(iii) If $C_1, C_2, \ldots, C_m$ is an ear decomposition of M then $r(D_i) - r(D_{i-1}) = |\tilde{C}_i| - 1$ for all $2 \leq i \leq m$.

□

Given a looped simple graph G, it will be convenient to refer to an ear decomposition $C_1, C_2, \ldots, C_m$ of $\mathcal{M}_{lc}(G)$ as an ear decomposition $H_1, H_2, \ldots, H_m$ of G where H_i is the $\mathcal{M}_{lc}$-circuit of G induced by C_i for $1 \leq i \leq m$. See Figure 7 for an example giving two distinct ear decompositions of the graph G drawn on the far left. The ear decomposition drawn in the middle has a flexible $\mathcal{M}_{lc}$-circuit K_4 whereas the circuits of the ear decomposition drawn on the right are all rigid $\mathcal{M}_{lc}$-circuits. The following lemma tells that a rigid $\mathcal{M}_{lc}$-connected graph always has an ear decompositon into rigid $\mathcal{M}_{lc}$-circuits.

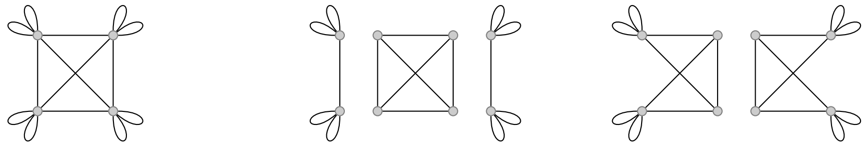

Fig. 7. A graph G on the left, and two ear decompositions of G in the middle and on the right.

Lemma 6.2. Let G be an $\mathcal{M}_{lc}$-connected looped simple graph with at least one loop. Then G has an ear decomposition into rigid $\mathcal{M}_{lc}$-circuits. □

Proof. Let ℓ be a loop of G. Since G is $\mathcal{M}_{lc}$-connected there exists an $\mathcal{M}_{lc}$-circuit H_1 containing ℓ. Then H_1 is a rigid $\mathcal{M}_{lc}$-circuit and, if $G = H_1$, then we are done. So suppose $G \neq H_1$. Extend H_1 to an ear decomposition $H_1, H_2, \dots, H_m$ of G such that each H_i is a rigid $\mathcal{M}_{lc}$-circuit for $1 \leq i \leq k$ and k is as large as possible.

Suppose $k < m$. Then we may choose an edge or loop f in H_{k+1} which does not belong to $\bigcup_{i=1}^{k} H_i$. Since $\bigcup_{i=1}^{k+1} H_i$ is $\mathcal{M}_{lc}$-connected, there exists an $\mathcal{M}_{lc}$-circuit $H'_{k+1} \subseteq \bigcup_{i=1}^{k+1} H_i$ such that ℓ, f are in H'_{k+1}. Then H'_{k+1} is a rigid $\mathcal{M}_{lc}$-circuit and $H_1, H_2, \dots, H_k, H'_{k+1}$ is a partial ear decomposition of G. Since every partial ear decomposition can be extended to a 'full' ear decomposition, this contradicts the maximality of k. ■

Lemma 6.2 and the fact that the union of two redundantly rigid graphs is redundantly rigid immediately give the following result.

Corollary 6.3. Let G be an $\mathcal{M}_{lc}$-connected looped simple graph. Then G is redundantly rigid if and only if G has a loop. □

Lemma 6.4. Let $H_1, H_2, \dots, H_m$ be an ear decomposition of a looped simple graph G into rigid $\mathcal{M}_{lc}$-circuits with $m \geq 2$. Let $H_i = (V_i, E_i, L_i)$ and $C_i = E_i \cup L_i$ for $1 \leq i \leq m$. Let $Y = V_m \setminus \bigcup_{i=1}^{m-1} V_i$ and $X = V_m \setminus Y$. Then
(i) $|\tilde{C}_m| = 2|Y| + 1$;
(ii) if $Y \neq \emptyset$ then every edge and loop in $\tilde{C}_m$ is incident to Y, X is mixed critical in H_m and $G[Y]$ is connected;
(iii) if G is balanced, $Y \neq \emptyset$ and $\tilde{C}_m$ contains no loops, then Y has at least three neighbours in X. □

Proof. (i) Let $G_{m-1} = \bigcup_{i=1}^{m-1} H_i$ and $D_{m-1} = \bigcup_{i=1}^{m-1} C_i$. Then $E(G_{m-1}) \cup L(G_{m-1}) = D_{m-1}$. Lemma 6.1(i) implies that G_{m-1} is $\mathcal{M}_{lc}$-connected. Since the $\mathcal{M}_{lc}$-circuits H_i are rigid, Corollary 6.3 now implies that G_{m-1} and G are rigid so $r(D_{m-1}) = 2|V \setminus Y|$ and $r(E \cup L) = 2|V|$. Hence by Lemma 6.1(iii) we have $|\tilde{C}_m| = r(E \cup L) - r(D_{m-1}) + 1 = 2|V| - 2|V \setminus Y| + 1 = 2|Y| + 1$.
(ii) Suppose $Y \neq \emptyset$. Let k be the number of edges and loops in $\tilde{C}_m$ which have all their endvertices in X. Since H_m is a rigid $\mathcal{M}_{lc}$-circuit, part (i) implies that

$$i_{E_m \cup L_m}(X) = |C_m| - |\tilde{C}_m| + k = 2|X \cup Y| + 1 - (2|Y| + 1) + k = 2|X| + k.$$

Since X is a proper subset of V_m we must have $k = 0$ and X is mixed critical in H_m.

Assume $G[Y]$ is disconnected. Let $Y_1, Y_2, \dots, Y_k$ be the vertex sets of the connected components of $G[Y]$. Since H_m is an $\mathcal{M}_{lc}$-circuit, $k \geq 2$ and X is mixed critical, we have $i_{E_m \cup L_m}(X \cup Y_i) - i_{E_m \cup L_m}(X) \leq 2|X \cup Y_i| - 2|X| = 2|Y_i|$. This implies that

$$|\tilde{C}_m| = \sum_{i=1}^{k} (i_{E_m \cup L_m}(X \cup Y_i) - i_{E_m \cup L_m}(X)) \leq \sum_{i=1}^{k} 2|Y_i| = 2|Y|,$$

contradicting part (i).
(iii) Let X' be the set of vertices in X which are adjacent to Y. Then $G - X'$ has a component with no loops. The fact that G is balanced and $\mathcal{M}_{lc}$-connected now implies that $|X'| \geq 3$. ■

Lemma 6.5. Let $G = (V, E, L)$ be an $\mathcal{M}_{lc}$-connected looped simple graph which contains a loop. Suppose that G' is obtained from G by an edge or loop addition, or a 1-extension. Then G' is $\mathcal{M}_{lc}$-connected. □

Proof. Note that G is redundantly rigid by Corollary 6.3.

First suppose that G' is obtained from G by adding a new edge or loop f. Since G is rigid, there exists an $\mathcal{M}_{lc}$-circuit C in G' with $f \in C$. The $\mathcal{M}_{lc}$-connectivity of G' now follows from the transitivity of the relation that defines $\mathcal{M}_{lc}$-connectivity.

Next suppose G' is obtained from G by a 1-extension operation that deletes an edge or loop f and adds a new vertex v with three incident edges or loops f_1, f_2, f_3. By transitivity, it will suffice to show that every $e \in (E \cup L) - f$ belongs to an $\mathcal{M}_{lc}$-circuit of G' containing f_1, f_2 and f_3. Since G is $\mathcal{M}_{lc}$-connected, there exists an $\mathcal{M}_{lc}$-circuit C in G containing e and f. Choose a base B of $\mathcal{M}_{lc}(G - f)$ with $C - f$ contained in B. Then $G[B]$ is rigid since G is rigid and $e \in C \subseteq B + f$. Hence $B' = B - e + f$ is another base of $\mathcal{M}_{lc}(G)$ and $G[B']$ is rigid. Then $B'' = B' - f + f_1 + f_2 + f_3$ is a base of $\mathcal{M}_{lc}(G')$ and $G'[B'']$ is rigid, since $G'[B'']$ is obtained by performing a 1-extension on $G[B']$. Hence $B'' + e$ contains a unique $\mathcal{M}_{lc}$-circuit C'. Since B'' is independent we have $e \in C'$. If $f_1 \notin C'$ then, since v is incident with exactly three edges, we would have $C \subseteq B$, contradicting the fact that B is independent. Hence f_1 is in C' and, since v is incident to exactly three edges, we also have $f_2, f_3 \in C'$. ■

We next consider the inverse operations to edge/loop addition and 1-extension. To do this we extend our definition of admissibility from an $\mathcal{M}_{lc}$-circuit to an $\mathcal{M}_{lc}$-connected graph $G = (V, E, L)$. We say that an edge or loop $f \in E \cup L$ is *admissible* if $G - f$ is $\mathcal{M}_{lc}$-connected, and a node $v \in V$ is *admissible* if there exists a 1-reduction at v which results in an $\mathcal{M}_{lc}$-connected graph.

We will show that every balanced $\mathcal{M}_{lc}$-connected graph G other than $K_1^{[3]}$ has an admissible edge or node. The main idea is to find an admissible edge/loop or node in the last $\mathcal{M}_{lc}$-circuit of an ear decomposition of G. We will need some additional terminology for nodes in rigid $\mathcal{M}_{lc}$-circuits to do this.

Suppose v is a node in a rigid $\mathcal{M}_{lc}$-circuit $G = (V, E, L)$ and $X \subseteq V$ is a critical set in G. We say that X is *node critical for* v if $N(v) - z \subseteq X \subseteq V \setminus \{v, z\}$ for some $z \in N(v)$ with $d^\dagger(z) \geq 4$. We will also refer to a critical set X as being *node critical* when it is *node critical* for some (unspecified) node. Recall that the graph F obtained by deleting all loops from $G[V^3]$ is a forest by Lemma 5.2. We say that v is a *leaf node* of G if $d_F(v) \in \{0, 1\}$, a *series node* if $d_F(v) = 2$ and a *branching node* if $d_F(v) = 3$. Note that a 1-reduction at a branching node cannot be admissible since the resulting graph will have a vertex v with $d^\dagger(v) = 2$.

Theorem 6.6. Let $G = (V, E, L)$ be a balanced, $\mathcal{M}_{lc}$-connected, looped simple graph distinct from $K_1^{[3]}$. Then G has an admissible edge, loop, or node. □

Proof. We proceed by contradiction. Suppose that G has no admissible edges, loops or nodes. Since G is balanced, G has at least one loop. Lemma 6.2 now implies that G has an ear decomposition $C_1, C_2, \ldots, C_m$ such that $H_i = G[C_i]$ is a rigid $\mathcal{M}_{lc}$-circuit for all $1 \leq i \leq m$. Let $H_i = (V_i, E_i, L_i)$ for all $1 \leq i \leq m$ and $H = \bigcup_{i=1}^{m-1} H_i$. Then H is $\mathcal{M}_{lc}$-connected by Lemma 6.1(i). Put $Y = V_m \setminus \cup_{i=1}^{m-1} V_i$ and $X = V_m \setminus Y$. Note that if $m = 1$ then $Y = V$ and $X = \emptyset$. If $Y = \emptyset$ then $H = G - e$ for some $e \in E \cup L$ and e is admissible in G since H is $\mathcal{M}_{lc}$-connected. Hence we may assume that $Y \neq \emptyset$. Lemma 6.4(ii) now implies that X is mixed critical in H_m. Hence Y contains at least one node of H_m by Lemma 5.3.

Claim 6.1. No node of H_m in Y is admissible in H_m, and hence every node of H_m in Y has at least two distinct neighbours in G. □

Proof. Suppose some node $v \in Y$ is admissible in H_m. Let C'_m be the edge set of the rigid $\mathcal{M}_{lc}$-circuit obtained from H_m by performing an admissible 1-reduction at v. Then v is a node in G and the graph obtained from G by performing the same 1-reduction at v will be $\mathcal{M}_{lc}$-connected since it will have an ear decomposition $C_1, C_2, \ldots, C_{m-1}, C'_m$. Hence v will be admissible in G, contradicting the fact that G has no admissible nodes. We can now use Lemma 5.4(c) to deduce that v has at least two distinct neighbours. ■

Claim 6.2. Suppose v is a node of H_m in Y. Then v has at least two neighbours in Y. □

Proof. We proceed by contradiction. Consider the following three cases.
Case 1: $N(v) \cap Y = \emptyset$.

Lemma 6.4(ii) implies that $Y = \{v\}$ and all edges and loops of $\tilde{C}_m$ are incident to v. If $G[N(v)]$ is not complete, then we may choose $p, q \in N(v)$ with $pq \notin E$. Lemma 6.5 and the fact that H is $\mathcal{M}_{lc}$-connected now imply that the 1-reduction at v which deletes v and adds the edge pq is admissible in G. This contradicts the assumption that G has no admissible nodes. Hence $G[N(v)]$ is complete. Then $G - v = H$ (since all edges and loops of $\tilde{C}_m$ are incident to v), so $G - v$ is $\mathcal{M}_{lc}$-connected. Since $G - e$ is a 1-extension of H for any edge e in $G[N(v)]$, it is also $\mathcal{M}_{lc}$-connected by Lemma 6.5. Hence e is admissible in G. This contradicts the assumption that G has no admissible edges.

Case 2: $N(v) = \{r, s, t\}$ with $\{r, t\} \subseteq X$, $s \in Y$.

Since v is not admissible in H_m by Claim 6.1, there exist critical sets R, T in H_m such that $\{s, r\} \subseteq T \subseteq V_m \setminus \{v, t\}$ and $\{s, t\} \subseteq R \subseteq V_m \setminus \{v, r\}$ by Lemma 5.4. We may suppose that R, T have been chosen to be the minimal critical sets with these properties. If R, T are both pure critical, then (R, S, X) would be an unbalanced flower on v in H_m, and we could use Lemma 5.7 to contradict the hypothesis that G is balanced. Relabelling R, T if necessary we may assume that R is mixed critical, but not pure critical, in H_m. The minimality of R and the fact that R is not pure critical now imply that $st \notin E(H_m)$. Lemma 5.8 now implies that (X, R, T) is a strong or weak flower on v in H_m. Let H'_m denote the graph obtained from H_m by applying a 1-reduction at v adding the edge st. Then H'_m contains a unique $\mathcal{M}_{lc}$-circuit D. The minimality of R and the fact that R is not pure critical now imply that $D = H_m[R] + st$ and D is a rigid $\mathcal{M}_{lc}$-circuit. Let $C'_m = E(D) \cup L(D)$.

We will show that v is admissible in G by verifying that $G' = G - v + st$ is $\mathcal{M}_{lc}$-connected. Since H is $\mathcal{M}_{lc}$-connected, it will suffice to show that $G' = H \cup D$ and that D contains an edge or loop of H. We first verifiy that D contains an edge of H. Since each edge of $\tilde{C}_m$ is incident with Y by Lemma 6.4(ii), this is equivalent to showing that D contains an edge of $H_m[X]$. This follows because R and X are mixed critical in H_m, so $R \cap X$ is mixed critical and nonempty by Lemma 5.5(b), and all edges of $H_m[R \cap X]$ belong to D. It remains to show that $G' = D \cup H$ i.e. $(\tilde{C}_m \setminus \{vr, vs, vt\}) + st \subseteq C'_m$. Lemma 5.6(a) implies that all edges of H_m which are not incident with v are induced by either X or R, and no edge of $\tilde{C}_m$ is induced by X by Lemma 6.4(ii). Since $C'_m = E(D) \cup L(D)$ and $D = H_m[R] + st$, we have $(\tilde{C}_m \setminus \{vr, vs, vt\}) + st \subseteq C'_m$ and $G' = D \cup H$. This implies that G' is $\mathcal{M}_{lc}$-connected and v is admissible in G, contradicting the assumption that G has no admissible nodes.

Case 3: $N(v) = \{r, s\}$ with $r \in X$ and $s \in Y$.

Since v is not admissible in H_m, Lemma 5.4 implies that there exists a pure critical set R in H_m with $\{r, s\} \subseteq R \subseteq V_m - v$ and a mixed critical set S with $s \in S \subseteq V_m \setminus \{r, v\}$. We may assume that R and S have been chosen to be the minimal such sets.

Suppose that $|R \cap X| \geq 2$. Then $X \cup R$ is mixed critical, $X \cap R$ is pure critical, $i_L(R \setminus X) = 0$ and $H_m - v = H_m[X] \cup H_m[R]$ by Lemma 5.5(c). Note that Lemma 5.5(b) applied to X and S tells us that $rs \notin E$. Let $H'_m = H_m - v + rs$. Then H'_m contains a unique $\mathcal{M}_{lc}$-circuit D. The minimality of R now implies that $E(D) = E(H_m[R]) + rs$ and $L(D) = \emptyset$. Let $G' = G - v + rs$. We will show that G' is $\mathcal{M}_{lc}$-connected. The facts that $H_m - v = H_m[X] \cup H_m[R]$, $X \subseteq V(H)$ and $i_L(R \setminus X) = 0$, imply that all edges of G' belong to H or D. Since H and D are both $\mathcal{M}_{lc}$-connected and have at least one edge in common (as $X \cap R$ is a critical set with at least two vertices in H_m), G' is $\mathcal{M}_{lc}$-connected and hence v is admissible in G. This contradicts the assumption that G has no admissible nodes so we must have $R \cap X = \{r\}$.

Since R is pure-critical, $R \cap X = \{r\}$ and $s \in R$, we can find a path P in $H_m[R]$ from r to s which avoids $X - r$. Since X and S are mixed critical in H_m and $v \notin X \cup S$, Lemma 5.5(b) implies there are no edges in H_m from $X \setminus S$ to $S \setminus X$, and Lemma 5.6(a) gives $X \cup S = V_m - v$. Since $r \in X \setminus S$ and $s \in S \setminus X$, this contradicts the existence of the path P. ■

Claim 6.3. Let v be a node of H_m in Y with three distinct neighbours. Then there is no unbalanced flower on v in H_m. □

Proof. Let $N(v) = \{w, u, z\}$ and suppose there exists an unbalanced flower (W, U, Z) on v in H_m with $u, z \in W \subseteq V_m \setminus \{v, w\}$, $w, z \in U \subseteq V_m \setminus \{v, u\}$, and $w, u \in Z \subseteq V_m \setminus \{v, z\}$. Relabelling W, U, Z if necessary, we may use Lemma 5.7 to deduce that W is mixed critical, U and Z are pure critical, $W \cap Z = \{u\}$, $W \cap U = \{z\}$, $U \cap Z = \{w\}$, $H_m - v = H_m[U] \cup H_m[W] \cup H_m[Z]$ and no vertex of $U \cup Z \setminus \{u, z\}$ is incident with a loop in H_m. Since X is mixed critical, each connected component of $H_m[X]$ contains a loop and hence $X \subseteq W$. This contradicts the hypothesis that G is balanced since the component of $G - \{u, z\}$ which contains $\{v, w\}$ will be loopless. ■

Claim 6.4. We can choose a node v of H_m in Y and a critical set X_v in H_m such that X_v is mixed node critical for v and $X \subseteq X_v$. □

Proof. Let F be the graph obtained by deleting all loops from $H_m[V^3 \cap Y]$. Then F is a forest by Lemma 5.2 and we may choose a vertex v of F such that $d_F(v) \leq 1$. Since v has at least two neighbours in Y by Claim 6.2, v has a neighbour z in Y with $d^\dagger(z) \geq 4$.

We first consider the case when $N(v) = \{y, z\}$. Then $y \in Y$ by Claim 6.2. Since v is not admissible in H_m by Claim 6.1, Lemma 5.4 implies that there exists a mixed critical set S in H_m with $y \in S \subseteq V_m \setminus \{v, z\}$. Then $S \cup X$ is a mixed node critical set for v by Lemma 5.5(b).

We next consider the case when $N(v) = \{w, u, z\} \subseteq Y$. Since $d_F(v) \leq 1$ we may assume that $d^\dagger(u) \geq 4$. Since v is not admissible in H_m, Lemma 5.8 and Claim 6.3 imply that there exists a strong or weak flower

(W, U, Z) on v in H_m with $u, z \in W \subseteq V_m \setminus \{v, w\}$, $w, z \in U \subseteq V_m \setminus \{v, u\}$, and $w, u \in Z \subseteq V_m \setminus \{v, z\}$. Then either Z, or U, is mixed critical and hence either $Z \cup X$, or $U \cup X$, is a mixed node critical set for v.

It remains to consider the case when $N(v) = \{w, u, z\} \not\subseteq Y$. Since v has at least two neighbours in Y we may assume that $u \in Y$ and $w \in X$. Since v is not admissible in H_m, Lemma 5.8 and Claim 6.3 imply that there exists a strong or weak flower (W, U, Z) on v in H_m with $u, z \in W \subseteq V_m \setminus \{v, w\}$, $w, z \in U \subseteq V_m \setminus \{v, u\}$, and $w, u \in Z \subseteq V_m \setminus \{v, z\}$. If Z is mixed critical or $|X \cap Z| \geq 2$ then $Z \cup X$ is a mixed node critical set for v. Hence we may assume that Z is not mixed critical and $X \cap Z = \{w\}$.

Thus (W, U, Z) is a weak flower. Lemma 5.6 and the fact that Z is not mixed critical now imply that W is mixed critical and $i_L(W^*) = 0 = d(W^*, Z^*)$. Furthermore, since $X \cap Z = \{w\}$, we have $X - w \subseteq Z^*$. Since $w \in W^*$, this implies that w is an isolated vertex of $H_m[X]$ and is not incident with any loops. This contradicts the fact that X is mixed critical in H_m. ■

Choose a node v of H_m in Y and a mixed node critical set X_v for v in H_m which satisfy the conditions of Claim 6.4 and are such that $|X_v|$ is as large as possible over all such choices of v and X_v. Let $Y_v = V_m \setminus X_v$. Since $X \subseteq X_v$ we have $v \in Y_v \subseteq Y$. Since X_v is mixed node critical for v, $|Y_v| \geq 2$. Let F_v be the graph obtained by deleting all loops from $H_m[V^3 \cap Y_v]$. By Lemma 5.3 we can choose a node z of H_m with $z \in Y_v - v$ such that $d_{F_v}(z) \leq 1$. By Claim 6.1, z has at least two distinct neighbours in G.

Suppose z has exactly two neighbours, say a, b, in G. If $\{a, b\} \cap X_v \neq \emptyset$ then the set $X'_v = X_v + z$ would contradict the maximality of X_v. Hence $\{a, b\} \subset Y_v$ and since $d_{F_v}(z) \leq 1$ we may assume that $d^\dagger(b) \geq 4$. Since z is not admissible in H_m we have $a \in X_z \subseteq V \setminus \{z, b\}$ for some mixed critical set X_z. Then $X'_z = X_z \cup X_v$ is mixed node critical for z and contradicts the maximality of X_v. Hence z has three distinct neighbours in G.

The facts that H_m is an $\mathcal{M}_{lc}$-circuit, X_v is mixed critical and $v \notin X_v + z$ imply that z has at most two neighbours in X_v. If z had exactly two neighbours in X_v, then $X_v \cup \{z\}$ would be a mixed node critical set for v and would contradict the maximality of X_v. Hence z has at most one neighbour in X_v. Since $d_{F_v}(z) \leq 1$, this implies that z is either a series node or a leaf node in H_m. Consider the following two cases.

Case 1: z is a series node in H_m.

Let $N(z) = \{p, q, t\}$. Since z is a series node in H_m and $d_{F_v}(z) \leq 1$, z has exactly one neighbour, say t, in X_v and t is a node in H_m. Without loss of generality, we may assume $d^\dagger(p) = 3$ and $d^\dagger(q) \geq 4$. Since z is not admissible, Lemma 5.4(a) implies there exists a (pure or mixed) critical set X_z with $\{t, p\} \subseteq X_z \subseteq V_m \setminus \{z, q\}$. Since the graph obtained by deleting all loops from $H_m[V^3]$ is a forest by Lemma 5.2, we have $pt \notin E_m$ and hence $X_z \neq \{t, p\}$. Since X_v, X_z are critical, t is a node, $t \in X_v \cap X_z$ and $z \notin X_v \cup X_z$, all neighbours of t other than z belong to $X_v \cap X_z$. Lemma 5.5(b) or (c) now implies that $X_v \cup X_z$ is mixed critical. Since $\{t, p\} \subseteq X_v \cup X_z \subseteq V \setminus \{z, q\}$ and $d^\dagger(q) \geq 4$, $X_v \cup X_z$ is mixed node critical for z in H_m. This contradicts the maximality of X_v.

Case 2: z is a leaf node in H_m.

Let $N(z) = \{z_1, z_2, z_3\}$. Since z is not admissible, Lemma 5.8 and Claim 6.3 imply there is either a strong or weak flower (Z_1, Z_2, Z_3) on z in H_m with $Z_i \subseteq V_m \setminus \{z, z_i\}$ for $1 \leq i \leq 3$. We have $z_i z_j \notin E_m$ for all $1 \leq i < j \leq 3$ by Lemma 5.6(b) (since $z_i \in Z_i^*$). Since z is a leaf node in H_m, we may suppose that $d^\dagger_{H_m}(z_1) \geq 4$ and $d^\dagger_{H_m}(z_2) \geq 4$. Since neither z_1 nor z_2 are nodes in H_m, Z_1 and Z_2 are two node critical sets for z with $Z_1 \cup Z_2 = V_m - z$ (by Lemma 5.6(a)) and at least one of them, say Z_1, is mixed critical. Then $Z_1 \cup X_v$ is mixed critical. If $z_1 \notin X_v$ then $Z_1 \cup X_v$ would be a mixed node critical set for z which would be larger than X_v since $z_2, z_3 \in Z_1$ and $|N(z) \cap X_v| \leq 1$. This would contradict the maximality of X_v so we must have $z_1 \in X_v$. The fact that $|N(z) \cap X_v| \leq 1$ now implies that $z_2, z_3 \notin X_v$. It follows that, if either Z_2 is mixed critical or $|Z_2 \cap X_v| \geq 2$, then $Z_2 \cup X_v$ would be a mixed node critical set for z which is larger than X_v. Hence Z_2 is pure critical and $Z_2 \cap X_v = \{z_1\}$.

We complete the proof by using a similar argument to the last paragraph of the proof of Claim 6.4. Lemma 5.6 and the fact that Z_2 is not mixed critical imply that (Z_1, Z_2, Z_3) is a weak flower, and $i_L(Z_1^*) = 0 = d(Z_1^*, Z_2^*)$. Furthermore, since $X_v \cap Z_2 = \{z_1\}$, we have $X_v - z_1 \subseteq Z_2^*$. This implies that z_1 is an isolated vertex of $H_m[X_v]$ and contradicts the fact that X_v is mixed critical in H_m. ■

Recursive constructions

We close this section by using Theorem 6.6 to obtain a recursive construction for rigid $\mathcal{M}_{lc}$-connected graphs. It uses the special case of the 2-sum operation in which one side of the 2-sum is a copy of K_4. We will refer to this operation and its inverse as a K_4-*extension* and a K_4-*reduction*, respectively.

We will need the following result on admissible reductions of loopless $\mathcal{M}_{lc}$-connected graphs and an extension of Lemma 5.9 to $\mathcal{M}_{lc}$-connected graphs.

Theorem 6.7. Let $G = (V, E)$ be an $\mathcal{M}_{lc}$-connected simple graph which is distinct from K_4 and $uv, xy \in E$.
(a) If G is an $\mathcal{M}_{lc}$-circuit then some vertex of $V \setminus \{u, v, x\}$ is an admissible node in G.

(b) If G is not an $\mathcal{M}_{lc}$-circuit then either some edge of $E \setminus \{uv, xy\}$ is admissible in G, or some vertex of $V \setminus \{u, v, x, y\}$ is an admissible node in G. □

Proof. Part (a) follows immediately from [1, Theorem 3.8]. To see (b) we choose an $\mathcal{M}_{lc}$-circuit H_1 in G containing uv, xy and then extend H_1 to an ear decomposition $H_1, H_2, \ldots, H_m$ of G. Then [12, Theorem 5.4] implies that either some edge of H_m distinct from uv, xy is admissible in G or some vertex of $H_m - \bigcup_{i=1}^{m-1} H_{m-1}$ is an admissible node of G. ■

An *unbalanced* 2-*separator* of a looped simple graph G is a pair of vertices $\{u, v\}$ such that $G - \{u, v\}$ has a component with no loops. Note that we allow this loopless component to be equal to $G - \{u, v\}$. An *unbalanced* 2-*separation* of G is an ordered pair of subgraphs (G_1, G_2) such that $G = G_1 \cup G_2$, $|V(G_1) \cap V(G_2)| = 2 < |V(G_2)|$ and $E(G_1) \cap E(G_2) = \emptyset = L(G_2)$.

Lemma 6.8. Let $G = (V, E, L)$ be a looped simple graph with $L \neq \emptyset$, and (G_1, G_2) be an unbalanced 2-separator of G such that $V(G_1) \cap V(G_2) = \{u, v\}$ and $uv \notin E$. Then the following statements are equivalent:
(a) $G + uv$ is $\mathcal{M}_{lc}$-connected;
(b) $G_1 + uv$ and $G_2 + uv$ are both $\mathcal{M}_{lc}$-connected;
(c) G is $\mathcal{M}_{lc}$-connected. □

Proof. Let $G_i + uv = (V_i, E_i, L_i)$ for $i = 1, 2$. By symmetry we may suppose $L_1 \neq \emptyset = L_2$. Choose $f_1 \in L_1$ and $g_2 \in E_2 - uv$.

(a)⇒(b). Suppose that $G + uv$ is $\mathcal{M}_{lc}$-connected. Then there exists an $\mathcal{M}_{lc}$-circuit C in $G + uv$ containing f_1, g_2. Lemma 5.9 now implies that C is the 2-sum of two $\mathcal{M}_{lc}$-circuits C_1, C_2 with $f_1, uv \in C_1 \subseteq G_1 + uv$ and $g_2, uv \in C_2 \subseteq G_2 + uv$. Transitivity now implies that $G_2 + uv$ is $\mathcal{M}_{lc}$-connected and that all loops of $G_1 + uv$ belong to the same $\mathcal{M}_{lc}$-connected component H_1 of $G_1 + uv$ as uv. To complete the proof we choose an edge $g_1 \in E_1$ and show g_1 is in H_1. Since $G + uv$ is $\mathcal{M}_{lc}$-connected, there exists an $\mathcal{M}_{lc}$-circuit C' in $G + uv$ containing f_1, g_1. If $C' \subseteq G_1 + uv$ then we are done. On the other hand, if $C' \nsubseteq G_1 + uv$ then Lemma 5.9 implies that $(C' \cap G_1) + uv$ is an $\mathcal{M}_{lc}$-circuit in $G_1 + uv$ containing f_1, g_1, uv.

(b)⇒(c). Suppose $G_1 + uv$ and $G_2 + uv$ are both $\mathcal{M}_{lc}$-connected. Then there exists an $\mathcal{M}_{lc}$-circuit C_i in $G_i + uv$ such that C_1 contains f_1, uv and C_2 contains g_2, uv. By Lemma 5.9, $C = (C_1 - uv) \cup (C_2 - uv)$ is an $\mathcal{M}_{lc}$-circuit in G. Transitivity now implies that all loops of G_1 and all edges of G_2 belong to the same $\mathcal{M}_{lc}$-connected component H of G. To complete the proof we choose an edge $g_1 \in E_1 - uv$ and show g_1 is in H_1. Since $G_1 + uv$ is $\mathcal{M}_{lc}$-connected, there exists an $\mathcal{M}_{lc}$-circuit C_1' in $G + uv$ containing f_1, g_1. If $C_1' \subseteq G$ then we are done. On the other hand, if $C_1' \nsubseteq G_1 + uv$ then $uv \in C_1'$ and Lemma 5.9 implies that $(C_1' - uv) \cup (C_2 - uv)$ is an $\mathcal{M}_{lc}$-circuit in G containing f_1, g_1.

(c)⇒(a). This follows immediately from Lemma 6.5. ■

We can now give our recursive construction.

Theorem 6.9. A looped simple graph is rigid and $\mathcal{M}_{lc}$-connected if and only if it can be obtained from $K_1^{[3]}$ by recursively applying the operations of 1-extension, K_4-extension and adding a new edge or loop. □

Proof. Sufficiency follows from Lemmas 6.5 and 6.8, and the facts that $K_1^{[3]}$ and K_4 are $\mathcal{M}_{lc}$-connected. To prove necessity it will suffice to show that every rigid $\mathcal{M}_{lc}$-connected graph G with at least two vertices can be reduced to a smaller rigid $\mathcal{M}_{lc}$-connected graph by applying either a 1-reduction, a K_4-reduction or an edge deletion. If G is balanced then Theorem 6.6 implies there is an edge, loop or node such that the corresponding edge/loop deletion or 1-reduction gives a rigid $\mathcal{M}_{lc}$-connected graph.

Hence we may assume that G is unbalanced. By Lemma 6.8, G is the 2-sum of an $\mathcal{M}_{lc}$-connected graph G_1 with at least one loop and a $\mathcal{M}_{lc}$-connected graph G_2 with no loops along an edge uv. We may assume that G_2 is 3-connected by choosing a 2-sum such that G_2 is as small as possible. If $G_2 = K_4$, then G can be reduced to G_1 by applying the K_4-reduction operation.

Hence we may assume that $G_2 \neq K_4$. Then Theorem 6.7 implies that G_2 contains either an admissible edge distinct from uv, or an admissible node x distinct from u, v. The graph G' obtained from G by performing the same reduction operation will then be $\mathcal{M}_{lc}$-connected since it is the 2-sum of G_1 and G_2' along uv. ■

Figure 8 provides an illustration of Theorem 6.9. In this figure a thick edge or loop means the next step in the construction will be a 1-extension which deletes that edge or loop, and a dashed edge or loop means the next step in the construction will add that edge or loop.

Theorem 6.9 implies in particular that all rigid $\mathcal{M}_{lc}$-circuits can be constructed from $K_1^{[3]}$ by applying the operations of 1-extension, K_4-extension and adding a new edge or loop. Since every rigid $\mathcal{M}_{lc}$-circuit

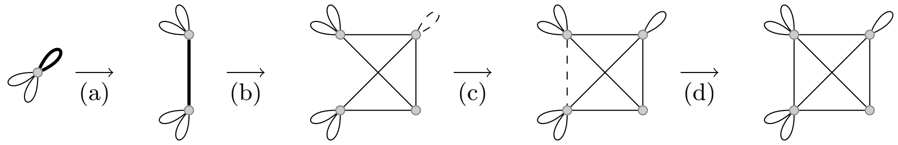

Fig. 8. A construction of the rigid $\mathcal{M}_{lc}$-connected graph drawn on the far right from a copy of $K_1^{[3]}$, by (a) 1-extension on a loop, (b) K_4-extension, (c) loop addition and (d) edge addition.

$G = (V, E, L)$ satisfies $|E| + |L| = 2|V|$ we can never use the edge/loop addition operation when constructing a rigid circuit from $K_1^{[3]}$. This gives the following result.

Theorem 6.10. A looped simple graph is a rigid $\mathcal{M}_{lc}$-circuit if and only if it can be obtained from $K_1^{[3]}$ by recursively applying the 1-extension and K_4-extension operations. □

7 Balanced $\mathcal{M}_{lc}$-connected graphs

We next consider reductions that preserve balance as well as $\mathcal{M}_{lc}$-connectivity. We will need two more structural results in unbalanced separations.

Lemma 7.1. Let G be $\mathcal{M}_{lc}$-connected and let $\{u, v\}$ be an unbalanced 2-separator in G. Then u and v each have at least four incident edges or loops in G. Furthermore, if G' is obtained from G by performing an edge/loop deletion or 1-reduction, then $\{u, v\}$ is an unbalanced 2-separator in G'. □

Proof. Let (G_1, G_2) be an unbalanced 2-separation in G with $V(G_1) \cap V(G_2) = \{u, v\}$. Then $G_1 + uv$ and $G_2 + uv$ are $\mathcal{M}_{lc}$-connected by Lemma 6.8. Hence u and v are each incident with at least three edges or loops in $G_i + uv$ for $i = 1, 2$. This implies that u and v each have at least four incident edges or loops in G. In particular, neither u nor v is a node of G. It is now straightforward to check that $\{u, v\}$ is an unbalanced 2-separator in G'. ■

Our next lemma shows that a rigid looped simple graph cannot contain two 'crossing' unbalanced 2-separations.

Lemma 7.2. Suppose that $G = (V, E, L)$ is a rigid looped simple graph and $(G_1, G_2), (G_1', G_2')$ are two unbalanced 2-separations in G with $G_i = (V_i, E_i, L_i)$ and $G_i' = (V_i', E_i', L_i')$ for $i = 1, 2$, $L_2 = \emptyset = L_2'$, $V_1 \cap V_2 = \{u, v\}$, and $V_1' \cap V_2' = \{u', v'\}$. Then $\{u', v'\} \subseteq V_i$ for some $i \in \{1, 2\}$. □

Proof. The lemma is trivialy true if $\{u, v\} \cap \{u', v'\} \neq \emptyset$ so we may assume that $\{u, v\} \cap \{u', v'\} = \emptyset$. Suppose for a contradiction that $u' \in V_1$ and $v' \in V_2$. If $\{u, v\} \subseteq V_i'$ for some $i \in \{1, 2\}$ then either $G - u'$ or $G - v'$ would have a loopless component. This would contradict the hypothesis that G is rigid and hence we may assume that $u \in V_1'$ and $v \in V_2'$.

Let $H_1 = G_1[V_1 \cap V_1']$, $H_2 = G_2[V_1' \cap V_2]$, $H_3 = G_2[V_2 \cap V_2']$, $H_4 = G_2'[V_1 \cap V_2']$, and put $n_i = |V(H_i)|$ for $1 \leq i \leq 4$, see Figure 9. Then H_1, H_2, H_3, H_4 cover $E \cup L$ and H_2, H_3, H_4 are loopless. This gives

$$r(G) \leq \sum_{i=1}^{4} r(H_i) \leq 2n_1 + \sum_{i=2}^{4} (2n_i - 3) = 2|V| - 1$$

and again contradicts the hypothesis that G is rigid. ■

An edge or loop f in a balanced $\mathcal{M}_{lc}$-connected graph G is *feasible* if $G - f$ is balanced and $\mathcal{M}_{lc}$-connected, and a node v is *feasible* if there exists a 1-reduction at v which results in a balanced $\mathcal{M}_{lc}$-connected graph. Figure 10 illustrates the difference between admissibility and feasibility. On the far left, the vertex x is not admissible in H. Each 1-reduction at x that adds a loop creates a vertex of degree 2, and the 1-reduction at x which adds an edge results in a graph with only 3 loops, none of which is contained in an $\mathcal{M}_{lc}$-circuit. The only admisible 1-reduction at the vertex y in the graph G is the one that adds the edge y_1y_2. However, the graph $G - y + y_1y_2$ is not balanced since $(G - y + y_1y_2) - \{u, v\}$ has no loops. Therefore y is admisible but not feasible in G. The 1-reduction at y_1 which adds the edge vy results in a balanced rigid $\mathcal{M}_{lc}$-circuit, so the vertex y_1 is feasible in G.

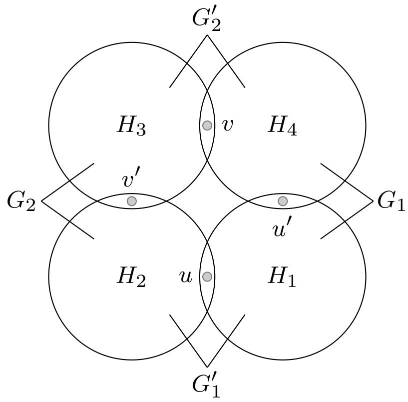

Fig. 9. The subgraphs H_1, H_2, H_3, H_4 in the proof of Lemma 7.2.

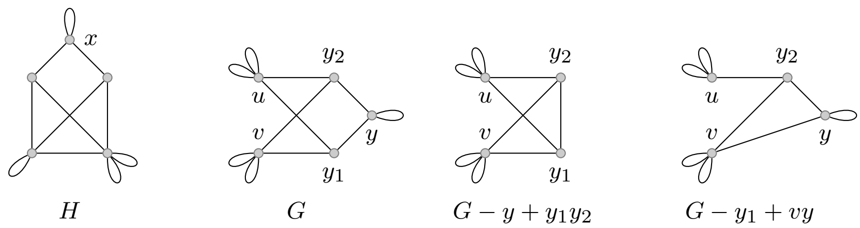

Fig. 10. The vertex x is not admissible in H, the vertex y is admissible but not feasible in G, and the vertex y_1 is feasible in G.

Theorem 7.3. Let $G = (V, E, L)$ be a balanced, $\mathcal{M}_{lc}$-connected looped simple graph distinct from $K_1^{[3]}$. Then some edge, loop, or node of G is feasible. □

Proof. Suppose, for a contradiction, that all possible edge/loop deletions and node 1-reductions of G fail to be either $\mathcal{M}_{lc}$-connected or balanced. Theorem 6.6 implies that G contains either an admissible node w or an admissible edge or loop f. By assumption neither w nor f is feasible. Let G' be the result of deleting f or performing an admissible 1-reduction at w. Then G' is $\mathcal{M}_{lc}$-connected and not balanced. Hence G' contains an unbalanced 2-separation H_1, H_2 where H_1 is loopless and $V(H_1) \cap V(H_2) = \{u, v\}$. We may suppose that the pair $(r, \{u, v\})$, where $r \in \{w, f\}$, has been chosen so that $X = V(H_1) \setminus \{u, v\}$ is as small as possible.

Let $H_1^+ = H_1 + uv$ and $H_2^+ = H_2 + uv$.

Claim 7.1. H_1^+ and H_2^+ are $\mathcal{M}_{lc}$-connected and H_2^+ is redundantly rigid. □

Proof. Lemma 6.8 implies that H_1^+ and H_2^+ are $\mathcal{M}_{lc}$-connected. Corollary 6.3 now implies that H_2^+ is redundantly rigid. ■

Note that the minimality of X implies that H_1^+ is 3-connected.

Claim 7.2. $uv \notin E$. □

Proof. Suppose for contradiction that $uv \in E$. We will show that uv is a feasible edge of G. Since G' is $\mathcal{M}_{lc}$-connected and $\{u, v\}$ is an unbalanced 2-separator in G', Lemma 6.8 implies that $G' - uv$ is $\mathcal{M}_{lc}$-connected. Since $G - uv$ is obtained from $G' - uv$ by an edge addition or a 1-extension, $G - uv$ is $\mathcal{M}_{lc}$-connected by Lemma 6.5. It remains to show that $G - uv$ is balanced.

Suppose $\{u', v'\}$ is an unbalanced 2-separator in $G - uv$. Since G is balanced, u and v belong to two different connected components of $(G - uv) - \{u', v'\}$ and at least one of these two components is loopless. Furthermore, $\{u', v'\}$ is an unbalanced 2-separator in $G' - uv$ by Lemma 7.1. This implies that every uv-path in $G' - uv$ contains either u' or v'. Since H_1^+ is 3-connected, there are two internally disjoint uv-paths in $H_1^+ - uv$. Hence $u', v' \in X$ and u and v belong to different components J_u and J_v of $H_2^+ - uv$, see Figure 11 for an illustration. Note that J_u and J_v both contain loops since G' is rigid and H_1 is loopless. This contradicts the fact that either

u or v belongs to a loopless component of $(G-uv)-\{u',v'\}$ since $G-uv$ is obtained from $G'-uv$ by applying a 1-extension or edge/loop addition. ■

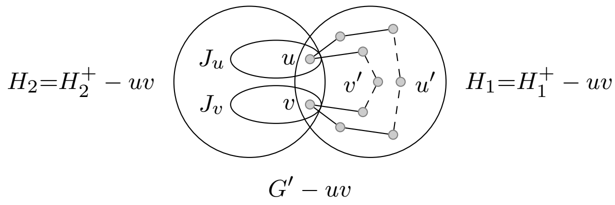

Fig. 11. An illustration of the structure of $G'-uv$ in the proof of Claim 7.2. The 3-connectivity of H_1^+ implies there are two disjoint uv-paths within $H_1 = H_1^+ - uv$, and hence removing u' and v' must destroy these paths.

Our strategy in the remainder of the proof is to show that some edge or node of G in H_1 is feasible in G. We have to be careful when considering the edges and nodes of H_1 since not all of them are edges or nodes in G. In addition a vertex which is a node in both H_1 and G may be incident with different edges in each graph. We use the following notation to handle this. We first put $E^\dagger(H_1) = E(H_1)\cap E(G)$. If $r=w$, we let $V^\dagger(H_1) = X\setminus N_G(w)$ and, if $r=f$ and $f=yz$, then we let $V^\dagger(H_1) = X\setminus\{y,z\}$. If the reduction operation which converts G to G' adds an edge e between two vertices of H_1 we put $\theta = e$. Otherwise there is a unique vertex x of X which is incident/adjacent to r and we put $\theta = x$.

Claim 7.3. No edge $e\in E^\dagger(H_1)$ is admissible in G. □

Proof. Suppose, for a contradiction, that $G-e$ is $\mathcal{M}_{lc}$-connected for some edge $e=ab\in E^\dagger(H_1)$. Then $G-e$ is not balanced so there exists an unbalanced 2-separator $\{u',v'\}$ in $G-e$. Since G is balanced, a and b are in different components of $(G-e)-\{u',v'\}$.

We can apply Lemma 7.1 to $G-e$ and G' respectively to deduce that $\{u',v'\}$ and $\{u,v\}$ are unbalanced 2-separators in $G''=G'-e$. Since G' is $\mathcal{M}_{lc}$-connected and contains a loop, it is redundantly rigid by Corollary 6.3. Hence $G''=G'-e$ is rigid. Lemma 7.2 now implies that $\{u',v'\}\subseteq V(H_i)$ for some $i\in\{1,2\}$. Since H_1^+ is 3-connected, H_1-e is a connected subgraph of G'' which contains $\{a,b\}$. On the other hand, $\{u',v'\}$ separates $\{a,b\}$ in $G-e$ and hence also in G''. This implies that $\{u',v'\}\cap V(H_1)\neq\emptyset$ and hence $\{u',v'\}\subseteq V(H_1)$.

Let X' be the vertex set of a loopless component of $(G-e)-\{u',v'\}$. Since G is balanced, each component of $H_2-\{u,v\}$ contains a loop and hence $X'\cap(V(H_2)\setminus S)=\emptyset$. This implies that $X'\cup\{u',v'\}\subseteq X\cup\{u,v\}\cup\{r\}$ so we may contradict the minimality of X by showing that at most one elelment of $\{a,b,r\}$ belongs to $X'\cup\{u',v'\}$. We have seen that a,b belong to different components of $(G-e)-\{u',v'\}$ so at most one of a,b belongs to $X'\cup\{u',v'\}$. If r is an edge then we trivially have $r\notin X'\cup\{u',v'\}$ so we may assume that r is a node. Note that $r\notin\{u',v'\}$ since $r\notin V(G'')$ and $\{u',v'\}\subset V(G'')$. If r is only adjacent to vertices of H_1, then the fact that G is balanced implies that r is incident with a loop and we have $r\notin X'$ since X' induces a loopless component of $(G-e)-\{u',v'\}$. On the other hand, if r is adjacent to some vertex of $V(H_2)\setminus S$, then the facts that $X'\cap(V(H_2)\setminus S)=\emptyset$ and $\{u',v'\}$ is a 2-separator of $G-e$ imply that $r\notin X'$. ■

Claim 7.4. No edge $e\in E^\dagger(H_1)$ is admissible in H_1^+. □

Proof. Suppose, for a contradiction, that H_1^+-e is $\mathcal{M}_{lc}$-connected. Then $G'-e$ is the 2-sum of H_1^+-e and H_2^+, and $G'-e$ is $\mathcal{M}_{lc}$-connected by Lemma 6.8. Lemma 6.5 now implies that $G-e$ is $\mathcal{M}_{lc}$-connected, contradicting Claim 7.3. ■

Claim 7.5. No node p of H_1^+ in $V^\dagger(H_1)$ is admissible in G. □

Proof. Suppose $p\in V^\dagger(H_1)$ is a node of H_1^+ with $N(p)=\{q,s,t\}$ and $G-p+st$ is $\mathcal{M}_{lc}$-connected. Then $G-p+st$ is not balanced, so has an unbalanced 2-separator $\{u',v'\}$. Since G is balanced, st and q are in different components of $(G-p+st)-\{u',v'\}$.

We can apply Lemma 7.1 to $G-p+st$ and G' respectively to deduce that $\{u',v'\}$ and $\{u,v\}$ are unbalanced 2-separators in $G''=G'-p+st$. Since G' is $\mathcal{M}_{lc}$-connected and contains loops it is redundantly rigid by Corollary 6.3. Hence $G'-pq$ is rigid. Since $G'-p$ is obtained from $G'-pq$ by deleting a vertex with two incident edges, it is rigid. Since G'' is obtained from $G'-p$ by an edge addition, it is also rigid. Lemma 7.2 now implies that $\{u',v'\}\subseteq V(H_i)$ for some $i\in\{1,2\}$. Since H_1^+ is 3-connected, H_1-p+st is a connected subgraph of G'' which contains $\{q,st\}$. On the other hand, $\{u',v'\}$ separates $\{q,st\}$ in $G-p+st$ and hence also in G''. This implies that $\{u',v'\}\cap V(H_1)\neq\emptyset$ and hence $\{u',v'\}\subseteq V(H_1)$.

Let X' be the vertex set of a loopless component of $(G-p+st)-\{u',v'\}$. Since G is balanced, each component of $H_2-\{u,v\}$ contains a loop and hence $X'\cap(V(H_2)\setminus\{u,v\})=\emptyset$. This implies that $X'\cup\{u',v'\}\subseteq (X\cup\{u,v\})+r-p$ so we may contradict the minimality of X by showing that r does not belong to $X'\cup\{u',v'\}$. This is trivially true if r is an edge, so we may assume that r is a node. Note that $r\notin\{u',v'\}$ since $r\notin V(G'')$ and $\{u',v'\}\subset V(G'')$. If r is only adjacent to vertices of H_1, then the fact that G is balanced implies that r is incident with a loop and we have $r\notin X'$ since X' induces a loopless component of $(G-p+st)-\{u',v'\}$. On the other hand, if r is adjacent to some vertex of $V(H_2)\setminus\{u,v\}$, then the facts that $X'\cap(V(H_2)\setminus S)=\emptyset$ and $\{u',v'\}$ is a 2-separator of $G-p+st$ imply that $r\notin X'$. ■

Claim 7.6. No node p of H_1^+ in $V^\dagger(H_1)$ is admissible in H_1^+. □

Proof. Suppose, for a contradiction, that p is a node with $N(p)=\{q,s,t\}$ and H_1^+-p+st is $\mathcal{M}_{lc}$-connected. Then $G'-p+st$ is the 2-sum of H_1^+-p+st and H_2^+ and $G'-p+st$ is $\mathcal{M}_{lc}$-connected by Lemma 6.8. It follows from Lemma 6.5 that $G-p+st$ is $\mathcal{M}_{lc}$-connected, contradicting Claim 7.5. ■

Claim 7.7. H_1^+ is an $\mathcal{M}_{lc}$-circuit. □

Proof. This follows from Theorem 6.7(b), Claims 7.1, 7.4 and 7.6, and the definition of $V^\dagger(H_1)$. ■

Claim 7.8. H_1^+ is isomorphic to K_4. □

Proof. Suppose H_1^+ is not isomorphic to K_4. By Claim 7.6, no node of H_1^+ in $V^\dagger(H_1)$ is admissible in H_1^+. Theorem 6.7(a), Claim 7.7 and the definition of $V^\dagger(H_1)$ now imply that $G'=G-w+xy$, for some node w of G with $x,y\in N_G(x)\cap X$ and u,v,x,y are the only admissible nodes in H_1^+.

Since x is an admissible node of H_1^+, H_1^+-x+st is $\mathcal{M}_{lc}$-connected for some $s,t\in N_{H_1^+}(x)$. Let $N_{H_1^+}(x)=\{q,s,t\}$. Since xy is an edge of H_1^+ and y is a node of H_1^+, we must have $y\in\{s,t\}$. Without loss of generality suppose $y=t$. Since $G'-x+sy$ is the 2-sum of H_1^+-x+sy and H_2^+, and H_2^+ is $\mathcal{M}_{lc}$-connected by Claim 7.1, Lemma 6.8 implies that $G'-x+sy$ is $\mathcal{M}_{lc}$-connected. Since $G-x+sw$ is a 1-extension of $G'-x+sy$, Lemma 6.5 now implies that $G-x+sw$ is $\mathcal{M}_{lc}$-connected.

Since x is not feasible in G, $G-x+sw$ is not balanced. Let $\{u',v'\}$ be an unbalanced 2-separator in $G-x+sw$. Since G is balanced, $\{u',v'\}$ separates sw and q. We can apply Lemma 7.1 to $G-x+sw$ and G' respectively to deduce that $\{u',v'\}$ and $\{u,v\}$ are unbalanced 2-separators in $G''=G'-x+sy$. Since G' is $\mathcal{M}_{lc}$-connected and contains a loop, Corollary 6.3 implies that G' is redundantly rigid, and hence $G'-xq$ is rigid. Since $G'-x$ is obtained from $G'-xq$ by deleting a vertex with two incident edges, it is rigid. Since $G''=G'-x+sy$ is obtained from $G'-x$ by an edge addition, it is also rigid. Lemma 7.2 now implies that $\{u',v'\}\subseteq V(H_i)$ for some $i\in\{1,2\}$. Since H_1^+ is 3-connected, H_1-x+sy is a connected subgraph of G'' which contains $\{q,s\}$. On the other hand, $\{u',v'\}$ separates $\{q,s\}$ in $G-x+sw$ and hence also in G''. This implies that $\{u',v'\}\cap V(H_1)\neq\emptyset$ and hence $\{u',v'\}\subseteq V(H_1)$.

Let X' be the vertex set of a loopless component of $(G-x+sw)-\{u',v'\}$. Since G is balanced, each component of $H_2^+-\{u,v\}$ contains a loop and hence $X'\cap(V(H_2)\setminus\{u',v'\})=\emptyset$. This implies that $X'\cup\{u',v'\}\subseteq(X\cup\{u,v\})+w-x$, so we may contradict the minimality of X by showing that w does not belong to $X'\cup\{u',v'\}$. Note that $w\notin\{u',v'\}$ by Lemma 7.1. If w is only adjacent to vertices of H_1, then the fact that G is balanced implies that w is incident with a loop and we have $w\notin X'$ since X' induces a loopless component of $(G-x+sw)-\{u',v'\}$. On the other hand, if w is adjacent to some vertex of $V(H_2)\setminus\{u,v\}$, then the facts that $X'\cap(V(H_2)\setminus\{u,v\})=\emptyset$ and $\{u',v'\}$ is a 2-separator of $(G-x+sw)$ imply that $w\notin X'$. ■

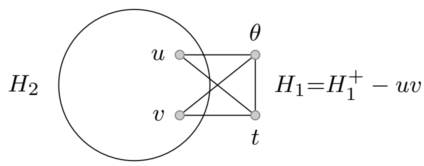

Fig. 12. The structure of G' in the proof of Claim 7.9.

Claim 7.9. $G'=G-w+xy$ for some node w of G with $x,y\in N(w)\cap V(H_1)$, and hence $\theta=xy\in E(H_1)$. □

Proof. Suppose that the claim is false. Then θ is a vertex in X, $V(H_1) = \{u, v, \theta, t\}$ and t is a node of G, see Figure 12. We will show that $G - t + uv$ is balanced and $\mathcal{M}_{lc}$-connected. Note that $uv \notin E$ by Claim 7.2. Note also that $G - t$ can be obtained from H_2^+ by either a a 1-extension (in the case that $G' = G - f$) or two 1-extensions (in the case when $G' = G - w + z_1z_2$ for some, not necessarily distinct, $z_1, z_2 \notin X$). Since H_2^+ is $\mathcal{M}_{lc}$-connected by Claim 7.1, it follows from Lemma 6.5 that $G - t + uv$ is $\mathcal{M}_{lc}$-connected. Since θ is adjacent to u and v and G is balanced there is no unbalanced 2-separation separating θ from uv in $G - t + uv$. Thus $G - t + uv$ is balanced. ■

Claim 7.10. $X \neq \{x, y\}$. □

Proof. Suppose, for a contradiction, that $X = \{x, y\}$. Then x, y are nodes of G. We shall show that $G - x + wv$ is $\mathcal{M}_{lc}$-connected and balanced. Note that $wv \notin E$ since, if w has a neighbour z distinct from x, y, then $z \in V(H_2) \setminus \{u, v\}$. Note further that $G - x + wv$ can be obtained from H_2^+ by a sequence of two 1-extensions. Since H_2^+ is $\mathcal{M}_{lc}$-connected by Claim 7.1, Lemma 6.5 implies that $G - x + wv$ is $\mathcal{M}_{lc}$-connected. Suppose that $G - x + wv$ is not balanced. Since G is balanced, there is an unbalanced 2-separator T in $G - x + wv$ that separates u and wv. Since u, w and v are all neighbours of y in $G - x + wv$, we must have $y \in T$. Since y is a node in $G - x + wv$, this contradicts Lemma 7.1. Thus $G - x + wv$ is balanced. ■

We can now complete the proof of the theorem. Claims 7.9 and 7.10 allow us to assume, after relabelling if necessary, that $X = \{x, t\}$ and $v = y$. Thus x is a node of G. We will show that $G - x + wt$ is $\mathcal{M}_{lc}$-connected and balanced. Note that $wt \notin E$ since, if w has a neighbour z distinct from x, y, then $z \in V(H_2) \setminus \{u, v\}$. Note further that $G - x + wt$ can be obtained from H_2^+ by a sequence of two 1-extensions. Since H_2^+ is $\mathcal{M}_{lc}$-connected by Claim 7.1, it follows from Lemma 6.5 that $G - x + wt$ is $\mathcal{M}_{lc}$-connected. Suppose that $G - x + wt$ is not balanced. Since G is balanced there is an unbalanced 2-separator T in $G - x + wt$ separating u and wt. Since ut is an edge of $G - x + wt$ we must have $t \in T$. Since t is a node in $G - x + wt$, this contradicts Lemma 7.1. Thus $G - x + wv$ is balanced. ■

Lemma 6.5 and Lemma 7.1 imply that the operations of edge/loop addition and 1-extension preserve the properties of being $\mathcal{M}_{lc}$-connected and balanced. Combined with Theorem 7.3, this immediately gives the following recursive construction.

Theorem 7.4. A looped simple graph is balanced and $\mathcal{M}_{lc}$-connected if and only if it can be obtained from $K_1^{[3]}$ by recursively applying the operations of performing a 1-extension and adding a new edge or loop. □

Consider the balanced, $\mathcal{M}_{lc}$-connected graph G drawn on the far right in Figure 13. We gave a construction of G from $K_1^{[3]}$ in Figure 8. However, the second step in this construction, where we use K_4-extension, resulted in an unbalanced graph. In Figure 13, we show that we can obtain G from $K_1^{[3]}$ by using only 1-extensions and edge or loop additions.

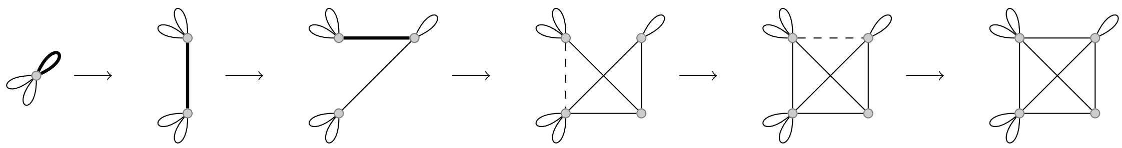

Fig. 13. An illustration of the recursive construction given in Theorem 7.4.

8 Global rigidity

We will use our recursive construction to characterise generic global rigidity. We first need a lemma which shows that the properties of redundant rigidity and $\mathcal{M}_{lc}$-connectedness are equivalent for connected balanced graphs.

Lemma 8.1. Let G be a balanced looped simple graph. Then G is $\mathcal{M}_{lc}$-connected if and only if G is connected and redundantly rigid. □

Proof. Necessity follows from Corollary 6.3 and the assumption that G is $\mathcal{M}_{lc}$-connected.

To prove sufficiency, we suppose, for a contradiction that G is connected and redundantly rigid but not $\mathcal{M}_{lc}$-connected. Let $H_1, H_2, \ldots, H_m$ be the $\mathcal{M}_{lc}$-components of G. Let $V_i = V(H_i)$, $X_i = V_i \setminus \bigcup_{j\neq i} V_j$ and $Y_i = V_i \setminus X_i$. Since G is connected, $|Y_i| \geq 1$, and since G is balanced, $|Y_i| \geq 3$ when H_i is loopless.

We may assume that $H_1, H_2, \ldots, H_s$ are loopless and $H_{s+1}, H_{s+2}, \ldots, H_m$ are not. Then

$$\begin{aligned} r(G) &= \sum_{i=1}^{s}(2|V_i|-3) + \sum_{i=s+1}^{m} 2|V_i| \\ &= \sum_{i=1}^{s}(2|X_i|+2|Y_i|-3) + \sum_{i=s+1}^{m}(2|X_i|+2|Y_i|) \\ &\geq \sum_{i=1}^{m}(2|X_i|+|Y_i|) + m - s, \end{aligned}$$

where the final inequality follows from the fact that $|Y_i| \geq 3$ for $1 \leq i \leq s$ and $|Y_i| \geq 1$ for $s+1 \leq i \leq m$. Since the X_i are disjoint we have $\sum_{i=1}^m |X_i| = |\bigcup_{i=1}^m X_i|$. Also each element of Y_i is contained in at least one other Y_j with $j \neq i$. Hence we have $\sum_{i=1}^m |Y_i| \geq 2|\bigcup_{i=1}^m Y_i|$. In addition the hypothesis that G is balanced implies that at least one H_i contains a loop so $m > s$. Hence

$$r(G) \geq 2(|\bigcup_{i=1}^{m} X_i| + |\bigcup_{i=1}^{m} Y_i|) + m - s > 2|V(G)|.$$

This contradicts the fact that $r(G) \leq 2|V(G)|$. ■

We may use Lemma 8.1 to restate Theorem 7.4 as follows.

Theorem 8.2. A looped simple graph is balanced, connected and redundantly rigid if and only if it can be obtained from $K_1^{[3]}$ by recursively applying the operations of performing a 1-extension and adding a new edge or loop. □

We can now characterise global rigidity for 2-dimensional generic linearly constrained frameworks.

Theorem 8.3. Suppose G is a connected looped simple graph with at least two vertices and (G, p, q) is a generic realisation of G as a linearly constrained framework in $\mathbb{R}^2$. Then the following statements are equivalent:
(a) (G, p, q) is globally rigid;
(b) G is balanced and redundantly rigid;
(c) (G, p, q) has a full rank equilibrium stress. □

Proof. The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) follow from Theorems 3.2 and 4.5, respectively. It remains to prove that (b) $\Rightarrow$ (c). We use induction on the number of vertices of G to show that (G, p, q) has a full rank equilibrium stress whenever G is connected, balanced and redundantly rigid. It is straighforward to check that every generic realisation of the smallest redundantly rigid looped simple graph $K_1^{[3]}$ has a full rank equilibrum stress (given by a 1×1 stress matrix of rank zero). The induction step now follows by using Lemma 4.6, and Theorems 4.7 and 8.2. ■

We can use Theorem 8.3 and the fact that a linearly constrained framework is globally rigid if and only if each of its connected components is globally rigid to deduce our next result.

Theorem 8.4. Suppose (G, p, q) is a generic linearly constrained framework in $\mathbb{R}^2$. Then (G, p, q) is globally rigid if and only if G is balanced and each connected component of G is either a single vertex with two loops or is redundantly rigid. □

Theorem 2.1 implies that redundant rigidity can be checked efficiently by graph orientation or pebble game type algorithms [2, 20]. Since we can also check the property of being balanced in polynomial time, Theorem 8.4 gives rise to an efficient algorithm to decide whether a given looped simple graph is generically globally rigid in $\mathbb{R}^2$.

We conclude this section by mentioning a possible direction for future research. As mentioned in the introduction, [8] gives a characterisation of generic rigidity for linearly constrained frameworks in $\mathbb{R}^d$ when each vertex is constrained to lie in an affine subspace of sufficiently small dimension compared to d. It would be interesting to obtain an analogous characterisation for generic global rigidity.

9 The number of equivalent realisations

We will extend Theorem 8.4 by determining the number of distinct frameworks which are equivalent to a given generic linearly constrained framework (G,p,q) when $G=(V,E,L)$ is a rigid $\mathcal{M}_{lc}$-connected looped simple graph. For $u,v\in V$, let $b(u,v)$ be the number of loopless connected components of $G-\{u,v\}$ and put $b(G)=\sum_{u,v\in V} b(u,v)$.

Theorem 9.1. Suppose (G,p,q) is a generic linearly constrained framework in $\mathbb{R}^2$ and that G is rigid and $\mathcal{M}_{lc}$-connected. Then there are exactly $2^{b(G)}$ distinct frameworks which are equivalent to (G,p,q). □

Our proof of Theorem 9.1 is similar to the proof of an analogous result for bar-joint frameworks [16, Theorem 8.2]. We will indicate below how the latter can be adapted to prove Theorem 9.1.

We first need a result on generic points in $\mathbb{R}^n$. We will denote the algebraic closure of a field $\mathbb{K}$ by $\overline{\mathbb{K}}$.

Lemma 9.2. Let $f:\mathbb{R}^n\to\mathbb{R}^n$ by $f(\mathbf{x})=(f_1(\mathbf{x}),f_2(\mathbf{x}),\ldots,f_n(\mathbf{x}))$, where $f_i(\mathbf{x})$ is a polynomial with coefficients in some extension field $\mathbb{K}$ of $\mathbb{Q}$ for all $1\leq i\leq n$. Suppose that $\max_{\mathbf{x}\in\mathbb{R}^n}\{\text{rank } df|_{\mathbf{x}}\}=n$. If either $\mathbf{x}$ or $f(\mathbf{x})$ is generic over $\mathbb{K}$, then $\mathbf{x}$ and $f(\mathbf{x})$ are both generic over $\mathbb{K}$ and $\overline{\mathbb{K}(\mathbf{x})}=\overline{\mathbb{K}(f(\mathbf{x}))}$. □

The proof of Lemma 9.2 is the same as that for [16, Lemmas 3.1, 3.2]. The only difference being that we work with polynomials with coefficients in $\mathbb{K}$ rather than $\mathbb{Z}$.

Let $G=(V,E,L)$ be a looped simple graph with $E=\{e_1,e_2,\ldots,e_m\}$, $F=\{\ell_1,\ell_2,\ldots,\ell_s\}$ and $q:L\to\mathbb{R}^2$. The *rigidity map* $f_{G,q}:\mathbb{R}^{2|V|}\to\mathbb{R}^{|E\cup L|}$ is defined by putting

$$f_{G,q}(p)=(f_{e_1}(p),\ldots,f_{e_m}(p),f_{\ell_1}(p)\ldots,f_{\ell_s}(p))$$

where $f_{e_i}(p)=\|p(u)-p(v)\|^2$ when $e_i=uv$ and $f_{\ell_j}(p)=q(\ell_j)\cdot p(v)$ when ℓ_j is incident to v.

Lemma 9.3. Let (G,p,q) be a generic, rigid, linearly constrained framework in $\mathbb{R}^2$, H be a minimally rigid spanning subgraph of G and (G,p',q) be an equivalent framework to (G,p,q). Then $\overline{\mathbb{Q}(p,q)}=\overline{\mathbb{Q}(f_{H,q}(p))}=\overline{\mathbb{Q}(p',q)}$. □

Proof. This follows from Lemma 9.2 by putting $f=f_{H,q}$ and $\mathbb{K}=\mathbb{Q}(q)$. ■

Given a linearly constrained framework (G,p,q) in $\mathbb{R}^2$, we say that two vertices u,v of G are *globally linked in* (G,p,q) if $\|p(u)-p(v)\|=\|p'(u)-p'(v)\|$ whenever (G,p',q) is equivalent to (G,p,q).

Lemma 9.4. Let (G,p,q) be a generic linearly constrained framework in $\mathbb{R}^2$ and v be a node of G such that $N_G(v)=\{u,w,x\}$ and $G-v$ is rigid. Then u,w are globally linked in (G,p,q). □

The proof of Lemma 9.4 is the same as that of [16, Lemma 4.1]. The only difference being that we use Lemma 9.3 instead of [16, Lemmas 3.3, 3.4].

Lemma 9.5. Suppose G is a rigid, $\mathcal{M}_{lc}$-connected looped simple graph and $\{u,v\}$ is an unbalanced 2-separation in G. Then u,v are globally linked in every generic realisation of G as a linearly constrained framework in $\mathbb{R}^2$. □

Proof. We use induction on $|E(G)|$. Choose an unbalanced 2-separation (H_1,H_2) of G such that H_2 is simple, $u,v\in V(H_1)$ and $|V(H_2)|$ is as small as possible. Let $V(H_1)\cap V(H_2)=\{u',v'\}$. Then $H_2+u'v'$ is 3-connected. In addition $H_1+u'v'$ is rigid, and $H_1+u'v'$ and $H_2+u'v'$ are both $\mathcal{M}_{lc}$-connected by Lemma 6.8.

Suppose $H_2+u'v'\neq K_4$. By Theorem 6.7, $H_2+u'v'$ has an admissible edge e distinct from $u'v'$ or an admissible node w distinct from both u' and v'. Let $H_2'=H_2-e$ in the former case and otherwise let $H_2'=H_2-w+xy$ be obtained by performing an admissible 1-reduction at w. Then $G'=(H_1\cup H_2')-u'v'$ is rigid and $\mathcal{M}_{lc}$-connected by Lemma 6.8. By induction, u,v are globally linked in every generic realisation of G'. This immediately implies that u,v are globally linked in every generic realisation of G if $G=G'+f$. Hence we may suppose that $G=G'-w+xy$. Since G is redundantly rigid and w is a node of G, $G-w$ is rigid. We can now use Lemma 9.4 to deduce that x,y are globally linked in G. The fact that u,v are globally linked in every generic realisation of G' now implies that u,v are globally linked in every generic realisation of G.

It remains to consider the case when $H_2+u'v'=K_4$. Choose $w\in V(H_2)\setminus\{u',v'\}$. Since G is redundantly rigid and w is a node of G, $G-w$ is rigid. Lemma 9.4 now implies that u',v' are globally linked in every generic realisation of G. If $\{u,v\}=\{u',v'\}$ then we are done so we may assume this is not the case. Then $\{u,v\}$ is an unbalanced 2-separation of $H_1+u'v'$. Since $H_1+u'v'$ is rigid and $\mathcal{M}_{lc}$-connected by Lemma 6.8, we may use induction to deduce that u,v are globally linked in every generic realisation of $H_1+u'v'$. The fact that u',v' are globally linked in every generic realisation of G, now implies that u,v are globally linked in every generic realisation of G. ■

Proof of Theorem 9.1

We use induction on $b(G)$. If $b(G) = 0$ then the result follows from Lemma 8.1 and Theorem 8.3. Hence we may suppose that $b(G) \geq 1$. Let (H_1, H_2) be an unbalanced 2-separation in G where H_2 is loopless and $|V(H_2)|$ is as small as possible. Let $V(H_1) \cap V(H_2) = \{u, v\}$. Then $H_2 + uv$ is 3-connected and we have $b(G) = b(H_1 + uv) + 1$ by Lemma 7.2. In addition, $H_1 + uv$ is rigid, and $H_1 + uv$ and $H_2 + uv$ are both $\mathcal{M}_{lc}$-connected by Lemma 6.8. Since $(H_2 + uv, p|_{H_2})$ is globally rigid as a bar-joint framework by Theorem 1.1 and u, v are globally linked in (G, p, q) by Lemma 9.5, the number of linearly constrained frameworks which are equivalent to (G, p, q) is exactly twice the number of linearly constrained frameworks which are equivalent to $(H_1 + uv, p|_{H_1}, q|_{H_1})$ (each equivalent framework to $(H_1 + uv, p|_{H_1}, q|_{H_1})$ gives rise to two equivalent frameworks to (G, p, q) which are related by reflecting H_2 in the line through u, v). We can now use induction to deduce that the number of linearly constrained frameworks which are equivalent to (G, p, q) is $2 \times 2^{b(H_1+uv)} = 2^{b(G)}$. ■

We close this section by noting that the problem of counting the number of non-congruent frameworks which are equivalent to a given generic bar-joint framework in $\mathbb{R}^2$ can be converted to that of counting the number of distinct frameworks which are equivalent to a related linearly constrained framework in $\mathbb{R}^2$. Given a simple graph G we construct a looped simple graph G^* by choosing an edge uv of G and adding two loops at both u and v. It is not difficult to see that the number of distinct linearly constrained frameworks which are equivalent to a generic rigid (G^*, p, q) is exactly twice the number of non-congruent bar-joint framework frameworks which are equivalent to (G, p). In particular, we can use this construction to deduce Theorem 1.1 from Theorem 9.1.

10 Spherically constrained frameworks

A d-dimensional *spherically constrained framework*, is a triple (G, p, s) where $G = (V, E, L)$ is a looped simple graph, $p : V \to \mathbb{R}^d$ and $s : L \to \mathbb{R}^d$. Two spherically constrained frameworks (G, p, s) and $(G, \tilde{p}, s)$ are *equivalent* if they have the same edge lengths and, for all $v \in V$ and all loops ℓ incident to v, both $p(v)$ and $\tilde{p}(v)$ lie on the same $(d-1)$-dimensional sphere centred on $s(\ell)$. We say that (G, p, s) is *globally rigid* if no other sperically constrained framework is equivalent to (G, p, s) and *rigid* if there exists an $\epsilon > 0$ such that every equivalent framework $(G, \tilde{p}, s)$ distinct from (G, p, s) satisfies $\|p - \tilde{p}\| > \epsilon$. The Jacobian matrix for the system of constraints defined by a spherically constrained framework (G, p, s) will be identical to the Jacobian matrix for the linearly constrained framework (G, p, q) when $q : L \to \mathbb{R}^d$ is defined by $q(\ell) = p(v) - s(\ell)$ for all $v \in V$ and all loops ℓ incident to V. This implies that a looped simple graph G will be generically rigid as a d-dimensional linearly constrained framework if and only of it is generically rigid as a d-dimensional spherically constrained framework.

The situation for global rigidity is different. Consider a looped simple graph G with one vertex and d loops. We have seen that every generic realisation of G as a d-dimensional linearly constrained framework is globally rigid. Since any set of d generic $(d-1)$-dimensional spheres in $\mathbb{R}^d$, which have a non-empty intersection, intersect in two distinct points, no generic realisation of G as a d-dimensional spherically constrained framework is globally rigid. We conjecture, however, that this graph G is the only connected looped simple graph with the property that the global rigidity of its d-dimensional generic realisations as generic linearly constrained or spherically constrained frameworks differ. We will verify this conjecture for $d = 2$. We first give a construction which relates the global rigidity of a given spherically constrained framework to that of an associated linearly constrained framework.

Suppose $d \geq 2$ is an integer and $G = (V, E, L)$ is a looped simple graph. Let $G^{+d} = (V \cup V^+, E \cup E^+, L^+)$ be the looped simple graph obtained from G by applying a d-dimensional 1-extension operation which deletes a loop ℓ at a vertex x and adds a new vertex v_ℓ, a new edge xv_ℓ and d new loops at v_ℓ, for *all* loops $\ell \in L$. Thus $V^+ = \{v_\ell : \ell \in L\}$ and $E^+ = \{xv_\ell : \ell \in L \text{ and } \ell \text{ is incident to } x\}$. Given maps $p : V \to \mathbb{R}^d$ and $s : L \to \mathbb{R}^d$ we define $p^+ : V \cup V^+ \to \mathbb{R}^d$ and $q : L^+ \to \mathbb{R}^d$ by putting $p^+(v) = p(v)$ for all $v \in V$, $p^+(v_\ell) = s(\ell)$ for all $v_\ell \in V^+$, and choosing any q such that, for all $v_\ell \in V^+$, the set of vectors $\{q(\ell^+) : \ell^+ \in L^+ \text{ and } \ell^+ \text{ is incident to } v_\ell\}$ spans $\mathbb{R}^d$. It is straightforward to check that (G, p, s) and $(G, \tilde{p}, s)$ are equivalent spherically constrained frameworks if and only if (G^{+d}, p^+, q) and $(G^{+d}, \tilde{p}^+, q)$ are equivalent linearly constrained frameworks. This immediately gives us the following result.

Lemma 10.1. (G, p, s) is globally rigid as a d–dimensional spherically constrained framework if and only if (G^{+d}, p^+, q) is globally rigid as a d–dimensional linearly constrained framework. □

Lemma 4.6 implies that the property of having an infinitesimally rigid realisation with a full rank equilibrium stress is a generic property of d-dimensional linearly constrained frameworks. We next show that this property holds for generic realisations of a looped simple graph G in $\mathbb{R}^d$ if and only if it holds for generic realisations of G^{+d}.

Lemma 10.2. Suppose G is a looped simple graph. Then G has a generic realisation as an infinitesimally rigid linearly constrained framework in $\mathbb{R}^d$ with a full rank equilibrium stress if and only if G^{+d} has a generic realisation as an infinitesimally rigid linearly constrained framework in $\mathbb{R}^d$ with a full rank equilibrium stress. □

Proof. Necessity follows immediately from Theorem 4.7 since G^+ is obtained from G by a sequence of d-dimensional 1-extension operations. To prove sufficiency it will suffice to show that the d-dimensional 1-reduction operation which deletes a vertex v incident to d loops and one edge vx from a looped simple graph $H = (V, E, L)$ and then adds a new loop ℓ_v at x to create $H^- = (V^-, E^-, L^-)$, preserves the property of having a generic realisation which is infinitesimally rigid and has a full rank equilibrium stress.

Suppose (H, p, q) is infinitesimally rigid and has a full rank equilibrium stress. Let (H^-, p^-, q^-) be defined by putting $p^-(u) = p(u)$ for all $u \in V^-$, $q^-(\ell) = q(\ell)$ for all $\ell \in L^- \setminus \{\ell_v\}$, and $q^-(\ell_v) = p(v) - p(x)$. It is straightforward to check that (H^-, p^-, q^-) is infinitesimally rigid and that, if (ω, λ) is a full rank equilibrium stress for (H, p, q) then (ω^-, λ^-) is a full rank equilibrium stress for (H^-, p^-, q^-), where $\omega^-(e) = \omega(e)$ for all $e \in E^-$, $\lambda^-(\ell) = \lambda(\ell)$ for all $\ell \in L^- \setminus \{\ell_v\}$, and $\lambda^-(\ell_v) = \omega(vx)$. ■

We can combine these lemmas with Theorem 8.3 to characterise generic global rigidity for spherically constrained frameworks in $\mathbb{R}^2$. This answers a question posed by Robert Connelly during the 2019 workshop "Geometric constraint systems: rigidity, flexibility and applications" at Lancaster University.

Theorem 10.3. Let (G, p, s) be a generic 2-dimensional spherically constrained framework. Then (G, p, s) is globally rigid if and only if G is balanced and redundantly rigid. □

Proof. We can assume without loss of generality that G is connected and has at least two vertices. Let (G^+, p^+, q) be the 2-dimensional linearly constrained framework associated to (G, p, s). Since the choice of q is arbitrary we may assume that (G^+, p^+, q) is generic. Then (G, p, s) is globally rigid if and only if (G^+, p^+, q) is globally rigid by Lemma 10.1. Theorem 8.3 tells us that (G^+, p^+, q) is globally rigid if and only if (G^+, p^+, q) is infinitesimally rigid and has a full rank equilibrium stress. Lemma 10.2 now tells us that this occurs if and only if G has a generic realisation as an infinitesimally rigid linearly constrained framework with a full rank equilibrium stress. We can now apply Theorem 8.3 again to deduce that this occurs if and only if G is balanced and redundantly rigid. ■

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