Strategic Contagion and Diffusion in Financial and Social Networks

A thesis submitted in partial fulfilment of the requirements of the degree of Doctor of Philosophy (Ph.D.) in Economics

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September 2019

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Declaration

I, Maya Jalloul, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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Details of collaboration and publications: Chapter 2 is a joint work with my supervisor, Nizar Allouch.

Abstract

This thesis examines contagion in networks in two settings: narratives infection in social networks and default contagion in financial networks. In the first chapter, we consider the choice of a politician of the timing of disclosure of a narrative, to maximise its adoption at a given deadline. Voters update their opinions following a variation of the average-based De Groot learning concept. With a certain probability a voter adopts an alternative narrative, and the politician faces a trade-off between early and late disclosure. We find the optimal timing of disclosure for disconnected and connected voters in certain networks. Finally, we consider the impact of homophily on timing of disclosure and explore comparative statics.

Chapter two presents a model of default in financial networks where the decision by one agent on whether or not to default impacts the incentives of other agents to escape default. Agents' payoffs are determined by the clearing mechanism introduced by the seminal contribution of Eisenberg and Noe (2001). We first show the existence of a Nash equilibrium of this default game. Next, we develop an algorithm to find all Nash equilibria that relies on the financial network structure. Finally, we explore some policy implications to achieve efficient coordination.

In the last chapter, we extend the model of exposure of financial institutions through liabilities linkages introduced by Eisenberg and Noe (2001), and we explore a second channel of exposure through credit lines or promised payments. While the liabilities create direct risk of default, credit lines can cause the default of their potential receivers. We prove the existence of a unique payment equilibrium and establish a fictitious default algorithm that computes the payment equilibrium and the chain of defaults. Finally, we investigate approaches to mitigate risks through multilateral netting and central clearing and by targeting banks with cash injections.

Acknowledgements

First and foremost, I would like to thank my supervisor Professor Nizar Allouch. I am deeply grateful for the invaluable guidance and support he provided throughout this period. His continuous advice and help allowed me to grow as a researcher. I am also thankful to Professor Marco Mariotti and Professor Sujoy Mukerji for their constructive suggestions and insights.

I thank the Microeconomic Theory group and my fellow PhD students at Queen Mary University of London for their helpful comments.

I am also very grateful to Mirela, Serena and Silvia, for being my support system during this life-changing experience and for all the priceless moments we shared together.

A special thanks goes to Omar. Words cannot express how grateful I am for all his ceaseless support, sacrifices and encouragement in every moment of this PhD. I consider myself extremely lucky to have him by my side.

Finally, I want to thank my parents, Hiam and Mohieddine, for inspiring me and for always believing in me. They have my greatest gratitude for their unconditional love and support that helped me get this far. I dedicate this thesis to them.

Introduction

Individual and institutional interactions impact several economic and social phenomena that range from learning, opinion formation, technology adoption, spread of diseases and epidemics up to trade of goods and services and financial risks. Being interconnected, agents and entities influence and get influenced by each other continuously and repeatedly in many different ways. In this context, the study of social and economic networks, which relies on graph theory, has been an instructive method to investigate the diffusion of ideas, information and risks through links and connections. In this thesis, we study two frameworks of economic networks and investigate contagion among different agents and institutions: the diffusion of narratives in social networks, and the contagion of the risk of default in financial networks.

Narrative adoption and strategic timing of disclosure

Chapter 1 investigates narrative adoption and opinion formation in social networks. A politician strategically chooses the timing to disclose a narrative in order to maximise its adoption by voters at an exogenously given deadline – for instance, an election. After disclosure, a voter can be influenced, with some probability, by an alternative narrative opposing the politician's. A voter updates his opinion by repeatedly averaging the politician's narrative, the alternative narrative and his neighbours' beliefs, as well as his own beliefs, in a De Groot learning manner. We determine the optimal timing of disclosure for the cases of one voter and a group of voters. We also examine connections among voters over symmetric networks. We consider two types of voters – supporters who are not influenced by the alternative narrative, and non-supporters, who switch their attention from the politician to the alternative narrative with a strictly positive probability. We show that as the number of supporters increases, the optimal number of learning periods (that is equivalently the length of the learning phase) increases, and the optimal timing of disclosure is earlier. Next, we study the impact of homophily: the phenomenon of having agents more connected to agents of their own type. We find that with higher homophily, the narrative adoption of the supporter is higher and that of the non-supporter is lower.

Moreover, we show that the optimal number of learning periods increases with the weight that the politician assigns to the supporter's opinion. Conversely, the optimal number of learning periods decreases with the probability of rejecting the politician's narrative. We conclude by illustrating our findings in a particular symmetric network – which is the circle.

Default and coordination in financial networks

In Chapter 2, we introduce strategic interactions in financial networks and consider a default coordination game. We model a financial system where agents are linked to each other through liabilities, and where the decision of a given agent on defaulting (or not) is affected by the corresponding decisions of other agents. As a result of complementarities and interdependencies of payments, the setting will generate a coordination game. The liabilities repayments are determined following the model of Eisenberg and Noe (2001). We prove the existence of a pure-strategy Nash equilibrium of the default game, and we show that the multiplicity of equilibria, which is a general feature of coordination games, is connected in our setting to the presence of cyclical obligations. Next, we establish an algorithm that computes all Nash equilibria, based on the network structure. An important issue that arises in coordination games is inefficient coordination, which results in our setting in a suboptimal number of agents defaulting. We define the best equilibrium as the Nash equilibrium with the smallest number of defaults. In this regard, we investigate the introduction of a central clearing counterparty as a policy to reach efficient coordination and the implementation of the best equilibrium.

Credit lines contagion in financial networks

Systemic risk, or the extent of contagion via domino effect and its impact on financial stability, has been the subject of ongoing interest given its significant spillover effects on the entire economy, generating long periods of low growth rates, unemployment and excessive levels of accumulated debts. In Chapter 3, we consider a financial system in which banks are connected via two exposure channels: liabilities, which are the obligations they owe to each other; and credit lines which take the form of payments they have promised to make in the future, or the debt they have promised to issue. We assume that liabilities and credit lines under consideration, have the same maturity date, and we examine the risk of default of different institutions. We extend the approach introduced by the seminal work of Eisenberg and Noe (2001) and use a fixed-point argument to prove the existence of a unique payment equilibrium. Afterwards, we present an algorithm that can compute the payment equilibrium and the possible chains of defaults. We also consider possible policy implications in order to mitigate the risk of default. One way is multilateral netting, which can be achieved through a central clearing counterparty that operates in the credit lines network or in the liabilities network or in both. A second approach is the intervention of a regulator and cash injections to particular banks in order to increase the interbank payments across the financial system.

Chapter 1

Narrative adoption and strategic timing of disclosure

1.1 Introduction

How do people form their opinions and how can communication strategies affect public opinion? These questions have been important for policymakers in choosing how to disclose their policies, marketing managers in deciding when to advertise a new product and researchers in understanding learning dynamics, along with many other contexts.

In politics, the choice of adequate communication and persuasion strategies has always been a key issue. Furthermore, the interest in narratives, stories and fake news has spiked in recent years, especially with the role that social media outlets have been playing in channelling them. In this chapter, we consider a model of adoption of narratives with naive average-based learning in an electoral framework. A group of non-strategic agents, voters, tend to adopt a narrative proposed by a strategic politician. Starting with a benchmark of one politician and one voter, the results are later generalised to a large number of voters. Understandably, this setting is not limited to an election or a political framework and can rather be applied to a variety of frameworks where one or more agents try to convey their opinion to a group of people or to persuade the group of people to believe something or take a certain action. The results can be naturally generalised to any context with an influencer and followers, an instructor and learners, and many similar situations where one agent is conveying a story or an idea to a group of individuals.

The main novelty in our model is that it explores learning while focusing on the timing in a finite framework. Often, in many real-life situations, there is a deadline for the opinion formation process, and the public opinion matters most at a particular point in time. For instance, opinions regarding political issues generally matter most at the time of an election from the politician's point of view. Broadly, the aim is to focus on two aspects. First, the communication strategies of politicians ahead of an election are not always based on promoting well-defined policies. Instead, there is evidence that electoral campaigns can revolve around advancing a certain opinion, narrative or story. Second, most of the opinion formation models focus on infinite learning processes and look at the convergence of opinions. However, in practice, there is a time horizon a deadline determined exogenously by several possible factors. An election is one example.

We study a setting in finite time where a given politician has a certain fixed opinion or story that he aims to convey to the voters, by a deadline that is for instance the date of an election. Nevertheless, after the politician's announcement of his opinion, the voter can replace the announcement with an alternative opinion, and abandon the politician's. In particular, we investigate the case where with some positive probability, the voter can abandon, disbelieve or move away from the narrative. This is similar to epidemics models, which generally model susceptible agents who can be infected with a certain positive probability, and recover afterwards with a positive probability. Here, we allow for partial infection. In other words, a voter can accept the narrative only partially. Consequently, the politician encounters a trade-off: disclosing his narrative early and bringing the voter closer, or disclosing later to avoid the narrative being replaced by the alternative.

We also explore social interactions through a network of voters. Networks have been studied and applied to different concepts in economics. Furthermore, it has been well established that the network structure and the distribution of social ties affect the opinion formation process. Individuals are influenced by their neighbours, peers, families and friends in various decisions they make in their life, from what they consume to their political inclination and opinions regarding specific issues. Here, a given individual learns about the narrative not only from the sender or the source of the narrative (who, in our model, is the politician), but also from friends, family, neighbours and colleagues – in other words, people who form his social network. We investigate how narrative adoption can be affected by their social network, and how this will impact the average opinion at the end of the learning process, and consequently the disclosure strategy of the politician.

1.1.1 Discussion: assumptions and motivation

Narratives There exist several definitions for narratives, which essentially agree that a narrative is a story that is told, but differ on the purposes for which that story is told and how it is presented. For instance, Bruner (1991) discusses how narratives not only represent, but also construct reality. In his paper on narrative economics, Shiller (2017) defines a narrative as a simple story, which explains events that are brought up in conversation between people and on news and social media. Furthermore, in his book (2019), he adds that "a story may also be a song, joke, theory, explanation, or plan that has emotional resonance and that can easily be conveyed in casual conversation". On the other hand, Benabou, Falk and Tirole (2018) describe narratives as arguments that people share to justify moral actions. Here, we use a generic definition: a narrative is a story, an idea or an argument used to explain facts, which can be but is not necessarily true, and which people discuss and might fully or partially adopt or reject.

It is discussed that narratives in societies date back to centuries ago. They have impacted the way people have thought about various phenomena throughout history, and they have also been present in economics as well as in sociology, history, political science and many other fields. In economics, for instance, they were widely used to provide reasons for economic crises from the Great Depression up until the recent European debt crisis.

Epidemics of narratives As suggested by Shiller (2017), the contagion of narratives can be approached using an epidemics model, in particular the seminal work by Kermack-McKendrick (1927), which introduced the SIR model, whereby they split a given population into three categories: susceptibles, infected and recovered. In our model, we start with a population of susceptible individuals at the time of disclosure of the narrative, where each individual becomes "infected" with some probability. Infection here means including the narrative in part of the individual's opinion of the next period. In other words, an individual can be partially infected by the narrative. Thereupon, the individual recovers with a positive probability, replacing the politician's narrative with another one, or with his initial opinion. When plotting the opinion of an individual over time, we observe that it will have a bell shape similar to the shape of the curve of the number of infected and contagious over time, in an epidemic, increasing then decreasing.

Average-based naive learning Social interaction generally comprises repeated sharing of ideas, information and opinions among connected individuals who form complex networks. That is mainly why the setting adopted in this chapter has the sense of the De Groot naive learning model based on imitation and repeated averaging of beliefs of neighbours. Golub and Jackson (2010) study when agents using simple updating rules, such as De Groot updating, correctly aggregate information and reach the true state of the world, in a similar way to the fully rational learning processes.

Jadbabaie, Molavi, Sandroni and Tahbaz-Salehi (2012) investigate a model where individuals communicate with their neighbours about a certain parameter via a simple updating rule and show that they aggregate information correctly if they consider their personal signals in a Bayesian way. Molavi, Tahbaz-Salehi and Jadbabaie (2018) provide behavioural foundations for non-Bayesian models of learning on social networks.

Types of voters A voter can be characterised by several features: his initial opinion, his network connections and weights on his neighbours, his attention and the weight he places on the politician, i.e. the probability of accepting the narrative of the politician, and the probability of recovery of the narrative. Notably, the attention can be regarded as the trust a given voter has in the politician's narrative or in the alternative narrative. Consequently, with some probability, the voter switches his attention, and replaces the politician's opinion with an alternative one. We consider the case where the alternative narrative is equivalent to the initial opinion of the voter. With this assumption, the probability of replacing the politician's narrative would represent the probability of rejecting the politician's narrative and moving back to the initial opinion.

Types of politicians Politicians can have different objectives, which in turn determine their strategy and choices when campaigning and addressing the people. A politician who worships an ideology and principles has naturally a different objective from a politician who cares about winning an election or reaching office. While the former focuses on citizens and voters who adopt or have a tendency to adopt the same political principles, who are known as "core voters", the latter directs his attention and action towards "swing" or "median" voters, who are less politically inclined. These are two extreme examples of politicians: ideology-motivated versus office-motivated. Nevertheless, in real life, politicians have both objectives and care about winning elections as well as having their principles adopted, but not equally. We look at different types of politicians and explore how the objective of the politician affects the timing of disclosure and the narrative adoption.

Homophily Individuals are generally inclined to associate and connect with others of similar types, whether that is in relation to their profession, age, gender, race, religion or any other characteristic. This favouritism towards similar individuals is known as homophily, and it is a common feature of social networks. With the emergence of social media outlets, homophily has become more observable and measurable and the interest in studying it has increased accordingly. Halberstam and Knight (2016) use data on connections between Twitter users and find that they are exposed to like-minded information disproportionately.

1.1.2 Results

First, we examine a benchmark model with one politician and only one voter. The voter updates his opinion in discrete time periods, by splitting his attention between his own opinion and an external opinion. This external opinion is the politician's narrative at the time of disclosure, but can be replaced after disclosure with an alternative narrative with some positive probability. The deadline $T \in \mathbb{N}_+$, which can be for instance an election, is equivalently the maximal number of learning periods. The politician's main objective is to choose a strategy that guarantees the voter's opinion at time T to be the closest to her proposed narrative.

We determine an optimal strategy for the timing of disclosure of the narrative, which is a function of the voter's attention to the politician and of the probability of replacing the narrative of the politician. We extend the setting to include more voters, and we find the optimal time of disclosure for the case of n disconnected voters, and for special networks such as networks with symmetric connections, the circle and the complete network. We also explore the existence of different types of voters (supporters and non-supporters), and

the impact of homophily on the dynamics of opinions.
We examine the case of a balanced network, which we define as a network represented by a double-stochastic matrix. We show that in this case if the politician targets the average opinion and weights the voters equally, the network has no effect on the optimal timing of disclosure. Moreover, we show that as the number of supporters increases, the average opinion and the optimal number of learning periods increase.

One main result is relating networks with symmetric structures, where we consider a network of two groups of voters, connected with an equal number of links to voters of the same type and with an equal number of links to voters of the other type. We show that when the politician targets the weighted average opinion, the belief of a supporter increases with a higher level of homophily, while that of a non-supporter decreases. Furthermore, the weighted average opinion increases with the weight assigned to supporters, and decreases with the probability of a non-supporter replacing the politician's narrative. Consequently, if the politician assigns a larger weight to the supporters' opinions, the optimal number of learning periods increases as homophily increases, and the politician's strategy would be to disclose earlier.

The chapter is organised as follows: Section 1.2 is dedicated to the literature review. We present the model and investigate the one voter benchmark in Section 1.3. We study the settings with n disconnected and connected voters in Sections 1.4 and 1.5 and Section 1.6. Iterations and computations of opinions are gathered in Appendix A, while proofs are in Appendix B.

1.2 Related literature

The chapter builds mainly on two strands of literature: narratives and average-based learning in networks.
Narratives have become a topic of interest in several fields of study. Shiller (2017) investigates how economic fluctuations are affected by narrative epidemics, in contrast to imperatives. He shows how several major economic events were considered to be the result of popular narratives at different times. He suggests that many narratives are either based on false ideas or have no factual basis, yet still propagate widely. Moreover, he suggests that the prominent model introduced by Kermack and McKendrick (1927) and known as the SIR model, which divides a population into susceptibles, infected and recovered individuals, can be applied to the contagion of narratives: an "infected" individual, who is convinced by a narrative, can infect a susceptible individual, who in turn would be infected after being exposed to a certain number of other infected individuals. While adopting a

somewhat different definition for narratives than the one we present here, Benabou, Falk and Tirole (2018) study narratives as justifications for moral behaviour.

We introduce here a model of average-based learning of networks. The seminal paper by De Groot (1974) provided a foundation for average-based learning. He established a model where a group of individuals, each having a certain probability distribution, interact to estimate some parameter. His main result was providing conditions for convergence to a consensus among agents. This model has been examined, developed and extended by a large number of research papers, which applied it to different settings and frameworks.
DeMarzo, Vayanos and Zwiebel (2003) introduce a phenomenon in opinion formation that they call persuasion bias and define as the failure of individuals to consider repeated information that they receive. They investigate two implications of persuasion bias: social influence, which is the importance of connections in communication networks in shaping group opinions; and unidimensional opinions, where they show that in the case of a set of issues, individuals' opinions converge to one opinion on the "left-right" spectrum. They also present some empirical applications. Golub and Jackson (2007) study networks with homophily and the dynamics of learning in a model where agents use simple updating rules such as De Groot updating. They show that agents correctly aggregate information and reach the true state of the world, in a similar way to the fully rational learning processes. They also identify networks where beliefs fail to converge to the rational limit. Acemoglu and Ozdaglar (2011) discuss the literature on opinion dynamics in networks. Golub and Jackson (2012) study average-based learning in networks characterised by homophily. They explore learning dynamics and measure speed of convergence to a consensus and how it relates to the degree of homophily in the network.

Jadbabaie, Molavi and Tahbaz-Salehi (2018) provide behavioural foundations for non-Bayesian models of learning on social networks.

Grabisch et al. (2018) investigate a model with two strategic agents aiming to influence non-strategic agents who update their options following De Groot learning. Each strategic agent's strategy consists of forming a link with one non-strategic agent in order to alter the average opinion. They provide a characterisation of the equilibrium that underlines influenceability of targets and their centrality, and they also introduce a new centrality measure. Moreover, they show that in the case of strategic agents with similar impact, the equilibria are symmetric. Rusinowska and Taalaibekova (2019) extend this model and introduce a third centrist persuader. They discuss the effect of the centrist persuader on the consensus and equilibria.

Finally, we also investigate timing in finite learning. In this regard, there are few papers that have focused on the timing in opinion formation. Gratton, Holden and Kolotilin (2018) study release of information of a sender about her private type and show that in equilibrium a bad sender releases later than a good sender. They also find empirical evidence to their prediction using data from U.S. presidential elections and timing of scandals.

1.3 Benchmark: one Voter

1.3.1 Two narratives

Define a setting with a politician (she) and a voter (he). Let x^P be the politician's narrative and x^0 the initial opinion of the voter such that x^P and x^0 belong to $\mathbb{R}_+$, and assuming x^P is sufficiently larger than x^0. We call t_d the time of disclosure of the narrative – that is, the decision variable of the politician. On the other hand, the number of learning periods k – that is, the difference between the date of the election T and the time of disclosure t_d is the length of the learning process: $k := T - t_d$. Consequently, the problem of determining the optimal timing of disclosure t_d is equivalent to determining the length of the opinion formation process k.
The expected belief of the voter at a given time t is $x(t)$, and for all $t \leq t_d$ it is assumed to be equal to the initial opinion x^0. This belief will correspond to the level of adoption or acceptance of the narrative by the voter. At the time of disclosure, the voter observes the narrative and weighs it in his opinion by splitting his attention between the narrative of the politician and his own opinion (from the previous period) respectively, such that $(1-\theta)$ is the weight he assigns to the politician's narrative. After that, with a probability $(1-p)$, the voter rejects the narrative and adopts instead an alternative narrative x^C in $\mathbb{R}_+$ while keeping the same weight $(1-\theta)$ on the new narrative, which can be identical to, or less than, x^0 (for now $x^C \leq x^0$). The main assumption in this setting is that the two narratives are not symmetrically weighted in the opinion updating rule the voter is following. Once a voter turns his attention $(1-\theta)$ towards the alternative narrative instead of the politician's, he never looks back and updates with the alternative as the only external source.

The decision of the politician regarding the optimal time of disclosure corresponds therefore to the optimal length of the opinion formation process, and more precisely the number of learning periods. Let $x(k)$ denote the expectation of the opinion of the voter at $t_d + k$, k periods after the time of disclosure t_d. The politician's objective is to determine the optimal number of periods k. The law of motion of the voter's opinion[1] over three learning

[1] In the case of a standard De Groot updating, where a voter observes both narratives and updates accordingly, weighting $p(1-\theta)$ and $(1-p)(1-\theta)$ as the politician's and the alternative narrative respectively, the updating rule will be such that $x(k) = p(1-\theta)x^P + (1-p)(1-\theta)x^C + \theta x(k-1) = (1-\theta^k)(px^P + (1-p)x^C) + \theta^k x^0$ which converges to $px^P + (1-p)x^C$ as k goes to infinity.

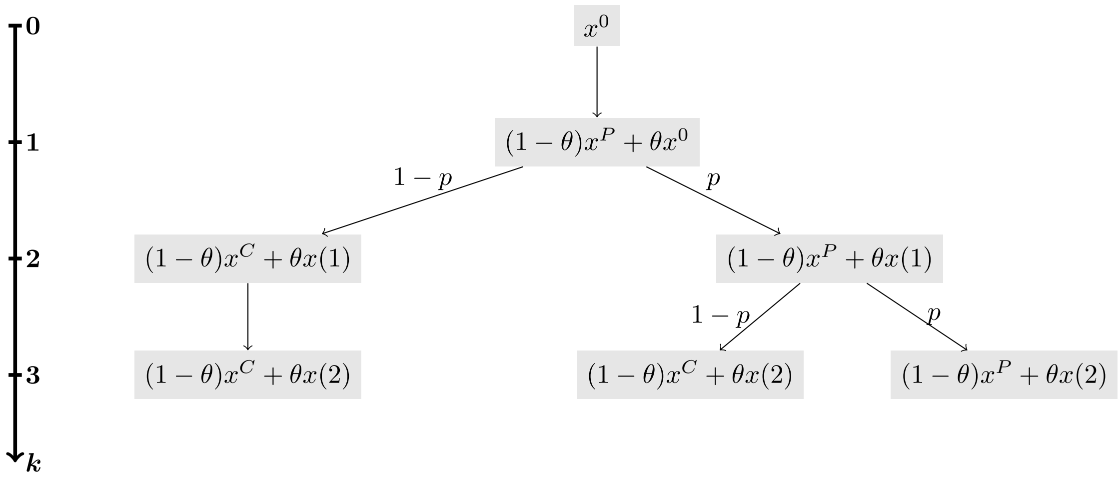


Figure 1.3.1: Learning process over three periods

periods is illustrated in Figure 1.3.1 and can be formally (shown in Appendix A) written for $k \geq 2$, as

$$x\left(k\right) = (1-p^{k-1})(1-\theta)x^C + p^{k-1}(1-\theta)x^P + \theta x\left(k-1\right) \tag{1.3.1}$$

Assuming $\theta \neq p$ and following a series of iterations, it can be rewritten as a function of the parameters $\theta, p, x^C$and x^P:

$$x\left(k\right) = \left(1 - \theta^{k-1} - \frac{p^{k-1} - \theta^{k-1}}{p-\theta} p(1-\theta)\right) x^C + \frac{p^k - \theta^k}{p-\theta}(1-\theta)x^P + \theta^k x^0$$

Lemma 1.1. *As the number of learning periods k goes to infinity, the expected opinion of the voter converges to the alternative narrative x^C with strictly positive $(1-p)$. In other words, if the politician discloses very early, before the election, the voter will adopt the alternative narrative, if he has a positive probability of replacing the politician's narrative.*

Proof. The proof of this lemma as well as all other proofs will be provided in Appendix B. □

The updating rule stated in equation 1.3.1 is an averaging rule where coefficients sum up to one, and where some of them are time-dependent. The weights the voter assigns to the politician's narrative and to the alternative narrative change with the number of learning periods. More specifically, if p is strictly less than one, meaning that with some strictly positive probability the voter will switch to x^C, or in other words the voter assigns a strictly positive weight to the alternative narrative, then as k increases, the weight on x^C increases and the weight on x^P decreases. Subsequently, the voter's opinion converges to x^C.

1.3.2 Absence of alternative narrative: standard De Groot updating

Here we suppose that the politician is the only source of influence and the probability of abandoning the narrative $(1-p)$ is equal to zero. This setting is equivalent to the regular average-based model by De Groot, with two agents (politician and voter), where the politician strategically observes only her opinion x^P and thus her opinion is fixed, while the voter updates by averaging between x^P weighting $(1-\theta)$ and his own opinion weighting θ. The opinion updating rule can be amended[2] as $x(k) = (1-\theta)x^P + \theta x(k-1)$. Using recursion, the expected opinion of the voter k periods after disclosure can be rewritten as: $x(k) = \sum_{l=0}^{k-1} \theta^l (1-\theta)x^P + \theta^k x^0 = (1-\theta^k)x^P + \theta^k x^0$. A similar convergence result to the one in the standard De Groot is present here. More precisely, we show that in this case, it is optimal for the politician to disclose her narrative as early as possible to maximise its adoption by the voter.

By considering the limit when k goes to infinity, the voter's opinion converges to the politician's narrative x^P: $\lim_{k\to\infty}\left[(1-\theta^k)x^P + \theta^k x^0\right] = x^P$. On the other hand, if we suppose that the initial opinion is $x^0 = 0$ and x^p is sufficiently large, the politician's objective will be to maximise $x(k) = \left(1-\theta^k\right)x^P$.

Lemma 1.2. *When the probability of replacing the politician's narrative is null, it is optimal for the politician to disclose as early as possible to maximise the voter's narrative adoption.*

Proof. See Appendix B. □

1.3.3 Narrative adoption

In what follows, we assume that, after disclosure, instead of adopting an alternative narrative with a certain probability, the voter abandons the politician's narrative and moves back to his initial opinion x^0 with a positive probability. More precisely, one period after disclosure, with a probability $(1-p)$, the voter rejects the narrative and moves back closer to his initial opinion. This is slightly similar to the SIR epidemics (Susceptible-Infected-Recovered) models, where the voter gets "infected" by the narrative with probability $(1-\theta)$, and then "recovers" afterwards with probability $(1-p)$, in a De Groot learning setting. In other words, the opinion of a given voter, after k learning periods, depicts the degree to which the voter is infected by the narrative of the politician, and how close his opinion is to x^P. This can be formally written, as shown in Appendix A:

$$x(k) = x^0 + (p^k - \theta^k)\frac{1-\theta}{p-\theta}\left(x^P - x^0\right)$$

[2]Noting here that the coefficients in the updating rule are not time-varying.

Next, and for the remainder of the chapter, we assume that the initial opinion is equal to 0. The problem of the politician can be reduced thereafter to maximising the expectation of the voter's belief after k periods from disclosure, as follows:

$$\max_k x\left(k\right) = \left(p^k - \theta^k\right) \frac{1-\theta}{p-\theta} x^P$$

Figure 1.3.2: Opinion as a function of learning periods in the case of one voter, with $\theta = 0.55$ and $p = 0.7$

Proposition 1.1. *For $p \neq \theta$ and $0 < \theta, p < 1$, the optimal number of learning periods that maximises the opinion of the voter is $k = \frac{\ln\left(\frac{\ln\theta}{\ln p}\right)}{\ln\frac{p}{\theta}}$.*

Proof. See Appendix B. □

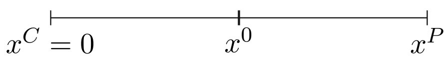
We examine now how the optimal number of learning periods k in the benchmark case of one voter changes with the level of attention to the politician $(1-\theta)$, and with the change of probability of narrative rejection $(1-p)$. To this end, we check the sign of the derivatives of k with respect to θ and p. The sign of these derivatives will dictate how k changes as θ or p increase. Conversely, k will change in the opposite direction as $(1-\theta)$ or $(1-p)$ increase.

Proposition 1.2. *(Comparative Statics) For $\theta \neq p$, the optimal number of learning periods k increases with p and with θ. Equivalently, k decreases with the probability of rejecting the narrative $(1-p)$ and with the attention to the politician $(1-\theta)$.*

Proof. See Appendix B. □

Non-zero initial opinion We examine here the case where there is an alternative narrative $x^S = 0$ that is different from the initial opinion $x^0 \neq 0$. Here, the voter's initial opinion is between the alternative narrative (from the left) and the politician's narrative. The opinion updating process becomes:

$$x(k) = (p^k - \theta^k)\frac{1-\theta}{p-\theta}x^P + \theta^k x^0$$



Corollary 1.1. *For $p \neq \theta$ and $0 < \theta, p < 1$, for a sufficiently large narrative x^P, alternative zero narrative $x^C = 0$ and a non-zero initial opinion $x^0 \neq 0$, the optimal number of learning periods to maximise the politician's narrative adoption is given by*

$$k = \frac{\ln\left[\frac{\ln\theta}{\ln p}\left(1 + \frac{\theta-p}{1-\theta}\frac{x^0}{x^P}\right)\right]}{\ln\frac{p}{\theta}}$$

Proof. Appendix B. □

Comparing the optimal timing of disclosure between the case where the alternative narrative is the initial opinion and the case where they are different, we get $k\rfloor_{x^0 \neq 0} \geq k\rfloor_{x^0=0}$ whenever $\theta > p$. This is true whenever $1 - \theta < 1 - p$, meaning that the weight assigned to the narrative (first the politician's and then the alternative narrative with probability $1 - p$) must be smaller than the probability of replacing the politician's narrative.

1.4 Case of n disconnected voters

We generalise the setting to $N = \{1, 2, ..., n\}$ voters with θ_i, voter i's attention to his opinion –and his neighbours' opinions if he is part of a network –, $(1 - p_i)$ the probability of rejecting the politician's narrative and x_i^0, i's initial opinion. The politician's objective remains maximising the adoption of her narrative by the voters. We consider two possible objective functions for the politician. One is the average opinion $\bar{x}(k) := \frac{\sum_i x_i(k)}{n}$, k periods after disclosure, which assumes that she values the opinions of all voters equally. Another possibility is a weighted average opinion $\tilde{x}(k) := \sum_i a_i x_i(k)$ where $a_i \in [0, 1]$ is the weight the politician assigns to the opinion of voter i, such that $\sum_i a_i = 1$.

In what follows, we will adopt a simplifying assumption that attention levels are identical among voters such that $\theta_i = \theta$ and $p_i = p$ for all $i \in N$. We will also look at a particular setting with two types of voters: supporters and non-supporters.

Definition 1.1. A voter is a *supporter* if his probability of rejecting the politician's narrative is null such that $p_i = 1$. He is a *non-supporters* if he rejects the politician's narrative with a strictly positive probability and has $p_i = p$ strictly less than one.

We define accordingly the sets of supporters and non-supporters as $S = \{i \in N \mid p_i = 1\}$ and $N \setminus S = \{i \in N \mid p_i = p < 1\}$ respectively, with $s = |S|$. The main advantage of these assumptions is that they will allow us to have a tractable and solvable model, and to achieve some comparative statics.

In this section we investigate the case where voters are disconnected and do not influence each other's opinions, whereas in the section after we examine networks of voters.

1.4.1 Different initial opinions

We consider here a case of voters with distinct initial opinions x_i^0, all strictly positive, while there is an alternative narrative $x^C = 0$ and x^P is assumed to be sufficiently large, as before. Given that the n voters do not interact nor influence each other's opinions, the opinion of each voter i at k periods after disclosure is given by

$$x_i(k) = (p^k - \theta^k)\frac{1-\theta}{p-\theta}x^P + \theta^k x_i^0 \tag{1.4.1}$$

Given that the influence of the politician and that of the alternative narrative over the voters are identical through θ and p, this setting shows that voters follow a similar opinion updating rule but start from different initial positions.

(a) Politician targets average opinion Let the average initial opinion be $\bar{\mathbf{x}}^0 = \frac{\sum_i x_i^0}{n}$. Supposing that the politician targets the average opinion $\bar{x}(k)$, we determine the optimal time of disclosure that will be a function of θ, p, x^P and the average initial opinion $\bar{\mathbf{x}}^0$. The approach and the result are similar to the case of one voter with an initial opinion different from zero (as in Corollary 1.1), with the average initial opinion instead of the voter's initial opinion.

Lemma 1.3. *In the case of n disconnected voters with different initial opinions, the optimal number of learning periods is given by* $k = \frac{\ln\left[\frac{\ln\theta}{\ln p}\left(1+\frac{\theta-p}{1-\theta}\frac{\bar{x}^0}{x^P}\right)\right]}{\ln\frac{p}{\theta}}$.

Proof. See Appendix B. □

(b) Politician targets weighted average opinion Let $\tilde{x}^0 = \sum_i a_i x_i^0$ be weighted average opinion, the politician maximises $\tilde{x}(k) = \sum_i a_i\left((p^k - \theta^k)\frac{1-\theta}{p-\theta}x^P + \theta^k x_i^0\right) = (p^k - \theta^k)\frac{1-\theta}{p-\theta}x^P + \theta^k\tilde{x}^0$. Following a similar reasoning to the case where the politician targets the average opinion, as in Lemma 1.3, the optimal time of disclosure is $k = \frac{\ln\left[\frac{\ln\theta}{\ln p}\left(1+\frac{\theta-p}{1-\theta}\frac{\tilde{x}^0}{x^P}\right)\right]}{\ln\frac{p}{\theta}}$ which is a function of $\tilde{x}^0$. The result is also similar to the one voter framework in Corollary 1.1.

1.4.2 Same initial opinions, different types

Here we look at the case where the alternative narrative is identical to the voter's initial opinion $x^C = x_i^0 = 0$ for every $i \in N$. Assuming all voters have the same attention to their own opinion θ, the opinions after k learning periods of a supporter and a non-supporter respectively will be :

$$x_s(k) = (1-\theta^k)x^P$$

$$x_m(k) = (p^k - \theta^k)\frac{1-\theta}{p-\theta}x^P$$

(a) Politician targets average opinion In the general case where voters have beliefs $x_i(k)$, attention θ and probability p_i, the updating rule for each voter is given by

$$x_i(k) = p_i^{k-1}(1-\theta)x^P + \theta x_i(k-1)$$

The average opinion can be therefore written as

$$\bar{x}(k) = \frac{\sum_i p_i^{k-1}}{n}(1-\theta)x^P + \theta\bar{x}(k-1) \tag{1.4.2}$$

In case of two types of voters, the politician's problem becomes to choose k that maximises $\bar{x}(k) = \frac{s}{n}x_s(k) + \frac{n-s}{n}x_m(k)$.

By substituting for $x_s(k)$ and $x_m(k)$ from the above equations, $\bar{x}(k)$ can be written as

$$\bar{x}(k) = \frac{s}{n}(1-\theta^k)x^P + \frac{n-s}{n}\frac{p^k-\theta^k}{p-\theta}(1-\theta)x^P \tag{1.4.3}$$

Lemma 1.4. *In the case of n disconnected voters of two different types, supporters and non-supporters, all with initial opinions at* 0*, if the politician maximises the average opinion, the optimal number of learning periods is* $k = \frac{\ln\left(\frac{\ln\theta}{\ln p}\left(\frac{s(p-\theta)}{(n-s)(1-\theta)}+1\right)\right)}{\ln\frac{p}{\theta}}$.

Proof. Appendix B. □

(b) Politician targets weighted average opinion Assuming voters are of two types, the politician's objective is to maximise $\tilde{x}(k) = ax_s(k) + (1-a)x_m(k)$.

Lemma 1.5. *In the case of n disconnected voters of two different types, supporters and non-supporters, all with initial opinions at* 0*, if the politician maximises the weighted av-*

erage opinion, the optimal number of learning periods would be $k = \frac{\ln\left(\frac{\ln\theta}{\ln p}\left(\frac{a(p-\theta)}{(1-a)(1-\theta)}+1\right)\right)}{\ln\frac{p}{\theta}}$.

Proof. See Appendix B. □

1.5 Case of n connected voters

We suppose now that the voters are connected, and we define $\gamma_{ij} \in [0;1]$ as the weight i assigns to j's opinion such that $\sum_{j\in N}\gamma_{ij} = 1.\gamma_{ij}$ is equally the influence of voter j's opinion on voter i's opinion. We assume that the network is exogenous and time-invariant, with Γ the corresponding $n \times n$ adjacency matrix.
Given x_i^0 the initial opinion of i, x^P the politician's narrative and x^C the alternative narrative, the updating rule of opinions can be written for $k \geq 1$ as

$$x_i(k) = x^C(1-p_i^{k-1})(1-\theta_i) + x^P p_i^{k-1}(1-\theta_i) + \theta_i \sum_{j\in N}\gamma_{ij}x_j(k-1)$$

(see Appendix A). Figure 1.5.1 illustrates the learning process of an individual i over three periods.
More generally, this can be written in matrix notation:

$$\mathbf{x}(k) = (\mathbf{I}-\Theta)(\mathbf{1}-\mathbf{p}^{(k-1)})x^C + (\mathbf{I}-\Theta)\mathbf{p}^{(k-1)}x^P + \Theta\Gamma\mathbf{x}(k-1) \tag{1.5.1}$$

where $\mathbf{x}(k)$ is the vector of opinions k periods after disclosure, Θ is the diagonal matrix with entries $\Theta_{ii} = \theta_i \forall i$, $\mathbf{I}$ is the identity matrix, $\mathbf{1}$ is the vector with all elements equal to 1 and $\mathbf{p}^{(k-1)}$ is the probabilities vector with elements p_i^{k-1}.

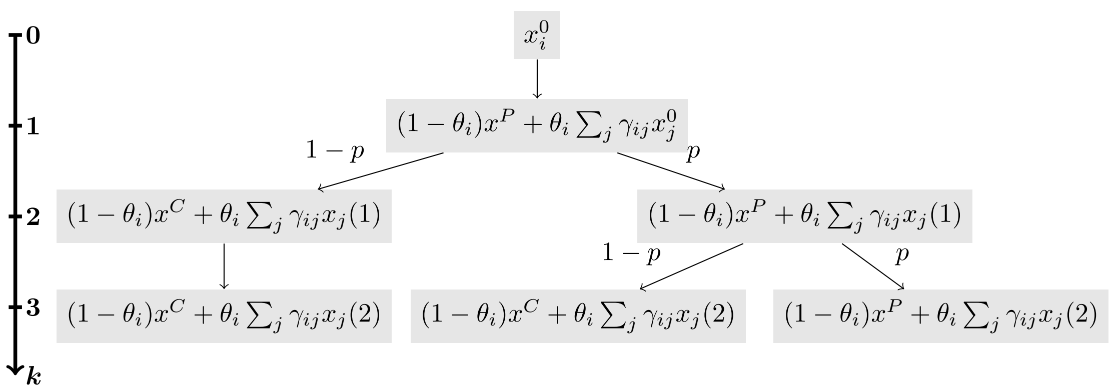


Figure 1.5.1: Learning process over three periods in the case of n connected voters

Grabisch et al. (2018) investigate a framework where the weights do not vary with time, and they prove that the opinions converge to a certain steady state that they determine.

Define two vectors $\boldsymbol{\alpha}^{(k-1)} = (\mathbf{I} - \Theta)\mathbf{p}^{(k-1)}$ and $\boldsymbol{\beta}^{(k-1)} = (\mathbf{I} - \Theta)(\mathbf{1}_{(n)} - \mathbf{p}^{(k-1)})$, and a matrix

$$\mathbf{M}^{(k-1)} = \left(\begin{array}{cc|c} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \hline \boldsymbol{\alpha}^{(k-1)} & \boldsymbol{\beta}^{(k-1)} & \boldsymbol{\Theta\Gamma} \end{array}\right).$$

$\mathbf{M}^{(k-1)}$ is the augmented updating matrix and is row-stochastic, time-varying and of size $(n+2) \times (n+2)$. Let the augmented opinion vector be the opinion vector including the voters, the politician and the alternative narrative source, which is formally defined as

$$\mathbf{X}(t) = \begin{pmatrix} x^P \\ x^C \\ \mathbf{x}(t) \end{pmatrix}$$

noting that x^P and x^C are fixed over time and not updated. The augmented updating rule can be written as:

$$\mathbf{X}(k) = \mathbf{M}^{(k-1)}\mathbf{X}(k-1)$$

$$\mathbf{X}(k) = \mathbf{M}^{(k-1)}\mathbf{M}^{(k-2)}\mathbf{X}(k-2) = \mathbf{M}^{(k-1)}\mathbf{M}^{(k-2)}...\mathbf{M}^{(1)}\mathbf{X}(1)$$

$$= \Pi_{l=0}^{k-1}\mathbf{M}^{(l)}\mathbf{X}(0)$$

DeMarzo, Vayanos and Zwiebel (2003) study a similar framework, and define the influence of j over i after k periods of communication as $\left[\Pi_{l=0}^{k-1}\mathbf{M}^{(l)}\right]_{ij}$. They prove that if the adjacency matrix of the network is strongly connected and assuming that the time-dependent weights (common to all agents) sum up to infinity, beliefs converge to a consensus. These conditions do not apply to our setting here for two main reasons. First, the adjacency matrix is not strongly connected, since the politician and the source of the alternative narrative are isolated and have fixed opinions. Second, the time-varying weights (p_i^{k-1} for every i) are different for different voters.

We define an $n \times 2$ matrix $\boldsymbol{\Lambda}^{(k-1)}$ whose columns are the two vectors $\boldsymbol{\alpha}^{(k-1)}$ and $\boldsymbol{\beta}^{(k-1)}$; the augmented updating matrix can therefore be written as

$$\mathbf{M}^{(k-1)} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \boldsymbol{\Lambda}^{(k-1)} & \boldsymbol{\Theta\Gamma} \end{pmatrix}$$

The product of the two matrices $\mathbf{M}^{(k-1)}$ and $\mathbf{M}^{(k-2)}$ is given by

$$\mathbf{M}^{(k-1)}\mathbf{M}^{(k-2)} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{\Lambda}^{(k-1)} & \mathbf{\Theta\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{\Lambda}^{(k-2)} & \mathbf{\Theta\Gamma} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{\Lambda}^{(k-1)} + \mathbf{\Theta\Gamma\Lambda}^{(k-2} & (\mathbf{\Theta\Gamma})^2 \end{pmatrix}$$

Moreover, the product $\Pi_{l=0}^{k-1}\mathbf{M}^{(l)}$ can be written as

$$\Pi_{l=0}^{k-1}\mathbf{M}^{(l)} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \sum_{l=0}^{k-1}(\mathbf{\Theta\Gamma})^l\mathbf{\Lambda}^{(k-1-l)} & (\mathbf{\Theta\Gamma})^{k-1} \end{pmatrix}$$

The updating rule becomes:

$$\mathbf{X}(k) = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \sum_{l=0}^{k-1}(\mathbf{\Theta\Gamma})^l\mathbf{\Lambda}^{(k-1-l)} & (\mathbf{\Theta\Gamma})^{k-1} \end{pmatrix} \mathbf{X}(0) \tag{1.5.2}$$

We show next that if the learning process has no time limit and no deadline, the beliefs of the voters converge eventually.

Proposition 1.3. *As the number of learning periods k goes to infinity, the beliefs of the voters converge.*

Proof. See Appendix B. □

Politician's objectives If the politician targets the average opinion $\bar{x}(k)$, her maximisation problem is

$$\max_k \bar{x}(k) = \frac{1}{n}\sum_{i\in N}\left[(1-p_i^{k-1})(1-\theta_i)x^C + p_i^{k-1}(1-\theta_i)x^P + \theta_i\sum_{j\in N}\gamma_{ij}x_j(k-1)\right]$$

In what follows, we consider specific well-behaved networks with two groups of voters S and $N\setminus S$, in cases where the politician targets the average opinion $\bar{x}(k)$ or the weighted average opinion $\tilde{x}(k) = ax_s(k) + (1-a)x_m(k)$.

1.5.1 Balanced network

We suppose that the alternative narrative and the initial opinions of the voters are such as $x^C = x^0 = 0$, and that the voters have the same level of attention θ but different p_i's. We define below a specific type of network, which we call a balanced network, as a directed network where the sum of weights assigned to every agent by the others is equal to 1. The corresponding adjacency matrix will be column-stochastic with the sum of the entries of every column of the equal to 1, as well as row-stochastic. Several networks with symmetries have this property, such as the complete network and the circle, which we will investigate later on.

Definition 1.2. A *balanced network* is one that is described by an adjacency matrix that is double stochastic (column and row stochastic). Formally, for every $i, j \in N$, it holds that $\sum_{j \in N} \gamma_{ij} = \sum_{i \in N} \gamma_{ij} = 1$.

We show below that for this simple type of network, if the politician targets the average opinion, then her problem in this case is identical to the case where voters are disconnected.

Proposition 1.4. *If the network is balanced, the optimal time of disclosure that maximises the average opinion is identical to the case of a disconnected setting where the network is not present; in other words, the network has no role in shaping the average narrative adoption.*

Proof. See Appendix B. □

Proposition 1.4 implies that $\bar{x}(k)$ in case of a balanced network will be as stated in equation 1.4.2 .

For tractability, we consider again the case of two types of voters – supporters in $S \subset N$ whose $p_i = 1$, and non-supporters in $N \setminus S \subset N$ who reject the narrative with a positive probability $(1 - p_i)$ – such that $p_i = p < 1$ and the average opinion will be, similarly to equation 1.4.3

$$\bar{x}(k) = \frac{s}{n}(1 - \theta^k)x^P + \frac{n-s}{n}\frac{p^k - \theta^k}{p - \theta}(1 - \theta)x^P$$

Consequently, the optimal number of learning periods that maximises narrative adoption at the deadline is $k = \frac{\ln\left(\frac{\ln\theta}{\ln p}\left(\frac{n-s-n\theta+sp}{(n-s)(1-\theta)}\right)\right)}{\ln(\frac{p}{\theta})}$.

Next, we examine how k varies as the number of supporters increases. Intuitively, given that supporters do not replace the politician's narrative with an alternative one, their level of adopting it, and consequently their opinion, should be higher than that of a non-supporter. Moreover, the trade-off the politician faces in regard to the timing of disclosure can be associated with the types of voters: supporters drive early disclosure and a longer learning period, while non-supporters push in the opposite direction given that they might replace x^P. A higher number of supporters implies a higher impact of the supporters' opinions on the weighted average opinion and dictates an earlier disclosure.

Proposition 1.5. *In a balanced network with s supporters and $n - s$ non-supporters, the optimal number of learning periods increases with the number of supporters.*

Proof. See Appendix B. □

1.5.2 Complete network

A complete network is a network where every node is connected to every other node. In particular, Jackson (2010) defines the complete network as "one where all possible links are present" between different nodes.

Given here that our model includes the self-loop, we consider a slight variation of the complete network which allows the voter to update including his own opinion. Each voter updates his opinion by averaging opinions of the other $n-1$ voters and his own. We look at an undirected unweighted graph, such that the degree of each voter will be equal to n including the $n-1$ other voters and a self-loop. The opinion of a given voter i is therefore given by

$$x_i(k) = p_i^{k-1}(1-\theta)x^P + \frac{\theta \sum_{j \in N} x_j(k-1)}{n}$$

which is equivalent to

$$x_i(k) = p_i^{k-1}(1-\theta)x^P + \theta \bar{x}(k-1) \tag{1.5.3}$$

The opinion of each individual at a time t is therefore a function of the average opinion of the network in the previous period.

(a) Politician targets average opinion The average opinion at each point in time is as follows:

$$\bar{x}(k) = \frac{\sum_i p_i^{k-1}(1-\theta)x^P}{n} + \frac{\sum_i \theta \bar{x}(k-1)}{n} = \frac{\sum_i p_i^{k-1}(1-\theta)x^P}{n} + \theta \bar{x}(k-1)$$

Reiterating, we can rewrite it as:

$$\bar{x}(k) = \frac{1}{n} \sum_{l=0}^{k-1} \sum_i p_i^{k-1-l} \theta^l (1-\theta)x^P = \frac{1}{n} \sum_i \frac{p_i^k - \theta^k}{p_i - \theta}(1-\theta)x^P$$

By taking the derivative with respect to k, we get

$$\frac{\partial \bar{x}(k)}{\partial k} = \frac{1}{n} \sum_i \frac{p_i^k \ln p - \theta^k \ln \theta}{p_i - \theta}(1-\theta)x^P = 0$$

which is a transcendental[3] equation that does not have a closed-form solution but can be solved numerically. Instead, if we consider the case of two types of voters defined above,

[3] A transcendental equation is an equation that contains a transcendental function that cannot be analytically expressed using algebraic operations; such as the logarithmic and exponential functions and the trigonometric functions (see Chapter 11 in Bashirov (2014)).

we can solve for k:

$$\bar{x}(k) = \frac{1}{n}\sum_{l=0}^{k-1}\left(s + (n-s)p^{k-1-l}\right)\theta^l(1-\theta)x^P$$

$$= \frac{s}{n}\frac{1-\theta^k}{1-\theta}(1-\theta)x^P + \frac{n-s}{n}(1-\theta)x^P\frac{1-\left(\frac{\theta}{p}\right)^k}{1-\frac{\theta}{p}}p^{k-1}$$

$$\bar{x}(k) = \frac{1-\theta}{n}x^P\left[\frac{s(1-\theta^k)}{1-\theta} + (n-s)\frac{p^k-\theta^k}{p-\theta}\right] \tag{1.5.4}$$

The corresponding first-order condition is:

$$\frac{1-\theta}{n}x^P\left[\frac{-s\theta^k\ln\theta}{1-\theta} + (n-s)\frac{p^k\ln p - \theta^k\ln\theta}{p-\theta}\right] = 0$$

$$\iff \frac{n-s}{p-\theta}p^k\ln p = (\frac{n-s}{p-\theta} + \frac{s}{1-\theta})\theta^k\ln\theta$$

$$\iff \left(\frac{p}{\theta}\right)^k = \frac{\ln\theta}{\ln p}\frac{\frac{n-s}{p-\theta}+\frac{s}{1-\theta}}{\frac{n-s}{p-\theta}} = \frac{\ln\theta}{\ln p}\left(1+\frac{s(p-\theta)}{(n-s)(1-\theta)}\right)$$

$$k = \frac{\ln\left[\frac{\ln\theta}{\ln p}\left(1+\frac{s(p-\theta)}{(n-s)(1-\theta)}\right)\right]}{\ln\frac{p}{\theta}}$$

Given that the complete network is a balanced network as per definition 1.2, the result obtained here is identical to the optimal timing of disclosure reached in the case of balanced network.

(b) Politician targets weighted average opinion Consider the complete network with the same two types of voters defined above, such that every voter is linked to every other voter, and the updating rule follows equation 1.5.3. The politician targets the weighted average opinion that can be written as[4]:

$$\tilde{x}(k) = \left(a + (1-a)p^{k-1}\right)(1-\theta)x^P + \theta\bar{x}(k-1).$$

Using equation 1.5.4, we substitute for $\bar{x}(k-1)$ and write

$$\tilde{x}(k) = \left(a + (1-a)p^{k-1}\right)(1-\theta)x^P + \theta\frac{1-\theta}{n}x^P\left[\frac{s(1-\theta^{k-1})}{1-\theta} + (n-s)\frac{p^{k-1}-\theta^{k-1}}{p-\theta}\right] \tag{1.5.5}$$

[4] $\tilde{x}(k) = ax_s + (1-a)x_m = (a + (1-a)p^{k-1})(1-\theta)x^P + \frac{\theta}{n}\left[asx_s(k-1) + a(n-s)x_m(k-1) + (1-a)sx_s(k-1) + (1-a)(n-s)x_m(k-1)\right]$

Taking the first-order condition, we find the optimal k:

$$\frac{p^k}{\theta^k} = \frac{p}{n}\frac{\frac{s}{1-\theta}+\frac{n-s}{p-\theta}}{1-a+\frac{\theta(n-s)}{n(p-\theta)}}\frac{\ln\theta}{\ln p}$$

$$\iff k = \frac{\ln\left(\frac{p}{n}\frac{\frac{s}{1-\theta}+\frac{n-s}{p-\theta}}{1-a+\frac{\theta(n-s)}{n(p-\theta)}}\frac{\ln\theta}{\ln p}\right)}{\ln\left(\frac{p}{\theta}\right)}$$

1.5.3 Networks with symmetric structures

We examine in this section a symmetric network with two groups, representing each type of voter. Letting $\mathcal{N}_i$ be the set of neighbours of i and $d_i = |\mathcal{N}_i|$ the degree of i, we define an equitable partition as follows.

Definition 1.3. An *equitable partition (see Powers and Sulaiman (1982))* is a partition of a set N composed of non-empty and pairwise disjoint groups $G_1, ..., G_H$ such that $H \geq 2$, and if any i and j belong to the same group G_h then $|\mathcal{N}_i \cap G_l| = |\mathcal{N}_j \cap G_l| \, \forall l = 1, ..., H$.

In such networks, agents will all have a similar number of links to agents of similar type, and a similar number of links to agents of different types. We consider here an equitable partition of N, formed of two groups S and $N \setminus S$, supporters and non-supporters respectively, such that a given voter has q links (including himself) with the same type, and r links with the other type. Equivalently, for $i \in S$ and $j \in N \setminus S$, we have $|\mathcal{N}_i \cap S| = |\mathcal{N}_j \cap (N \setminus S)| = q$ and $|\mathcal{N}_i \cap (N \setminus S)| = |\mathcal{N}_j \cap S| = r$. It follows that each voter i will have a degree $d_i = q+r$. Further, assuming the network is undirected and unweighted, the weight a given voter puts on each of her neighbours is $\frac{1}{q+r}$. Say G_i is the group a voter i belongs to and G_{-i} is the other group; the opinion updating rule of a voter of group i is:

$$x_i(k) = p_i^{k-1}(1-\theta)x^P + \frac{\theta}{q+r}\left[q x_i(k-1) + r x_{-i}(k-1)\right]$$

where $x_i(t)$ and $x_{-i}(t)$ are the opinions of a voter of group i and a voter of the other group $-i$ respectively.

Specifically, the beliefs of a supporter and of a non-supporter, $x_s(k)$ and $x_m(k)$ respectively, are given by:

$$x_s(k) = (1-\theta)x^P + \frac{\theta q}{q+r}x_s(k-1) + \frac{\theta r}{q+r}x_m(k-1)$$

$$x_m(k) = p^{k-1}(1-\theta)x^P + \frac{\theta q}{q+r}x_m(k-1) + \frac{\theta r}{q+r}x_s(k-1)$$

A network where voters have more links with voters of his own type than with voters of the other type, exhibits homophily and has $q > r$. Homophily is generally defined as the tendency of individuals to connect more to individuals of their own type, which can be their age, race, gender, profession, political preferences, etc. In this context, homophily is the tendency of supporters to connect more and assign more weight to their fellow supporters rather than non-supporters, and vice versa. Define $\rho := \frac{q}{r}$ as the ratio of links within the same type over links across types; ρ will depict homophily in the network. A network with homophily has therefore $q > r$ and $\rho > 1$. This modifies the updating processes[5] such that

$$x_s(k) = (1-\theta)x^P + \frac{\theta\rho}{\rho+1}x_s(k-1) + \frac{\theta}{\rho+1}x_m(k-1) \tag{1.5.6}$$

$$x_m(k) = p^{k-1}(1-\theta)x^P + \frac{\theta\rho}{\rho+1}x_m(k-1) + \frac{\theta}{\rho+1}x_s(k-1) \tag{1.5.7}$$

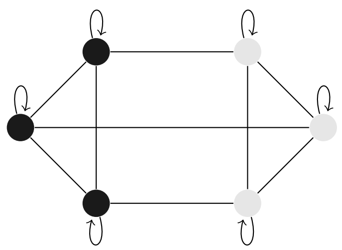
(a) Politician targets average opinion Assuming the two groups are of the same size, the objective function of the politician becomes:

$$\bar{x}(k) = \frac{x_s(k)+x_m(k)}{2} = \frac{1+p^{k-1}}{2}(1-\theta)x^P + \theta\frac{x_s(k-1)+x_m(k-1)}{2}$$

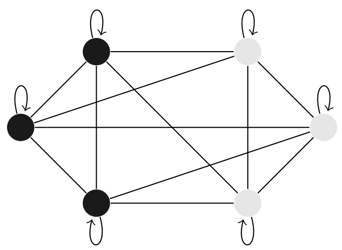
The network here will be a balanced network, and if the politician maximises the average opinion, the optimal number of learning periods is identical to the one in that case, with $\frac{s}{n} = \frac{n-s}{n} = \frac{1}{2}$ such that

$$k = \frac{\ln\left(\frac{\ln\theta}{\ln p}\frac{1-2\theta+p}{1-\theta}\right)}{\ln\frac{p}{\theta}}$$

An illustration of a network with equitable partition with two types of voters is in Figure 1.5.2. In Figure 1.5.2a, each voter has $q = 3$ links to the same type (including the self-loop) and $r = 1$ link to a voter of another type, while in figure 1.5.2b, $q = 3$ and $r = 2$. There is higher homophily in the network (a) on the left, and there are more inter-type connections in the network (b) on the right.



(a) Case of $\rho = 3$



(b) Case of $\rho = \frac{3}{2}$

Figure 1.5.2: Network with equitable partition with $n = 6$ and $s = 3$

[5] $q = \rho r$ and $q + r = \frac{q(\rho+1)}{\rho}$

(b) Politician targets weighted average opinion We look here at the case of a population split equally between groups of individuals of two types, represented by a network with equitable partition. The opinions of a supporter or a non-supporter are given respectively by equations 1.5.6 and 1.5.7.

The weighted average can be written as

$$\tilde{x}(k) = C_0 + C_1 p^{k-1} - C_2 \theta^{k-1} + (1 - C_0 - C_1 + C_2) \left(\frac{\theta(\rho - 1)}{\rho + 1} \right)^{k-1} \tag{1.5.8}$$

where C_0, C_1 and C_2 are constants and defined as $C_0 := (1-\theta) x^P \frac{\rho+1}{\rho+1-\theta(\rho-1)} \left(a + \frac{\theta}{(\rho+1)(1-\theta)} \right)$, $C_1 := (1-\theta) x^P \frac{p(\rho+1)}{p(\rho+1)-\theta(\rho-1)} \left(1 - a + \frac{\theta}{(p-\theta)(\rho+1)} \right)$ and $C_2 := (1-\theta) x^P \frac{\theta}{2} \left(\frac{p+1-2\theta}{(1-\theta)(p-\theta)} \right)$.
The first-order condition is given by:

$$\frac{\partial \tilde{x}(k)}{\partial k} = 0 = C_1 p^{k-1} \ln p - C_2 \theta^{k-1} \ln \theta + (1 - C_0 - C_1 + C_2) \left(\frac{\theta(\rho - 1)}{\rho + 1} \right)^{k-1} \ln \left(\frac{\theta(\rho - 1)}{\rho + 1} \right)$$

This equation is transcendental and has no closed form solution for k which can be determined numerically.

In what follows, we examine the change of $\tilde{x}(k)$ with the degree of homophily ρ. First, we show that the opinion of a supporter $x_s(k)$ is larger than the opinion of a non-supporter $x_m(k)$. In particular, the two opinions are equal for $k = 0$ and 1, whereas for any $k > 1$, $x_s(k)$ is strictly greater than $x_m(k)$.

Proposition 1.6. *For a given level of homophily ρ, the belief (adoption of the narrative) is higher for a supporter than for a non-supporter: $x_s(k) \geq x_m(k)$.*

Proof. See Appendix B. □

Proposition 1.7. *In a network with equitable partition where voters are equally split between supporters and non-supporters, the belief of a supporter (non-supporter) increases (decreases) with homophily ρ.*

Proof. See Appendix B. □

Proposition 1.7 reveals that an increase in the level of homophily has opposite effects on the opinions of supporters and non-supporters; the former increases while the latter decreases. This will have direct implications for the politician's strategy regarding the timing of disclosure, which depends on the weight she assigns to each type. This is investigated further in Proposition 1.8.

Proposition 1.8. *(Comparative Statics) The weighted average opinion $\tilde{x}(k)$ increases as the weight the politician places on the supporters' opinion increases, and as the probability of a non-supporter replacing the narrative decreases. Consequently, in both cases, the optimal number of learning periods k^* increases, and the timing of disclosure is earlier.*

Proof. See Appendix B. □

Proposition 1.9. *(Convergence) As k goes to infinity, the opinions of a supporter and a non-supporter and the weighted average opinion converge to* $\frac{1+\rho(1-\theta)}{1+\theta+\rho(1-\theta)}x^P$, $\frac{\theta}{1+\theta+\rho(1-\theta)}x^P$ *and* $\frac{a(1-\theta+\rho(1-\theta))+\theta}{1+\theta+\rho(1-\theta)}$ *respectively.*

Proof. See Appendix B. □

Notably, when considering the opinions of supporters and non-supporters as the number of learning periods k goes to infinity, we conclude that they converge to different beliefs, and no consensus is reached. The limiting level of narrative adoption of the supporter is always higher than that of a supporter. Furthermore, the belief of the supporter is smaller than x^P in contrast to the case where he does not communicate with non-supporters.

Illustration We consider an illustration of a network with equitable partition of supporters and non-supporters. We suppose that the parameters are given by $x^P = 9$, $\theta = 0.8, 0.9$ and $p = 0.9, 0.7$. In each figure, we plot the weighted average opinion $\tilde{x}(k)$ with respect to the number of learning periods k, for two levels of homophily $\rho = 8$ and $\rho = 0.5$. Figures 1.5.3 and 1.5.5 illustrate the case of a politician who assigns more weight to supporters such that $a = 0.9$; while Figures 1.5.4 and 1.5.6 illustrate the case of a politician who assigns more weight to non-supporters such that $a = 0.1$. As predicted by Propositions 1.7 and 1.8, the weighted average opinion with more weight on the supporters' opinion ($a = 0.9$), is higher for higher homophily $\rho = 0.8$; whereas when more weight is assigned to non-supporters, the weighted average opinion is lower for higher homophily.

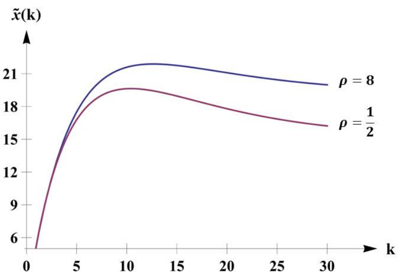


Figure 1.5.3: Weighted average opinion over time with $\theta = 0.8$, $p = 0.9$ and $a = 0.9$

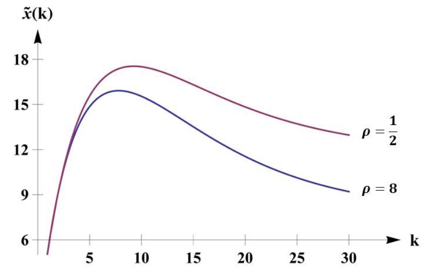


Figure 1.5.4: Weighted average opinion over time with $\theta = 0.8$, $p = 0.9$ and $a = 0.1$

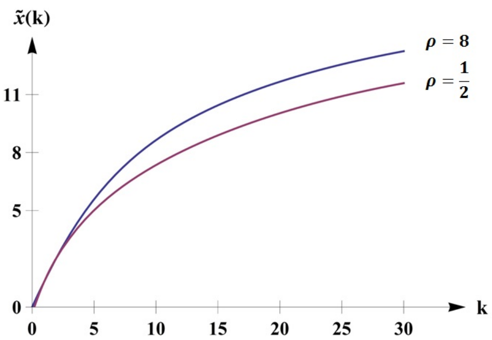


Figure 1.5.5: Weighted average opinion over time with $\theta = 0.9$, $p = 0.7$ and $a = 0.9$

Extreme types of politicians: ideology versus office motivated We consider here two specific types of politicians with extreme motivations and objectives. Whether a politician cares about the ideology of her supporters or about the beliefs of the entire population will dictate her objective function and consequently her choice of timing of disclosure. Assume here that voters update their opinions following equations 1.5.6 and 1.5.7 in a similar way to the case of two unequal groups. The two objective functions of the politician considered here are the weighted average opinions with two possible extreme weights: $a = 1$, assigning all the weight to the opinion of the supporters; and $a = 0$, assigning all the weight to the opinion of the non-supporters.

In the first case, a politician, who is solely motivated by ideology, aims to convey her narrative and maximise its adoption by a subgroup of supporters. Formally, the politician maximises $\tilde{x}(k)$ at $a = 1$: $\max_k \tilde{x}(k)\rfloor_{a=1} = x_s(k)$.

In contrast to the ideology-motivated politician, an office-motivated politician cares about

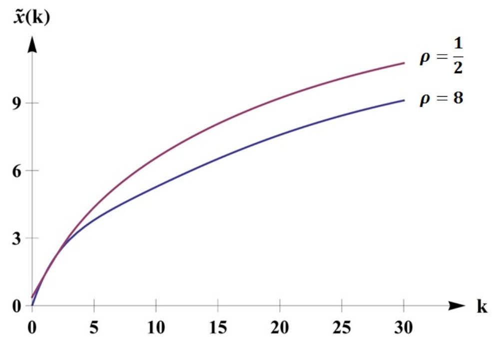


Figure 1.5.6: Weighted average opinion over time with $\theta = 0.9$, $p = 0.7$ and $a = 0.1$

winning an election by getting the maximum possible adoption of her narrative by all of the population. In particular, we assume that this type of politician considers that supporters will choose her for office in any situation, and consequently targets only the non-supporters. Here, the politician maximises $\tilde{x}(k)$ at $a = 0$: $\max_k \tilde{x}(k)\rfloor_{a=0} = x_m(k)$.
Proposition 1.8 shows that with a higher a, the weighted average opinion increases and so does the optimal number of learning periods k, while the timing of disclosure is earlier. Therefore, we can notice that these two types of politicians represent the two opposing extremes. The ideology-motivated politician with $a = 1$ will have the earliest optimal timing of disclosure, while the office-motivated politician will have the latest.

1.5.4 Circle

A circle is a network that is formed of one cycle, such that every node has two neighbours. In order to include the self-loop, we consider a variation of the standard circle network where n voters are placed on an undirected unweighted circle, such that each is connected to three voters (two neighbours and a self-loop), as illustrated in figure 1.5.7. A formal definition of the circle is given by the $n \times n$ adjacency matrix

$$\Gamma = \begin{pmatrix} 1 & 1 & 0 & ... & 0 & 1 \\ 1 & 1 & 1 & 0 & ... & 0 \\ 0 & 1 & 1 & 1 & ... & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & ... & 1 & 1 & 1 \\ 1 & 0 & ... & 0 & 1 & 1 \end{pmatrix}$$

Each voter i has a degree $d_i = 3$ and a probability of replacing the narrative of $(1 - p_i)$.

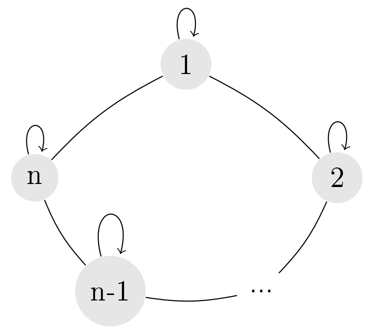


Figure 1.5.7: Undirected unweighted circle

(a) Politician targets average opinion Starting with the initial condition $x_i(t_d) = 0$ and $x_i(1) = (1-\theta)x^P$ for every $i \in N$, we can write the average opinion in the circle (Appendix A) as

$$\bar{x}(k) = (1-\theta)\frac{x^P}{n}\left(n\theta^{k-1} + \sum_{l=0}^{k-2}\theta^l \sum_{i=1}^{n} p_i^{k-1-l}\right)$$

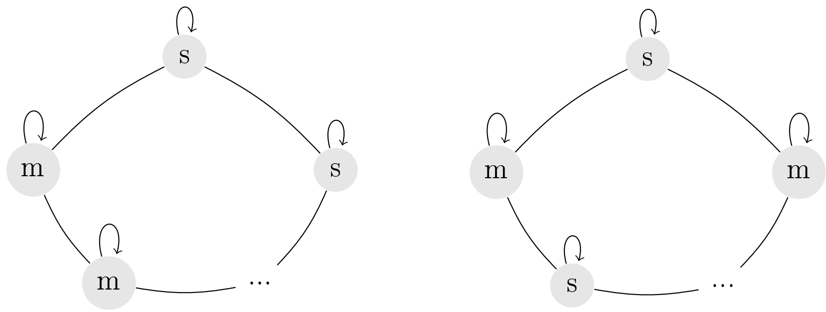
In this unweighted circle, every voter listens to three other voters and influences three other voters as well. Hence, it is a balanced network as per definition 1.2. The optimal timing of disclosure that maximises the average opinion will be similarly $k = \frac{1}{\ln\frac{p}{\theta}}\ln\left(\frac{\ln\theta}{\ln p}\frac{sp-n\theta+(n-s)}{(n-s)(1-\theta)}\right)$.

(b) Politician targets weighted average opinion A circle network with an even number of nodes is a specific network where each voter has three links: the self-loop and two others. It would be a network with equitable partition if we assume that the two types of voters behave symmetrically and have the same size; each has two links (including his) to the same type $q = 2$ and one to the other type $r = 1$ (case A), or the other way around $q = 1$ and $r = 2$ (case B), as in Figure 1.5.8. Consequently we compare between $\rho = 2$ and $\rho = \frac{1}{2}$.

Case A $\rho = 2$ When the network is characterised with homophily then $\rho = 2$ (Figure 1.5.8a) – that is, the weight on one's own type is double that on the other type, the opinions of a supporter and of a non-supporter are given by

$$x_s(k) = (1-\theta)x^P + \frac{2\theta}{3}x_s(k-1) + \frac{\theta}{3}x_m(k-1)$$

$$x_m(k) = p^{k-1}(1-\theta)x^P + \frac{2\theta}{3}x_m(k-1) + \frac{\theta}{3}x_s(k-1)$$



(a) Case A: $\rho = 2$ (b) Case B: $\rho = \frac{1}{2}$

Figure 1.5.8: Two types of circle network

The weighted average opinion can be written as (Appendix A)

$$\tilde{x}(k) = (1-\theta)x^P \left[C_0 + C_1 p^k - C_2 \theta^{k-1} + (1 - C_0 - C_1 + C_2) \left(\frac{\theta}{3}\right)^{k-1} \right]$$

where $C_0 = \frac{3}{3-\theta}\left(a + \frac{\theta}{3(1-\theta)}\right)$, $C_1 = \left(1 - a + \frac{\theta}{3(p-\theta)}\right)\frac{3}{3p-\theta}$ and $C_2 = \frac{p+1-2\theta}{2(1-\theta)(p-\theta)}\theta$.
The first-order condition:

$$\frac{\partial \tilde{x}(k)}{\partial k} = 0 = (1 - C_0 - C_1 + C_2)\left(\frac{\theta}{3}\right)^{k-1} \ln\frac{\theta}{3} + C_1 p^k \ln p - C_2 \theta^{k-1} \ln\theta$$

This is a transcendental equation that can be solved numerically.

Case B $\rho = \frac{1}{2}$ This case represents a circle where each voter is connected to two other voters of opposite type, which results in only one-third of the weight being assigned to his own opinion and type (figure 1.5.8b). It is an illustration of non-homophily. The opinions of supporters and non-supporters are

$$x_s(k) = (1-\theta)x^P + \frac{\theta}{3}x_s(k-1) + \frac{2\theta}{3}x_m(k-1)$$

$$x_m(k) = p^{k-1}(1-\theta)x^P + \frac{\theta}{3}x_m(k-1) + \frac{2\theta}{3}x_s(k-1)$$

The weighted average opinion can be written as

$$\tilde{x}(k) = (1-\theta)x^P \left[C_0 + C_1 p^k - C_2 \theta^{k-1} + (1 - C_0 - C_1 + C_2) \left(-\frac{\theta}{3}\right)^{k-1} \right]$$

where $C_0 = \left(a + \frac{2\theta}{3(1-\theta)}\right)\frac{3}{3+\theta}$, $C_1 = \left(1 - a + \frac{1}{p-\theta}\right)\frac{3}{3p+\theta}$ and $C_2 = \frac{p+1-2\theta}{(1-p)(p-\theta)}\frac{3}{4}$.
The first-order condition is given by

$$\frac{\partial \tilde{x}(k)}{\partial k} = 0 = (1 - C_0 - C_1 + C_2)\left(-\frac{\theta}{3}\right)^{k-1} \ln\frac{\theta}{3} + C_1 p^k \ln p - C_2 \theta^{k-1} \ln\theta$$

How does time of disclosure change with homophily in the circle? In order to determine how the optimal number of periods k changes with the degree of homophily ρ, we start by looking at how $x_s(k)$ and $x_m(k)$ change with ρ.

Let $\Delta x_s(k) := x_s(k)\rfloor_{\rho=2} - x_s(k)\rfloor_{\rho=\frac{1}{2}}$ and $\Delta x_m(k) = x_m(k)\rfloor_{\rho=2} - x_m(k)\rfloor_{\rho=\frac{1}{2}}$.

Proposition 1.10. *In the circle, the opinion of a supporter (non-supporter) k periods after disclosure $x_s(k)$ ($x_m(k)$) is higher (lower) for a degree of homophily $\rho = 2$ than for $\rho = \frac{1}{2}$.*

Proof. See Appendix B. □

We show next that if the politician is ideology-motivated and cares about the supporters, for higher $\rho = 2$, it is optimal to disclose earlier than for $\rho = \frac{1}{2}$. If, on the other hand, he is office-motivated and cares only about the non-supporters, it is optimal to disclose later for higher ρ.

Proposition 1.11. *In the circle, the number of periods that maximises the belief of a supporter $x_s(k)$ is higher when for degree of homophily $\rho = 2$ than for $\rho = \frac{1}{2}$. Conversely, k that maximises the belief of a non-supporter $x_m(k)$ is lower for $\rho = 2$ than for $\rho = \frac{1}{2}$.*

Proof. See Appendix B. □

Corollary 1.2. *In the case of a circle, a politician whose objective is to maximise $\tilde{x}(k)$ discloses earlier (later) for higher homophily $\rho = 2$ than for $\rho = \frac{1}{2}$ if he puts more weight on the supporter $a > \frac{1}{2}$ (non-supporter $1 - a > \frac{1}{2}$).*

Proof. See Appendix B. □

1.5.5 Two unequal groups

We investigate here the case of two groups of voters (supporters and non-supporters) that are of different sizes. Assuming that voters update their opinions following equations 1.5.6 and 1.5.7, the politician's problem changes according to s, $n - s$ the number of each type of voters.

(a) Politician targets average opinion The average opinion here can be defined as: $\bar{x}(k) = \frac{s x_s(k) + (n-s) x_m(k)}{n}$. Substituting for $x_s(k)$ and $x_m(k)$, we get: $\bar{x}(k) = \frac{s+(n-s)p^{k-1}}{n}(1-\theta)x^P + \frac{\theta}{\rho+1}(\rho s + n - s)x_s(k-1) + \frac{\theta}{\rho+1}(s + \rho(n-s))x_m(k-1)$.

(b) Politician targets weighted average opinion Consider a network of voters who belong to two distinct groups of different sizes, supporters and non-supporters. As before, we assume that every voter i has q links with voters of the same type and r of those of a different type, and that the degree of homophily is $\rho = \frac{q}{r}$. The opinions of the voters will update following equations (1.5.6) and (1.5.7). If the politician's objective is to maximise the weighted average opinion, the weight on the opinion of the supporter $a \in [0;1]$ will represent the weight on the group of supporters, and the same will apply for the non-supporters. Accordingly, if s is the number of supporters, define σ_s as the weight that the politician puts on the opinion of each supporter such that $\sigma_s = \frac{a}{s}$. For the $n-s$ non-supporters, the weight the politician puts on each of them will be σ_m defined as $\sigma_m = \frac{1-a}{n-s}$. Note that it holds that $s\sigma_s + (n-s)\sigma_m = 1$, and consequently, σ_m can be written as $\sigma_m = \frac{1-\sigma_s s}{n-s}$. The weighted average opinion is therefore

$$\tilde{x}(k) = s\sigma_s x_s(k) + (n-s)\sigma_m x_m(k)$$

An evident implication[6] is the effect of the group sizes and weights on $\tilde{x}(k)$ and consequently on the optimal timing of disclosure k^*. The latter will increase (decrease) with the number of supporters s (non-supporters $n-s$), and it will increase (decrease) with the σ_s (σ_m).

1.6 Conclusion

In this chapter, we have presented a model of narrative adoption in an election setting where voters update following a variation of De Groot average-based learning. The voters, who have a certain initial opinion, split their attention between their network and a narrative proposed by a politician, which they replace later on, with some probability, with an alternative narrative or with their initial opinion. The objective of the politician, the strategic agent in this model, is to maximise the adoption of this narrative at a given, exogenous, time limit or deadline, which we assume here to be an election. For this purpose, the politician chooses strategically the optimal timing to disclose her narrative.

We suppose that the politician either targets the average opinion, viewing all the voters as equal, or targets a weighted average opinion, assigning different weights to different voters or types of voters. We solve this problem for a benchmark with one voter, n disconnected voters and n voters connected over networks with particular structures.

We also investigate homophily and its effects on opinions and timing of disclosure. In the case of two types of voters – supporters who do not reject the politician's narrative, and non-supporters who do with a positive probability – such that the two types are equally sized with symmetric connections (the case of a network with equitable partition), the belief

[6]Taking the derivatives w.r.t. s and σ_s: $\frac{\partial \tilde{x}(k)}{\partial s} = \sigma_s x_s(k) - \frac{1-s\sigma_s-1+n\sigma_s}{n-s} x_m(k) = \sigma_s\left(x_s(k-1) - x_m(k-1)\right) \geq 0$; $\frac{\partial \tilde{x}(k)}{\partial \sigma_s} = s\left(x_s(k-1) - x_m(k-1)\right) \geq 0$.

of a supporter increases with homophily, whereas the belief of a non-supporter decreases. Furthermore, the optimal timing of disclosure is earlier and the weighted average opinion increases as the weight that the politician assigns to the opinion of supporters increases and as the probability of a non-supporter replacing the narrative decreases.

This setting can be extended to more general and inclusive situations. One possibility is considering other types of voters, notably voters who oppose the politician and update their opinions without paying any attention to her. However, a voter of this type can still be exposed to the politician's narrative indirectly through his neighbours, which will consequently affect the politician's strategy. Another extension is the presence of other strategic politicians, advocating different narratives. In political competition, as well as in marketing competition and other influence situations, a considerable number of strategic agents compete in order to convince and to gain the support of non-strategic agents. Adding more strategic agents to our approach would constitute an interesting aspect to consider and would provide more explanations for the decisions of politicians regarding narrative disclosure.

Appendices

Appendix 1.A Computations

Opinion of one voter in presence of an alternative narrative x^C

The initial opinion of the voter at the time of disclosure of the narrative is given by $x(0) = x^0$, it evolves such that

$$x(1) = (1-\theta)x^P + \theta x^0$$

$$x(2) = (1-p)\left[(1-\theta)x^C + \theta x(1)\right] + p\left[(1-\theta)x^P + \theta x(1)\right]$$

$$= (1-p)(1-\theta)x^C + p(1-\theta)x^P + \theta x(1)$$

$$x(3) = (1-p)(1-\theta)x^C + p(1-p)(1-\theta)x^C + p^2(1-\theta)x^P + \theta x(2)$$

$$\ldots$$

$$x(k) = \sum_{l=0}^{k-2} p^l(1-p)(1-\theta)x^C + p^{k-1}(1-\theta)x^P + \theta x(k-1)$$

$$= (1-\theta)x^C + p^{k-1}(1-\theta)(x^P - x^C) + \theta x(k-1)$$

By solving recursively, this expression can be rewritten as a function of x^C, x^P and x^0:

$$x(k) = \sum_{l=0}^{k-1} \theta^l(1-\theta)x^C + \sum_{l=0}^{k-1} \theta^l p^{k-1-l}(1-\theta)(x^P - x^C) + \theta^k x^0$$

Solving for the geometric series we get

$$x(k) = \left(1 - \theta^{k-1} - \frac{p^{k-1} - \theta^{k-1}}{p-\theta} p(1-\theta)\right) x^C + \frac{p^k - \theta^k}{p-\theta}(1-\theta)x^P + \theta^k x^0$$

Opinion of one voter in the case of narrative infection

$$x(0) = x^0$$

$$x(1) = (1-\theta)x^P + \theta x^0$$

$$x(2) = (1-p)\left[(1-\theta)x^0 + \theta x(t_d+1)\right] + p\left[(1-\theta)x^P + \theta x(1)\right]$$

$$= (1-p)(1-\theta)x^0 + p(1-\theta)x^P + \theta x(1)$$

$$x(k) = \sum_{l=0}^{k-2} p^l(1-p)(1-\theta)x^0 + p^{k-1}(1-\theta)x^P + \theta x(k-1)$$

$$= (1-\theta)x^0 + p^{k-1}(1-\theta)(x^P - x^0) + \theta x(k-1)$$

Computing recursively as above:

$$x(k) = (1-\theta^k)x^0 + (p^k - \theta^k)\frac{1-\theta}{p-\theta}\left(x^P - x^0\right) + \theta^k x^0$$

$$= x^0 + (p^k - \theta^k)\frac{1-\theta}{p-\theta}\left(x^P - x^0\right)$$

Opinions of n voters in an arbitrary network

$$x_i(0) = x_i^0$$

$$x_i(1) = \theta_i x^P + (1-\theta_i)\sum_{j\in N}\gamma_{ij}x_j(0)$$

$$x_i(2) = p\left[\theta_i x^C + (1-\theta_i)\sum_{j\in N}\gamma_{ij}x_j(1)\right] + (1-p)\left[\theta_i x^P + (1-\theta_i)\sum_{j\in N}\gamma_{ij}x_j(1)\right]$$

$$x_i(k) = \sum_{l=0}^{k-2}(1-p)^l p\theta_i x^C + (1-p)^{k-1}\theta_i x^P + (1-\theta_i)\sum_{j\in N}\gamma_{ij}x_j(k-1)$$

$$= \theta_i x^C + (1-p)^{k-1}\theta_i(x^P - x^C) + (1-\theta_i)\sum_{j\in N}\gamma_{ij}x_j(k-1)$$

Networks with symmetric structures, $\tilde{x}(k)$

The weighted average opinion of n voters is given by

$$\tilde{x}(k) = ax_s(k) + (1-a)x_m(k)$$

$$= \left[a + (1-a)p^{k-1}\right](1-\theta)x^P + \frac{\theta}{\rho+1}\left[\left(a\rho + 1 - a\right)x_s(k-1) + \left(a + \rho(1-a)\right)x_m(k-1)\right]$$

(See footnote [7])

$$= \left[a + (1-a)p^{k-1}\right](1-\theta)x^P + \frac{\theta}{\rho+1}\left[2\bar{x}(k-1) + (\rho-1)\tilde{x}(k-1)\right]$$

Iterating and using $\bar{x}(k) = \left(\frac{1-\theta^k}{1-\theta} + \frac{p^k-\theta^k}{p-\theta}\right)\frac{1-\theta}{2}x^P$, we get

$$= (1-\theta)x^P \left[\sum_{l=0}^{k-1}\left(a + (1-a)p^{k-1-l}\right)\left(\frac{\theta(\rho-1)}{\rho+1}\right)^l\right.$$

$$\left. + \frac{\theta}{\rho+1}\sum_{l=0}^{k-2}\left(\frac{\theta(\rho-1)}{\rho+1}\right)^l\left(\frac{1-\theta^{k-1-l}}{1-\theta} + \frac{p^{k-1-l}-\theta^{k-1-l}}{p-\theta}\right)\right]$$

$$= (1-\theta)x^P\left\{\frac{\rho+1}{\rho+1-\theta(\rho-1)}\left(a + \frac{\theta}{(\rho+1)(1-\theta)}\right)\right.$$

$$+ \left(\frac{\theta(\rho-1)}{\rho+1}\right)^{k-1}\left[1 - \frac{\rho+1}{\rho+1-\theta(\rho-1)}\left(a + \frac{\theta}{(\rho+1)(1-\theta)}\right)\right.$$

$$\left. - \frac{p(\rho+1)}{p(\rho+1)-\theta(\rho-1)}\left(1 - a + \frac{\theta}{(p-\theta)(\rho+1)}\right) + \frac{\theta}{2}\frac{p+1-2\theta}{(1-\theta)(p-\theta)}\right]$$

$$\left. + \frac{p(\rho+1)}{p(\rho+1)-\theta(\rho-1)}\left(1 - a + \frac{\theta}{(p-\theta)(\rho+1)}\right)p^{k-1} - \frac{\theta}{2}\left(\frac{p+1-2\theta}{(1-\theta)(p-\theta)}\right)\theta^{k-1}\right\}$$

$$= C_0 + C_1 p^{k-1} - C_2\theta^{k-1} + (1 - C_0 - C_1 + C_2)\left(\frac{\theta(\rho-1)}{\rho+1}\right)^{k-1}$$

where C_0, C_1 and C_2 are constants, independent of k.

Circle with $\rho = 2$, $\tilde{x}(k)$

$$\tilde{x}(k) = \left(a + (1-a)p^{k-1}\right)(1-\theta)x^P$$

$$+ \left(\frac{2}{3}a + \frac{1}{3}(1-a)\right)\theta x_s(k-1) + \left(\frac{2}{3}(1-a) + \frac{1}{3}a\right)\theta x_m(k-1)$$

$$= \left(a + (1-a)p^{k-1}\right)(1-\theta)x^P + \frac{1}{3}\theta\left[\tilde{x}(k-1) + 2\bar{x}(k-1)\right]$$

$$= \sum_{l=0}^{k-2}\left(\frac{\theta}{3}\right)^l\left[a + (1-a)p^{k-1-l}\right](1-\theta)x^P + \sum_{l=0}^{k-2}\left(\frac{\theta}{3}\right)^l\frac{2\theta}{3}\bar{x}(k-l-1) + \left(\frac{\theta}{3}\right)^{k-1}\tilde{x}(1)$$

where $\tilde{x}(1) = (1-\theta)x^P$, then

$$\tilde{x}(k) = \sum_{l=0}^{k-1}\left(\frac{\theta}{3}\right)^l\left[a + (1-a)p^{k-1-l}\right](1-\theta)x^P + \frac{2\theta}{3}\sum_{l=0}^{k-2}\left(\frac{\theta}{3}\right)^l\bar{x}(k-l-1)$$

[7] $(a\rho+1-a)\,x_s(k-1) + (a+\rho(1-a))\,x_m(k-1) = (a\rho-a)x_s(k-1) + (a-a\rho)x_m(k-1) + x_s(k-1) + \rho x_m(k-1) + x_m(k-1) - x_m(k-1) = a(\rho-1)x_s(k-1) + (1-a)(\rho-1)x_m(k-1) + 2\bar{x}(k)$.

However, the average opinion is given by

$$\bar{x}(k) = \frac{x_s(k) + x_m(k)}{2} = \frac{1 + p^{k-1}}{2}(1-\theta)x^P + \theta\frac{x_s(k-1) + x_m(k-1)}{2}$$

$$= \frac{1 + p^{k-1}}{2}(1-\theta)x^P + \theta\bar{x}(k-1) = \sum_{l=0}^{k-1}\frac{(1 + p^{k-1-l})}{2}\theta^l(1-\theta)x^P$$

$$= \left(\frac{1-\theta^k}{1-\theta} + \frac{p^k - \theta^k}{p-\theta}\right)\frac{(1-\theta)}{2}x^P$$

Substituting we get

$$\tilde{x}(k) = \sum_{l=0}^{k-1}\left(\frac{\theta}{3}\right)^l\left[a + (1-a)p^{k-1-l}\right](1-\theta)x^P$$

$$+\frac{2\theta}{3}\sum_{l=0}^{k-2}\left(\frac{\theta}{3}\right)^l\left(\frac{1-\theta^{k-1-l}}{1-\theta} + \frac{p^{k-1-l} - \theta^{k-1-l}}{p-\theta}\right)\frac{(1-\theta)}{2}x^P$$

$$= (1-\theta)x^P\left[a\sum_{l=0}^{k-1}\left(\frac{\theta}{3}\right)^l + (1-a)p^{k-1}\sum_{l=0}^{k-1}\left(\frac{\theta}{3p}\right)^l + \frac{\theta}{3(1-\theta)}\sum_{l=0}^{k-2}\left(\frac{\theta}{3}\right)^l\right.$$

$$\left.-\frac{\theta^k}{3(1-\theta)}\sum_{l=0}^{k-2}\left(\frac{1}{3}\right)^l + \frac{\theta p^{k-1}}{3(p-\theta)}\sum_{l=0}^{k-2}\left(\frac{\theta}{3p}\right)^l - \frac{\theta^k}{3(p-\theta)}\sum_{l=0}^{k-2}\left(\frac{1}{3}\right)^l\right]$$

$$= (1-\theta)x^P\left[\left(\frac{\theta}{3}\right)^{k-1} + \left(a + \frac{\theta}{3(1-\theta)}\right)\frac{1-\left(\frac{\theta}{3}\right)^{k-1}}{1-\frac{\theta}{3}}\right.$$

$$\left.+\left(1 - a + \frac{\theta}{3(p-\theta)}\right)p^{k-1}\frac{1-\left(\frac{\theta}{3p}\right)^{k-1}}{1-\frac{\theta}{3p}} - \left(\frac{1}{1-\theta} + \frac{1}{p-\theta}\right)\frac{\theta^k}{3}\frac{1-\left(\frac{1}{3}\right)^{k-1}}{1-\frac{1}{3}}\right]$$

$$= (1-\theta)x^P\left[\left(\frac{\theta}{3}\right)^{k-1}\left(1 - \frac{3}{3-\theta}(a + \frac{\theta}{3(1-\theta)}) - (1 - a + \frac{\theta}{3(p-\theta)})\frac{3p}{p-\theta} + \frac{p+1-2\theta}{(1-\theta)(p-\theta)}\frac{\theta}{2}\right)\right.$$

$$\left.+\frac{3}{3-\theta}\left(a + \frac{\theta}{3(1-\theta)}\right) - \frac{p+1-2\theta}{2(1-\theta)(p-\theta)}\theta^k + \left(1 - a + \frac{\theta}{3(p-\theta)}\right)\frac{3p^k}{3p-\theta}\right]$$

$$= (1-\theta)x^P\left[C_0 + C_1p^k - C_2\theta^k + (1 - C_0 - C_1 + C_2)\left(\frac{\theta}{3}\right)^{k-1}\right]$$

where C_0, C_1 and C_2 are constants that depend on the parameters of the model.

Circle with $\rho = \frac{1}{2}$, $\tilde{x}(k)$

$$\tilde{x}(k) = \left[a + (1-a)p^{k-1}\right](1-\theta)x^P$$

$$+\left(\frac{1}{3}a + \frac{2}{3}(1-a)\right)\theta x_s(k-1) + \left(\frac{2}{3}a + \frac{1}{3}(1-a)\right)\theta x_m(k-1)$$

$$= \left[a + (1-a)p^{k-1}\right](1-\theta)x^P$$

$$+\frac{\theta}{3}\left[2x_s(k-1) - ax_s(k-1) + (a-1)x_m(k-1) + 2x_m(k-1)\right]$$

$$= \left[a + (1-a)p^{k-1}\right](1-\theta)x^P + \frac{\theta}{3}\left[4\bar{x}(k-1) - \tilde{x}(k-1)\right]$$

$$= \sum_{l=0}^{k-1}\left(-\frac{\theta}{3}\right)^l\left[a + (1-a)p^{k-1-l}\right](1-\theta)x^P + \sum_{l=0}^{k-2}\frac{4\theta}{3}\left(-\frac{\theta}{3}\right)^l\bar{x}(k-1-l)$$

substituting for $\bar{x}(k-1-l)$,

$$= \sum_{l=0}^{k-1}\left(-\frac{\theta}{3}\right)^l\left[a + (1-a)p^{k-1-l}\right](1-\theta)x^P$$

$$+\frac{4\theta}{3}\sum_{l=0}^{k-2}\left(-\frac{\theta}{3}\right)^l\left(\frac{1-\theta^{k-1-l}}{1-\theta} + \frac{p^{k-1-l}-\theta^{k-1-l}}{p-\theta}\right)\frac{1-\theta}{2}x^P$$

$$= (1-\theta)x^P\left[a\sum_{l=0}^{k-1}\left(-\frac{\theta}{3}\right)^l + (1-a)p^{k-1}\sum_{l=0}^{k-1}\left(-\frac{\theta}{3p}\right)^l + \frac{2\theta}{3(1-\theta)}\sum_{l=0}^{k-2}\left(-\frac{\theta}{3}\right)^l\right.$$

$$\left.-\frac{2\theta^k}{3(1-\theta)}\sum_{l=0}^{k-2}\left(-\frac{1}{3}\right)^l + \frac{2\theta p^{k-1}}{3(p-\theta)}\sum_{l=0}^{k-2}\left(-\frac{\theta}{3p}\right)^l - \frac{2\theta^k}{3(p-\theta)}\sum_{l=0}^{k-2}\left(-\frac{1}{3}\right)^l\right]$$

$$= (1-\theta)x^P\left[\left(-\frac{\theta}{3}\right)^{k-1} + \frac{1-\left(-\frac{\theta}{3}\right)^{k-1}}{1+\frac{\theta}{3}}\left(a + \frac{2\theta}{3(1-\theta)}\right)\right.$$

$$\left.+\frac{1-\left(-\frac{\theta}{3p}\right)^{k-1}}{1+\frac{\theta}{3p}}p^{k-1}\left(1-a+\frac{2\theta}{3(p-\theta)}\right) - \frac{1-\left(-\frac{1}{3}\right)^{k-1}}{1+\frac{1}{3}}\frac{2\theta^k}{3}\left(\frac{1}{1-\theta} + \frac{1}{p-\theta}\right)\right]$$

$$= (1-\theta)x^P\left[\left(-\frac{\theta}{3}\right)^{k-1}\left(1 - \left(a + \frac{2\theta}{3(1-\theta)}\right)\frac{3}{3+\theta} - (1-a+\frac{1}{p-\theta})\frac{3}{3p+\theta} + \frac{3}{4}\frac{p+1-2\theta}{(1-p)(p-\theta)}\right)\right.$$

$$\left.+\left(a + \frac{2\theta}{3(1-\theta)}\right)\frac{3}{3+\theta} + \left(1-a+\frac{1}{p-\theta}\right)\frac{3}{3p+\theta}p^k - \theta^{k-1}\frac{p+1-2\theta}{(1-p)(p-\theta)}\frac{3}{4}\right]$$

$$= (1-\theta)x^P\left[C_0 + C_1p^k - C_2\theta^{k-1} + (1-C_0-C_1+C_2)\left(-\frac{\theta}{3}\right)^{k-1}\right]$$

Appendix 1.B Proofs

Proof of Lemma 1.1

$$\lim_{k\to\infty} x(k) = \lim_{k\to\infty}\left(1-\theta^{k-1} - \frac{p^{k-1}-\theta^{k-1}}{p-\theta}p(1-\theta)\right)x^C + \frac{p^k-\theta^k}{p-\theta}(1-\theta)x^P + \theta^k x^0 = x^C$$

Proof of Lemma 1.2

The first order condition of the politician's maximisation problem is as follows:

$$-\theta^k \ln \theta x^P = 0 \iff \theta^k = 0$$

This holds when k goes to infinity.

Proof of Proposition 1.1

The first order condition is:

$$(p^k \ln p - \theta^k \ln \theta)\frac{1-\theta}{p-\theta}x^P = 0 \iff \left(\frac{p}{\theta}\right)^k = \frac{\ln \theta}{\ln p}$$

$$\iff k = \frac{\ln\left(\frac{\ln \theta}{\ln p}\right)}{\ln \frac{p}{\theta}}$$

k is well-defined for $p \neq \theta$ and $0 < \theta, p < 1$. We show below that k will always be positive. We take the second order condition to verify that this is a maximum:

$$\frac{\partial^2 x(k)}{\partial k^2} = (p^k(\ln p)^2 - \theta^k(\ln \theta)^2)\frac{1-\theta}{p-\theta}x^P$$

for which the sign is the sign of

$$\frac{p^k(\ln p)^2 - \theta^k(\ln \theta)^2}{p-\theta}$$

Finding the roots:

$$\frac{p^k(\ln p)^2 - \theta^k(\ln \theta)^2}{p-\theta} = 0 \iff \frac{\theta^k\left[\left(\frac{p}{\theta}\right)^k \ln^2 p - \ln^2 \theta\right]}{p-\theta} = 0$$

substituting for $\left(\frac{p}{\theta}\right)^k$we get:

$$\frac{\theta^k\left[\frac{\ln \theta}{\ln p}\ln^2 p - \ln^2 \theta\right]}{p-\theta} = \frac{\theta^k\left[\ln \theta(\ln p - \ln \theta)\right]}{p-\theta} = 0$$

This is not defined at $p = \theta$ and is negative elsewhere since:

if $p > \theta$:$p - \theta > 0$; $\ln p > \ln \theta$ and $\ln \theta < 0$.

if $p < \theta$:$p - \theta < 0$; $\ln p < \ln \theta$ and $\ln \theta < 0$.

Sign of k

- For $\theta > p$ then:$\ln\left(\frac{p}{\theta}\right)$ and $\ln\left[\frac{\ln \theta}{\ln p}\right] > 0 \implies k > 0$.

- For $\theta < p$ then: $\ln\left(\frac{p}{\theta}\right)$ and $\ln\left[\frac{\ln\theta}{\ln p}\right] < 0 \implies k > 0$

Proof of Proposition 1.2

First, derivative of k with respect to θ and p will give an opposite sign to the derivative of k with respect to $1-\theta$ and $1-p$ respectively since:

$$\frac{\partial k}{\partial(1-\theta)} = \frac{\partial f}{\partial\theta}\frac{\partial\theta}{\partial(1-\theta)} = \frac{\partial f}{\partial\theta}\frac{1}{\frac{\partial(1-\theta)}{\partial\theta}} = -\frac{\partial k}{\partial\theta}$$

and similarly for the derivative with respect to p.

Now, taking the derivative of k with respect to p, we get:

$$\frac{\partial k}{\partial p} = -\frac{\frac{1}{\ln p}\ln\frac{p}{\theta} + \ln\frac{\ln\theta}{\ln p}}{p(\ln\frac{p}{\theta})^2}$$

$$\therefore sign\left(\frac{\partial k}{\partial p}\right) = sign\left[-\left(\frac{1}{\ln p}\ln\frac{p}{\theta} + \ln\frac{\ln\theta}{\ln p}\right)\right]$$

$$= sign\left[-\left(1 - \frac{\ln\theta}{\ln p} + \ln\frac{\ln\theta}{\ln p}\right)\right] = sign\left(-1 + \frac{\ln\theta}{\ln p} - \ln\frac{\ln\theta}{\ln p}\right)$$

The function $-1 + \frac{\ln\theta}{\ln p} - \ln\frac{\ln\theta}{\ln p}$ has the form of the transcendental equation $x - 1 - \ln x$ whose minimum is at $x = 1$ and is equal to 0. Therefore this function is non-negative and $\frac{\partial k}{\partial p} > 0$ for $p \neq \theta$. Consequently, $\frac{\partial k}{\partial(1-p)} < 0$.

Finally, consider the derivative of k with respect to θ which gives:

$$\frac{\partial k}{\partial\theta} = \frac{\frac{1}{\ln\theta}\ln\frac{p}{\theta} + \ln\frac{\ln\theta}{\ln p}}{\theta(\ln\frac{p}{\theta})^2}$$

$\therefore sign\left(\frac{\partial k}{\partial\theta}\right) = sign\left(\frac{1}{\ln\theta}\ln\frac{p}{\theta} + \ln\frac{\ln\theta}{\ln p}\right) = sign\left(\frac{\ln p}{\ln\theta} - 1 + \ln\frac{\ln\theta}{\ln p}\right) = sign\left(\frac{\ln p}{\ln\theta} - 1 - \ln\frac{\ln p}{\ln\theta}\right)$

Similarly to the above, the function $\frac{\ln p}{\ln\theta} - 1 - \ln\frac{\ln p}{\ln\theta}$ is a transcendental equation of the form $x - 1 - \ln x$. Thus, $\frac{\partial k}{\partial\theta} > 0\ for\ p \neq \theta$ and $\frac{\partial k}{\partial(1-\theta)} < 0$.

Proof of Corollary 1.1

First order condition:

$$(p^k\ln p - \theta^k\ln\theta)\frac{1-\theta}{p-\theta}x^P + \theta^k\ln\theta x^0 = 0$$

$$p^k\ln p\frac{1-\theta}{p-\theta}x^P = \theta^k\ln\theta\left(\frac{1-\theta}{p-\theta}x^P - x^0\right)$$

$$\left(\frac{p}{\theta}\right)^k = \frac{\ln\theta}{\ln p}\frac{\left(\frac{(1-\theta)x^P-(p-\theta)x^0}{p-\theta}\right)}{\frac{1-\theta}{p-\theta}x^P} = \frac{\ln\theta}{\ln p}\frac{(1-\theta)x^P-(p-\theta)x^0}{(1-\theta)x^P} = \frac{\ln\theta}{\ln p}\left(1 + \frac{\theta-p}{1-\theta}\frac{x^0}{x^P}\right)$$

$$\iff k = \frac{\ln\left[\frac{\ln\theta}{\ln p}\left(1 + \frac{\theta - p}{1-\theta}\frac{x^0}{x^P}\right)\right]}{\ln\frac{p}{\theta}}$$

The second derivative is given by $\frac{\partial^2 x(k)}{\partial k^2} = (p^k(\ln p)^2 - \theta^k(\ln\theta)^2)\frac{1-\theta}{p-\theta}x^P + \theta^k(\ln\theta)^2 x^0$, which can be written as $\theta^k \ln\theta\left(\frac{1-\theta}{p-\theta}x^P - x^0\right)(\ln p - \ln\theta)$ using the first order condition that gives $\left(\frac{p}{\theta}\right)^k = \frac{\ln\theta}{\ln p}\left(1 + \frac{\theta-p}{1-\theta}\frac{x^0}{x^P}\right)$.

- If $p > \theta$ then $\frac{1-\theta}{p-\theta} > 1 \iff \frac{1-\theta}{p-\theta}x^P - x^0 > x^P - x^0 > 0$ and $\ln p - \ln\theta > 0$ while $\ln\theta < 0$.
- If $p < \theta$ then $\frac{1-\theta}{p-\theta}x^P < 0 \iff \frac{1-\theta}{p-\theta}x^P - x^0 < 0$, $\ln p - \ln\theta < 0$ and $\ln\theta < 0$.

Therefore for any $p \neq \theta$, the second derivative is negative and k is a maximum.

Proof of Lemma 1.3

The politician maximises the average opinion $\bar{x}(k) = \frac{\sum_i x_i(k)}{n} = (p^k - \theta^k)\frac{1-\theta}{p-\theta}x^P + \frac{\theta^k}{n}\sum_{i\in N} x_i^0 = (p^k - \theta^k)\frac{1-\theta}{p-\theta}x^P + \theta^k \bar{x}^0$. The first order condition is:

$$\frac{\partial \bar{x}(k)}{\partial k} = (p^k \ln p - \theta^k \ln\theta)\frac{1-\theta}{p-\theta}x^P + \theta^k \ln\theta \bar{x}^0 = 0$$

$$\iff \theta^k \ln\theta\left(\frac{1-\theta}{p-\theta}x^P - \bar{x}^0\right) = p^k \ln p \frac{1-\theta}{p-\theta}x^P$$

$$k = \frac{\ln\frac{\ln\theta}{\ln p}\left(1 - \frac{p-\theta}{1-\theta}\frac{\bar{x}^0}{x^P}\right)}{\ln\frac{p}{\theta}}$$

Second order condition:

$$\frac{\partial \bar{x}^2(k)}{\partial k^2} = \left(p^k(\ln p)^2 - \theta^k(\ln\theta)^2\right)\frac{1-\theta}{p-\theta}x^P + \theta^k(\ln\theta)^2\bar{x}^0$$

By substituting from the first order condition for $p^k \ln p$, we get

$$= \theta^k \ln\theta\left(\frac{1-\theta}{p-\theta}x^P - \bar{x}^0\right)(\ln p - \ln\theta)$$

which is negative since for $p > \theta$, $\ln p - \ln\theta > 0$ and $1 - p < 1 - \theta \implies p - \theta < 1 - \theta$ and $\frac{1-\theta}{p-\theta}x^P - \bar{x}^0 > 0$. While for $p < \theta$, $\frac{1-\theta}{p-\theta}x^P < 0$ and $\ln p - \ln\theta < 0$. Hence, k is a maximum.

Proof of Lemma 1.4

Solving the politician's problem:

$$\max_k \bar{x}(k) = \frac{s}{n}(1-\theta^k)x^P + \frac{n-s}{n}\frac{p^k - \theta^k}{p-\theta}(1-\theta)x^P$$

The first order condition:

$$\frac{\partial \bar{x}(k)}{\partial k} = -\frac{s}{n}\theta^k \ln\theta x^P + \frac{n-s}{n}\frac{1-\theta}{p-\theta}p^k \ln p x^P - \frac{n-s}{n}\frac{1-\theta}{p-\theta}\theta^k \ln\theta x^P = 0$$

$$\iff \frac{n-s}{n}\frac{1-\theta}{p-\theta}p^k \ln p = \left(\frac{s}{n} + \frac{n-s}{n}\frac{1-\theta}{p-\theta}\right)\theta^k \ln\theta$$

$$\iff p^k \ln p = \left(\frac{s(p-\theta)}{(n-s)(1-\theta)} + 1\right)\theta^k \ln\theta$$

$$\iff \left(\frac{p}{\theta}\right)^k = \frac{\ln\theta}{\ln p}\left(\frac{s(p-\theta)}{(n-s)(1-\theta)} + 1\right)$$

$$k = \frac{\ln\left(\frac{\ln\theta}{\ln p}\left(\frac{s(p-\theta)}{(n-s)(1-\theta)} + 1\right)\right)}{\ln\frac{p}{\theta}}$$

k exists whenever $\left(\frac{s(p-\theta)}{(n-s)(1-\theta)} + 1\right) > 0$, and equivalently $n(1-\theta) - s(1-p) > 0$.

The second order condition:

$$\frac{\partial \bar{x}(k)^2}{\partial k^2} = -\frac{s}{n}\theta^k (\ln\theta)^2 x^P + \frac{n-s}{n}\frac{1-\theta}{p-\theta}p^k (\ln p)^2 x^P - \frac{n-s}{n}\frac{1-\theta}{p-\theta}\theta^k (\ln\theta)^2 x^P$$

Substituting for $p^k \ln p = \theta^k \ln\theta\left(\frac{s(p-\theta)}{(n-s)(1-\theta)} + 1\right)$ from the first order condition, we get

$$\frac{\partial^2 \bar{x}(k)}{\partial k^2} = \theta^k \ln\theta x^P \left(\frac{s}{n} + \frac{n-s}{n}\frac{1-\theta}{p-\theta}\right)(\ln p - \ln\theta)$$

This can be simplified into

$$\frac{\theta^k \ln\theta x^P}{n}\left(s(p-\theta) + (n-s)(1-\theta)\right)\frac{(\ln p - \ln\theta)}{p-\theta}$$

$$= \frac{\theta^k \ln\theta x^P}{n}\left(n(1-\theta) - s(1-p)\right)\frac{(\ln p - \ln\theta)}{p-\theta}$$

Clearly, $\frac{(\ln p - \ln\theta)}{p-\theta}$ is positive for any p and θ; and $\frac{\theta^k \ln\theta x^P}{n} < 0$ due to $\ln\theta$. The second derivative is therefore negative since $n(1-\theta) - s(1-p) > 0$ for existence, and k is a maximum.

Proof of Lemma 1.5

The first order condition is given by

$$\frac{\partial \tilde{x}(k)}{\partial k} = -a\theta^k \ln\theta x^P + (1-a)(p^k \ln p - \theta^k \ln\theta)\frac{1-\theta}{p-\theta}x^P = 0$$

$$\iff \frac{p^k}{\theta^k} = \frac{\ln\theta}{\ln p}\left(\frac{a(p-\theta)}{(1-a)(1-\theta)} + 1\right)$$

$$k = \frac{\ln\left[\frac{\ln\theta}{\ln p}\left(\frac{a(p-\theta)}{(1-a)(1-\theta)}+1\right)\right]}{\ln\frac{p}{\theta}}$$

This exists whenever $\frac{a(p-\theta)}{(1-a)(1-\theta)} + 1 > 0$, which holds for $1-\theta-a(1-p) > 0$.

The second derivative is

$$\frac{\partial^2 \tilde{x}(k)}{\partial k^2} = -a\theta^k(\ln\theta)^2 x^P + (1-a)\left(p^k(\ln p)^2 - \theta^k(\ln\theta)^2\right)\frac{1-\theta}{p-\theta}x^P$$

substituting for $p^k \ln p$ from the first order condition we get

$$= \theta^k \ln\theta x^P\left[-a\ln\theta + \ln p\left(a + (1-a)\frac{1-\theta}{p-\theta}\right)\right]$$

$$= \theta^k \ln\theta x^P \frac{\ln p - \ln\theta}{p-\theta}\left(1-\theta-(1-p)a\right)$$

which is negative whenever k exists. Hence, k is a maximum.

Proof of Proposition 1.3

$$\mathbf{X}(k) = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \sum_{l=0}^{k-1}(\boldsymbol{\Theta\Gamma})^l\boldsymbol{\Lambda}^{(k-1-l)} & (\boldsymbol{\Theta\Gamma})^{k-1}\end{pmatrix}\mathbf{X}(0)$$

First, we show that $(\boldsymbol{\Theta\Gamma})^{k-1}$ is convergent. By definition, the adjacency matrix of the voters network is row-stochastic. We define $\theta^{\max}$ as the highest θ_i across all voters: $\theta^{\max} := \max\{\theta_i, i = 1, 2, ..., n\}$. This implies that $\boldsymbol{\Theta\Gamma} \leq \theta\boldsymbol{\Gamma}$.

The spectrum of $\boldsymbol{\Theta\Gamma}$ is the set of its eigenvalues, call it $spec(\boldsymbol{\Theta\Gamma})$. Its spectral radius is defined by $\rho(\boldsymbol{\Theta\Gamma}) := \max\{|\lambda| : \lambda \in spec(\boldsymbol{\Theta\Gamma})\}$. Given that $\boldsymbol{\Gamma}$ is row-stochastic, then $\rho(\boldsymbol{\Gamma}) = 1$ and $\rho(\boldsymbol{\Theta\Gamma}) \leq \theta\rho(\boldsymbol{\Gamma}) = \theta < 1$ (see Meyer (2000)). This implies that $(\boldsymbol{\Theta\Gamma})^{k-1}$ converges to zero as k goes to infinity. Furthermore, $\sum_{k=0}^{\infty}(\boldsymbol{\Theta\Gamma})^k = (\mathbf{I} - \boldsymbol{\Theta\Gamma})^{-1}$.

Second, we investigate $\sum_{l=0}^{k-1}(\boldsymbol{\Theta\Gamma})^l\boldsymbol{\Lambda}^{(k-1-l)}$. This can be written as

$$\boldsymbol{\Lambda}^{(k-1)} + (\boldsymbol{\Theta\Gamma})\boldsymbol{\Lambda}^{(k-2)} + ... + (\boldsymbol{\Theta\Gamma})^{k-2}\boldsymbol{\Lambda}^{(1)} + (\boldsymbol{\Theta\Gamma})^{k-1}\boldsymbol{\Lambda}^{(0)}$$

We can show that

$$\boldsymbol{\Lambda}^{(k-1)} = (\mathbf{I} - \Theta)\begin{pmatrix} p_1^{k-1} & 1 - p_1^{k-1} \\ \vdots & \\ & \\ p_n^{k-1} & 1 - p_n^{k-1}\end{pmatrix}$$

$$= \begin{pmatrix} (1-\theta_1)\, p_1^{k-1} & (1-\theta_1)(1-p_1^{k-1}) \\ \vdots & \\ & \\ (1-\theta_n)\, p_n^{k-1} & (1-\theta_n)(1-p_n^{k-1}) \end{pmatrix} \leq \begin{pmatrix} (1-\theta_1) & (1-\theta_1) \\ \vdots & \\ & \\ (1-\theta_n) & (1-\theta_n) \end{pmatrix}$$

which holds for any k. Let $\varphi := \begin{pmatrix} (1-\theta_1) & (1-\theta_1) \\ \vdots & \\ & \\ (1-\theta_n) & (1-\theta_n) \end{pmatrix}$, then we can write

$$\mathbf{\Lambda}^{(k-1)} + (\mathbf{\Theta\Gamma})\,\mathbf{\Lambda}^{(k-2)} + ... + (\mathbf{\Theta\Gamma})^{k-1}\mathbf{\Lambda}^{(0)} \leq \left[\mathbf{I} + (\mathbf{\Theta\Gamma}) + ... + (\mathbf{\Theta\Gamma})^{k-1}\right]\varphi$$

As k goes to infinity, the right-hand side converges to $(\mathbf{I} - \mathbf{\Theta\Gamma})^{-1}\varphi$. This implies that :

$$\sum_{l=0}^{k-1}(\mathbf{\Theta\Gamma})^{l}\mathbf{\Lambda}^{(k-1-l)} \leq (\mathbf{I} - \mathbf{\Theta\Gamma})^{-1}\varphi$$

Hence $\sum_{l=0}^{k-1}(\mathbf{\Theta\Gamma})^{l}\mathbf{\Lambda}^{(k-1-l)}$ is convergent. Therefore, the matrix

$$\Pi_{l=0}^{k-1}\mathbf{M}^{(l)} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \sum_{l=0}^{k-1}(\mathbf{\Theta\Gamma})^{l}\mathbf{\Lambda}^{(k-1-l)} & (\mathbf{\Theta\Gamma})^{k-1} \end{pmatrix}$$

is also convergent.

Proof of Proposition 1.4

To show that the average opinion after k learning periods does not depend on the network, we show that $\bar{x}(k)$ does not depend on γ_{ij} for any $i, j \in N$.

$$\bar{x}(k) = \frac{1}{n}\sum_i x_i(k) \text{ where } x_i(k) = p_i^{k-1}(1-\theta)x^P + \theta\sum_j \gamma_{ij}x_j(k-1)$$

$$\bar{x}(k) = \frac{1}{n}\sum_i \left[p_i^{k-1}(1-\theta)x^P + \theta\sum_j \gamma_{ij}x_j(k-1)\right]$$

$$= \frac{\sum_i p_i^{k-1}}{n}(1-\theta)x^P + \frac{\theta}{n}\sum_i\sum_j \gamma_{ij}x_j(k-1)$$

$$= \frac{\sum_i p_i^{k-1}}{n}(1-\theta)x^P + \frac{\theta}{n}\sum_j\left(\sum_i \gamma_{ij}\right)x_j(k-1)$$

Since Γ is column stochastic $\sum_i \gamma_{ij} = 1$ for all $j \in N$, hence

$$\bar{x}(k) = \frac{\sum_i p_i^{k-1}}{n}(1-\theta)x^P + \frac{\theta}{n}\sum_i x_i(k-1)$$

$$= \frac{\sum_i p_i^{k-1}}{n}(1-\theta)x^P + \theta\bar{x}(k-1)$$

which does not depend on γ_{ij} and is identical to the average opinion when voters are disconnected.

Proof of Proposition 1.5

Taking the derivative of k with respect to s we get:

$$\frac{\partial k}{\partial s} = \frac{\ln\frac{\ln\theta}{\ln p}}{\ln\frac{p}{\theta}}\frac{p-\theta}{1-\theta}\frac{n}{(n-s)^2} = \frac{n(\theta-p)}{(n-s)(s(1-p)-n(1-\theta))\ln\frac{p}{\theta}}$$

$\frac{\theta-p}{\ln\frac{p}{\theta}}$ is always negative since if $\theta > p$, $\frac{p}{\theta} < 1$ and $\ln\frac{p}{\theta} < 0$, while if $\theta < p$, $\theta - p < 0$ and $\ln\frac{p}{\theta} > 0$. Hence, $sign\left(\frac{\partial k}{\partial s}\right) = -sign\left(s(1-p) - n(1-\theta)\right)$.

$$s(1-p) - n(1-\theta) < 0 \iff n - s - n\theta + sp > 0$$

which is the condition of existence for k in the balanced network. Therefore, it always holds that $\frac{\partial k}{\partial s} > 0$.

Conversely, if this condition does not hold, $\bar{x}(k)$ is increasing and the earliest disclosure is optimal.

Proof of Proposition 1.6

We consider the difference between the belief of a supporter and that of a non-supporter

$$x_s(k) - x_m(k) =$$

$$(1-p^{k-1})(1-\theta)x^P + \frac{\theta(\rho-1)}{\rho+1}\left(x_s(k-1) - x_m(k-1)\right)$$

$$= \sum_{l=0}^{k-1}\left(\frac{\theta(\rho-1)}{\rho+1}\right)^l (1-p^{k-1-l})(1-\theta)x^P$$

$$= \sum_{l=0}^{k-1}\left(\frac{\theta(\rho-1)}{\rho+1}\right)^l (1-\theta)x^P - \sum_{l=0}^{k-1}\left(\frac{\theta(\rho-1)}{p(\rho+1)}\right)^l p^{k-1}(1-\theta)x^P$$

Define x and y such that $x := \frac{\theta(\rho-1)}{\rho+1}$ and $y := \frac{\theta(\rho-1)}{p(\rho+1)}$. We notice that $p = \frac{x}{y}$ and we substitute by x and y:

$$= \sum_{l=0}^{k-1} x^l (1-\theta)x^P - \sum_{l=0}^{k-1} y^l \left(\frac{x}{y}\right)^{k-1} (1-\theta)x^P$$

$$= (1-\theta)x^P \sum_{l=0}^{k-2} \left[x^l - \left(\frac{x}{y}\right)^{k-1} y^l \right]$$

$$= (1-\theta)x^P \left[1 + x + ... + x^{k-1} - \left(\frac{x}{y}\right)^{k-1} \left(1 + y + ... + y^{k-1}\right) \right]$$

$$= (1-\theta)x^P \left[1 - \left(\frac{x}{y}\right)^{k-1} + x\left(1 - \left(\frac{x}{y}\right)^{k-2}\right) + ... + x^{k-2}\left(1 - \frac{x}{y}\right) \right]$$

$$= (1-\theta)x^P \left[1 - p^{k-1} + \frac{\theta(\rho-1)}{\rho+1}\left(1 - p^{k-2}\right) + ... + \left(\frac{\theta(\rho-1)}{\rho+1}\right)^{k-2} (1-p) \right]$$

Comparing every two consecutive terms, we notice that their sum is positive:

$$1 - p^{k-1} > 1 - p^{k-2} \text{ and } 1 > \frac{\theta(\rho-1)}{\rho+1} \forall \rho$$

$$1 - p^{k-3} > 1 - p^{k-4} \text{ and } \left(\frac{\theta(\rho-1)}{\rho+1}\right)^{k-2} > \left(\frac{\theta(\rho-1)}{\rho+1}\right)^{k-3} \forall \rho$$

This clearly holds when the number of terms in the sum is even. While when it is odd, the $\left(\frac{\theta(\rho-1)}{\rho+1}\right)^{k-2}$ in the last term will have a positive power $k-2$ (since in the first term it has power 0), therefore it is also positive. Hence $x_s(k) - x_m(k) > 0$ for any ρ.

Proof of Proposition 1.7

First, we present the following Lemma that we will use to complete the proof of Proposition 1.7.

Lemma 1.6. *The changes of beliefs of a supporter and a non-supporter with the level of homophily ρ are such that $\frac{\partial x_s(k)}{\partial \rho} + \frac{\partial x_m(k)}{\partial \rho} = 0$.*

Proof. $\frac{\partial x_s(k)}{\partial \rho} = \frac{\theta}{(\rho+1)^2}\left(x_s(k-1) - x_m(k-1)\right) + \frac{\theta\rho}{\rho+1}\frac{\partial x_s(k-1)}{\partial \rho} + \frac{\theta}{\rho+1}\frac{\partial x_m(k-1)}{\partial \rho}$
$\frac{\partial x_m(k)}{\partial \rho} = \frac{\theta}{(\rho+1)^2}\left(x_m(k-1) - x_s(k-1)\right) + \frac{\theta\rho}{\rho+1}\frac{\partial x_m(k-1)}{\partial \rho} + \frac{\theta}{\rho+1}\frac{\partial x_s(k-1)}{\partial \rho}$

$$\frac{\partial x_s(k)}{\partial \rho} + \frac{\partial x_m(k)}{\partial \rho} = \theta\left(\frac{\partial x_s(k-1)}{\partial \rho} + \frac{\partial x_m(k-1)}{\partial \rho}\right)$$

$$= \theta^2\left(\frac{\partial x_s(k-2)}{\partial \rho} + \frac{\partial x_m(k-2)}{\partial \rho}\right) = \theta^k\left(\frac{\partial x_s(0)}{\partial \rho} + \frac{\partial x_m(0)}{\partial \rho}\right) = 0$$

□

Now, we can show using the mean value theorem that, for two functions $f(.)$ and $g(.)$ continuous at x_0 and x_1 such that $x_0 < x_1$ and $f(x_0) \geq g(x_0)$, then $f'(x) > g'(x)$ if and only if $f(x_1) > g(x_1)$. Suppose here that $x_s(k) = f(\rho)$ and $x_m(k) = g(\rho)$.

First, at $\rho = 0$, $f(0) = (1-\theta)x^P + \theta x_m(k-1)$ and $g(0) = p^{k-1}(1-\theta)x^P + \theta x_s(k-1)$, we can show that $f(0) > g(0)$:

$$f(0) - g(0) = \frac{(1-\theta)x^P \left(p - p^k + (1-p)(-\theta)^k + \theta(1-p^k)\right)}{(1+\theta)(p+\theta)}$$

This is positive since: if k is even, all terms are positive; while when k is odd $p - p^k + (1-p)(-\theta)^k + \theta(1-p^k) = p(1-p^{k-1}) - \theta^k + p\theta^k + \theta - \theta p^k = p(1-p^{k-1}) + \theta(1-\theta^{k-1}) + \theta p(\theta^{k-1} - p^{k-1})$ which is positive since $p(1-p^{k-1}) > \theta p(\theta^{k-1} - p^{k-1})$.

Since we have shown that $f(\rho) > g(\rho)$, then $\frac{\partial f(\rho)}{\partial \rho} > \frac{\partial g(\rho)}{\partial \rho}$.

However we have shown that $\frac{\partial f(\rho)}{\partial \rho} + \frac{\partial g(\rho)}{\partial \rho} = 0$. This leads to $\frac{\partial f(\rho)}{\partial \rho} - \frac{\partial g(\rho)}{\partial \rho} = 2\frac{\partial f(\rho)}{\partial \rho} > 0 \implies \frac{\partial f(\rho)}{\partial \rho} > 0$ and $\frac{\partial g(\rho)}{\partial \rho} < 0$. Therefore $\frac{\partial x_s(k)}{\partial \rho} + \frac{\partial x_m(k)}{\partial \rho} = 0$, $\frac{\partial x_s(k)}{\partial \rho} > 0$ and $\frac{\partial x_m(k)}{\partial \rho} < 0$.

Proof of Proposition 1.8

First we investigate how $\tilde{x}(k)$ changes with a:

$$\frac{\partial \tilde{x}(k)}{\partial a} = x_s(k) - x_m(k)$$

which is weakly positive.

Second, in order to find how the weighted average opinion changes with p, we check how the opinions of supporters and non-supporters change with p:

$$\frac{\partial x_s(k)}{\partial p} = \frac{\theta}{\rho+1}\left[\rho\frac{\partial x_s(k-1)}{\partial p} + \frac{\partial x_m(k-1)}{\partial p}\right]$$

$$\frac{\partial x_m(k)}{\partial p} = (k-1)p^{k-2}(1-\theta)x^P + \frac{\theta}{\rho+1}\left[\frac{\partial x_s(k-1)}{\partial p} + \rho\frac{\partial x_m(k-1)}{\partial p}\right]$$

We use induction to show that both are positive. It holds for $k = 3$[8]:

$$\frac{\partial x_s(3)}{\partial p} = \frac{(1-\theta)\theta x^P}{1+\rho} > 0 \text{ and } \frac{\partial x_m(3)}{\partial p} = \frac{(1-\theta)x^P(\theta\rho + 2p(1+\rho)}{1+\rho} > 0$$

Assuming it holds for $k-1$, then $\frac{\partial x_s(k)}{\partial p}$ and $\frac{\partial x_m(k)}{\partial p}$ are both positive, implying that $\tilde{x}(k)$ is positive as well.

[8]For $k = 2, \frac{\partial x_s(3)}{\partial p} = 0$ and $\frac{\partial x_m(3)}{\partial p} = x^P(1-\theta) > 0$.

Proof of Proposition 1.9

We take the limit of the opinions as k goes to infinity $\lim_{k\to+\infty} x_s(k) = \frac{x^P}{2}\left\{\frac{2(1+\theta+\rho-\theta\rho)-2\theta}{1+\theta+\rho-\theta\rho}\right\} = \frac{1+\rho(1-\theta)}{1+\theta+\rho(1-\theta)}x^P$

$\lim_{k\to+\infty} x_m(k) = \frac{\theta}{1+\theta+\rho(1-\theta)}x^P$[9]

$\lim_{k\to\infty} \tilde{x}(k) = a\lim_{k\to+\infty} x_s(k) + (1-a)\lim_{k\to+\infty} x_m(k) = \frac{a(1+\rho(1-\theta)-\theta)+\theta}{1+\theta+\rho(1-\theta)}$

Proof of Proposition 1.10

First, in order to complete this proof, we prove the following lemma which applies to any network with equitable partition.

Lemma 1.7. *The changes of beliefs of a supporter and a non-supporter between two levels of homophily $\rho_1 > \rho_2$ are such that $\Delta x_s(k) + \Delta x_m(k) = 0$.*

Proof. This can be done by induction. First, it is assumed that $x_s(0) = x_m(0) = 0$; this implies that $x_s(1) = x_m(1) = (1-\theta)x^P$. At $k = 2$, $x_s(2) = (1+\theta)(1-\theta)x^P$ and $x_m(2) = (p+\theta)(1-\theta)x^P$. Thus, for $k = 0, 1, 2$, the homophily parameter does not appear and $\Delta x_s(k) = \Delta x_m(k) = 0$. For $k = 3$, homophily has a role in the beliefs and we have:

$$x_s(3) = (1+\theta^2)(1-\theta)x^P + \theta\frac{\rho+p}{\rho+1}(1-\theta)x^P$$

and

$$x_m(3) = (p^2+\theta^2)(1-\theta)x^P + \theta\frac{1+\rho p}{\rho+1}(1-\theta)x^P$$

For $\rho_1 > \rho_2$,

$$\Delta x_s(3) = \theta(1-\theta)x^P\left(\frac{\rho_1+p}{\rho_1+1} - \frac{\rho_2+p}{\rho_2+1}\right) = \theta(1-\theta)x^P\frac{(1-p)(\rho_1-\rho_2)}{(\rho_1+1)(\rho_2+1)}$$

$$= -\theta(1-\theta)x^P\frac{(1-p)(\rho_2-\rho_1)}{(\rho_1+1)(\rho_2+1)} = -\Delta x_m(3)$$

.

Assume this is true for $k-1$: $\Delta x_s(k-1) + \Delta x_m(k-1) = 0$. Want to show that it holds for k:

$$\Delta x_s(k) + \Delta x_m(k)$$

$$= \theta\left\{\frac{1}{\rho_1+1}\left[\rho_1 x_s(k-1)\rfloor_{\rho_1} + x_m(k-1)\rfloor\rho_1\right] - \frac{1}{\rho_2+1}\left[\rho_2 x_s(k-1)\rfloor_{\rho_2} + x_m(k-1)\rfloor\rho_2\right]\right.$$

$$\left. + \frac{1}{\rho_1+1}\left[x_s(k-1)\rfloor_{\rho_1} + \rho_1 x_m(k-1)\rfloor\rho_1\right] - \frac{1}{\rho_2+1}\left[x_s(k-1)\rfloor_{\rho_2} + \rho_2 x_m(k-1)\rfloor\rho_2\right]\right\}$$

[9] Both limits of $x_s(k)$ and $x_m(k)$ as k goes to infinity are bounded below by 0 and bounded above by x^P.

$$= \theta\left[\Delta x_s(k-1) + \Delta x_m(k-1)\right] = 0$$

□

Next, We prove $\Delta x_s(k) > 0$, then given that $\Delta x_m(k) = -\Delta x_s(k)$ we can deduce that $\Delta x_m(k) < 0$.

This is done by induction. For $k = 0, 1, 2$, $\Delta x_s(k) = 0$.

For $k = 3$, $\Delta x_s(3) = \theta\frac{2+p}{3}(1-\theta)x^P - \theta\frac{\frac{1}{2}+p}{\frac{1}{2}+p}(1-\theta)x^P = \frac{1}{3}(1-p)\theta(1-\theta)x^P > 0$.

For $k = 4$, $\Delta x_s(4) = \frac{1}{3}(1-p^2)\theta(1-\theta)x^P > 0$.

We assume that it is true for $k-1$ and k, and $\Delta x_s(k-1), \Delta x_s(k) > 0$. Want to show that it holds for $k+1$.

From the definition of $x_s(k)$ we can write,

$$\Delta x_s(k) = x_s(k)\rfloor_{\rho=2} - x_s(k)\rfloor_{\rho=\frac{1}{2}}$$

$$= \frac{\theta}{3}\left(\Delta x_s(k-1) + \Delta x_m(k-1) + x_s(k-1)\rfloor_{\rho=2} - x_m(k-1)\rfloor_{\rho=\frac{1}{2}}\right)$$

$$= \frac{\theta}{3}\left[x_s(k-1)\rfloor_{\rho=2} - x_m(k-1)\rfloor_{\rho=\frac{1}{2}}\right] > 0$$

as per the assumption above. This is equivalent to $x_s(k-1)\rfloor_{\rho=2} - x_m(k-1)\rfloor_{\rho=\frac{1}{2}} > 0$.

At $k+1$, $\Delta x_s(k+1) = \frac{\theta}{3}\left[x_s(k)\rfloor_{\rho=2} - x_m(k)\rfloor_{\rho=\frac{1}{2}}\right]$

$$= \frac{\theta}{3}\left\{(1-p^k)(1-\theta)x^P + \frac{\theta}{3}\left(2x_s(k-1)\rfloor_{\rho=2} + x_m(k-1)\rfloor_{\rho=2}\right)\right.$$

$$\left. -\frac{2\theta}{3}\left(x_s(k-1)\rfloor_{\rho=\frac{1}{2}} + \frac{1}{2}x_m(k-1)\rfloor_{\rho=\frac{1}{2}}\right)\right\}$$

$$= \frac{\theta}{3}\left\{(1-p^k)(1-\theta)x^P + \frac{\theta}{3}\left(2\Delta x_s(k-1) + \Delta x_m(k-1)\right)\right\}$$

$$= \frac{\theta}{3}\left\{(1-p^k)(1-\theta)x^P + \frac{\theta}{3}\left(\Delta x_s(k-1)\right)\right\} > 0$$

Proof of Proposition 1.11

Using the mean value theorem, it can be shown that for two functions $f(.)$ and $g(.)$ continuous at x_0 and x_1 such that $x_0 < x_1$ and $f(x_0) \geq g(x_0)$, then $f'(x) > g'(x)$ if and only if $f(x_1) > g(x_1)$. Consider $x_s(k)\rfloor_{\rho=2} = f(k)$ and $x_s(k)\rfloor_{\rho=1} = g(k)$. We know that for $k = 0, 1, 2$, $f(k) = g(k)$. Since we have shown that $\Delta x_s(k) > 0$ for $k > 3$, then $\frac{\partial x_s(k)\rfloor_{\rho=2}}{\partial k} > \frac{\partial x_s(k)\rfloor_{\rho=1}}{\partial k}$. This implies that when $x_s(k)\rfloor_{\rho=1}$ is maximised, $x_s(k)\rfloor_{\rho=2}$ will be positive and will reach its maximum later at a higher k.

Proof of Corollary 1.2

Let $\Delta\tilde{x}(k) = \tilde{x}(k)\rfloor_{\rho=1} - \tilde{x}(k)\rfloor_{\rho=2}$. We determine first how the sign of $\Delta\tilde{x}(k)$ changes and then we look at the optimal k for each degree of homophily ρ.

$$\Delta\tilde{x}(k) = ax_s(k)\rfloor_{\rho_1} + (1-a)x_m(k)\rfloor_{\rho_1} - ax_s(k)\rfloor_{\rho_2} - (1-a)x_m(k)\rfloor\rho_2$$

$$= a\Delta x_s(k) + (1-a)\Delta x_m(k)$$

$$= (2a-1)\Delta x_s(k)$$

which is positive whenever $a > \frac{1}{2}$ and negative for $a < \frac{1}{2}$. For $k = 0, 1, 2$, $\Delta\tilde{x}(k) = 0$ since $\Delta x_s(k) = 0$, and for $a > \frac{1}{2}$, $\Delta x_s(3) > 0$, implying that $\Delta\tilde{x}(3) > 0$. Therefore, for $a > \frac{1}{2}$, $\frac{\partial\tilde{x}(k)\rfloor_{\rho=2}}{\partial k} > \frac{\partial\tilde{x}(k)\rfloor_{\rho=1}}{\partial k}$.

Chapter 2

Default and coordination in financial networks

2.1 Introduction

Financial institutions carry out various transactions with each other, including risk–sharing and insurance. The architecture of the network of transactions between institutions can support financial stability because it enables them to share funding or transfer risk. But these linkages can also facilitate the diffusion of shocks through the system, due to chains of default and the domino effect. This is referred to as systemic risk. Systemic risk is costly for individuals, institutions and economies, as demonstrated by the last financial crisis. The obvious need for a stable financial system has led to a significant interest in policies that could reduce systemic risk and mitigate contagion.

This paper introduces a model of default in financial networks. We study a two-period economy where agents have a positive endowment in each period. The endowment represents agents' cash flows from outside the financial system. We assume that agents hold each other's financial liabilities and that this constitutes the network between them. These liabilities mature in the second period, and we assume that agents' second-period endowments are small and deterministic, so that they face a risk of default. More specifically, the liabilities structure results in cyclical payments interdependencies that are simultaneously computed according to the clearing mechanism described in the seminal contribution of Eisenberg and Noe (2001). The clearing vector satisfies three criteria:

- debt absolute priority, which stipulates that liabilities are paid in full in order to have positive equity;
- limited liability, which means that the payment made by each agent cannot exceed its inflows;
- equal seniority of all creditors, which implies pro rata repayments.

Agents can avoid default by storing part of their first-period endowment. Due to complementarities in the payments, the decision taken by one agent to store part of his endowment exerts a positive externality on the other agents to whom he is connected.[1] We show that the strategic interactions in the financial system modelled here can be investigated as a coordination game, called the default game, where agents' decisions are simply whether to default or not. It is well known in the literature that coordination games will in general yield multiple pure–strategy Nash equilibria and that the set of pure–strategy Nash equilibria has a lattice structure — in particular, there are two extreme pure–strategy Nash equilibria. In our setting, the best equilibrium is the one where the largest number of

[1]The non-storage in our model can be equivalently interpreted as a bank run in the influential Diamond–Dybvig model.

agents choose the maximal action Non-Default and the worst equilibrium is the one where the largest number of agents choose the minimal action Default. In the paper, we relate the multiplicity of Nash equilibria to the presence of a cycle of financial obligations. Then, we develop a simple algorithm for finding all Nash equilibria of the default game. While there are easy algorithms for finding the maximal and minimal equilibria and relatively easy algorithms to compute all Nash equilibria in coordination games such as the default game (see Echenique (2007)), the advantage of the algorithm developed in this paper is that it relies on the financial network structure to inform the computation of Nash equilibria. Algorithms that exploit the financial network structure such as the algorithm developed in this paper, as well as quickly computing all Nash equilibria, provide useful information on the strategic interactions between agents.

In this paper, we show that the problem of inefficient coordination may arise in financial networks. Similar to other areas in economics, the strategic complementarities of payments due to the cyclical financial interconnections allow for the existence of multiple Nash equilibria. This gives rise to the question of which one of these equilibria will be the outcome of the underlying default game. From a policy perspective, given that inefficient coordination might pose a severe economic problem, there is a need for financial institutions fostering efficient coordination of agents' decisions. Recently, central clearing has become the cornerstone of policy reform in financial markets since it limits the scope of default contagion. Our analysis shows that introducing a central clearing counterparty (henceforth, CCP) also allows agents playing different actions at different Nash equilibria to coordinate on the efficient equilibrium at no additional cost. As a consequence, our result reinforces the key role CCP's play in stabilising financial markets.

This paper is structured as follows. In Section 2.2, we go over the related literature. Then we describe the model and show the existence of a Nash equilibrium in Section 2.3. We develop an algorithm to find all Nash equilibria in Section 2.4 and Section 2.5 provides some policy implications of central clearing. Section 2.6 concludes the paper, followed by an appendix devoted to the proofs.

2.2 Related Literature

The impact of the financial network structure on economic stability has been a subject of ongoing interest since the last financial crisis (of 2008). The seminal contributions of Allen and Gale (2000) and Eisenberg and Noe (2001) were first to acknowledge that the financial network structure determines default contagion, and would serve as a basis for many subsequent contributions.

Allen and Gale (2000) investigate how symmetric financial networks lead to contagion, where links represent sharing agreements. Their key finding is that incomplete financial networks are less resilient and more vulnerable to contagion than their complete counterparts. Eisenberg and Noe (2001) develop a static model of default contagion in a financial network where agents hold each other's financial liabilities and the activities and operations of each agent are condensed into one value: the operational cash flow. The repayment of liabilities will be interdependent, since whether an agent defaults or not is a result of his operational cash flow as well as the payments he receives from other agents. Eisenberg and Noe first prove the existence of a clearing payment vector that is unique under mild conditions. They also provide an algorithm to compute the clearing vector, which is important to predict chains of defaults.

Acemoglu, Ozdaglar and Tahbaz-Salehi (2015) extend the Eisenberg and Noe model to accommodate agent exposure to outside shocks. They establish that up to a certain magnitude of shocks, the more connected the financial network is, the more stable it is; beyond this threshold, the connectedness of the network makes it more prone to contagion and thus more fragile. Elliott, Golub and Jackson (2014) introduce two concepts of cross-holdings that have distinctive and non-monotonic impact on default cascades. Integration, which measures the dependence on counterparties, expands the extent of default contagion but reduces the probability of the first failure; while diversification, which measures the heterogeneity of cross-holdings, increases the propagation of failure cascades but decreases the exposure level among pairs of financial institutions. Cabrales, Gottardi and Vega-Redondo (2017) investigate the optimal network structure that maximizes risk-sharing benefits among interconnected firms while decreasing their risk exposure. Other recent contributions include Teteryatnikova (2014) and Csóka and Herings (2016).

Several approaches have been investigated to mitigate the domino effect in the financial network, such as central clearing and identifying the most systemically relevant financial institutions and then targeting them through cash injections. For instance, Demange (2018), following a similar approach to Eisenberg and Noe (2001), develops a new measure, called the *threat index*, which identifies the most systemically relevant agents for optimal targeted cash injection.

2.3 The Model

Consider a two-period ($t = 1, 2$) economy with $N = \{1, 2, ..., n\}$ agents. Agent i's endowment in the first period is $z_i^1 \geq 0$ and in the second period is $z_i^2 > 0$. The endowment of agent i in each period denotes the cash flows arriving from outside the financial system.

We assume that agents hold each other's liabilities, which mature in the second period. More specifically, given two agents $i, j \in N$, let $L_{ij} \in \mathbb{R}^+$ denote the liability that agent i owes agent j. Then, agent i's total liabilities are $L_i = \sum_{j \in N} L_{ij}$. Meanwhile, $\sum_{j \in N} L_{ji}$ is the total assets of agent i. Let $\boldsymbol{\alpha} = (\alpha_{ij})_{i,j \in N}$ denote the matrix of relative liabilities, with entries $\alpha_{ij} = \frac{L_{ij}}{L_i}$ representing the ratio of the liability agent i owes to agent j over the total amount of agent i's liabilities.

Each agent i can store an amount $x_i \in \left[0, z_i^1\right]$ from his first-period endowment and receives an interest rate $r > 0$ on his storage. Given the storage strategies of agents $\mathbf{x} = (x_i)_{i \in N}$, let $\boldsymbol{\pi}^{\mathbf{x}} = (\pi_i^{\mathbf{x}})_{i \in N}$ denote the clearing payment vector, uniquely defined as in Eisenberg and Noe (2001), such that for each agent i it holds that

$$\pi_i^{\mathbf{x}} = \min \left\{ z_i^2 + (1+r)\, x_i + \sum_{j=1}^{n} \alpha_{ji} \pi_j^{\mathbf{x}} ; L_i \right\}.$$

This means that $z_i^1 - x_i$ denotes the equity of agent i in the first period and

$$z_i^2 + (1+r) x_i + \sum_{j=1}^{n} \alpha_{ji} \pi_j^{\mathbf{x}} - \pi_i^{\mathbf{x}}$$

denotes the equity of agent i in the second period.

The utility function of agent i is $U_i(e_i^1, e_i^2) = e_i^1, +e_i^2$, where e_i^1 is the equity of agent i at $t = 1$ and e_i^2 is the equity of agent i at $t = 2$. Therefore, the utility function of agent i, given the storage strategies of agents $\mathbf{x} = (x_i, x_{-i})$, is

$$U_i(z_i^1 - x_i, z_i^2 + (1+r)\, x_i + \sum_{j=1}^{n} \alpha_{ji} \pi_j^{\mathbf{x}} - \pi_i^{\mathbf{x}}) = z_i^1 + z_i^2 + r x_i + \sum_{j=1}^{n} \alpha_{ji} \pi_j^{\mathbf{x}} - \pi_i^{\mathbf{x}}.$$

2.4 Nash Equilibria of the Default Game

First, we investigate further the economy introduced above. Observe that each agent will choose to store a positive amount of his first-period endowment if and only if he prefers (is better off) not to default; otherwise he will store nothing. If he prefers not to default, since his utility is linear and the interest rate $r > 0$ he will store his entire first-period endowment. Similarly, it is only the decision of an agent to default or not, rather than the amount of storage, that affects the other agents. This is because, if he defaults, he will pay out his total second-period equity and, if he does not default, he will pay his total liability, neither of which is directly affected by his level of storage.

Therefore, the strategic interaction of agents in the economy can be investigated as a binary coordination game with two actions (Default) $= 0$ and (Non-Default) $= 1$ among which agents must choose. Now, define a threshold $T_i(\mathbf{a}_{-i})$ as the minimum amount agent i must pay in the second period to avoid default, given other agents' actions $\mathbf{a}_{-i}$.

Proposition 2.1. *The threshold $T_i(\mathbf{a}_{-i})$ is well-defined and decreasing in $\mathbf{a}_{-i}$.*

Proof. The proof of Proposition 2.1, together with all our other proofs, appears in the Appendix. □

Proposition 2.1 shows that the threshold $T_i(\mathbf{a}_{-i})$ is well-defined. Observe that agent i will choose to play 1 whenever

$$(1+r)z_i^1 - T_i(\mathbf{a}_{-i}) \geq z_i^1.$$

Therefore, the best reply function of agent i can be written as follows:

$$\Psi_i(\mathbf{a_{-i}}) = \begin{cases} 1 & \text{if } rz_i^1 - T_i(\mathbf{a}_{-i}) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

A profile of actions $\mathbf{a}^* \in \{0,1\}^N$ is a Nash equilibrium if $a_i^* = \Psi_i(\mathbf{a}^*_{-\mathbf{i}})$.

The default game introduced above corresponds to a binary game of strategic complements. As defined in Topkis (1979) and Bulow, Geanakoplos and Klemperer (1985), strategic complementarities arise if an increase in one agent's strategy increases the optimal strategies of the other agents.[2]

Theorem 2.1. *There exists a pure–strategy Nash equilibrium of the default game.*

Theorem 2.1 shows the existence of a pure–strategy Nash equilibrium. Understandably, the existence of a pure–strategy Nash equilibrium follows from the strategic complementarities between agents' actions, since the decision of an agent not to default makes it easier for other agents not to default too.

It is well known in the literature that a binary game of strategic complements will in general have multiple pure–strategy Nash equilibria with a lattice structure. In particular, this class of games has two extreme equilibria: the best equilibrium is the equilibrium where the largest number of agents choose the maximal action (Non-Default) $= 1$ and the worst equilibrium is the equilibrium where the largest number of agents choose the minimal action (Default) $= 0$.

For simplicity, for the remainder of this paper, we assume that at a Nash equilibrium of the default game, no agent is indifferent between (Non-Default) $= 1$ and (Default) $= 0$,

[2]See, Sobel (1988), Milgrom and Roberts (1990), Vives (1990), Echenique and Sabarwal (2003), Amir (2005), Echenique (2007) and Barraquer (2013) for other economic applications of games of strategic complements.

which is likely to be the case.[3] The following result highlights the connection between the multiplicity of equilibria and the structure of the financial network.

Proposition 2.2. *If the default game has multiple Nash equilibria then, the financial network has cyclical obligations.*

Proposition 2.2 shows that the presence of a cycle of financial obligations is generically necessary for the multiplicity of Nash equilibria. Eisenberg and Noe (2001) term this phenomenon cyclical interdependence and illustrate it as follows: "*A default by Firm A on its obligations to Firm B may lead B to default on its obligations to C. A default by C may, in turn have a feedback effect on A.*"

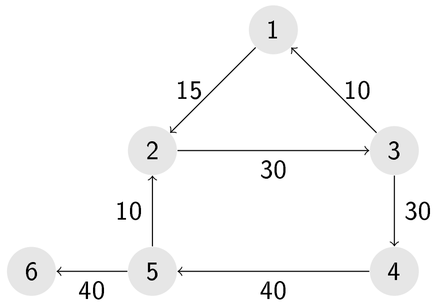


Figure 2.4.1: Cyclical Obligations

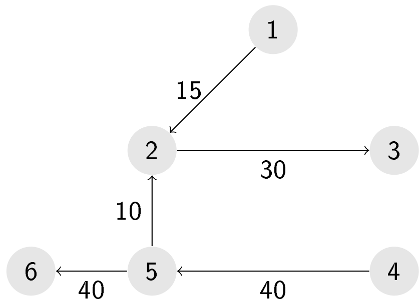


Figure 2.4.2: Unidirectional obligations

In the following, we will show that the close relationship between the multiplicity of Nash equilibria and the cyclical financial interconnections is useful to solve for pure–strategy Nash equilibria of the default game. More specifically, we will provide an algorithm to find all pure–strategy Nash equilibria of the default game.

Recall that the financial network is strongly connected if there is a path of obligations between all pairs of agents. A strongly connected component (henceforth, SCC) of the financial network is a maximal[4] strongly connected subnetwork.

[3]That is, this always holds except for a finite set of first-period endowments.
[4]In the sense that it is not properly contained in a larger SCC.

2.4.1 A financial network with a unique SCC

First, for simplicity, we consider the case of a financial network with a unique strongly connected component. We will use the following notion of *ear decomposition* of a network, which is useful given its close relationship to network connectivity. An ear decomposition of a network is a partition of the set of agents into an ordered collection of agent-disjoint simple paths, called ears. More precisely, an ear decomposition of a network is a partition of the agents into $E_0, E_1, \ldots, E_p$ such that

- $E_0 = \{v_0\}$ is a single agent;
- for each $h = 1, \ldots, p$, it holds that $E_h = \{v_{1_h}, \cdots, v_{k_h}\}$ is a directed path such that the endpoints of each E_h—that is, v_{1_h} and v_{k_h}—are in $E_1 \cup \ldots \cup E_{h-1}$ but the internal agents of E_h—that is, $v_{2_h}, \cdots, v_{(k-1)_h}$—are not in $E_1 \cup \ldots \cup E_{h-1}$.[5]

A financial network is strongly connected if and only if it has an ear decomposition. In the following, we will refine further the concept of ear decomposition. Given an ear E_h, we say a subset of consecutive internal agents $R_{t_h} = \{v_{t_h}, \cdots, v_{s_h}\}$ is a *rim* of the ear if $v_{(t-1)_h}$ is an ear's first agent and v_{s_h} is either an ear's first agent or E_h's penultimate agent and none of the other agents in the rim is an ear's first agent. Hence the internal agents of each ear can be partitioned into a collection of rims. Observe also that the last ear always has a unique rim. In interpretation, the concept of rim represents a useful refinement of the ear decomposition since the decision of agents outside a rim are affected only by the last non–defaulting agent in the rim.

In the following, we will rely on this refinement of the ear decomposition to provide an algorithm to find all pure–strategy Nash equilibria of the default game of a financial network with a unique SCC.

The algorithm, which we call USCCNE, goes as follows:

1. For each rim in the network, *assume* that each agent in the rim is the last non–defaulting agent or that all agents in the rim are defaulting.
2. For every case in (1), start from the last ear E_p and repeat the following until reaching the first ear E_0: for each ear delete the internal agents and update the inflows of the affected (intercepting) agents.
3. For every case of assumed actions in (1), start from the single agent v_0 in E_0 and move along all agents in every ear in the opposite direction; for each agent compute, the optimal action while taking feedback into consideration.[6]

[5] Each E_h ($h = 1, \ldots, p$) is called an ear.

[6] That is, for agent i it holds that inflow$_i = a_i \pi_i + b_i$, which vary according to the case considered.

In interpretation, the USCCNE algorithm assumes for each rim that a particular agent is the last non-defaulting agent or that all agents in the rim default. Then start from the last ear and repeat the following until reaching the first ear: delete all the internal agents of each ear and update the inflows of all affected (intercepting) agents. Finally, the algorithm navigates every ear in the opposite direction computing the optimal actions of all agents. The Nash equilibria correspond to the iterations where all the *assumed* actions are satisfied.

The key feature of the USCCNE is that it exploits the fact that, in a given rim, all nodes except the first one have only one in-going link. The algorithm consists of computing the different possible inflows to each last node in every rim, by considering the different actions by the previous nodes, its debtors. For every case that represents a set of assumptions, we compute the action of the last node, starting from the last ear E_p and moving towards E_0. After computing the actions of these nodes, we move in the opposite direction, starting from E_0 to check whether the assumptions in each case are verified. When they are, it is a Nash Equilibrium.[7]

The next example illustrates the default game.

Example 1. Consider an economy of ten agents connected through their ownership of each other's liabilities, among which only the first nine agents are strategically relevant. Agents' endowments in the first period are $\mathbf{z}^1 = (25, 25, 40, 40, 60, 40, 40, 70, 24)$ and in the second period are $\mathbf{z}^2 = (3, 3, 3, 3, 3, 3, 3, 3, 3)$ and the interest rate is $r = 0.1$. All agents have the same utility function $U_i(e_i^1, e_i^2) = e_i^1 + e_i^2$. The financial liabilities of agents to each other are illustrated in the financial network in Figure 2.4.3.

This financial network contains a unique SCC, $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, which has four ears, $E_0 = \{1\}$; $E_1 = \{1, 2, 3, 4, 5, 1\}$; $E_2 = \{3, 6, 7, 8, 2\}$; and $E_3 = \{7, 9, 1\}$, and five rims, $R_2 = \{2, 3\}$; $R_4 = \{4, 5\}$; $R_6 = \{6, 7\}$; $R_8 = \{8\}$; and $R_9 = \{9\}$.

In order to compute the Nash equilibria, we apply the USCCNE algorithm described above. To illustrate this for a particular case we assume that all agents in all rims are defaulting except $R_4 = \{4, 5\}$, where 5 is the last non-defaulter. Deleting the internal agents of ear E_3 and updating the inflows of the (intercepting) agent 1 it holds that

$$\text{Inflow}_1^{E_3} = \frac{1}{9}\pi_7 + 3.$$

Deleting the internal agents of ear E_2 and updating the inflows of the (intercepting) agents

[7] Observe that the USCCNE algorithm also provides a bound on the number of Nash equilibria.

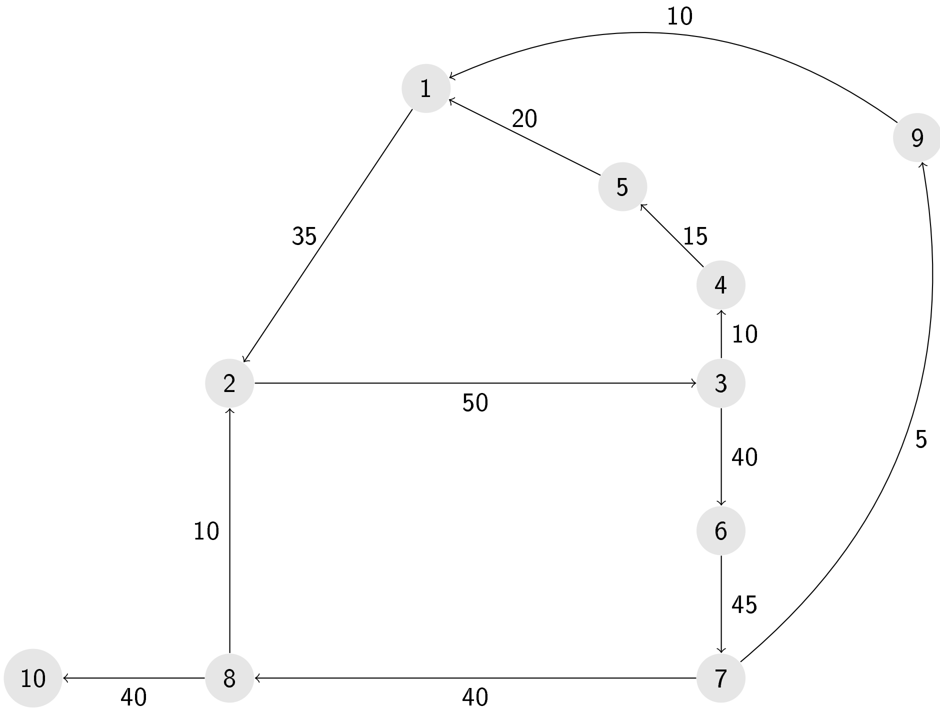


Figure 2.4.3: A financial network with ten agents

1 and 2 it holds that

$$\text{Inflow}_1^{E_3,E_2} = \frac{4}{45}\pi_3 + \frac{11}{3} \text{ and } \text{Inflow}_2^{E_3,E_2} = \pi_1 + \frac{8}{45}\pi_3 + \frac{31}{15}.$$

Finally, deleting the internal agents of ear E_1 and updating the inflows of the (intercepting) agent 1 it holds that

$$\text{Inflow}_1^{E_3,E_2,E_1} = \frac{4}{37}\pi_1 + 24.53$$

Since $z_1^2 = 3$ and $L_1 = 35$ it follows that the threshold agent 1 must at least pay in the second period in order to avoid default is 3.68. Since $rz_1^1 = 2.5 < 3.68$, it follows that agent 1 chooses to default and in this case

$$\pi_1 = \text{Inflow}_1^{E_3,E_2,E_1} + z_1^2 = \frac{4}{37}\pi_1 + 27.53 = 30.86.$$

Repeating the same procedure, will allow us to compute the actions and the payments of all other agents in the economy and check whether we have a Nash equilibrium.
Considering all the cases, we find three Nash equilibria: the best equilibrium $1, 1, 1, 1, 1, 1, 1, 1, 1$, the intermediate equilibrium $0, 0, 0, 1, 1, 0, 0, 0, 0$, and the worst equilibrium $0, 0, 0, 0, 0, 0, 0, 0, 0$, which we illustrate in Figures 2.4.4-2.4.6.

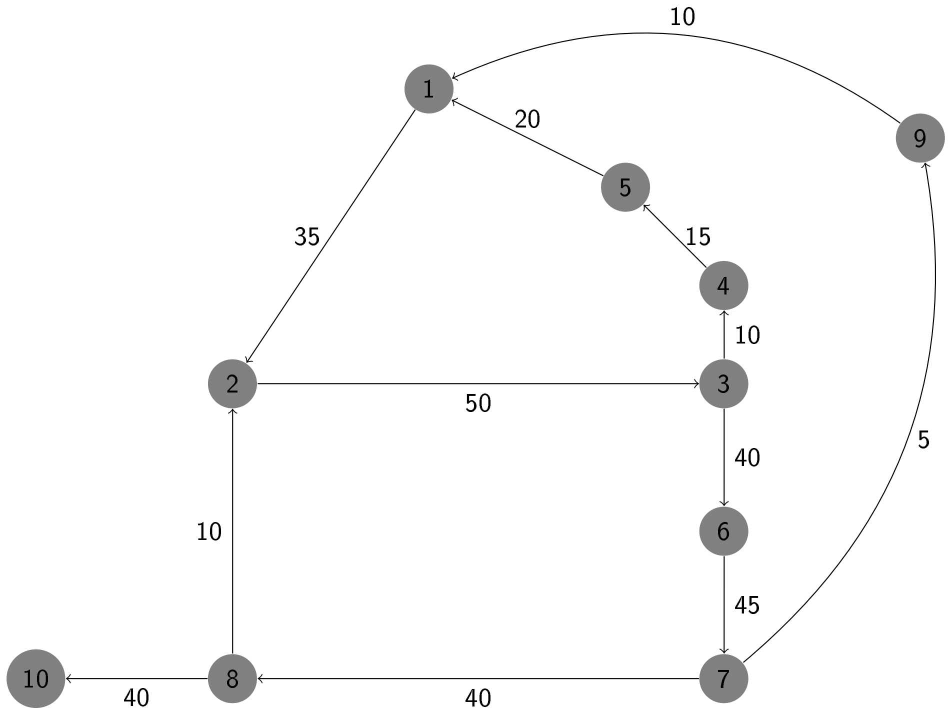


Figure 2.4.4: The best equilibrium

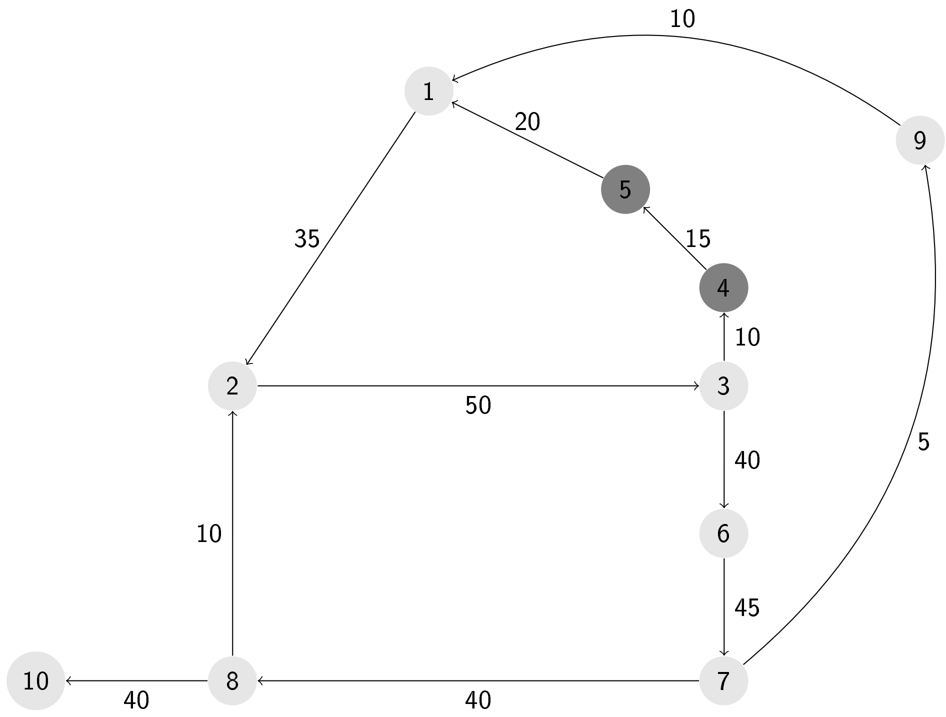


Figure 2.4.5: The intermediate equilibrium

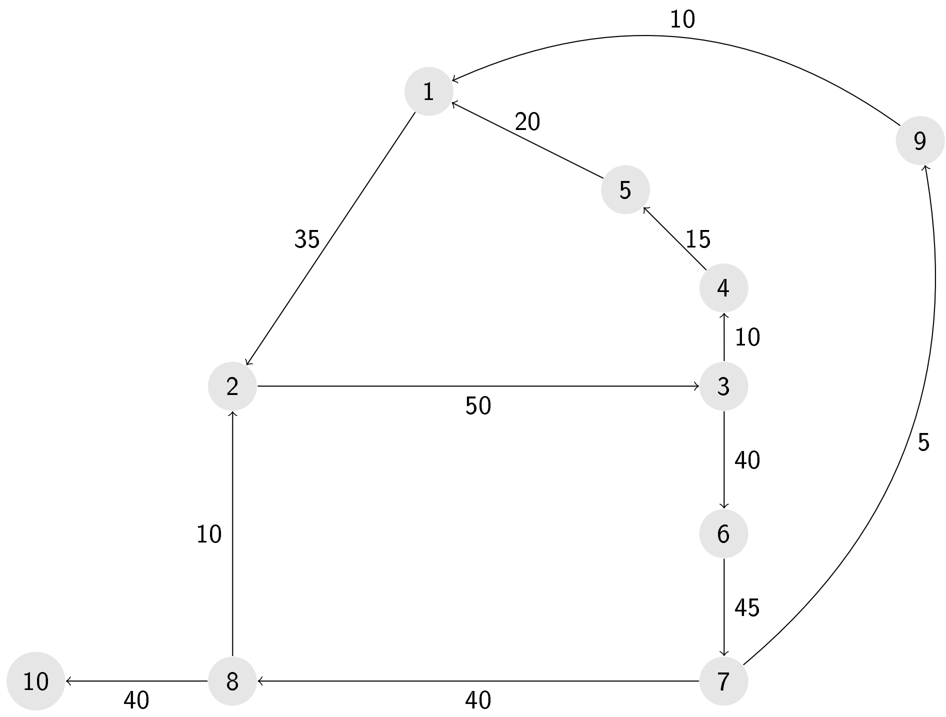


Figure 2.4.6: The worst equilibrium

2.4.2 Arbitrary financial network

Now we investigate the case of an arbitrary financial network. Recall that an arbitrary financial network can be transformed into a *directed acyclic graph* (henceforth, DAG)—that is, a network with no cycles–by contracting each SCC into a single large node (see Figures 2.4.7-2.4.8).

The algorithm described here (MSCCNE) is a generalisation of USCCNE. It consists of applying the USCCNE to each SCC in any given arbitrary network starting by the SCCs with no incoming link from any outside node or group of nodes, which are the SCCs that are not impacted by the other nodes in the network, and moving along the chain of SCCs.

In the following, we will rely on *transitive reduction*, which is a uniquely defined operation on a DAG, to compute the pure–strategy Nash equilibria of a financial network with multiple SCCs. A transitive reduction of a DAG is the network representation with the fewest possible links that preserves the chains of default of the original financial network(see Figure 8). It is hence constructed by removing all the links that are unnecessary for the chain of default to be realised and only the nodes which were connected by a path in the original network remain connected in the transitively reduced network. For instance, if A links to B, and B links to C, then the transitive reduction removes the link from A to C, if it exists.

Observe that, from the minimality of links in the transitive reduction, there exists a unique partition of the set of agents $\mathcal{W} = \{W_1, \ldots, W_k\}$ such that W_1 corresponds to the

SCCs with no incoming links, W_2 corresponds to the SCCs with only incoming links from W_1, W_3 corresponds to the SCCs with only incoming links from $W_1 \cup W_2$, and so on.

Then, the algorithm USCCNE can be easily extended to compute the Nash equilibria with multiple SCCs. The algorithm, which we call MSCCNE, goes as follows:

1. Apply USCCNE to find all Nash equilibria for each SCC in W_1.

2. For each product of Nash equilibria of SCCs in W_1, apply USCCNE to find all Nash equilibria for each SCC in W_2.

3. For each product of Nash equilibria of SCCs in $W_1 \cup W_2$, apply USCCNE to find all Nash equilibria for each SCC in W_3.

4. Repeat the procedure until visiting all the elements of the partition $\mathcal{W}$.

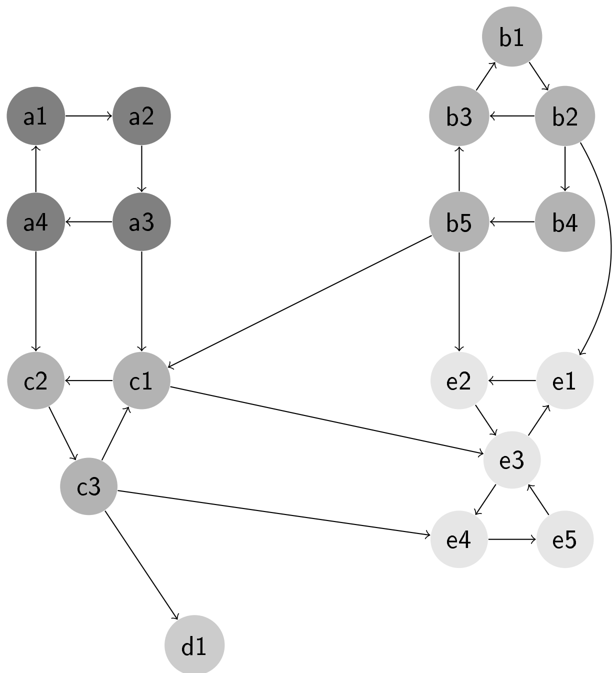


Figure 2.4.7: Example of a DAG

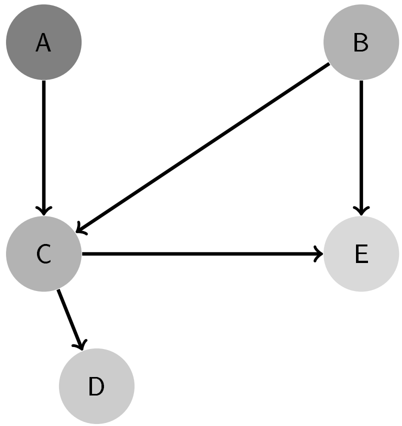


Figure 2.4.8: Condensation of the DAG

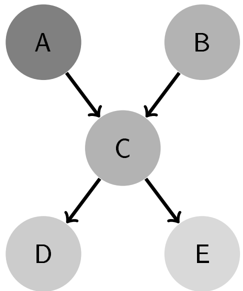


Figure 2.4.9: Transitive reduction of the DAG

The MSCCNE algorithm is a simple algorithm that exploits a network decomposition technique to find all the pure–strategy Nash equilibria of a financial network. It is worth noting that the MSCCNE algorithm can be easily adapted to compute the clearing payment vector of Eisenberg and Noe (2001).

Corollary 2.1. *Assume that the first-period endowment of each agent i is zero—that is, $z_i^1 = 0$. Then the MSCCNE algorithm computes the clearing payment vector in Eisenberg and Noe (2001).*

Recall that the clearing payment vector of Eisenberg and Noe (2001) is unique under mild conditions. Hence the existence of cyclical financial interconnections, while necessary for multiple equilibria, is not sufficient.

At the heart of the seminal contribution of Eisenberg and Noe (2001) lies the elegant *fictitious default algorithm* that computes the unique clearing payment vector. The fictitious default algorithm goes as follows. First, determine the set of agents who cannot fulfil their obligation, even when we assume that all agents receive their due payments. These agents will be called the *first wave of default.* Then, assume that the agents in the first wave of default pay their liabilities pro rata and the new defaulting agents will be called the *second wave of default* and so on until the algorithm terminates. In this way, the fictitious default algorithm produces a natural measure of systemic risk, which is the number of waves required to induce a given agent to default.

Echenique (2007) provides the most efficient algorithm for computing all pure–strategy Nash equilibria in the class of games of strategic complements, of which the default game is a special case. The algorithm elegantly checks whether there is another Nash equilibrium once the smallest and largest pure–strategy Nash equilibria are computed from classical algorithms (for example, Topkis (1979)).

While each of the above algorithms is clearly interesting in many aspects, arguably, the advantage of the MSCCNE algorithm developed in this paper is that it relies on the financial network architecture to compute the Nash equilibria. Generally, algorithms that exploit the financial network structure such as the algorithm developed in this paper, as well as having a clear computational advantage, provide valuable information on the strategic interactions among agents, as we will show below.

2.5 Policy Implications of central clearing

From a policy perspective, in view of the multiplicity of Nash equilibria of the default game, there is the central policy question of equilibrium selection. In particular, it may be desirable to implement the best equilibrium in order to achieve financial stability and minimise the cost of default.

Given the best and the worst equilibria, agents in the network can be classified into three types:[8]

1. agents that choose 0 in the worst equilibrium and 1 in the best equilibrium;
2. agents that choose 0 in the worst equilibrium and 0 in the best equilibrium;
3. agents that choose 1 in the worst equilibrium and 1 in the best equilibrium.

Note that agents of type (2) and (3) are not strategically relevant since they play the same action in the worst and the best equilibrium. Actually, we could construct a *reduced financial network* containing only agents of type (1). To do so, we first eliminate all outgoing links emanating from agents of type (3) and, since none of them defaults, add their liabilities pro rata to the cash flow of the agents intercepting their outgoing links. As for agents of type (2), given that they default and pay their inflows—i.e. their cash flow and the payments they receive from their debtors—they can be eliminated from the network

[8] Obviously, it is not possible for an agent to choose 1 in the worst equilibrium and 0 in the best.

by adding their cash flow to the cash flow of their creditors pro rata and by extending their ingoing liabilities links to their creditors pro rata so that the new liabilities directly link between their debtors and their creditors.

Recently, CCP has become increasingly the cornerstone of policy reform in financial markets. Introducing a CCP in the financial network modifies the structure of the financial network: each liability between a debtor and a creditor is erased and replaced by two new liabilities—one liability between the debtor and the CCP, and another one between the CCP and the creditor. As a consequence, one of the key benefits of central clearing is that, by breaking down the cyclical connections of financial liabilities, it reduces the aggregate level of default exposure, which in turn reduces default contagion.

There is a growing literature which investigates the benefits of central clearing. Duffie and Zhu (2011) show that CCP's reduce significantly the counterparty risk even when clearing across multiple derivative classes. Zawadowski (2013) suggests that a CCP eliminates *ex ante* own default externalities by making banks contribute to the insurance of counterparty risk in the form of a guarantee fund. In other respect, Tirole (2011) argues that centralisation should be encouraged and CCP's enhance transparency and allow for multilateral netting. Acharya and Bisin (2014) study how the lack of transparency between agents sharing default risk produce counterparty risk externality and show that this externality disappears when introducing a centralized clearing mechanism which ensures transparency. They prove that the main advantage of central clearing is enhancing the aggregation of information.

The following proposition points out another potential benefit of introducing central clearing in financial markets.

Proposition 2.3. *Introducing a CCP in each SCC of the reduced financial network achieves the best equilibrium in the default game at no additional cost.*

Proposition 2.3 shows that when a CCP intermediates the liabilities of each SCC of the reduced financial network,[9] the best equilibrium is achieved and the CCP is budget neutral. As a consequence, in addition to reducing default contagion by eliminating the cyclical financial interconnections, central clearing can also serve as a coordination device that achieves the best equilibrium of the default game.

The following example illustrates this point.

Example 2 Consider an economy of six agents connected through their ownership of each other's liabilities, among which only the first five agents are strategically relevant. Agents' endowments in the first period are $\mathbf{z}^1 = (22, 22, 75, 170, 100)$ and in the second period are

[9]That is, the financial network with only strategic relevant agents.

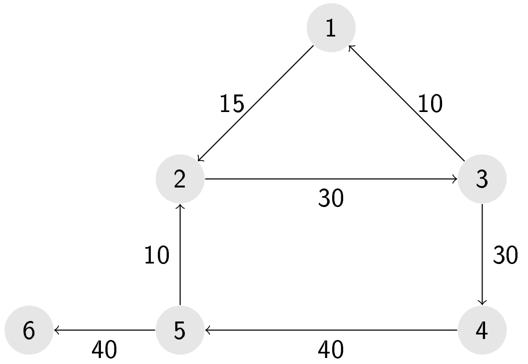


Figure 2.5.1: A financial network with five agents

$\mathbf{z}^2 = (3, 3, 3, 3, 3)$ and the interest rate is $r = 0.1$. All agents have the same utility function $U_i(e_i^1, e_i^2) = e_i^1 + e_i^2$. The financial liabilities of agents to each other are illustrated in the network in Figure 2.5.1.

This financial network contains a unique SCC $\{1, 2, 3, 4, 5\}$. To compute the Nash equilibria, we apply the USCCNE algorithm described above. We find three Nash equilibria—the best equilibrium $1, 1, 1, 1, 1$, the intermediate equilibrium $0, 0, 0, 1, 1$, and the worst equilibrium $0, 0, 0, 0, 0$—which we illustrate in Figures 2.5.2-2.5.4.

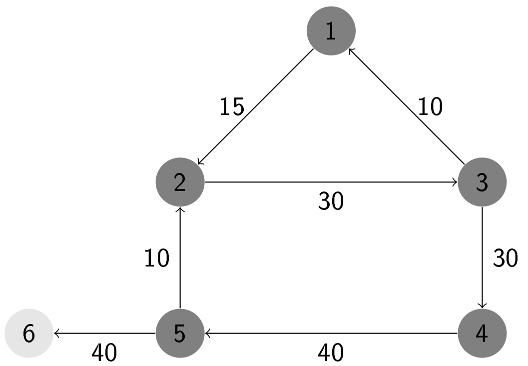


Figure 2.5.2: The best equilibrium

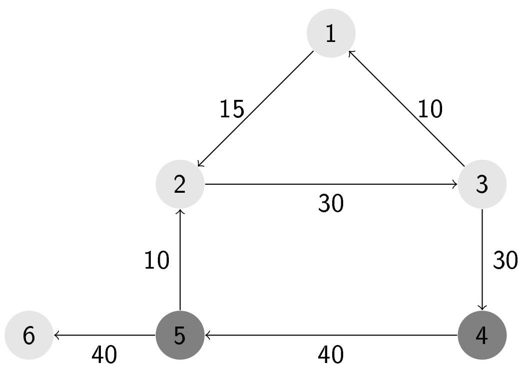


Figure 2.5.3: The intermediate equilibrium

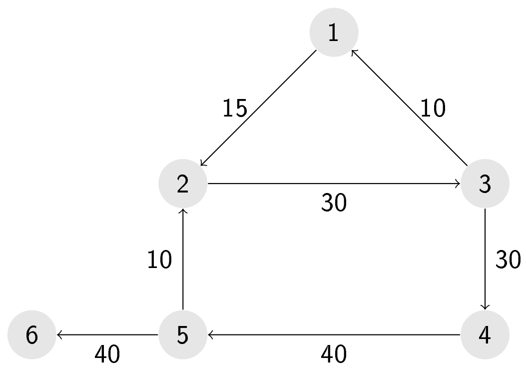


Figure 2.5.4: The worst Equilibrium

Adding a CCP will result in a new financial network as shown in Figure 2.5.5, with the following liabilities vector:

$$\tilde{\mathbf{L}} = (5, 5, 10, 10, 10, -40).$$

Given that there are no feedback effects in the presence of the CCP, the minimum cash flow for an agent i to escape default is equal to the new liability $\tilde{L}_i$. Therefore, after the introduction of a CCP, it is easy to check that the best equilibrium is implemented at no additional cost since the inflows and outflows of CCP are equal.

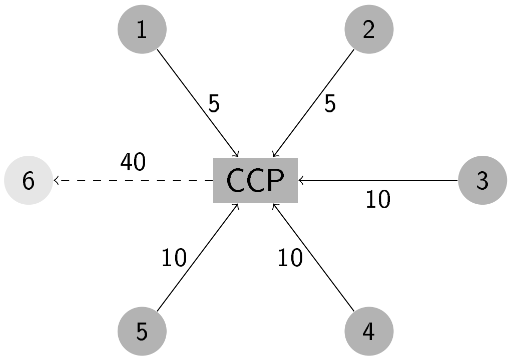


Figure 2.5.5: Adding a CCP

2.6 Conclusion

This paper shows that the introduction of a CCP allows agents playing different actions at different Nash equilibria to achieve the best equilibrium at no additional cost. As a consequence, central clearing can serve as a coordination device in financial markets. While our result reinforces the key role CCP plays in financial markets, as highlighted in several important contributions by Duffie and Zhu (2011), Tirole (2011), Zawadowski (2013) and Acharya and Bisin (2014), it remains to be seen whether other policies can be designed to minimise the number of defaults, such as identifying key agents and targeting them through either cash injection or minimum endowment requirement.

Appendix

Proof of Proposition 2.1

Recall that the default game corresponds to a binary coordination game with two actions (Default) $= 0$ and (Non-Default) $= 1$ among which agents must choose.

First, for each agent i we will show that $T_i(\mathbf{a}_{-i})$ is well-defined given other agents' actions $\mathbf{a}_{-i} \in \{0,1\}^{N-1}$. To do so, for each agent i we consider an auxiliary economy with a modified network of liabilities, where we eliminate all outgoing links emanating from agent i and add his liabilities pro rata to the cash flow of the agents intercepting his outgoing links. Hence, the matrix of relative liabilities in the auxiliary economy is $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_{kj})_{k,j \in N}$, where $\hat{\alpha}_{kj} = \alpha_{kj}$ if $k \neq i$ and $\hat{\alpha}_{kj} = 0$ otherwise. Moreover, the (augmented) second-period endowment of agent j in the auxiliary economy is $\hat{z}_j^2 = z_j^2 + \alpha_{ij} L_i$.

Now, given other agents' actions $\mathbf{a}_{-i}$, let $\mathbf{x}^{\mathbf{a}_{-i}} = (x_j^{\mathbf{a}_{-i}})_{j \in N}$ denote the agents' storage strategies, where $x_j^{\mathbf{a}_{-i}} = z_j^1$ for each agent $j \neq i$ such that $a_j = 1$, and $x_j^{\mathbf{a}_{-i}} = 0$ otherwise. Let also $\boldsymbol{\pi}^{\mathbf{x}^{\mathbf{a}_{-i}}} = (\pi_j^{\mathbf{x}^{\mathbf{a}_{-i}}})_{j \in N}$ denote the clearing payment vector, uniquely defined as in Eisenberg and Noe (2001), such that for each agent j it holds that

$$\pi_j^{\mathbf{x}^{\mathbf{a}_{-i}}} = \min \left\{ \hat{z}_j^2 + (1+r)\, x_j^{\mathbf{a}_{-i}} + \sum_{k=1}^{n} \hat{\alpha}_{kj} \pi_k^{\mathbf{x}^{\mathbf{a}_{-i}}} ; L_j \right\}.$$

Therefore, since $x_i^{\mathbf{a}_{-i}} = 0$ it holds that

$$T_i(\mathbf{a}_{-i}) = \max \left\{ L_i - z_i^2 - \sum_{j=1}^{n} \hat{\alpha}_{ji} \pi_j^{\mathbf{x}^{\mathbf{a}_{-i}}} ; 0 \right\}. \tag{2.6.1}$$

Hence, the threshold $T_i(\mathbf{a}_{-i})$ is well-defined.

Moreover, it follows from Lemma 5 in Eisenberg and Noe (2001) (see, also, Theorem 6 in Milgrom and Roberts (1990)) that $\boldsymbol{\pi}^{\mathbf{x}^{\mathbf{a}_{-i}}}$ is increasing in $\mathbf{x}^{\mathbf{a}_{-i}}$, which, in turn, is increasing in $\mathbf{a}_{-i}$. Hence, it follows from (2.6.1) that the threshold $T_i(\mathbf{a}_{-i})$ is decreasing in $\mathbf{a}_{-i}$.

Proof of Theorem 2.1

Since the threshold $T_i(\mathbf{a}_{-i})$ is decreasing in $\mathbf{a}_{-i}$ it follows that the best reply function of agent i

$$\Psi_i(\mathbf{a_{-i}}) = \begin{cases} 1 & \text{if } r z_i^1 - T_i(\mathbf{a}_{-i}) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is increasing in $\mathbf{a_{-i}}$. By the Knaster–Tarski Theorem, there exists a fixed point of the following map:

$$\Psi : \{0,1\}^N \longrightarrow \{0,1\}^N$$

$$\Psi(\mathbf{a}) = (\Psi_1(\mathbf{a_{-1}}), ..., \Psi_n(\mathbf{a_{-n}})),$$

which will be a Nash equilibrium of the default game.

Proof of Proposition 2.2

Suppose not—that is, the default game has multiple equilibria and the financial network does not have cyclical obligations. Let R denote the set of agents who play 0 in the worst Nash equilibrium and 1 in the best Nash equilibrium. Then the subnetwork induced by R contains an agent i that does not have any ingoing link. As a consequence, the inflow of agent i does not change between the worst equilibrium and the best equilibrium, and as a result agent i will not change his choice in the worst equilibrium and the best equilibrium. This is a contradiction.

Proof of Proposition 2.3

Adding a CCP in the middle of the financial network will net out the liabilities and will sort agents into two types: debtors and creditors to the CCP. Let node 0 represent the CCP, and $\tilde{L}_{i0}$ the liabilities to/from the CCP such that

$$\tilde{L}_{i0} = \sum_{j \in N} L_{ij} - \sum_{j \in N} L_{ji}.$$

Hence, if $\tilde{L}_{i0}$ is positive (resp. negative), agent i is a debtor (resp. creditor) to the CCP.

Since the best equilibrium can be reached, it follows that whenever agent i receives all the liabilities from his debtors, he will choose not default. Therefore, it holds that

$$z_i^2 + (1+r)\, z_i^1 + \sum_{j \in N} L_{ji} \geq \sum_{j \in N} L_{ij},$$

which implies

$$z_i^2 + (1+r)\, z_i^1 \geq \tilde{L}_{i0}.$$

Hence, the non-default condition is satisfied for each agent in the network with liabilities intermediated by the CCP and the best equilibrium is reached.

Chapter 3

Credit lines contagion in financial networks

3.1 Introduction

Financial crises, and in particular the crisis of 2008, have revealed that the composition of the financial systems is of paramount importance to financial stability. Furthermore, interconnections and links among different financial institutions, such as banks (commercial and investment banks), hedge funds, insurance and reinsurance companies, and various financial intermediaries, can be conducive to the diffusion of shocks and defaults via domino effect. This risk of shock-waves contagion and the possible collapse of the system, caused by the default of one or more financial institutions is known as systemic risk. This kind of failure has damaging short-term and long-term effects that could possibly hit the economy as a whole. For these reasons, systemic risk has been the subject of ongoing interest for researchers as well as policymakers.

Different institutions in the financial sector are connected to each other through claims, obligations and cross-holdings. Interbank markets allow banks to borrow from each other when suffering from liquidity shortages and negative shocks. Essentially, the value of a bank issuing a liability depends not only on its own financial health, but also on the payments it receives from its debtors, which in turn depend on the payments they receive from their own debtors, even though the bank, when making a decision, can only assess the situation of its immediate debtors, and not that of other institutions whose defaults and failure to repay debts might lead to its own default. In this sense, interdependencies within the financial system are key to economic and financial outcomes.

The aim of this chapter is to investigate the multiplicity of channels of exposure among financial institutions that are part of a single financial network. An important channel of systemic risk and shock contagion, albeit one that has not been investigated to the same extent as liabilities, is credit lines – that is, promised payments or future potential liabilities. A negative shock to an organisation will compromise its ability to honour its commitments and to issue new obligations. The potential debtor-institutions, receivers of credit lines, take financial or operational decisions based on the contingent credit lines they were granted, and cutting back such funding opportunities would be costly for them.

The systemic importance of credit lines arises from their fragility in view of their dependence not only on the failure of potential creditors, but also on minor shocks, contrary to obligations, which, because of debt priority rules, are lost only in the event that the debtor defaults. The main objective is to explore the role and to determine the payments among banks in the network while following the standard conditions of priority of creditors and potential debtors over shareholders, and limited liability. We also assume that banks have equal priority – that is, no preferred creditor over others – and thus we adopt the proportional payment mechanism. Moreover, we will drop the all-or-nothing payment scheme: in the event that a bank fails to pay in full, it should distribute its available resources proportionately to cover at least a part of its due payments.

We introduce a fictitious default algorithm analogous to the one introduced by the fundamental paper of Eisenberg and Noe (2001) by incorporating credit lines in the interbank network. This algorithm aims to determine which institutions default, and is a way to assess each institution's exposure. Furthermore, it permits us to identify the systemically relevant organisations and to evaluate the threat they pose to the stability of the financial network, namely the consequences of their default on the other (close and distant) institutions.

At the level of policies that aim to reduce systemic risk and financial contagion, we will explore central clearing before going over a motivation for targeting policies. First, introducing a central clearing counterparty that executes multilateral netting of the interbank liabilities and credit lines has a significant impact on the financial network architecture. The position of every node, and the weight and direction of its due payments, is changed and the payment mechanism, the equilibrium computation and the resulting defaults change accordingly. With respect to targeting policies, we study granting a small amount of cash to the systemically relevant financial institutions.

There is a growing literature that studies different aspects and features of financial contagion and systemic risk. Static financial networks have been studied using two approaches: either by considering a regular or symmetric network structure that follows some particular distribution, deriving general results and executing simulations to assess contagion (Nier et al.(2007), Upper (2011)), or by examining how a given network fails and exploring its properties (Eisenberg and Noe (2001), Cifuentes et al. (2005)). This chapter belongs to the second category and considers a financial networks approach that uses graph theory and its applications. In particular, we build on the model of Eisenberg and Noe (2001), which introduces the notion of a clearing mechanism for the simultaneous payment of liabilities in a static financial network using a fixed-point argument. Their results show that a clearing vector always exists and is unique under mild conditions. In addition, they propose an algorithm that computes the clearing vector and determines the chains of default.

Elliott, Golub and Jackson (2014) study how interdependencies between banks lead to cascading defaults and failure. They establish two different concepts representing two aspects of the interbank links and cross-holdings. The first one is integration, which identifies the dependence on the counterparties and consequently how much a financial institution is susceptible to others and their risks; the second concept is diversification, which represents the heterogeneity of the cross-holdings for each financial institution – in other words, how many counterparties it has. These two concepts have distinctive and non-monotonic impacts on cascades: integration expands the extent of contagion and hence defaults, but reduces the probability of the first failure, while diversification increases the propagation of cascades throughout the network and at the same time decreases the exposure level among pairs of financial institutions.

Acemoglu, Ozdaglar and Tahbaz-Salehi (2015) build on the model by Eisenberg and

Noe (2001) to assess interbank exposure. They show that there is a certain threshold of magnitude of shocks, such that below it, the increase in the connectedness in the financial network makes it more stable. In contrast, when the shock exceeds this level, a higher degree of connectedness reduces stability and makes the system weaker and more exposed to contagion and risk.

Many papers focus on interbank liabilities as a channel of risk propagation without analysing credit lines or other possible channels among financial institutions. A pertinent paper by Muller (2006) investigates the credit lines in the Swiss interbank market as a channel of financial contagion in addition to the liabilities exposure channel. Relying on data and a simulated default situation, the findings indicate that both channels are relevant and constitute a source of systemic risk. The author supposes that each bank can have an opportunity to raise a credit line if it is short on liquidity and the corresponding counterparties can provide it. In this way, she introduces this new channel of contagion of defaults. In our chapter, the payment of the credit line is now enforced whenever the potential creditor resources allow that, in contrast to Muller's assumption that the benefiting party can only use their credit line if they are defaulting or illiquid. We define an interbank network with two types of links, liabilities and credit lines, which both take part in the clearing mechanism. The payment of credit lines is computed simultaneously with the repayments of the debt, which results in one clearing vector.

The other part of the chapter examines possible policies that can be employed in order to mitigate this domino effect in the financial network: central clearing and targeting policies. The paper by Demange (2018) explores welfare improving policies in the interbank liabilities network. Her approach follows Eisenberg and Noe (2001) in studying clearing vectors and extends it by introducing a measure of the threat that a particular bank poses to the level of repayments in the network, called the "threat index", which identifies the most systemically relevant banks for which a targeted cash injection would be optimal.

This chapter also contributes to the literature studying the role of central clearing counterparties (henceforth CCP) in financial systems. Several articles and papers published by researchers and Central Banks have examined the efficiency of CCPs. For instance, two papers issued by the Bank of England present a numerical approach by treating the financial resources and the loss-allocation rules of a central counterparty, while other research, such as that by Koeppl and Monnet (2010) and Duffie and Zhu (2011), has shown the theoretical efficiency of CCPs. Amini, Filipovic and Minca (2014) highlighted several theoretical implications of the introduction of a central clearing counterparty to the network of interbank liabilities. Their main results emphasise the positive impact of a CCP on aggregate surplus in the network; nevertheless, its impact on systemic risk is uncertain. They provide conditions on the guarantee fund of the CCP to enhance reduction of systemic risk. Acharya and Bisin (2014) discuss the role of central clearing in improving information aggregation.

The rest of this chapter is organised as follows. We start by describing the theoretical framework by defining a financial network for liabilities and credit lines in Section 3.2 and we prove the existence of a payment equilibrium. Section 3.3 describes the computation of the payment vector using a fictitious default algorithm. In Section 3.4 we explore central clearing for liabilities alone, credit lines alone, and liabilities and credit lines. Section 3.5 discusses a motivation for cash injection, and Section 3.6 concludes the chapter.

3.2 The model

3.2.1 Defining the network

Consider an economy $\mathcal{E}$ constituted of a set $N = \{1, 2, ..., n\}$ of distinct financial institutions which we call from now on banks for simplicity. The economy will be formalised by a weighted directed network with n nodes. Each bank has a cash flow z_i that is a result of operations conducted outside the financial system, which is assumed to be exogenous. We assume that there is no external debt; in this sense, z_i is non-negative and a bank does not hold any liability outside the banking sector. Furthermore, the banks forming the financial network are connected via two types of links: liabilities and credit lines. Initially, banks exchange debt contracts within the network and make promises of providing credit lines. We suppose that all liabilities and credit lines have the same maturity and all corresponding payments should be offset at the same point in time. It can also be thought of in terms of banks signing debt contracts and promises of credit lines payments at any date prior to the maturity date under consideration, and that this setting does not include any other possible liabilities or credit lines maturing at future dates.

The interbank liabilities structure is represented by the $n \times n$ matrix $L = (L_{ij})$ where L_{ij} is the magnitude of i's nominal liability to j, such that $L_{ii} = 0\ \forall i$. Each bank will have a sum of their total liabilities – $\bar{L}_i$ – that they will have to repay: $\bar{L}_i = \sum_j L_{ij}$.

The relative liability is defined by taking the ratio of the liability from a given bank i to a bank j over the total amount of liabilities of i to determine the proportion of i's debt to j of i's total debt:

$$\alpha_{ij} = \begin{cases} \frac{L_{ij}}{\bar{L}_i} & \text{if } \bar{L}_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

with α being the corresponding $n \times n$ relative liabilities matrix .

Similarly, the interbank credit lines structure is represented by the $n \times n$ matrix $C = (C_{ij})$ where C_{ij} is the magnitude of i's promised payment to j, and $C_{ii} = 0\ \forall i$.

The total amount of credit lines of bank i is $\bar{C}_i = \sum_j C_{ij}$. In a similar way, we define the relative credit lines ratio for every node i as follows:

$$\beta_{ij} = \begin{cases} \frac{C_{ij}}{\bar{C}_i} & \text{if } \bar{C}_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

with β the $n \times n$ relative credit lines matrix.

Note that we have $\forall i$, $\sum_{j=1}^{n} \alpha_{ij} = 1$ and $\sum_{j=1}^{n} \beta_{ij} = 1$. In matrix notation, it is written as $\alpha \mathbf{1} = \mathbf{1}$ and $\beta \mathbf{1} = \mathbf{1}$.
The liability and credit line structures are not symmetric. That is, L_{ij} and L_{ji} are not necessarily equal, and the same is true for C_{ij} and C_{ji}. Moreover, both can be simultaneously positive.
While the above definitions of liabilities and credit lines seem similar, the two concepts are quite different. Liabilities are the debts a bank holds for other banks in the network, which, if not settled, lead to default. Credit lines, on the other hand, are potential obligations banks have promised to issue for other banks, which if not met, their issuers do not default.

Definition 3.1. A *financial network* in an economy $\mathcal{E}$ associated with the financial structure $\left(\alpha, \beta, \bar{L}, \bar{C}, z\right)$ is a directed multi-graph with two types of links, where the nodes are the financial institutions and the edges are of two types: liabilities and credit lines. A directed edge from vertex i to j represents the debt of i to j, that is, $i \rightarrow j$ if $L_{ij} > 0$, or equivalently, $\alpha_{ij} > 0$, and a dashed directed edge from vertex i to vertex j represents the promised credit line by i to j, that is, $i \dashrightarrow j$ if $C_{ij} > 0$, or equivalently, $\beta_{ij} > 0$.

Equivalently, it is a set of n nodes representing financial institutions and constituting two overlapping networks, each of them defined by a particular type of links. Despite the fact that a node is part of the two networks and might have positive liabilities and credit lines simultaneously, its characteristics such as position in each network and centrality, may differ.

Assumptions In order to determine the clearing payment vector, we will establish the following standard assumptions.

(i) **Debt seniority**. Meeting the liabilities is prior to paying the credit lines and to distributing shareholders' equity. If a bank does not have enough resources to pay its liabilities and credit lines in total, it should use the available resources to fulfil its liabilities.

(ii) **Equal seniority.** Liabilities have the same seniority, and credit lines as well; repayments are proportional in case of default. For liabilities, no creditor is more important than another: if the debtor is defaulting and is not able to repay his obligations in full, he should pay his creditors pro rata. Furthermore, for credit lines, all the potential debtors have equal priority and are paid proportionately in case the given bank is incapable of settling all the promised amounts.

(iii) **Credit lines seniority over equity.** Credit lines must be paid before paying equity. If a bank meets its liabilities, it pays its credit lines and then distributes equity; if that is not feasible, it spends what is left of its inflows to settle a part of its promised credit lines.

(iv) **Non-negative equity.** The total payment made by each bank should not exceed the inflows the bank is receiving.

Payment vector Based on the above assumptions, each bank will pay at most the total value of its liabilities and credit lines. Let $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ denote the vector of payments, where π_i is the total payment made by bank i which is bounded above by the total amount of liabilities and credit lines of i such that $\pi_i \in \left[0, \bar{L}_i + \bar{C}_i\right]$. Let π_{ij} be the payment that i will make to j. It can include both liabilities and credit lines payments:

$$\pi_{ij} \in \left[0, L_{ij} + C_{ij}\right].$$

We define f_i as bank i's inflows. They constitute all the resources available for i that can be used to make its due payments and to distribute equity to shareholders if possible. It will be the sum of all the payments that i receives from other banks whether in the form of liabilities or credit lines, added to i's cash flow:

$$f_i = z_i + \sum_j \pi_{ji}$$

State of a bank It is possible to distinguish between three realisable states for a given bank in this setting.

(i) Defaulter: the resources it has at hand are not enough to complete the payment of its liabilities; hence, it defaults and repays part of them proportionately without extending any credit lines as promised:

$$D = \left\{i \in N : \pi_i < \bar{L}_i\right\}$$

(ii) Distressed: the resources are enough to cover all liabilities debt but only a part of the credit lines; it does not default but it is unable to meet the total amount of credit lines:

$$G = \left\{i \in N : \bar{L}_i \leq \pi_i < \bar{L}_i + \bar{C}_i\right\}$$

(iii) Safe: the available resources cover all the due payments for both liabilities and credit lines:

$$S = \left\{i \in N : \pi_i = \bar{L}_i + \bar{C}_i\right\}$$

3.2.2 Payment equilibrium

Clearing payment vector existence and uniqueness Using a fixed-point argument, we will show the existence of a clearing payment vector $\pi^* = (\pi_1^*, \pi_2, \ldots, \pi_n^*)$ with entries:

$$\pi_i^* = \sum_j \pi_{ij}^*, \forall i.$$

Given α the relative liabilities matrix, β the relative credit lines matrix, $\bar{L} = (\bar{L}_1, \bar{L}_2, \ldots, \bar{L}_n)$ the vector of total liabilities for each node, $\bar{C} = (\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_n)$ the vector of total credit lines for each node and $z = (z_1, z_2, \ldots, z_n)$ the vector of cash flows, we define the map:

$$\Phi\colon \left[0 \ , \ \bar{L} + \bar{C}\right] \mapsto \left[0 \ , \ \bar{L} + \bar{C}\right]$$

by

$$\Phi(\pi) \equiv ((z + \pi^T) \wedge (\bar{L} + \bar{C})).$$

The clearing payment vector $\pi^* = (\pi_1^*, \pi_2^*, \ldots, \pi_n^*)$, consistent with the conditions of the payment mechanism, is a fixed point of the map $\Phi(\cdot)$. Each bank will pay the minimum between its inflows f_i that are constituted by its cash flow z_i and the payments it receives $\sum_j \pi_{ji}^*$, and the total value of its obligations and promised credit lines $\bar{L}_i + \bar{C}_i$.

$$\pi_i^* = \min\left\{f_i \ ; \ \bar{L}_i + \bar{C}_i\right\} = \min\{z_i + \sum_j \pi_{ji}^* \ ; \ \bar{L}_i + \bar{C}_i\}$$

The payment made by bank i is the sum of its payments to all the other nodes in the system could be formally written as follows:

$$\pi_{ij}^* = \begin{cases} \alpha_{ij} f_i & \text{if } f_i \in \left[0 \ ; \ \bar{L}_i\right] \\ L_{ij} + \beta_{ij}(f_i - \bar{L}_i) & \text{if } f_i \in \left[\bar{L}_i \ ; \ \bar{L}_i + \bar{C}_i\right] \\ L_{ij} + C_{ij} & \text{if } f_i \in \left[\bar{L}_i + \bar{C}_i \ ; \infty\right) \end{cases}$$

In the first case, the bank defaults and pays all its inflows as part of its scheduled liabilities; whereas in the second case, the bank is distressed and despite not defaulting, it faces trouble meeting credit lines payments. In the third case, the bank is safe and reimburses as promised. This can be summarised in a one-vector characterisation as stated by Proposition 3.1.

Proposition 3.1. *The payment made by a bank i to a bank j, which includes a liability and/or a credit line payment, can be formally written as:*

$$\pi_{ij}^* = \min\{\alpha_{ij} \min\{z_i + \sum_j \pi_{ji}^* \ ; \ \bar{L}_i\} + \beta_{ij}(z_i + \sum_j \pi_{ji}^* - \bar{L}_i)^+ \ ; \ L_{ij} + C_{ij}\}. \tag{3.2.1}$$

Proof. See Appendix B.

□

3.2.3 Characterisation of two separate payment vectors

Since liabilities and credit lines do not have the same seniority, and according to the aforementioned assumption a bank must repay all its liabilities before starting to pay its promised credit lines, the payment vector π is formed by two components and can be divided into two separate vectors: one for the liabilities payment a, and one for the credit lines payment b.
For every node i:

$$\pi_i = a_i + b_i = \sum_j a_{ij} + \sum_j b_{ij},$$

where a_i and b_i are i's liabilities and credit lines payments respectively. Here a_{ij} is the liabilities payment from i to j; and b_{ij} is the credit line payment from i to j.
We have $a_{ij} \in [0, L_{ij}]$; $a_i \in [0, \bar{L}_i]$; $b_{ij} \in [0, L_{ij}]$; $b_i \in [0, \bar{C}_i]$.
Since the payment should be made made pro rata, then we will have: $a_{ij} = \alpha_{ij} a_i$ and $b_{ij} = \beta_{ij} b_i$.
The relationship between a and b can be depicted by the assumption of priority of liability over credit line, which implies that if the liabilities payment for each bank i is less than the total amount due, the credit line payment is equal to zero.

$$\forall i \text{ such that } a_{ij} \leq L_{ij} \text{ for some } j \implies b_{ij} = 0,$$

or, equivalently,

$$a_i \leq \bar{L}_i \implies b_i = 0.$$

The credit line payment is only positive when the owed liabilities are reimbursed in total:

$$b_i \geq 0 \iff a_i = \bar{L}_i.$$

Moreover, a defaulting node will pay the part of the obligations that it can afford pro rata, and will not pay any of the credit lines.
Cash flows to bank i can therefore be redefined as follows:

$$f_i = z_i + \sum_j a_{ji} + \sum_j b_{ji} = z_i + \sum_j \alpha_{ij}^T a_j + \sum_j \beta_{ij}^T b_j$$

This can be written in matrix notation: $f = z + \alpha^T a + \beta^T b$.

Definition 3.2. The value of equity of node i, e_i is defined to be the inflows to i less the payments made by i:

$$e_i = f_i - \pi_i = z_i + \sum_j \pi_{ji} - \pi_i$$

$$e_i = f_i - a_i - b_i = z_i + \sum_j \alpha_{ij}^T a_j + \sum_j \beta_{ij}^T b_j - a_i - b_i.$$

The same separation argument formerly used allows us to show that two clearing vectors a^* and b^* exist, such that the payment of liabilities from i to j is determined by comparing the total value L_{ij}, which is equal to $\alpha_{ij}\bar{L}_i$, with i's inflows:

$$a^*_{ij} = \alpha_{ij}\left[\min\left\{\bar{L}_i \ ; \ f_i\right\}\right].$$

For the credit lines payment, we must compare the value of the $\bar{C}_i$ with what is left from the inflows after settling the liabilities, and rule out the case when $f_i \leq \bar{L}_i$:

$$b^*_{ij} = \beta_{ij}\left[\min\left\{\bar{C}_i \ ; \ (f_i - \bar{L}_i)\right\}\right]^+.$$

Proposition 3.2. *A bank i's payment consists of a payment of liabilities*

$$a^*_i = \sum_j a^*_{ij} = \min\{\bar{L}_i \ ; \ f_i\} = \min\left\{\bar{L}_i \ ; \ z_i + \sum_j \alpha^T_{ij} a^*_j + \sum_j \beta^T_{ij} b^*_j\right\}$$

and a payment of credit lines

$$b^*_i = \sum_j b^*_{ij} = \left[\min\left\{\bar{C}_i \ ; \ f_i - \bar{L}_i\right\}\right]^+ = \min\left\{\bar{C}_i \ ; \ z_i + \sum_j \alpha^T_{ij} a^*_j + \sum_j \beta^T_{ij} b^*_j - \bar{L}_i\right\}^+.$$

Proof. See Appendix B. ∎

The clearing payment vector is a fixed point $\begin{pmatrix} a^* \\ b^* \end{pmatrix}$ of the map:

$$\Psi\left(.,. \ ; \alpha, \beta, \bar{L}, \bar{C}, z\right): \ \left[0, \bar{L}\right] \times \left[0, \bar{C}\right] \mapsto \left[0, \bar{L}\right] \times \left[0, \bar{C}\right]$$

defined by

$$\Psi\left(\begin{pmatrix} a \\ b \end{pmatrix} \ ; \alpha, \beta, \bar{L}, \bar{C}, z\right) \equiv \begin{pmatrix} \bar{L} \wedge \left(\alpha^T a + \beta^T b + z\right) \\ \left[\bar{C} \wedge \left(\alpha^T a + \beta^T b + z - \bar{L}\right)\right] \vee 0 \end{pmatrix}.$$

The following results establish existence and uniqueness of a clearing payment vector.

Theorem 3.1. *Given any $\alpha, \beta, \bar{L}, \bar{C}$ and z,*

(1) There exist a greatest and a least clearing payment vectors π^+ and π^-.

(1') Similarly, if we consider two separate payment vectors, there exist greatest and least clearing liabilities and credit lines payment vectors a^+ and a^-, and b^+ and b^- respectively.

(2) Under all clearing vectors, the value of equity at each node of the financial system is the same:
if $\begin{pmatrix} a' \\ b' \end{pmatrix}$ *and* $\begin{pmatrix} a'' \\ b'' \end{pmatrix}$ *are two clearing vectors:*

$$\left(z + \alpha^T a' + \beta^T b' - \bar{L} - \bar{C}\right)^+ = \left(z + \alpha^T a'' + \beta^T b'' - \bar{L} - \bar{C}\right)^+ .$$

Proof. See Appendix B.

□

Corollary 3.1. *The clearing (total) payment vector* π^* *is unique.*

Proof. See Appendix B.

□

3.3 Fictitious default algorithm

3.3.1 Intuition

In this section, we present a fictitious default algorithm that serves to compute the payments of liabilities and credit lines in the financial system. By adding the credit lines contagion channel, we will study the defaults that are caused by the failure of some banks to pay their promised credit lines. As mentioned above, there are three possible situations for a bank: either it can fail to pay its due liabilities and hence its credit lines as well, in which case it defaults; or it can pay all its liabilities and all its credit lines; or it is able to pay all its liabilities and only a part of its credit lines. The last case captures the defaults induced by the interbank credit lines: despite the fact that a bank that is paying only a part of its credit lines does not default, it might cause the default of another bank.

The first step of the simulation is to assume that all the nodes pay all their due liabilities and credit lines – that is, $\forall i$, $\pi_i = \bar{L}_i + \bar{C}_i$. Now, for each node, take the equity value, difference between what it received and what it paid: $e_i = z_i + \sum_j \pi_{ji} - \pi_i$. If this equity value is negative, compute $z_i + \sum_j \pi_{ji} - \bar{L}_i$. If it is negative, then this node is defaulting. If it is positive, this node is able to pay its liabilities and does not default, but it can pay only part of its credit lines.

If some nodes have negative equity values, compute the values again by assuming only the nodes that defaulted in the first step are now defaulting and will pay part of their obligations pro rata, and that only the non-defaulting nodes with negative equity values are paying part of their credit lines, also pro rata, while all the other nodes pay in full.

This allows us to detect the new defaults caused by the defaults and the partial credit lines payments from the first step. We repeat the same process until no new defaults occur.

This algorithm determines which banks default, and is a way to assess each bank's exposure. It will also enable us to identify the systemically important banks and evaluate the threat that they pose to the stability of the financial network, namely the consequences of their default on the other banks.

3.3.2 Algorithm

The fixed-point operator Φ has a set of super-solutions $\mathcal{S}$:

$$\mathcal{S} = \left\{\pi \in \left[0 \ ; \ \bar{L} + \bar{S}\right] : \Phi(\pi) \leq \pi\right\}.$$

It is the set of the payment vectors under which some node is paying other nodes more than its total inflows.

We define the default set $D(\pi)$ under π for all $\pi \in \mathcal{S}$ as:

$$D(\pi) = \left\{i : \Phi(\pi)_i < \bar{L}_i\right\}.$$

We recall that if a node pays its obligations but does not pay its promised credit lines, it does not default. The default only occurs if a node is not able to pay its total liabilities.

Define $\Lambda(\pi)$ the $n \times n$ diagonal matrix such that:

$$\Lambda(\pi)_{ij} = \begin{cases} 1 & i = j \text{ and } i \in D(\pi) \\ 0 & \text{otherwise} \end{cases}.$$

The matrix $I - \Lambda(\pi)$ converts the entries corresponding to defaulting nodes under π to 0. The non-defaulting nodes under π represented by $I - \Lambda(\pi)$ will pay their total liabilities. We still need to investigate their credit lines payments.

For a fixed π', the non-defaulting nodes $I - \Lambda(\pi')$ will pay either $\bar{L} + \bar{C}$ or $\bar{L}$ and a part of $\bar{C}$, which will be less than $\bar{L} + \bar{C}$.

Hence, it will be convenient to define a set of nodes that are not able to pay their promised credit lines fully but will pay their full liabilities:

$$G(\pi) = \left\{i : \bar{L}_i < \Phi(\pi)_i < \bar{L}_i + \bar{C}_i\right\}.$$

And we define accordingly the $n \times n$ matrix $\Gamma(\pi)$ such that:

$$\Gamma(\pi)_{ij} = \begin{cases} 1 & i = j \text{ and } i \in G(\pi) \\ 0 & \text{otherwise} \end{cases}.$$

The matrix $I - \Gamma(\pi)$ represents the diagonal matrix with entries 1 for the nodes that are able to pay their liabilities and credit lines in total.

For fixed $\pi' \in S$, we define the map:

$$F_{\pi'} : \pi \to F_{\pi'}(\pi)$$

such that

$$\begin{aligned} F_{\pi'}(\pi) := \quad & \Lambda(\pi')\left\{\alpha^T\left[\Lambda(\pi')\pi + (I-\Lambda(\pi'))\left(\Gamma(\pi')\bar{L} + \left(I-\Gamma(\pi')\bar{L}\right)\right)\right] + \right. \\ & \beta^T\left[\Lambda(\pi')\times 0 + (I-\Lambda(\pi'))\left(\Gamma(\pi')\left(\pi-\bar{L}\right) + (I-\Gamma(\pi'))\right)\bar{C}\right] + z\} \\ & + (I-\Lambda(\pi'))\left\{\Gamma(\pi')\left[\bar{L} + \left(\pi-\bar{L}\right)\right] + (I-\Gamma(\pi'))\left[\bar{L}+\bar{C}\right]\right\} \end{aligned}$$

This map shows that all defaulting nodes under π' will pay π and it will be a liabilities payment, while the non-defaulting ones will pay the total amount of their liabilities but will be divided into two categories: those that will pay all their credit lines (above their liabilities), and those that will pay all the liabilities but only part of their credit lines, which will amount to $\pi - \bar{L}$. $F_{\pi'}$ has a unique fixed point $f(\pi')$.
The sequence of payment vectors which represents the fictitious default sequence is

- $\pi^0 = \bar{L} + \bar{C}$, since we start by assuming that all nodes pay all their liabilities and credit lines in the first step.
- $\pi^j = f(\pi^{j-1})$.

3.4 Policy implication: central clearing

The transmission of systemic risk is costly for individuals, institutions and economies. The interest in generating strategies and tools to reduce systemic risk has thrived due to the need for a stable and efficient financial system. Policymakers attempt to design regulations and laws to prevent the outbreak and propagation of shocks and preserve stability. In this context, regulators have tackled the idea of creating central counterparty clearing houses in an effort to ameliorate financial market infrastructure. Central clearing can be thought of as a risk-sharing or an insurance arrangement. The main functions of a central clearing counterparty (CCP) are to improve the aggregate surplus by assuming the bilateral obligations, reducing the number of obligations and guaranteeing their settlement. The concept of CCP was applied to the interbank liabilities structure as an effective tool to induce efficiency, as shown by Duffie and Zhu (2011), Zawadowski (2013) and Acharya and Bisin (2014). We explore the introduction of central clearing to the credit lines interbank network structure. A central clearing institution for credit lines would clear out bilateral credit lines and consequently reduce counterparty risk. By adding a central clearing institution to the credit lines network, we obtain a star-shaped network with the CCP as the central node. Figure 3.4.1 illustrates the case of a network of five banks.

The direction of the edge between a given node and the central clearing institution depends on the net liability between the two and more precisely on the net liabilities of a given node – that is, on the difference between what it owes to others and what others owe to it. This states whether a bank is a credit line receiver from or a payer to the CCP.

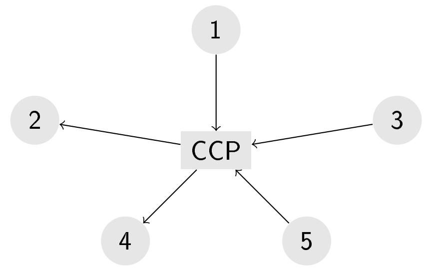


Figure 3.4.1: Central clearing for a network of five banks

3.4.1 Motivation: numerical example

Consider the simple case of a network of liabilities, constituted of three nodes nodes – A, B and C – having circular interdependences. Suppose the interbank liabilities are as displayed in Figure 3.4.2. A and B are linked by $L_{AB} = 100$ and $L_{BA} = 40$, B and C by $L_{BC} = 75$ and $L_{CB} = 30$ and C and A by $L_{CA} = 65$ and $L_{AC} = 5$.

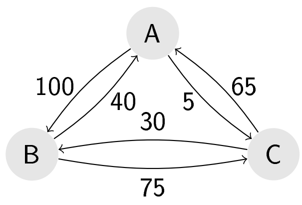


Figure 3.4.2: Over-the-counter liabilities structure

Figure 3.4.3 shows the network after bilateral netting between every pair of banks. There is a directed link between every two nodes representing the net liability resulting from the two-way liabilities. The liabilities structure becomes $L_{AB} = 60$, $L_{BC} = 45$ and $L_{CA} = 60$.

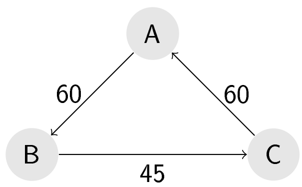


Figure 3.4.3: Bilateral netting

To examine the degree of contagion in the network and the spread of defaults, we take as an example $z = (7, 10, 5)$ as the cash flow vector. Following the fictitious default algorithm established in the preceding section, we notice that in the first step, C will default, since its inflows (after assuming that all liabilities are settled) are $5 + 45 = 50 < L_{CA} = 60$. And this default will prompt A's default in the next step: its inflows will be $50 + z_A = 57 < 60$. The algorithm will end here with two defaults as a result.

Adding a CCP Suppose now that we add at the centre of the network a central clearing counterparty CCP that will execute a multilateral netting. A should pay 60 to B and receive 60 from C; it will have a net liability of 0 in the presence of the CCP. B has to pay 45 to C and receive 60 from A, and thus it will owe the CCP $60 - 45 = 15$; and C will have to pay 15 to the CCP.

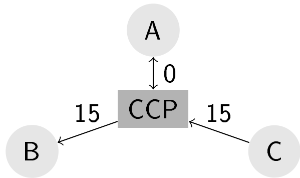


Figure 3.4.4: Multilateral netting: adding a CCP

Comparing the two networks before and after adding the CCP (Figure 3.4.3 and Figure 3.4.4), we notice that the CCP can have an important impact on the systemic risk in the network. If we suppose that bank A is hit by a shock that causes its default and failure to pay its debt to B in full, this will put B at risk. Moreover, take the case where A's payment to B added to B's cash flow is strictly less than L_{BC} – in this case, B also defaults and puts C at risk. When we have central multilateral clearing, a shock hitting A does not affect the other banks, and thus the propagation of the shock throughout the network is avoided.

If we go back to the numerical example of the above fictitious algorithm with the vector of cash flow $z = (7, 10, 5)$, now with the presence of the CCP, the number of defaults in the system is reduced: C defaults and cannot pay its debt to the CCP, but A is not affected. In this way, the algorithm will terminate in one step.

On the other hand, even if a node that is supposed to make a payment to the CCP, defaults and pays only part of this liability, the CCP can cover up using several margins and its guarantee fund to prevent its default.

However, if the CCP fails to cover up and its reserves and funds are not sufficient, causing it to default, its default could increase the systemic risk in the system.

This basic yet representative model can be generalised to a broader and more complicated network of n financial institutions.

In what follows, we will apply the central clearing concept to our model: first by adding a CCP to the liabilities network while keeping the credit lines network as it is; and second by adding a CCP for both networks simultaneously.

3.4.2 Clearing with a CCP for liabilities

Framework

Consider again the economy, $\mathcal{E}$, with $N = \{1, 2, ..., n\}$ distinct financial institutions, where the liabilities structure is represented by the $n \times n$ matrix $L = (L_{ij})$ and the credit lines structure is represented by the $n \times n$ matrix $C = (C_{ij})$. Now suppose we introduce a CCP for liabilities.

We add a new node, denoted by 0, to the network representing the CCP.

Consider now an auxiliary economy, $\mathcal{E}_{\text{CCP}}$, with $\tilde{N} = N \bigcup \{0\} = \{0, 1, 2, \ldots, n\}$ distinct financial institutions. Each node i in $\{1, 2, \ldots, n\}$ has a cash flow z_i as in $\mathcal{E}$, while z_0 denotes the guarantee fund of the CCP. As usual, z_0 is assumed to be positive, meaning that the CCP does not hold any liability outside the banking sector.

In the auxiliary economy, $\mathcal{E}_{\text{CCP}}$, the liabilities structure is represented by the $(n+1) \times (n+1)$ matrix $L_{\text{CCP}} = (\tilde{L}_{ij})$, defined as follows:

$$\tilde{L}_{ij} = -\tilde{L}_{ji} = \begin{cases} 0 & \text{if } i \text{ and } j \neq 0 \\ \sum_{k=1}^{n} L_{ik} - \sum_{k=1}^{n} L_{ki} & \text{if } j = 0 \end{cases}.$$

Similarly, the interbank credit lines structure in the auxiliary economy, $\mathcal{E}_{\text{CCP}}$, is represented by the $(n+1) \times (n+1)$ matrix $C_{\text{CCP}} = (\tilde{C}_{ij})$, defined as follows:

$$\tilde{C}_{ij} = \begin{cases} 0 & \text{if } i \text{ or } j = 0 \\ C_{ij} & \text{otherwise} \end{cases}.$$

The liabilities network in the auxiliary economy is a star-shaped network where the CCP occupies the central position and the other banks are on the periphery. Each bank has no liabilities to any other bank, while its (net) liability to the CCP corresponds to the difference between the total liabilities that the node should pay and the receivable that the node should receive. Observe that in the auxiliary economy, $\mathcal{E}_{\text{CCP}}$, the CCP's liabilities are equal to the CCP's receivables, which simply highlights the role of the CCP of clearing liabilities.

The credit lines network remains the same for all banks with the addition of having the CCP as an isolated node.

We assume that in the event that the CCP defaults, the payment will be executed proportionately. Hence, our approach in Section 3 above computes the clearing payment vector of the auxiliary economy, $\mathcal{E}_{\text{CCP}}$. Indeed, after treating the CCP as a node of the network, we can apply the preceding findings from Sections 2 and 3 to this network of $n + 1$ nodes: $\{0, 1, ..., n\}$.

Proposition 3.3. *There exists a unique clearing payment vector* $\pi^* = (\pi_0^*, \pi_1^*, \ldots, \pi_n^*)$.

Proof. See Appendix B. □

This proposition can be naturally extended to the settings below where the CCP is clearing the credit lines market, and where it is clearing both markets of the liabilities and credit lines simultaneously. Here, since the liabilities network in the auxiliary economy,$\mathcal{E}_{\text{CCP}}$, is a star-shaped network and there are no direct interbank liabilities payments anymore, then the defaults emerging from this network can be directly computed by checking the repayments to the node 0 representing the CCP, while the contagion and the multi-step default algorithm take place in the standard credit lines network. Consequently, the presence of the CCP, even if its multilateral netting is restricted to the liabilities network, will accelerate the convergence of the fictitious default algorithm. As in the previous three-nodes example, in the general case of n nodes the number of links in the network is considerably diminished: instead of having several links (up to $n-1$ links) with the other nodes of the liabilities network, each node would have at most one link with the CCP. This reduces the total computational load in any practical calculation, in particular the fictitious default algorithm.

3.4.3 Clearing with a CCP for credit lines

Framework

We examine here central clearing for credit lines only. The auxiliary economy, $\mathcal{E}_{\text{CCP}}$, is the set $\tilde{N} = N\bigcup\{0\} = \{0,1,2,\ldots,n\}$ of financial institutions where $\{0\}$ is the CCP. Similarly to before, each node $i \in \tilde{N}$ has a cash flow z_i as in $\mathcal{E}$, while z_0 denotes the guarantee fund of the CCP. In the auxiliary economy, $\mathcal{E}_{\text{CCP}}$, the liabilities structure is represented by the $(n+1)\times(n+1)$ matrix $L_{\text{CCP}} = (\tilde{L}_{ij})$, defined as follows:

$$\tilde{L}_{ij} = -\tilde{L}_{ji} = \begin{cases} 0 & \text{if } i \text{ or } j = 0 \\ L_{ij} & \text{otherwise} \end{cases}.$$

Credit lines, meanwhile, are represented by the $(n+1)\times(n+1)$ matrix $C_{\text{CCP}} = (\tilde{C}_{ij})$, defined by

$$\tilde{C}_{ij} = \begin{cases} 0 & \text{if } i \neq 0 \text{ and } j \neq 0 \\ \sum_{k\in N} C_{ik} - \sum_{k\in N} C_{ki} & \text{if } j = 0 \end{cases}.$$

In this case, the credit lines are covered by the CCP, and furthermore, they cease to constitute a channel of default contagion. If one or more banks fail to meet their promised credit lines, other banks do not bear the distress. If the source of risk in a given financial system is at the level of the credit lines exposure channel rather than the liabilities channel, this risk is eliminated by the introduction of the CCP. In other words, the CCP acts as a lender of last resort when it issues new liabilities and fulfils the credit lines that the distressed banks failed to meet. For instance, if there is a low economic conjecture, for a period of time, the intervention of the CCP, by extending the promised credit lines, can buy time for the banks and the financial institutions, and the economy as a whole, to recover while avoiding default and failure of distressed institutions.

3.4.4 Clearing with a CCP for liabilities and credit lines

Understandably, default contagion may still occur even in the presence of clearing with a CCP for liabilities, since the interbank network structure for credit lines is not cleared; it is equally the case when a CCP is clearing the credit lines market only. This brings about the policy question of whether to extend the role of CCP to clear both liabilities and credit lines. We will try to investigate this case in what follows while maintaining the assumption of priority of the liabilities over the credit lines for the CCP. Moreover, z_0, which represents the guarantee fund of the CCP, is assumed to be quite large in the way that the CCP hardly defaults.

Example

Consider an economy of four financial institutions: A, B, C and D. Suppose the interbank liabilities and credit lines network is as described in Figure 3.4.5 below.

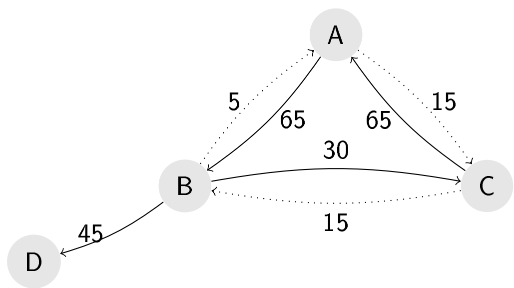


Figure 3.4.5: Network of four banks

Suppose the cash flow vector is $z = (5, 5, 5, 5)$. Following the default algorithm designed in Section 3.3, bank C will default in the first step. In step two, A and B will default and the algorithm will end.

First, we add a CCP for liabilities while keeping the credit lines network as it is (Figure 3.4.6).

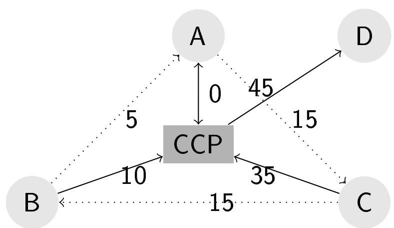


Figure 3.4.6: Central clearing for the liabilities network

After adding the CCP to liabilities, we notice that with the same cash flow vector used above, C still defaults in the first step, while in the second step, A is not affected, but B defaults due to C's default and its failure to extend its promised credit line to B. This compels us to apply central clearing to the credit lines network as well (Figure 3.4.7).

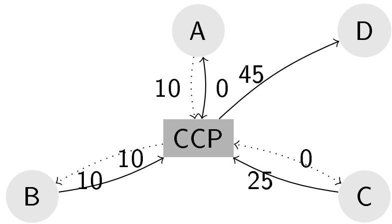


Figure 3.4.7: Central clearing for the credit lines and liabilities networks

In this case, with the same cash flow vector as before, $z = (5, 5, 5, 5)$, only C defaults in the first step and the algorithm terminates. The contagion effect is therefore reduced with the introduction of CCP to this four nodes network. The assumption that z_0 is large is important to prevent the CCP from defaulting and in this way the default algorithm becomes trivial.

General framework

In this section, we extend our analysis to a general network of $n+1$ nodes, n banks and a CCP for credit lines and liabilities. For simplicity, we will introduce only one CCP for both, and it will be at the centre of the two overlapping networks.
The liabilities and credit lines amounts that node i should pay to all other nodes in the network and here to the CCP are respectively $L_i = \sum_{j=1}^{n} L_{ij}$ and $C_i = \sum_{j=1}^{n} C_{ij}$.
Consider again the auxiliary economy $\mathcal{E}_{\text{CCP}}$. Also, as before, each node i in $\{1, 2, \ldots, n\}$ has a cash flow z_i that is assumed to be positive as in $\mathcal{E}$, while z_0 denotes the guarantee fund of the CCP.
The liabilities structure is represented by the $(n+1)\times(n+1)$ matrix $L_{\text{CCP}} = (\tilde{L}_{ij})$, defined as before.
Similarly, the interbank credit lines structure in the auxiliary economy is represented by the $(n+1)\times(n+1)$ matrix $C_{\text{CCP}} = (\tilde{C}_{ij})$, and is now defined as follows:

$$\tilde{C}_{ij} = -\tilde{C}_{ji} = \begin{cases} 0 & \text{if } i \text{ and } j \neq 0 \\ \sum_{k=1}^{n} C_{ik} - \sum_{k=1}^{n} C_{ki} & \text{if } j = 0 \end{cases}.$$

In line with what was previously explained concerning central clearing in the liabilities network, the auxiliary economy $\mathcal{E}_{\text{CCP}}$ still takes the form of a star-shaped network where the CCP is the central node. In addition, none of the banks have any liabilities or credit lines to any other bank, while the (net) liability and the (net) credit line to the CCP represent the difference between what it should be and what it should receive. As above, the CCP's liabilities are equal to the CCP's receivables (liabilities), and the CCP's payable

credit lines are equal to the CCP's receivable credit lines.
The existence and the uniqueness of the clearing payment vector in this framework of the model can be promptly derived using the same mathematical methodology. It is important to emphasise that all the payments of liabilities and credit lines occur directly and bilaterally between each bank and the system, in contrast with the previous scenario, where payments were done between banks themselves in a more complicated way.
With regard to the default algorithm, the number of links in the network diminishes even more when we apply central clearing to the credit lines network: instead of having up to $n-1$ liabilities links and $n-1$ credit lines links for every node, it would have at most two links with the CCP (one liability and one credit line). Therefore, the convergence of the fictitious default algorithm will be even more accelerated.

3.5 Policy implication: cash injection

Our analysis may also pave the way for policy intervention, which may take the form of either funds redistribution or cash injection, in order to trigger increases in debt reimbursement, credit line payment and the net cash flow in the network. Understandably, such interventions may potentially involve designing incentive-compatible solidarity policies to transfer funds from safe organisations to risky organisations, which would prevent them from defaulting and as a consequence the whole network from bearing the costs of such defaults.

In the following, we will discuss the possibility of cash injection which can be regarded as a positive increase in the cash flow of the banks receiving it. For this purpose, we suppose that the objective is to maximise the total payments in the financial system subject to the constraints of limited liability, and the payment of every bank i not exceeding its total due payments. This problem can be formally stated as

$$\mathcal{P}: \ \max \sum_i \pi_i$$

$$\text{s.t. } \pi_i \in \left[0, \bar{L}_i + \bar{C}_i\right] \text{ for all } i$$

$$\text{and } \pi_i \leq \sum_j \pi_{ji} + z_i \text{ for all } i$$

3.5.1 Examples

An evident example of the effectiveness of cash injection policies and the importance of targeting the right nodes is the star network – also known as the core-periphery network (Figure 3.5.1).

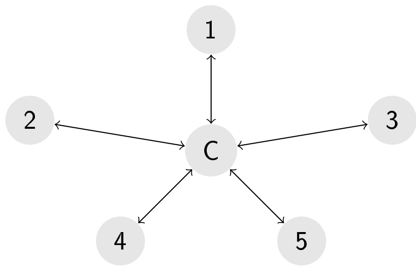


Figure 3.5.1: Star network

Suppose a shock hits the central node and as a result does not pay promised credit lines to the peripheries, which may result in the default of some of them. The question here is: should cash injection target the central node or defaulting nodes?

An extension of the star network, which is also representative of the contagion effect, is the tree (Figure 3.5.2).

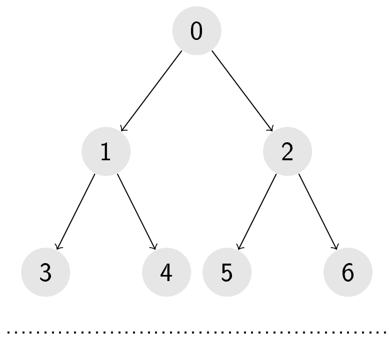


Figure 3.5.2: Tree

3.5.2 Motivation

A regulator has a certain amount of cash which she can provide to some banks in order to increase the payments in the financial system.

Cash injection to safe banks that are already paying their entire debt and promised credit would not affect the overall payments in the network unless it is stipulated that cash injection is compounded with a matching rate $m > 0$ for credit lines. That is, the target bank receiving the cash injection is forced to increase its credit lines by $(1+m)$. This may have a different policy effect, as an increase in the cash inflows would improve the rate of reimbursement of liabilities and the extension of credit lines in the financial network.

However, here we consider that a cash injection targets either defaulting or distressed banks.

Suppose that distressed banks are targeted by an injection of a small amount of cash, $\epsilon \in \mathbb{R}^+$. First, let π^D, π^G and π^S be the payment vectors of the defaulters, distressed banks and safe banks respectively. Accordingly, let $\alpha^{A,B}$ be the matrix of relative liabilities from banks in set A to banks in set B such that $A, B \in \{D, G, S\}$. Moreover, $L^D, L^G$and L^S are the corresponding liabilities vectors. Define $\beta^{A,B}$ and C^S in a similar way.

After the cash injection, the payment of distressed banks increases as follows: $\pi_i^G = \min\{z_i + \epsilon + \sum_{j\in D} \alpha_{ji}\pi_j^D + \sum_{j\in N\setminus D} \alpha_{ji}\bar{L}_j + \sum_{j\in G} \beta_{ji}(f_j - \bar{L}_j) + \sum_{j\in S} \beta_{ji}\bar{C}_j; \bar{L}_i + \bar{C}_i\}$. We suppose that ϵ is small enough such that it is still insufficient for i to fulfil all its promised credit lines $\bar{C}_i$ and

$$\pi_i^G = z_i + \epsilon + \sum_{j\in D} \alpha_{ji}\pi_j^D + \sum_{j\in N\setminus D} \alpha_{ji}\bar{L}_j + \sum_{j\in S} \beta_{ji}\bar{C}_j + \sum_{j\in G} \beta_{ji}(f_j - \bar{L}_j)$$

A cash injection to bank $i \in G$ has two effects on the payments in the network and on i in particular: initially, it directly increases the payment made by i and the inflows to the receivers of i's credit lines; additionally, it increases the inflows to i if i is on a cycle of defaulters or receivers of credit lines. If for instance a bank $i \in G$ receives a cash injection, and j is such that $C_{ij} > 0$ and there is a bank k such that $L_{jk}, j \in D$ and/or $C_{jk} > 0, j \in G$ and $L_{ki}, k \in D$ and/or $C_{ki} > 0, k \in G$, then inflows to i from k increase as i's credit line payment increases.

In matrix notation:

$$\pi^G = z + \epsilon + \alpha^{D,G}\pi^D + \alpha^{G,G}L^G + \alpha^{S,G}L^S + \beta^{G,G}(f - L^G) + \beta^{S,G}C^S$$

In order to assess the impact that an increase in credit lines payment has on the overall system, we consider the two separate payment vectors approach, with **a** and **b** being the payment vectors of liabilities and credit lines respectively. After injecting a small amount ϵ to $i \in G$, such that i remains in G after receiving the cash, the credit lines payment of $i \in G$ becomes:

$$b_i^\epsilon = z_i + \epsilon + \sum_{j\in D} \alpha_{ji}a_j + \sum_{j\in N\setminus D} \alpha_{ji}\bar{L}_j + \sum_{j\in G} \beta_{ji}\bar{C}_j + \sum_{j\in G} \beta_{ji}b_j^\epsilon - \bar{L}_i$$

In matrix notation:

$$\mathbf{b}^{G,\epsilon} = \mathbf{z}^G + \epsilon\mathbf{1}_G + \alpha^{D,G}\mathbf{a}^D + \alpha^{G,G}L^G + \alpha^{S,G}L^S + \beta^{S,G}C^S + \beta^{G,G}\mathbf{b}^{G,\epsilon}$$

such that $\mathbf{b}^{G,\epsilon}$ is the $g \times 1$ vector of credit lines payments of distressed banks after injecting an amount ϵ into each of them, $\mathbf{z}^G$ the $g \times 1$ vector of cash flows of distressed banks, $\mathbf{1}_G$

the $g \times 1$ vector with entries 1, $\alpha^{D,G}$, $\alpha^{G,G}$ and $\alpha^{S,G}$ the relative liabilities matrices from defaulting $(d \times g)$, distressed $(g \times g)$ and safe $((n-d-g) \times g)$ banks to distressed banks respectively, $\beta^{S,G}$and $\beta^{G,G}$ the corresponding relative credit lines matrices, $\mathbf{a}^D$ the $d \times 1$ liabilities payment vector of defaulting banks, L^G and L^S the liabilities vectors of banks in G and S, and C^S the credit lines vector of banks in S.

3.5.3 Cash injection

When considering two channels of systemic risk contagion, a given bank faces two types of risks: the risk of defaulting as a consequence of not receiving promised credit lines, and the risk of defaulting as a consequence of its debtors not paying their due liabilities. Equivalently, the two types of banks that are perturbed are defaulters as well as distressed.

Demange (2018) defines a threat index that takes into account the spillover effects of a defaulting bank and shows that a cash injection would be optimal if the receivers have the largest threat indices whenever the set of defaulting banks is kept unchanged, meaning that no defaulting bank escapes default after the injection. She also proves that cash injection for the banks that have a higher probability of defaulting might be suboptimal.

We extend this result here and apply it to our setting with two channels of defaults. We define two threat indices: the first one is that of defaulters, which takes into consideration spillover effects that emerge via the liabilities channel, while the second index is that of distressed banks, which accounts for spillover effects through the credit lines channel.

More precisely, the maximisation problem $\mathcal{P}$, stated at the beginning of Section 3.5, can be split into to problems each to maximise one type of payments as follows. If a regulator's objective is to maximise the payments of liabilities

$$\mathcal{P}^a : \max \sum_i a_i$$

$$\text{s.t. } a_i \in \left[0, \bar{L}_i\right] \text{ for all } i$$

$$\text{and } a_i \leq \sum_j \alpha_{ji} a_j + \sum \beta_{ji} b_j + z_i \text{ for all } i$$

Whereas if the regulator targets the payments of credit lines, the problem solved would be

$$\mathcal{P}^b : \max \sum_i b_i$$

$$\text{s.t. } b_i \in \left[\bar{L}_i, \bar{L}_i + \bar{C}_i\right] \text{ for all } i$$

$$\text{and } b_i \leq \sum_j \alpha_{ji} a_j + \sum \beta_{ji} b_j + z_i - \bar{L}_i \text{ for all } i$$

Accordingly, and following Demange's (2018) definition of threat index, we define $\boldsymbol{\mu}$ as the vector of liabilities threat indices such that for every $i \in N \setminus D$, $\mu_i = 0$ and for $i \in D$, $\mu_i = 1 + \sum_{j \in D} \alpha_{ij}\mu_j$; and we define $\boldsymbol{\delta}$ as the vector of credit lines threat indices such that for every $i \in N \setminus G$, $\delta_i = 0$ and for $i \in G$, $\delta_i = 1 + \sum_{j \in D} \beta_{ij}\delta_j$. These indices will be the Lagrange multipliers connected to the second constraint in each problem, which stands for limited liability. The multipliers capture the sensitivity of the objective function, the sum of payments, to changes in these constraints. We assume that the cash injection is small enough that it does not change the state of the bank, that is it does not reduce the multiplier to 0, and the constraint is still binding. In other words, the sets S, G and D are unchanged. However, with the increase in cash received which increases the inflows of the receiver, the limited liability constrained is less binding, and the repayments increase. We can show that an optimal cash injection that maximises the total payments would be one that targets the highest threat index among liabilities and credit lines.

Proposition 3.4. *An injection of a small amount of cash, that does not change the states of the banks, is optimal, if it targets the bank with the highest liabilities threat index in case the objective is to maximise liabilities repayments, and the bank with the highest credit lines threat index in case the objective is to maximise credit lines payments.*

Proof. See Appendix B. □

3.6 Conclusion

In this chapter, we explored two channels of contagion of financial risk of default, credit lines and liabilities. We built on Eisenberg and Noe's model by extending the proof of existence of a unique payment equilibrium and establishing a fictitious default algorithm. The introduction of a central clearing counterparty for the network of interbank liabilities alone, the network of credit lines alone and the networks of liabilities and credit lines together, was investigated and proved to have important impacts. More work can be done on this topic, especially regarding the regulations and management of the CCP and its guarantee funds. Another type of policy intervention that has been motivated here is targeting policies and cash injection. This can be further investigated in many ways. In particular, one can explore the optimal strategies for a regulator or a policymaker, regarding the choice of which channel of exposure to target and which type of banks at risk to save.

Appendices

Appendix 3.A Basic Notions

We recall some basic notions and concepts from lattice theory that we use along with other definitions and theorems.

A lattice is a partially ordered set such that every two elements have a least upper bound and a greatest lower bound. For any two vectors x and $y \in \mathbb{R}^n$ such that $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, the lattice operations are defined as follows:

The meet, or greatest lower bound (infimum) which is the greatest element of $\mathbb{R}^n$ lying below both x and y:

$$x \wedge y \equiv (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots, \min\{x_n, y_n\}).$$

The join, or least upper bound (supremum) which is the least element of $\mathbb{R}^n$ lying above both x and y:

$$x \vee y \equiv (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \ldots, \max\{x_n, y_n\}).$$

We also define the vector x^+:

$$x^+ \equiv (\max\{x_1, 0\}, \max\{x_2, 0\}, \ldots, \max\{x_n, 0\}).$$

And ordering any two vectors x and y:

$$x \leq y \iff x_i \leq y_i \text{ for all } i = 1, \ldots, n.$$

Also, $\forall x \in \mathbb{R}^n$, l^1-norm on $\mathbb{R}^n$

$$\|x\| \equiv \sum_{i=1}^{n} |x_i|.$$

Tarski's fixed point theorem

Let $(L, \leq)$ be any complete lattice. Suppose the map $f\colon L \mapsto L$ is monotone increasing i.e. $\forall x, y \in L,\ x \leq y \implies f(x) \leq f(y)$. Then the set of fixed points of f, denoted by $\mathrm{F} = \{x \in L \mid f(x) = x\}$, is a complete lattice with respect to $\leq$. Consequently, f has a greatest fixed point $\bar{u}$ and a least fixed point $\underline{u}$. Moreover, $\forall x \in L, x \leq f(x) \implies x \leq \bar{u}$ while $f(x) \leq x \implies \underline{u} \leq x$.

Appendix 3.B Proofs

Proof of Proposition 3.1

Since the payment depends on the inflows f_i to i, we must consider the three following possible cases for f_i.

1. Inflows do not cover the total liabilities of i, $f_i \in \big[0, \bar{L}_i\big]$ and i defaults. Then $f_i - \bar{L}_i \leq 0 \implies (f_i - \bar{L}_i)^+ = \max\big\{(f_i - \bar{L}_i)\ ;\ 0\big\} = 0$ and $\min\big\{f_i\ ;\ \bar{L}_i\big\} = f_i \leq \bar{L}_i \leq \bar{L}_i + \bar{C}_i$. Therefore:
$$\pi^*_{ij} = \min\{\alpha_{ij} f_i\ ;\ L_{ij} + C_{ij}\} = \alpha_{ij} f_i$$
since $\alpha_{ij} f_i \leq L_{ij} \leq L_{ij} + C_{ij}$. Thus, i pays only part of the liabilities on a pro rata basis and $i \in D$.

2. The inflows cover the liabilities but not the credit lines, $f_i \in \big[\bar{L}_i, \bar{L}_i + \bar{C}_i\big]$. This implies that $f_i - \bar{L}_i \geq 0$ and $(f_i - \bar{L}_i)^+ = f_i - \bar{L}_i$. Also $\min\big\{f_i\ ;\ \bar{L}_i\big\} = \bar{L}_i$. But $f_i \leq \bar{L}_i + \bar{C}_i \implies f_i - \bar{L}_i \leq \bar{C}_i \implies \beta_{ij}(f_i - \bar{L}_i) \leq C_{ij}$, therefore:
$$\pi^*_{ij} = \alpha_{ij}\bar{L}_i + \beta_{ij}(f_i - \bar{L}_i) = L_{ij} + \beta_{ij}(f_i - \bar{L}_i).$$
In this case, i will pay their total amount of liability and will not default, but will pay part of their promised credit lines pro rata following the priority of liabilities over credit lines; $i \in \mathcal{T}$.

3. Inflows are enough to cover all due payments: $f_i \in \big[\bar{L}_i + \bar{C}_i, \infty\big)$. This implies that $\min\big\{f_i\ ;\ \bar{L}_i\big\} = \bar{L}_i$ and $f_i - \bar{L}_i \geq 0$. Since $f_i \geq \bar{L}_i + \bar{C}_i$ thus $f_i - \bar{L}_i \geq \bar{C}_i \implies \beta_{ij}(f_i - \bar{L}_i) \geq C_{ij}$. Therefore:
$$\pi^*_{ij} = L_{ij} + C_{ij}.$$
i will pay the total amount of liabilities and credit lines for every j; $i \in \mathcal{S}$.

Proof of Proposition 3.2

Take each of the three possible cases of bank i's inflows f_i.

1. Defaulter: $f_i \in \left[0\ ;\ \bar{L}_i\right]$

   Since liabilities are prioritised over credit lines and credit lines over equity, and the inflows are not enough to pay all the liabilities, then i will default and will pay the inflows for its debtors pro rata.

$$a_i^* = f_i = z_i + \sum_j \alpha_{ij}^T a_j^* + \sum_j \beta_{ij}^T b_j^* \text{ and } b_i^* = 0.$$

2. Distressed: $f_i \in \left[\bar{L}_i\ ;\ \bar{L}_i + \bar{C}_i\right]$

   Bank i will pay all its liabilities $\bar{L}_i$ and then a part of its promised credit lines pro rata.

$$a_i^* = \bar{L}_i \text{ and } b_i^* = (f_i - \bar{L}_i)^+ = f_i - \bar{L}_i > 0.$$

3. Safe: $f_i \in \left[\bar{L}_i + \bar{C}_i\ ;\ \infty\right)$

   This is the trivial case where i will pay all its due amounts:

$$a_i^* = \bar{L}_i \text{ and } b_i^* = \bar{C}_i$$

   since $f_i \geq \bar{L}_i + \bar{C}_i$.

Proof of Theorem 3.1

The proofs of Theorem 3.1 and of the following Corollary 3.1 generalise the proofs of existence and uniqueness of the clearing vector in Eisenberg and Noe (2001) to our setting that incorporates credit lines.
First, we must prove that the map Φ positive, increasing, concave and non expansive.
Positivity and monotonicity and concavity follow from the fact that the map is a composition of two maps: the map $q \to q^T + z$ which is positive, increasing and affine, and the map $q \to q^T \wedge \left(\bar{L} + \bar{C}\right)$ which is positive, increasing, and concave .
As for non-expansiveness, we first recall the following definition. A map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is l^1-non-expansive if $\forall x \in \mathbb{R}^n$, $\| \Phi(x) - \Phi(y) \| \leq \| x - y \|$.
Also:

$$\forall x, y, \text{ and } z, \|x \wedge z - y \wedge z\| \leq \|x - y\|$$

Thus:

$$\left\|\Phi(\pi)-\Phi(\pi')\right\| = \left\|\left(\pi^T+z\right)\wedge\left(\bar{L}+\bar{C}\right)-\left(\pi'^T+z\right)\wedge\left(\bar{L}+\bar{C}\right)\right\| \leq \left\|\pi^T-\pi'^T\right\|$$

Let $F(\Phi)$ be the set of fixed points of Φ. Since Φ is increasing, $\Phi(0)\geq 0$ and $\Phi(\bar{L}+\bar{C})\leq \bar{L}+\bar{C}$, then by Tarski fixed point theorem $F(\Phi)$ is non-empty and has a greatest and least element. (1) is proved and (1') follows immediately.
(2) can be proved following the same proof proposed by Eisenberg and Noe (2001).

Proof of Corollary 3.1

First we define a *surplus set* $S\subset N$ as the set of nodes that have no liability or credit line to any node outside the set and it has a positive cash flows. Precisely, if S^C is the complement of S,

$$\forall\,(i,j)\in S\times S^C, \alpha_{ij}=0, \beta_{ij}=0 \text{ and } \sum_{i\in S} z_i>0$$

which is equivalent to

$$\iff L_{ij}=0, C_{ij}=0 \text{ and } \sum_{i\in S} z_i>0$$

Lemma 3.1. *If π^* is a clearing vector, then it is not possible for all nodes in a surplus set to have zero equity value.*

Proof. Let S be a surplus set and P_i^+ be the sum of payments received by a node $i\in S$ from nodes outside S such that $P_i^+=\sum_{j\in S^C}\pi_{ji}$.
Since S is a surplus set, then nodes in S make no payments to nodes in S^C. Next, suppose that all nodes in S have *equity value* $=0$ then for every $i\in S$ it holds that

$$f_i-\pi_i=0$$

$$\iff z_i+\sum_{j\in N}\pi_{ji}-\pi_i=0$$

$$\iff z_i+\sum_{j\in S}\pi_{ji}+\sum_{j\in S^C}\pi_{ji}-\pi_i=0$$

$$\iff \pi_i=z_i+\sum_{j\in S}\pi_{ji}+P_i^+$$

Summing over all the nodes in S, we get

$$\sum_{i\in S}\pi_i=\sum_{i\in S}z_i+\sum_{i\in S}\left(\sum_{j\in S}\pi_{ji}\right)+\sum_{i\in S}P_i^+=\sum_{j\in S}\left(\sum_{i\in S}\pi_{ji}\right)+\sum_{i\in S}\left[z_i+P_i^+\right]$$

However, the sum of payments of nodes in S can be written as

$$\sum_{i\in S}\pi_{ji}=\sum_{i\in N}\pi_{ji}-\sum_{i\in S^C}\pi_{ji}=\pi_j-\sum_{i\in S^C}\pi_{ji}$$

When summing over j:

$$\sum_{j\in S}\left(\sum_{i\in S}\pi_{ji}\right)=\sum_{j\in S}\left(\pi_j-\sum_{i\in S^C}\pi_{ji}\right)=\sum_{j\in S}\pi_j-\sum_{j\in S}\sum_{i\in S^C}\pi_{ji}$$

Since S is a surplus set, then $\forall j\in S$, we have $\sum_{i\in S^C}\pi_{ji}=0$ and therefore:

$$\sum_{j\in S}\sum_{i\in S^C}\pi_{ji}=0$$

This implies

$$\sum_{i\in S}\pi_i=\sum_{j\in S}\pi_j+\sum_{i\in S}\left[z_i+P_i^+\right]$$

$$\iff \sum_{i\in S}\left[z_i+P_i^+\right]=0$$

Contradiction with the assumption $\sum_{i\in S}z_i>0$.

□

Now define the *Risk Orbit* of a node as the set of all nodes connected to it through a directed path whether through liabilities edges or credit lines.

$$o(i)=\{j\in N:\exists\text{ a directed path from i to j}\}=\{j\in N:\ L_{ij}>0\text{ and/or }C_{ij}>0\}$$

Lemma 3.2. *Let π^*be a clearing vector for $\left(\alpha,\beta,\bar{L},\bar{C},z\right)$ and let $o(i)$ be a risk orbit such that $\sum_{j\in o(i)}z_j>0$, then under π^*at least one node of $o(i)$ has positive equity value*

$$\exists j\in o(i)\text{ such that }\bar{L}_j+\bar{C}_j<z_j+\sum_k\pi_{kj}^*$$

Proof. We will show that $o(i)$ is a surplus set.
Suppose it is not. Let $k\in o(i)$ such that k owed something to $j\in o(i)^C$.Take the path from i to k and add to it the edge from k to j. We get a path from i to j.Thus $j\in o(i)\implies j\notin o(i)^C$.
Contradiction. Thus $o(i)$ is a surplus set.

□

By lemma 3.1, every surplus set contains a node with positive equity.
Define a financial system as *regular* if every orbit $o(i)$ is a surplus set. Then, if the financial system is regular, the greatest and the least clearing vectors are the same $\pi^+=\pi^-$implying the clearing vector is unique by Theorem 2 proved by Eisenberg and Noe (2001).

Proof of Proposition 3.3

The financial network in the auxiliary economy with the CCP, is a special case of the general network before central clearing. Here, the number of links is smaller, such that the liabilities network is star-shaped with the CCP at the core. Thus, existence and uniqueness of a payment clearing vector in this economy are proved by Theorem 3.1 and Corollary 3.1.

Proof of Proposition 3.4

First given the assumption that the cash injection keeps the state of banks unchanged, we can state that the choice of the regulator of the objective function, and problem to solve will determine a targeted set of banks. In particular, if the regulator maximises the liabilities payments and solves problem $\mathcal{P}^a$, given that $\mu_{N\setminus D} = \mu_G = \mu_S = 0$, the targeted banks are defaulters. In this case, the result proved by Demange (2018) can be trivially applied. In case the regulator targets the credit lines payments and solves problem $\mathcal{P}^b$, the threat indices which represent the impact of an increase or a decrease in payment of a given bank, are null for defaulters and safe banks. That is because an increase in inflows, that does not change the state of the receiver, will not change the payment of credit lines by a defaulter which remains null, nor that by a safe banks which pays in full. The targeted set are the distressed bank, and the previous result can be extended.

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