HAUSDORFF-YOUNG INEQUALITY FOR ORLICZ SPACES ON COMPACT HOMOGENEOUS MANIFOLDS

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Abstract. We prove the classical Hausdorff-Young inequality for Orlicz spaces on compact homogeneous manifolds.

1. Introduction

The classical Hausdorff-Young inequality is one of the fundamental inequalities in the theory of Fourier analysis on groups. For a locally compact abelian group G, $1 \leq p \leq 2$ and $p' = \frac{p}{p-1}$, the classical Hausdorff-Young inequality says, " If $f \in L^1(G) \cap L^p(G)$ then the Fourier transform satisfies $\widehat{f} \in L^{p'}(\widehat{G})$ and $\|\widehat{f}\|_{p'} \leq \|f\|_p$. " This inequality was first given by Hausdorff in 1923 for torus $\mathbb{T}$. Hausdorff was inspired by the work of W. H. Young in 1912 who proved a similar result but did not formulate his result in terms of inequalities. For a historical discussion on this inequality we refer to [6, 11]. The Hausdorff-Young inequality for a compact group G is given in Hewitt and Ross [11, 10]. Later, Kunze [17] extended it to unimodular groups. On Lebesgue spaces, the Hausdorff-Young inequality is proved by using the Riesz convexity complex interpolation theorem between $p = 1$ and $p = 2$. It is well known that between any two Lebesgue spaces there is an Orlicz space which is not a Lebesgue space. M. M. Rao [19] studied the Hausdorff-Young inequality for Orlicz spaces on locally compact abelian groups. In fact, the celebrated work of M. M. Rao in the context of Orlicz spaces on locally compact groups (see [19, 21, 22, 23]) motivated the first author to study the Hausdorff-Young inequality for Orlicz spaces on compact hypergroups [14, 13, 15]. Recently, the second author with his collaborators established non-commutative version of the aforementioned inequality and of Hardy-Littlewood inequalities on compact homogeneous manifolds and locally compact groups [1, 3, 2, 4]. In this article, we study the classical Hausdorff-Young inequality with

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an enlargement of the space, namely, an Orlicz space on compact homogeneous manifolds. It is to be noted that the Riesz convexity theorem is useful for the L^p-spaces only. For Orlicz spaces results are obtained first by extending a key inequality of Hausdorff-Young in the form of Hardy-Littlewood [9, p. 170]. It is worth mentioning here that we apply the method of Hausdorff-Hardy-Littlewood [9].

In Section 2, we present the basics of compact homogeneous manifolds and of Orlicz spaces in the form we use in the sequel. In Section 3, we prove a key lemma which occupies a major part of this section, and finally, we prove the Hausdorff-Young inequality for Orlicz spaces on compact homogeneous manifolds.

2. Preliminaries

2.1. **Fourier analysis on compact homogeneous manifolds.** Let G be a compact Lie group and let K be a closed subgroup of G. The left coset space G/K can be seen as a homogeneous manifold with respect to the action of G on G/K given by the left multiplication. The homogeneous manifold G/K has a unique normalized G-invariant positive Radon measure μ such that the Weyl formula holds. There exists a unique differential structure for the quotient G/K. Examples of compact homogeneous manifolds are spheres $\mathbb{S}^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$, real projective spaces $\mathbb{RP}^n \cong \mathrm{SO}(n+1)/\mathrm{O}(n)$, complex projective spaces $\mathbb{CP}^n \cong \mathrm{SU}(n+1)/\mathrm{SU}(1) \times \mathrm{SU}(n)$ and more generally Grassmannians $\mathrm{Gr}(r,n) \cong \mathrm{O}(n)/\mathrm{O}(n-r) \times \mathrm{O}(r)$.

Let us denote by $\widehat{G}_0$ the subset of $\widehat{G}$, of representations in G, that are of class I with respect to the subgroup K. This means that $\pi \in \widehat{G}_0$ if there exists at least one non-zero invariant vector a in the representation space $\mathcal{H}_\pi$ with respect to K, i.e., $\pi(h)a = a$ for every $h \in K$. Let us denote by B_π the vector space of these invariant vectors and let $k_\pi = \dim B_\pi$. We fix the an orthonormal basis of $\mathcal{H}_\pi$ so that the first k_π vectors are the basis of B_π. We note that if $K = \{e\}$, then G/K is equal to the Lie group G and in this case $k_\pi = d_\pi$ for all $\pi \in \widehat{G}$. On the other hand, if K is a massive subgroup then $k_\pi = 1$. This is the case for the sphere $\mathbb{S}^n$. Other examples can be found in [26].

For a function $f \in C^\infty(G/K)$ we can write the Fourier series of its canonical lifting $\tilde{f} := f(gK)$ to G, $\tilde{f} \in C^\infty(G)$, so that the Fourier coefficients satisfy $\widehat{\tilde{f}} = 0$ for all representations $\pi \notin \widehat{G}_0$. Moreover, for class I representations we have $\widehat{\tilde{f}}(\pi)_{ij} = 0$ for

$i > k_\pi$. We will often drop the tilde for the simplicity and agree that for a distribution $f \in \mathcal{D}'(G/K)$ we have $\widehat{f}(\pi) = 0$ for $\pi \notin \widehat{G}_0$ and $\widehat{f}(\pi)_{ij} = 0$ for $i > k_\pi$. In order to shorten the notation, for $\pi \in \widehat{G}_0$ it make sense to set $\pi(x)_{ij} = 0$ for $j > k_\pi$. We can write the Fourier series of f (or of $\tilde{f}$) in terms of the spherical functions $\pi_{ij}, 1 \leq j \leq k_\pi$, of the representation $\pi \in \widehat{G}_0$, with respect to the subgroup K. The Fourier series of $f \in C^\infty(G/K)$ is given by

$$f(x) = \sum_{\pi \in \widehat{G}_0} d_\pi \sum_{i=1}^{d_\pi} \sum_{j=1}^{k_\pi} \widehat{f}(\pi)_{ji} \pi_{ij}(x),$$

which can also be written in a compact form

$$f(x) = \sum_{\pi \in \widehat{G}_0} d_\pi \mathrm{Tr}(\widehat{f}(\pi)\pi(x)).$$

For the future reference we note that with these conventions the matrix $\pi(x)\pi(x)^*$ is diagonal matrix with first k_π diagonal entries equal to one and others are equal to zero. Therefore, we have $\sum_{i=1}^{d_\pi} \sum_{j=1}^{k_\pi} |\pi(x)_{ij}|^2 = \mathrm{Tr}(\pi(x)\pi(x)^*) = k_\pi^{\frac{1}{2}}$. The ℓ^p-spaces on $\widehat{G}_0$ can be defined similar to the spaces $\ell^p(\widehat{G})$ defined in [25] (see also [12]). First, for the space of Fourier coefficients of functions on G/K we set

$$\Sigma(G/K) = \{\sigma : \pi \mapsto \sigma(\pi) \in \mathbb{C}^{d_\pi \times d_\pi} : [\pi] \in \widehat{G}_0, \ \sigma(\pi)_{ij} = 0 \, \mathrm{for} \, i > k_\pi\}. \tag{1}$$

Now, we define the Lebesgue spaces $\ell^p(\widehat{G}_0) \subset \Sigma(G/K)$ by the condition

$$\|\sigma\|_{\ell^p(\widehat{G}_0)} := \left(\sum_{[\pi] \in \widehat{G}_0} d_\pi k_\pi^{p(\frac{1}{p} - \frac{1}{2})} \|\sigma(\pi)\|_{HS}^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \tag{2}$$

and

$$\|\sigma\|_{\ell^\infty(\widehat{G}_0)} := \sup_{[\pi] \in \widehat{G}_0} k_\pi^{-\frac{1}{2}} \|\sigma(\pi)\|_{HS}.$$

The following embedding properties hold for these spaces:

$$\ell^{p_1}(\widehat{G}_0) \subset \ell^{p_2}(\widehat{G}_0) \text{ and } \|\sigma\|_{\ell^{p_2}(\widehat{G}_0)} \leq \|\sigma\|_{\ell^{p_1}(\widehat{G}_0)}, \ \ 1 \leq p_1 \leq p_2 \leq \infty.$$

The Hausdorff-Young inequality for Lebesgue spaces on compact homogeneous manifolds is proved in [18] which is stated in the following theorem.

Theorem 2.1. *Let G/K be a compact homogeneous manifold with normalized Haar measure μ and let $f \in L^p(G/K)$ for $1 \leq p \leq 2$. Suppose $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\|\hat{f}\|_{\ell^q(\widehat{G}_0)} \leq \|f\|_{L^p(G/K)}. \tag{3}$$

Here we would like to mention that there is a well-established theory of another family of Lebesgue spaces ℓ^p_{sch} on $\widehat{G}_0$ defined using Schatten p-norm $\|\cdot\|_{S^p}$ instead of Hilbert-Schmidt norm $\|\cdot\|_{HS}$ on the space of $(d_\pi \times d_\pi)$-dimensional matrices. Indeed, the space $\ell^p_{\text{sch}}(\widehat{G}_0) \subset \Sigma(G/H)$ is defined by the norm

$$\|\sigma\|_{\ell^p_{\text{sch}}(\widehat{G}_0)} := \left(\sum_{[\pi]\in \widehat{G}_0} d_\pi \|\sigma(\pi)\|^p_{S^p} \right)^{\frac{1}{p}} \quad \sigma \in \Sigma(G/K),\ 1 \leq p < \infty, \tag{4}$$

and

$$\|\sigma\|_{\ell^\infty_{\text{sch}}(\widehat{G}_0)} := \sup_{[\pi]\in\widehat{G}_0} \|\sigma(\pi)\|_{\mathcal{L}(\mathcal{H}_\pi)} \quad \sigma \in \Sigma(G/K).$$

The following proposition presents the relation between both norms on $\ell^p(\widehat{G}_0)$.

Proposition 2.2. *For $1 \leq p \leq 2$, we have the following continuous embeddings as well as the estimates: $\ell^p(\widehat{G}_0) \hookrightarrow \ell^p_{sch}(\widehat{G}_0)$ and $\|\sigma\|_{\ell^p_{sch}(\widehat{G}_0)} \leq \|\sigma\|_{\ell^p(\widehat{G}_0)}$ for all $\sigma \in \Sigma(G/K)$. For $2 \leq p \leq \infty$, we have $\ell^p_{sch}(\widehat{G}_0) \hookrightarrow \ell^p(\widehat{G}_0)$ and $\|\sigma\|_{\ell^p(\widehat{G}_0)} \leq \|\sigma\|_{\ell^p_{sch}(\widehat{G}_0)}$ for all $\sigma \in \Sigma(G/K)$.*

Proof. The proof of proposition can be found in [8]. □

The space $\ell^p(\widehat{G}_0)$ and the Hausdorff-Young inequality for it become useful for convergence of Fourier series and characterization of Gevrey-Roumieu ultradifferentiable functions and Gevrey-Beurling ultradifferentiable functions on compact homogeneous manifolds [7]. For more detail on Fourier analysis on compact homogeneous manifolds we refer to [26, 18, 5, 7].

2.2. **Basics of Orlicz spaces.** For basics of Orlicz spaces one can refer to excellent monographs [27, 20, 22] and articles [19, 21, 23, 13]. However we present a few definitions and results here in the form we need.

A non-zero convex function $\Phi : [0,\infty) \to [0,\infty]$ is called a *Young function* if $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$. For any given Young function Φ the *complimentary function* Ψ of Φ is given by

$$\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\} \quad (y \geq 0),$$

which is also a Young function. If Ψ is the complementary function of Φ then Φ is the complimentary function of Ψ; the pair (Ψ, Φ) is called a *complementary pair*. In fact, a complementary pair of Young functions satisfies

$$xy \leq \Phi(x) + \Psi(y) \;\; (x, y \geq 0).$$

If a complimentary pair of Young functions (Φ, Ψ) satisfies $\Phi(1) + \Psi(1) = 1$ then the pair (Φ, Ψ) is called a *normalized complimentary pair*.

Let G/K be a compact homogeneous manifold with a left Haar measure μ. Denote the set of all complex valued μ-measurable functions on G/K by $L^0(G/K)$. Given a Young function Φ, the *modular function* $\rho_\Phi : L^0(G/K) \to \mathbb{R}$ is defined by $\rho_\Phi(f) := \int_{G/K} \Phi(|f|)\, d\mu$ and the *Orlicz space* is defined by

$$L^\Phi(G/K) := \left\{ f \in L^0(G/K) : \rho_\Phi(af) < \infty \text{ for some } a > 0 \right\}.$$

Then the Orlicz space is a Banach space with respect to the norm $N_\Phi(\cdot)$, called as *Luxumburg norm*, defined by

$$N_\Phi(f) := \inf \left\{ \lambda > 0 : \int_{G/K} \Phi\left(\frac{|f|}{\lambda}\right) d\mu \leq \Phi(1) \right\}.$$

One can also define the norm $\|\cdot\|_\Phi$, called *Orlicz norm* on $L^\Phi(G/K)$ by

$$\|f\|_\Phi := \sup \left\{ \int_{G/K} |fv| d\mu : \int_{G/K} \Psi\left(|v|\right) d\mu \leq \Phi(1) \right\}.$$

These two norms are equivalent: $\Phi(1)N_\Phi(\cdot) \leq \|\cdot\|_\Phi \leq 2N_\Phi(\cdot)$. Also, it is known that $N_\Phi(f) \leq 1$ if and only if $\rho_\Phi(f) \leq 1$. We recover the Lebesgue spaces L^p, $1 \leq p < \infty$, by considering the Young function $\Phi(x) = \frac{x^p}{p}$. The complementary young function Ψ corresponding to Φ is given by $\Psi(x) = \frac{x^q}{q}$, where $q = \frac{p}{p-1}$. Other examples of Young functions are $e^x - x - 1$, $\cosh x - 1$ and $x^p \ln(x)$.

The following Hölder inequality holds for Orlicz spaces: for any complementary pair (Φ, Ψ) and for $f \in L^\Phi(G/K),\ g \in L^\Psi(G/K)$, we have

$$\int_{G/K} |fg|\, d\mu \leq \min\{N_\Phi(f)\|g\|_\Psi,\ \|f\|_\Phi N_\Psi(g)\}.$$

If (Φ, Ψ) is a normalized complimentary pair of Young functions then the above Hölder inequality becomes [20, P. 58 and P. 62] (see also [22, P. 27]):

$$\left| \int_{G/K} fg\, d\mu \right| \leq N_\Phi(f)\, N_\Psi(g). \tag{5}$$

Let $C(G/K)$ denote the space of complex valued continuous functions on G/K. It can be easily proved that $L^\Phi(G/K) \subset L^1(G/K)$ (for example see [13]) as G/K is compact. Therefore, the closure of $C(G/K)$ inside $L^\Phi(G/K)$ denoted by $M^\Phi(G/K)$ is same as $L^\Phi(G/K)$ which is not the case in general. A Young function Φ satisfies the Δ_2 condition if there exist a constant $C > 0$ and $x_0 \geq 0$ such that $\Phi(2x) \leq C\Phi(x)$ for all $x \geq x_0$. In this case we write $\Phi \in \Delta_2$. If $\Phi \in \Delta_2$, then it follows that $(L^\Phi)^* = L^\Psi$. If, in addition, $\Psi \in \Delta_2$, then the Orlicz space L^Φ is a reflexive Banach space.

To define the space $\ell^\Phi(\widehat{G}_0)$ we follow the similar construction as in ℓ^p. The Orlicz space $\ell^\Phi(\widehat{G}_0) \subset \Sigma(G/K)$ is defined by the norm

$$N_\Phi(\sigma) = \inf \left\{ \lambda > 0 : \sum_{\pi \in \widehat{G}_0} \Phi\left(\frac{k_\pi^{-\frac{1}{2}} \|\sigma(\pi)\|_{HS}}{\lambda} \right) k_\pi d_\pi \leq \Phi(1) \right\}. \tag{6}$$

The partial order $\prec$ on th set of all Young functions is defined as: $\Phi_1 \prec \Phi_2$ whenever $\Phi_1(ax) \leq b\Phi_2(x)$ for $|x| \geq x_0 > 0$ and $\Phi_2(cx) \leq d\Phi_1(x)$ for all $|x| \leq x_1$, where a, b, c, d, x_0 and x_1 are fixed positive constants independent of x. In particular, for L^p-spaces with $p \geq 1$ we can see that $a = b = c = d = 1, x_1 \geq 1$ and $x_0 \geq 1$. With the help of this ordering we can define inclusion relation in Orlicz spaces: if Φ_1, Φ_2 are continuous Young functions and $\Phi_1 \prec \Phi_2$ then $L^{\Phi_2}(G/K) \subset L^{\Phi_1}(G/K)$ and $N_{\Phi_1}(\cdot) \leq \alpha\, N_{\Phi_2}(\cdot)$ for some $\alpha > 0$.

Also, for $\widehat{G}_0$, the space $L^\Phi(\widehat{G}_0)$ becomes $\ell^\Phi(\widehat{G}_0)$ under the identification of a matrix valued function $\sigma \in \ell^\Phi(\widehat{G}_0)$ with a non-negative function $\widetilde{\sigma}(\pi)$ defined on $\widehat{G}_0$ by $\widetilde{\sigma}(\pi) := k_\pi^{-\frac{1}{2}} \|\sigma(\pi)\|_{HS}$ and in this case $\Phi_1 \prec \Phi_2$ implies that $\ell^{\Phi_1}(\widehat{G}_0) \subset \ell^{\Phi_2}(\widehat{G}_0)$ and $N_{\Phi_2}(\cdot) \leq \beta\, N_{\Phi_1}(\cdot)$ for some $\beta > 0$. The following result is well-known (see [22, Lemma 1, p. 209]).

Lemma 2.3. *Let (Φ_i, Ψ_i), $i = 1, 2$ be complementary pairs of continuous Young functions and let $\Phi_1 \prec \Phi_2$. Then $\Psi_2 \prec \Psi_1$.*

3. The Main result

Throughout this section, we assume that G/K is a compact homogeneous manifold and $\widehat{G}_0$ be the set of type I irreducible inequivalent continuous representations of G for our convenience. From now onwards, we assume that the pair of complimentary continuous Young functions (Φ, Ψ) is a normalized pair. Note that continuity of a Young function guaranties the existence of its derivative [20, Corollary 2]. Also, since Φ is a positive continuous convex function on $[o,\infty)$, it is increasing.

For $f \in L^\Phi(G/K)$ define $F_f : \widehat{G}_0 \to [0,\infty)$ by

$$F_f^2(\pi) = \sum_{i=1}^{d_\pi}\sum_{j=1}^{k_\pi} \frac{|\widehat{f}(\pi)_{i,j}|^2}{k_\pi} = \frac{\mathrm{Tr}(\widehat{f}(\pi)^*\widehat{f}(\pi))}{k_\pi} \text{ for } \pi \in \widehat{G}_0. \tag{7}$$

Now, define the gauge norm of F_f by

$$N_\Phi(F_f) := \inf\left\{\lambda > 0 : \sum_{\pi\in\widehat{G}_0} \Phi\left(\frac{F_f(\pi)}{\lambda}\right) k_\pi d_\pi \le \Phi(1)\right\}. \tag{8}$$

We note here that for $f \in L^\Phi(G/K)$, $\widehat{f} : \widehat{G}_0 \to \bigcup_{\pi\in\widehat{G}_0} \mathbb{C}^{d_\pi\times d\pi}$. So, the norm $N_\Phi(F_f)$ is same as the norm $N_\Phi(\widehat{f})$ on the non-commutative Orlicz space $\ell^\Phi(\widehat{G}_0)$ as $F_f(\pi) = k_\pi^{-\frac{1}{2}}\|\widehat{f}(\pi)\|_{HS}$. The space $(\ell^\Phi(\widehat{G}_0), N_\Phi(\cdot))$ is a Banach space. If Φ is continuous then there exists $\lambda_0 := N_\Phi(F_f)$ such that inequality in (8) is an equality with $\lambda = \lambda_0$ on the left, i.e., $\sum_{\pi\in\widehat{G}_0} \Phi\left(\frac{F_f(\pi)}{\lambda_0}\right) d_\pi\, k_\pi = \Phi(1)$.

The proof of our main result depends on the following key lemma which is an extension of an important inequality in the case of L^p due to Hardy and Littlewood [9].

Lemma 3.1. *Let G/K be a compact homogeneous manifold with the normalized measure μ and let (Φ,Ψ) be a pair of continuous normalized Young functions such that*

(i) $\Phi \prec \Phi_0$, *where* $\Phi_0(t) = \frac{1}{2}|t|^2$,

(ii) $\Psi'(t) \le c_0\, t^p$, $\forall\, t \ge 0$, *for some* $p \ge 1$, *and for some* $c_0 > 0$.

Suppose Λ is a finite subset of $\widehat{G}_0$. Define $f_\Lambda : G/K \to \mathbb{C}$ by

$$f_\Lambda(x) := \sum_{\pi\in\Lambda} d_\pi \sum_{i=1}^{d_\pi}\sum_{j=1}^{k_\pi} c^\pi_{i,j}\pi_{i,j}(x), \tag{9}$$

where $c^\pi_{i,j} \in \mathbb{C}$. If $F_\Lambda = F_f$ is as in (7) with $f = f_\Lambda$, then

$$N_\Psi(F_\Lambda) \le \bar{r}_0\, N_\Phi(f_\Lambda), \tag{10}$$

where $\bar{r_0} > 0$ depends only on Φ and the ordering $\prec$.

Proof. Let Λ be a finite subset of $\widehat{G}_0$. If f_Λ is as in the statement of the lemma, $\hat{f}_\Lambda(\pi)_{i,j} = c^\pi_{i,j}\,\chi_\Lambda(\pi)$, where χ_Λ is characteristic function of the subset Λ in $\widehat{G}_0$. For simplicity of expressions, we set $S_\Phi(f_\Lambda) = N_\Phi(F_{f_\Lambda})$. For a non-zero $f \in L^\Phi(G/K)$, the Fourier coefficients

$\hat{f}(\pi)_{i,j}$ of f are denoted by $\tilde{c}^{\pi}_{i,j}$. Let $\tilde{f}_\Lambda$ be the function f_Λ given by (9) with $c^{\pi}_{i,j} = \tilde{c}^{\pi}_{i,j}$. Following an idea of Hardy and Littlewood [9], we define,

$$M = M_\Phi(\Lambda) := \sup\left\{ \frac{S_\Psi(\tilde{f}_\Lambda)}{N_\Phi(f)} : f \neq 0 \right\}. \tag{11}$$

We prove the lemma in three steps.

STEP I. $M < \infty$.

Since M is described by a ratio of norms, without loss of generality we assume that $S_\Psi(\tilde{f}_\Lambda) = 1$ to find a bound on M. It follows using continuity of Ψ and the definition of the gauge norm (with $\lambda_0 = 1$, see the discussion in the paragraph below the equation (8)) that

$$\sum_{\pi \in \widehat{G}_0} \Psi(F_{\tilde{f}_\Lambda}(\pi)) k_\pi d_\pi = \Psi(1). \tag{12}$$

Since $k_\pi \geq 1$, $F_{\tilde{f}_\Lambda}(\pi) = 0$ for $\pi \in \widehat{G}_0 \backslash \Lambda$ and $0 < \Psi(1) < 1$, at least one term on the left hand side of (12) is greater than or equal to $\frac{\Psi(1)}{\#(\Lambda)}$, where $\#(\Lambda)$ is the cardinality of Λ. If this term is for $\pi = \pi' \in \Lambda$ then we have

$$0 < \Psi^{-1}\left[\frac{\Psi(1)}{\#(\Lambda)\, k_{\pi'}\, d_{\pi'}} \right] \leq F_{\tilde{f}_\Lambda}(\pi'). \tag{13}$$

Next,

$$\begin{aligned} F_{\tilde{f}_\Lambda}(\pi) = \left(\frac{1}{k_\pi} \sum_{i=1}^{d_\pi} \sum_{j=1}^{k_\pi} |\tilde{c}^{\pi}_{i,j}|^2 \right)^{\frac{1}{2}} &\leq \frac{1}{k_\pi^{\frac{1}{2}}} \sum_{i=1}^{d_\pi} \sum_{j=1}^{k_\pi} |\tilde{c}^{\pi}_{i,j}| \\ &\leq \frac{1}{k_\pi^{\frac{1}{2}}} \int_{G/K} |f(x)| \sum_{i=1}^{d_\pi} \sum_{j=1}^{k_\pi} |\pi(x)_{i,j}|\, d\mu(x). \end{aligned}$$

Using Cauchy-Schwartz inequality, we get

$$F_{\tilde{f}_\Lambda}(\pi) \leq \frac{1}{k_\pi^{\frac{1}{2}}} \int_{G/K} |f(x)| \left(\sum_{i=1}^{d_\pi} \sum_{j=1}^{k_\pi} |\pi(x)_{i,j}|^2 \right)^{\frac{1}{2}} d_\pi^{\frac{1}{2}} k_\pi^{\frac{1}{2}}\, d\mu(x).$$

Using $\mathrm{Tr}(\pi(x)\pi(x)^*) = k_\pi$, we get

$$F_{\tilde{f}_\Lambda}(\pi) \leq \int_{G/K} |f(x)|\, d_\pi^{\frac{1}{2}} k_\pi^{\frac{1}{2}}\, d\mu(x) = d_\pi^{\frac{1}{2}} k_\pi^{\frac{1}{2}} \int_{G/K} |f(x)|\, d\mu(x). \tag{14}$$

Now, by the Hölder inequality (5)

$$F_{\tilde{f}_\Lambda}(\pi) \leq d_\pi^{\frac{1}{2}} k_\pi^{\frac{1}{2}}\, N_\Phi(f). \tag{15}$$

Now by combining (13) and (15), we have

$$\frac{1}{N_\Phi(f)} \leq \left(\frac{d_{\pi'}^{\frac{1}{2}} k_{\pi'}^{\frac{1}{2}}}{\Psi^{-1}\left(\frac{\Psi(1)}{\#(\Lambda) k_{\pi'} d_{\pi'}}\right)} \right) < \infty. \tag{16}$$

Since the right hand side of (16) is independent of f, we have $M < \infty$.

STEP II. M is independent of Λ.

For f_Λ as in (9), define g by

$$g(x) := \Psi' \left(\frac{|f_\Lambda(x)|}{N_\Psi(f_\Lambda)} \right) \operatorname{sgn}(f_\Lambda(x)), \tag{17}$$

where $\operatorname{sgn}(z) = z/|z|$ for $z \neq 0$ and 0 for $z = 0$. Since Φ is continuous and $\mu(G/K) = 1$ it follows from the discussion in [27, p. 175] (see also [22, Chapter VI]) that $N_\Phi(g) = 1$, and that the Hölder's inequality (5) is an equality for the functions f_Λ and g, that is,

$$N_\Psi(f_\Lambda) = N_\Phi(g) N_\Psi(f_\Lambda) = \left| \int_{G/K} g(x) f_\Lambda(x)\, d\mu(x) \right|.$$

Using the Parseval formula, we have

$$\begin{aligned}
N_\Psi(f_\Lambda) &= \left| \sum_{\pi\in\Lambda} d_\pi \sum_{i=1}^{d_\pi} \sum_{j=1}^{k_\pi} \hat{g}(\pi)_{i,j} \overline{\hat{f}(\pi)_{i,j}} \right| \quad \text{(by using } \sum_l a_l b_l \leq \left(\sum a_l^2\right)^{\frac{1}{2}} \left(\sum b_l^2\right)^{\frac{1}{2}}\text{)} \\
&\leq \sum_{\pi\in\Lambda} d_\pi \left(\sum_{i=1}^{d_\pi} \sum_{j=1}^{k_\pi} |\hat{f}(\pi)_{ij}|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{d_\pi} \sum_{j=1}^{k_\pi} |\hat{g}(\pi)_{ij}|^2 \right)^{\frac{1}{2}} \\
&= \sum_{\pi\in\Lambda} k_\pi d_\pi\, F_{f_\Lambda}(\pi)\, F_{\tilde{g}_\Lambda}(\pi) \quad \text{(by Hölder's inequality)} \\
&\leq N_\Phi(F_{f_\Lambda}) N_\Psi(F_{\tilde{g}_\Lambda}) = S_\Phi(f_\Lambda) S_\Psi(\tilde{g}_\Lambda).
\end{aligned}$$

By STEP I, we know that $S_\Psi(\tilde{g}_\Lambda) \leq M N_\Phi(g) = M$ and therefore

$$\frac{N_\Psi(f_\Lambda)}{S_\Phi(f_\Lambda)} \leq S_\Psi(\tilde{g}_\Lambda) \leq M. \tag{18}$$

Note that $M \geq 1$. In fact, for $f = 1$, we note that $N_\Phi(1) = 1$ as the measure μ is normalized. By continuity of Φ,

$$\sum_{\pi\in\widehat{G}_0} \Phi \left(\frac{F_{\tilde{f}_\Lambda}(\pi)}{S(\tilde{f}_\Lambda)} \right) k_\pi d_\pi = \Phi(1).$$

Now, by choosing $\pi' \in \Lambda$ such that (13) holds, we have

$$S_\Psi(\tilde{f}_\Lambda) \geq \frac{1}{\left[\Phi^{-1}\left(\frac{\Phi(1)}{k_{\pi'} d_{\pi'}}\right)\right]} = r_0 \text{ (say)}. \tag{19}$$

Note that $r_0 \geq 1$. Indeed, since Φ is increasing and $0 < \Phi(1) < 1$, we have $r_0 = \frac{1}{\left[\Phi^{-1}\left(\frac{\Phi(1)}{k_{\pi'}d_{\pi'}}\right)\right]} \geq \frac{1}{\left[\Phi^{-1}\left(\frac{1}{k_{\pi'}d_{\pi'}}\right)\right]} \geq 1$. Therefore, $M \geq \frac{S_\Psi(\tilde{f}_\Lambda)}{N_\Phi(f)} \geq r_0 \geq 1$. By continuity of norms, there is a function f_Λ such that $M = \frac{N_\Psi(f_\Lambda)}{S_\Phi(f_\Lambda)}$. Consequently, from (18) we get $S_\Phi(\tilde{g}_\Lambda) = M$. We fix this f_Λ and set g as in (17) for the remaining part of this step.

Let us denote S_Φ by S_2 for the Young function $\Phi(x) = \frac{|x|^2}{2}$. Using the Bessel inequality, we get

$$S_2^2(\tilde{g}_\Lambda) = \sum_{\pi\in\Lambda} d_\pi^2 \sum_{i=1}^{d_\pi}\sum_{j=1}^{k_\pi} |\hat{g}(\pi)_{i,j}|^2 \leq \int_{G/K} |g|^2\,d\mu \leq N_\Psi(g^2), \tag{20}$$

where the last inequality follows from Hölder's inequality (since $N_\Phi(1) = 1$). Set $\Psi_1(t) = \Psi(t^2)$. Then Ψ_1 is a Young function satisfying $\Psi \prec \Psi_1$. Since g is a bounded function, by setting $a^2 = N_\Psi(g^2)(< \infty)$, we get

$$\Psi_1(1) = \Psi(1) = \int_{G/K} \Psi\left(\frac{|g|^2}{a^2}\right) d\mu(x) = \int_{G/K} \Psi_1\left(\frac{|g|}{a}\right) d\mu(x),$$

whence $a = N_{\Psi_1}(g)$. Thus, by (20), we have

$$S_2(\tilde{g}_\Lambda) \leq N_{\Psi_1}(g). \tag{21}$$

Now, we find an absolute bound for M. If $a = N_{\Psi_1}(g)$ then, by definition, there exists $b_0 > 0$ such that

$$1 = \int_{G/K} \Psi_1\left(\frac{g}{ab_0}\right) d\mu(x) = \int_{G/K} \Psi_1\left[\frac{1}{ab_0}\Psi'\left(\frac{|f_\Lambda|}{N_\Psi(f_\Lambda)}\right)\right] d\mu(x).$$

Since $\Psi'(t) \leq c_0\, t^p$ for some $p \geq 1$, we get

$$1 \leq \int_{G/K} \Psi_1\left[\frac{c_0}{b_0 a}\left(\frac{|f_\Lambda|}{N_\Psi(f_\Lambda)}\right)^p\right] d\mu(x) = \int_{G/K} \Psi_2\left[b_1\frac{|f_\Lambda|}{N_\Psi(f_\Lambda)}\right] d\mu(x), \tag{22}$$

where $\Psi_2(t) = \Psi_1(t^p)$ and $b_1 = \left(\frac{c_0}{b_0 a}\right)^{\frac{1}{p}} > 0$. Thus Ψ_2 is a Young function satisfying $\Psi \prec \Psi_1 \prec \Psi_2$. Then (22) gives the following important inequality: there exists a constant b_2 depending only on Ψ_2 and independent of f_Λ, such that

$$N_{\Psi_2}\left(\frac{b_1 f_\Lambda}{N_\Psi(f_\Lambda)}\right) \geq b_2 > 0, \tag{23}$$

(see [20, Theorem 2, Chapter III]). Since $N_{\Psi_2}(\cdot)$ is a norm, by the definition of b_1, we get

$$\left[\frac{N_{\Psi_2}(f_\Lambda)}{N_\Psi(f_\Lambda)}\right]^p \geq b_2^p\left(\frac{b_0}{c_0}\right) N_{\Psi_1}(g) = b_3 N_{\Psi_1}(g), \tag{24}$$

where $b_3 = b_2^p \frac{b_0}{c_0}$. Since we have $\Psi_0 \prec \Psi \prec \Psi_1 \prec \Psi_2$, by Lemma 2.3, $\Phi_2 \prec \Phi_1 \prec \Phi \prec \Phi_0$ so that $\ell^{\Phi_2} \subset \ell^{\Phi_1} \subset \ell^{\Phi} \subset \ell^{\Phi_0} = \ell^2 \subset \ell^{\Psi} \subset \ell^{\Psi_1} \subset \ell^{\Psi_2}$. Now, for some $r_2 > 0$, we get the following inequalities:

$$1 \leq M = M_\Phi = S_\Psi(\tilde{g}_\Lambda) \quad \leq \quad r_2 S_2(\tilde{g}_\Lambda) \leq \frac{r_2}{b_3}\left[\frac{N_{\Psi_2}(f_\Lambda)}{N_\Psi(f_\Lambda)}\right]^p \leq \frac{r_2}{b_3}\left[\frac{N_{\Psi_3}(f_\Lambda)}{N_\Psi(f_\Lambda)}\right]^p, \tag{25}$$

where $\Psi_3(t) = \frac{c_0}{p+1} t^{2p(p+1)}$, so $\Psi_2 \prec \Psi_3$, (Since $\Psi'(t) \leq c_0\, t^p$ so $\Psi(t) = \frac{c_0}{p+1} t^{p+1}$ and therefore $\Psi_2(t) = \Psi_1(t^p) = \Psi(t^{2p}) \leq \frac{c_0}{p+1} t^{2p(p+1)} = \Psi_3(t)$). Let Φ_3 be the complementary function of Ψ_3. Then $S_{\Phi_3}(f_\Lambda) \leq b_4 S_\Phi(f_\Lambda)$ for some $b_4 > 0$ depending only on Φ_3 and Φ only. Therefore, by (25), we have

$$1 \leq M = M_\Phi \quad \leq \quad \frac{r_2}{b_3}\left[\frac{M_{\Phi_3} S_{\Phi_3}(f_\Lambda)}{M_\Phi S_\Phi(f_\Lambda)}\right]^p \leq \frac{r_2}{b_3}\left[b_4 \frac{M_{\Phi_3}}{M_\Phi}\right]^p. \tag{26}$$

Hence,

$$1 \leq M_\Phi^{p+1} \leq r_3^p\, M_{\Phi_3}^p, \tag{27}$$

for some positive constant $r_3 > 0$ which depend only on Φ, Φ_2, Φ_3 and the ordering constants. But note that $L^{\Psi_3}(G/K) = L^{p'}(G/K)$, where $p' = 2p(p+1) \geq 2$. It follows from Theorem 2.1 that $M_{\Phi_3} \leq r_4 < \infty$, for a positive constant r_4 depending on c_0 and r. Therefore (27) gives

$$1 \leq M_\Phi \leq r_5 < \infty$$

where $r_5 = (r_3 r_4)^{\frac{p}{p+1}}$, which is independent of Λ.

STEP III. By setting $\bar{r}_0 = r_5$, equations (11) immediately give the required inequality in (10). Since Ψ_2 and Ψ_3 depend on the complimentary Young function Ψ of Φ, all the constants involve depend on Φ and the ordering $\Phi \prec \Phi_0$, and perhaps on c_0 and p. This completes the proof of the lemma. □

Now, we are ready to prove the Hausdorff-Young inequality for Orlicz spaces on compact homogeneous manifolds.

Theorem 3.2. *Let G/K be a compact homogeneous manifold with the normalized measure μ and let (Φ, Ψ) be a pair of continuous normalized Young functions such that*

(i) $\Phi \prec \Phi_0$, *where* $\Phi_0(t) = \frac{1}{2}t^2$,

(ii) $\Psi'(t) \leq c_0\, t^p$, $t \geq 0$, *for some* $p \geq 1$, *where c_0 is a positive constant. If $f \in L^\Phi(G/K)$ then there is $r_0 \geq 1$ such that*

$$N_\Psi(F_f) \leq r_0\, N_\Phi(f).$$

Proof. Let $f \in L^\Phi(G/K)$ and let Λ be a finite subset of $\widehat{G}_0$. Suppose $\tilde{f}_\Lambda$ is given by (9) where $c_{i,j}^\pi = \hat{f}(\pi)_{i,j}$. The set $\{\tilde{f}_\Lambda : \Lambda \subset \widehat{G}_0\}$ is the collection of all simple functions which, in particular, contains the set of all matrix coefficients of type I representation of G and therefore it is dense in $L^\Phi(G/K) \subset L^2(G/K)$. We have $\lim_{\Lambda \subset \widehat{G}_0} N_\Phi(f - \tilde{f}_\Lambda) = 0$ and therefore, $\tilde{c}_{ij}^\pi = \widehat{f}_\Lambda(\pi)_{ij} \to \widehat{f}(\pi)_{ij} = \tilde{c}_{ij}^\pi$. Consequently, we have

$$\lim_{\Lambda \subset \widehat{G}_0} N_\Psi(F_{\tilde{f}_\Lambda}) = N_\Psi(F_f),$$

where the limit as Λ varies in $\widehat{G}_0$ is taken using the partial order defined by inclusion of subsets of $\widehat{G}_0$. Now, by using this with the inequality $N_\Psi(F_{f_\Lambda}) \leq \bar{r}_0\, N_\Phi(f_\Lambda)$ of (10) of Lemma 3.1, we get $N_\Psi(F_f) \leq r_0\, N_\Phi(f)$, where $r_0 = \bar{r}_0$. This completes the proof. □

Remark 1. (i) For the Lebesgue space, we have that constant $r_0 = 1$ (see Theorem 2.1); however it is clear from the proof of Lemma 3.1 that the constant r_0 in Theorem 3.2 is greater or equal to 1.

(ii) We give an example of a pair of Young functions (Φ, Ψ) satisfying the condition of Theorem 3.2 such that the corresponding Orlicz spaces are not Lebesgue spaces. This example is taken from the [22] which was originally discovered by Riordan [24]. For $1 < p < 2$, we define function $\Phi(x) = \frac{x^p}{p} \ln(x) \ln(\ln x)$ for $x \geq x_0$ and $\Phi(x) = \frac{x^p}{p} \ln(\frac{1}{x}) \ln(\ln \frac{1}{x})$ for $x \leq x_1$. We choose x_0 large enough and x_1 small enough so that $\Phi(x)$ gives a convex function by joining the points $(x_1, \Phi(x_1))$ and $(x_0, \Phi(x_0))$ by a straight line. If q denotes the Lebesgue conjugate of p, that is, $q = \frac{p}{p-1}$ then the function Ψ is given by $\Psi(x) = \frac{x^q}{q} L(x)^{\frac{q}{p}}$, where $L(x) = \ln(x) \ln(\ln x)$ for $x \geq x_0'$ and $\ln(\frac{1}{x}) \ln(\ln \frac{1}{x})$ for $x \leq x_1'$. Then we again choose x_0'' large enough and x_1' small enough so that $\Phi(x)$ gives a convex function by joining the points $(x_1, \Phi(x_1))$ and $(x_0, \Phi(x_0))$ by a straight line. Although, Ψ may not be the complementary function but it is equivalent to the complimentary function.

(iii) For $1 < p \leq 2$, if $\Phi(x) = \frac{x^p}{p}$ and $\Psi(x) = \frac{x^q}{q}$ with $q = \frac{p}{p-1}$, then using the above method we can get usual Hausdorff-Young inequality for Lebesgue spaces on compact homogeneous manifolds. In fact, this method was used to solve maximal problem in the case for compact groups by Hirschman [12] using the same norm on Lebesgue spaces as we considered in this paper .

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References

[1] R. Akylzhanov, E. Nursultanov and M. Ruzhansky, Hardy-Littlewood-Paley inequalities and Fourier multipliers on $SU(2)$, *Studia Mathematica* 234 (2016) 1-29.
[2] R. Akylzhanov, E. Nursultanov and M. Ruzhansky, Hardy-Littlewood, Hausdorff-Young-Paley inequalities, and $L^p - L^q$ Fourier multipliers on compact homogeneous manifolds, *Journal of Mathematical Analysis and Applications* 479(2) (2019) 1519-1548. arXiv:1504.07043
[3] R. Akylzhanov and M. Ruzhansky, Fourier multipliers and group von Neumann algebras, *Comptes Rendus Mathematique* 354(8) (2016) 766-770.
[4] R. Akylzhanov and M. Ruzhansky, L^p-L^q multipliers on locally compact groups, *Journal of Functional Analysis* (2019). DOI: https://doi.org/10.1016/j.jfa.2019.108324
[5] D. Applebaum, *Probability on Compact Lie Groups*, With a foreword by Herbert Heyer. Probability Theory and Stochastic Modelling, vol. 70. Springer, Cham (2014).
[6] S. D. Chatterji, Remarks on the Hausdorff-Young inequality, *Enseign. Math.(2)* 46(3-4), 339348, (2000).
[7] A. Dasgupta and M. Ruzhansky, Gevrey functions and ultradistributionson compact Lie groups and homogeneous spaces, *Bull. Sci. Math.* 138(6), 756-782 (2014).
[8] V. Fischer and M. Ruzhansky, *Quantization on nilpotent Lie groups*, Progress in Mathematics, 314. Birkhuser/Springer, [Cham], (2016) xiii+557 pp.
[9] G. H. Hardy and J. E. Littlewood, Some new properties of Fourier constant, *Math. Annalen* 97 (1927) 159-209.
[10] E. Hewitt and K. A. Ross, Rearrangement of L^r Fourier series on compact abelian groups, *Proc. Lond. Math. Soc.*, 29(3) (1974) 519-540.
[11] E. Hewitt and K. A. Ross, *Abstract Harmonic analysis Vol. II: Structure and analysis for compact groups Analysis on locally compact Abelian groups.* Die Grundlehren der mathematischen Wissenschaften, Band 152 Springer-Verlag, New York-Berlin (1970).
[12] I. I. Hirschman, Jr., A maximal problem in harmonic analysis-II, *Pacific J. Math.*, 9, 525-540 (1959).

[13] V. Kumar, R. Sarma and N. S. Kumar, Orlicz spaces on hypergroups, *Publ. Math. Debrecen*, 94(1-2) (2019) 3147.

[14] V. Kumar and R. Sarma, The Hausdorff-Young inequality for Orlicz spaces on compact hypergroups, (to appear in) *Colloquium Mathematicum* (2019).

[15] V. Kumar, Orlicz spaces and amenability of hypergroups, *Bull. Iran. Math. Soc.* (2019). https://doi.org/10.1007/s41980-019-00310-7

[16] V. Kumar, Pseudo-differential operators on homogeneous spaces of compact and Hausdorff groups, *Forum Math.* 31(2), 275-282 (2019).

[17] R. A. Kunze, L^p Fourier transform on locally compact unimoduler groups, *Trans. Amer. Math. Soc.*, 89 (1958).

[18] E. Nursultanov, M. Ruzhansky and S. Tikhonov, Nikolskii inequality and Besov, Triebel-Lizorkin, Wiener and Beurling spaces on compact homogeneous manifolds, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 16(3), 981-1017 (2016).

[19] M.M. Rao, Extension of the Hausdorff-Young Theorem, *Israel J. Math.*, 6 (1967) 133-149.

[20] M.M. Rao and Z.D. Ren, *Theory of Orlicz spaces*, Dekker, New York (1991).

[21] M. M. Rao, Convolution of vector fields-II : random walk models, *Nonlinear Analysis*, 47 (2001) 3599-3615.

[22] M. M. Rao and Z. D. Ren, *Applications of Orlicz spaces* Monographs and Textbooks in Pure and Applied Mathematics, 250, Marcel Dekker Inc., New York, (2002).

[23] M. M. Rao, Convolutions of vector fields-III: Amenability and spectral properties, *Real and stochastic analysis, Trends Math., Birkhuser Boston, Boston, MA*, (2004) 375-401.

[24] W. J. Riordan, *On the interpolations of operator,*, Ph. D. Thesis, University of Chicago, (1957).

[25] M. Ruzhansky and V. Turunen, *Pseudo-differential Operators and Symmetries: Background Analysis and Advanced Topics*. Birkhaüser-Verlag, Basel (2010).

[26] N. Ja. Vilenkin and A.U. Klimyk, *Representation of Lie Groups and Special Functions. Vol. 1. Simplest Lie Groups, Special Functions and Integral Transforms,* Kluwer Academic Publishers Group, Dordrecht (1991).

[27] A. Zygmund, *Trigonometry series* Vol. *I* and *II* combined, third edition, with a foreword by Robert A. Fefferman, Cambridge mathematical library, Cambridge University Press, Cambridge, (2002).

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