RECONSTRUCTION AND QUANTIZATION OF RIEMANNIAN STRUCTURES

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Abstract. We use algebraic methods to obtain a Cartan-type formula $\nabla_\omega\eta = \frac{1}{2}(\delta(\omega\eta) - (\delta\omega)\eta + \omega\delta\eta + \omega \perp \mathrm{d}\eta + \mathrm{d}\omega \perp \eta + \mathrm{d}(\omega \perp \eta))$ for the Levi-Civita connection on a classical Riemannian manifold M in the direction of a 1-form ω (i.e. the usual Levi-Civita connection along the corresponding vector field via the metric). Here $\perp$ denotes a degree -2 bidirectional interior product built from the metric and δ is the divergence or codifferential. We also recover that δ obeys a 7-term relation making the exterior algebra into a Batalin-Vilkovisky algebra. These formulae arise naturally from a novel view of Riemannian structures as cocycles governing the central extension of the classical exterior algebra to a quantum one, motivated by ideas for quantum gravity. The approach also works when the initial exterior algebra is already quantum, allowing us to construct examples of quantum Riemannian structures, including quantum Levi-Civita connections, as cocycle data. Combining with the semidirect product of a differential graded algebra by the quantum differential algebra $\Omega(t, \mathrm{d}t)$ in one variable, we recover a differential quantisation of $M \times \mathbb{R}$ associated to any conformal Killing vector field on a Riemannian manifold M.

1. Introduction

We describe in this paper a novel approach to Riemannian geometry and its generalisation that is motivated from quantum gravity in the form of the following geometric question: can a Riemannian structure on a manifold M be usefully reconstructed from the algebraic properties of the divergence or codifferential δ on the exterior algebra $\Omega(M)$ of differential forms? This is not unlike the famous question of can a manifold be reconstructed from its Laplacian (the answer is no) or from its Dirac operator (the answer is yes, an observation at the heart of Connes 'spectral triple' approach to noncommutative geometry[11]). In our case this is not really in doubt and our starting point in Section 2 is to observe that indeed the failure of δ to commute with functions allows one to recover the metric (Lemma 2.2) after which the Levi-Civita connection is of course determined by the Koszul formula. What is less obvious and which we find is that the Levi-Civita connection and its properties have a direct expression in these terms in the style of the famous Cartan formula $\mathcal{L}_v = \lfloor_v \mathrm{d} + \mathrm{d}\lfloor_v$ for the Lie derivative on forms along a vector field v. Here $\lfloor_v$ denotes the usual interior product. In Theorem 2.8 we find a similar formula, as stated in the abstract, for $\nabla_\omega\eta$ along a 1-form ω. We work with forms but one can view ω as a vector field via the metric, i.e. we work with the 'index raised' version of the Levi-Civita connection. In fact we see two parts to it:

$$\nabla_\omega\eta - \nabla_\eta\omega = L_\delta(\omega, \eta)$$

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is the 'Leibnizator' measuring the failure of δ to be a derivation (as defined in (1.4) in the Preliminaries). It is known cf.[10] that this Leibnizator is closely related to the Schouten bracket of alternating multivector fields [9], so the above can be seen as something like an expression of zero torsion. The other ingredient is the inverse metric $(\ ,\ )$ on 1-forms extended to a degree -2 'product' on $\Omega(M)$ as a bi-graded-derivation, which we denote $\perp$ (see (2.10) for the precise definition). If either side is a 1-form then this is just a usual left or right-handed interior product albeit 'index raised' along 1-forms. Then

$$\nabla_\omega\eta+\nabla_\eta\omega=\omega\perp \mathrm{d}\eta+\mathrm{d}\omega\perp\eta+\mathrm{d}(\omega\perp\eta)$$

where the outer two terms could be thought of as a 'form Lie derivative' $\mathcal{L}_\omega$.

A further comment is that the Leibnizator of δ is itself a graded-derivation, which amounts in Corollary 2.5 to a 7-term triple product identity

$$\delta(\omega\eta\zeta)=(\delta(\omega\eta))\zeta-(\delta\omega)\eta\zeta-(-1)^{|\omega|}\omega(\delta\eta)\zeta+(-1)^{|\omega|}\omega\delta(\eta\zeta)$$
$$+(-1)^{(|\omega|-1)|\eta|}\eta\delta(\omega\zeta)-(-1)^{|\omega|+|\eta|}\omega\eta\delta\zeta$$

for all forms ω,η,ζ. This makes the exterior algebra into a Batalin-Vilkovisky (BV) algebra underlying the Schouten bracket as its associated Gerstenhaber algebra. This is a known observation [19, 21] on identifying alternating multivector fields there with differential forms via the metric. As a further modest application of our codifferential approach to Riemannian geometry, we show in Section 2.5 when $(\ ,\ )$ has an inverse, g, that

$$\mathrm{Ricci}=-\frac{1}{2}\Delta g$$

as announced in [14], where Δ extends canonically to 1-1-forms. Although such a view of Ricci is known in specially adapted Gaussian coordinates, this formula via the extended Hodge Laplacian $\Delta=\mathrm{d}\delta+\delta\mathrm{d}$ puts it on a coordinate-free footing and also better exhibits the sense in which the vacuum Einstein equation is like a wave equation. Also note that by focussing on forms, we only refer to δ and $(\ ,\ )$ without requiring the 1-1-form metric itself, which is potentially a generalisation useful for the degenerate case.

We believe that these results should be of interest to geometers in their own right, and once formulated they are not too hard to prove directly (as we illustrate in Section 2.5). We have found them, however, by means of an algebraic approach to geometry coming out of ideas for quantum gravity, as follows. Thus, it is now commonly accepted that quantum gravity effects mean that spacetime could be better modelled as an effectively noncommutative or 'quantum' one where the coordinate algebra A need not be commutative. In this case one can still do differential geometry and a common feature of several (but not all) approaches to such 'noncommutative geometry' is to express the differential structure by means of a differential graded algebra (DGA) $(\Omega(A),\mathrm{d})$ of 'quantum differential forms'. This is weaker than a classical exterior algebra on a manifold as it need not be 'graded-commutative'. It also need not be that Ω is generated by $A,\mathrm{d}A$, but if it is then we say that it is 'standard'. If it is standard and graded-commutative then we are basically in the classical case and we say that $\Omega(A)$ is of classical type. The Preliminaries provides more information, but suffice it to say that one can define a generalised metric as $g\in\Omega^1\otimes_A\Omega^1$ with inverse $(\ ,\ ):\Omega^1\otimes_A\Omega^1\to A$, typically quantum symmetric if $\wedge(g)=0$, and we can define what we mean by a quantum Levi-Civita bimodule connection, curvature etc. We adopt here a 'constructive approach' as featured in [5, 6, 15, 8] and related works and motivated by (but not limited to) quantum groups as key examples, in contrast to the approach of [11] and others coming from cyclic cohomology and K-homology.

Now consider the following question. If spacetime truly has a 'quantum differential structure' then it must formally recover the classical $\Omega(M)$ as the Planck scale parameter $\to 0$ in our effective description. Turning this around, what data controls the extension of the classical exterior algebra to a quantum one? The answer in the limited but precise formulation of the present paper turns out to be a Riemannian structure, i.e. Riemannian geometry and the above formulae for the Levi-Civita connection arise naturally as the data for a particularly simple class of 'central extensions' of the classical differential structure to a quantum one, i.e. from little other than the Leibniz rule and its interaction with non-commutativity. We use the term 'extension' here rather than 'quantisation' in a deformation sense since in practice there is typically an obstruction or 'quantum anomaly for differentiation'[4] that forces the quantum differential calculus to have a higher dimension if one wants to preserve (quantum) symmetries. In general, the data for the deformation-quantisation of classical differential structures in [7] needs, for associativity of $\Omega(A)$, that a certain Poisson-compatible connection is flat, which typically is not the case. There are thus two orthogonal resolutions to this obstruction: one is to move to 'nonassociative differential geometry' and the other is to absorb the anomaly in a higher dimension. Here we explore the second option, and to keep things simple we focus on the 'cleft' case were the coordinate algebra remains unchanged and the 'quantum' aspect appears in noncommutativity of functions with differentials.

This is the topic of Section 3 where, motivated by the above, we introduce a precise theory of cleft central extensions of a differential graded algebra by an additional graded-central 1-form θ' with $\mathrm{d}\theta' = 0$. We then show that cleft central extensions of the classical exterior algebra $(\Omega(M), \mathrm{d})$ correspond to a class of possibly degenerate metric-connection pairs where the metric-compatibility tensor and torsion are matched (Proposition 3.16). Within this theory of cleft central extensions, we consider those which are isomorphic to ones where d is not changed, which we call 'flat'. In the classical case, this lands us on the classical Levi-Cevita connection where the torsion is zero. Thus we put Riemannian geometry into a more general context where we now think of a metric-connection pair as equivalent to the extension or 'cocycle' data $(\Delta, [\![\ ,\ ]\!])$, where $[\![\ ,\ ]\!]$ encodes ∇ and the interior product, and the flat case corresponds to $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$ for some degree -1 map δ which becomes the codifferential. The 'homologically trivial' case where also $[\![\omega, \eta]\!] = L_\delta(\omega, \eta)$ and the extension is isomorphic to a tensor product is not relevant to us but is of interest as an interpretation of stochastic differentials on Riemannian manifolds [1]. Another remark is that our formula for $\nabla_\omega \eta$ applies equally well, with signs, to all degrees of ω, η (see Corollary 3.17).

Our algebraic approach also works when the initial DGA $\Omega(A)$ (the one being centrally extended) is already non-graded-commutative or 'quantum' on a possibly noncommutative algebra A. This amounts then to a new construction for quantum Riemannian geometries as cleft extension data for $\Omega(A)$ (see Proposition 3.6). Our main new result at this level of general $\Omega(A)$ is an explicit construction of a cocycle for a flat cleft central extension in Theorem 3.12 starting with the assumption of a degree -2 product $\perp$ obeying a 4-term identity

$$(-1)^{|\eta|}(\omega\eta) \perp \zeta + (\omega \perp \eta)\zeta = \omega \perp (\eta\zeta) + (-1)^{|\omega|+|\eta|}\omega(\eta \perp \zeta),\ \forall \omega, \eta, \zeta \in \Omega$$

and compatible δ. In the quantum case $\perp$ on 1-forms is not exactly the quantum metric $(\ ,\ )$ but is closely related, and its value on degree 1 does not automatically extend to all forms as a biderivation (this would depend on the relations of $\Omega(A)$); rather we consider it as metric-like data subject to the weaker 4-term relation above. Then Theorem 3.12 plays the role of the Koszul formula in giving the connection from this data $\perp$. We include an illustrative non-graded-commutative Example 3.15

on a set of two points. Proposition 3.20 concludes Section 3 with a further extension where we allow $\mathrm{d}\theta' \neq 0$, and here again the 4-term relation emerges as the solution to the algebraic extension problem. We denote the two extended DGAs by $\tilde\Omega(A)$ and $\tilde{\tilde\Omega}(A)$ respectively.

Section 4 is a specific application to a Riemannian manifold M equipped with a conformal Killing vector field, but using our new δ-based framework of Section 2. We introduce the corresponding notion that a 1-form τ is 'δ-conformal' if

$$[\delta, \mathcal{L}_\tau]\omega = \alpha\delta\omega + (|\omega| - \beta)\mathrm{i}_{\mathrm{d}\alpha}\omega, \quad \forall \omega \in \Omega^1(M),$$

for some function α and some constant β. Lemma 4.3 shows that in the classical setting this is equivalent to more conventional notions of conformal Killing 1-forms [20]. We show (Proposition 4.1) that this data gives us an action of the noncommutative DGA $\Omega(t, \mathrm{d}t)$ in one variable by graded-derivations on $\tilde{\tilde\Omega}(M)$ and the resulting semidirect product $\tilde{\tilde\Omega}(M) \rtimes \Omega(t, \mathrm{d}t)$ is a noncommutative differential version or 'quantisation' of $\Omega(M \times \mathbb{R})$. The degree 0 algebra quantises a subalgebra of $C^\infty(M \times \mathbb{R})$ (namely polynomial in t) to a semidirect product with commutation relations $[f, t] = \lambda\tau(f)$ for all $f \in C^\infty(M)$ and now with τ the corresponding vector field via the metric, and what we achieve is the natural differential exterior algebra of this quantisation. This extends a construction in [13, Sec. 3] from 1-forms to forms of all degree, although not quite in the full generality as used there to 'quantise' the Schwarzschild black-hole.

This is the final version of my preprint arXiv:1307.2778(math.QA). Compared to previous versions, the main addition is Section 2.5 containing a direct proof of some of our results in the classical case. It is also the case that the algebraic approach to differential geometry used in most of the paper is by now more established as relevant to quantum gravity, e.g. [16]. Meanwhile, aside from the conference announcement [14], the work [18] provided a fully worked example of the results of the present paper applied to the important case of the bicrossproduct model quantum spacetime $[x, t] = \imath\lambda_P x$. We also note [2], which picks up on the idea of a cross product of a DGA as used in Section 4.

1.1. **Preliminaries.** Our approach to calculations works for a 'coordinate algebra' A over a field k of characteristic not 2. For the main application to manifolds, the field could be taken to be $\mathbb{R}$ and the algebra could be taken to be smooth functions on a smooth manifold. We require enough differentiable structure so as to have an associative 'differential graded algebra' (DGA) of differential forms, $\Omega(A) = \oplus_n \Omega^n$ where $\Omega^0 = A$, equipped with a graded-derivation $\mathrm{d} : \Omega^i \to \Omega^{i+1}$ with $\mathrm{d}^2 = 0$. We say that a DGA is *standard* if Ω^1 is spanned by elements of the form $a\mathrm{d}b$ for $a, b \in A$ and Ω is generated by degrees 0,1 over A. We are mainly interested in the case of *classical type* where $\Omega(A)$ is graded-commutative, standard, and given by the tensor algebra over A of Ω^1 modulo relations of antisymmetry. This is intended to keep us close to the classical situation and ensures in particular that antisymmetric module maps descend to $\Omega(A)$.

We will always work with differential forms, but once we have a 'metric inner product', by which we mean a bimodule map $(\ ,\ ) : \Omega^1 \otimes_A \Omega^1 \to A$, we will have an associated 'vector field' $(\omega,\ ) : \Omega^1 \to A$ or $X_\omega = (\omega, \mathrm{d}(\ )) : A \to A$ for any $\omega \in \Omega^1$ and this will be relevant to the motivation behind some of the definitions in the paper. The meaning of $(\ ,\ )$ non-degenerate is the obvious one and one way to achieve it is the existence of a central element $g \in \Omega^1 \otimes_A \Omega^1$ 'the metric' such that $(\omega, g^1)g^2 = \omega = g^1(g^2, \omega)$ for all $\omega \in \Omega^1$. Here $g = g^1 \otimes g^2$ (a sum of such terms understood) is a notation. This is the normal set-up in noncommutative differential geometry in the approach of [6, 15, 8] but is also useful in the 'classical' case, where we normally also require that $(\ ,\ )$ is symmetric.

By a (left) algebraic connection on a DGA in noncommutative geometry one normally means $\nabla:\Omega^1\to\Omega^1\otimes_A\Omega^1$ in degree 1 or more generally $\Omega^m\to\Omega^1\otimes_A\Omega^m$ such that $\nabla(a\eta)=a\nabla\eta+\mathrm{d}a\otimes\nabla\eta$ for all $a\in A$ and $\eta\in\Omega^m$. The nicest case is that of a 'bimodule connection' where in addition we have $\nabla(\eta a)=(\nabla\eta)a+\sigma(\omega\otimes_A\mathrm{d}a)$ for some map $\sigma:\Omega^m\otimes_A\Omega^1\to\Omega^1\otimes_A\Omega^m$, called the 'generalised braiding'. Such a map if it exists is uniquely determined, so this is a really a property that a left connection can further have. The notion goes back to [12] and is a further ingredient the approach of [6, 15, 8].

One departure, we shall more often be interested in directly defining a '1-form covariant derivative' $\nabla_\omega:\Omega^m\to\Omega^m$ for all $\omega\in\Omega^1$ with analogous properties given by evaluation against a map $(\ ,\ ):\Omega^1(A)\otimes_A\Omega^1(A)\to A$, namely

$$\nabla_{a\omega}=a\nabla_\omega,\quad \nabla_\omega(a\eta)=\nabla_{\omega a}\eta+(\omega,\mathrm{d}a)\eta,\quad \forall\omega\in\Omega^1,\ \eta\in\Omega^m. \tag{1.1}$$

and this is a bimodule covariant derivative if

$$\nabla_\omega(\eta a)=(\nabla_\omega\eta)a+\sigma_\omega(\eta\otimes_A\mathrm{d}a);\quad \sigma:\Omega^1\otimes_A\Omega^m\otimes_A\Omega^1\to\Omega^m \tag{1.2}$$

for some bimodule map σ. Again this is a property of a covariant derivative rather than additional structure. Also for any covariant derivative we have the 'half curvature'

$$\rho(\omega\otimes_A\eta)=\nabla_\omega\nabla_\eta-\nabla_{\nabla_\omega\eta},\quad \forall\omega,\eta\in\Omega^1$$

and it is a nice check from the above properties that this depends only on $\omega\otimes_A\eta$:

$$\begin{aligned}\nabla_\omega\nabla_{a\eta}-\nabla_{\nabla_\omega(a\eta)}&=\nabla_\omega(a\nabla_\eta\ )-\nabla_{\nabla_{\omega a}\eta+(\omega,\mathrm{d}a)\eta}\\&=\nabla_{\omega a}\nabla_\eta-\nabla_{\nabla_{\omega a}\eta}+(\omega,\mathrm{d}a)\nabla_\eta-\nabla_{(\omega,\mathrm{d}a)\eta}=\nabla_{\omega a}\nabla_\eta-\nabla_{\nabla_{\omega a}\eta}\end{aligned}$$

As a result when $(\ ,\ )$ is invertible, the 'Laplace-Beltrami' operator

$$\Delta_{LB}=\nabla_{g^1}\nabla_{g^2}-\nabla_{\nabla_{g^1}g^2} \tag{1.3}$$

is well-defined. We will not go deeply into noncommutative differential geometry but some of our constructions will be no harder in the possibly noncommutative case and this is one of them.

We will often be interested in the failure of the Leibniz rule. For this it is usual to define for any degree $|B|$ linear map $B:\Omega(A)\to\Omega(A)$, the 'Leibnizator'

$$L_B(\omega,\eta)=B(\omega\eta)-(B\omega)\eta-(-1)^{|B||\omega|}\omega B\eta,\quad \forall\omega,\eta\in\Omega(A). \tag{1.4}$$

2. Reconstruction from a codifferential

In the case of a Riemannian manifold (or pseudo-Riemannian, of any signature) one has a divergence or codifferential $\delta:\Omega^i\to\Omega^{i-1}$. It will be immediately clear that we can recover the Riemannian structure from δ because we can recover the metric inner product $(\ ,\ )$ according to Definition 2.1. However, this point of view turns out to give natural formulae for all of the ensuing structures and these formulae are exactly what are needed for the quantisation in Section 3.

2.1. **Interior product.**

Definition 2.1. Let $\Omega(A)$ be a standard DGA. We say that a degree -1 linear map is *regular* if there exist degree -1 bimodule maps $\mathrm{i}:\Omega^1\otimes_A\Omega\to\Omega$ and $\grave{\mathrm{i}}:\Omega\otimes_A\Omega^1\to\Omega$ such that

$$\delta(a\omega)=a\delta\omega+\mathrm{i}_{\mathrm{d}a}\omega,\quad \delta(\omega a)=(\delta\omega)a+\omega\grave{\mathrm{i}}_{\mathrm{d}a}$$

where $\grave{\mathrm{i}}$ acts from the right. In this case we refer to the associated bimodule map $(\ ,\ ):\Omega^1\otimes_A\Omega^1\to A$ defined by

$$(\omega,\eta)=\frac{1}{2}(\omega\grave{\mathrm{i}}_\eta+\mathrm{i}_\omega\eta),\quad \omega,\eta\in\Omega^1$$

as the associated 'metric inner product'.

Note that if these maps exist, they are uniquely determined by δ.

Lemma 2.2. *Let δ be regular.*

(1) δ anticommutes with $\mathrm{i}_\eta,\grave{\mathrm{i}}_\eta$ for all $\eta\in\Omega^1$ iff δ^2 is a bimodule map. In this case $\mathrm{i}_\eta,\grave{\mathrm{i}}_\omega$ mutually anticommute for all $\eta,\omega\in\Omega^1$.

(2) $\mathrm{i}_{\mathrm{d}a},\grave{\mathrm{i}}_{\mathrm{d}a}$ are graded-derivations iff

$$L_\delta(a\omega,\eta)=aL_\delta(\omega,\eta)+(-1)^{|\omega|}\omega\mathrm{i}_{\mathrm{d}a}\eta,\quad L_\delta(\omega,\eta a)=L_\delta(\omega,\eta)a+(\omega\grave{\mathrm{i}}_{\mathrm{d}a})\eta,\quad \forall\omega,\eta\in\Omega,\ a\in A.$$

Proof. (1) For any $a\in A,\omega\in\Omega$, $\delta\mathrm{i}_{\mathrm{d}a}\omega=\delta(\delta(a\omega)-a\delta\omega)=\delta^2(a\omega)-a\delta^2\omega-\mathrm{i}_{\mathrm{d}a}\delta\omega$ and similarly for $\grave{\mathrm{i}}$. When quasi-nilpotency holds, we then have

$$\mathrm{i}_{\mathrm{d}a}(\omega\grave{\mathrm{i}}_{\mathrm{d}b})=\mathrm{i}_{\mathrm{d}a}(\delta(\omega b)-(\delta\omega)b)=-\delta((\mathrm{i}_{\mathrm{d}a}\omega)b)+(\delta\mathrm{i}_{\mathrm{d}a}\omega)b=-(\mathrm{i}_{\mathrm{d}a}\omega)\grave{\mathrm{i}}_{\mathrm{d}b}.$$

This implies that $(\mathrm{i}_{(\mathrm{d}a)b}\omega)\grave{\mathrm{i}}_{f\mathrm{d}h}=(\mathrm{i}_{\mathrm{d}a}b\omega f)\grave{\mathrm{i}}_{\mathrm{d}h}=-\mathrm{i}_{\mathrm{d}a}(b\omega f\grave{\mathrm{i}}_{\mathrm{d}h})=-\mathrm{i}_{(\mathrm{d}a)b}(\omega\grave{\mathrm{i}}_{f\mathrm{d}h})$ for all $a,b,f,h\in A$ by the bimodule map properties, hence the result applies to arbitrary 1-forms. (2) We also have

$$L_\delta(a\omega,\eta)-aL_\delta(\omega,\eta)=\delta(a\omega\eta)-(\delta(a\omega))\eta-a\delta(\omega\eta)+a(\delta\omega)\eta=\mathrm{i}_{\mathrm{d}a}(\omega\eta)-(\mathrm{i}_{\mathrm{d}a}\omega)\eta$$

for all $\omega,\eta\in\Omega$. Hence $\mathrm{i}_{\mathrm{d}a}$ is a right derivation iff the first stated condition holds. Similarly for $\grave{\mathrm{i}}_{\mathrm{d}a}$ (as a left derivation). The conditions amount to 6-term conditions on the behaviour of δ on a triple product where one factor is in A. □

In the graded-commutative case the left and right 'interior products' coincide. It follows if we assume that δ^2 is a tensorial (a module map) that $\mathrm{i}_\omega^2=0$ for all $\omega\in\Omega^1(A)$. Similarly, the derivation property extends to i_ω for all $\omega\in\Omega^1(A)$. It is also convenient for contact with classical differential geometry (but not essential as we will see in Section 3) to suppose that

$$\mathrm{i}_{\mathrm{d}a}(\mathrm{d}b)=\mathrm{i}_{\mathrm{d}b}(\mathrm{d}a),\quad \forall a,b\in A. \tag{2.1}$$

Definition 2.3. We say that δ is of *classical type* if it is regular, the two conditions of Lemma 2.2 apply and symmetry in the form (2.1) holds. When both $\Omega(A)$ and δ are of classical type we say that the pair $(\Omega(A),\delta)$ is of classical type.

In this case the anticommutativity of i means that we can extend it to $\mathrm{i}_{\omega_1\cdot\omega_m}=\mathrm{i}_{\omega_1}\cdots\mathrm{i}_{\omega_m}$ where $\omega_i\in\Omega^1$, to give a well-defined degree $-m$ linear map on $\Omega(A)$. Note, however, that this map is in general only a graded-derivation when $m=1$. For example one may easily compute

$$L_{\mathrm{i}_{\omega_1\omega_2}}(\omega,\eta)=(-1)^{|\omega|}\left((\mathrm{i}_{\omega_1}\omega)\mathrm{i}_{\omega_2}\eta-(\mathrm{i}_{\omega_2}\omega)\mathrm{i}_{\omega_1}\eta\right),\quad \forall\omega,\eta\in\Omega \tag{2.2}$$

and similar formulae in general. Using this notation, the mutual anticommutativity of $\mathrm{i}_{\mathrm{d}a}, \delta$ in the proof of Lemma 2.2 is readily seen to generalise to

$$\delta \mathrm{i}_\omega + \mathrm{i}_\omega \delta = \mathrm{i}_{\mathrm{d}\omega}, \quad \forall \omega \in \Omega^1. \tag{2.3}$$

(This implies a similar formulae for all degrees of ω but with a graded-commutator on the left.)

Lemma 2.4. *Let $(\Omega(A), \delta)$ be of classical type. Then*

$$\begin{aligned}\mathrm{i}_\zeta L_\delta(\omega,\eta) &= -L_\delta(\mathrm{i}_\zeta\omega,\eta) - (-1)^{|\omega|} L_\delta(\omega, \mathrm{i}_\zeta \eta) + L_{\mathrm{i}_{\mathrm{d}\zeta}}(\omega,\eta), \quad \forall \omega,\eta\in\Omega,\ \zeta\in\Omega^1\\ &= \mathrm{i}_\omega \mathrm{d}\mathrm{i}_\eta\zeta - \mathrm{i}_\eta \mathrm{d}\mathrm{i}_\omega\zeta - \mathrm{i}_\eta\mathrm{i}_\omega \mathrm{d}\zeta \qquad \text{if} \quad \omega,\eta\in\Omega^1.\end{aligned}$$

If, moreover, $(\ ,\ )$ is nondegenerate then $L_\delta(\omega,\ )$ is a degree $|\omega|-1$ derivation for all $\omega\in\Omega$ and

$$L_\delta(\omega_1\cdots\omega_m, \eta_1\cdots\eta_n) = \sum_{i,j} (-1)^{i+j} \omega_1\cdots\widehat{\omega_i}\cdots\omega_m L_\delta(\omega_i,\eta_j)\eta_1\cdots\widehat{\eta_j}\cdots\eta_n$$

where $\omega_i, \eta_j \in \Omega^1$. Here the hat denotes omission.

Proof. (1) We use (2.3) in the definition of L_δ. Thus

$$\begin{aligned}\mathrm{i}_\zeta L_\delta(\omega,\eta) &= \mathrm{i}_\zeta\delta(\omega\eta) - (\mathrm{i}_\zeta\delta\omega)\eta - (-1)^{|\omega|-1}(\delta\omega)\mathrm{i}_\zeta\eta - (-1)^{|\omega|}(\mathrm{i}_\zeta\omega)\delta\eta - \omega \mathrm{i}_\zeta\delta\eta\\ &= -\delta((\mathrm{i}_\zeta\omega)\eta + (-1)^{|\omega|}\omega \mathrm{i}_\zeta\eta) + \mathrm{i}_{\mathrm{d}\zeta}(\omega\eta) + (\delta \mathrm{i}_\zeta\omega)\eta - (\mathrm{i}_{\mathrm{d}\zeta}\omega)\eta\\ &\quad + (-1)^{|\omega|}(\delta\omega)\mathrm{i}_\zeta\eta - (-1)^{|\omega|}(\mathrm{i}_\zeta\omega)\delta\eta + \omega\delta \mathrm{i}_\zeta\eta - \omega \mathrm{i}_{\mathrm{d}\zeta}\eta\\ &= -L_\delta((\mathrm{i}_\zeta\omega),\eta) - (-1)^{|\omega|}L_\delta(\omega,\mathrm{i}_\zeta\eta) + \mathrm{i}_{\mathrm{d}\zeta}(\omega\eta) - (\mathrm{i}_{\mathrm{d}\zeta}\omega)\eta - \omega \mathrm{i}_{\mathrm{d}\zeta}\eta\end{aligned}$$

We also have $L_\delta(a,\omega) = L_\delta(\omega,a) = \mathrm{i}_{\mathrm{d}a}\omega$ for all $a\in A$ and $\omega\in\Omega$ after which our result implies the explicit formula in the case $\omega,\eta\in\Omega^1$. (1) Note that $L_{\mathrm{i}_{\mathrm{d}\zeta}}(\omega,)$ using (2.2) is a graded-derivation of degree ω and when $\omega\in\Omega^1$ our stated formula implies

$$\mathrm{i}_\zeta L_\delta(\omega,\eta) = -\mathrm{i}_{\mathrm{d}\mathrm{i}_\zeta\omega}(\eta) + L_\delta(\omega,\mathrm{i}_\zeta(\eta)) + L_{\mathrm{i}_{\mathrm{d}\zeta}}(\omega,\eta). \tag{2.4}$$

We prove by induction that $L_\delta(\omega,\ )$ is a derivation on a product $\eta\eta'$, assuming that the same is true on a product where either η,η' are replaced by a form of one less degree. Thus

$$\begin{aligned}\mathrm{i}_\zeta L_\delta(\omega,\eta\eta') &= -\mathrm{i}_{\mathrm{d}\mathrm{i}_\zeta\omega}(\eta\eta') + L_\delta(\omega,\mathrm{i}_\zeta(\eta\eta')) + L_{\mathrm{i}_{\mathrm{d}\zeta}}(\omega,\eta\eta')\\ &= -(\mathrm{i}_{\mathrm{d}\mathrm{i}_\zeta\omega}\eta)\eta' - (-1)^{|\eta|}\eta \mathrm{i}_{\mathrm{d}\mathrm{i}_\zeta\omega}(\eta') + L_{\mathrm{i}_{\mathrm{d}\zeta}}(\omega,\eta)\eta' + (-1)^{|\eta|}\eta L_{\mathrm{i}_{\mathrm{d}\zeta}}(\omega,\eta)\\ &\quad + L_\delta(\omega,(\mathrm{i}_\zeta\eta))\eta' + (\mathrm{i}_\zeta\eta)L_\delta(\omega,\eta') + (-1)^{|\eta|}L_\delta(\omega,\eta)\mathrm{i}_\zeta\eta' + (-1)^{|\eta|}\eta L_\delta(\omega,\mathrm{i}_\zeta\eta')\\ &= (\mathrm{i}_\zeta\eta)L_\delta(\omega,\eta') + (-1)^{|\eta|}\eta \mathrm{i}_\zeta L_\delta(\omega,\eta') + (\mathrm{i}_\zeta L_\delta(\omega,\eta))\eta' + (-1)^{|\eta|}L_\delta(\omega,\eta)\mathrm{i}_\zeta\eta'\\ &= \mathrm{i}_\zeta\left(L_\delta(\omega,\eta)\eta' + \eta L_\delta(\omega,\eta')\right)\end{aligned}$$

using (2.4) on the product and in reverse to recognise the answer. Now, if η or η' have degree 0 then the derivation property on $\eta\eta'$ reduces to part of Lemma 2.2, so this holds and provides the boundary condition for the induction. Thus if $(\ ,\ )$ is nondegenerate we see that $L_\delta(\omega,\ )$ is a derivation for all $\omega\in\Omega^1$. We now prove that if the DGA is graded-commutative and $L_\delta(\omega,)$ is a derivation for $\omega\in\Omega^1$ then $L_\delta(\omega,\ )$ is a graded-derivation of degree $|\omega|-1$ for all ω. The degree zero case $L_\delta(a,\ ) = \mathrm{i}_{\mathrm{d}a}$ is already assumed to be a graded-derivation of degree -1 in Lemma 2.2. We will need the tautological identity

$$L_\delta(\omega\eta,\zeta) + L_\delta(\omega,\eta)\zeta = L_\delta(\omega,\eta\zeta) + (-1)^{|\omega|}\omega L_\delta(\eta,\zeta) \tag{2.5}$$

which holds for the Leibnizator of any degree -1 linear map on any graded algebra (just write out the definitions on both sides), cf[10] in the graded-commutative case.

Suppose $L_\delta(\omega,\ )$ is a degree $|\omega|-1$ derivation for ω of some degree. Using (2.5) we deduce

$$\begin{aligned} L_\delta(\omega\eta,\zeta) &= L_\delta(\omega,\eta\zeta)+(-1)^{|\omega|}\omega L_\delta(\eta,\zeta)-L_\delta(\omega,\eta)\zeta\\ &=(-1)^{(|\omega|-1)|\eta|}\eta L_\delta(\omega,\zeta)+(-1)^{|\omega|}\omega L_\delta(\eta,\zeta) \end{aligned} \tag{2.6}$$

for the given ω and all η,ζ. We also suppose $L_\delta(\eta,\ )$ is a degree $|\eta|-1$ graded derivation for $|\eta|\le|\omega$ (it suffices to take $|\eta|=1$). Then,

$$\begin{aligned} L_\delta(\omega\eta,\xi\zeta) &= L_\delta(\omega,\eta\xi\zeta)+(-1)^{|\omega|}\omega L_\delta(\eta,\xi\zeta)-L_\delta(\omega,\eta)\xi\zeta\\ &=(-1)^{(|\omega|-1)(|\eta|+|\xi|)}\eta\xi L_\delta(\omega,\zeta)+L_\delta(\omega,\eta\xi)\zeta+(-1)^{|\omega|}(-1)^{(|\eta|-1)|\xi|}\omega\xi L_\delta(\eta,\zeta)\\ &\quad+(-1)^{|\omega|}\omega L_\delta(\eta,\xi)\zeta-L_\delta(\omega,\eta)\xi\zeta\\ &=(-1)^{(|\omega|-1)(|\eta|+|\xi|)}\eta\xi L_\delta(\omega,\zeta)+(-1)^{|\omega|}(-1)^{(|\eta|-1)|\xi|}\omega\xi L_\delta(\eta,\zeta)+L_\delta(\omega\eta,\xi)\zeta\\ &=(-1)^{(|\omega|+|\eta|-1)|\xi|}\Big((-1)^{(|\omega|-1)|\eta|}\xi\eta L_\delta(\omega,\zeta)+(-1)^{|\omega|}\xi\omega L_\delta(\eta,\zeta)\Big)+L_\delta(\omega\eta,\xi)\zeta\\ &=(-1)^{(|\omega|+|\eta|-1)|\xi|}\xi L_\delta(\omega\eta,\zeta)+L_\delta(\omega\eta,\xi)\zeta \end{aligned}$$

where we used (2.5), then our assumed derivation properties, (2.5) in reverse, graded-commutativity and the computation above, to recognise the answer. This proves the required graded-derivation property by induction. It follows from (2.6) in the graded-commutative case that if a degree -1 bilinear map L_δ obeys (2.5) and $L_\delta(\omega,\ )$ is a graded-derivation, then

$$L_\delta(\omega_1\cdots\omega_m,\ )=\sum_{i=1}^m(-1)^{i-1}\omega_1\cdots\widehat{\omega_i}\cdots\omega_m L_\delta(\omega_i,\ ),\quad\forall\omega_i\in\Omega^1 \tag{2.7}$$

leading to the formula stated. This is a general observation which we will also use for other maps obeying (2.5). □

The specific formula for ω,η of degree 1 confirms that $L_\delta(\omega,\eta)$ in the case of a classical manifold corresponds via the metric to the Lie bracket of vector fields. Thus, let $X_\omega=(\omega,\mathrm{d}(\ ))$ be the vector field associated to a 1-form and let $[X_\omega,X_\eta]$ be the usual Lie bracket of such vector fields viewed as a tensorial map on 1-forms. Then

$$[X_\omega,X_\eta](\zeta)=(\omega,\mathrm{d}(\eta,\zeta))-(\mathrm{d}(\omega,\zeta),\eta)-\mathrm{i}_\eta\mathrm{i}_\omega\mathrm{d}\zeta,\quad\forall\omega,\eta,\zeta\in\Omega^1 \tag{2.8}$$

in agreement with $\mathrm{i}_\zeta L_\delta(\omega,\eta)$ in the lemma. In this case it is clear from the formula on higher degrees that $L_\delta(\omega,\eta)$ corresponds to the Schouten bracket of alternating multivector fields cf. [10], and indeed the results of Lemma 2.4 can be seen as parallel to properties of this [9, 17]. There will also be a form of graded Jacobi identity which we have not elaborated here as we will not need it. On the other hand we view $L_\delta(\omega,\eta)$ as the primary object with its evaluations such as (2.8) defining the bracket as a linear map on ζ of appropriate degree even in the degenerate case. For example, by iterating Lemma 2.4 one has

$$\mathrm{i}_\zeta L_\delta(\omega,\eta)=\mathrm{i}_\omega\mathrm{d}\mathrm{i}_\eta\zeta-\mathrm{i}_\eta\mathrm{d}\mathrm{i}_\omega\zeta-\mathrm{i}_\eta\mathrm{i}_\omega\mathrm{d}\zeta,\quad\forall\omega\in\Omega^1,\ \eta,\zeta\in\Omega^2 \tag{2.9}$$

without assuming nondegeneracy, and similarly in general degree for η,ζ (we will need this only in degrees 1,2).

Corollary 2.5. *If $(\Omega(A),\delta)$ is of classical type and $(\ ,\ )$ is nondegenerate then*

$$\begin{aligned} \delta(\omega\eta\zeta) &= (\delta(\omega\eta))\zeta+(-1)^{|\omega|}\omega\delta(\eta\zeta)+(-1)^{(|\omega|-1)|\eta|}\eta\delta(\omega\zeta)\\ &\quad-(\delta\omega)\eta\zeta-(-1)^{|\omega|}\omega(\delta\eta)\zeta-(-1)^{|\omega|+|\eta|}\omega\eta\delta\zeta \end{aligned}$$

$\forall\omega,\eta,\zeta\in\Omega$. In other words, $(\Omega(A),\delta)$ is a Batalin-Vilkovisky algebra slightly generalised to allow δ^2 to be a left module map.

Proof. This is just the content of $L_\delta(\omega,\ )$ a graded-derivation, written out in terms of δ. □

The case of ω of degree 0 is the content of our classical type assumption (the derivation properties in Lemma 2.2) and the corollary says that in the nondegenerate case the stated identity then holds in all degrees, in keeping with the known fact that the divergence on multivector fields provides a BV algebra structure[19, 21].

Also in the case of classical type we now introduce an operation $\perp: \Omega\otimes_A\Omega\to\Omega$ of degree -2,

$$(\omega_1\cdots\omega_m)\perp(\eta_1\cdots\eta_n)=\sum_{i,j}(-1)^{i+j}(\omega_i,\eta_j)\omega_1\cdots\widehat{\omega_i}\cdots\omega_m\eta_1\cdots\widehat{\eta_j}\cdots\eta_n,\quad (2.10)$$

for all $\omega_i,\eta_j\in\Omega^1$. Classically i_ω in degree 1 is a graded derivation of degree -1 and in our case similarly

$$\omega_1\cdots\omega_m\perp(\ )=\sum_{j=1}^m(-1)^{j-1}\omega_1\cdots\widehat{\omega_j}\cdots\omega_m\mathrm{i}_{\omega_j}(\ ),$$

is a degree $m-2$ derivation, while

$$(\ )\perp\eta_1\cdots\eta_n=\sum_{i=1}^n(-1)^{i-1}(\mathrm{i}_{\eta_i}\ )\eta_1\cdots\widehat{\eta_i}\cdots\eta_n$$

is such that $((-1)^D(\ ))\perp\eta_1\cdots\eta_n$ is a degree $n-2$ right derivation, where D is the degree operator. It is also clear from the form of these expressions that they depend tensorially (they are A-module maps) and antisymmetrically and hence descend to $\Omega(A)$. In particular, if ω is degree 1 then $\omega\perp$ and $\perp\omega$ revert to the interior product by ω. We will particularly need

$$\omega_1\omega_2\perp=\omega_2\mathrm{i}_{\omega_1}-\omega_1\mathrm{i}_{\omega_2}.$$

Note that interior products are usually considered by vector fields and one could consider that the 1-form ω is being converted to a vector field $(\omega,\mathrm{d}(\ ))$ for this purpose. Later on, in Section 3, we shall generalise this construction so that $\perp$ need not be symmetric when restricted to degree 1, but for δ of classical type as here, $\perp$ just extends the metric inner product $(\ ,\ )$.

Using the interior product we define the 'form Lie derivative' as

$$\mathcal{L}_\omega=\mathrm{d}\mathrm{i}_\omega+\mathrm{i}_\omega\mathrm{d},\quad \omega\in\Omega^1$$

along the lines of the classical Cartan formula. Clearly it obeys

$$\mathcal{L}_{a\omega}\eta=a\mathcal{L}_\omega\eta+(\mathrm{d}a)\mathrm{i}_\omega\eta,\quad \mathcal{L}_\omega(a\eta)=a\mathcal{L}_\omega(\eta)+(\omega,\mathrm{d}a)\eta,\quad \forall\omega\in\Omega^1.\quad (2.11)$$

2.2. **Form covariant derivatives.** We similarly define a 'covariant derivative' $\nabla:\Omega^1\times\Omega\to\Omega$ to be a map characterised by (1.1) as explained in the Preliminaries. We will be interested in metric compatibility, which means vanishing of the tensor

$$C_\omega(\eta,\zeta)=(\omega,\mathrm{d}(\eta,\zeta))-(\nabla_\omega\eta,\zeta)-(\eta,\nabla_\omega\zeta),\quad \forall\omega,\eta,\zeta\in\Omega^1.\quad (2.12)$$

In the same vein we define

$$T(\omega,\eta)(\zeta)=(\omega,\nabla_\eta\zeta)-(\eta,\nabla_\omega\zeta)-\mathrm{i}_\omega\mathrm{i}_\eta\mathrm{d}\zeta,\quad \forall\omega,\eta,\zeta\in\Omega^1\quad (2.13)$$

as the torsion of a covariant derivative. Both maps are easily seen to be tensorial in all of their inputs. These formulae are dualizations of the usual formulae with vector fields and make sense in this form for any standard graded commutative $\Omega(A)$ of classical type, but note that we do not assume that $(\ ,\ )$ is nondegenerate.

Lemma 2.6. *Let $\Omega(A)$ be of classical type and equipped with a symmetric metric inner product $(\ ,\ )$ and ∇ a covariant derivative. Then*

$$T(\omega,\eta)(\zeta)+C_\eta(\omega,\zeta)-C_\omega(\eta,\zeta)=\mathrm{i}_\zeta\left(\nabla_\omega\eta-\nabla_\omega\eta\right)-\mathrm{i}_\omega\mathrm{d}\mathrm{i}_\eta\zeta+\mathrm{i}_\eta\mathrm{d}\mathrm{i}_\omega\zeta-\mathrm{i}_\omega\mathrm{i}_\eta\mathrm{d}\zeta$$
$$T(\zeta,\omega)(\eta)+T(\zeta,\eta)(\omega)-C_\zeta(\omega,\eta)=\mathrm{i}_\zeta\left(\nabla_\omega\eta+\nabla_\eta\omega-\mathcal{L}_\omega\eta-\mathcal{L}_\eta\omega+\mathrm{d}(\omega,\eta)\right)$$

for all $\omega,\eta,\zeta\in\Omega^1$.

Proof. For the first part we use (2.12) in each of the first two terms of the definition (2.13) of torsion. For the second part we use (2.13) on each term to compute

$$\begin{aligned}(\zeta,\nabla_\omega\eta+\nabla_\eta\omega)&=(\omega,\nabla_\zeta\eta)+T(\zeta,\omega)(\eta)+\mathrm{i}_\zeta\mathrm{i}_\omega\mathrm{d}\eta+(\eta,\nabla_\zeta\omega)+T(\zeta,\eta)(\omega)+\mathrm{i}_\zeta\mathrm{i}_\eta\mathrm{d}\omega\\&=T(\zeta,\omega)(\eta)+T(\zeta,\eta)(\omega)+\mathrm{i}_\zeta(\mathrm{d}(\omega,\eta)+\mathrm{i}_\omega\mathrm{d}\eta+\mathrm{i}_\eta\mathrm{d}\omega)-C_\zeta(\omega,\eta)\end{aligned}$$

using (2.12). We then recognise the answer in terms of a Lie derivative. □

This means that in the nondegenerate case a metric compatible torsion free covariant derivative, if it exists, is uniquely determined as its symmetric and antisymmetric parts are determined (as on a classical manifold). One can also treat the curvature evaluated against one-forms in a similar spirit. However, in the case where $(\ ,\ )$ comes from a δ of classical type one can do rather better:

Proposition 2.7. *Let $(\Omega(A),\delta)$ be of classical type and ∇ a covariant derivative. Then*

$$R(\omega,\eta)(\zeta):=\nabla_\omega\nabla_\eta\zeta-\nabla_\eta\nabla_\omega\zeta-\nabla_{L_\delta(\omega,\eta)}\zeta,\quad\forall\omega,\eta,\zeta\in\Omega^1$$

is tensorial in all its inputs and reduces to the usual curvature in the nondegenerate or algebraic cases. If the covariant derivative is $(\ ,\ )$-compatible then

$$T(\omega,\eta):=\nabla_\omega\eta-\nabla_\eta\omega-L_\delta(\omega,\eta),\quad\forall\omega,\eta\in\Omega^1$$

is tensorial in its inputs and evaluates via $(\ ,\ )$ to the torsion.

Proof. We now let ∇_ω be any covariant derivative and check

$$\begin{aligned}R(\omega,\eta)(a\zeta)&=\nabla_\omega(a\nabla_\eta\zeta)-\nabla_\eta(a\nabla_\omega\zeta)-a\nabla_{L_\delta(\omega,\eta)}\zeta\\&\quad+\nabla_\omega((\eta,\mathrm{d}a)\zeta)-\nabla_\eta((\omega,\mathrm{d}a)\zeta)-(L_\delta(\omega,\eta),\mathrm{d}a)\zeta\\&=aR(\omega,\eta)(\zeta)+(\omega,\mathrm{d}(\eta,\mathrm{d}a))-(\eta,\mathrm{d}(\omega,\mathrm{d}a))-(L_\delta(\omega,\eta),\mathrm{d}a)=aR(\omega,\eta)(\zeta)\\R(a\omega,\eta)(\zeta)&=a\nabla_\omega\nabla_\eta\zeta-\nabla_\eta(a\nabla_\omega\zeta)-a\nabla_{L_\delta(\omega,\eta)}\zeta-(\eta,\mathrm{d}a)\nabla_\omega\zeta=aR(\omega,\eta)(\zeta)\end{aligned}$$

for all $\omega,\eta,\zeta\in\Omega^1$, $a\in A$. For the first computation we used the defining property (1.1) of a covariant derivative followed by Lemma 2.4. For the second computation we used the covariant derivative property and Lemma 2.2. Similarly for the other input of the curvature. Note that at least in the nondegenerate case one can then evaluate the algebraic curvature $R_\nabla=(\mathrm{d}\otimes\mathrm{id}-(\wedge\otimes\mathrm{id})(\mathrm{id}\otimes\nabla))\nabla$ by applying $\mathrm{i}_\eta\mathrm{i}_\omega$, to obtain

$$R(\omega,\eta)(\zeta)=\nabla_\omega\nabla_\eta\zeta-\nabla_\eta\nabla_\omega\zeta-\nabla_{[\omega,\eta]}\zeta,\quad\forall\omega,\eta,\zeta\in\Omega^1$$

as a definition in this case, where the 'Lie bracket' on 1-forms is given by $[\omega,\eta]=L_\delta(\omega,\eta)$ or rather by its evaluation on 1-forms as explained above. That we have $\mathrm{i}_\eta\mathrm{i}_\omega R_\nabla$ in the case where $\nabla:\Omega^1\to\Omega^1\otimes_A\Omega^1$ is defined is part of the standard derivation of the algebraic expression for R_∇. Suffice for completeness to note by the covariant derivative property (1.1) that

$$\nabla(\nabla_\omega\zeta)=\mathrm{d}(\omega,\nabla^1\zeta)\otimes\nabla^2\zeta+(\omega,\nabla^1\zeta)\nabla\nabla^2\zeta$$

where we use a notation $\nabla\zeta=\nabla^1\zeta\otimes\nabla^2\zeta$. We use this in the evaluation of the 2nd term of R_∇ and Lemma 2.4 for the evaluation of the first term $\mathrm{i}_\eta\mathrm{i}_\omega\mathrm{d}$.

The torsion ∇ on 1-forms is likewise given by $\mathrm{i}_\eta\mathrm{i}_\omega$ against the algebraic torsion $T = \wedge\nabla - \mathrm{d} : \Omega^1 \to \Omega^2$ to give the map (2.13). In the important case where the covariant derivative is $(\ ,\ )$-compatible, we recognise the formula in Lemma 2.4 for $\mathrm{i}_\zeta L_\delta(\omega,\eta)$ in the expression for torsion in Lemma 2.6. We then take $T(\omega,\eta) \in \Omega^1$ as a definition applicable in the $(\ ,\ )$-compatible case. Tensoriality is from Lemma 2.2 and (1.1). □

2.3. **Levi-Civita covariant derivative.** We are now ready to state and prove our main result:

Theorem 2.8. *In the setting above with $(\Omega(A),\delta)$ of classical type, there is a covariant derivative*

$$\nabla_\omega\eta = \frac{1}{2}\left(L_\delta(\omega,\eta) + \mathcal{L}_\omega\eta + (\mathrm{d}\omega) \perp \eta\right), \quad \forall\omega\in\Omega^1,\ \eta\in\Omega.$$

which is torsion free and compatible with $(\ ,\ )$ and in the nondegenerate case acts as a derivation.

Proof. We will see in Section 3 how this formula arises as a requirement for quantisation; for the moment we verify directly that the stated expression is indeed a covariant derivative with the stated properties. Thus

$$\begin{aligned}
2\nabla_{a\omega}\eta &= \mathcal{L}_{a\omega}(\eta) + L_\delta(a\omega,\eta) + (\mathrm{d}(a\omega)) \perp \eta\\
&= a\mathcal{L}_\omega\eta + (\mathrm{d}a)\mathrm{i}_\omega\eta + aL_\delta(\omega,\eta) - \omega\mathrm{i}_{\mathrm{d}a}\eta + ((\mathrm{d}a)\omega) \perp \eta + a\mathrm{d}\omega \perp \eta = 2a\nabla_\omega\eta\\
2\nabla_\omega(a\eta) &= \mathcal{L}_\omega(a\eta) + L_\delta(\omega,a\eta) + (\mathrm{d}\omega) \perp a\eta\\
&= a\mathcal{L}_\omega\eta + (\omega,\mathrm{d}a)\eta + aL_\delta(\omega,\eta) + (\mathrm{i}_{\mathrm{d}a}\omega)\eta + a(\mathrm{d}\omega) \perp \eta = 2\nabla_\omega(\eta) + 2(\omega,\mathrm{d}a).
\end{aligned}$$

Next we note that acting in degree 1 this covariant derivative can be written as

$$\nabla_\omega\eta = \frac{1}{2}L_\delta(\omega,\eta) + \frac{1}{2}(\mathcal{L}_\omega\eta + \mathcal{L}_\eta\omega - \mathrm{d}(\omega,\eta))$$

so that

$$\nabla_\omega\eta - \nabla_\eta\omega = L_\delta(\omega,\eta), \quad \nabla_\omega\eta + \nabla_\eta\omega = \mathcal{L}_\omega\eta + \mathcal{L}_\eta\omega - \mathrm{d}(\omega,\eta), \quad \forall\omega,\eta\in\Omega^1 \quad (2.14)$$

which comparing with Lemma 2.6 and using the formula for $\mathrm{i}_\zeta L_\delta(\omega,\eta)$ in Lemma 2.4 implies that if $T = 0$ then $C = 0$.

It remains to prove that $T = 0$ in (2.13). For this we put in the particular form of our covariant derivative as found above. Then

$$\begin{aligned}
2T(\omega,\eta)(\zeta) &= (\omega, L_\delta(\eta,\zeta) - \mathcal{L}_\eta\zeta + \mathrm{i}_\zeta\mathrm{d}\eta + 2\mathrm{d}\mathrm{i}_\zeta\eta) - (\eta, \mathcal{L}_\omega\zeta + L_\delta(\omega,\zeta) + \mathrm{i}_\zeta\mathrm{d}\omega)\\
&= (\omega, L_\delta(\eta,\zeta)) - (\eta, L_\delta(\omega,\zeta)) + \mathrm{i}_\omega\mathrm{i}_\zeta\mathrm{d}\eta + \mathrm{i}_\omega\mathrm{d}\mathrm{i}_\zeta\eta - \mathrm{i}_\eta\mathrm{d}\mathrm{i}_\omega\zeta - \mathrm{i}_\eta\mathrm{i}_\zeta\mathrm{d}\omega\\
&= -\mathrm{i}_{\mathrm{d}(\omega,\eta)}\eta + \mathrm{i}_{\mathrm{d}\omega}(\eta\zeta) + \mathrm{i}_{\mathrm{d}(\eta,\zeta)}\omega - \mathrm{i}_{\mathrm{d}\eta}(\omega\zeta) + \mathrm{i}_\omega\mathrm{i}_\zeta\mathrm{d}\eta + \mathrm{i}_\omega\mathrm{d}\mathrm{i}_\zeta\eta - \mathrm{i}_\eta\mathrm{d}\mathrm{i}_\omega\zeta - \mathrm{i}_\eta\mathrm{i}_\zeta\mathrm{d}\omega\\
&= 0.
\end{aligned}$$

We used Lemma 2.4 for L_δ and symmetry of same-degree interior products to cancel.

The Lie derivative and $\mathrm{d}\omega \perp$ act by derivations which covers the symmetric part, and in the nondegenerate case Lemma 2.4 tells us that the antisymmetric part also acts as a derivation. □

This provides a natural 'Levi-Civita' covariant derivative in our setting. In principle its curvature and other geometrical properties can be computed in terms of δ. We also at the same time defined ∇ naturally on all degrees. In the nondegenerate case we know from Lemma 2.6 that it is unique for $C = T = 0$ and in this case we also have a picture of L_δ as Lie bracket of vector fields, so in this case we have a formula for ∇ that depends only on the metric, akin to the familiar Koszul formula. One also has torsion freeness and metric-compatibility on all degrees and in a suitable sense.

2.4. **Divergence operator.** We conclude by supplying the partial inverse to Theorem 2.8.

Proposition 2.9. *Let $\Omega(A)$ be of classical type and $(\ ,\ )$ the inverse of a symmetric metric g with metric compatible and torsion free covariant derivative ∇ extending as a derivation to Ω. Then the 'divergence operator'*

$$\delta_\nabla = \mathrm{i}_{g^1}\nabla_{g^2}$$

is of classical type and application of Theorem 2.8 recovers $(\ ,\ ),\nabla$.

Proof. Here $\delta = \delta_\nabla$ as stated is well-defined as ∇_ω is tensorial in ω. Clearly

$$\delta(a\omega) = \mathrm{i}_{g^1}((g^2, \mathrm{d}a)\omega + a\nabla_{g^2}\omega) = a\delta\omega + \mathrm{i}_{g^1}(\omega)(g^2, \mathrm{d}a) = a\delta\omega + \mathrm{i}_{\mathrm{d}a}\omega$$

applies with the interior product provided by the given metric inner product, which is indeed a graded-derivation and symmetric. Hence Lemma 2.2 applies and δ is of classical type provided we can show that δ^2 is an A-module map. As in the proof of Lemma 2.2 (worked in reverse), this amounts to showing that $\mathrm{i}_{\mathrm{d}a}$ and δ anticommute for all $a \in A$. We do this in two steps. First, we observe that for any $\omega, \eta \in \Omega^1$,

$$[\mathrm{i}_\eta, \nabla_\omega] = -\mathrm{i}_{\nabla_\omega \eta} \tag{2.15}$$

holds as operations on Ω. Indeed, using the Leibniz property of ∇_ω and the graded-Leibniz property of interior one can check that the left hand side of (2.15) is a degree -1 graded-derivation. The right hand side is also a graded-derivation and (2.15) holds in degree 1 by metric compatibility (and is trivial in degree 0). Here for $\omega, \eta \in \Omega^1$,

$$[\mathrm{i}_\eta, \nabla_\omega]\zeta = (\eta, \nabla_\omega\zeta) - (\omega, \mathrm{d}(\eta, \zeta)) = -(\zeta, \nabla_\omega \eta)$$

by metric compatibility (2.12). Next, using (2.15), we compute for all $a \in A, \omega \in \Omega$,

$$(\delta \mathrm{i}_{\mathrm{d}a} + \mathrm{i}_{\mathrm{d}a}\delta)(\omega) = \mathrm{i}_{g^1}\nabla_{g^2}\mathrm{i}_{\mathrm{d}a}\omega - \mathrm{i}_{g^1}\mathrm{i}_{\mathrm{d}a}\nabla_{g^2}\omega = \mathrm{i}_{g^1}\mathrm{i}_{\nabla_{g^2}\mathrm{d}a}(\omega) = 0$$

since $g^1\nabla_{g^2}(\mathrm{d}a) = 0$ as an expression of zero torsion. Indeed, if $T = 0$ then

$$\mathrm{i}_\eta \mathrm{i}_\omega(g^1\nabla_{g^2}\zeta) = \mathrm{i}_\eta(\nabla_\omega\zeta - g^1\mathrm{i}_\omega\nabla_{g^2}\zeta) = \mathrm{i}_\eta\nabla_\omega\zeta - \mathrm{i}_\omega\nabla_\eta\zeta = \mathrm{i}_\eta\mathrm{i}_\omega\mathrm{d}\zeta.$$

This concludes our proof that $(\Omega(A), \delta_\nabla)$ is of classical type. Now consider the covariant derivative defined by Theorem 2.8. It clearly coincides with the given ∇ on degree 1 since both are metric compatible and torsion free and $(\ ,\ )$ is non-degenerate (see Lemma 2.6). Both covariant derivatives are derivations, in the case of the one in Theorem 2.8 by Lemma 2.4 and since $\mathcal{L}_\omega + \mathrm{d}\omega \perp$ is a derivation, hence the two covariant derivatives coincide in all degrees. □

This implies in particular that every invertible metric (and associated covariant derivative) is in the image of the construction of Theorem 2.8 for some choice of δ. The same result applies more generally to $\nabla = \nabla^1 \otimes \nabla^2$ an algebraic connection and $(\ ,\ )$ possibly degenerate, where we take $\delta_\nabla = \mathrm{i}_{\nabla^1\eta}(\nabla^2\eta)$. We now complete the picture by analysing when different δ give the same metric and connection. For this we need the notion of a vector field which in algebraic terms when $\Omega(A)$ is of classical type just means a tensorial map $v : \Omega^1(A) \to A$. We let interior product $\lfloor_v$ be its extension to $\Omega(A)$ as a degree -1 graded derivation. It is easy to see that $\lfloor_v$ anticommutes with i_ω for all $\omega \in \Omega^1$.

Proposition 2.10. *Let $\Omega(A)$ and δ be of classical type and $v:\Omega^1(A)\to A$ a vector field. Then $\delta'=\delta+\lfloor_v$ is also of classical type and results in the same ∇ and $(\ ,\ )$ as δ. Conversely, if δ,δ' are both of classical type and result in the same ∇, $(\ ,\ )$ then they differ by $\lfloor_v$ along some vector field v.*

Proof. For the first part, as $\lfloor_v$ is a degree -1 graded derivation, $\delta'=\delta+\lfloor_v$ has the same Leibnizator as δ. Hence δ' is regular with the same interior products such that $L_\delta(\omega,a)=L_{\delta'}(\omega,a)=\mathrm{i}_{\mathrm{d}a}$ and has the same associated metric by Lemma 2.2. Moreover,

$$(\delta\lfloor_v+\lfloor_v\delta)(a\omega)=\delta(a\lfloor_v\omega)+\lfloor_v(a\delta\omega+\mathrm{i}_{\mathrm{d}a}\omega)=a(\delta\lfloor_v+\lfloor_v\delta)\omega+\{\mathrm{i}_{\mathrm{d}a},\lfloor_v\}\omega=a(\delta\lfloor_v+\lfloor_v\delta)\omega$$

so δ'^2 is a module map if δ^2 is. $L_\delta=L_{\delta'}$ also means that the associated covariant derivatives in Theorem 2.8 have the same first term and we already know that they have the same remaining terms as these depend only on $\mathrm{d},\mathrm{i}_\omega$. Hence the covariant derivatives are the same in all degrees.

Conversely, suppose δ,δ' are degree -1 maps of classical type and lead to the same metric inner product and covariant derivative. Then δ,δ' have the same Leibnizator if one argument is in degree 0, as this is the interior product. More generally, as they result in the same covariant derivative on all degrees in Theorem 2.8 we conclude that $L_\delta(\omega,\eta)=L_{\delta'}(\omega,\eta)$ for all $\omega\in\Omega^1$ and all $\eta\in\Omega$. Next we recall the tautological identity (2.5) for L_δ, and the same for $L_{\delta'}$. It follows by induction on the degree of the first argument that $L_\delta=L_{\delta'}$ in all degrees. Hence $\delta'-\delta$ is a degree -1 graded-derivation. A degree -1 graded-derivation is determined by its value on degree 1 as a vector field $\Omega^1(A)\to A$ and takes the form of an interior product along it. ☐

This also applies in the invertible metric case where $\mathrm{i}_\omega=\lfloor_{(\omega,\ )}$ are equivalent constructions. Thus Riemannian geometries are equivalent to δ modulo the addition of an interior product along a vector field. Note that in this case $(\delta+\mathrm{i}_\omega)^2=\delta^2+\delta\mathrm{i}_\omega+\mathrm{i}_\omega\delta=\delta^2+\mathrm{i}_{\mathrm{d}\omega}$ by (2.3). So adding an interior product in general changes δ^2 but not if the corresponding ω is closed.

2.5. **Classical case of a smooth manifold.** Here we illustrate what the above specialises to in more conventional and less algebraic terms in the case of a smooth Riemannian manifold (M,g).

(i) We start with a direct proof of the 7-term relation in Corollary 2.5 in the case of smooth manifold. This is also known from [19, 21] in an equivalent form. In our case, we write the metric, interior product and Levi-Civita connection $g=g_{\mu\nu}\mathrm{d}x^\mu\otimes\mathrm{d}x^\mu$, $\mathrm{i}^\mu=\mathrm{i}_{\mathrm{d}x^\mu}(=g^{\mu\nu}i_{\frac{\partial}{\partial x^\nu}})$ and $\nabla^\mu=\nabla_{\mathrm{d}x^\mu}(=g^{\mu\nu}\nabla_{\frac{\partial}{\partial x^\nu}})$ as shorthand in local coordinates x^μ. The second expressions are in terms of usual interior product and covariant derivative along vector fields. We take the divergence $\delta=g_{\mu\nu}\mathrm{i}^\mu\nabla^\nu)=g^{\mu\nu}\mathrm{i}_\mu\nabla_\nu)$ and

$$\begin{aligned}\delta(\omega\eta)&=g_{\mu\nu}\mathrm{i}^\mu\nabla^\nu(\omega\eta)=g_{\mu\nu}\mathrm{i}^\mu((\nabla^\nu\omega)\eta+\omega\nabla^\nu\eta)\\&=(\delta\omega)\eta+(-1)^{|\omega|}\omega\delta\eta+g_{\mu\nu}\Big((-1)^{|\omega|}(\nabla^\nu\omega)\mathrm{i}^\mu\eta+(\mathrm{i}^\mu\omega)\nabla^\nu\eta\Big)\end{aligned}\tag{2.16}$$

on expanding out using the derivation property of ∇ and the graded-derivation property of i. From this we deduce that

$$\begin{aligned}\delta((\omega\eta)\zeta)&=(\delta\omega\eta)\zeta+g_{\mu\nu}(-1)^{|\omega|+|\eta|}(\nabla^\nu(\omega\eta))\mathrm{i}^\mu\zeta+g_{\mu\nu}(\mathrm{i}^\mu(\omega\eta))\nabla^\nu\zeta+(-1)^{|\omega|+|\eta|}\omega\delta\zeta\\&=(\delta\omega\eta)\zeta+g_{\mu\nu}(-1)^{|\omega|+|\eta|}((\nabla^\nu\omega)+\omega\nabla\eta))\mathrm{i}^\mu\zeta\\&\qquad+g_{\mu\nu}(\mathrm{i}^\mu(\omega)\eta+(-1)^{|\omega|}\omega\mathrm{i}^\mu\eta)\nabla^\nu\zeta+(-1)^{|\omega|+|\eta|}\omega\delta\zeta\\&=(\delta\omega\eta)\zeta+(-1)^{|\omega|+|\eta|}\omega\delta\zeta+g_{\mu\nu}(-1)^{|\eta|}((-1)^{|\omega|}(\nabla^\nu\omega)\eta\mathrm{i}^\mu\zeta+(-1)^{|}\eta|(\mathrm{i}^\mu\omega)\eta\nabla^\nu\zeta)\end{aligned}$$

$$+ g_{\mu\nu}(-1)^{|\omega|}\omega\left((\mathrm{i}^\mu\eta)\nabla^\nu\zeta+(-1)^{|\eta|}(\nabla^\nu\eta)\mathrm{i}^\mu\zeta\right)$$
$$=(\delta\omega\eta)\zeta+(-1)^{|\omega|+|\eta|}\omega\delta\zeta+(-1)^{|\eta|(|\omega|-1)}\eta g_{\mu\nu}\left((-1)^{|\omega|}(\nabla^\nu\omega)\mathrm{i}^\mu\zeta+(\mathrm{i}^\mu\omega)\nabla^\nu\zeta\right)$$
$$+(-1)^{|\omega|}\omega g_{\mu\nu}\left((-1)^{|\eta|}(\nabla^\nu\eta)\mathrm{i}^\mu\zeta+(\mathrm{i}^\mu\eta)\nabla^\nu\zeta\right)$$

where the first equality is (2.16) with $\omega\eta$, ζ in the role of ω,η. We expand out the connection and then the interior product for the second equality. We then combine the $g_{\mu\nu}$ terms vertically to obtain the third equality and move η to the left in one of the groups for the fourth. We now use (2.16) in reverse on the bracketed expressions to complete the direct proof of the relation in Corollary 2.5 in the present case.

Similarly, from (2.16), we have

$$\begin{aligned}L_\delta(\mathrm{d}x^\alpha,\mathrm{d}x^\beta)&+\mathcal{L}_{\mathrm{d}x^\alpha}\mathrm{d}x^\beta+\mathrm{d}\mathrm{d}x^\alpha\perp\mathrm{d}x^\beta\\
&=L_\delta(\mathrm{d}x^\alpha,\mathrm{d}x^\beta)+\mathrm{i}_{\mathrm{d}x^\alpha}\mathrm{d}\mathrm{d}x^\beta+\mathrm{d}\mathrm{i}^\alpha\mathrm{d}x^\beta\\
&=g_{\mu\nu}\left(-(\nabla^\nu\mathrm{d}x^\alpha)\mathrm{i}^\mu\mathrm{d}x^\beta+(\mathrm{i}^\mu\mathrm{d}x^\alpha)\nabla^\nu\mathrm{d}x^\beta\right)+\mathrm{d}g^{\alpha\beta}\\
&=-\nabla^\beta\mathrm{d}x^\alpha+\nabla^\alpha\mathrm{d}x^\beta+\mathrm{d}g^{\alpha\beta}\\
&=\left(\Gamma^{\alpha\beta}{}_\mu-\Gamma^{\beta\alpha}{}_\mu+\frac{\partial}{\partial x^\mu}g^{\alpha\beta}\right)\mathrm{d}x^\mu=-2\Gamma^{\beta\alpha}{}_\mu\mathrm{d}x^\mu=2\nabla^\alpha\mathrm{d}x^\beta\end{aligned}$$

where $\nabla_\nu\mathrm{d}x^\mu=-\Gamma^\mu{}_{\nu\rho}\mathrm{d}x^\rho$ defines the Christoffel symbols as usual and we used $\mathrm{d}^2=0$ and, in the penultimate step, the standard formulae

$$\Gamma^\mu{}_{\nu\rho}=\frac{1}{2}g^{\mu\alpha}(g_{\alpha\nu,\rho}+g_{\alpha\rho,\nu}-g_{\nu\rho,\alpha}),\quad g^{\alpha\beta}{}_{,\rho}=-g^{\alpha\mu}g_{\mu\nu,\rho}g^{\nu\beta}.$$

This directly verifies that Theorem 2.8 correctly recovers the classical Levi-Civita connection from the classical divergence.

(ii) Now suppose we start with M an orientable smooth manifold and $\star$ the Hodge duality defined by $(\omega,\eta)\mathrm{Vol}=\omega\star\eta$ for the metric inner product (,) extended to forms of the same degree. Define δ by $\delta(\omega)=(-1)^{|\omega|+1}\star^{-1}\mathrm{d}\star(\omega)$. This implies

$$\delta(a\omega)=(-1)^{|\omega|+1}\star^{-1}\mathrm{d}\star(a\omega)=a\delta\omega+(-1)^{|\omega|+1}\star^{-1}((\mathrm{d}a)\star\omega)$$

so that

$$\mathrm{i}_\omega\eta=(-1)^{|\eta|-1}\star^{-1}(\omega(\eta^\star)),\quad\forall\omega\in\Omega^1,\ \eta\in\Omega,$$

which is a known formula for the interior product and known to provide a left-derivation, so Lemma 2.2 applies. These conventions mean that δ is adjoint to $-\mathrm{d}$ in the sense of Hodge theory, which is a known convention though not necessarily the most popular one. In this convention the Hodge Laplacian and the Laplace-Beltrami operators coincide in degree 0 rather than with a minus sign. Our degree -2 map $\perp$ extending the interior product is not usually considered in Hodge theory and provides a new ingredient.

Clearly Theorem 2.8 again recovers the Levi-Civita covariant derivative and provides a formula for it in terms of the Lie derivative, interior product and the failure of δ to be a graded-derivation. Because the same applies to δ_∇ in Proposition 2.9, we conclude that the two coincide possibly up to interior product along a vector field. In fact there is no such vector field as $\delta=\delta_\nabla$ is equivalent, given the above, to $\mathrm{d}\omega^\star=g^1(\nabla_{g^2}\omega)^\star$ for all $\omega\in\Omega$. From the formula in Theorem 2.8, it is easy to see that $\star$ commutes with ∇ (there are also other easy ways to see this) so we require $\mathrm{d}\omega=g^1\nabla_{g^2}\omega$ for all $\omega\in\Omega$. But on degree 1 this is just the content of zero torsion (see the proof of Proposition 2.9) and hence holds in all degrees by derivation properties of both sides.

We can also recover the Leibniz property of the Hodge-Laplacian $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$. We note the tautological identity

$$L_\Delta(\omega,\eta) = \mathrm{d}L_\delta(\omega,\eta) + L_\delta(\mathrm{d}\omega,\eta) + (-1)^{|\omega|}L_\delta(\omega,\mathrm{d}\eta),\quad \forall\omega,\eta\in\Omega \tag{2.17}$$

valid for any degree -1 linear map δ on any DGA on writing out the definitions of all the terms, and as observed in [10] in the present graded-commutative context. A special case is $\Delta(a\omega) = (\Delta a)\omega + a\Delta(\omega) + L_\delta(\mathrm{d}a,\omega) + \mathcal{L}_{\mathrm{d}a}\omega$ for all $a\in A$ and $\omega\in\Omega$. By Theorem 2.8, the last two terms are $2\nabla_{\mathrm{d}a}\omega$, giving a 2nd order Leibniz rule normally proven by other means.

2.6. **Ricci tensor.** Here we give a more substantial application of our formula for the Levi-Civita covariant derivative in Theorem 2.8. We suppose that $(\Omega(A),\delta)$ is of classical type with $(\ ,\ )$ invertible, with inverse g, and we let $\Delta = \mathrm{d}\delta+\delta\mathrm{d}$. We assume that $\delta=\delta_\nabla$ as we know can be arranged in the classical setting, see Remark 2.5.

Lemma 2.11. *Let B be a degree 0 linear map on Ω such that $L_B(a,\omega) = 2\nabla_{\mathrm{d}a}\omega$ for all $a\in A$, $\omega\in\Omega$. Then B extends canonically to tensor products as*

$$B(\omega\otimes_A\eta) = B\omega\otimes_A\eta + \omega\otimes_A B\eta + 2\nabla_{g^1}\omega\otimes_A\nabla_{g^2}\eta$$

for all $\omega,\eta\in\Omega$.

Proof. First note that the construction is depends tensorially on g, i.e. only on $g\in\Omega^1\otimes_A\Omega^1$ so it is well-defined. With $\otimes=\otimes_A$,

$$\begin{aligned}B(\omega a\otimes\eta) &= B\omega\otimes a\eta + 2\nabla_{\mathrm{d}a}\omega\otimes\eta + \omega\otimes(Ba)\eta + \omega\otimes aB\eta + 2\nabla_{g^1}a\omega\otimes\nabla_{g^2}\eta\\ &= B\omega\otimes a\eta + 2\nabla_{\mathrm{d}a}\omega\otimes\eta + \omega\otimes B(a\eta) - \omega\otimes 2\nabla_{\mathrm{d}a}\eta\\ &\quad +\nabla_{g^1}\omega\otimes\nabla_{g^2}\eta + 2\omega\otimes\nabla_{\mathrm{d}a}\eta\\ &= B\omega\otimes a\eta + 2\nabla_{\mathrm{d}a}\omega\otimes\eta + \omega\otimes B(a\eta) + 2\nabla_{g^1}\omega\otimes\nabla_{g^2}a\eta - 2\nabla_{\mathrm{d}a}\omega\otimes\eta\\ &= B(\omega\otimes a\eta)\end{aligned}$$

so that the construction stated descends to a map on $\Omega\otimes_A\Omega$. □

The Leibnizator here is characteristic of a covariant second-order operator and holds for the Hodge-Laplacian Δ by Remark 2.5 (the explanation is valid for any $(\Omega(A),\delta)$ of classical type).

Proposition 2.12. *The 'Laplace-Beltrami' operator (1.3) obeys*

$$L_{\Delta_{LB}}(\omega,\eta) = 2(\nabla_{g^1}\omega)\nabla_{g^2}\eta$$

for all $\omega,\eta\in\Omega$. In particular, it is of the type in Lemma 2.11. Moreover,

(1) $W := \Delta_{LB}-\Delta$ is tensorial (an A-module map) and zero in degree 0.

(2) $\Delta_{LB}(g)=0$.

Proof. We have already explained the definition of Δ_{LB} in the Preliminaries in full generality. In our case we compute

$$\begin{aligned}\Delta_{LB}(\omega\eta) &= \nabla_{g^1}\left((\nabla_{g^2}\omega)\eta + \omega\nabla_{g^2}\eta\right) - (\nabla_{\nabla_{g^1}g^2}\omega)\eta - \omega\nabla_{\nabla_{g^1}g^2}\eta\\ &= (\Delta_{LB}\omega)\eta + \omega\Delta_{LB}\eta + (\nabla_{g^1}\omega)\nabla_{g^2}\eta + (\nabla_{g^2}\omega)\nabla_{g^1}\eta.\end{aligned}$$

The last two terms are the same since they depend tensorially on g and hence we can use its symmetry. As a special case, we see that $L_{\Delta_{LB}}(a,\omega) = 2\nabla_{\mathrm{d}a}\omega$ for all $a\in A$, $\omega\in\Omega$, so Lemma 2.11 applies.

Next, (1) In degree 0, $\Delta a = \delta\mathrm{d}a = \mathrm{i}_{g^1}\nabla_{g^2}\mathrm{d}a = \nabla_{g^2}\mathrm{i}_{g^1}\mathrm{d}a - \mathrm{i}_{\nabla_{g^2}g^1}\mathrm{d}a = \mathrm{i}_{g^2}\mathrm{d}\mathrm{i}_{g^1}\mathrm{d} - \mathrm{i}_{\nabla_{g^2}g^1}\mathrm{d}a = \Delta_{LB}a$ for all $a\in A$, using metric compatibility and symmetry of the metric. Hence W is zero on degree 0. Since both Δ, Δ_{LB} have the same Leibnizator when one argument is in A, we then have $W(a\omega) = W(a)\omega + aW(\omega) = aW(\omega)$.

(2) We evaluate half of the desired expression against $\omega \otimes \eta$,

$$\begin{aligned}
&(\omega, \nabla_{g^1} \nabla_{g^2} g'^1)(\eta, g'^2) + (\omega, \nabla_{g^1} g'^1)(\eta, \nabla_{g^2} g'^2)\\
&\quad= (\omega, \nabla_{g^1}((\eta, g'^2)\nabla_{g^2} g'^1)) - (g^1, \mathrm{d}(\eta, g'^2))(\omega, \nabla_{g^2} g'^1)\\
&\qquad +(\omega, \nabla_{g^1} g'^1)(-(\nabla_{g^2}\eta, g'^2) + (g^2, \mathrm{d}(\eta, g'^2))\\
&\quad= (\omega, (\nabla_{g^1}((\eta, g'^2)\nabla_{g^2} g'^1)) - (\omega, \nabla_{g^1} g'^1)(\nabla_{g^2}\eta, g'^2)\\
&\quad= -(\omega, \nabla_{g^1}((g^2, \mathrm{d}(\eta, g'^2))g'^1)) + (\omega, g'^1)(g^1, \mathrm{d}(\nabla_{g^2}\eta, g'^2))\\
&\quad= -(\omega, \nabla_{\mathrm{d}(\eta, g'^2)} g'^1) + (\omega, g'^1)(g^1, \mathrm{d}(\nabla_{g^2}\eta, g'^2)) - (\omega, g'^1)(g^1, \mathrm{d}(g^2, \mathrm{d}(\eta, g'^2)))\\
&\quad= -(\omega, \nabla_{\mathrm{d}(\eta, g'^2)} g'^1) + (g^1, \mathrm{d}(\nabla_{g^2}\eta, \omega)) - (\nabla_{\mathrm{d}(\omega, g'^1)}\eta, g'^2)\\
&\qquad -(g^1, \mathrm{d}(g^2, (\omega, g'^1)\mathrm{d}(\eta, g'^2))) + (\mathrm{d}(\omega, g'^1), \mathrm{d}(\eta, g'^2)))
\end{aligned}$$

where for the first equality we moved a scalar factor inside a covariant derivative and compensated and we used metric compatibility to move over to an action on η. We repeat the first process so as to be able to cancel a metric with its inverse, and repeat this principle. A similar calculation for the other half gives

$$\begin{aligned}
-(\eta, \nabla_{\mathrm{d}(\omega, g'^1)} g'^2) + (g^1, \mathrm{d}(\nabla_{g^2}\omega, \eta)) - (\nabla_{\mathrm{d}(\eta, g'^2)}\omega, g'^1)\\
- (g^1, \mathrm{d}(g^2, (\eta, g'^2)\mathrm{d}(\omega, g'^1))) + (\mathrm{d}(\eta, g'^2), \mathrm{d}(\omega, g'^1)))
\end{aligned}$$

Adding these together using metric compatibility and the Leibniz rule for d gives zero. □

Finally, we prove the relationship with the Ricci tensor. We define the Ricci map by $\widetilde{\mathrm{Ricci}}(\omega) = R(\omega, g^1)g^2$ for all $\omega \in \Omega^1$, where R is the Riemann curvature. This is well-defined and tensorial by the tensoriality of Riemann. The Ricci tensor itself is then defined by $\mathrm{Ricci} = g^1 \otimes_A \widetilde{\mathrm{Ricci}}(g^2) \in \Omega^1 \otimes_A \Omega^1$.

Corollary 2.13. *At least in the case of a classical Riemannian manifold* (M, g), $\mathrm{Ricci} = -\frac{1}{2}\Delta(g)$.

Proof. At least in the classical case one knows that $W = \widetilde{\mathrm{Ricci}}$ on Ω^1 and also that Ricci is symmetric. Then

$$\begin{aligned}
\Delta(g) &= \Delta(g^1) \otimes g^2 + g^1 \otimes \Delta(g^2) + 2\nabla_{g^1} g'^1 \otimes \nabla_{g^2} g'^2\\
&= -W(g^1) \otimes g^2 - g^1 \otimes W(g^2) + \Delta_{LB}(g) = -2\mathrm{Ricci}
\end{aligned}$$

using the symmetry proven. This then provides the stated formula for the Ricci tensor. □

We expect the same result for all $(\Omega(A), \delta)$ of classical type, using the methods as above. The required symmetry of Ricci is straightforward to prove but the calculation of W using our particular methods appears to be more tedious.

3. Riemannian structures induced by central extensions

In this section we see how a metric and covariant derivative arise naturally from an extension problem in noncommutative geometry, including how the datum δ in Section 2 arises naturally.

3.1. **Central extensions of DGAs.** We first formulate the required notion of a 'central extension' of a general DGA $\Omega(A)$ in degree 1 by the algebra $\Omega_{\theta'} = k[\theta']/\langle\theta'^2\rangle$ viewed as a trivial DGA with θ' of degree 1 and $\mathrm{d}\theta' = 0$.

Definition 3.1. By central extension of a DGA $\Omega(A)$ we mean a DGA $\tilde\Omega(A)$ such that

$$\tilde\Omega(A) = \Omega_{\theta'} \otimes \Omega(A)$$

as a vector space and

$$0 \to \Omega_{\theta'} \to \tilde\Omega(A) \to \Omega(A) \to 0$$

as maps of DGA's, where the maps come from the canonical inclusion in the tensor product and by setting $\theta' = 0$. We also require that $\Omega_{\theta'}$ here is graded-central,

$$\theta'\omega = (-1)^{|\omega|}\omega\theta'$$

in $\tilde\Omega(A)$. A morphism of extensions $\Phi : \tilde\Omega(A) \to \tilde\Omega'(A)$ means a map of DGA's such that

$$\begin{array}{ccc} & \tilde\Omega(A) & \\ \Omega_{\theta'} \nearrow\!\!\searrow & \downarrow \Phi & \searrow\!\!\nearrow \Omega(A), \\ & \tilde\Omega'(A) & \end{array} \quad \Phi(\theta') = \theta', \quad \Phi(\omega) = \omega - \frac{\lambda}{2}\theta'\delta(\omega)$$

By a (left) *cleft* central extension we mean a central extension where the canonical linear inclusion of $\Omega(A)$ coming from the tensor product form is a left A-module map.

Clearly the exterior derivative and product of $\tilde\Omega(A)$ must necessarily have the form

$$\omega\cdot\eta = \omega\eta - \frac{\lambda}{2}\theta'[\![\omega,\eta]\!], \quad \mathrm{d}.\omega = \mathrm{d}\omega - \frac{\lambda}{2}\theta'\Delta\omega, \quad \omega,\eta\in\Omega(A)$$

for a bilinear map $[\![\ ,\ ]\!]$ of degree -1 and a linear map Δ of degree 0. This form is necessary since θ' has degree 1. The $\lambda/2$ is a parameter which we insert here in the normalisations of the maps as it may be relevant to a future deformation analysis, but for our purposes we think of it as a non-zero element of the ground field and can set it to 1. The extension is cleft precisely when $[\![a,\ ]\!] = 0$ for all $a \in A$.

Proposition 3.2. *Let $\Omega(A)$ be a DGA on an algebra A. Degree 0,-1 maps Δ : $\Omega(A) \to \Omega(A)$ and $[\![\ ,\ ]\!] : \Omega(A)\otimes\Omega(A) \to \Omega(A)$ respectively define a central extension $\tilde\Omega(A;\Delta,[\![\ ,\ ]\!])$ iff*

$$[\![\omega\eta,\zeta]\!] + [\![\omega,\eta]\!]\zeta = [\![\omega,\eta\zeta]\!] + (-1)^{|\omega|}\omega[\![\eta,\zeta]\!]. \tag{3.1}$$

$$L_\Delta(\omega,\eta) = \mathrm{d}[\![\omega,\eta]\!] + [\![\mathrm{d}\omega,\eta]\!] + (-1)^{|\omega|}[\![\omega,\mathrm{d}\eta]\!] \tag{3.2}$$

for all $\omega,\eta,\zeta\in\Omega(A)$, and $[\Delta,\mathrm{d}] = 0$.

Proof. For associativity we compute

$$\begin{aligned}(\omega\cdot\eta)\cdot\zeta &= \left(\omega\eta + \frac{\lambda}{2}(-1)^{|\omega|+|\eta|}[\![\omega,\eta]\!]\theta'\right)\cdot\zeta \\ &= \omega\eta\zeta + \frac{\lambda}{2}(-1)^{|\omega|+|\eta|+|\zeta|}[\![\omega,\eta]\!]\zeta\theta' + \frac{\lambda}{2}(-1)^{|\omega|+|\eta|+|\zeta|}[\![\omega\eta,\zeta]\!]\theta' \\ \omega\cdot(\eta\cdot\zeta) &= \omega\cdot\left(\eta\zeta + \frac{\lambda}{2}(-1)^{|\eta|+|\zeta|}[\![\eta,\zeta]\!]\theta'\right) \\ &= \omega\eta\zeta + \frac{\lambda}{2}(-1)^{|\eta|+|\zeta|}\omega[\![\eta,\zeta]\!]\theta' + \frac{\lambda}{2}(-1)^{|\omega|+|\eta|+|\zeta|}[\![\omega,\eta\zeta]\!]\theta'\end{aligned}$$

Comparing, we see that we need (3.1). Next, for the Leibniz rule we compute

$$\mathrm{d}.(\omega\cdot\eta) = \mathrm{d}.\left(\omega\eta + \frac{\lambda}{2}(-1)^{|\omega|+|\eta|}[\![\omega,\eta]\!]\theta'\right)$$

$$= \mathrm{d}(\omega\eta) - \frac{\lambda}{2}(-1)^{|\omega|+|\eta|}(\Delta(\omega\eta))\theta' + \frac{\lambda}{2}(-1)^{|\omega|+|\eta|}(\mathrm{d}[\![\omega,\eta]\!])\theta'$$

$$(\mathrm{d}.\omega)\cdot\eta + (-1)^{|\omega|}\omega\cdot\mathrm{d}.\eta = \left(\mathrm{d}\omega - \frac{\lambda}{2}(-1)^{|\omega|}(\Delta\omega)\theta'\right)\cdot\eta + (-1)^{|\omega|}\omega\cdot\left(\mathrm{d}\eta - \frac{\lambda}{2}(-1)^{|\eta|}(\Delta\eta)\theta'\right)$$

$$= (\mathrm{d}\omega)\eta + (-1)^{|\omega|}\omega\mathrm{d}\eta + \frac{\lambda}{2}(-1)^{|\omega|+1+|\eta|}[\![\mathrm{d}\omega,\eta]\!]\theta' + \frac{\lambda}{2}(-1)^{|\eta|+1}[\![\omega,\mathrm{d}\eta]\!]\theta'$$

$$-\frac{\lambda}{2}(-1)^{|\omega|+|\eta|}(\Delta\omega)\eta\theta' - \frac{\lambda}{2}(-1)^{|\eta|+|\omega|}\omega(\Delta\eta)\theta'$$

where we used $\theta'^2 = 0$ and $\mathrm{d}.\theta' = 0$. Comparing, we see that we need (3.2). We also need

$$\mathrm{d}.\mathrm{d}.\omega = \mathrm{d}.(\mathrm{d}\omega - \frac{\lambda}{2}(-1)^{|\omega|}(\Delta\omega)\theta') = \mathrm{d}^2\omega - \frac{\lambda}{2}(-1)^{|\omega|+1}(\Delta\mathrm{d}\omega)\theta' - \frac{\lambda}{2}(-1)^{|\omega|}(\mathrm{d}\Delta\omega)\theta' = 0$$

which requires $[\Delta, \mathrm{d}] = 0$. □

We refer to the pair $(\Delta, [\![\ ,\ ]\!])$ obeying (3.1)-(3.2) and $[\Delta, \mathrm{d}] = 0$ as a *2-cocycle* on the DGA in analogy with the way that central extensions of groups are defined by 2-cocycles. To complete this picture:

Lemma 3.3. *Two cocycles* $(\Delta, [\![\ ,\ ]\!])$ *and* $(\Delta', [\![\ ,\ ]\!]')$ *give isomorphic central extensions iff*

$$\Delta' = \Delta + \mathrm{d}\delta + \delta\mathrm{d}, \quad [\![\omega,\eta]\!]' = [\![\omega,\eta]\!] + L_\delta(\omega,\eta)$$

for some degree -1 linear map $\delta : \Omega(A) \to \Omega(A)$.

Proof. Any map $\Phi : \tilde\Omega(A; \Delta, [\![\ ,\ ]\!]) \to \tilde\Omega(A; \Delta', [\![\ ,\ ]\!]')$ that commutes with the canonical inclusions and projections (a morphism of extensions) must have the form:

$$\omega \mapsto \omega - \frac{\lambda}{2}\theta'\,\delta\omega, \quad \theta' \mapsto \theta', \quad \forall \omega \in \Omega(A),$$

for some degree -1 map δ. One may easily verify from the definitions that this is an isomorphism iff the difference between the $\cdot$ products and d. in the two cases have the form stated. □

We refer to a cocycle of the form

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d}, \quad [\![\omega,\eta]\!] = L_\delta(\omega,\eta), \quad \forall \omega,\eta \in \Omega. \tag{3.3}$$

associated to any degree -1 linear map δ as its *coboundary* and in this case Lemma 3.3 says that central extensions up to isomorphism are classified by cocycles $(\Delta, [\![\ ,\ ]\!])$ up to such coboundaries, i.e. by a form of 2-cohomology. We have not placed the cohomology here into a general context but this is parallel to central extensions of a group being classified by its 2-cohomology.

We have already observed in Section 2 that (3.3) tautologically solves (3.1)-(3.2) for any degree -1 linear map δ, cf [10] in the graded-commutative case. This 'homologically trivial' case is not interesting from our point of view of 'quantisation' as it does not change the DGA but it can still be of interest[1].

We now restrict this degree of equivalence by focussing on cleft extensions, and will then see how Riemannian geometry 'emerges' from this restricted extension problem.

Lemma 3.4. *Let* $\Omega(A)$ *be a standard DGA. In a cleft central extension the* $[\![\ ,\ ]\!]$ *part of the cocycle is uniquely determined by the* Δ *part.*

Proof. Here we are supposing that $[\![a,\ ]\!]=0$ for all $a\in A$. In this case we have

$$L_\Delta(a,\eta)=[\![\mathrm{d}a,\eta]\!],\quad \forall a\in A,\quad \eta\in\Omega(A) \tag{3.4}$$

is a special case of (3.2). Next, we specialise (3.1) to $a\in A$, $\omega,\eta,\zeta\in\Omega(A)$ as

$$[\![a\eta,\zeta]\!]=a[\![\eta,\zeta]\!] \tag{3.5}$$

$$[\![\omega a,\zeta]\!]+[\![\omega,a]\!]\zeta=[\![\omega,a\zeta]\!] \tag{3.6}$$

$$[\![\omega\eta,a]\!]+[\![\omega,\eta]\!]a=[\![\omega,\eta a]\!]+(-1)^{|\omega|}\omega[\![\eta,a]\!] \tag{3.7}$$

We now proceed as follows. From L_Δ we define $[\![\mathrm{d}a,\eta]\!]$ for all η. By (3.5) and the assumption that the DGA is surjective (the standard case) we see that we have defined $[\![\omega,\eta]\!]$ for all $\omega\in\Omega^1$ and all η. Now suppose for a fixed $\zeta\in\Omega(A)$ that $[\![\eta,\zeta]\!]$ has been defined up to η of some degree. Then (3.1) defines $[\![\omega\eta,\zeta]\!]$ for any ω. In this way, assuming $\Omega(A)$ is generated by degree 0 and 1, we have defined $[\![\eta,\zeta]\!]$ for all η. The initial case for the induction where η has degree 1 was already specified for any ζ earlier in the construction. □

For completeness we also list a remaining special case of (3.2),

$$L_\Delta(\omega,a)=\mathrm{d}[\![\omega,a]\!]+[\![\mathrm{d}\omega,a]\!]+(-1)^{|\omega|}[\![\omega,\mathrm{d}a]\!],\quad \forall\omega\in\Omega,\ a\in A \tag{3.8}$$

which will need later.

3.2. **Bimodule covariant derivatives associated to cleft extensions.**

Definition 3.5. We say that a cleft extension $(\Delta,[\![\ ,\ ]\!])$ on a standard DGA $\Omega(A)$ is *n-regular* if

$$\mathrm{j}_\omega(a\mathrm{d}b)=\frac{1}{2}[\![\omega a,b]\!],\quad \forall\omega\in\Omega,\ a,b\in A$$

is a well-defined degree -1 map $\mathrm{j}:\Omega^i\otimes_A\Omega^1\to\Omega^{i-1}$ for $i\le n$. We say that the cleft extension is regular is it is regular for all degrees. We refer to j as 'interior product' and its restriction $(\ ,\ ):\Omega^1\otimes_A\Omega^1\to A$ to degree 1 as 'metric'.

The following is stated for regular extensions but we need only 1-regularity for the metric and a covariant derivative to be defined and 2-regularity for this to be a bimodule covariant derivative acting on degree 1 (which is the main case of interest for Riemannian geometry).

Proposition 3.6. *If $(\Delta,[\![\ ,\ ]\!])$ is a regular cleft extension on a standard DGA $\Omega(A)$ then j is a bimodule map and*

$$\nabla_\omega\eta=\frac{1}{2}[\![\omega,\eta]\!],\quad \forall\omega\in\Omega^1,\quad \eta\in\Omega$$

is a bimodule covariant derivative on Ω with respect to $(\ ,\ )$. Here

$$\sigma:\Omega^1\otimes_A\Omega\otimes_A\Omega^1\to\Omega,\quad \sigma_\omega(\eta\otimes_A\zeta)=\mathrm{j}_{\omega\eta}(\zeta)+\omega\mathrm{j}_\eta(\zeta),\quad \forall\omega,\zeta\in\Omega^1,\ \eta\in\Omega.$$

Moreover, if $(\ ,\ )$ is invertible with metric g then ∇_ω acts on tensor products,

$$\nabla_\omega(\eta\otimes_A\zeta)=\nabla_\omega\eta\otimes_A\zeta+\sigma_\omega(\eta\otimes_A g^1)\otimes_A\nabla_{g^2}\zeta,\quad \forall\eta,\zeta\in\Omega. \tag{3.9}$$

Proof. On 0-forms (3.5) tells us that $\mathrm{j}_{a\omega}=a\mathrm{j}_\omega$ and (3.6) combined with the Leibniz rule tells us that $\mathrm{j}_\omega((\mathrm{d}a)b)=\mathrm{j}_\omega(\mathrm{d}a)b$ so that j becomes a bimodule map. On 1-forms (3.5) tells us that $\nabla_{a\omega}=a\nabla_\omega$. Meanwhile (3.6) tells us that ∇_ω is a covariant derivative in the sense (1.1) given the definition of $(\ ,\ )$. Given this, (3.7) tells us that we have a bimodule covariant derivative in the sense (1.2) provided we define σ as stated. That σ is a bimodule map is immediate from the properties of j. If $(\ ,\ )$ is invertible then a bimodule covariant derivative induces an algebraic bimodule connection and these act on tensor products as in (3.9). □

One might expect in the invertible case that the above action on tensor products is compatible with $\wedge$, which is to say:

$$\nabla_\omega(\eta\zeta) = (\nabla_\omega\eta)\zeta + \sigma_\omega(\eta\otimes_A g^1)\nabla_{g^2}\zeta, \quad \forall\eta,\zeta\in\Omega \tag{3.10}$$

and this can be the case but does not appear always to be true. However, there is always a different kind of generalised Leibniz rule:

Proposition 3.7. *Given a regular cleft extension, the general* $\nabla_\omega = \frac{1}{2}[\![\omega,\ ]\!]$ *for all* $\omega\in\Omega$ *is a left-covariant derivative in the sense*

$$\nabla_{a\omega} = a\nabla_\omega, \quad \nabla_\omega(a\eta) = \nabla_{\omega a}\eta + \mathrm{j}_\omega(\mathrm{d}a)\eta$$

and a bimodule covariant derivative in the sense (1.2) but now with

$$\sigma:\Omega\otimes_A\Omega\otimes_A\Omega^1\to\Omega, \quad \sigma_\omega(\eta\otimes_A\zeta) = \mathrm{j}_{\omega\eta}(\zeta) - (-1)^{|\omega|}\omega\mathrm{j}_\eta(\zeta).$$

Moreover,

$$\nabla_\omega(\eta\zeta) = (\nabla_\omega\eta)\zeta - (-1)^{|\omega|}\omega\nabla_\eta\zeta + \nabla_{\omega\eta}\zeta, \quad \forall\omega,\eta,\zeta\in\Omega. \tag{3.11}$$

$$\frac{1}{2}L_\Delta(\omega,\eta) = (\mathrm{d}\nabla_\omega + (-1)^{|\omega|}\nabla_\omega\mathrm{d})\eta + \nabla_{\mathrm{d}\omega}\eta, \quad \forall\omega,\eta,\in\Omega. \tag{3.12}$$

Proof. That we have a generalised bimodule covariant derivative follows exactly the same argument as the proof of Proposition 3.6, just now using more general forms in (3.5)-(3.7). Moreover, the general (3.1) and (3.2) now become the two displayed equations (3.11) and (3.12) respectively. □

Also observe that when (,) is invertible and noting that in this case the metric g is necessarily central, it is easy to see that there is a potentially different higher-form bimodule covariant derivative

$$\nabla'_\omega = \mathrm{j}_\omega(g^1)\nabla_{g^2}, \quad \sigma'_\omega(\eta\otimes_A\zeta) = \mathrm{j}_\omega(g^1)\sigma_{g^2}(\eta\otimes_A\zeta)$$

which coincides with ∇_ω for $\omega\in\Omega^1$.

Proposition 3.8. *When (,) is invertible, the following are equivalent*
(1) $(\nabla'_\omega,\sigma'_\omega)$ *obey the braided-Leibniz rule (3.10) for all* $\omega\in\Omega^1$.
(2) $(\nabla_\omega,\sigma_\omega)$ *obey the braided-Leibniz rule (3.10) for all* $\omega\in\Omega^1$.
(3)$\nabla_\omega = \nabla'_\omega$ *for all* $\omega\in\Omega$.
In this case the braided-Leibniz rules also hold for all $\omega\in\Omega$ *and*

$$\mathrm{j}_{\omega\eta} = \mathrm{j}_\omega(g^1)(\mathrm{j}_{g^2\eta} + g^2\mathrm{j}_\eta) + (-1)^{|\omega|}\omega\mathrm{j}_\eta, \qquad \forall\omega,\eta\in\Omega.$$

Proof. That (∇',σ') obeys (3.10) is

$$\mathrm{j}_\omega(g^1)\nabla_{g^2}(\eta\zeta) = \mathrm{j}_\omega(g^1)((\nabla_{g^2}\eta)\zeta + \sigma_{g^2}(\eta\otimes\bar g^1)\nabla_{\bar g^2}\zeta)$$

where $\bar g$ is another copy of g. Putting in the properties of ∇ along 1-forms and cancelling, our condition is

$$\mathrm{j}_\omega(g^1)(g^2\nabla_\eta\zeta + \nabla_{g^2\eta}\zeta) = \mathrm{j}_\omega(g^1)(\mathrm{j}_{g^2\eta}(\bar g^1) + g^2\mathrm{j}_\eta(\bar g^1))\nabla_{\bar g^2}\zeta$$

which holds for all $\omega\in\Omega$ (and all η,ζ understood) iff it holds for all $\omega\in\Omega^1$, where it reduces to the condition

$$\nabla_{\omega\eta} - \mathrm{j}_{\omega\eta}(g^1)\nabla_{g^2} = (-1)^{|\omega|}\omega(\nabla_\eta - \mathrm{j}_\eta(g^1)\nabla_{g^2}), \quad \forall\omega\in\Omega^1, \quad \eta\in\Omega$$

On the other hand, this is the condition that (∇,σ) obeys (3.10) given the form of σ and moreover, if it holds for all $\omega\in\Omega^1$ then by induction on the degree of ω, we conclude that $\nabla_\omega - \nabla'_\omega = 0$ for all degrees of ω (given that this vanishes on ω of degree 1). In this case the last displayed condition vanishes identically for all degrees of ω, so (∇,σ) obeys (3.10) for all degrees of ω. Finally, if $\nabla_\omega = \nabla'_\omega$ then $\sigma_\omega = \sigma'_\omega$ since these are uniquely determined, which is the stated condition on j. □

We also see that when the braided-Leibniz rule does hold, ∇_ω along higher forms is given in terms of ∇ along forms of lower degree and hence inductively in terms of the 1-form covariant derivative. Example 3.15 below is an instance where the braided-Leibniz rule holds and the above applies.

We conclude by studying some basic elements of the noncommutative geometry for this class of bimodule covariant derivatives.

Proposition 3.9. *Let $(\nabla_\omega, \sigma_\omega)$ be a bimodule 1-form covariant derivative as in Proposition 3.6 and $(\ ,\ )$ be invertible. Then the algebraic torsion $T = g^1\nabla_{g^2} - \mathrm{d} : \Omega \to \Omega$ is a bimodule map (one says[5] 'torsion compatible') iff*

$$g^1g^2\mathrm{j}_\omega(\zeta) + g^1\mathrm{j}_{g^2\omega}(\zeta) = (-1)^{|\omega|}\omega\zeta, \quad \forall \omega \in \Omega,\ \zeta \in \Omega^1.$$

If this holds and if the braided-Leibniz rule (3.10) holds then T is a derivation.

Proof. The torsion is already a left module map by the connection property and centrality of the metric. For the right module property we use the form of σ to compute

$$\begin{aligned} T(\omega a) &= g^1((\nabla_{g^2}\omega)a + \sigma_{g^2}(\omega \otimes \mathrm{d}a)) - (\mathrm{d}\omega)a - (-1)^{|\omega|}\omega\mathrm{d}a \\ &= (T\omega)a + g^1\mathrm{j}_{g^2\omega}(\mathrm{d}a) + g^1g^2\mathrm{j}_\omega(\mathrm{d}a) - (-1)^{|\omega|}\omega\mathrm{d}a \end{aligned}$$

so we require the condition stated. If this holds and (3.10) holds then

$$\begin{aligned} T(\omega\eta) &= g^1((\nabla_{g^2}\omega)\eta + \sigma_{g^2}(\omega \otimes_A \bar{g}^1)\nabla_{\bar{g}^2}\eta) - \mathrm{d}(\omega\eta) \\ &= (T\omega)\eta - (-1)^{|\omega|}\omega\mathrm{d}\eta + g^1(\mathrm{j}_{g^2\omega}(\bar{g}^1) + g^2\mathrm{j}_\omega(\bar{g}^1)\nabla_{\bar{g}^2} \\ &= (T\omega)\eta - (-1)^{|\omega|}\omega\mathrm{d}\eta + (-1)^{|\omega|}\omega g^1\nabla_{g^2}\eta = (T\omega)\eta + (-1)^{|\omega|}\omega T\eta \end{aligned}$$

by the right-module condition. □

Proposition 3.10. *Let $(\nabla_\omega, \sigma_\omega)$ be a bimodule 1-form covariant derivative as in Proposition 3.6 and $(\ ,\ )$ be invertible. The Laplace-Beltrami operator (1.3) has Leibnizator*

$$L_{\Delta_{LB}}(a, \omega) = \nabla_{2\mathrm{d}a + \mathrm{j}_{g^1g^2}(\mathrm{d}a)}\omega.$$

If $\wedge(g) = 0$ (one says that g is 'quantum symmetric') then $\Delta_{LB} - \Delta$ is a left-module map and

$$\mathrm{Ricci}_\Delta := g^1 \otimes_A (\Delta_{LB} - \Delta)(g^2)$$

is well-defined.

Proof. For 1-form covariant derivative on a DGA Ω and $(\ ,\ )$ invertible, one has Δ_{LB} defined as explained in the Preliminaries. When we have a bimodule covariant derivative then we also have

$$\begin{aligned} \Delta_{LB}(a\omega) &= \nabla_{g^1}\nabla_{g^2}(a\omega) - \nabla_{\nabla_{g^1}g^2}(a\omega) \\ &= \nabla_{g^1}\nabla_{g^2a}\omega - \nabla_{(\nabla_{g^1}g^2)a}\omega + \nabla_{g^1}((\mathrm{j}_{g^2}(\mathrm{d}a)\omega) - \mathrm{j}_{\nabla_{g^1}g^2}(\mathrm{d}a)\omega \\ &= \nabla_{g^1}\nabla_{g^2a}\omega - \nabla_{\nabla_{g^1}(g^2a)}\omega + \nabla_{\sigma_{g^1}(g^2\otimes_A\mathrm{d}a)} \\ &\quad +\nabla_{g^1\mathrm{j}_{g^2}(\mathrm{d}a)}\omega + \mathrm{j}_{g^1}(\mathrm{d}\mathrm{j}_{g^2}(\mathrm{d}a))\omega - \mathrm{j}_{\nabla_{g^1}g^2}(\mathrm{d}a)\omega \\ &= a\Delta_{LB}\omega + (\Delta_{LB}a)\omega + \nabla_{\mathrm{d}a+\sigma_{g^1}(g^2\otimes_A\mathrm{d}a)}\omega \end{aligned}$$

where we used the definition, the bimodule covariant derivative properties and the fact (see the Preliminaries) that the half-curvature ρ depends on the element of $\Omega^1 \otimes_A \Omega^1$, so that $\rho(g^1 \otimes_A g^2a) = \rho(ag^1 \otimes_A g^2) = a\rho(g)$ by centrality of the metric and the covariant derivative properties. We see that

$$L_{\Delta_{LB}}(a, \omega) = \nabla_{\mathrm{d}a+\sigma_{g^1}(g^2\otimes_A\mathrm{d}a)}\omega, \quad \forall a \in A,\ \omega \in \Omega \tag{3.13}$$

quite generally. Putting in the specific form of σ in our case, we immediately obtain the expression stated. Meanwhile, from (3.4) we see that $L_\Delta(a,\omega)=2\nabla_{\mathrm{d}a}\omega$ so of $g^1g^2=0$ then the difference $\Delta_{LB}-\Delta$ is a left A-module map and in that case we can define Ricci_Δ as stated. □

This should be viewed as a working definition and novel approach to the Ricci tensor in noncommutative geometry, motivated by our classical calculations in Section 2.6. It does not necessarily connect up to the trace of the Riemann curvature in general, but does do so in the classical Riemannian manifold case.

3.3. **Flat cleft central extensions and their construction.** To proceed further we say that a central extension is *flat* if Δ (but not necessarily the whole cocycle) is cohomologous to zero. This means that up to an isomorphism we can take $\Delta=0$, which is clearly a natural restriction. According to our analysis of morphisms in Lemma 3.3, this is equivalent to the existence of a degree -1 linear map δ such that $\Delta=\mathrm{d}\delta+\delta\mathrm{d}$. Meanwhile, we have seen that a cleft extension is controlled entirely by Δ and hence now by δ.

Proposition 3.11. *In a flat cleft extension, if δ is regular in the sense of Definition 2.1 then the cleft extension is 1-regular in the sense of Definition 3.5 and (,) coincides with the metric associated to δ.*

Proof. From (3.4) we have

$$\begin{aligned}[\![\mathrm{d}a,b]\!]=L_\Delta(a,b)&=\Delta(ab)-(\Delta a)b-a\Delta(b)=\delta\mathrm{d}(ab)-(\delta\mathrm{d}a)b-a\delta\mathrm{d}b\\&=\delta((\mathrm{d}a)b)+\delta(a\mathrm{d}b)-(\delta\mathrm{d}a)b-a\delta\mathrm{d}b=\mathrm{d}a\grave{\mathrm{i}}_{\mathrm{d}b}+\mathrm{i}_{\mathrm{d}a}\mathrm{d}b=2(\mathrm{d}a,\mathrm{d}b).\end{aligned}$$

where (,) is from Definition 2.1. Then from (3.6) we have

$$2J_{\mathrm{d}a}(b\mathrm{d}c)=[\![(\mathrm{d}a)b,c]\!]=[\![\mathrm{d}a,bc]\!]-[\![\mathrm{d}a,b]\!]c=2(\mathrm{d}a,\mathrm{d}(bc))-2(\mathrm{d}a,\mathrm{d}b)c=2(\mathrm{d}a,b\mathrm{d}c)$$

for all a,b,c, using the bimodule properties of (,). Hence j on degree 1 is well-defined and agrees with (,) from δ. □

For higher degrees one needs a derivation property as in Lemma 2.2 which works well in the graded-commutative case covered later, or a comparable assumption in the general context as in the following theorem.

Theorem 3.12. *Let $\perp$ be a degree -2 bilinear map on a standard DGA $\Omega(A)$ such that $\perp a=a\perp=0$ for all $a\in A$ and*

$$(-1)^{|\eta|}(\omega\eta)\perp\zeta+(\omega\perp\eta)\zeta=\omega\perp(\eta\zeta)+(-1)^{|\omega|+|\eta|}\omega(\eta\perp\zeta),\quad\forall\omega,\eta,\zeta\in\Omega$$

and let δ be regular with

$$\delta(a\omega)-a\delta\omega=\mathrm{d}a\perp\omega,\quad\forall a\in A,\ \omega\in\Omega.$$

Then there is a regular flat cleft extension with

$$\Delta=\mathrm{d}\delta+\delta\mathrm{d},\quad[\![\omega,\eta]\!]=L_\delta(\omega,\eta)+\omega\perp\mathrm{d}\eta-(-1)^{|\omega|}\mathrm{d}\omega\perp\eta-(-1)^{|\omega|}\mathrm{d}(\omega\perp\eta),\quad\forall\omega,\eta\in\Omega.$$

Proof. We first observe that special cases of the $\perp$ identity when one of the forms is in degree 0 tell us that $\perp:\Omega\otimes_A\Omega\to\Omega$ and that this is a bimodule map. Moreover,

$$\Delta_0=0,\quad[\![\ ,\ ]\!]_0=\omega\perp\mathrm{d}\eta-(-1)^{|\omega|}\mathrm{d}\omega\perp\eta-(-1)^{|\omega|}\mathrm{d}(\omega\perp\eta)$$

provide a flat extension. For this we check

$$\begin{aligned}&[\![\omega\eta,\zeta]\!]+[\![\omega,\eta]\!]\zeta-[\![\omega,\eta\zeta]\!]-(-1)^{|\omega|}\omega[\![\eta,\zeta]\!]\\&=(\omega\eta)\perp\mathrm{d}\zeta-(-1)^{|\omega|+|\eta|}\mathrm{d}((\omega\eta)\perp\zeta)-(-1)^{|\omega|+|\eta|}(\mathrm{d}(\omega\eta))\perp\zeta\end{aligned}$$

$$
\begin{aligned}
&+(\omega\perp \mathrm{d}\eta)\zeta-(-1)^{|\omega|}(\mathrm{d}(\omega\perp\eta))\zeta-(-1)^{|\omega|}(\mathrm{d}\omega\perp\eta)\zeta\\
&-\omega\perp \mathrm{d}(\eta\zeta)+(-1)^{|\omega|}\mathrm{d}(\omega\perp(\eta\zeta))+(-1)^{|\omega|}\mathrm{d}\omega\perp(\eta\zeta)\\
&-(-1)^{|\omega|}\omega(\eta\perp \mathrm{d}\zeta)+(-1)^{|\omega|+|\eta|}\omega \mathrm{d}(\eta\perp\zeta)+(-1)^{|\omega|+|\eta|}\omega(\mathrm{d}\eta\perp\zeta)\\
=&(\omega\eta)\perp \mathrm{d}\zeta-(-1)^{|\omega|+|\eta|}\mathrm{d}((\omega\eta)\perp\zeta)-(-1)^{|\omega|+|\eta|}((\mathrm{d}\omega)\eta)\perp\zeta-(-1)^{|\eta|}(\omega \mathrm{d}\eta)\perp\zeta\\
&+(\omega\perp \mathrm{d}\eta)\zeta-(-1)^{|\omega|}\mathrm{d}((\omega\perp\eta)\zeta)+(-1)^{|\omega|}(\omega\perp\eta)\mathrm{d}\zeta-(-1)^{|\omega|}(\mathrm{d}\omega\perp\eta)\zeta\\
&-\omega\perp((\mathrm{d}\eta)\zeta)-\omega\perp(\eta \mathrm{d}\zeta)+(-1)^{|\omega|}\mathrm{d}(\omega\perp(\eta\zeta))+(-1)^{|\omega|}\mathrm{d}\omega\perp(\eta\zeta)\\
&-(-1)^{|\omega|}\omega(\eta\perp \mathrm{d}\zeta)+(-1)^{|\omega|+|\eta|}\mathrm{d}(\omega(\eta\perp\zeta))-(-1)^{|\omega|+|\eta|}(\mathrm{d}\omega)(\eta\perp\zeta)\\
&+(-1)^{|\omega|+|\eta|}\omega(\mathrm{d}\eta\perp\zeta)
\end{aligned}
$$

vanishes. There are 16 terms and they cancel in groups of 4 under application of the 4-term condition on $\perp$ assumed in the statement of the theorem when applied to appropriate elements. For example, the leading term is in a group of 4 which cancel by application to $\omega,\eta,\mathrm{d}\zeta$. Next, for any degree -1 map δ we add its coboundary according to Lemma 3.3 to obtain the stated extension $(\Delta,[\![\ ,\ ]\!])$. This is cleft iff δ obeys the condition stated given that $\perp a=a\perp=0$ for all $a\in A$. This is also half of the assumed regularity of δ (namely that $\mathrm{i}_{\mathrm{d}a}=\mathrm{d}a\perp$). Also, we find $[\![\omega a,b]\!]=L_\delta(\omega a,b)+\omega\perp a\mathrm{d}b)$ so for this to depend only on $a\mathrm{d}b$ we need the other half of the regularity assumption on δ, namely $L_\delta(\omega a,b)=(\omega a)\grave{\mathrm{i}}_{\mathrm{d}b}=\omega\grave{\mathrm{i}}_{a\mathrm{d}b}$. Then we define $\mathrm{j}_\omega(a\mathrm{d}b)=\frac{1}{2}(L_\delta(\omega a,b)+\omega\perp(a\mathrm{d}b))$, or

$$
\mathrm{j}_\omega(\zeta)=\frac{1}{2}(\omega\grave{\mathrm{i}}_\zeta+\omega\perp\zeta),\quad \forall\omega\in\Omega,\ \zeta\in\Omega^1. \tag{3.14}
$$

According to Proposition 3.6, $\nabla_\omega=\frac{1}{2}[\![\omega,\ ]\!]$ then gives us a bimodule covariant derivative, in fact extended to a covariant derivative along ω of all degrees. □

Lemma 3.13. *Let $\Omega(A)$ be a standard DGA and $(\perp,\delta)$ a solution for the data in Theorem 3.12. Then*

$$
\omega\perp'\eta=\omega\perp\eta+(-1)^{|\omega|+1}L_B(\omega,\eta),\qquad \delta'=\delta+B\mathrm{d}-\mathrm{d}B
$$

is also a solution, for any degree -2 bimodule map B. This leaves Δ and $[\![\ ,\ ]\!]$ in Theorem 3.12 and hence the induced metric and covariant derivative unchanged.

Proof. This is a matter of direct verification that $\perp'$ still obeys the 4-term relation in Theorem 3.12. Moreover.

$$
\begin{aligned}
\delta'(a\omega)-a\delta'&=\mathrm{d}a\perp\omega+B\mathrm{d}(a\omega)-\mathrm{d}B(a\omega)-a(B\mathrm{d}-\mathrm{d}B)\omega\\
&=\mathrm{d}a\perp\omega+B((\mathrm{d}a)\omega)-(\mathrm{d}a)B\omega=\mathrm{d}a\perp'\omega
\end{aligned}
$$

for all $a\in A$, $\omega\in\Omega$. Hence the flatness condition is maintained as is the left half of the regularity of δ'. We also have

$$
\begin{aligned}
L_{\delta'}(\omega a,b)&=L_\delta(\omega a,b)+B\mathrm{d}(\omega ab)-\mathrm{d}B(\omega ab)-(B\mathrm{d}(\omega a))b-(\mathrm{d}B(\omega a))b\\
&=L_\delta(\omega a,b)+B(\mathrm{d}(\omega ab)-(\mathrm{d}(\omega a))b)-\mathrm{d}((B(\omega a))b))+(\mathrm{d}B(\omega a))b\\
&=L_\delta(\omega a,b)+(-1)^{|\omega|}B(\omega a\mathrm{d}b)-(-1)^{|\omega|}(B\omega)a\mathrm{d}b
\end{aligned}
$$

using that B is a bimodule map. The last two terms depend only on $a\mathrm{d}b$ so the other half of the required regularity condition is also maintained with $\grave{\mathrm{i}}'_\omega(\eta)=\grave{\mathrm{i}}_\omega(\eta)+(-1)^{|\omega|}(B(\omega\eta)-B(\omega)\eta)$. Since the extension is cleft, $[\![\ ,\ ]\!]$ is determined by Δ and hence is unchanged as the latter is unchanged by the addition of $\mathrm{d}(B\mathrm{d}-\mathrm{d}B)+(B\mathrm{d}-\mathrm{d}B)\mathrm{d}=0$. □

In particular, we can start with the zero solution, then any degree -2 bimodule map B generates a 'coboundary' solution but with trivial end product.

3.4. **Noncommutative inner flat cleft extensions.** Before we do the classical case we present a class of 'quantum' or noncommutative examples. We focus on the inner case, which is not possible classically, where we assume the existence of a 1-form $\theta \in \Omega^1$ such that $\mathrm{d}\omega = \theta\omega - (-1)^{|\omega|}\omega\theta$ for all $\omega \in \Omega(A)$. The latter is assumed to be of standard type.

Proposition 3.14. *If $\Omega(A)$ is of standard type and inner via $\theta \in \Omega^1$ and if $\perp$ solves the 4-term condition in Theorem 3.12 then $\delta = \theta \perp$ completes the data for a regular flat cleft extension. Here*

$$j_\omega(\zeta) = \frac{1}{2}\omega \perp \zeta, \quad \Delta = 2\nabla_\theta - \theta^2 \perp$$

$$\nabla_\omega = -\frac{1}{2}L_{\perp\theta}(\omega,\ ), \quad \sigma_\omega(\eta \otimes_\zeta) = \frac{1}{2}\Big((\omega\eta) \perp \zeta - (-1)^{|\omega|}\omega(\eta \perp \zeta)\Big)$$

for all $\omega, \eta \in \Omega$ and $\zeta \in \Omega^1$. The cocycle is $[\![\omega,\ ]\!] = 2\nabla_\omega$.

Proof. We have assumed $\perp$ and clearly $\delta(a\omega) - a\delta\omega = \theta a \perp \omega - a\theta \perp \omega = \mathrm{d}a \perp \omega$. Similarly $L_\delta(\omega, a) = \delta(\omega a) - (\delta\omega)a = (\theta \perp \omega a) - (\theta \perp \omega)a = 0$ by the bimodule properties of $\perp$. Hence δ is regular and we have a regular flat cleft extension. Clearly $\mathrm{j}_\omega(\mathrm{d}a) = \frac{1}{2}\omega \perp \mathrm{d}a$. To compute the covariant derivative,

$$\begin{aligned} &\omega \perp \mathrm{d}\eta - (-1)^{|\omega|}\mathrm{d}\omega \perp \eta - (-1)^{|\omega|}\mathrm{d}(\omega \perp \eta) \\ &\quad = \omega \perp (\theta\eta) - (-1)^{|\eta|}\omega \perp (\eta\theta) - (-1)^{|\omega|}(\theta\omega) \perp \eta + (\omega\theta) \perp \eta \\ &\qquad -(-1)^{|\omega|}\theta(\omega \perp \eta) + (-1)^{|\eta|}(\omega \perp \eta)\theta \\ &\quad = (\omega \perp \theta)\eta - (\omega\eta) \perp \theta - \theta \perp (\omega\eta) + (-1)^{|\omega|}\omega(\theta \perp \eta) \\ &\qquad +(\theta \perp \omega)\eta + (-1)^{|\omega|}\omega(\eta \perp \theta) \\ &\quad = -L_{\theta\perp}(\omega,\eta) - L_{\perp\theta}(\omega,\eta) \end{aligned}$$

using the definition of d in the inner case and applying the 4-term identity to the terms pairwise, 3 times. The generalised braiding is from Proposition 3.7. For the Hodge Laplacian,

$$\begin{aligned} \Delta\omega &= \theta(\theta \perp \omega) + (-1)^{|\omega|}(\theta \perp \omega)\theta + \theta \perp (\theta\omega) - (-1)^{|\omega|}\theta \perp (\omega\theta) \\ &= -\theta^2 \perp \omega + (\theta \perp \theta)\omega - (\theta\omega) \perp \theta - \theta(\omega \perp \theta) = -\theta^2 \perp \omega - L_{\perp\theta}(\theta,\omega) \end{aligned}$$

using the definition of Δ, d and two applications of the 4-term identity for $\perp$. □

We have covered the case of ∇_ω along forms of all degrees but $\omega \in \Omega^1$ corresponds to a usual covariant derivative. Note also that $(\ ,\ ) = \frac{1}{2}\perp$ in this class of examples and Lemma 3.13 provides a construction for $\perp$.

Example 3.15. We let $A = k(\{x,y\}) = k \oplus k$, the algebra of functions on 2 points and $\Omega(A)$ its universal calculus. Here for any unital algebra, $\Omega^n \subset A^{\otimes(n+1)}$ is the sub-bimodule such that the product applied to any two adjacent copies is zero. The exterior derivative is $\mathrm{d}(a_0 \otimes \cdots \otimes a_n) = \sum_i (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_n$. In practice in our case it is better to think of A as functions on the group $\mathbb{Z}_2$. The universal calculus on a group is bicovariant and hence has a basis of left-invariant 1-forms, and also the calculus is inner. In our case it is generated by A and a single 1-form θ, so in degree n the n-form θ^n forms a basis. If $f \in A$, let $\bar{f}(x) = f(y)$, $\bar{f}(y) = f(x)$. Then the relations of the DGA are

$$\theta f = \bar{f}\theta, \quad \mathrm{d}f = (\bar{f} - f)\theta, \quad \mathrm{d}\theta^n = (1 - (-1)^n)\theta^{n+1}.$$

We next solve the condition in Theorem 3.12. Since $\perp$ is a bimodule map it is enough to define it on the invariant forms, i.e. on powers of θ, and we take for example

$$\theta^m \perp \theta^n = 2(-1)^{m+1}mn\theta^{m+n-2},$$

where the 2 also fixes a particular normalisation that we will need.

According to Proposition 3.14 we now have a regular flat cleft extension and the associated codifferential, bimodule covariant derivative, invertible metric and Laplacian are

$$\delta(f\theta^n)=2\bar f n\theta^{n-1},\quad \nabla_\theta(f\theta^n)=(f-(-1)^n\bar f)\theta^n,\quad \sigma_\theta(f\theta^n\otimes f'\theta)=(-1)^n\bar f\theta^n\bar f'$$

$$(f\theta,f'\theta)=f\bar f',\quad g=\theta\otimes\theta,\quad \Delta\omega=2(\nabla_\theta\omega+2|\omega|\omega),\quad \forall f,f'\in A,\ \omega\in\Omega.$$

These computations are immediate from the general structure in Proposition 3.14 applied in our case. We find now that the connection is torsion free and metric-compatible. Thus, the torsion is

$$T(f\theta^n)=\theta\nabla_\theta(f\theta^n)-(\bar f-(-1)^n f)\theta^{n+1}=0,$$

while from the action (3.9) of tensor products we have

$$\nabla_\theta g=\nabla_\theta(\theta\otimes\theta)=\nabla_\theta\theta\otimes\theta+\sigma_\theta(\theta\otimes\theta)\otimes_A\nabla_\theta\theta=0$$

using $\nabla_\theta=2\theta$ and $\sigma_\theta(\theta\otimes\theta)=-\theta$.

One also sees that δ^2 is not zero but commutes with functions and that, more surprisingly, the canonical Laplace-Beltrami operator vanishes,

$$\Delta_{LB}=\nabla_{g^1}\nabla_{g^2}-\nabla_{\nabla_{g^1}g^2}=(\nabla_\theta)^2-2\nabla_\theta=0.$$

This happens to be tensorial in our example, which is possible as the metric is not quantum symmetric. One also finds that the noncommutative Riemann curvature vanishes. Using the algebraic form, this is

$$R(\theta^n)=(\mathrm{d}\otimes\mathrm{id}-(\wedge\otimes\mathrm{id})(\mathrm{id}\otimes\nabla))\nabla\theta^n=\mathrm{d}2\theta\otimes\theta^n-2\theta2\theta\otimes\theta^n=0$$

for all n odd (and zero in any case if n is even).

Finally, one may similarly compute from $L_{\perp\theta}(\theta^m,\theta^n)$ that

$$\nabla_{\theta^m}(\theta^n)=(-1)^{m+1}m\theta^{m-1}\nabla_\theta=j_{\theta^m}(\theta)\nabla_\theta$$

so that the braided-Leibniz rule applies by Proposition 3.8. One can verify this directly as a check.

3.5. **Classical type flat cleft extensions.** Finally we specialise to the case where $\Omega(A)$ is graded-commutative case and of 'classical type'. First we consider the general theory of cleft extensions of classical type.

Proposition 3.16. *For a regular cleft extension, on $\Omega(A)$ of classical type, the associated $(\ ,\ )$ is symmetric, the interior product is a graded derivation and the covariant derivative in Proposition 3.6 has symmetric part*

$$\nabla_\omega\eta+\nabla_\eta\omega=\mathrm{j}_{\mathrm{d}\omega}(\eta)+\mathrm{j}_{\mathrm{d}\eta}(\omega)+\mathrm{d}(\omega,\eta),\quad \forall\omega,\eta\in\Omega^1$$

and has torsion and metric-compatibility tensor obeying

$$T(\zeta,\omega)(\eta)+T(\zeta,\eta)(\omega)=C_\zeta(\omega,\eta),\quad \forall\omega,\eta,\zeta\in\Omega^1.$$

In the invertible case the Laplace-Beltrami operator obeys

$$L_{\Delta_{LB}}(a,\omega)=2\nabla_{\mathrm{d}a}\omega,\quad a\in A,\ \omega\in\Omega$$

so that Ricci_Δ *is defined.*

Proof. From (3.4) we have $L_\Delta(a,b)=[\![\mathrm{d}a,b]\!]=2(\mathrm{d}a,\mathrm{d}b)$ which is symmetric by graded-commutativity. Next, comparing (3.6),(3.7) and using the graded-commutativity and (3.5) we see that $[\![\ ,a]\!]$ and hence $\mathrm{j}_{(\ )}(\mathrm{d}a)$ is a graded derivation. This is also the map $\mathrm{i}_{\mathrm{d}a}$ in the general theory of form-covariant derivatives in Section 2.2. Similarly, from (3.8) and (3.4) and graded-commutativity we have

$$\nabla_{\mathrm{d}a}\omega = \frac{1}{2}L_\Delta(a,\omega) = \frac{1}{2}L_\Delta(\omega,a) = -\nabla_\omega \mathrm{d}a + \mathrm{j}_{\mathrm{d}\omega}(\mathrm{d}a) + \mathrm{d}\mathrm{j}_\omega(\mathrm{d}a)$$

from which we conclude the stated symmetry of the covariant derivative. The torsion result is then immediate from Lemma 2.6 after allowing for the change of notation. Also, since g is symmetric, the Laplace-Beltrami operator in Proposition 3.10 has the stated Leibnizator (same as the Hodge Laplacian). □

The generalised covariant derivatives along higher forms also apply as in the general case, but with σ trivial in the sense $\sigma_\omega(\eta\otimes_A\zeta) = \mathrm{j}_\omega(\zeta)\eta$. Thus,

Corollary 3.17. *For a regular cleft extension on $\Omega(A)$ of classical type the 'extended covariant derivative'*

$$\nabla_\omega := \frac{1}{2}[\![\omega,\ ]\!]$$

on Ω has degree $|\omega|-1$ and obeys

$$\nabla_{a\omega} = a\nabla_\omega,\quad \nabla_\omega(a\eta) = a\nabla_\omega\eta + \mathrm{j}_\omega(\mathrm{d}a)\eta,\quad \nabla_1\eta = \nabla_\omega 1 = 0,$$

$$\frac{1}{2}L_\Delta(\omega,\ ) = [\mathrm{d},\nabla_\omega\} + \nabla_{\mathrm{d}\omega},\quad \forall a\in A,\ \omega,\eta\in\Omega.$$

If, moreover, $(\ ,\ )$ is invertible and ∇_ω is a derivation for all $\omega\in\Omega^1$ then ∇_ω is a graded-derivation for all $\omega\in\Omega$ and

$$\nabla_{\omega_1\cdots\omega_m} = \sum_{i=1}^m (-1)^{i-1}\omega_1\cdots\widehat{\omega_i}\cdots\omega_m\nabla_{\omega_i},\quad \forall\omega_i\in\Omega^1,$$

where the hat denotes omission.

Proof. We bring together some of the properties of the solution for $[\![\ ,\ ]\!]$ obtained in the course of Theorem 3.18. These include (3.1)-(3.2) by definition of $\nabla_\omega = \frac{1}{2}[\![\omega,\ ]\!]$. In this case the explicit formula follows from

$$\nabla_{\omega\eta} = (-1)^{(|\omega|-1)|\eta|}\eta\nabla_\omega + (-1)^{|\omega|}\omega\nabla_\eta,\quad \forall\omega,\eta\in\Omega$$

deduced from (3.1) when ∇_ω is a graded-derivation. We use Proposition 3.8. □

We now consider the converse direction. In Section 2 we had a notion of δ of classical type and we assume this now except *without* the δ^2 tensorial and *without* the symmetry of δ. Thus we assume that δ is regular, see Definition 2.1, and that the conditions in part (2) of Lemma 2.2 apply.

Theorem 3.18. *Let $\Omega(A)$ be of classical type and δ a regular degree -1 map obeying the derivation conditions in Lemma 2.2. Then $\perp$ defined on degree 1 by $\mathrm{d}a\perp\omega = \delta(a\omega) - a\delta\omega$ extends as a graded-derivation of appropriate degree and Theorem 3.12 provides a regular flat cleft extension*

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d},\quad [\![\omega,\eta]\!] = L_\delta(\omega,\eta) + \mathcal{L}_\omega\eta - (-1)^{|\omega|}(\mathrm{d}\omega)\perp\eta,\quad \forall\omega,\eta\in\Omega(A)$$

with associated metric $(\omega,\eta) = \frac{1}{2}(\omega\perp\eta + \eta\perp\omega)$ and $\nabla_\omega = \frac{1}{2}[\![\omega,\ ]\!]$. Here

$$\mathcal{L}_\omega\eta = \omega\perp\mathrm{d}\eta - (-1)^{|\omega|}\mathrm{d}(\omega\perp\eta),\quad \forall\omega,\eta\in\Omega$$

extends the usual Lie derivative as a degree $|\omega|-1$ derivation. The above provides the unique cleft central extension with the given Δ. If δ is of classical type then the covariant derivative is torsion free (and hence metric compatible) and is a derivation.

Proof. We let $\perp\colon \Omega^1\otimes_A\Omega^1\to A$ be defined by $\delta(a\omega)-a\delta\omega=\mathrm{d}a\perp\omega$ (so that $\eta\perp\omega=\mathfrak{i}_\eta(\omega)$ for $\omega,\eta\in\Omega^1$ in the notation of Section 2.1). This extends to to a map

$$\omega_1\cdots\omega_m\perp\eta_1\cdots\eta_n=\sum_{i,j}(-1)^{i+j}(\omega_i\perp\eta_j)\omega_1\cdots\widehat{\omega_i}\cdots\omega_m\eta_1\cdots\widehat{\eta_j}\cdots\eta_n$$

much as in Section 2. This is antisymmetric in the ω_i factors and the η_i factors, and hence is well-defined on $\Omega\otimes_A\Omega$. Also as in Section 2, $\omega\perp$ is a degree $|\omega|-2$ derivation and $\perp\eta$ similarly obeys

$$(\omega\omega')\perp\eta=(-1)^{|\omega'|(|\eta|-1)}(\omega\perp\eta)\omega'+(-1)^{|\omega|}\omega(\omega'\perp\eta),\quad\forall\omega,\omega',\eta\in\Omega.\tag{3.15}$$

Then the 4-term relation required in Theorem 3.12 holds as

$$\begin{aligned}(-1)^{|\eta|}(\omega\eta)\perp\zeta-(-1)^{|\omega|+|\eta|}\omega(\eta\perp\zeta)&=(-1)^{|\eta|}(-1)^{|\eta|(|\zeta|-1)}(\omega\perp\zeta)\eta\\&=(-1)^{|\eta||\zeta|}(\omega\perp\zeta)\eta=(-1)^{|\eta||\omega|}(-1)^{|\eta|(|\omega|+|\zeta|-2)}(\omega\perp\zeta)\eta\\&=(-1)^{|\eta|(|\omega|-2)}\eta(\omega\perp\zeta)=\omega(\eta\perp\zeta)-(\omega\perp\eta)\zeta.\end{aligned}$$

Next, by the derivation assumption in Lemma 2.2 we know that δ is such that $L_\delta(a,\omega)=\mathrm{d}a\perp\omega$ for all $\omega\in\Omega$, as needed for flatness. This also meets the regularity requirement and we have a regular flat cleft extension. The interior product is $\mathfrak{j}_\omega(\mathrm{d}a)=\frac{1}{2}(L_\delta(\omega,a)+\omega\perp\mathrm{d}a)=\frac{1}{2}(L_\delta(a,\omega)+\omega\perp\mathrm{d}a)$ due to the graded-commutativity. Thus

$$\mathfrak{j}_\omega(\zeta)=\frac{1}{2}(\zeta\perp\omega+\omega\perp\zeta),\quad\omega\in\Omega,\ \zeta\in\Omega^1.\tag{3.16}$$

Using (3.15), it follows that $\mathfrak{j}_{(\ )}(\zeta)$ is a degree -1 derivation for all $\zeta\in\Omega^1$ extending the metric $(\ ,\ )$. It therefore coincides with the map $\mathfrak{i}_\zeta$ in Section 2.2 which developed the general theory of covariant derivatives on $\Omega(A)$ of classical type. The covariant derivative also has precisely the form in Theorem 2.8 but generalises it as we have not required the symmetry condition on δ nor that δ^2 to be tensorial. Because we have not assumed the symmetry condition, the map $\mathfrak{i}$ associated to δ in Section 2.1 is no longer the interior product associated to the metric as used in Section 2.2; we are denoting the latter now as $\mathfrak{j}$. When we do have δ of classical type we see that $(\ ,\ )=\perp$ on degree 1 and the theory of Section 2 applies directly, with $\perp$ now having the same meaning as in Section 2.1. Uniqueness is from Lemma 3.4.

Finally, we prove that

$$\mathcal{L}_\omega(\eta\eta')=(\mathcal{L}_\omega\eta)\eta'+(-1)^{(|\omega|-1)|\eta|}\eta\mathcal{L}_\omega\eta',\quad\forall\omega,\eta,\eta'\in\Omega.$$

This follows straightforwardly from

$$\mathcal{L}_\omega(\eta\eta')=\omega\perp\Big((\mathrm{d}\eta)\eta'+(-1)^{|\eta|}\eta\mathrm{d}\eta'\Big)-(-1)^{|\omega|}\mathrm{d}(\omega\perp(\eta\eta'))$$

on computing further via $\omega\perp$ a degree $|\omega|-2$ derivation and then comparing with the right hand side computed from the definition. This justifies our term 'extended Lie derivative'. Note that $\mathcal{L}_\omega=\omega\perp\mathrm{d}-(-1)^{|\omega|}\mathrm{d}\omega\perp$ reduces to the Cartan formula for the usual Lie derivative when $\omega\in\Omega^1$. □

This puts the Levi-Civita covariant derivative of Section 2 into a wider context coming from the theory of flat cleft extensions. This more general theory applies to more general BV structures on $\Omega(A)$ of classical type as we do not assume that δ is symmetric. Note that by using the freedom in Lemma 3.13 one could change the antisymmetric part of $\perp$ on degree 1 and hence potentially make it symmetric but not necessarily obtaining or retaining other desired properties of δ.

The uniqueness in Theorem 3.18 is an analogue of the way that in Riemannian geometry the Levi-Civita covariant derivative is uniquely determined by the metric, in the present case encoded in the choice of $\perp$ in degree 1 or more precisely by δ giving this. Theorem 3.18 also achieves our goal of putting in a proper context formulae in [13, Sec. 2] where $A = C^\infty(M)$ on a Riemannian manifold (M, g).

Corollary 3.19. *Every classical Riemannian manifold M has an 'almost commutative' exterior algebra $\tilde\Omega(M)$ with extension data*

$$[\![a, \eta]\!] = 0, \quad [\![\omega, a]\!] = 2(\mathrm{d}a, \omega), \quad [\![\omega, \eta]\!] = 2\nabla_\omega \eta, \quad \forall a \in C^\infty(M),\ \omega, \eta \in \Omega^1(M)$$

and Δ the Hodge Laplacian. We obtain the relations and differential

$$[a, \omega] = \lambda(\mathrm{d}a, \omega)\theta', \quad \{\omega, \eta\} = \lambda(\mathcal{L}_\omega \eta + \mathrm{i}_\eta \mathrm{d}\omega)\theta' \quad [a, \theta'] = \{\omega, \theta'\} = \theta'^2 = 0$$
$$\mathrm{d}.a = \mathrm{d}a - \tfrac{\lambda}{2}(\Delta_{LB} a)\theta', \quad \mathrm{d}.\omega = \mathrm{d}\omega + \tfrac{\lambda}{2}((\Delta_{LB} - \mathrm{Ricci})\omega)\theta'$$

for all $a \in C^\infty(M)$, $\omega, \eta \in \Omega^1$, essentially as in [13, Sec. 2] *in the case* $\mathrm{d}\theta' = 0$.

Proof. This follows from Theorem 3.18 specialised to the classical case as in Remark 2.5. The $\cdot$ product and d. from Proposition 3.2 then give the commutation relations and derivative as stated. This is not quite the generality in [13] where we did not assume that $\mathrm{d}\theta' = 0$ (we come to this later) and we have used (2.14) to simplify in terms of the Lie derivative. There is also a change of sign of λ compared to [13] and we did not give $\mathrm{d}.\omega$ so explicitly as we do now. The conversion from the Hodge-Laplacian to the Laplace-Beltrami is standard but note that by the 2nd order Leibniz rule in Remark 2.5 the leading part of Δ in our conventions agrees with the Laplace-Beltrami operator, after which the coefficient of Ricci can be fixed by the identity $[\Delta_{LB}, \mathrm{d}]a = \mathrm{Ricci}(\mathrm{d}a)$ used in [13] as being equivalent to $[\Delta, \mathrm{d}] = 0$. □

Moreover, since $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$, this cocycle extension is isomorphic to a non-cleft central extension of $\Omega(M)$ by cocycle

$$\Delta = 0, \quad [\![\omega, \eta]\!] = \mathcal{L}_\omega \eta - (-1)^{|\omega|}(\mathrm{d}\omega) \perp \eta, \quad \forall \omega, \eta \in \Omega(M) \tag{3.17}$$

which has the same commutation relations but explicitly does not deform d.

3.6. **Further extension of a flat cleft extension.** Here to cover the case $\mathrm{d}\theta' \neq 0$, which turns out to be necessary for later sections.

Proposition 3.20. *Let $\Omega(A)$ be a standard DGA. Its flat central extension constructed in Theorem 3.12 has a further extension $\tilde{\tilde\Omega}(A) = \Omega(A) \oplus \Omega(A)\theta' \oplus \Omega(A)\mathrm{d}\theta'$ with product and exterior derivative*

$$\omega \cdot \eta = \omega\eta + \frac{\lambda}{2}(-1)^{|\omega|+|\eta|}[\![\omega, \eta]\!]\theta' - \frac{\lambda}{2}(-1)^{|\omega|}(\omega \perp \eta)\mathrm{d}\theta'$$
$$\mathrm{d}.\omega = \mathrm{d}\omega - \frac{\lambda}{2}(-1)^{|\omega|}(\Delta\omega)\theta' + \frac{\lambda}{2}(\delta\omega)\mathrm{d}\theta', \quad \forall \omega, \eta \in \Omega(A),$$

where $\theta'^2 = \theta'\mathrm{d}\theta' = (\mathrm{d}\theta')\theta' = \{\omega, \theta'\} = 0$. Moreover $\tilde{\tilde\Omega}(A) \to \tilde\Omega(A) \to \Omega(A)$ are surjections of DGAs successively setting $\mathrm{d}\theta' = 0$ and $\theta' = 0$.

Proof. We define of course $\mathrm{d}.\theta' = \mathrm{d}\theta'$ as the notation suggests. For the Leibniz rule we recompute the proof of Proposition 3.2,

$$\mathrm{d}.(\omega \cdot \eta) = \mathrm{d}.\left(\omega\eta + \frac{\lambda}{2}(-1)^{|\omega|+|\eta|}[\![\omega, \eta]\!]\theta' - \frac{\lambda}{2}(-1)^{|\omega|}(\omega \perp \eta)\mathrm{d}\theta'\right)$$
$$= \mathrm{d}(\omega\eta) - \frac{\lambda}{2}(-1)^{|\omega|+|\eta|}(\Delta(\omega\eta))\theta' + \frac{\lambda}{2}(\delta(\omega\eta))\mathrm{d}\theta'$$

$$+\frac{\lambda}{2}(-1)^{|\omega|+|\eta|}(\mathrm{d}[\![\omega,\eta]\!]\theta'+(-1)^{|\omega|+|\eta|-1}[\![\omega,\eta]\!]\mathrm{d}\theta')-\frac{\lambda}{2}(-1)^{|\omega|}(\mathrm{d}(\omega\perp\eta))\mathrm{d}\theta'$$

$$\begin{aligned}
&(\mathrm{d}\cdot\omega)\cdot\eta+(-1)^{|\omega|}\omega\cdot\mathrm{d}.\eta\\
&=\left(\mathrm{d}\omega-\frac{\lambda}{2}(-1)^{|\omega|}(\Delta\omega)\theta'+\frac{\lambda}{2}(\delta\omega)\mathrm{d}\theta'\right)\cdot\eta+(-1)^{|\omega|}\omega\cdot\left(\mathrm{d}\eta-\frac{\lambda}{2}(-1)^{|\eta|}(\Delta\eta)\theta'+\frac{\lambda}{2}(\delta\eta)\mathrm{d}\theta'\right)\\
&=(\mathrm{d}\omega)\eta+(-1)^{|\omega|}\omega\mathrm{d}\eta+\frac{\lambda}{2}(-1)^{|\omega|+1+|\eta|}[\![\mathrm{d}\omega,\eta]\!]\theta'+\frac{\lambda}{2}(-1)^{|\eta|+1}[\![\omega,\mathrm{d}\eta]\!]\theta'\\
&-\frac{\lambda}{2}(-1)^{|\omega|+1}(\mathrm{d}\omega)\perp\eta\mathrm{d}\theta'-\frac{\lambda}{2}(\omega\perp\mathrm{d}\eta)\mathrm{d}\theta'-\frac{\lambda}{2}(-1)^{|\omega|+|\eta|}(\Delta\omega)\eta\theta'\\
&-\frac{\lambda}{2}(-1)^{|\eta|+|\omega|}\omega(\Delta\eta)\theta'+\frac{\lambda}{2}(\delta\omega)\eta\mathrm{d}\theta'+\frac{\lambda}{2}(-1)^{|\omega|}\omega\delta\eta\mathrm{d}\theta'
\end{aligned}$$

where we used $\theta'^2=0$ and $\theta'\mathrm{d}\theta'=0$. We already have equality for all terms except those with $\mathrm{d}\theta'$ which had been ignored before. Equality of these is exactly the formula for $[\![\omega,\eta]\!]$ in Theorem 3.12. We also need

$$\begin{aligned}
\mathrm{d}.\mathrm{d}.\omega&=\mathrm{d}.(\mathrm{d}\omega-\frac{\lambda}{2}(-1)^{|\omega|}(\Delta\omega)\theta'+\frac{\lambda}{2}(\delta\omega)\mathrm{d}\theta')\\
&=\mathrm{d}^2\omega-\frac{\lambda}{2}(-1)^{|\omega|+1}(\Delta\mathrm{d}\omega)\theta'+\frac{\lambda}{2}(\delta\mathrm{d}\omega)\mathrm{d}\theta'\\
&\quad-\frac{\lambda}{2}(-1)^{|\omega|}((\mathrm{d}\Delta\omega)\theta'+(-1)^{|\omega|}(\Delta\omega)\mathrm{d}\theta')+\frac{\lambda}{2}(\mathrm{d}\delta\omega)\mathrm{d}\theta'=0
\end{aligned}$$

where the new $\mathrm{d}\theta'$ terms cancel in view of the definition of Δ as the Hodge Laplacian. Last but not least, we have to check associativity for the new product. Referring to the proof of Proposition 3.2 the new terms are

$$\begin{aligned}
(\omega\cdot\eta)\cdot\zeta&=\cdots-\frac{\lambda}{2}(-1)^{|\omega|+|\eta|}(\omega\eta)\perp\zeta\mathrm{d}\theta'-\frac{\lambda}{2}(-1)^{|\omega|}(\omega\perp\eta)\zeta\mathrm{d}\theta'\\
\omega\cdot(\eta\cdot\zeta)&=\cdots-\frac{\lambda}{2}(-1)^{|\omega|}\omega\perp(\eta\zeta)\mathrm{d}\theta'-\frac{\lambda}{2}(-1)^{|\eta|}\omega(\eta\perp\zeta)\mathrm{d}\theta'
\end{aligned}$$

Equality holds by exactly the 4-term identity for $\perp$ in Theorem 3.12. □

When specialised to $\Omega(M)$ in the case of a smooth manifold as in Corollary 3.19, we have a DGA $\tilde{\tilde{\Omega}}(M)$ with relations that essentially recover the more general case of [13, Sec. 2] at low degree, now defined for all degrees.

4. Semidirect products of differential graded algebras

In this section we want to apply some of the theory above. Most of the work will be in classical type, so basically classical Riemannian geometry but done using our codifferential approach. We start with some generalises that include the non-graded-commutative case.

4.1. **Semidirect product by the DGA in one variable.** Let $(\Omega(A),\mathrm{d})$ be a DGA equipped with a derivation τ, i.e. a derivation of $\Omega(A)$ as an algebra which respects degree and commutes with d. Let $\Omega(t,\mathrm{d}t)$ be the general bicovariant calculus of the additive line. This is a super-Hopf algebra with a parameter λ and relations and coproduct

$$[\mathrm{d}t,t]=\lambda\mathrm{d}t,\quad \mathrm{d}t\wedge\mathrm{d}t=0,\quad \Delta t=t\otimes1+1\otimes t,\quad \Delta\mathrm{d}t=\mathrm{d}t\otimes1+1\otimes\mathrm{d}t.$$

We denote by $A_t=A\rtimes k[t]$ the semidirect product of A by t with relations $[t,a]=\lambda\tau(a)$ for all $a\in A$. We recall that a DGA is inner if there is a distinguished 1-form θ that generates d by graded-commutator.

Proposition 4.1. *Given a derivation τ commuting with* d*, the super-Hopf algebra $\Omega(t, \mathrm{d}t)$ acts on $\Omega(A)$ and the super-semidirect product $\Omega(A_t) = \Omega(A) \rtimes \Omega(t, \mathrm{d}t)$ is an inner DGA. The relations and exterior derivative are*

$$[t, \omega] = \lambda(\tau - |\omega|)\omega, \quad \forall \omega \in \Omega(A), \quad [\mathrm{d}t, \ \} = \lambda \mathrm{d}$$

where we use the graded-commutator.

Proof. We define an action of t and $\mathrm{d}t$ by

$$t \triangleright \omega = \lambda(\tau - |\omega|)\omega, \quad \mathrm{d}t \triangleright \omega = \lambda \mathrm{d}\omega, \quad \forall \omega \in \Omega(A).$$

Clearly t acts as a derivation and $\mathrm{d}t$ as a graded-derivation (since d does). Moreover $[\mathrm{d}t, t] = \mathrm{d}t \triangleright \lambda(\tau(\omega) - |\omega|\omega) - t \triangleright \lambda \mathrm{d}\omega = \lambda^2(\mathrm{d}\tau(\omega) - |\omega|\mathrm{d}\omega) - \lambda^2(\tau(\mathrm{d}\omega) - (|\omega|+1)\mathrm{d}\omega) = \lambda^2 \mathrm{d}\omega = \lambda \mathrm{d}t \triangleright \omega$ hence we have an action of $\Omega(t, \mathrm{d}t)$ on $\Omega(A)$ as a super-module algebra. Hence the semidirect product is a super-algebra. Its algebra structure is

$$(\omega\phi)(\eta\psi) = (-1)^{|\phi_{(2)}||\eta|}\omega(\phi_{(1)} \triangleright \eta)\phi_{(2)}\psi, \quad \forall \omega, \eta \in \Omega(A), \ \phi, \psi \in \Omega(t, \mathrm{d}t).$$

where $\phi_{(1)} \otimes \phi_{(2)}$ is the super-coproduct of $\Omega(t, \mathrm{d}t)$. Putting in the form of the coproduct, we have the relations as stated. More explicitly, writing $\Omega(A_t) = \Omega(A)k[t] \oplus \Omega(A)k[t]\mathrm{d}t$ and keeping everything normal ordered with $t, \mathrm{d}t$ to the right, we have

$$\omega\phi(t)\eta\psi = \omega(\eta\phi(\lambda(\tau - D) + t))\psi$$

$$\omega(\phi(t)\mathrm{d}t)\eta\psi = (-1)^{|\eta|}\omega(\eta\phi(\lambda(\tau - D) + t))(\mathrm{d}t)\psi + \lambda\omega((\mathrm{d}\eta)\phi(\lambda(\tau - D) + t))\psi$$

where the $\lambda(\tau - D)$ is understood to act to the *left* and D is the degree operator. Finally, we define d as the graded-commutator in this algebra with $\theta = \mathrm{d}t$ and verify that this reduces to $\mathrm{d}t$ on t and $\mathrm{d}\omega$ on $\omega \in \Omega(A)$, i.e. extends the given ones. Equivalently, we specify

$$\mathrm{d}(\omega\phi) = (\mathrm{d}\omega)\phi + (-1)^{|\omega|}\omega \mathrm{d}\phi, \quad \omega \in \Omega(A), \ \phi \in \Omega(t, \mathrm{d}t).$$

One may verify that this defines a graded-derivation. □

For example, one can take $\tau = 0$. Then any DGA on A has a canonical extension to $\Omega(A[t]) = \Omega(A) \rtimes \Omega(t, \mathrm{d}t)$ as a DGA on the central extension $A[t]$. Here the cross relations are

$$[\omega, t] = \lambda|\omega|\omega, \quad [\mathrm{d}t, \omega\} = \lambda \mathrm{d}\omega, \quad \forall \omega \in \Omega(A).$$

One can check that these relations also hold for $\omega \in \Omega(A[t])$, i.e. the canonical degree derivation D and exterior graded-derivation d on $\Omega(A)$ are rendered inner by t and $\mathrm{d}t$ respectively. In particular, for any smooth manifold M we have a non-graded-commutative DGA $\Omega(M) \rtimes \Omega(t, \mathrm{d}t)$ on $C^\infty(M)[t]$, the latter being a classical commutative algebra of functions on $M \times \mathbb{R}$.

4.2. **Conformal 1-forms.** Here we study the notion of a 'conformal Killing' vector field or rather 1-form, τ, on (Ω, δ) of classical type as in Section 2. In terms of the metric inner product the usual notion that the metric is invariant up to scale is the 'metric-conformal' condition

$$(\mathcal{L}_\tau \omega, \eta) + (\omega, \mathcal{L}_\tau \eta) = (\tau, \mathrm{d}(\omega, \eta)) + \alpha(\omega, \eta), \quad \forall \omega, \eta \in \Omega^1, \tag{4.1}$$

for some function $\alpha \in A$. Classically this is equivalent to the 'conformal Killing' condition, e.g. [20],

$$\nabla_\omega \tau = \frac{1}{2} \mathrm{i}_\omega \mathrm{d}\tau + \frac{\alpha}{2}\omega, \quad \forall \omega \in \Omega^1. \tag{4.2}$$

in terms of the Levi-Civita covariant derivative, which is useful for extending the concept to higher degree forms. In both cases the value of α classically is determined. The special case with $\alpha = 0$ corresponds in classical geometry to a Killing 1-form. In our case we are taking a coderivation δ as the starting point and ∇ from Theorem 2.8. At this level we define:

Definition 4.2. Let $(\Omega(A),\delta)$ be of classical type and let D be the degree operator. We say that a 1-form $\tau\in\Omega^1$ is *δ-conformal* if there is some $\alpha\in A$ and some constant β such that

$$[\delta,\mathcal{L}_\tau]=\alpha\delta+\mathrm{i}_{\mathrm{d}\alpha}(D-\beta)$$

holds acting on degree $D=1$. We say that τ is *strongly δ-conformal* if this holds acting on all of Ω.

The motivation and relationship between these concepts is clarified by the following lemma.

Lemma 4.3. *Let $(\Omega(A),\delta)$ be of classical type and $\tau\in\Omega^1$.*

(1) τ conformal Killing for some function α implies τ metric-conformal and if $(\ ,\ )$ is nondegenerate then these two notions are equivalent.

(2) τ conformal Killing implies $\mathcal{L}_\tau=\alpha\tau$ and if $(\ ,\ )$ is invertible then $\alpha=\delta_\nabla\tau/\beta$ and $\beta=\frac{1}{2}(\ ,\ )(g)$.

(3) τ conformal Killing with $\alpha=\delta\tau/\beta$ for some constant β and $\delta^2=0$ implies τ is δ-conformal.

(4) τ δ-conformal for some function α and some constant β implies τ is metric-conformal with factor α, and in the nondegenerate case, $(\tau,\mathrm{d}(\alpha-\delta\tau/\beta))=0$.

Proof. (1) From our formula for the Levi-Civita covariant derivative in Theorem 2.8 the requirement of a conformal Killing form in the sense (4.2) becomes

$$\delta(\omega\tau)-(\delta\omega)\tau+\mathcal{L}_\tau\omega=\omega(\alpha-\delta\tau),\quad\forall\omega\in\Omega^1. \tag{4.3}$$

From the formula for $\mathrm{i}_\zeta L_\delta$ in Lemma 2.4 we have

$$(\zeta,\omega)\alpha=\mathrm{i}_\zeta L_\delta(\omega,\tau)+(\zeta,\mathcal{L}_\tau\omega)=(\omega,\mathrm{d}\mathrm{i}_\tau\zeta+\mathrm{i}_\tau\mathrm{d}\zeta)-\mathrm{i}_\tau\mathrm{d}\mathrm{i}_\omega\zeta+(\zeta,\mathcal{L}_\tau\omega)$$

for all $\omega,\zeta\in\Omega^1$, which is (4.1). If $(\ ,\ )$ is nondegenerate then we can push this backwards to conclude (4.3).

(2) Setting $\omega=\tau$ in (4.3), we have $\mathcal{L}_\tau\tau=\alpha\tau$. If $(\ ,\ )$ is invertible then using (4.2) we have

$$\delta_\nabla(\tau)=\mathrm{i}_{g^1}\nabla_{g^2}\tau=\frac{1}{2}\left(\mathrm{i}_{g^1}\mathrm{i}_{g^2}\mathrm{d}\tau+\alpha\mathrm{i}_{g^1}g^2\right).$$

The first term vanishes by symmetry of g, giving the value of α. This assumes $\Omega(A)$ is of classical type but does not refer to our given δ.

(3) Assuming the stated form of α, the equation (4.2) becomes

$$\delta(\omega\tau)-(\delta\omega)\tau+\mathcal{L}_\tau\omega=\omega\alpha(1-\beta),\quad\forall\omega\in\Omega^1$$

and we apply δ to this to give

$$-(\delta\omega)\delta\tau-\mathrm{i}_\tau\mathrm{d}(\delta\omega)+\delta\mathcal{L}_\tau\omega=\alpha(1-\beta)\delta\omega+(1-\beta)\mathrm{i}_{\mathrm{d}\alpha}\omega$$

for all $\omega\in\Omega^1$, which is the condition to be δ-conformal provided $\delta\tau=\alpha\beta$.

(4) Now using the condition to be δ-conformal in Definition 4.2, we compute for any $a\in A$,

$$\begin{aligned}&\delta\mathcal{L}_\tau(a\omega)-\mathcal{L}_\tau\delta(a\omega)-\alpha\delta(a\omega)-(1-\beta)\mathrm{i}_{\mathrm{d}\alpha}(a\omega)\\&\quad=\delta((\tau,\mathrm{d}a)\omega+a\mathcal{L}_\tau\omega)-\mathcal{L}_\tau(a\delta\omega+\mathrm{i}_{\mathrm{d}a}\omega)-a\alpha\delta(\omega)-\alpha\mathrm{i}_{\mathrm{d}a}\omega-a(1-\beta)\mathrm{i}_{\mathrm{d}\alpha}\omega\\&\quad=(\tau,\mathrm{d}a)\delta\omega+\mathrm{i}_{\mathrm{d}(\tau,\mathrm{d}a)}\omega+\mathrm{i}_{\mathrm{d}a}\mathcal{L}_\tau\omega-(\tau,\mathrm{d}a)\delta\omega-\mathcal{L}_\tau\mathrm{i}_{\mathrm{d}a}\omega-\alpha\mathrm{i}_{\mathrm{d}a}\omega\end{aligned}$$

$$= \mathrm{i}_{\mathrm{d}(\tau,\mathrm{d}a)}\omega + \mathrm{i}_{\mathrm{d}a}\mathcal{L}_\tau\omega - \mathcal{L}_\tau \mathrm{i}_{\mathrm{d}a}\omega - \alpha \mathrm{i}_{\mathrm{d}a}\omega$$

where we used Lemma 2.2. Now the initial expression $= 0$ if τ is δ-conformal as $a\omega$ is another 1-form, so we deduce that

$$[\mathrm{i}_{\mathrm{d}a}, \mathcal{L}_\tau]\omega = \alpha \mathrm{i}_{\mathrm{d}a}\omega - \mathrm{i}_{\mathcal{L}_\tau \mathrm{d}a}\omega, \quad \forall a \in A,\ \omega \in \Omega^1.$$

We then consider this same identity for $\eta = \sum b_i \mathrm{d}a_i$. Using that i is A-linear and the Leibniz property of $\mathcal{L}_\tau$ we deduce that the identity holds for all $\eta \in \Omega^1$ in place of $\mathrm{d}a$, which is (4.1). In the nondegenerate case we conclude by (1) and (2) that $\mathcal{L}_\tau\tau = \alpha\tau$ and in this case the condition for a δ-conformal 1-form and Lemma 2.2 tells us that

$$\mathrm{i}_{\mathrm{d}\alpha}\tau = \delta(\alpha\tau) - \alpha\delta\tau = \delta\mathcal{L}_\tau\tau - \alpha\delta\tau = (\tau, \mathrm{d}\delta\tau) + (1-\beta)\mathrm{i}_{\mathrm{d}\alpha}\tau$$

or $(\tau, \mathrm{d}(\beta\alpha)) = (\tau, \mathrm{d}\delta\tau)$ which we write suggestively as stated. □

It follows that in the case of a classical Riemannian manifold all three notions are equivalent. This is well-known for (4.1)-(4.2) whereas our notion of δ-conformal in Definition 4.2 is the one we will want and appears to be new. We will need several more properties of δ-conformal 1-forms as follows.

Lemma 4.4. *Let $(\Omega(A), \delta)$ be of classical type and τ δ-conformal. Then*

$$\mathcal{L}_\tau(\omega \perp \eta) + \alpha\omega \perp \eta = (\mathcal{L}_\tau\omega) \perp \eta + \omega \perp \mathcal{L}_\tau\eta$$

$$\mathcal{L}_\tau S(\omega, \eta) + \alpha S(\omega, \eta) - S(\mathcal{L}_\tau\omega, \eta) - S(\omega, \mathcal{L}_\tau\eta) = (-1)^{|\omega|}(\mathrm{d}\alpha)(\omega \perp \eta)$$

for all $\omega, \eta \in \Omega$. Here $S(\omega, \eta) = \omega \perp \mathrm{d}\eta - (-1)^{|\omega|}\mathrm{d}(\omega \perp \eta) - (-1)^{|\omega|}(\mathrm{d}\omega) \perp \eta$.

Proof. (1) The smallest nontrivial case of the first statement is with ω, η of degree 1 and is just the metric conformality (4.1), so this case holds by Lemma 4.3. We let $\eta \in \Omega^1$ so that $\perp \eta = \mathrm{i}_\eta$, then we want to prove

$$[\mathrm{i}_\eta, \mathcal{L}_\tau] = \alpha \mathrm{i}_\eta - \mathrm{i}_{\mathcal{L}_\tau\eta} \tag{4.4}$$

acting on Ω of all degrees. This is easy to see by induction and the (graded)-derivation properties of the ingredients, and establishes the first statement of the lemma for all $\omega \in \Omega$ and $\eta \in \Omega^1$. We now show that if the statement holds for the pairs (ω, η) and (ω, η') then it holds for $(\omega, \eta\eta')$ which then establishes the result by induction on the degree of η. We use again that $\mathcal{L}_\tau$ is a derivation and now that $\omega \perp$ is a derivation of degree $|\omega| - 2$ to break down both sides and compare, using graded-commutativity where needed.These steps are all straightforward and details are again omitted. (2) We next apply the operations in the first displayed statement to each term of $S(\omega, \eta)$. Since d commutes with $\mathcal{L}_\tau$ we have equality for each term except one, where we have $\alpha \mathrm{d}(\omega \perp \eta) = \mathrm{d}(\alpha\omega \perp \eta) - (\mathrm{d}\alpha)\omega \perp \eta$. The second term results in the right hand side of the second displayed statement. □

Proposition 4.5. *Let $(\Omega(A), \delta)$ be of classical type and τ δ-conformal. The following are equivalent*

(1) τ is strongly δ-conformal.

(2)

$$\mathcal{L}_\tau L_\delta(\omega, \eta) + \alpha L_\delta(\omega, \eta) - L_\delta(\mathcal{L}_\tau\omega, \eta) - L_\delta(\omega, \mathcal{L}_\tau\eta) = -(-1)^{|\omega|}|\omega|\omega \mathrm{i}_{\mathrm{d}\alpha}\eta - |\eta|(\mathrm{i}_{\mathrm{d}\alpha}\omega)\eta \tag{4.5}$$

for all $\omega, \eta \in \Omega$. In this case

$$[\Delta, \mathcal{L}_\tau] = \alpha\Delta + (D - \beta)\mathcal{L}_{\mathrm{d}\alpha} + (\mathrm{d}\alpha)\delta + \mathrm{i}_{\mathrm{d}\alpha}\mathrm{d} \tag{4.6}$$

where $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$. Moreover, these identities all apply if $(\ ,\ )$ is nondegenerate.

Proof. (1) We let $C(\omega,\eta)$ first denote the LHS of (4.5). Then

$$\begin{aligned}
\mathrm{i}_\zeta C(\omega,\eta) &= \mathrm{i}_\zeta\mathcal{L}_\tau L_\delta(\omega,\eta)+\alpha\mathrm{i}_\zeta L_\delta(\omega,\eta)-\mathrm{i}_\zeta L_\delta(\mathcal{L}_\tau\omega,\eta)-\mathrm{i}_\zeta L_\delta(\omega,\mathcal{L}_\tau\eta)\\
&=(\mathcal{L}_\tau+2\alpha)\mathrm{i}_\zeta L_\delta(\omega,\eta)-\mathrm{i}_{(\mathcal{L}_\tau\zeta)}L_\delta(\omega,\eta)-\mathrm{i}_\zeta L_\delta(\mathcal{L}_\tau\omega,\eta)-\mathrm{i}_\zeta L_\delta(\omega,\mathcal{L}_\tau\eta)\\
&=-(\mathcal{L}_\tau+2\alpha)\Big(L_\delta(\mathrm{i}_\zeta\omega,\eta)+(-1)^{|\omega|}L_\delta(\omega,\mathrm{i}_\zeta\eta)-L_{\mathrm{i}_{\mathrm{d}\zeta}}(\omega,\eta)\Big)\\
&\quad+L_\delta(\mathrm{i}_{(\mathcal{L}_\tau\zeta)}\omega,\eta)+(-1)^{|\omega|}L_\delta(\omega,\mathrm{i}_{(\mathcal{L}_\tau\zeta)}\eta)-L_{\mathrm{i}_{\mathrm{d}\mathcal{L}_\tau\zeta}}(\omega,\eta)\\
&\quad+L_\delta(\mathrm{i}_\zeta\mathcal{L}_\tau\omega,\eta)+(-1)^{|\omega|}L_\delta(\mathcal{L}_\tau\omega,\mathrm{i}_\zeta\eta)-L_{\mathrm{i}_{\mathrm{d}\zeta}}(\mathcal{L}_\tau\omega,\eta)\\
&\quad+L_\delta(\mathrm{i}_\zeta\omega,\mathcal{L}_\tau\eta)+(-1)^{|\omega|}L_\delta(\omega,\mathrm{i}_\zeta\mathcal{L}_\tau\eta)-L_{\mathrm{i}_{\mathrm{d}\zeta}}(\omega,\mathcal{L}_\tau\eta)\\
&=-(\mathcal{L}_\tau+2\alpha)\Big(L_\delta(\mathrm{i}_\zeta\omega,\eta)+(-1)^{|\omega|}L_\delta(\omega,\mathrm{i}_\zeta\eta)\Big)\\
&\quad+L_\delta(\mathcal{L}_\tau\mathrm{i}_\zeta\omega,\eta)+L_\delta(\alpha\mathrm{i}_\zeta\omega,\eta)+(-1)^{|\omega|}L_\delta(\mathcal{L}_\tau\omega,\mathrm{i}_\zeta\eta)\\
&\quad+L_\delta(\mathrm{i}_\zeta\omega,\mathcal{L}_\tau\eta)+(-1)^{|\omega|}L_\delta(\omega,\mathcal{L}_\tau\mathrm{i}_\zeta\eta)+(-1)^{|\omega|}L_\delta(\omega,\alpha\mathrm{i}_\zeta\eta)\\
&=-(\mathcal{L}_\tau+\alpha)\Big(L_\delta(\mathrm{i}_\zeta\omega,\eta)+(-1)^{|\omega|}L_\delta(\omega,\mathrm{i}_\zeta\eta)\Big)\\
&\quad+L_\delta(\mathcal{L}_\tau\mathrm{i}_\zeta\omega,\eta)+(-1)^{|\omega|-1}(\mathrm{i}_\zeta\omega)\mathrm{i}_{\mathrm{d}\alpha}\eta+(-1)^{|\omega|}L_\delta(\mathcal{L}_\tau\omega,\mathrm{i}_\zeta\eta)\\
&\quad+L_\delta(\mathrm{i}_\zeta\omega,\mathcal{L}_\tau\eta)+(-1)^{|\omega|}L_\delta(\omega,\mathcal{L}_\tau\mathrm{i}_\zeta\eta)+(-1)^{|\omega|}(\mathrm{i}_{\mathrm{d}\alpha}\omega)\mathrm{i}_\zeta\eta\\
&=-C(\mathrm{i}_\zeta\omega,\eta)-(-1)^{|\omega|}C(\omega,\mathrm{i}_\zeta\eta)-(-1)^{|\omega|}(\mathrm{i}_\zeta\omega)\mathrm{i}_{\mathrm{d}\alpha}\eta+(-1)^{|\omega|}(\mathrm{i}_{\mathrm{d}\alpha}\omega)\mathrm{i}_\zeta\eta
\end{aligned}$$

where we used (4.4) to swap the derivative with the interior product and used the first part of Lemma 2.4 to break down L_δ. We used

$$[\mathrm{i}_\eta,\mathcal{L}_\tau]=2\alpha\mathrm{i}_\eta-\mathrm{i}_{\mathcal{L}_\tau\eta},\quad \forall\eta\in\Omega^2,$$

easily deduced from (4.4), to cancel the $L_{\mathrm{i}_{\mathrm{d}\zeta}}$ and $L_{\mathrm{i}_{\mathcal{L}_\tau\mathrm{d}\zeta}}$ terms arising from Lemma 2.4 (where $\mathcal{L}_\tau$ commutes with d). Finally, we used Lemma 2.2 to move α out from inside L_δ leaving a residue as shown. We have thus obtained an 'induction formula' for $C(\omega,\eta)$ from smaller degrees and this is solved by the right hand side of (4.5). The initial values are $C(a,\omega)=C(\omega,a)=0$ for $a\in A$ and if $(\ ,\ )$ is nondegenerate then the induction step completely determines $C(\omega,\eta)$ as equal to the stated RHS of (4.5). Hence (4.5) holds in the non-degenerate case. (2) Next, let D be the degree operator and observe that the Leibnizator of $\mathrm{i}_{\mathrm{d}\zeta}D$ is

$$\begin{aligned}
L_{\mathrm{i}_{\mathrm{d}\alpha}D}(\omega,\eta)&=\mathrm{i}_{\mathrm{d}\alpha}D(\omega\eta)-(\mathrm{i}_{\mathrm{d}\alpha}D\omega)\eta-(-1)^{|\omega|}\omega\mathrm{i}_{\mathrm{d}\alpha}D\eta\\
&=(|\omega|+|\eta|)((\mathrm{i}_{\mathrm{d}\alpha}\omega)\eta+(-1)^{|\omega|}\omega\mathrm{i}_{\mathrm{d}\alpha}\eta)-|\omega|(\mathrm{i}_{\mathrm{d}\alpha}\omega)\eta-(-1)^{|\omega|}|\eta|\omega\mathrm{i}_{\mathrm{d}\alpha}\eta\\
&=(-1)^{|\omega|}|\omega|\omega\mathrm{i}_{\mathrm{d}\alpha}\eta+|\eta|(\mathrm{i}_{\mathrm{d}\alpha}\omega)\eta=-C_\alpha(\omega,\eta)
\end{aligned}$$

where now $C_\alpha(\omega,\eta)$ refers to the expression on the RHS of (4.5). The operator $B=\mathrm{i}_{\mathrm{d}\alpha}(D-\beta)$ will have the same Leibnizator since it differs by a constant multiple of i and the latter is a derivation, so $L_B(\omega,\eta)=-C_\alpha(\omega,\eta)$. Now suppose that the condition in Definition 4.2 holds on ω,η, then we show that it holds on $\omega\eta$ provided (4.5) holds:

$$\begin{aligned}
\delta\mathcal{L}_\tau(\omega\eta)&=\delta((\mathcal{L}_\tau\omega)\eta)+\delta(\omega\mathcal{L}_\tau\eta)\\
&=L_\delta(\mathcal{L}_\tau\omega,\eta)+L_\delta(\omega,\mathcal{L}_\tau\eta)+(\delta\mathcal{L}_\tau\omega)\eta+(-1)^{|\omega|}(\mathcal{L}_\tau\omega)\delta\eta+(\delta\omega)\mathcal{L}_\tau\eta+(-1)^{|\omega|}\omega\delta\mathcal{L}_\tau\eta\\
&=L_\delta(\mathcal{L}_\tau\omega,\eta)+L_\delta(\omega,\mathcal{L}_\tau\eta)+(\mathcal{L}_\tau\delta\omega)\eta+(-1)^{|\omega|}(\mathcal{L}_\tau\omega)\delta\eta+(\delta\omega)\mathcal{L}_\tau\eta+(-1)^{|\omega|}\omega\mathcal{L}_\tau\delta\eta\\
&\quad+\alpha(\delta\omega)\eta+(B\omega)\eta+(-1)^{|\omega|}\alpha\omega\delta\eta+(-1)^{|\omega|}\omega B\eta\\
&=L_\delta(\mathcal{L}_\tau\omega,\eta)+L_\delta(\omega,\mathcal{L}_\tau\eta)+(\mathcal{L}_\tau\delta\omega)\eta+(-1)^{|\omega|}(\mathcal{L}_\tau\omega)\delta\eta+(\delta\omega)\mathcal{L}_\tau\eta+(-1)^{|\omega|}\omega\mathcal{L}_\tau\delta\eta\\
&\quad+\alpha\delta(\omega\eta)-\alpha L_\delta(\omega,\eta)+B(\omega\eta)+C_\alpha(\omega,\eta)\\
&=\mathcal{L}_\tau((\delta\omega)\eta)+(-1)^{|\omega|}\mathcal{L}_\tau(\omega\delta\eta)+\alpha\delta(\omega\eta)+B(\omega\eta)+\mathcal{L}_\tau L_\delta(\omega,\eta)\\
&=\mathcal{L}_\tau\delta(\omega\eta)+\alpha\delta(\omega\eta)+B(\omega\eta)
\end{aligned}$$

as required. We used the derivation property of $\mathcal{L}$ and the 'compensated derivation' property of δ, B where we correct with the respective Leibnizator. For the 3rd equality we swapped the order of $\delta, \mathcal{L}_\tau$ under the inductive assumption of the condition in Definition 4.2 and for the 5th equality we use (4.5) and the derivation property of $\mathcal{L}_\tau$ in reverse. Hence (4.5) implies the condition in Definition 4.2 in all degrees. Clearly this proof can be reversed, i.e. if the condition in Definition 4.2 holds on all degrees then so does (4.5). (3) Finally,

$$\begin{aligned}[\Delta, \mathcal{L}_\tau] &= [\delta \mathrm{d} + \mathrm{d}\delta, \mathcal{L}_\tau] = [\delta, \mathcal{L}_\tau]\mathrm{d} + \mathrm{d}[\delta, \mathcal{L}_\tau] \\ &= \alpha\delta\mathrm{d} + \mathrm{i}_{\mathrm{d}\alpha}(D-\beta)\mathrm{d} + \mathrm{d}(\alpha\delta(\ )) + \mathrm{d}\mathrm{i}_{\mathrm{d}\alpha}(D-\beta) \\ &= \alpha\Delta - \alpha\mathrm{d}\delta + \mathrm{d}(\alpha\delta(\ )) + (D-\beta)\mathcal{L}_{\mathrm{d}\alpha} + \mathrm{i}_{\mathrm{d}\alpha}\mathrm{d}\end{aligned}$$

which we recognise as stated in (4.6). □

To be sure of all these identities we will require τ to be strongly δ-conformal. But we see that if $(\ ,\ )$ is nondegenerate then δ-conformal implies strongly δ-conformal.

4.3. **Quantisation by a δ-conformal 1-form.** We now combine the two preceding subsections. When $(\Omega(A), \delta)$ is of classical type we have a canonical extension to $\tilde\Omega(A)$ by Theorem 3.18 and a canonical further extension of that to $\tilde{\tilde\Omega}(A)$ by Proposition 3.20.

Proposition 4.6. *Let $(\Omega(A), \delta)$ be of classical type and $\tau \in \Omega^1$ strongly δ-conformal. Then τ defines a derivation on $\tilde{\tilde\Omega}(A)$ in Proposition 3.20 by*

$$\tau(\theta') = \alpha\theta', \quad \tau(\mathrm{d}\theta') = (\mathrm{d}\alpha)\theta' + \alpha\mathrm{d}\theta',$$

$$\tau(\omega) = \mathcal{L}_\tau\omega + \frac{\lambda}{2}(-1)^{|\omega|}(|\omega| - \beta)\mathrm{i}_{\mathrm{d}\alpha}\omega\theta', \quad \forall \omega \in \Omega$$

The semidirect product DGA $\tilde{\tilde\Omega} \rtimes \Omega(t, \mathrm{d}t)$ by Proposition 4.1 in the case of a Riemannian manifold recovers the basic case of the calculus in [13, Thm 3.1].

Proof. We verify that τ as defined is a derivation for the $\cdot$ product in Proposition 3.20 that commutes with d.. We have already established all needed identities in Proposition 4.5. Thus

$$\begin{aligned}\tau(\mathrm{d}.\omega) &= \tau\left(\mathrm{d}\omega - \frac{\lambda}{2}(-1)^{|\omega|}(\Delta\omega)\theta' + \frac{\lambda}{2}(\delta\omega)\mathrm{d}\theta'\right) \\ &= \mathcal{L}_\tau\mathrm{d}\omega + \frac{\lambda}{2}(-1)^{|\omega|+1}(|\omega| + 1 - \beta)\mathrm{i}_{\mathrm{d}\alpha}\mathrm{d}\omega\theta' - \frac{\lambda}{2}(-1)^{|\omega|}\mathcal{L}_\tau\Delta\omega\theta' + \frac{\lambda}{2}\mathcal{L}_\tau\delta\omega\mathrm{d}\theta' \\ &\quad - \frac{\lambda}{2}(-1)^{|\omega|}(\Delta\omega)\alpha\theta' + \frac{\lambda}{2}(\delta\omega)\mathrm{d}\alpha\theta' + \frac{\lambda}{2}(\delta\omega)\alpha\mathrm{d}\theta' \\ &= \mathrm{d}\mathcal{L}_\tau\omega + \frac{\lambda}{2}(-1)^{|\omega|+1}(|\omega - \beta)\mathrm{i}_{\mathrm{d}\alpha}\mathrm{d}\omega\theta' - \frac{\lambda}{2}(-1)^{|\omega|}\Delta\mathcal{L}_\tau\omega\theta' + \frac{\lambda}{2}\delta\mathcal{L}_\tau\omega\mathrm{d}\theta' \\ &\quad + \frac{\lambda}{2}(-1)^{|\omega|}(|\omega| - \beta)\mathcal{L}_{\mathrm{d}\alpha}\omega\theta' - \frac{\lambda}{2}(|\omega| - \beta)\mathrm{i}_{\mathrm{d}\alpha}\omega\mathrm{d}\theta' \\ &= \mathrm{d}\mathcal{L}_\tau\omega - \frac{\lambda}{2}(-1)^{|\omega|}\Delta\mathcal{L}_\tau\omega\theta' + \frac{\lambda}{2}\delta\mathcal{L}_\tau\omega\mathrm{d}\theta' \\ &\quad + \frac{\lambda}{2}(-1)^{|\omega|}(|\omega - \beta)\mathrm{d}\mathrm{i}_{\mathrm{d}\alpha}\omega\theta' - \frac{\lambda}{2}(|\omega| - \beta)\mathrm{i}_{\mathrm{d}\alpha}\omega\mathrm{d}\theta' \\ &= \mathrm{d}.\left(\mathcal{L}_\tau\omega + \frac{\lambda}{2}(-1)^{|\omega|}(|\omega| - \beta)\mathrm{i}_{\mathrm{d}\alpha}\omega\theta'\right) = \mathrm{d}.\tau(\omega)\end{aligned}$$

for all $\omega \in \Omega^1$. We used the definitions then strong-δ-conformality and (4.6) to reorder. We used graded-commutativity on $(\delta\omega)\mathrm{d}\alpha$ and we used the Cartan formula for $\mathcal{L}_{\mathrm{d}\alpha}$, before recognising the result. Next, we note that $[\![\omega,\eta]\!] = L_\delta(\omega,\eta)+S(\omega,\eta)$ according to Theorem 3.18 and we write $B = \mathrm{i}_{\mathrm{d}\alpha}(D-\beta)$. Then

$$\begin{aligned}
\tau(\omega\cdot\eta) &= \tau\left(\omega\eta + \frac{\lambda}{2}(-1)^{|\omega|+|\eta|}[\![\omega,\eta]\!]\theta' - \frac{\lambda}{2}(-1)^{|\omega|}(\omega \perp \eta)\mathrm{d}\theta'\right)\\
&= \mathcal{L}_\tau(\omega\eta) + (-1)^{|\omega|+|\eta|}\frac{\lambda}{2}B(\omega\eta)\theta'\\
&\quad+\frac{\lambda}{2}(-1)^{|\omega|+|\eta|}\mathcal{L}_\tau[\![\omega,\eta]\!]\theta' - \frac{\lambda}{2}(-1)^{|\omega|}\mathcal{L}_\tau(\omega\perp\eta)\mathrm{d}\theta'\\
&\quad+\frac{\lambda}{2}(-1)^{|\omega|+|\eta|}[\![\omega,\eta]\!]\alpha\theta' - \frac{\lambda}{2}(-1)^{|\omega|}(\omega\perp\eta)(\mathrm{d}\alpha\theta' + \alpha\mathrm{d}\theta')\\
&= \mathcal{L}_\tau(\omega\eta) + \frac{\lambda}{2}(-1)^{|\omega|+|\eta|}B(\omega\eta)\theta' + \frac{\lambda}{2}(-1)^{|\omega|+|\eta|}([\![\mathcal{L}_\tau\omega,\eta]\!] + [\![\omega,\mathcal{L}_\tau\eta]\!])\theta'\\
&\quad-\frac{\lambda}{2}(-1)^{|\omega|}((\mathcal{L}_\tau\omega)\perp\eta + \omega\perp\mathcal{L}_\tau\eta)\mathrm{d}\theta' + \frac{\lambda}{2}(-1)^{|\omega|+|\eta|}C_\alpha(\omega,\eta)\theta'\\
&= (\mathcal{L}_\tau\omega)\eta + \omega\mathcal{L}_\tau\eta + \frac{\lambda}{2}(-1)^{|\omega|+|\eta|}([\![\mathcal{L}_\tau\omega,\eta]\!] + [\![\omega,\mathcal{L}_\tau\eta]\!])\theta'\\
&\quad-\frac{\lambda}{2}(-1)^{|\omega|}((\mathcal{L}_\tau\omega)\perp\eta + \omega\perp\mathcal{L}_\tau\eta)\mathrm{d}\theta' + \frac{\lambda}{2}(-1)^{|\omega|+|\eta|}B(\omega)\eta\theta' + \frac{\lambda}{2}(-1)^{|\eta|}\omega B(\eta)\theta'\\
&= \left(\mathcal{L}_\tau\omega + \frac{\lambda}{2}(-1)^{|\omega|}B(\omega)\theta'\right)\cdot\eta + \omega\cdot\left(\mathcal{L}_\tau\eta + \frac{\lambda}{2}(-1)^{|\eta|}B(\eta)\theta'\right)\\
&= \tau(\omega)\cdot\eta + \omega\cdot\tau(\eta)
\end{aligned}$$

We use the definitions, then the second statement of Lemma 4.4 and (4.5) tell us $\mathcal{L}_\tau[\![\omega,\eta]\!]$ while the first of Lemma 4.4 tells us $\mathcal{L}_\tau(\omega\perp\eta)$. We then use from the proof Proposition 4.5 that $L_B(\omega,\eta) = -C_\alpha(\omega,\eta)$ and recognise the result.

We then apply Proposition 4.1 to obtain a DGA $\tilde{\tilde{\Omega}}(A){>\!\!\!\triangleleft}\Omega(t,\mathrm{d}t)$ over A_t and compute the cross relations for $a \in A$ and $\omega\in\Omega^1$,

$$[t,\theta'] = \lambda\alpha\theta', \quad [t,\omega] = \lambda(\mathcal{L}_\tau\omega - \omega) + \lambda^2\left(\frac{n-2}{4}\right)(\mathrm{d}\alpha,\omega)\theta', \quad [\mathrm{d}t, a] = \lambda\mathrm{d}a$$

$$[t,\mathrm{d}\theta'] = \lambda\mathrm{d}((\alpha-1)\theta'), \quad \{\mathrm{d}t,\omega\} = \lambda\mathrm{d}\omega, \quad \{\mathrm{d}t,\theta'\} = \lambda\mathrm{d}\theta', \quad \forall\omega\in\Omega^1$$

which, remembering the change of sign of λ, is the calculus in [13, Thm 3.1] and [13, Prop 3.6] in the special case $\beta=\zeta=0$ in the notation used there and in the case of a Riemannian manifold. □

We have thus put [13] into a proper framework and to all degrees: we start with a classical Riemannian manifold and usual exterior algebra $\Omega(M)$, construct an extension $\tilde{\tilde{\Omega}}(M)$ from Proposition 3.20 and then apply the semidirect product construction Proposition 4.1 using a δ-conformal 1-form τ to give a 'extended quantum spacetime' DGA $\tilde{\tilde{\Omega}}(M){>\!\!\!\triangleleft}\Omega(t,\mathrm{d}t)$. The special case covered quantises the wave operator for the direct product metric on $M\times\mathbb{R}$ whereas the full generality in [13] can be expected to require a cocycle semidirect product version of Proposition 4.1.

We also note that starting with any standard DGA $\Omega(A)$ it is straightforward to formulate a general version of Proposition 3.20 with $\Delta,\delta,[\![\ ,\ ]\!],\perp$ graded maps of the appropriate degree used to define an associated product $\cdot$ and differential d. and to write out the requirements on these maps to arrive at a DGA. This is basically the composition of Theorem 3.12 and Proposition 3.20 and for this reason the details are omitted. It is clear from the form of the commutation relations that Proposition 4.6 provides a non-graded-commutative example where the role of $\Omega(A)$ is now played by the standard inner DGA $\Omega(A)\rtimes\Omega(t, \mathrm{d}t)$. This is because we can use the commutation relations to move $\theta', \mathrm{d}\theta'$ to the far right.

5. Concluding remarks

First of all, we have obtained some identities for classical Riemannian geometry involving the divergence or codifferential δ. We showed how Riemannian geometries are equivalent to giving δ up to an inner derivation such that $(\ ,\ )$ is invertible. It was also natural to proceed without needing $(\ ,\ )$ to be invertible provided we stick to the ∇ as an 'index raised' connection along 1-forms.

Moreover, these results arise naturally from a novel view of a Riemannian structure as cocycle data for the cleft flat central extension of the exterior algebra $\Omega(M)$ as a quantum one. We do not know actual quantum gravity models where exactly such extensions feature – rather we see this as the mathematically simplest in a class of extensions. We also showed that more general cleft central extensions correspond to Laplacians Δ (no longer given by a codifferential) and ∇ with both torsion and non-metric compatibility but in a controlled way, see Proposition 3.16. In general, we explained in the introduction that fundamental obstructions in quantum geometry mean that *some kind* of extension in the quantum calculus may be inevitable if we wish to remain in an associative setting, a fact which can be read backwards as the emergence of a wave operator as the partial derivative associated to an extra cotangent direction θ' out of the hypothesis that spacetime is a quantum geometry. This was the 'wave operator approach' to quantising the Schwarzschild black hole in [13] where the extension needed is already a little beyond the semidirect product form in Section 4. The deeper structure of this phenomenon should be an interesting direction for further work given the importance of the wave operator for physics.

It should also be interesting to introduce matter fields into our cocycle picture, extending the analogy with group cocycles. Indeed, a significant gap in quantum geometry remains the lack of an appropriate noncommutative variational principle and an associated Noether theorem for conserved currents associated to matter fields. A proper understanding of this on a quantum spacetime would be needed for an understanding of suitable stress-energy tensors and the quantum version of Einstein's equation, possibly emerging (as we have seen for the wave operator) from the algebraic constraints of quantum spacetime as a model of quantum gravity effects. Concretely, a conserved current would also be needed for a proper understanding of cosmological particle creation, such as on the integer line in [16].

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