An efficient algorithm for 3D bi-modulus structures

Qinxue Pan[1*], Jianlong Zheng[1], P.H. Wen[2]

[1] *Key Laboratory of Road Structure and Material of Ministry of Transport (Changsha), School of Traffic and Transportation Engineering, Changsha University of Science & Technology, Changsha 410114, Hunan, PR China*

[2] *School of Engineering and Materials Science, Queen Mary University of London, London E1 4NS, UK*

Abstract: The bi-modulus material is a classical model to describe the elastic behavior of materials with tension-compression asymmetry. Due to the inherent nonlinear properties of bi-modular materials, the traditional iteration methods suffer from low convergence efficiency and poor adaptability for large scale structures in engineering. In this paper, a novel 3D complemented algorithm is established through complementing three shear moduli of constitutive equation in principal stress coordinates. Comparing to the existing 3D shear modulus constructed based on the experience, the shear modulus in this paper is derived theoretically through a limit process. Then a theoretically self-consistent complemented algorithm is established and implemented in ABAQUS via UMAT, whose good stability and convergence efficiency are verified by benchmark examples. Numerical analysis shows that the calculation error for the bi-modulus structure using the traditional linear elastic theory is large, which is not in line with the reality.

Keywords: Elastic theory; Bi-modulus material; 3D complemented algorithm; Finite element method; Generalized elastic law; General 3D shear modulus

1. Introduction

A large number of experimental studies [1, 2] show that the tensile modulus and compressive modulus are different, such as polymethyl methacrylate [3], polyester acrylic plastics, concrete etc. [4]. For example, the composite material glass fiber AC-30 (20℃) has tension and compression modulus of 1390 MPa and 200 MPa, respectively, i.e. the ratio E^{+}/E^{-} reaches 7 [5]. Some special phenomena, such as membrane folding and cell sensing, can also be perfectly explained or predicted by the bi-modulus theory [6]. Therefore, it is necessary to study the bi-modulus problems in science and engineering.

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* Corresponding to: Qinxue Pan, School of Traffic and Transportation Engineering, Changsha University of Science & Technology, Changsha 410114, Hunan, PR China; E-mail address: pqx123456789pqx@163.com .

In 1941, Timoshenko proposed the concept of bi-modulus material [7]. In 1982, Ambartsumyan published the first monograph about bi-modulus problems and the constitutive theory based on the difference between tensile modulus and compressive modulus [8, 9]. Since then, it attracted many researchers in the world to carry out the investigation on this issue. For example, the constitutive relation was improved from different perspectives [10-13], and the analytic solutions of bi-modulus problem for the simple cases were obtained [4, 14, 15]. Recently, the important progress in the bi-modulus theory is that the traditional variational principle for smooth constitutive relation is extended to the systems with non-smooth constitutive relations, and the characters of the solution on bi-modulus elastic problem are observed, which is helpful for constructing efficient numerical solution algorithms [16].

In general, the analytical solutions of 3D bi-modulus elasticity problems are difficult to be obtained, especially the geometry and loading condition are not regular. Therefore, the numerical methods are necessary in analysis of structures. However, due to the jump of the Young's modulus in constitutive equation, most algorithms have low convergence efficiency and poor adaptability [17-20]. The parametric variational principle (PVP) algorithm [21] turns bi-modulus problems into complementary problem based on the parameters variational principle in order to avoid the iterative update of stiffness matrix with considerable convergence efficiency. However, for large-scale structures, the convergence efficiency of PVP algorithm is restricted due to the parameter variables dimensionality. In order to overcome such difficulty, the authors have used the continuous model and the meshless method to demonstrated the degree of accuracy and convergence of the proposed technique by comparing with the analytical solutions [22].

Recently, Du et al. [6] proved that the potential energy functional of bi-modulus is a strict convex function with uniqueness and semi-linearity of the solutions. They found that the reason for the poor convergence of traditional iterative algorithm is the adoption of secant stiffness matrix. Then the alternative tangential stiffness algorithm (2D and 3D) and 2D complemented stiffness algorithm are established and implemented in ABAQUS with the subroutine UMAT. Numerical results show that those algorithms have the second-order convergence rate like Newton-Raphson algorithm, which is of great significance to promote the application of bi-modulus theory in engineering. Recently, they investigated the topology optimization design for bi-modulus materials with the use of the algorithms [5]. However, the realization of tangent algorithm will be more complex for 3D problems for bi-modulus materials than the complemented algorithm. Unfortunately, they only deduced shear modulus of bi-modulus material in 2D case rather than 3D shear modulus of general case. Therefore, it is of theoretical significance to study the shear modulus in 3D cases. Due to the degree of difficulty, the existing bi-modulus researches focus mainly on the simple structures and simple boundary conditions. It is necessary to develop a strong adaptability, high efficiency and easy to implement 3D numerical algorithm in engineering.

In this paper, a new 3D complemented algorithm is established first time through complementing three shear moduli of constitutive equation in principal stress coordinates and the 3D shear modulus is derived theoretically by the limit principle. Therefore, the complemented algorithm established is theoretically self-consistent, which provide an excellent convenient complemented algorithm in engineering. This paper is organized as follows. In section 2, the constitutive equation in Cartesian coordinate is presented. In section 3, a self-consistent shear modulus general term is derived and a 3D complemented algorithm is proposed by using subroutine UMAT in ABAQUS. In section 4, the efficiency of the algorithm are verified by three 3D examples. Comparison analysis between bi-modulus theory and traditional linear elastic theory is demonstrated.

2. Generalized elastic law of bi-modulus elasticity theory

2.1 Bi-modulus elasticity theory

The object of the bi-modulus is considered as continuous, homogeneous and isotropic with small deformation. Ambartsumyan [4] noticed that the curves of relationship between the stress and strain for the bi-modulus materials can be described by two straight lines. While for three-dimensional case, the bi-modulus constitutive relation divides the principal stress space into eight subregions according to different stress states. The constitutive equation in principal stress directions is written as follows:

$$\boldsymbol{\varepsilon}_I = \boldsymbol{A}_I \boldsymbol{\sigma}_I, \quad \boldsymbol{\varepsilon}_I = \begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \end{Bmatrix}, \quad \boldsymbol{\sigma}_I = \begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \end{Bmatrix}, \quad \boldsymbol{A}_I = \begin{bmatrix} \dfrac{1}{E^\alpha} & -\dfrac{\mu^\beta}{E^\beta} & -\dfrac{\mu^\gamma}{E^\gamma} \\ -\dfrac{\mu^\alpha}{E^\alpha} & \dfrac{1}{E^\beta} & -\dfrac{\mu^\gamma}{E^\gamma} \\ -\dfrac{\mu^\alpha}{E^\alpha} & -\dfrac{\mu^\beta}{E^\beta} & \dfrac{1}{E^\gamma} \end{bmatrix} \tag{1}$$

where ε_α, ε_β, ε_γ are principal strains, σ_α, σ_β, σ_γ are principal stresses, $\boldsymbol{A_I}$ is flexibility matrix. The modulus E and Poisson's ratio μ are functions of the principal stresses. For example, if $\sigma_\alpha > 0$, the modulus E^α and Poisson's ratio μ^α are taken as E^+ and μ^+ respectively. Conversely, they are taken E^- and μ^- if $\sigma_\alpha < 0$. It can be seen from the above equation that when all three principal stresses are either positive or negative, their constitutive equations are the same as that for classical elastic theory, which is defined as the first type of bi-modulus material. Otherwise, the constitutive equations are different, which is defined as the second type of bi-modulus material. For example, when $\sigma_\alpha > 0, \sigma_\beta < 0, \sigma_\gamma > 0$, the flexibility matrix $\boldsymbol{A_I}$ is:

$$\boldsymbol{A}_I = \begin{bmatrix} \dfrac{1}{E^+} & -\dfrac{\mu^-}{E^-} & -\dfrac{\mu^+}{E^+} \\ -\dfrac{\mu^+}{E^+} & \dfrac{1}{E^-} & -\dfrac{\mu^+}{E^+} \\ -\dfrac{\mu^+}{E^+} & -\dfrac{\mu^-}{E^-} & \dfrac{1}{E^+} \end{bmatrix} \tag{2}$$

To ensure that $\boldsymbol{A_I}$ is a symmetric matrix, $\mu^+/E^+ = \mu^-/E^-$ is required.

2.2 Generalized elasticity law of bi-modulus materials

It is necessary to convert the constitutive equation in the principal stress coordinate into global Cartision coordinate. The direction cosines between coordinate axes and principal stress directions are shown in Table 1.

Table 1 Direction cosine between the principal stress and coordinate axis

	α	β	γ
x	l_1	m_1	n_1
y	l_2	m_2	n_2
z	l_3	m_3	n_3

According to the formula of stress and strain tensors in Appendix A in different coordinate system, the following constitutive equation can be obtained:

$$\begin{cases}\varepsilon_x = \dfrac{l_1^2\sigma_\alpha}{2G_\alpha} + \dfrac{m_1^2\sigma_\beta}{2G_\beta} + \dfrac{n_1^2\sigma_\gamma}{2G_\gamma} + A_\alpha\sigma_\alpha + A_\beta\sigma_\beta + A_\gamma\sigma_\gamma \\ \varepsilon_y = \dfrac{l_2^2\sigma_\alpha}{2G_\alpha} + \dfrac{m_2^2\sigma_\beta}{2G_\beta} + \dfrac{n_2^2\sigma_\gamma}{2G_\gamma} + A_\alpha\sigma_\alpha + A_\beta\sigma_\beta + A_\gamma\sigma_\gamma \\ \varepsilon_z = \dfrac{l_3^2\sigma_\alpha}{2G_\alpha} + \dfrac{m_3^2\sigma_\beta}{2G_\beta} + \dfrac{n_3^2\sigma_\gamma}{2G_\gamma} + A_\alpha\sigma_\alpha + A_\beta\sigma_\beta + A_\gamma\sigma_\gamma \\ \gamma_{xy} = \dfrac{l_1 l_2\sigma_\alpha}{G_\alpha} + \dfrac{m_1 m_2\sigma_\beta}{G_\beta} + \dfrac{n_1 n_2\sigma_\gamma}{G_\gamma} \\ \gamma_{yz} = \dfrac{l_2 l_3\sigma_\alpha}{G_\alpha} + \dfrac{m_2 m_3\sigma_\beta}{G_\beta} + \dfrac{n_2 n_3\sigma_\gamma}{G_\gamma} \\ \gamma_{xz} = \dfrac{l_1 l_3\sigma_\alpha}{G_\alpha} + \dfrac{m_1 m_3\sigma_\beta}{G_\beta} + \dfrac{n_1 n_3\sigma_\gamma}{G_\gamma}\end{cases} \tag{3}$$

where:

$$\left.\begin{aligned} G_\alpha = E^\alpha/[2(1+\mu^\alpha)],\ A_\alpha = -\mu^\alpha/E^\alpha \\ G_\beta = E^\beta/[2(1+\mu^\beta)],\ A_\beta = -\mu^\beta/E^\beta \\ G_\gamma = E^\gamma/[2(1+\mu^\gamma)],\ A_\gamma = -\mu^\gamma/E^\gamma \end{aligned}\right\} \tag{4}$$

If the tensile modulus and compressive modulus are equal, we have

$$\left.\begin{aligned} G_\alpha = G_\beta = G_\gamma = E/[2(1+\mu)] \\ A_\alpha = A_\beta = A_\gamma = -\mu/E \end{aligned}\right\} \tag{5}$$

and Eq. (3) becomes the classical Hooke's law. If the tensile modulus and compressive modulus are not equal, Eq. (3) is similar to the classical Hooke's law for the first type of region. Only when $\sigma_\alpha > 0$, $\sigma_\beta > 0$, $\sigma_\gamma > 0$, then

$$\left.\begin{aligned} G_\alpha = G_\beta = G_\gamma = E^+ \big/ [2(1+\mu^+)] = G^+ \\ A_\alpha = A_\beta = A_\gamma = -\mu^+ \big/ E^+ = A^+ \end{aligned}\right\} \tag{6}$$

and when $\sigma_\alpha < 0$, $\sigma_\beta < 0$, $\sigma_\gamma < 0$, then

$$\left.\begin{aligned} G_\alpha = G_\beta = G_\gamma = E^- \big/ [2(1+\mu^-)] = G^- \\ A_\alpha = A_\beta = A_\gamma = -\mu^- \big/ E^- = A^- \end{aligned}\right\} \tag{7}$$

For the second type of region, i.e. $\sigma_\alpha < 0$, $\sigma_\beta < 0$, $\sigma_\gamma > 0$, the elastic constitutive equation can be arranged as follows:

$$\begin{cases} \varepsilon_x = \dfrac{\sigma_x}{2G^+} + A^+\Theta + (\dfrac{1}{G^-} - \dfrac{1}{G^+}) \dfrac{m_1^2 \sigma_\beta}{2} + (A^- - A^+)\sigma_\beta \\ \varepsilon_y = \dfrac{\sigma_y}{2G^+} + A^+\Theta + (\dfrac{1}{G^-} - \dfrac{1}{G^+}) \dfrac{m_2^2 \sigma_\beta}{2} + (A^- - A^+)\sigma_\beta \\ \varepsilon_z = \dfrac{\sigma_z}{2G^+} + A^+\Theta + (\dfrac{1}{G^-} - \dfrac{1}{G^+}) \dfrac{m_3^2 \sigma_\beta}{2} + (A^- - A^+)\sigma_\beta \\ \gamma_{xy} = \dfrac{\tau_{xy}}{G^+} + (\dfrac{1}{G^-} - \dfrac{1}{G^+}) m_1 m_2 \sigma_\beta \\ \gamma_{yz} = \dfrac{\tau_{yz}}{G^+} + (\dfrac{1}{G^-} - \dfrac{1}{G^+}) m_2 m_3 \sigma_\beta \\ \gamma_{xz} = \dfrac{\tau_{xz}}{G^+} + (\dfrac{1}{G^-} - \dfrac{1}{G^+}) m_1 m_3 \sigma_\beta \end{cases} \tag{8}$$

where Θ is the first invariant of stress tensor ($\Theta = \sigma_x + \sigma_y + \sigma_z = \sigma_\alpha + \sigma_\beta + \sigma_\gamma$). It can be seen from Eq. (3) and Eq. (8) that the constitutive equations with bi-modulus in the normal rectangular coordinate system are completely different from the classical constitutive equations. The relationship between the stress and the strain is nonlinear. In addition to the linear terms in classical elastic relations, there are also nonlinear terms as the coefficients of linear terms are no longer constants, which depend on the signs of the principal stress. Based on the above analysis, Eq. (3) can be arranged in Cartesian coordinate system as following.

$$\begin{aligned} \varepsilon_x = {} & [(\frac{l_1^2}{2G_\alpha} + A_\alpha) l_1^2 + (\frac{m_1^2}{2G_\beta} + A_\beta) m_1^2 + (\frac{n_1^2}{2G_\gamma} + A_\gamma) n_1^2] \sigma_x + 2[(\frac{l_1^2}{2G_\alpha} + A_\alpha) l_1 l_2 + (\frac{m_1^2}{2G_\beta} + A_\beta) m_1 m_2 + (\frac{n_1^2}{2G_\gamma} + A_\gamma) n_1 n_2] \tau_{xy} \\ & + [(\frac{l_1^2}{2G_\alpha} + A_\alpha) l_2^2 + (\frac{m_1^2}{2G_\beta} + A_\beta) m_2^2 + (\frac{n_1^2}{2G_\gamma} + A_\gamma) n_2^2] \sigma_y + 2[(\frac{l_1^2}{2G_\alpha} + A_\alpha) l_2 l_3 + (\frac{m_1^2}{2G_\beta} + A_\beta) m_2 m_3 + (\frac{n_1^2}{2G_\gamma} + A_\gamma) n_2 n_3] \tau_{yz} \\ & + [(\frac{l_1^2}{2G_\alpha} + A_\alpha) l_3^2 + (\frac{m_1^2}{2G_\beta} + A_\beta) m_3^2 + (\frac{n_1^2}{2G_\gamma} + A_\gamma) n_3^2] \sigma_z + 2[(\frac{l_1^2}{2G_\alpha} + A_\alpha) l_1 l_3 + (\frac{m_1^2}{2G_\beta} + A_\beta) m_1 m_3 + (\frac{n_1^2}{2G_\gamma} + A_\gamma) n_1 n_3] \tau_{xz} \end{aligned} \tag{9}$$

$$\begin{aligned} \varepsilon_y = {} & [(\frac{l_2^2}{2G_\alpha} + A_\alpha) l_1^2 + (\frac{m_2^2}{2G_\beta} + A_\beta) m_1^2 + (\frac{n_2^2}{2G_\gamma} + A_\gamma) n_1^2] \sigma_x + 2[(\frac{l_2^2}{2G_\alpha} + A_\alpha) l_1 l_2 + (\frac{m_2^2}{2G_\beta} + A_\beta) m_1 m_2 + (\frac{n_2^2}{2G_\gamma} + A_\gamma) n_1 n_2] \tau_{xy} \\ & + [(\frac{l_2^2}{2G_\alpha} + A_\alpha) l_2^2 + (\frac{m_2^2}{2G_\beta} + A_\beta) m_2^2 + (\frac{n_2^2}{2G_\gamma} + A_\gamma) n_2^2] \sigma_y + 2[(\frac{l_2^2}{2G_\alpha} + A_\alpha) l_2 l_3 + (\frac{m_2^2}{2G_\beta} + A_\beta) m_2 m_3 + (\frac{n_2^2}{2G_\gamma} + A_\gamma) n_2 n_3] \tau_{yz} \\ & + [(\frac{l_2^2}{2G_\alpha} + A_\alpha) l_3^2 + (\frac{m_2^2}{2G_\beta} + A_\beta) m_3^2 + (\frac{n_2^2}{2G_\gamma} + A_\gamma) n_3^2] \sigma_z + 2[(\frac{l_2^2}{2G_\alpha} + A_\alpha) l_1 l_3 + (\frac{m_2^2}{2G_\beta} + A_\beta) m_1 m_3 + (\frac{n_2^2}{2G_\gamma} + A_\gamma) n_1 n_3] \tau_{xz} \end{aligned} \tag{10}$$

$$
\begin{aligned}
\varepsilon_z = {} & [(\frac{l_3^2}{2G_\alpha}+A_\alpha)l_1^2+(\frac{m_3^2}{2G_\beta}+A_\beta)m_1^2+(\frac{n_3^2}{2G_\gamma}+A_\gamma)n_1^2]\sigma_x+2[(\frac{l_3^2}{2G_\alpha}+A_\alpha)l_1l_2+(\frac{m_3^2}{2G_\beta}+A_\beta)m_1m_2+(\frac{n_3^2}{2G_\gamma}+A_\gamma)n_1n_2]\tau_{xy} \\
& +[(\frac{l_3^2}{2G_\alpha}+A_\alpha)l_2^2+(\frac{m_3^2}{2G_\beta}+A_\beta)m_2^2+(\frac{n_3^2}{2G_\gamma}+A_\gamma)n_2^2]\sigma_y+2[(\frac{l_3^2}{2G_\alpha}+A_\alpha)l_2l_3+(\frac{m_3^2}{2G_\beta}+A_\beta)m_2m_3+(\frac{n_3^2}{2G_\gamma}+A_\gamma)n_2n_3]\tau_{yz} \\
& +[(\frac{l_3^2}{2G_\alpha}+A_\alpha)l_3^2+(\frac{m_3^2}{2G_\beta}+A_\beta)m_3^2+(\frac{n_3^2}{2G_\gamma}+A_\gamma)n_3^2]\sigma_z+2[(\frac{l_3^2}{2G_\alpha}+A_\alpha)l_1l_3+(\frac{m_3^2}{2G_\beta}+A_\beta)m_1m_3+(\frac{n_3^2}{2G_\gamma}+A_\gamma)n_1n_3]\tau_{xz}
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\gamma_{xy} = {} & [\frac{l_1l_2}{G_\alpha}l_1^2+\frac{m_1m_2}{G_\beta}m_1^2+\frac{n_1n_2}{G_\gamma}n_1^2]\sigma_x+[\frac{l_1l_2}{G_\alpha}l_2^2+\frac{m_1m_2}{G_\beta}m_2^2+\frac{n_1n_2}{G_\gamma}n_2^2]\sigma_y \\
& +[\frac{l_1l_2}{G_\alpha}l_3^2+\frac{m_1m_2}{G_\beta}m_3^2+\frac{n_1n_2}{G_\gamma}n_3^2]\sigma_z+[\frac{l_1l_2}{G_\alpha}2l_1l_2+\frac{m_1m_2}{G_\beta}2m_1m_2+\frac{n_1n_2}{G_\gamma}2n_1n_2]\tau_{xy} \\
& +[\frac{l_1l_2}{G_\alpha}2l_1l_3+\frac{m_1m_2}{G_\beta}2m_1m_3+\frac{n_1n_2}{G_\gamma}2n_1n_3]\tau_{xz}+[\frac{l_1l_2}{G_\alpha}2l_2l_3+\frac{m_1m_2}{G_\beta}2m_2m_3+\frac{n_1n_2}{G_\gamma}2n_2n_3]\tau_{yz}
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\gamma_{xz} = {} & [\frac{l_1l_3}{G_\alpha}l_1^2+\frac{m_1m_3}{G_\beta}m_1^2+\frac{n_1n_3}{G_\gamma}n_1^2]\sigma_x+[\frac{l_1l_3}{G_\alpha}l_2^2+\frac{m_1m_3}{G_\beta}m_2^2+\frac{n_1n_3}{G_\gamma}n_2^2]\sigma_y \\
& +[\frac{l_1l_3}{G_\alpha}l_3^2+\frac{m_1m_3}{G_\beta}m_3^2+\frac{n_1n_3}{G_\gamma}n_3^2]\sigma_z+[\frac{l_1l_3}{G_\alpha}2l_1l_2+\frac{m_1m_3}{G_\beta}2m_1m_2+\frac{n_1n_3}{G_\gamma}2n_1n_2]\tau_{xy} \\
& +[\frac{l_1l_3}{G_\alpha}2l_1l_3+\frac{m_1m_3}{G_\beta}2m_1m_3+\frac{n_1n_3}{G_\gamma}2n_1n_3]\tau_{xz}+[\frac{l_1l_3}{G_\alpha}2l_2l_3+\frac{m_1m_3}{G_\beta}2m_2m_3+\frac{n_1n_3}{G_\gamma}2n_2n_3]\tau_{yz}
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\gamma_{yz} = {} & [\frac{l_2l_3}{G_\alpha}l_1^2+\frac{m_2m_3}{G_\beta}m_1^2+\frac{n_2n_3}{G_\gamma}n_1^2]\sigma_x+[\frac{l_2l_3}{G_\alpha}l_2^2+\frac{m_2m_3}{G_\beta}m_2^2+\frac{n_2n_3}{G_\gamma}n_2^2]\sigma_y \\
& +[\frac{l_2l_3}{G_\alpha}l_3^2+\frac{m_2m_3}{G_\beta}m_3^2+\frac{n_2n_3}{G_\gamma}n_3^2]\sigma_z+[\frac{l_2l_3}{G_\alpha}2l_1l_2+\frac{m_2m_3}{G_\beta}2m_1m_2+\frac{n_2n_3}{G_\gamma}2n_1n_2]\tau_{xy} \\
& +[\frac{l_2l_3}{G_\alpha}2l_1l_3+\frac{m_2m_3}{G_\beta}2m_1m_3+\frac{n_2n_3}{G_\gamma}2n_1n_3]\tau_{xz}+[\frac{l_2l_3}{G_\alpha}2l_2l_3+\frac{m_2m_3}{G_\beta}2m_2m_3+\frac{n_2n_3}{G_\gamma}2n_2n_3]\tau_{yz}
\end{aligned}
\tag{14}
$$

Seeing from Eq. (9) to Eq. (14), we can observe that all coefficients do not contain principal stress or principal strain, so they characterized the relationships between stress and strain in normal rectangular coordinate system, namely, generalized elastic law. When the tensile modulus is equal, we have

$$
\left.
\begin{aligned}
G_\alpha = G_\beta = G_\gamma = G = E/[2(1+\mu)] \\
A_\alpha = A_\beta = A_\gamma = A = -\mu/E
\end{aligned}
\right\}
\tag{15}
$$

and

$$
\begin{aligned}
\varepsilon_x = {} & (\frac{l_1^2}{2G}+A)(l_1^2\sigma_x+l_2^2\sigma_y+l_3^2\sigma_z+2l_1l_2\tau_{xy}+2l_1l_3\tau_{xz}+2l_2l_3\tau_{yz})+(\frac{m_1^2}{2G}+A)(m_1^2\sigma_x+m_2^2\sigma_y+m_3^2\sigma_z \\
& +2m_1m_2\tau_{xy}+2m_1m_3\tau_{xz}+2m_2m_3\tau_{yz})+(\frac{n_1^2}{2G}+A)(n_1^2\sigma_x+n_2^2\sigma_y+n_3^2\sigma_z+2n_1n_2\tau_{xy}+2n_1n_3\tau_{xz}+2n_2n_3\tau_{yz}) \\
= {} & (\frac{l_1^2}{2G}+A)\sigma_\alpha+(\frac{m_1^2}{2G}+A)\sigma_\beta+(\frac{n_1^2}{2G}+A)\sigma_\gamma=\frac{1}{2G}(l_1^2\sigma_\alpha+m_1^2\sigma_\beta+n_1^2\sigma_\gamma)+A(\sigma_\alpha+\sigma_\beta+\sigma_\gamma) \\
= {} & \frac{\sigma_x}{2G}+A(\sigma_x+\sigma_y+\sigma_z)=\frac{\sigma_x}{E}-\frac{\mu\sigma_y}{E}-\frac{\mu\sigma_z}{E}
\end{aligned}
\tag{16}
$$

which is Hook’s law for classical elasticity. The other five elasticity equations can be obtained in the same way.

2.3 Discussion on mechanical properties of bi-modulus problem

The stress state in the structure with bi-modulus materials can be classified into three groups: 1) The three principal stresses of the point are all positive or negative. The constitutive equation is same as the isotropic constitutive equation. 2) Three principal stress signs are not the same, but the principal stress direction is exactly the same as the coordinate axis direction. The constitutive equation of this point can be simplified into the original constitutive equation defined in Eq. (1), which is similar to the constitutive equation of orthogonal anisotropy. 3) Three principal stress signs are not exactly the same, and the principal stress direction does not coincide with the coordinate axis direction also. In the complex stress state, such areas generally account for the vast majority, and the constitutive equations are shown form Eq. (9) ~ Eq. (14). The magnitude and direction of the principal stress in such region are generally different. It can be seen that the generalized constitutive equations are the same in the form, but the corresponding coefficients are not equal. In the constitutive equations, the normal strains are not only related to the normal stresses, but also related to the shear stresses. In the same way, the shear strains depend not only on shear stresses, but also on the three normal stresses. Therefore, the generalized elastic constitutive equations in this region are similar to the constitutive equations of anisotropy.

It is clear that the constitutive relations for bi-modulus materials are of the linear elastic form. However, the constitutive equations of the bi-modulus elastic system depend on the directions of the principal stresses. Therefore, the mechanical behavior of structures composed of bi-modulus materials is function of the stress state in the field, which results the non-linearity and anisotropy.

3. General shear modulus and complemented algorithm in 3D case

As Ambartsymyan pointed out that the difference between bi-modulus theory and the classical linear elasticity theory lies in the constitutive relations. Therefore, the computational strategies of the finite element method with the bi-modulus materials are the same as that for the classical elastic materials except the elastic matrix ***D***, in another word, only the elastic matrix ***D*** and the stiffness matrix ***K*** needs to be modified.

3.1 Elastic matrix of bi-modulus theory

The transform equation on principal stress and principal strain with normal stress and normal strain are given:

$$\boldsymbol{\sigma}_I = \boldsymbol{L}_\sigma \boldsymbol{\sigma}\ ,\quad \boldsymbol{\varepsilon}_I = \boldsymbol{L}_\varepsilon \boldsymbol{\varepsilon} \tag{17}$$

$$\boldsymbol{L}_\sigma = \begin{bmatrix} l_1^2 & m_1^2 & n_1^2 & 2m_1n_1 & 2n_1l_1 & 2l_1m_1 \\ l_2^2 & m_2^2 & n_2^2 & 2m_2n_2 & 2n_2l_2 & 2l_2m_2 \\ l_3^2 & m_3^2 & n_3^2 & 2m_3n_3 & 2n_3l_3 & 2l_3m_3 \end{bmatrix},\quad \boldsymbol{L}_\varepsilon = \begin{bmatrix} l_1^2 & m_1^2 & n_1^2 & m_1n_1 & n_1l_1 & l_1m_1 \\ l_2^2 & m_2^2 & n_2^2 & m_2n_2 & n_2l_2 & l_2m_2 \\ l_3^2 & m_3^2 & n_3^2 & m_3n_3 & n_3l_3 & l_3m_3 \end{bmatrix}. \tag{18}$$

The strain energy per unit volume is expressed by principal strains as:

$$U = \frac{1}{2}\boldsymbol{\varepsilon}_I{}^T \boldsymbol{D}_I \boldsymbol{\varepsilon}_I \tag{19}$$

where $\boldsymbol{D}_I$ is the elastic matrix in principal directions ($\boldsymbol{D}_I$=$\boldsymbol{A}_I^{-1}$, $\boldsymbol{A}_I$ is gievn in Eq. (1)). Substituting Eq. (17) into Eq. (19) gives:

$$U = \frac{1}{2}\boldsymbol{\varepsilon}^T \boldsymbol{L}_\varepsilon{}^T \boldsymbol{D}_I \boldsymbol{L}_\varepsilon \boldsymbol{\varepsilon} \tag{20}$$

The strain energy per unit volume is expressed in terms of normal strain in Cartesian coordinate system as:

$$U = \frac{1}{2}\boldsymbol{\varepsilon}^T \boldsymbol{D}\boldsymbol{\varepsilon} \tag{21}$$

Since the energy is independent of the selection of coordinate system, we have:

$$\boldsymbol{D} = \boldsymbol{L}_\varepsilon{}^T \boldsymbol{D}_I \boldsymbol{L}_\varepsilon \tag{22}$$

where $\boldsymbol{D}$ is the elastic matrix of bi-modulus materials in Cartesian coordinate system. Therefore, the finite element stiffness matrix of bi-modulus theory can be obtained by:

$$\boldsymbol{K} = \int_V \boldsymbol{B}^T \boldsymbol{L}_\varepsilon{}^T \boldsymbol{D}_I \boldsymbol{L}_\varepsilon \boldsymbol{B} dV \tag{23}$$

3.2 Shear modulus and complement elastic matrix in 3D

The constitutive equation in principal stress coordinate system adopted in the traditional iterative algorithm is given as follows:

$$\begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \end{Bmatrix} = \boldsymbol{D}_I \begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \end{Bmatrix}, \quad \boldsymbol{D}_I = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} = \boldsymbol{A}_I^{-1} \tag{24}$$

In fact, for 3D problems, the elastic matrix should be a 6×6 order matrix as:

$$\boldsymbol{D} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \\ d_{41} & d_{42} & d_{43} & d_{44} & d_{45} & d_{46} \\ d_{51} & d_{52} & d_{53} & d_{54} & d_{55} & d_{56} \\ d_{61} & d_{62} & d_{63} & d_{64} & d_{65} & d_{66} \end{bmatrix} \tag{25}$$

Since the traditional constitutive matrix does not give values of the other terms coefficients, them will be defaulted to zero and the elastic matrix in the principal stress directions is expressed:

$$\begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \\ \tau_{\alpha\beta} \\ \tau_{\beta\gamma} \\ \tau_{\alpha\gamma} \end{Bmatrix} = \boldsymbol{D}_I \begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \\ \gamma_{\alpha\beta} \\ \gamma_{\beta\gamma} \\ \gamma_{\alpha\gamma} \end{Bmatrix}, \quad \boldsymbol{D}_I = \begin{bmatrix} d_{11} & d_{12} & d_{13} & 0 & 0 & 0 \\ d_{21} & d_{22} & d_{23} & 0 & 0 & 0 \\ d_{31} & d_{32} & d_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{26}$$

According to Eq. (25) and Eq. (26), it is obvious that even if the shear stress and shear strain in principal stress direction are assumed to be zero, it does not mean that the corresponding elastic coefficient terms are zeros, at least in terms of d_{44}, d_{55} and d_{66}, *i.e*. the so-called shear modulus is not zero. He et al, Zhang, et al. [19, 20] proposed the empirical shear modulus in order to improve the stability and convergence of the algorithm. However, the convergence efficiency is still unsatisfied. The reason is that the completed shear modulus does not satisfy the self-consistency.

Based on certain assumptions, the self-consistent shear modulus terms in 3D case are deduced by the limit principle of stress and strain, and a self-consistent 3D complemented algorithm is proposed in this paper. The proposed algorithm has efficient convergence efficiency for general cases, and is easy to be implemented with commercial finite element software.

It is assumed that the elastic matrix in principal stress direction for bi-modulus problems is in the same form of orthogonal anisotropy, and the principal stress axis is coincident with the principal strain axis. Then the elastic matrix and flexibility matrix based on the principal stress direction gives:

$$\boldsymbol{D}=\begin{bmatrix} d_{11} & d_{12} & d_{13} & 0 & 0 & 0 \\ d_{21} & d_{22} & d_{23} & 0 & 0 & 0 \\ d_{31} & d_{32} & d_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{66} \end{bmatrix}, \quad \boldsymbol{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix} \tag{27}$$

It can be seen from elastic mechanics and the matrix principle, $\boldsymbol{D}=\boldsymbol{A}^{-1}$, $d_{44}=1/a_{44}=G_{\alpha\beta}$, $d_{55}=1/a_{55}=G_{\beta\gamma}$, $d_{66}=1/a_{66}=G_{\alpha\gamma}$. $G_{\alpha\beta}$, $G_{\beta\gamma}$ and $G_{\alpha\gamma}$ are the shear moduli in the principal stress directions. However, since it is assumed that the principal strain axis is coincident with the principal stress axis, both shear stress and shear strain are zero.

Assuming that the axes x, y, and z tend to be the axis α, β and γ respectively, then we have:

$$\begin{cases} G_{\alpha\beta} = \lim\limits_{\substack{l_1,m_2,n_3\to 1 \\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} G_{xy} = \lim\limits_{\substack{l_1,m_2,n_3\to 1 \\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} \dfrac{\tau_{xy}}{\gamma_{xy}} \\ G_{\alpha\gamma} = \lim\limits_{\substack{l_1,m_2,n_3\to 1 \\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} G_{xz} = \lim\limits_{\substack{l_1,m_2,n_3\to 1 \\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} \dfrac{\tau_{xz}}{\gamma_{xz}} \\ G_{\beta\gamma} = \lim\limits_{\substack{l_1,m_2,n_3\to 1 \\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} G_{yz} = \lim\limits_{\substack{l_1,m_2,n_3\to 1 \\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} \dfrac{\tau_{yz}}{\gamma_{yz}} \end{cases} \tag{28}$$

According to the rotating formula of stress and strain, we hold:

$$\begin{cases} \tau_{xy} = l_1 l_2 \sigma_\alpha + m_1 m_2 \sigma_\beta + n_1 n_2 \sigma_\gamma \\ \gamma_{xy} = 2(l_1 l_2 \varepsilon_\alpha + m_1 m_2 \varepsilon_\beta + n_1 n_2 \varepsilon_\gamma) \end{cases} \tag{29}$$

When the coordinate axis changes from an infinitesimal angle from principal stress direction, the corresponding direction cosines have an infinitesimal change, and the new direction cosines yield l_1, m_2, $n_3 \to 1$; l_2, l_3, m_1, m_3, n_1, $n_2 \to 0$.

When $\sigma_\alpha = \sigma_\beta \neq \sigma_\gamma$, it can be obtained from cosine equations $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$, thus $n_1 n_2 = -(l_1 l_2 + m_1 m_2)$. Eq.(28) gives:

$$
\begin{aligned}
G_{\alpha\beta} &= \lim_{\substack{l_1, m_2, n_3 \to 1 \\ l_2, l_3, m_1, m_3, n_1, n_2 \to 0}} \frac{l_1 l_2 \sigma_\alpha + m_1 m_2 \sigma_\beta + n_1 n_2 \sigma_\gamma}{2(l_1 l_2 \varepsilon_\alpha + m_1 m_2 \varepsilon_\beta + n_1 n_2 \varepsilon_\gamma)} \\
&= \lim_{n_1, n_2 \to 0} \frac{-n_1 n_2 \sigma_\alpha + n_1 n_2 \sigma_\gamma}{2(-n_1 n_2 \varepsilon_\alpha + n_1 n_2 \varepsilon_\gamma)} \\
&= \frac{\sigma_\alpha - \sigma_\gamma}{2(\varepsilon_\alpha - \varepsilon_\gamma)} = G_{\alpha\gamma} = G_{\beta\gamma}
\end{aligned}
\tag{30}
$$

Similarly, if $\sigma_\alpha \neq \sigma_\beta = \sigma_\gamma$, then $\varepsilon_\alpha = \varepsilon_\beta \neq \varepsilon_\gamma$, one has:

$$G_{\alpha\beta} = G_{\alpha\gamma} = G_{\beta\gamma} = \frac{\sigma_\alpha - \sigma_\beta}{2(\varepsilon_\alpha - \varepsilon_\beta)} \tag{31}$$

if $\sigma_\alpha = \sigma_\gamma \neq \sigma_\beta$, then $\varepsilon_\alpha = \varepsilon_\gamma \neq \varepsilon_\beta$, one has:

$$G_{\alpha\beta} = G_{\alpha\gamma} = G_{\beta\gamma} = \frac{\sigma_\beta - \sigma_\gamma}{2(\varepsilon_\beta - \varepsilon_\gamma)} \tag{32}$$

if $\sigma_\alpha \neq \sigma_\beta \neq \sigma_\gamma$, then $\varepsilon_\alpha \neq \varepsilon_\gamma \neq \varepsilon_\beta$, we have:

$$G_{\alpha\beta} = \lim_{\substack{l_1, m_2, n_3 \to 1 \\ l_2, l_3, m_1, m_3, n_1, n_2 \to 0}} \frac{(l_2 / m_1)(\sigma_\alpha - \sigma_\gamma) + (\sigma_\beta - \sigma_\gamma)}{2[(l_2 / m_1)(\varepsilon_\alpha - \varepsilon_\gamma) + (\varepsilon_\beta - \varepsilon_\gamma)]} \tag{33}$$

$$G_{\alpha\gamma} = \lim_{\substack{l_1, m_2, n_3 \to 1 \\ l_2, l_3, m_1, m_3, n_1, n_2 \to 0}} \frac{(l_3 / n_1)(\sigma_\alpha - \sigma_\beta) + (\sigma_\gamma - \sigma_\beta)}{2[(l_3 / n_1)(\varepsilon_\alpha - \varepsilon_\beta) + (\varepsilon_\gamma - \varepsilon_\beta)]} \tag{34}$$

$$G_{\beta\gamma} = \lim_{\substack{l_1, m_2, n_3 \to 1 \\ l_2, l_3, m_1, m_3, n_1, n_2 \to 0}} \frac{(m_3 / n_2)(\sigma_\beta - \sigma_\alpha) + (\sigma_\gamma - \sigma_\alpha)}{2[(m_3 / n_2)(\varepsilon_\beta - \varepsilon_\alpha) + (\varepsilon_\gamma - \varepsilon_\alpha)]} \tag{35}$$

The ratios of l_2 / m_1, m_3 / n_2 and n_1 / l_3 are proved to -1 in Appendix B. Then, substituting these values into Eq. (33) ~ Eq. (35) yields:

$$G_{\alpha\beta} = \frac{\sigma_\alpha - \sigma_\beta}{2(\varepsilon_\alpha - \varepsilon_\beta)} \tag{36}$$

$$G_{\alpha\gamma} = \frac{\sigma_\alpha - \sigma_\gamma}{2(\varepsilon_\alpha - \varepsilon_\gamma)} \tag{37}$$

$$G_{\beta\gamma} = \frac{\sigma_\beta - \sigma_\gamma}{2(\varepsilon_\beta - \varepsilon_\gamma)} \tag{38}$$

There are eight cases of shear modulus in principal stress coordinate as follows in Table 2:

Table 2 Shear moduli in eight type of principal stress state

1	$\sigma_\alpha \geq 0$, $\sigma_\beta \geq 0$, $\sigma_\gamma \geq 0$		$G_{\alpha\beta} = G_{\alpha\gamma} = G_{\beta\gamma} = \frac{E^+}{2(1+\mu^+)} = G^+$
2	$\sigma_\alpha < 0$, $\sigma_\beta < 0$, $\sigma_\gamma < 0$		$G_{\alpha\beta} = G_{\alpha\gamma} = G_{\beta\gamma} = \frac{E^-}{2(1+\mu^-)} = G^-$
3	$\sigma_\alpha < 0$	$\sigma_\alpha \neq \sigma_\beta \neq \sigma_\gamma$	$G_{\alpha\beta} = \frac{\sigma_\alpha - \sigma_\beta}{\sigma_\alpha / G^- - \sigma_\beta / G^+}$, $G_{\alpha\gamma} = \frac{\sigma_\alpha - \sigma_\gamma}{\sigma_\alpha / G^- - \sigma_\gamma / G^+}$, $G_{\beta\gamma} = G^+$

	$\sigma_\beta \ge 0$ $\sigma_\gamma \ge 0$	$\sigma_\alpha \ne \sigma_\beta = \sigma_\gamma$	$G_{\alpha\beta}=G_{\alpha\gamma}=G_{\beta\gamma}=\dfrac{\sigma_\alpha-\sigma_\beta}{\sigma_\alpha/G^- - \sigma_\beta/G^+}$
4	$\sigma_\alpha \ge 0$ $\sigma_\beta < 0$ $\sigma_\gamma < 0$	$\sigma_\alpha \ne \sigma_\beta \ne \sigma_\gamma$	$G_{\alpha\beta}=\dfrac{\sigma_\alpha-\sigma_\beta}{\sigma_\alpha/G^+ - \sigma_\beta/G^-}$, $G_{\alpha\gamma}=\dfrac{\sigma_\alpha-\sigma_\gamma}{\sigma_\alpha/G^+ - \sigma_\gamma/G^-}$, $G_{\beta\gamma}=G^-$
		$\sigma_\alpha \ne \sigma_\beta = \sigma_\gamma$	$G_{\alpha\beta}=G_{\alpha\gamma}=G_{\beta\gamma}=\dfrac{\sigma_\alpha-\sigma_\beta}{\sigma_\alpha/G^+ - \sigma_\beta/G^-}$
5	$\sigma_\alpha \ge 0$ $\sigma_\beta < 0$ $\sigma_\gamma \ge 0$	$\sigma_\alpha \ne \sigma_\beta \ne \sigma_\gamma$	$G_{\alpha\beta}=\dfrac{\sigma_\alpha-\sigma_\beta}{\sigma_\alpha/G^+ - \sigma_\beta/G^-}$, $G_{\alpha\gamma}=G^+$, $G_{\beta\gamma}=\dfrac{\sigma_\beta-\sigma_\gamma}{\sigma_\beta/G^- - \sigma_\gamma/G^+}$
		$\sigma_\alpha = \sigma_\gamma \ne \sigma_\beta$	$G_{\alpha\beta}=G_{\alpha\gamma}=G_{\beta\gamma}=\dfrac{\sigma_\alpha-\sigma_\beta}{\sigma_\alpha/G^+ - \sigma_\beta/G^-}$
6	$\sigma_\alpha < 0$ $\sigma_\beta \ge 0$ $\sigma_\gamma < 0$	$\sigma_\alpha \ne \sigma_\beta \ne \sigma_\gamma$	$G_{\alpha\beta}=\dfrac{\sigma_\alpha-\sigma_\beta}{\sigma_\alpha/G^- - \sigma_\beta/G^+}$, $G_{\alpha\gamma}=G^-$, $G_{\beta\gamma}=\dfrac{\sigma_\beta-\sigma_\gamma}{\sigma_\beta/G^+ - \sigma_\gamma/G^-}$
		$\sigma_\alpha \ne \sigma_\beta = \sigma_\gamma$	$G_{\alpha\beta}=G_{\alpha\gamma}=G_{\beta\gamma}=\dfrac{\sigma_\alpha-\sigma_\beta}{\sigma_\alpha/G^- - \sigma_\beta/G^+}$
7	$\sigma_\alpha \ge 0$ $\sigma_\beta \ge 0$ $\sigma_\gamma < 0$	$\sigma_\alpha \ne \sigma_\beta \ne \sigma_\gamma$	$G_{\alpha\beta}=G^+$, $G_{\alpha\gamma}=\dfrac{\sigma_\alpha-\sigma_\gamma}{\sigma_\alpha/G^+ - \sigma_\gamma/G^-}$, $G_{\beta\gamma}=\dfrac{\sigma_\beta-\sigma_\gamma}{\sigma_\beta/G^+ - \sigma_\gamma/G^-}$
		$\sigma_\alpha = \sigma_\beta \ne \sigma_\gamma$	$G_{\alpha\beta}=G_{\alpha\gamma}=G_{\beta\gamma}=\dfrac{\sigma_\alpha-\sigma_\gamma}{\sigma_\alpha/G^+ - \sigma_\gamma/G^-}$
8	$\sigma_\alpha < 0$ $\sigma_\beta < 0$ $\sigma_\gamma \ge 0$	$\sigma_\alpha \ne \sigma_\beta \ne \sigma_\gamma$	$G_{\alpha\beta}=G^-$, $G_{\alpha\gamma}=\dfrac{\sigma_\alpha-\sigma_\gamma}{\sigma_\alpha/G^- - \sigma_\gamma/G^+}$, $G_{\beta\gamma}=\dfrac{\sigma_\beta-\sigma_\gamma}{\sigma_\beta/G^- - \sigma_\gamma/G^+}$
		$\sigma_\alpha = \sigma_\beta \ne \sigma_\gamma$	$G_{\alpha\beta}=G_{\alpha\gamma}=G_{\beta\gamma}=\dfrac{\sigma_\alpha-\sigma_\gamma}{\sigma_\alpha/G^- - \sigma_\gamma/G^+}$

When $\sigma_\alpha = \sigma_\gamma = \sigma_\beta$, it is hydraulic stress state. It is discussed in section 2.1 and the shear moduli are proved to be G^+ and G^-respectively.

3.3 Complemented algorithm with FEM

3.3.1 Finite element calculation process

Because the bi-modulus problem is a nonlinear problem as all elastic parameters in the field are functions of the stress state, the iterative technique is employed in this paper. By using the results from the previous calculation, the principal stress state is specified to determine the elastic matrix for the next step of iteration. The iteration format is as follows:

$$\boldsymbol{K}_{i-1}\boldsymbol{u}_i = \boldsymbol{F}_i \tag{39}$$

where $\boldsymbol{K}_{i-1}$ is the global stiffness matrix in the ***i***-1 iteration step, $\boldsymbol{u}_i$ is current displacement matrix and $\boldsymbol{F}_i$ is the vector of the force term.

The calculation of iteration can be described as follows:

Step 1. Set the mechanical property of structure as one modulus, i.e. the initial elastic parameters of the structure are specified as in either state of full tension or full compression (the initial elastic matrix is $\boldsymbol{D}^+$ or $\boldsymbol{D}^-$), then calculate the stresses and strains in each element.

Step 2. Determine the principal stress and their directions of each Gaussian integral point. Based on the principal stresses in each integral point, determine the compliance matrix ***A*** in the principal stress direction. Then, obtain the corresponding elastic matrix ***D*** of bi-modulus theory by Eq. (22) and table x, and the stiffness matrix ***K*** by Eq. (23).

Step 3. The stresses and strains of each element are calculated according to the new stiffness matrix.

Step 4. Calculating the displacement difference of each node or the stress difference of the unit integral point at the i+1 and i iteration. If the convergence is satisfied, the calculation is completed, Otherwise, let $i = i+1$, and go to step 2 for the next iteration.

The calculation procedure is described in the flow chart as following:

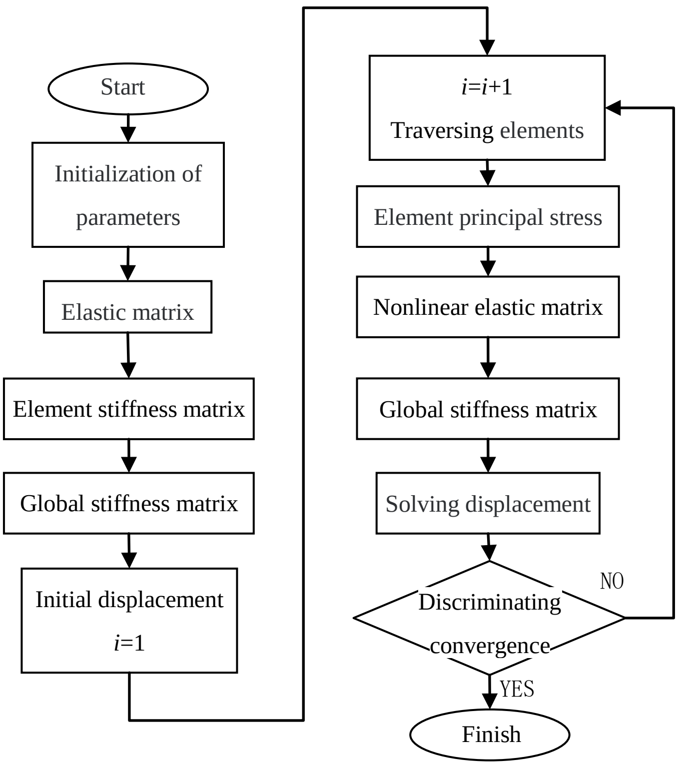


Fig.1. The flow chart of calculation for bi-modulus problem

The convergence criterion can be defined as:

(1) The difference between displacement at i time and i+1 time of each node, namely:

$$\left\|\boldsymbol{u}_i-\boldsymbol{u}_{i-1}\right\| \le \lambda_1 \text{ or } \frac{\left\|\boldsymbol{u}_i-\boldsymbol{u}_{i-1}\right\|}{\boldsymbol{u}_{i-1}} \le \lambda_2 \tag{40}$$

(2) The difference between stress at i time and i+1 time of each node, namely:

$$\left\|\boldsymbol{\sigma}_i-\boldsymbol{\sigma}_{i-1}\right\| \le \lambda_3 \text{ or } \frac{\left\|\boldsymbol{\sigma}_i-\boldsymbol{\sigma}_{i-1}\right\|}{\boldsymbol{\sigma}_{i-1}} \le \lambda_4 \tag{41}$$

The study [20, 23] shows that the above two controls have very small difference.

3.3.2 Implementation on complemented algorithm by ABAQUS

Based on the tangent algorithm proposed by Du and Zhang first [6, 23], we developed the 3D complemented algorithm with subroutine UMAT in ABAQUS. Since ABAQUS adopts the displacement method, namely taking the displacement as the unknown variable, the strain is transmitted to UMAT. Therefore, ABAQUS should judge the stress combination according to the following principles or formulas:

According to Eq. (1),we have

$$\begin{cases}\sigma_\alpha=\dfrac{1}{\Delta}\left[E_\alpha(1-\mu_\beta\mu_\gamma)\varepsilon_\alpha+E_\alpha\mu_\beta(1+\mu_\gamma)\varepsilon_\beta+E_\alpha\mu_\gamma(1+\mu_\beta)\varepsilon_\gamma\right]\\ \sigma_\beta=\dfrac{1}{\Delta}\left[E_\beta\mu_\alpha(1+\mu_\gamma)\varepsilon_\alpha+E_\beta(1-\mu_\alpha\mu_\gamma)\varepsilon_\beta+E_\beta\mu_\gamma(1+\mu_\alpha)\varepsilon_\gamma\right]\\ \sigma_\gamma=\dfrac{1}{\Delta}\left[E_\gamma\mu_\alpha(1+\mu_\beta)\varepsilon_\alpha+E_\gamma\mu_\beta(1+\mu_\alpha)\varepsilon_\beta+E_\gamma(1-\mu_\alpha\mu_\beta)\varepsilon_\gamma\right]\end{cases} \tag{42}$$

where $\Delta=1-2\mu_\alpha\mu_\beta\mu_\gamma-(\mu_\alpha\mu_\beta+\mu_\beta\mu_\gamma+\mu_\gamma\mu_\alpha)$. According to the requirements in the subroutine UMAT of ABAQUS, it only needs to determine stress states whether the following inequalities are satisfied.

(1) When $\sigma_\alpha\geq0$, $\sigma_\beta\geq0$, $\sigma_\gamma\geq0$,

Let $\Delta_1=\dfrac{1}{(1+\mu^+)(1-2\mu^+)}$, then we hold:

$$\begin{cases}\sigma_\alpha=\Delta_1\left[E^+(1-\mu^+)\varepsilon_\alpha+E^+\mu^+\varepsilon_\beta+E^+\mu^+\varepsilon_\gamma\right]\\ \sigma_\beta=\Delta_1\left[E^+\mu^+\varepsilon_\alpha+E^+(1-\mu^+)\varepsilon_\beta+E^+\mu^+\varepsilon_\gamma\right]\\ \sigma_\gamma=\Delta_1\left[E^+\mu^+\varepsilon_\alpha+E^+\mu^+\varepsilon_\beta+E^+(1-\mu^+)\varepsilon_\gamma\right]\end{cases} \tag{43}$$

where $\Delta_1=\dfrac{1}{(1+\mu^+)(1-2\mu^+)}$. Because the Poisson's ratio is less than 0.5, then Δ_1>0, Therefore, it only needs to judge whether the following inequality is satisfied:

$$\begin{cases}(1-\mu^+)\varepsilon_\alpha+\mu^+\varepsilon_\beta+\mu^+\varepsilon_\gamma\geq0\\ \mu^+\varepsilon_\alpha+(1-\mu^+)\varepsilon_\beta+\mu^+\varepsilon_\gamma\geq0\\ \mu^+\varepsilon_\alpha+\mu^+\varepsilon_\beta+(1-\mu^+)\varepsilon_\gamma\geq0\end{cases} \tag{44}$$

In the same way, the discriminant inequality of the rest cases can be obtained as follows:

(2) When $\sigma_\alpha<0$, $\sigma_\beta<0$, $\sigma_\gamma<0$,

$$\begin{cases}(1-\mu^-)\varepsilon_\alpha+\mu^-\varepsilon_\beta+\mu^-\varepsilon_\gamma<0\\ \mu^-\varepsilon_\alpha+(1-\mu^-)\varepsilon_\beta+\mu^-\varepsilon_\gamma<0\\ \mu^-\varepsilon_\alpha+\mu^-\varepsilon_\beta+(1-\mu^-)\varepsilon_\gamma<0\end{cases} \tag{45}$$

(3) When $\sigma_\alpha<0$, $\sigma_\beta\geq0$, $\sigma_\gamma\geq0$,

$$
\begin{cases}
(1-\mu^{+}\mu^{+})\varepsilon_{\alpha}+\mu^{+}(1+\mu^{+})\varepsilon_{\beta}+\mu^{+}(1+\mu^{+})\varepsilon_{\gamma}<0\\
\mu^{-}(1+\mu^{+})\varepsilon_{\alpha}+(1-\mu^{-}\mu^{+})\varepsilon_{\beta}+\mu^{+}(1+\mu^{-})\varepsilon_{\gamma}\geq 0\\
\mu^{-}(1+\mu^{+})\varepsilon_{\alpha}+\mu^{+}(1+\mu^{+})\varepsilon_{\beta}+(1-\mu^{-}\mu^{+})\varepsilon_{\gamma}\geq 0
\end{cases}
\tag{46}
$$

(4) When $\sigma_{\alpha}\geq 0$, $\sigma_{\beta}<0$, $\sigma_{\gamma}<0$,

$$
\begin{cases}
(1-\mu^{-}\mu^{-})\varepsilon_{\alpha}+\mu^{-}(1+\mu^{-})\varepsilon_{\beta}+\mu^{-}(1+\mu^{-})\varepsilon_{\gamma}\geq 0\\
\mu^{+}(1+\mu^{-})\varepsilon_{\alpha}+(1-\mu^{-}\mu^{+})\varepsilon_{\beta}+\mu^{-}(1+\mu^{+})\varepsilon_{\gamma}<0\\
\mu^{+}(1+\mu^{-})\varepsilon_{\alpha}+\mu^{-}(1+\mu^{-})\varepsilon_{\beta}+(1-\mu^{-}\mu^{+})\varepsilon_{\gamma}<0
\end{cases}
\tag{47}
$$

(5) When $\sigma_{\alpha}\geq 0$, $\sigma_{\beta}<0$, $\sigma_{\gamma}\geq 0$,

$$
\begin{cases}
(1-\mu^{-}\mu^{+})\varepsilon_{\alpha}+\mu^{-}(1+\mu^{+})\varepsilon_{\beta}+\mu^{+}(1+\mu^{-})\varepsilon_{\gamma}\geq 0\\
\mu^{+}(1+\mu^{+})\varepsilon_{\alpha}+(1-\mu^{+}\mu^{+})\varepsilon_{\beta}+\mu^{+}(1+\mu^{+})\varepsilon_{\gamma}<0\\
\mu^{+}(1+\mu^{-})\varepsilon_{\alpha}+\mu^{-}(1+\mu^{+})\varepsilon_{\beta}+(1-\mu^{+}\mu^{-})\varepsilon_{\gamma}\geq 0
\end{cases}
\tag{48}
$$

(6) When $\sigma_{\alpha}<0$, $\sigma_{\beta}\geq 0$, $\sigma_{\gamma}<0$,

$$
\begin{cases}
(1-\mu^{+}\mu^{-})\varepsilon_{\alpha}+\mu^{+}(1+\mu^{-})\varepsilon_{\beta}+\mu^{-}(1+\mu^{+})\varepsilon_{\gamma}<0\\
\mu^{-}(1+\mu^{-})\varepsilon_{\alpha}+(1-\mu^{-}\mu^{-})\varepsilon_{\beta}+\mu^{-}(1+\mu^{-})\varepsilon_{\gamma}\geq 0\\
\mu^{-}(1+\mu^{+})\varepsilon_{\alpha}+\mu^{+}(1+\mu^{-})\varepsilon_{\beta}+(1-\mu^{-}\mu^{+})\varepsilon_{\gamma}<0
\end{cases}
\tag{49}
$$

(7) When $\sigma_{\alpha}\geq 0$, $\sigma_{\beta}\geq 0$, $\sigma_{\gamma}<0$,

$$
\begin{cases}
(1-\mu^{+}\mu^{-})\varepsilon_{\alpha}+\mu^{+}(1+\mu^{-})\varepsilon_{\beta}+\mu^{-}(1+\mu^{+})\varepsilon_{\gamma}\geq 0\\
\mu^{+}(1+\mu^{-})\varepsilon_{\alpha}+(1-\mu^{+}\mu^{-})\varepsilon_{\beta}+\mu^{-}(1+\mu^{+})\varepsilon_{\gamma}\geq 0\\
\mu^{+}(1+\mu^{+})\varepsilon_{\alpha}+\mu^{+}(1+\mu^{+})\varepsilon_{\beta}+(1-\mu^{+}\mu^{+})\varepsilon_{\gamma}<0
\end{cases}
\tag{50}
$$

(8) When $\sigma_{\alpha}<0$, $\sigma_{\beta}<0$, $\sigma_{\gamma}\geq 0$,

$$
\begin{cases}
(1-\mu^{-}\mu^{+})\varepsilon_{\alpha}+\mu^{-}(1+\mu^{+})\varepsilon_{\beta}+\mu^{+}(1+\mu^{-})\varepsilon_{\gamma}<0\\
\mu^{+}(1+\mu^{+})\varepsilon_{\alpha}+(1-\mu^{-}\mu^{+})\varepsilon_{\beta}+\mu^{+}(1+\mu^{-})\varepsilon_{\gamma}<0\\
\mu^{+}(1+\mu^{-})\varepsilon_{\alpha}+\mu^{-}(1+\mu^{-})\varepsilon_{\beta}+(1-\mu^{-}\mu^{-})\varepsilon_{\gamma}\geq 0
\end{cases}
\tag{51}
$$

By observing the principal stresses, we can determine the shear modulus based on Table 2, then obtain the compliance matrix $\boldsymbol{A}$. Considering $\boldsymbol{D}_I=\boldsymbol{A}^{-1}$, the elastic matrix $\boldsymbol{D}_I$ can be obtained. Finally, the elastic matrix $\boldsymbol{D}$ in global coordinate for bi-modulus materials can be obtained by:

$$
\boldsymbol{D}=\bar{\boldsymbol{L}}_{\varepsilon}^{T}\boldsymbol{D}_I\bar{\boldsymbol{L}}_{\varepsilon}
\tag{52}
$$

where $\bar{\boldsymbol{L}}_{\varepsilon}$ is the transformation matrix:

$$
\bar{\boldsymbol{L}}_{\varepsilon}=\begin{bmatrix}
l_1^2 & m_1^2 & n_1^2 & m_1n_1 & n_1l_1 & l_1m_1\\
l_2^2 & m_2^2 & n_2^2 & m_2n_2 & n_2l_2 & l_2m_2\\
l_3^2 & m_3^2 & n_3^2 & m_3n_3 & n_3l_3 & l_3m_3\\
2l_1l_2 & 2m_1m_2 & 2n_1n_2 & m_1n_2+n_1m_2 & n_1l_2+l_2n_1 & l_1m_2+l_2m_1\\
2l_2l_3 & 2m_2m_3 & 2n_2n_3 & m_2n_3+n_2m_3 & n_2l_3+l_2n_3 & l_2m_3+l_3m_2\\
2l_3l_1 & 2m_3m_1 & 2n_3n_1 & m_3n_1+n_3m_1 & n_3l_1+l_3n_1 & l_3m_1+l_1m_3
\end{bmatrix}
\tag{53}
$$

The subsequent steps are the same for classical elasticity. The flow chat with UMAT of ABAQUS is shown in Fig. 3.

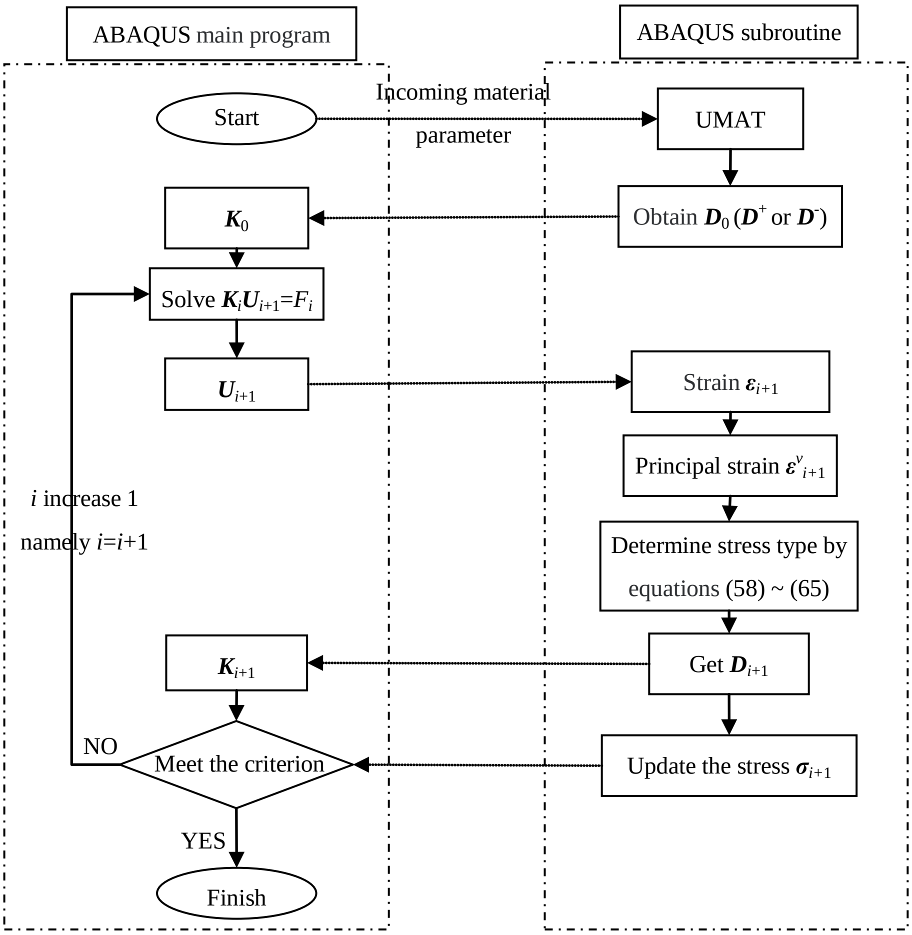


Fig.2. The calculation principle of complemented algorithm being applied to ABAQUS

The convergence criterion of ABAQUS is multi-index comprehensive control, among which the maximum iterative residual internal force R_a and the displacement correction c_a play major control roles. The standard value of ABAQUS's default convergence is that R_a is small than 0.5% of the average force on the structure, and c_a is less than 1% of the total incremental displacement $\|\Delta \boldsymbol{u}\|$, which are defined as:

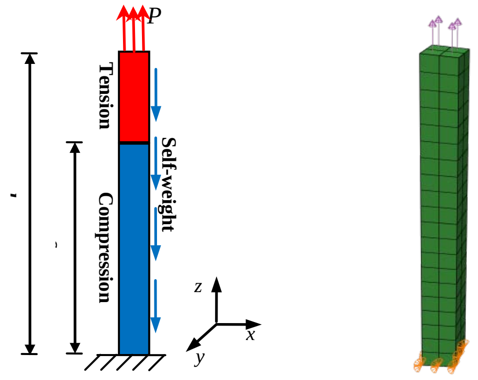
$$R_a = K_{i+1}U_{i+1} - F \ ; \quad c_a = \|\boldsymbol{u}_{i+1} - \boldsymbol{u}_i\| \ ; \quad \|\Delta \boldsymbol{u}\| = \|\boldsymbol{u}_{i+1} - \boldsymbol{u}_0\| \tag{54}$$

It has been shown that when the computation converges, $\|\Delta \boldsymbol{u}\|$ is generally less than 10^{-8} with the ABAQUS default convergence criterion. The accuracy requirements are fully met, and the error between the calculated results in this paper and those in existing literature is minimal. Therefore, the subsequent analysis in this paper will adopt ABAQUS software's default convergence criteria.

4. Numerical examples

4.1 A tensile column with gravity

As shown in Fig. 4, the length of column l is 10m, the cross-section is 1m×1m with 3D linear integrator element (C3D8). The uniformly distributed load P is 6Pa and the self-weight of material per unit volume γ is 2N /m^3. The fixed compressive modulus $E^-$is 5000Pa, and the tensile Poisson's ratio and compressive Poisson's ratio are all zeroes. The ratio of tensile modulus is free variable.



(*a*) Force schematic diagram (*b*) 3D finite element model

Fig.3. A tensile column analysis with gravity

According to the literature [4], the analytical solution of vertical displacement at any point along the z direction of this column is:

$$w_z = \begin{cases} \dfrac{pz}{E^-} - \dfrac{\gamma}{E^-}(lz - \dfrac{1}{2}z^2) & z < c \\ \dfrac{\gamma}{2}\left[\dfrac{(z-c)^2}{E^+} - \dfrac{c^2}{E^-}\right] & z > c \end{cases} \tag{55}$$

In Eq. (55), $c = l - p/\gamma$, that is demarcation point of the tensile and compressive stresses in the column. The calculation results are shown in Table 3 and the iteration numbers are shown in Table 4.

Table 3 The contrast between numerical solution and analytical solution of displacement

z(m)	E^-/E^+=1 theoretical value	E^-/E^+=1 numerical value	E^-/E^+=5 theoretical value	E^-/E^+=5 numerical value	E^-/E^+=10 theoretical value	E^-/E^+=10 numerical value
2	-4.80E-03	-4.800E-03	-4.80E-03	-4.800E-03	-4.80E-03	-4.800E-03
4	-8.00E-03	-8.000E-03	-8.00E-03	-8.000E-03	-8.00E-03	-8.000E-03
7	-9.80E-03	-9.800E-03	-9.80E-03	-9.800E-03	-9.80E-03	-9.800E-03
9	-9.00E-03	-9.000E-03	-5.80E-03	-5.800E-03	-1.80E-03	-1.800E-03

Table 4 Iteration numbers of different algorithms

E^-/E^+	z=10m analytical solution	z=10m numerical solution	Number of convergent iterations PVP	Tangent algorithm	complemented algorithm $E^0 = E^+$	complemented algorithm $E^0 = E^-$
2	-6.2E-03	-6.200E-03	15	2	2	2

5	-8.0E-04	-8.000E-04	39	2	2	2
10	8.2E-03	8.200E-03	82	2	2	2
50	8.02E-02	8.020E-02	/	2	2	2
100	1.702E-01	1.702E-01	/	2	2	2
1000	1.79	1.7902	/	14	2	2
5000	8.99	8.9902	/	82	2	2
10000	17.99	17.9902	/	286	2	2

4.2 Door-shaped frame with gravity

Consider a 3D door-shaped structure with gravity. The boundary conditions and dimension of structure are shown in Fig. 5. The bottom is fixed and 384 C3P8 linear elements in total are used. For the convenience to contrast, the same material parameters as those in literature [24] are adopted, i.e. the compressive modulus E^{-}=1800MPa, and the compressive Poisson’s ratio is 0.3. Then the tensile modulus $E^{+}$and the tensile Poisson's ratio μ^{+} satisfy $\mu^{+}/E^{+}=\mu^{-}/E^{-}$. The computational results and convergence efficiency are shown in Table 5 and Fig. 6 respectively.

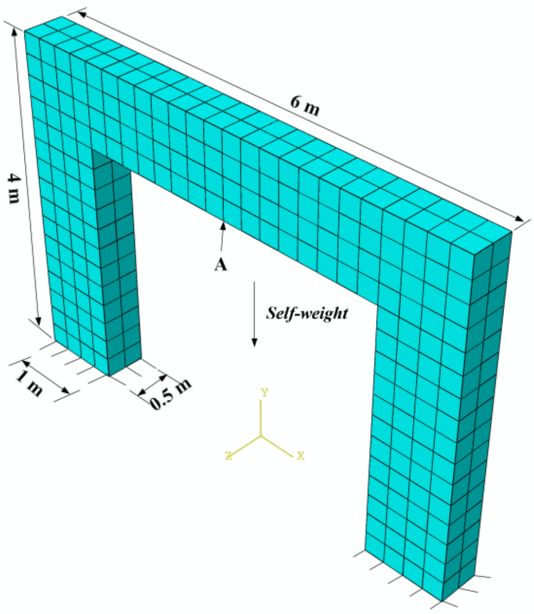


Fig.4. Sketch of a door-shaped frame

Table 5 Contrast to calculation results based on different E^{-}/E^{+}

E^{-}/E^{+}	pvp	tangent algorithm	complemented algorithm	iterations number of tangent algorithm	iterations number of complemented algorithm
1	5.462E-4	5.57896E-4	5.57896E-4	1	1
2	6.237E-4	6.38550E-4	6.38550E-4	4	4
3	6.827E-4	7.01622E-4	7.01622E-4	5	5

4	7.329E-4	7.55735E-4	7.55735E-4	5	5
5	7.775E-4	8.04274E-4	8.04274E-4	6	6
100	/	2.86946E-3	2.86946E-3	11	11
1000	/	1.56524E-2	1.56524E-2	24	24
5000	/	6.87581E-2	6.87581E-2	29	29
10000	/	1.34807E-1	1.34807E-1	42	42

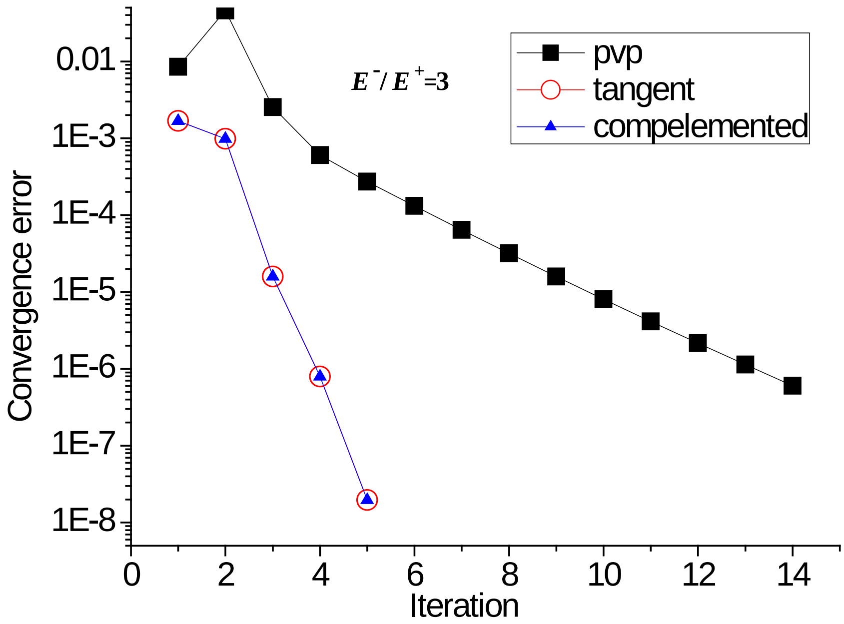


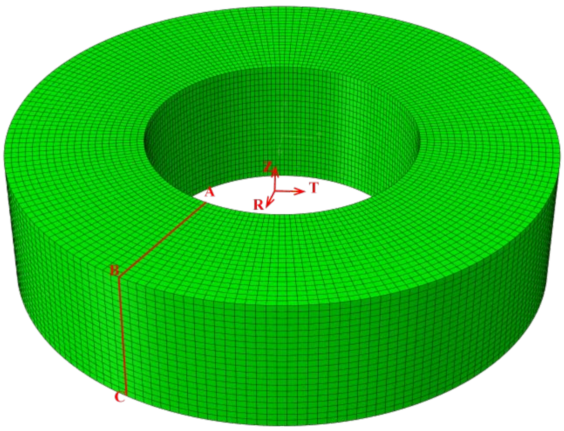
Fig.5. The displacement tolerance convergence curves of three algorithms when E^{-}/E^{+} is 3

These two examples show that the numerical results are in excellent agreement with the solutions in references.

In addition, compared with PVP algorithm, it can be seen from Table 5 and Fig. 5 that the complemented algorithm has fewer iterations with faster reduction of iteration tolerance and high convergence efficiency. Table 4 shows that the iterations number of completion algorithm are exactly the same in different initial guess. When the difference between tensile modulus and compressive modulus is small, the complemented algorithm is not much different from the tangent algorithm. Otherwise, the convergence efficiency of the complemented algorithm is slightly better than that of the tangent algorithm. We checked the iteration history of tangent algorithm and find out that when E^{-}/E^{+} greater than 1000, the stiffness matrix of the structure has negative eigenvalues during the iteration process in Example 1, but the completion algorithm does not appear. This maybe the reason for the iterations number of tangent algorithm increasing in some cases. In addition, as the tangent algorithm needs the derivative of the elastic matrix in order to obtain the tangent modulus matrix, the process is complicated and tedious. However, the complemented algorithm can be discriminated directly and be easy to be implemented.

4.3 A hollow cylinder with gravity

The hollow cylinder shown in Fig.6 and Fig.7 has an inner diameter of 1m, an outer diameter of 2m and a height of 1m. The inner pressure P_1 is 2Pa, the outer pressure P_2 is 1Pa, the uniform tension P_3 on the top surface is 1Pa, and the uniform pressure P_4 on the bottom surface is 1Pa. The self-weight of material per unit volume γ is 2N /m^3. Cylindrical coordinate (R is radial direction, T is circumference tangent direction, Z is axial direction) is adopted for structural calculation. The compressive modulus is selected as 100kPa, and the compressive Poisson's ratio was 0.2. The ratio (ω) of compressive modulus to tensile modulus is variable.



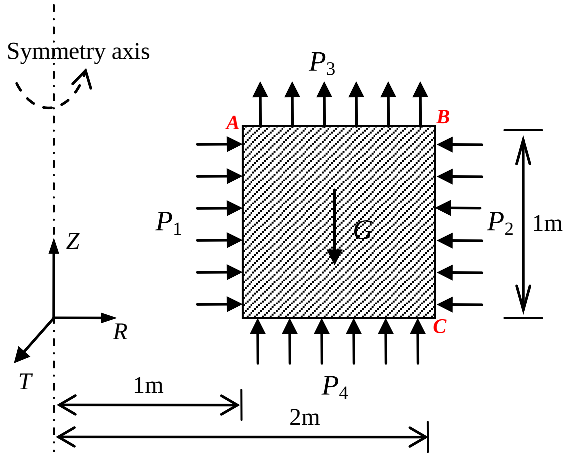


Fig.6. Schematic diagram of structure meshing **Fig.7.** Structural force and its size

Numerical and analytical solutions (*) [4]on the tangential stress (σ_T) and the tangential strain (ε_T) along AB are shown in Table 6, and the variations of strains with coordinate for different ratios ω are shown in Figure 8 ~ Figure 10.

Table 6 The contrast between numerical solution and analytical solution (*) of strain (10^{-6})

R/m	$\omega=E^-/E^+=1$				$\omega=E^-/E^+=4$			
	σ_T*/Pa	σ_T/Pa	ε_T */10^{-6}	ε_T /10^{-6}	σ_T*/Pa	σ_T/Pa	ε_T */10^{-6}	ε_T /10^{-6}
1.0	0.67	0.67	8.7	8.7	0.23	0.23	11.1	11.2
1.2	0.26	0.26	3.8	3.8	0.11	0.12	5.8	5.9
1.4	0.01	0.01	0.8	0.8	0.04	0.05	2.5	2.6
1.6	-0.15	-0.15	-1.1	-1.1	-0.01	0.00	0.3	0.4
2.0	-0.25	-0.26	-2.4	-2.4	-0.14	-0.13	-1.2	-1.2

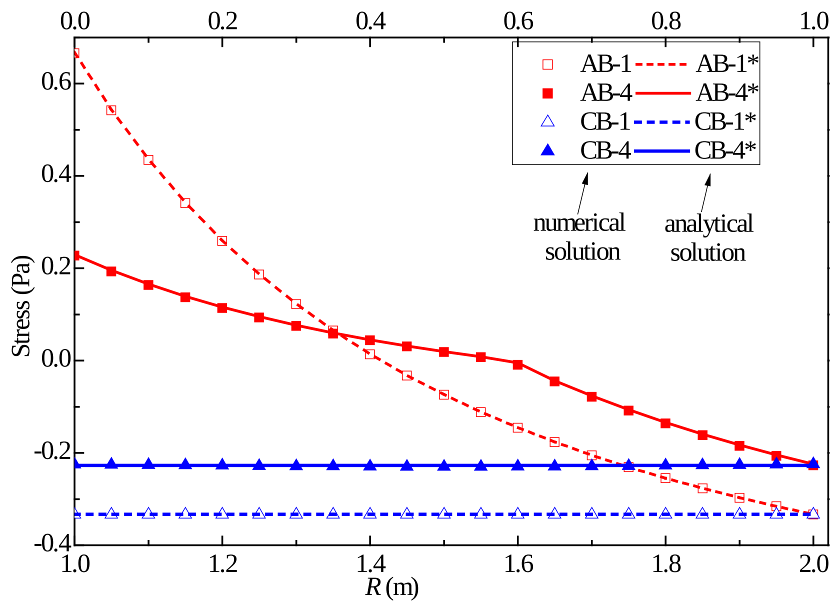


Fig.8. Changing laws of tangential stress (σ_T) of AB and CB (ω=1 & ω=4)

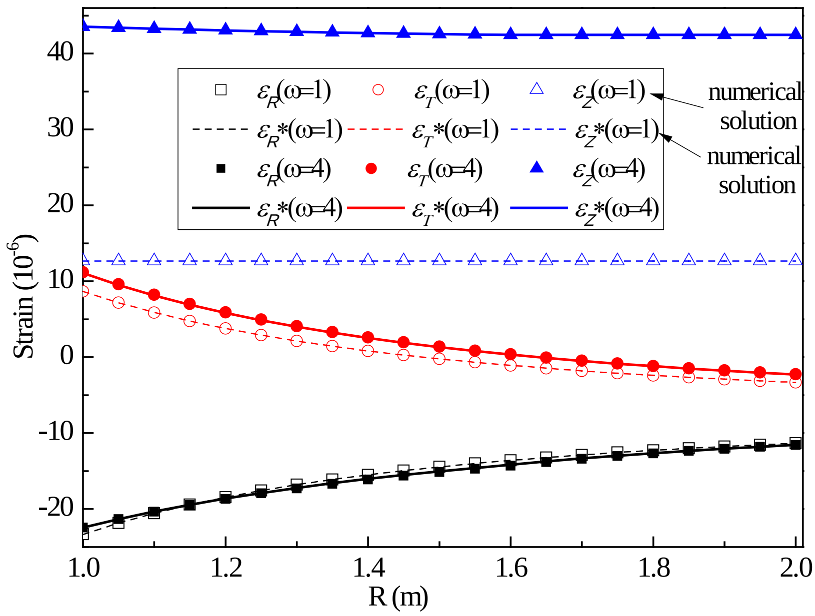


Fig.9. Changing laws of strain of AB (ω=1 & ω=4)

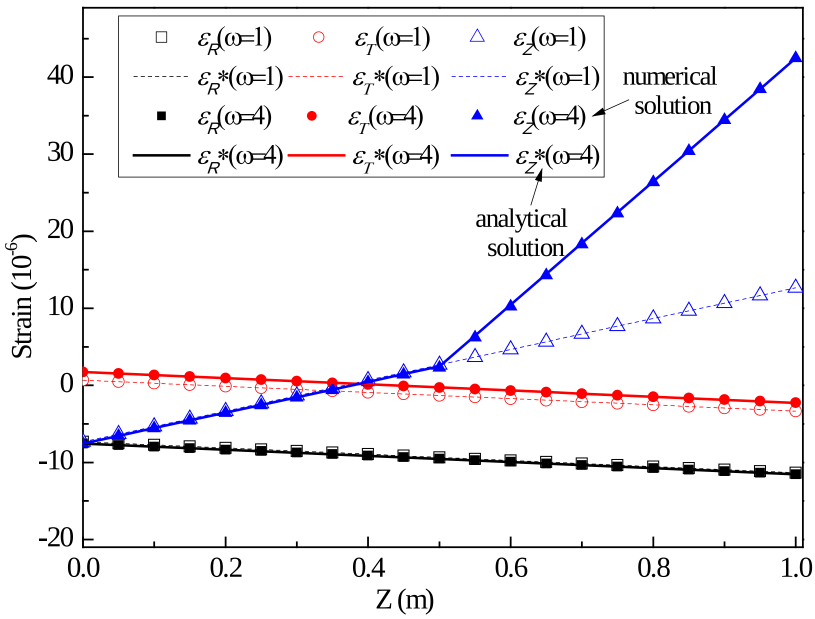


Fig.10. Changing laws of strain of CB (ω=1 & ω=4)

It can be seen from Table 6 that the numerical solution of hollow cylinder strain is consistent with the analytical solution with very high degree of accuracy. Seeing from Fig.8 ~ Fig.10, there are huge differences between these calculation results using the compression modulus (ω=1) and the bi-modulus (ω=4), i.e. the tangential strain ε_T(ω=4) along CB is 4 times of the tangential strain ε_T (ω =1), the nonlinear law of the radial stress σ_T on AB and the axial strain ε_z on CB. Therefore, the bi-modulus theory should be used to mechanical calculate of the bi-modulus structure in engineering to avoid large errors.

5. Conclusions

Based on the bi-modulus theory established by Ambartsumyan, The relationships between stress and strain in general rectangular coordinate system are studied, and a simple, efficient numerical algorithm applied to 3D bi-modulus structures is proposed in this paper. The main conclusions can be summarized as follows:

1) The constitutive equations of bi-modulus theory in general rectangular coordinate system, namely the generalized elastic law, are derived. Through the analysis on generalized elastic law, the anisotropy and nonlinear characteristics of structures composed with bi-modulus materials are observed.

2) The general 3D shear modulus formula in principal stress directions is deduced and a theoretical self-consistent complemented algorithm is proposed.

3) The 3D complemented algorithm is implemented in ABAQUS with the subroutine UMAT. The calculation results show that the algorithm is simple, good stability and convergence efficiency.

Numerical results show that the different ratios of the properties in tension and compression of bi-modulus materials have significantly influence on the structural mechanical responses. The dynamic analysis with bi-modular materials will be investigated in the future work.

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Appendix A. The deducing for constitutive equation Eq. (3)

There are relationships with direction cosines as follows.

$$\begin{cases} l_1^2+l_2^2+l_3^2=1 & l_1m_1+l_2m_2+l_3m_3=0 \\ m_1^2+m_2^2+m_3^2=1 & l_1n_1+l_2n_2+l_3n_3=0 \\ n_1^2+n_2^2+n_3^2=1 & m_1n_1+m_2n_2+m_3n_3=0 \\ l_1^2+m_1^2+n_1^2=1 & l_1l_2+m_1m_2+n_1n_2=0 \\ l_2^2+m_2^2+n_2^2=1 & l_1l_3+m_1m_3+n_1n_3=0 \\ l_3^2+m_3^2+n_3^2=1 & l_2l_3+m_2m_3+n_2n_3=0 \end{cases} . \tag{A. 1}$$

Strain components in coordinate system give:

$$\begin{cases} \varepsilon_x = l_1^2\varepsilon_\alpha + m_1^2\varepsilon_\beta + n_1^2\varepsilon_\gamma \\ \varepsilon_y = l_2^2\varepsilon_\alpha + m_2^2\varepsilon_\beta + n_2^2\varepsilon_\gamma \\ \varepsilon_z = l_3^2\varepsilon_\alpha + m_3^2\varepsilon_\beta + n_3^2\varepsilon_\gamma \\ \gamma_{xy} = 2(l_1l_2\varepsilon_\alpha + m_1m_2\varepsilon_\beta + n_1n_2\varepsilon_\gamma) \\ \gamma_{yz} = 2(l_2l_3\varepsilon_\alpha + m_2m_3\varepsilon_\beta + n_2n_3\varepsilon_\gamma) \\ \gamma_{xz} = 2(l_1l_3\varepsilon_\alpha + m_1m_3\varepsilon_\beta + n_1n_3\varepsilon_\gamma) \end{cases} . \tag{A. 2}$$

Substituting Eq. (1) into the constitutive Eq. (A.2), results:

$$
\begin{aligned}
\varepsilon_x &= l_1^2\varepsilon_\alpha + m_1^2\varepsilon_\beta + n_1^2\varepsilon_\gamma \\
&= l_1^2(a_{11}\sigma_\alpha + a_{12}\sigma_\beta + a_{13}\sigma_\gamma) + m_1^2(a_{21}\sigma_\alpha + a_{22}\sigma_\beta + a_{23}\sigma_\gamma) + n_1^2(a_{31}\sigma_\alpha + a_{32}\sigma_\beta + a_{33}\sigma_\gamma) \\
&= (a_{11}l_1^2 + a_{21}m_1^2 + a_{31}n_1^2)\sigma_\alpha + (a_{12}l_1^2 + a_{22}m_1^2 + a_{32}n_1^2)\sigma_\beta + (a_{13}l_1^2 + a_{23}m_1^2 + a_{33}n_1^2)\sigma_\gamma \\
&= a_{11}(l_1^2\sigma_\alpha + m_1^2\sigma_\beta + n_1^2\sigma_\gamma) + a_{22}(l_1^2\sigma_\alpha + m_1^2\sigma_\beta + n_1^2\sigma_\gamma) + a_{33}(l_1^2\sigma_\alpha + m_1^2\sigma_\beta + n_1^2\sigma_\gamma) \\
&\quad -[l_1^2(a_{22}+a_{33})\sigma_\alpha + m_1^2(a_{11}+a_{33})\sigma_\beta + n_1^2(a_{11}+a_{22})\sigma_\gamma] \\
&\quad +[(a_{21}m_1^2 + a_{31}n_1^2)\sigma_\alpha + (a_{12}l_1^2 + a_{32}n_1^2)\sigma_\beta + (a_{13}l_1^2 + a_{23}m_1^2)\sigma_\gamma] \\
&= (a_{11}+a_{22}+a_{33})\sigma_x - (a_{11}+a_{22}+a_{33})(l_1^2\sigma_\alpha + m_1^2\sigma_\beta + n_1^2\sigma_\gamma) + a_{11}l_1^2\sigma_\alpha + a_{22}m_1^2\sigma_\beta + a_{33}n_1^2\sigma_\gamma \\
&\quad +[(m_1^2+n_1^2)a_{21}\sigma_\alpha + (l_1^2+n_1^2)a_{12}\sigma_\beta + (l_1^2+m_1^2)a_{13}\sigma_\gamma] \\
&= a_{11}l_1^2\sigma_\alpha + a_{22}m_1^2\sigma_\beta + a_{33}n_1^2\sigma_\gamma - a_{21}l_1^2\sigma_\alpha - a_{12}m_1^2\sigma_\beta - a_{13}n_1^2\sigma_\gamma + a_{21}\sigma_\alpha + a_{12}\sigma_\beta + a_{13}\sigma_\gamma \\
&= (a_{11}-a_{21})l_1^2\sigma_\alpha + (a_{22}-a_{12})m_1^2\sigma_\beta + (a_{33}-a_{13})n_1^2\sigma_\gamma + a_{21}\sigma_\alpha + a_{12}\sigma_\beta + a_{13}\sigma_\gamma \\
&= \frac{l_1^2\sigma_\alpha}{2G_\alpha} + \frac{m_1^2\sigma_\beta}{2G_\beta} + \frac{n_1^2\sigma_\gamma}{2G_\gamma} + a_{21}\sigma_\alpha + a_{12}\sigma_\beta + a_{13}\sigma_\gamma
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{xy} &= 2(l_1l_2\varepsilon_\alpha + m_1m_2\varepsilon_\beta + n_1n_2\varepsilon_\gamma) \\
&= 2\left\{l_1l_2(a_{11}\sigma_\alpha + a_{12}\sigma_\beta + a_{13}\sigma_\gamma) + m_1m_2(a_{21}\sigma_\alpha + a_{22}\sigma_\beta + a_{23}\sigma_\gamma) + n_1n_2(a_{31}\sigma_\alpha + a_{32}\sigma_\beta + a_{33}\sigma_\gamma)\right\} \\
&= 2\left\{(a_{11}l_1l_2 + a_{21}m_1m_2 + a_{31}n_1n_2)\sigma_\alpha + (a_{12}l_1l_2 + a_{22}m_1m_2 + a_{32}n_1n_2)\sigma_\beta + (a_{13}l_1l_2 + a_{23}m_1m_2 + a_{33}n_1n_2)\sigma_\gamma\right\} \\
&= 2\left\{\begin{array}{l}
a_{11}(l_1l_2\sigma_\alpha + m_1m_2\sigma_\beta + n_1n_2\sigma_\gamma) + a_{22}(l_1l_2\sigma_\alpha + m_1m_2\sigma_\beta + n_1n_2\sigma_\gamma) \\
+a_{33}(l_1l_2\sigma_\alpha + m_1m_2\sigma_\beta + n_1n_2\sigma_\gamma) \\
-[l_1l_2(a_{22}+a_{33})\sigma_\alpha + m_1m_2(a_{11}+a_{33})\sigma_\beta + n_1n_2(a_{11}+a_{22})\sigma_\gamma] \\
+[(a_{21}m_1m_2 + a_{31}n_1n_2)\sigma_\alpha + (a_{12}l_1l_2 + a_{32}n_1n_2)\sigma_\beta + (a_{13}l_1l_2 + a_{23}m_1m_2)\sigma_\gamma]
\end{array}\right\} \\
&= 2\left\{\begin{array}{l}
(a_{11}+a_{22}+a_{33})\tau_{xy} - (a_{11}+a_{22}+a_{33})(l_1l_2\sigma_\alpha + m_1m_2\sigma_\beta + n_1n_2\sigma_\gamma) \\
+a_{11}l_1l_2\sigma_\alpha + a_{22}m_1m_2\sigma_\beta + a_{33}n_1n_2\sigma_\gamma \\
+[(m_1m_2 + n_1n_2)a_{21}\sigma_\alpha + (l_1l_2 + n_1n_2)a_{12}\sigma_\beta + (l_1l_2 + m_1m_2)a_{13}\sigma_\gamma]
\end{array}\right\} \\
&= 2\left\{a_{11}l_1l_2\sigma_\alpha + a_{22}m_1m_2\sigma_\beta + a_{33}n_1n_2\sigma_\gamma - a_{21}l_1l_2\sigma_\alpha - a_{12}m_1m_2\sigma_\beta - a_{13}n_1n_2\sigma_\gamma\right\} \\
&= 2(a_{11}-a_{21})l_1l_2\sigma_\alpha + 2(a_{22}-a_{12})m_1m_2\sigma_\beta + 2(a_{33}-a_{13})n_1n_2\sigma_\gamma \\
&= \frac{l_1l_2\sigma_\alpha}{G_\alpha} + \frac{m_1m_2\sigma_\beta}{G_\beta} + \frac{n_1n_2\sigma_\gamma}{G_\gamma}
\end{aligned}
$$

Similarly, the rest equations can be obtained in the same way.

Appendix B. The ratios of l_2/m_1, m_3/n_2 and n_1/l_3

It can be noticed that the direction cosines satisfy following equations:

$$
\begin{cases}
l_1m_1 + l_2m_2 + l_3m_3 = 0 & l_1n_1 + l_2n_2 + l_3n_3 = 0 \qquad m_1n_1 + m_2n_2 + m_3n_3 = 0 \\
l_1l_2 + m_1m_2 + n_1n_2 = 0 & l_1l_3 + m_1m_3 + n_1n_3 = 0 \qquad l_2l_3 + m_2m_3 + n_2n_3 = 0
\end{cases}. \tag{B. 1}
$$

When l_1, m_2, n_3→1; l_2, l_3, m_1, m_3, n_1, n_2→0, we have:

$$
\begin{cases}
m_1 + l_2 + l_3m_3 = 0 \ (a) & n_1 + l_2n_2 + l_3 = 0 \ \ (b) \quad m_1n_1 + n_2 + m_3 = 0 \ \ (c) \\
l_2 + m_1 + n_1n_2 = 0 \ (d) & l_3 + m_1m_3 + n_1 = 0 \ \ (e) \quad l_2l_3 + m_3 + n_2 = 0 \ \ (f)
\end{cases}. \tag{B. 2}
$$

From Eqs (a) (d) in Eq. (B.2), we have:

$$
l_3m_3 = n_1n_2 \quad \Rightarrow \quad m_3/n_2 = n_1/l_3 \tag{B. 3}
$$

and from Eqs (b) (c) in Eq. (B.2), it is obvious that:

$$l_2 n_2 = m_1 m_3 \quad \Rightarrow \quad l_2 / m_1 = m_3 / n_2 \, . \tag{B. 4}$$

Same to Eqs (*e*) (*f*) in Eq. (B.2), one has:

$$l_2 l_3 = m_1 n_1 \quad \Rightarrow l_2 / m_1 = n_1 / l_3 \, . \tag{B. 5}$$

Combined Eqs. (B.3) ~ (B.5), and let $\lim\limits_{\substack{l_1, m_2, n_3 \to 1 \\ l_2, l_3, m_1, m_3, n_1, n_2 \to 0}} l_2 / m_1 = k$, therefore:

$$\lim_{\substack{l_1, m_2, n_3 \to 1 \\ l_2, l_3, m_1, m_3, n_1, n_2 \to 0}} l_2 / m_1 = m_3 / n_2 = n_1 / l_3 = k \, . \tag{B. 6}$$

From Eq. (B.3) to Eq. (B.6), it can be seen that the three shear modulus have similar form with *k*. Consider,

$$\begin{cases} l_1^2 + l_2^2 + l_3^2 = 1 \quad (a') \quad m_1^2 + m_2^2 + m_3^2 = 1 \quad (b') \quad n_1^2 + n_2^2 + n_3^2 = 1 \quad (c') \\ l_1^2 + m_1^2 + n_1^2 = 1 \quad (d') \quad l_2^2 + m_2^2 + n_2^2 = 1 \quad (e') \quad l_3^2 + m_3^2 + n_3^2 = 1 \quad (f') \end{cases} \tag{B. 7}$$

Eqs (*a* ') and (*d* ') give:

$$l_2^2 + l_3^2 = m_1^2 + n_1^2 \, .$$

Substitute Eq. (B.6) into the above equation yields,

$$(k^2 - 1) m_1^2 = (k^2 - 1) l_3^2 \, .$$

Thus

$$k^2 = 1 \text{ or } m_1^2 = l_3^2 \, . \tag{B. 8}$$

Similarly, from Eqs (*b* ') (*e* '), we have:

$$k^2 = 1 \text{ or } m_1^2 = n_2^2 \tag{B. 9}$$

and from Eqs (*c* ') and (*f* ')

$$k^2 = 1 \text{ or } l_3^2 = n_2^2 \, . \tag{B. 10}$$

Combine Eqs (*a* ') and (*e* ') and take the limits, we can obtain:

$$l_3^2 = n_2^2 \, . \tag{B. 11}$$

Similarly, from Eqs (*a* ') and (*f* ')

$$l_2^2 = m_3^2 \tag{B. 12}$$

and from Eqs (*b* '), (*d* ') and from Eqs (*b* '), (*f* ')

$$n_1^2 = m_3^2 \, , \tag{B. 13}$$

$$m_1^2 = l_3^2 \, . \tag{B. 14}$$

Also from Eqs (*c* '), (*d* ') and from Eqs (*c* '), (*e* '), one has

$$m_1^2 = n_2^2 \, , \tag{B. 15}$$

$$n_1^2 = l_2^2 \, . \tag{B. 16}$$

Considering Eqs (B.6), (B.8) ~ (B.16) results

$$\begin{cases} m_1^2 = l_3^2 = n_2^2 \\ l_2^2 = n_1^2 = m_3^2 \end{cases} . \tag{B. 17}$$

It can be seen from the above formula that the direction cosines have the same signs (positive or negative) or different sign at the same time. l_3, m_1 and n_2 are infinitesimally small quantities of the same order, as well as are l_2, n_1 and m_3. In addition, the limit can be obtained:

$$\begin{aligned}\lim_{\substack{l_1,m_2,n_3\to 1\\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} \tau_{xy} &= \lim_{\substack{l_1,m_2,n_3\to 1\\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} l_1 l_2\sigma_\alpha + m_1 m_2\sigma_\beta + n_1 n_2\sigma_\gamma \\ &= \lim_{\substack{l_1,m_2,n_3\to 1\\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} l_2\sigma_\alpha + m_1\sigma_\beta + n_1 n_2\sigma_\gamma \quad . \\ &= \lim_{\substack{l_1,m_2,n_3\to 1\\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} l_2\sigma_\alpha + m_1\sigma_\beta\end{aligned} \tag{B. 18}$$

As the shear stress is zero on the principal stress surface, i.e.

$$\tau_{\alpha\beta} = l_1 m_1\sigma_x + l_2 m_2\sigma_y + l_3 m_3\sigma_z + (l_1 m_2 + l_2 m_1)\tau_{xy} + (l_2 m_3 + l_3 m_2)\tau_{yz} + (l_1 m_3 + l_3 m_3)\tau_{zx} = 0 \tag{B. 19}$$

and

$$\begin{aligned}&\lim_{\substack{l_1,m_2,n_3\to 1\\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} l_1 m_1\sigma_x + l_2 m_2\sigma_y + l_3 m_3\sigma_z + (l_1 m_2 + l_2 m_1)\tau_{xy} + (l_2 m_3 + l_3 m_2)\tau_{yz} + (l_1 m_3 + l_3 m_3)\tau_{zx} \\ &= \lim_{\substack{l_1,m_2,n_3\to 1\\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} m_1\sigma_x + l_2\sigma_y + l_3 m_3\sigma_z + (1 + l_2 m_1)\tau_{xy} + (l_2 m_3 + l_3)\tau_{yz} + (m_3 + l_3 m_3)\tau_{zx} \\ &= \lim_{\substack{l_1,m_2,n_3\to 1\\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} m_1\sigma_\alpha + l_2\sigma_\beta + l_3 m_3\sigma_\gamma + \tau_{xy} + l_3\tau_{yz} + m_3\tau_{xz} \\ &= \lim_{\substack{l_1,m_2,n_3\to 1\\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} m_1\sigma_\alpha + l_2\sigma_\beta + l_3 m_3\sigma_\gamma + \tau_{xy} \\ &= \lim_{\substack{l_1,m_2,n_3\to 1\\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} m_1\sigma_\alpha + l_2\sigma_\beta + \tau_{xy} = 0\end{aligned} \tag{B. 20}$$

Then, it can be obtained from Eq. (B.19) and Eq. (B.20):

$$\lim_{\substack{l_1,m_2,n_3\to 1\\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} m_1\sigma_\alpha + l_2\sigma_\beta = -\tau_{xy} . \tag{B. 21}$$

Combining Eq. (B.18) and Eq. (B.21) yields:

$$l_2\sigma_\alpha + m_1\sigma_\beta = -m_1\sigma_\alpha - l_2\sigma_\beta .$$

Therefore, we have:

$$(l_2 + m_1)(\sigma_\alpha + \sigma_\beta) = 0 . \tag{B. 22}$$

Since the above equation is true for any magnitude principal stress σ_α and σ_β, it means that

$$\lim_{\substack{l_1,m_2,n_3\to 1\\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} l_2 + m_1 = 0$$

which leads

$$\lim_{\substack{l_1,m_2,n_3\to 1\\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} l_2 / m_1 = -1 . \tag{B. 23}$$

In the same way, we have

$$\lim_{\substack{l_1,m_2,n_3\to 1\\ l_2,l_3,m_1,m_3,n_1,n_2\to 0}} m_3 / n_2 = l_3 / n_1 = -1 . \tag{B. 24}$$

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