# FRACTIONAL LOGARITHMIC INEQUALITIES AND BLOW-UP RESULTS WITH LOGARITHMIC NONLINEARITY ON HOMOGENEOUS GROUPS

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Abstract. In this paper, we prove the fractional logarithmic Sobolev inequality on homogeneous groups. Also, we establish fractional logarithmic Gagliardo-Nirenberg and Caffarelli-Kohn-Nirenberg inequalities on homogeneous groups. In addition, we show blow-up results for the fractional heat equation with logarithmic nonlinearity on homogeneous groups.

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## 1. Introduction

First of all, let us briefly give an introduction to the considered results. The obtained inequalities seem new even in the Euclidean setting.

### 1.1. Fractional Sobolev inequality.

Let $\Omega \subset \mathbb{R}^N$ be a measurable set and $1 < p < N$, then the (classical) Sobolev inequality is formulated as

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \ \ u \in C_0^\infty(\Omega), \tag{1.1}$$

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where $C = C(N,p) > 0$ is a positive constant, $p^* = \frac{Np}{N-p}$ and $\nabla$ is the standard gradient in $\mathbb{R}^N$. The Sobolev inequality is one of the most important tools in PDE and variational problems.

In [6] the authors obtained the fractional Sobolev inequality in the case $N > sp$, $1 < p < \infty$, and $s \in (0,1)$:

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C[u]_{s,p}, \tag{1.2}$$

for any measurable and compactly supported function $u$ where $C = C(N,p,s) > 0$ is a suitable constant,

$$[u]^p_{s,p} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy$$

is the Gagliardo seminorm and $p^* = \frac{Np}{N-sp}$. There is a number of generalisations and extensions of the above Sobolev's inequality. For example, in [1] the authors proved the following weighted fractional Sobolev inequality: Let $1 < p < \frac{N}{s}$ and $0 < \beta < \frac{N-ps}{2}$, then for all $u \in C_0^\infty(\mathbb{R}^N)$ one has

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{\frac{2\beta p^*}{p}}}dx\right)^{\frac{p}{p^*}} \leq C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}|x|^\beta|y|^\beta}dxdy, \tag{1.3}$$

where $C = C(N,p,s) > 0$ and $p^* = \frac{Np}{N-sp}$.

In the homogeneous group setting, the fractional Sobolev inequality was obtained in [12]. In [15], the author proved the logarithmic Sobolev inequality in the following form:

$$\int_{\mathbb{R}^N} \frac{|u|^p}{\|u\|^p_{L^p(\mathbb{R}^N)}} \log\left(\frac{|u|^p}{\|u\|^p_{L^p(\mathbb{R}^N)}}\right)dx \leq \frac{N}{p}\log\left(C\frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)}}{\|u\|^p_{L^p(\mathbb{R}^N)}}\right),\ 1 \leq p < \infty, \tag{1.4}$$

where $u, \nabla u \in L^p(\mathbb{R}^N)$.

Combining the above ideas in [9], the authors proved a version of the fractional logarithmic Sobolev inequality. In this paper one of our aims is to obtain an analogue of the fractional logarithmic Sobolev inequality on the homogeneous groups.

We refer to [7] for the original appearance of such groups, and to [24] for a recent comprehensive treatment. However, for convinience, we shortly review basic facts needed in this paper in Section 2.

1.2. **Fractional Gagliardo-Nirenberg inequality.** In the works of E. Gagliardo [8] and L. Nirenberg [16] (independently), they obtained the following (interpolation) inequality

$$\|u\|^p_{L^p(\mathbb{R}^N)} \leq C\|\nabla u\|^{N(p-2)/2}_{L^2(\mathbb{R}^N)}\|u\|^{(2p-N(p-2))/2}_{L^2(\mathbb{R}^N)},\ u \in H^1(\mathbb{R}^N), \tag{1.5}$$

where

$$\begin{cases} 2 \leq p \leq \infty \text{ for } N = 2, \\ 2 \leq p \leq \frac{2N}{N-2} \text{ for } N > 2. \end{cases}$$

The Gagliardo-Nirenberg inequality on the Heisenberg group $\mathbb{H}^n$ has the following form

$$\|u\|^p_{L^p(\mathbb{H}^n)} \leq C\|\nabla_{\mathbb{H}^n} u\|^{Q(p-2)/2}_{L^2(\mathbb{H}^n)}\|u\|^{(2p-Q(p-2))/2}_{L^2(\mathbb{H}^n)}, \tag{1.6}$$

where $\nabla_{\mathbb{H}}$ is a horizontal gradient and $Q = 2n + 2$ is a homogeneous dimension of $\mathbb{H}^n$. In [5], the authors determined the best constant for the sub-elliptic Gagliardo-Nirenberg inequality (1.6). Consequently, in [20] the best constants in Gagliardo-Nirenberg and Sobolev inequalities were also found for general hypoelliptic operators (Rockland operators) on graded groups.

In [17] the authors obtained a fractional version of the Gagliardo-Nirenberg inequality in the form:

$$\|u\|_{L^\tau(\mathbb{R}^N)} \leq C[u]^a_{s,p}\|u\|^{1-a}_{L^\alpha(\mathbb{R}^N)}, \ \forall u \in C^1_c(\mathbb{R}^N), \tag{1.7}$$

for $N \geq 1, \ s \in (0,1), \ p > 1, \ \alpha \geq 1, \ \tau > 0$, and $a \in (0,1]$ is such that

$$\frac{1}{\tau} = a\left(\frac{1}{p} - \frac{s}{N}\right) + \frac{1-a}{\alpha}.$$

Then this inequality was extended to homogeneous groups in [14]. Also, the logarithmic Gagliardo-Nirenberg inequality with $s = 1$ was proved in [15] and its fractional version was proved in [9]. In this paper we obtain the fractional logarithmic Gagliardo-Nirenberg inequality on the homogeneous groups.

Note that, to the best of our knowledge, in this direction systematic studies on the homogeneous groups started by the paper [18] in which homogeneous group versions of Hardy and Rellich inequalities were proved as consequences of universal identities resulting the recent book [24]. Also, in [13] Hardy-Littlewood-Sobolev and Stein-Weiss inequalities were obtained on homogeneous Lie groups.

1.3. **Fractional Caffarelli-Kohn-Nirenberg inequality.** In their pioneering work [2], L. Caffarelli, R. Kohn and L. Nirenberg established:

**Theorem 1.1.** *Let $N \geq 1$, and let $l_1$, $l_2$, $l_3$, $a$, $b$, $d$, $\delta \in \mathbb{R}$ be such that $l_1, l_2 \geq 1$, $l_3 > 0, \ 0 \leq \delta \leq 1$, and*

$$\frac{1}{l_1} + \frac{a}{N}, \ \frac{1}{l_2} + \frac{b}{N}, \ \frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} > 0. \tag{1.8}$$

*Then,*

$$\||x|^{\delta d + (1-\delta)b}u\|_{L^{l_3}(\mathbb{R}^N)} \leq C\||x|^a\nabla u\|^\delta_{L^{l_1}(\mathbb{R}^N)}\||x|^b u\|^{1-\delta}_{L^{l_2}(\mathbb{R}^N)}, \ \ u \in C^\infty_c(\mathbb{R}^N), \tag{1.9}$$

*if and only if*

$$\frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} = \delta\left(\frac{1}{l_1} + \frac{a-1}{N}\right) + (1-\delta)\left(\frac{1}{l_2} + \frac{b}{N}\right),$$
$$a - d \geq 0, \quad \text{if } \delta > 0,$$
$$a - d \leq 1, \quad \text{if } \delta > 0 \ \text{ and } \ \frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} = \frac{1}{l_1} + \frac{a-1}{N}, \tag{1.10}$$

*where $C$ is a positive constant independent of $u$.*

In [17], the authors proved the fractional analogues of the Caffarelli-Kohn-Nirenberg inequality in the weighted fractional Sobolev spaces. The fractional Caffarelli-Kohn-Nirenberg inequality for an admissible weight in $\mathbb{R}^N$ was obtained in [1]. The logarithmic analogue of the Caffarelli-Kohn-Nirenberg inequality was proved in [25].

Recently many different versions of Caffarelli-Kohn-Nirenberg inequalities have been obtained on different Lie groups, namely, in [26] on the Heisenberg groups, in [22] and [23] on stratified groups, in [19] and [21] on (general) homogeneous groups. On the homogeneous groups a fractional analogue of Caffarelli-Kohn-Nirenberg inequality was proved in [14]. One of the aims of this paper is to obtain the fractional logarithmic Caffarelli-Kohn-Nirenberg inequality on homogeneous groups.

1.4. **Blow-up result for the fractional heat equation with logarithmic nonlinearity.** In [3] the authors study the following initial boundary value problem:

$$\begin{cases} u_t(x,t) - \Delta_x u(x,t) = u\log|u|, \quad (x,t) \in \Omega\times(0,+\infty), \\ u(x,0) = u_0(x), \quad x\in\Omega, \\ u(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,+\infty). \end{cases} \tag{1.11}$$

So, they prove the following blow-up theorem (in the Euclidean setting):

**Theorem 1.2.** [3] *Assume that* $u_0 \in H_0^1(\Omega)$ *and*

$$J(u_0) = \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \frac{1}{2}\int_\Omega |u_0|^2 \log|u_0| dx + \frac{1}{4}\int_\Omega |u_0|^2 dx \le M, \tag{1.12}$$

*and*

$$I(u_0) = \int_\Omega |\nabla u_0|^2 dx - \int_\Omega |u_0|^2 \log|u_0| dx < 0. \tag{1.13}$$

*Then the weak solution of the problem (1.11) blows up at* $+\infty$.

Moreover, in [10] it is showed the condition $J(u_0) \le M$ is unnecessary to blow-up at infinity to a solution of the problem (1.11). We also refer to [4] and [11] to blow-up results for the pseudo-parabolic equations with logarithmic nonlinearity. In this paper, we considered the heat equation with the fractional sub-Laplacian with logarithmic nonlinearity and we obtain the blow-up result. That is, we extend the blow-up theorem from [10] to general homogeneous groups.

Summarising our main results of the present paper, we prove the following facts:

- An analogue of the fractional logarithmic Sobolev inequality on the homogeneous group $\mathbb{G}$;
- An analogue of the fractional logarithmic Gagliardo-Nirenberg inequality on $\mathbb{G}$;
- An analogue of the fractional logarithmic Caffarelli-Kohn-Nirenberg inequality on $\mathbb{G}$;
- Blow-up theorem for the fractional heat equation with the logarithmic nonlinearity on $\mathbb{G}$.

The paper is organised as follows. First we give some basic discussions on fractional Sobolev spaces and related facts on homogeneous groups, then in Section 3 we present the fractional logarithmic Sobolev, Gagliardo-Nirenberg, and Caffarelli-Kohn-Nirenberg inequalities on $\mathbb{G}$. In Section 4 we prove the blow-up theorem to the fractional heat equation with logarithmic nonlinearity on the homogeneous group $\mathbb{G}$.

## 2. Preliminaries

We recall that a Lie group (on $\mathbb{R}^n$) $\mathbb{G}$ with the dilation

$$D_\lambda(x) := (\lambda^{\nu_1}x_1, \ldots, \lambda^{\nu_n}x_n),\ \nu_1, \ldots, \nu_n > 0,\ D_\lambda : \mathbb{R}^n \to \mathbb{R}^n,$$

which is an automorphism of the group $\mathbb{G}$ for each $\lambda > 0$, is called a *homogeneous (Lie) group*. In this paper, for simplicity, we use the notation $\lambda x$ instead of the dilation $D_\lambda(x)$. The homogeneous dimension of the homogeneous group $\mathbb{G}$ is denoted by

$$Q := \nu_1 + \ldots + \nu_n.$$

A homogeneous quasi-norm on $\mathbb{G}$ is a continuous non-negative function

$$\mathbb{G} \ni x \mapsto |x| \in [0, \infty), \tag{2.1}$$

with the properties

i) $|x| = |x^{-1}|$ for all $x \in \mathbb{G}$,
ii) $|\lambda x| = \lambda |x|$ for all $x \in \mathbb{G}$ and $\lambda > 0$,
iii) $|x| = 0$ iff $x = 0$.

Moreover, the following polarisation formula on homogeneous groups will be used in our proofs: there is a (unique) positive Borel measure $\sigma$ on the unit quasi-sphere $\mathfrak{S} := \{x \in \mathbb{G} : |x| = 1\}$, so that for every $f \in L^1(\mathbb{G})$ we have

$$\int_{\mathbb{G}} f(x)dx = \int_0^\infty \int_{\mathfrak{S}} f(ry)r^{Q-1}d\sigma(y)dr. \tag{2.2}$$

We refer to [7] and a recent open access book [24] for more details.

Let us define quasi-ball centered at $x$ with radius $r$ in the following form:

$$B(x, r) := \{x \in \mathbb{G} : |x^{-1}y| < r\}. \tag{2.3}$$

Let $s \in (0, 1)$ and let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q$. For a (Haar) measurable and compactly supported function $u$ the fractional sub-Laplacian $(-\Delta_s)$ on $\mathbb{G}$ can be defined as

$$(-\Delta_s)u(x) := 2 \lim_{r \searrow 0} \int_{\mathbb{G}\setminus B(x,r)} \frac{u(x) - u(y)}{|y^{-1}x|^{Q+2s}} dy,\ \ x \in \mathbb{G}, \tag{2.4}$$

where $|\cdot|$ is a quasi-norm on $\mathbb{G}$ and $B(x, r)$ is a quasi-ball with respect to $|\cdot|$, with radius $r$ centered at $x \in \mathbb{G}$ .

Let $p > 1$. For a measurable function $u : \mathbb{G} \to \mathbb{R}$ we define the Gagliardo quasi-seminorm by

$$[u]_{s,p} := \left( \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|u(x) - u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \right)^{1/p}. \tag{2.5}$$

Now we recall the definition of the fractional Sobolev spaces on homogeneous groups denoted by $W^{s,p}(\mathbb{G})$. For $p \geq 1$ and $s \in (0, 1)$, the functional space

$$W^{s,p}(\mathbb{G}) = \{u \in L^p(\mathbb{G}) : u \text{ is measurable}, [u]_{s,p} < +\infty\}, \tag{2.6}$$

is called the fractional Sobolev space on $\mathbb{G}$.

Similarly, if $\Omega \subset \mathbb{G}$ is a Haar measurable set, we define the Sobolev space

$$W^{s,p}(\Omega) = \{u \in L^p(\Omega) : u \text{ is measurable},$$
$$[u]_{s,p,\Omega} = \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy\right)^{\frac{1}{p}} < +\infty\}. \quad (2.7)$$

Let us define $W_0^{s,p}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{s,p}(\Omega)} = [u]_{s,p,\Omega}. \quad (2.8)$$

In the case $p = 2$, for all $\varphi \in W_0^{s,2}(\Omega)$ we have

$$\langle(-\Delta_s)u, \varphi\rangle := \int_\Omega \int_\Omega \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|y^{-1}x|^{Q+2s}} dxdy. \quad (2.9)$$

Now we recall the definition of the weighted fractional Sobolev space on the homogeneous groups denoted by

$$W^{s,p,\beta}(\mathbb{G}) = \{u \in L^p(\mathbb{G}) : u \text{ is measurable},$$
$$[u]_{s,p,\beta} = \left(\int_\mathbb{G} \int_\mathbb{G} \frac{|x|^{\beta_1 p}|y|^{\beta_2 p}|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy\right)^{\frac{1}{p}} < +\infty\}, \quad (2.10)$$

where $\beta_1,\ \beta_2 \in \mathbb{R}$ with $\beta = \beta_1 + \beta_2$, that is, it depends on $\beta_1$ and $\beta_2$.

As above, for a Haar measurable set $\Omega \subset \mathbb{G}$ , $p \geq 1$, $s \in (0,1)$ and $\beta_1,\ \beta_2 \in \mathbb{R}$ with $\beta = \beta_1 + \beta_2$, we define the weighted fractional Sobolev space

$$W^{s,p,\beta}(\Omega) = \{u \in L^p(\Omega) : u \text{ is measurable},$$
$$[u]_{s,p,\beta,\Omega} = \left(\int_\Omega \int_\Omega \frac{|x|^{\beta_1 p}|y|^{\beta_2 p}|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy\right)^{\frac{1}{p}} < +\infty\}. \quad (2.11)$$

Obviously, taking $\beta = \beta_1 = \beta_2 = 0$ in (2.11), we recover (2.7).

## 3. Fractional logarithmic inequalities on $\mathbb{G}$

In this section we prove analogues of fractional logarithmic inequalities on the homogeneous groups. From now on, unless specified otherwise, $\mathbb{G}$ will be a homogeneous group of homogeneous dimension $Q$. Firstly, we show weighted Hölder's inequality on $\mathbb{G}$.

**Lemma 3.1.** *Suppose that* $1 < p \leq r \leq q \leq \infty$, $a \in [0,1]$, $\alpha \in \mathbb{R}$, $|x|^\alpha u \in L^p(\mathbb{G}) \cap L^q(\mathbb{G})$ *with*

$$\frac{1}{r} = \frac{a}{p} + \frac{1-a}{q}, \quad (3.1)$$

*then we have*

$$\||x|^\alpha u\|_{L^r(\mathbb{G})} \leq \||x|^\alpha u\|^a_{L^p(\mathbb{G})} \||x|^\alpha u\|^{1-a}_{L^q(\mathbb{G})}. \quad (3.2)$$

*Proof.* By using Hölder's inequality we obtain

$$\begin{aligned}\||x|^{\alpha}u\|^{r}_{L^{r}(\mathbb{G})} &= \int_{\mathbb{G}}|x|^{\alpha r}|u(x)|^{r}dx = \int_{\mathbb{G}}(|x|^{\alpha}|u(x)|)^{ar}(|x|^{\alpha}|u(x)|)^{(1-a)r}dx\\ &\leq \left(\int_{\mathbb{G}}|x|^{\alpha p}|u(x)|^{p}dx\right)^{\frac{ar}{p}}\left(\int_{\mathbb{G}}|x|^{\alpha q}|u(x)|^{q}dx\right)^{\frac{(1-a)r}{q}} \\ &= \||x|^{\alpha}u\|^{ar}_{L^{p}(\mathbb{G})}\||x|^{\alpha}u\|^{(1-a)r}_{L^{q}(\mathbb{G})},\end{aligned} \tag{3.3}$$

with

$$\frac{ar}{p}+\frac{(1-a)r}{q}=1. \tag{3.4}$$

□

Now let us present logarithmic Hölder's inequality.

**Lemma 3.2.** *Let $|x|^{\alpha}u \in L^{p}(\mathbb{G})\cap L^{q}(\mathbb{G})$ with some $\alpha\in\mathbb{R}$, $1<p<q\leq\infty$. Then we have*

$$\int_{\mathbb{G}}\frac{(|x|^{\alpha p}|u|^{p})}{\||x|^{\alpha}u\|^{p}_{L^{p}(\mathbb{G})}}\log\left(\frac{|x|^{\alpha p}|u|^{p}}{\||x|^{\alpha}u\|^{p}_{L^{p}(\mathbb{G})}}\right)dx \leq \frac{q}{q-p}\log\left(\frac{\||x|^{\alpha}u\|^{p}_{L^{q}(\mathbb{G})}}{\||x|^{\alpha}u\|^{p}_{L^{p}(\mathbb{G})}}\right). \tag{3.5}$$

*Proof.* Consider the function

$$F\left(\frac{1}{r}\right)=\log\left(\||x|^{\alpha}u\|_{L^{r}(\mathbb{G})}\right). \tag{3.6}$$

Now let us show that the function (3.6) is convex. To prove it we use Lemma 3.1, so we get

$$\begin{aligned}F\left(\frac{1}{r}\right) &= \log\left(\||x|^{\alpha}u\|_{L^{r}(\mathbb{G})}\right)\leq\log\left(\||x|^{\alpha}u\|^{a}_{L^{p}(\mathbb{G})}\||x|^{\alpha}u\|^{1-a}_{L^{q}(\mathbb{G})}\right)\\ &= \log\left(\||x|^{\alpha}u\|^{a}_{L^{p}(\mathbb{G})}\right)+\log\left(\||x|^{\alpha}u\|^{1-a}_{L^{q}(\mathbb{G})}\right)=aF\left(\frac{1}{p}\right)+(1-a)F\left(\frac{1}{q}\right),\end{aligned} \tag{3.7}$$

with $a\in[0,1]$ and $\frac{1}{r}=\frac{a}{p}+\frac{1-a}{q}$.

Since we have

$$F(r)=r\log\int_{\mathbb{G}}|x|^{\frac{\alpha}{r}}|u(x)|^{\frac{1}{r}}dx, \tag{3.8}$$

the derivative of (3.8) is

$$\begin{aligned}F'(r) &= \log\int_{\mathbb{G}}|x|^{\frac{\alpha}{r}}|u(x)|^{\frac{1}{r}}dx + r\left(\log\int_{\mathbb{G}}|x|^{\frac{\alpha}{r}}|u(x)|^{\frac{1}{r}}dx\right)'_{r}\\ &= \log\int_{\mathbb{G}}|x|^{\frac{\alpha}{r}}|u(x)|^{\frac{1}{r}}dx + r\frac{\left(\int_{\mathbb{G}}(|x|^{\alpha}u(x))^{\frac{1}{r}}dx\right)'_{r}}{\int_{\mathbb{G}}|x|^{\frac{\alpha}{r}}|u(x)|^{\frac{1}{r}}dx}\\ &= \log\int_{\mathbb{G}}|x|^{\frac{\alpha}{r}}|u(x)|^{\frac{1}{r}}dx - \frac{1}{r}\frac{\int_{\mathbb{G}}(|x|^{\alpha}u(x))^{\frac{1}{r}}\log(|x|^{\alpha}|u(x)|)dx}{\int_{\mathbb{G}}|x|^{\frac{\alpha}{r}}|u(x)|^{\frac{1}{r}}dx}.\end{aligned} \tag{3.9}$$

On the other hand, by (3.7) $F(r)$ is convex, therefore, we have

$$F'(r) \geq \frac{F(r') - F(r)}{r' - r}, \quad r' > r > 0. \tag{3.10}$$

With $r = \frac{1}{p}$ and $r' = \frac{1}{q}$ it yields

$$p\frac{\int_{\mathbb{G}} ||x|^{\alpha} u|^p \log |x|^{\alpha}|u| dx}{\int_{\mathbb{G}} |x|^{\alpha p}|u(x)|^p dx} - \log \int_{\mathbb{G}} |x|^{\alpha p}|u(x)|^p dx \leq \frac{qp}{q-p} \log \left( \int_{\mathbb{G}} \frac{\||x|^{\alpha} u\|_{L^q(\mathbb{G})}}{\||x|^{\alpha} u\|_{L^p(\mathbb{G})}} \right). \tag{3.11}$$

We have

$$\log \int_{\mathbb{G}} |x|^{\alpha p}|u(x)|^p dx = \frac{\log \int_{\mathbb{G}} |x|^{\alpha p}|u(x)|^p dx \int_{\mathbb{G}} |x|^{\alpha p}|u(x)|^p dx}{\int_{\mathbb{G}} |x|^{\alpha p}|u(x)|^p dx}$$
$$= \frac{\int_{\mathbb{G}} |x|^{\alpha p}|u(x)|^p \log \||x|^{\alpha} u\|^p_{L^p(\mathbb{G})} dx}{\int_{\mathbb{G}} |x|^{\alpha p}|u(x)|^p dx}. \tag{3.12}$$

Substituting it in (3.11) we obtain logarithmic Hölder's inequality

$$p\frac{\int_{\mathbb{G}} ||x|^{\alpha} u|^p \log |x|^{\alpha}|u| dx}{\int_{\mathbb{G}} |x|^{\alpha p}|u(x)|^p dx} - \log \int_{\mathbb{G}} |x|^{\alpha p}|u(x)|^p dx$$
$$= p\frac{\int_{\mathbb{G}} ||x|^{\alpha} u|^p \log |x|^{\alpha}|u| dx}{\int_{\mathbb{G}} |x|^{\alpha p}|u(x)|^p dx} - \frac{\int_{\mathbb{G}} |x|^{\alpha p}|u(x)|^p \log \||x|^{\alpha} u\|^p_{L^p(\mathbb{G})} dx}{\int_{\mathbb{G}} |x|^{\alpha p}|u(x)|^p dx}$$
$$= \frac{\int_{\mathbb{G}} ||x|^{\alpha} u|^p \log |x|^{\alpha p}|u|^p dx}{\int_{\mathbb{G}} |x|^{\alpha p}|u(x)|^p dx} - \frac{\int_{\mathbb{G}} |x|^{\alpha p}|u(x)|^p \log \||x|^{\alpha} u\|^p_{L^p(\mathbb{G})} dx}{\int_{\mathbb{G}} |x|^{\alpha p}|u(x)|^p dx}$$
$$= \int_{\mathbb{G}} \frac{(|x|^{\alpha p}|u|^p)}{\||x|^{\alpha}|u|\|^p_{L^p(\mathbb{G})}} \log \left( \frac{|x|^{\alpha p}|u|^p}{\||x|^{\alpha} u\|^p_{L^p(\mathbb{G})}} \right) \leq \frac{q}{q-p} \log \left( \frac{\||x|^{\alpha} u\|^p_{L^q(\mathbb{G})}}{\||x|^{\alpha} u\|^p_{L^p(\mathbb{G})}} \right). \tag{3.13}$$

□

To prove fractional logarithmic Sobolev's inequality we need the version of a fractional Sobolev inequality on the homogeneous groups.

**Theorem 3.3** ([12], Fractional Sobolev inequality)**.** *Let $\mathbb{G}$ be a homogeneous group with homogeneous dimension $Q$. Let $p > 1$, $s \in (0,1)$, $Q > sp$, and let $|\cdot|$ be a quasi-norm on $\mathbb{G}$. For any measurable and compactly supported function $u : \mathbb{G} \to \mathbb{R}$ there exists a positive constant $C = C(Q,p,s,q) > 0$ such that*

$$||u||_{L^{p^*}(\mathbb{G})} \leq C[u]_{s,p}, \tag{3.14}$$

*where $p^* = p^*(Q,s) = \frac{Qp}{Q-sp}$.*

Now we present the fractional logarithmic Sobolev inequality on $\mathbb{G}$.

**Theorem 3.4.** *Under the assumptions of Theorem 3.3 we have the fractional logarithmic Sobolev's inequality*

$$\int_{\mathbb{G}} \frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}} \log \left( \frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}} \right) dx \leq \frac{Q}{sp} \log \left( C \frac{[u]^p_{s,p}}{\|u\|^p_{L^p(\mathbb{G})}} \right), \tag{3.15}$$

*for any measurable and compactly supported function $u$. Here $C$ is a positive constant indefendent on $u$.*

*Proof.* By using weighted logarithmic Hölder's inequality (3.5) with $\alpha = 0$, we get

$$\int_{\mathbb{G}} \frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}} \log\left(\frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}}\right) dx \leq \frac{q}{q-p} \log\left(\frac{\|u\|^p_{L^q(\mathbb{G})}}{\|u\|^p_{L^p(\mathbb{G})}}\right). \tag{3.16}$$

By the assumption we have $1 \leq p < q = p^* = \frac{pQ}{Q-sp}$, that is,

$$\begin{aligned}\int_{\mathbb{G}} \frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}} \log\left(\frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}}\right) dx &\leq \frac{p^*}{p^*-p} \log\left(\frac{\|u\|^p_{L^{p^*}(\mathbb{G})}}{\|u\|^p_{L^p(\mathbb{G})}}\right) \\ &\leq \frac{p^*}{p^*-p} \log\left(C\frac{[u]^p_{s,p}}{\|u\|^p_{L^p(\mathbb{G})}}\right). \end{aligned} \tag{3.17}$$

Here we have

$$\frac{p^*}{p^*-p} = \frac{\frac{pQ}{Q-sp}}{\frac{pQ}{Q-sp} - p} = \frac{\frac{Q}{Q-sp}}{\frac{Q}{Q-sp} - 1} = \frac{Q}{sp}.$$

□

**Remark 3.5.** In the Abelian (Euclidean) case $\mathbb{G} = (\mathbb{R}^N, +)$, we have $Q = N$ and $|\cdot| = |\cdot|_E$ ($|\cdot|_E$ is the Euclidean distance), if $s \to 1^-$ and from (3.15) we get the logarithmic Sobolev inequality from [15].

Let us recall the fractional logarithmic Gagliardo-Nirenberg inequality

**Theorem 3.6** ([14], Fractional Gagliardo-Nirenberg inequality)**.** *Let $\mathbb{G}$ be a homogeneous group with homogeneous dimension $Q$. Assume that $Q \geq 2$, $s \in (0,1)$, $p > 1$, $q \geq 1$, $\tau > 0$, $a \in (0,1]$, $Q > sp$ and*

$$\frac{1}{\tau} = a\left(\frac{1}{p} - \frac{s}{Q}\right) + \frac{1-a}{q}.$$

*Then,*

$$\|u\|_{L^\tau(\mathbb{G})} \leq C[u]^a_{s,p}\|u\|^{1-a}_{L^q(\mathbb{G})}, \tag{3.18}$$

*for all measurable and compactly supported $u$ and $C = C(s,p,Q,a,\alpha) > 0$.*

We have the following logarithmic fractional Gagliardo-Nirenberg inequality.

**Theorem 3.7.** *Under the assumptions of Theorem 3.6 with the parameters $1 \leq p < \infty$, $1 < q < \infty$ with $q \leq p^*$, there exists $C = C(Q,p,s,q) > 0$ such that for all measurable and compactly supported $u$ we have*

$$\int_{\mathbb{G}} \frac{|u|^q}{\|u\|^q_{L^q(\mathbb{G})}} \log\left(\frac{|u|^q}{\|u\|^q_{L^q(\mathbb{G})}}\right) dx \leq \frac{1}{1-\frac{q}{\tau}} \log\left(C\frac{[u]^q_{s,p}}{\|u\|^q_{L^q(\mathbb{G})}}\right) dx. \tag{3.19}$$

*Proof.* Combining the fractional Gagliardo-Nirenberg inequality (3.18) and the logarithmic Hölder inequality (3.5), we obtain

$$\int_{\mathbb{G}} \frac{|u|^q}{\|u\|^q_{L^q(\mathbb{G})}} \log\left(\frac{|u|^q}{\|u\|^q_{L^q(\mathbb{G})}}\right) dx \leq \frac{1}{1-\frac{q}{\tau}} \log\left(\frac{\|u\|^q_{L^\tau(\mathbb{G})}}{\|u\|^q_q}\right)$$
$$\leq \frac{1}{1-\frac{q}{\tau}} \log\left(C\frac{[u]^{qa}_{s,p}\|u\|^{(1-a)q}_{L^q(\mathbb{G})}}{\|u\|^q_{L^q(\mathbb{G})}}\right) = \frac{a}{1-\frac{q}{\tau}} \log\left(C\frac{[u]^q_{s,p}}{\|u\|^q_{L^q(\mathbb{G})}}\right). \quad (3.20)$$

□

**Remark 3.8.** In the Abelian (Euclidean) case $\mathbb{G} = (\mathbb{R}^N, +)$, we have $Q = N$ and $|\cdot| = |\cdot|_E$ ($|\cdot|_E$ is the Euclidean distance), if $s \to 1^-$ and from (3.19) we get the logarithmic Sobolev inequality in [15].

Recall the fractional Caffarelli-Kohn-Nirenberg (CKN) inequality on homogeneous groups.

**Theorem 3.9** ([14], Fractional CKN inequality)**.** *Let $\mathbb{G}$ be a homogeneous group with homogeneous dimension $Q$. Assume that $Q \geq 2$, $s \in (0,1)$, $p > 1$, $q \geq 1$, $\tau > 0$, $a \in (0,1]$, $\beta_1$, $\beta_2$, $\beta$, $\mu$, $\gamma \in \mathbb{R}$, $\beta_1 + \beta_2 = \beta$ and*

$$\frac{1}{\tau} + \frac{\gamma}{Q} = a\left(\frac{1}{p} + \frac{\beta - s}{Q}\right) + (1-a)\left(\frac{1}{q} + \frac{\mu}{Q}\right). \quad (3.21)$$

*Assume in addition that, $0 \leq \beta - \sigma$ with $\gamma = a\sigma + (1-a)\mu$, and*

$$\beta - \sigma \leq s \text{ only if } \frac{1}{\tau} + \frac{\gamma}{Q} = \frac{1}{p} + \frac{\beta - s}{Q}. \quad (3.22)$$

*Then for all measurable and compactly supported $u$ we have*

$$\||x|^\gamma u\|_{L^\tau(\mathbb{G})} \leq C[u]^a_{s,p,\beta}\||x|^\mu u\|^{1-a}_{L^q(\mathbb{G})}, \quad (3.23)$$

*when $\frac{1}{\tau} + \frac{\gamma}{Q} > 0$, and for $u \in C^1_c(\mathbb{G}\setminus\{e\})$ we have*

$$\||x|^\gamma u\|_{L^\tau(\mathbb{G})} \leq C[u]^a_{s,p,\beta}\||x|^\mu u\|^{1-a}_{L^q(\mathbb{G})}, \quad (3.24)$$

*when $\frac{1}{\tau} + \frac{\gamma}{Q} < 0$. Here $e$ is the identity element of $\mathbb{G}$.*

Now we present the fractional logarithmic CKN type inequality on homogeneous groups.

**Theorem 3.10.** *Under the assumptions of Theorem 3.9 with*

$$\alpha = \beta = \mu,\ 1 < q < p^*,\ 1 < p < Q,\ \beta p + Q > 0,\ \beta q + Q > 0, \quad (3.25)$$

*there exists a positive constant $C$ such that*

$$\int_{\mathbb{G}} \frac{(|x|^\alpha |u|)^q}{\||x|^\alpha u\|^q_{L^q(\mathbb{G})}} \log\left(\frac{|x|^{\alpha q}|u|^q}{\||x|^\alpha u\|^q_{L^q(\mathbb{G})}}\right) dx \leq \frac{1}{1-\frac{q}{p^*}} \log\left(\frac{[u]_{s,p,\alpha}}{\||x|^\alpha u\|^q_{L^q(\mathbb{G})}}\right), \quad (3.26)$$

*for all measurable and compactly supported $u$.*

*Proof.* From the assumptions of Theorem 3.9 with $\alpha=\beta=\gamma$, we obtain that

$$\frac{1}{\tau}=\frac{a}{p^*}+\frac{1-a}{q}. \tag{3.27}$$

From this identity with $q<p^*$ we get $q<\tau$. By using these facts with weighted logarithmic Hölder's inequality and $\alpha=\beta=\gamma$ we obtain

$$\begin{aligned}\int_{\mathbb{G}}\frac{|x|^{\alpha q}|u|^q}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\log\left(\frac{|x|^{\alpha q}|u|^q}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\right)dx &\leq \frac{\tau}{\tau-q}\log\left(\frac{\||x|^{\alpha}u\|^q_{L^{\tau}(\mathbb{G})}}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\right)\\ &\leq \frac{\tau}{\tau-q}\log\left(C^a\frac{[u]^{aq}_{s,p,\alpha}\||x|^{\alpha}u\|^{(1-a)q}_{L^q(\mathbb{G})}}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\right)\\ &= \frac{a\tau}{\tau-q}\log\left(C\frac{[u]^q_{s,p,\alpha}}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\right).\end{aligned} \tag{3.28}$$

Since $\alpha=\beta=\gamma$, we have

$$\frac{a\tau}{\tau-q}=\frac{p^*}{p^*-q}. \tag{3.29}$$

□

**Remark 3.11.** In the Abelian (Euclidean) case $\mathbb{G}=(\mathbb{R}^N,+)$, we have $Q=N$ and $|\cdot|=|\cdot|_E$ ($|\cdot|_E$ is the Euclidean distance), if $s\to 1^-$ and from (3.26) we get the logarithmic Sobolev inequality in [25].

## 4. Blow-up theorem on $\mathbb{G}$

Let us consider the following Cauchy-Dirichlet fractional heat equation on the homogeneous group:

$$\begin{cases}\frac{\partial u(x,t)}{\partial t}+(-\Delta_s)u(x,t)=u(x,t)\log|u(x,t)|, & (x,t)\in\Omega\times(0,+\infty),\ \Omega\subset\mathbb{G},\\ u(x,t)=0,\ \ (x,t)\in\mathbb{G}\setminus\Omega\times(0,+\infty),\\ u(x,0)=u_0(x),\end{cases} \tag{4.1}$$

where $\Delta_s$ is the fractional sub-Laplacian with $s\in(0,1)$.

For simplicity, we introduce the notations $H^s_0(\Omega):=W^{s,2}_0(\Omega)$ and $[u]_s:=[u]_{s,2,\Omega}$. Let us recall the definition of a weak solution.

**Definition 4.1.** Let $T>0$. A function $u:\Omega\times[0,+\infty)\to\mathbb{R}$, $u=u(x,t)\in L^{\infty}(0,T;H^s_0(\Omega))$ with $\frac{\partial u}{\partial t}\in L^2(0,T;L^2(\Omega))$ is a called a weak solution of problem (4.1) in $\Omega\times[0,+\infty)$, if $u_0\in H^s_0(\Omega)$ and $u$ satisfies (4.1) in the sense of distribution,

$$\int_{\Omega}u_t\varphi dx+\langle(-\Delta_s)u,\varphi\rangle=\int_{\Omega}u\log|u|\varphi dx, \tag{4.2}$$

for any $\varphi\in H^s_0(\Omega)$, $t\in(0,+\infty)$.

**Definition 4.2.** Let $u(x,t)$ be a weak solution of (4.1). We say that $u(x,t)$ blows up at $+\infty$ if

$$\lim_{t\to+\infty}\|u(t,\cdot)\|^2_{L^2(\Omega)}=+\infty. \tag{4.3}$$

Let us consider the following energy functionals

$$J(u) = \frac{1}{2}[u]_s^2 - \frac{1}{2}\int_\Omega u^2 \log|u|dx + \frac{1}{4}\int_\Omega |u|^2 dx, \tag{4.4}$$

and

$$I(u) = [u]_s^2 - \int_\Omega u^2 \log|u|dx. \tag{4.5}$$

Then we have

$$J(u) = \frac{1}{2}I(u) + \frac{1}{4}\int_\Omega |u|^2 dx. \tag{4.6}$$

We have the following energy identity for (4.1).

**Lemma 4.3.** *Suppose that $u$ is a weak solution of the problem (4.1). Then we have*

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + J(u) = J(u_0), \quad \forall t \in (0, +\infty). \tag{4.7}$$

*Proof.* Multiplying by $u_t$ and integrating over $\Omega$ in (4.1), we get

$$\int_\Omega |u_t|^2 dx + \langle (-\Delta_s)u, u_t \rangle = \int_\Omega u_t u \log|u|dx. \tag{4.8}$$

For the second term on the left hand side of (4.8), we have

$$\begin{aligned}\langle (-\Delta_s)u, u_t \rangle &= \int_\Omega \int_\Omega \frac{(u(x,t) - u(y,t))(u_t(x,t) - u_t(y,t))}{|y^{-1}x|^{Q+2s}} dxdy \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega \int_\Omega \frac{|u(x,t) - u(y,t)|^2}{|y^{-1}x|^{Q+2s}} dxdy = \frac{1}{2}\frac{d[u(t)]_s^2}{dt}. \end{aligned} \tag{4.9}$$

On the right hand side of (4.8), we have

$$\frac{du^2 \log|u|}{dt} = 2u_t u \log|u| + uu_t, \tag{4.10}$$

then

$$\begin{aligned}\int_\Omega u_t u \log|u|dx &= \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \log|u|dx - \frac{1}{2}\int_\Omega uu_t dx \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \log|u|dx - \frac{1}{4}\frac{d}{dt}\int_\Omega u^2 dx. \end{aligned} \tag{4.11}$$

Plugging (4.9) and (4.11) in (4.8), we get

$$\begin{aligned}\int_\Omega |u_t|^2 dx + \frac{d}{dt}\left(\frac{1}{2}[u]_s^2 - \frac{1}{2}\int_\Omega u^2 \log|u|dx + \frac{1}{4}\int_\Omega u^2 dx\right) \\ = \int_\Omega |u_t|^2 dx + \frac{d}{dt}J(u) = 0. \end{aligned} \tag{4.12}$$

Integrating over $(0,t)$, we arrive at

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + \int_0^t \frac{dJ(u)}{d\tau} d\tau = 0, \tag{4.13}$$

that is,

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + J(u) = J(u_0). \tag{4.14}$$

□

Now we are in the position to present the main result of this section.

**Theorem 4.4.** *Suppose that $u$ is a weak solution of (4.1) with $u_0 \in H_0^s(\Omega)$ and $I(u_0) < 0$. Then*

$$\lim_{t\to+\infty} \|u(t,\cdot)\|^2_{L^2(\Omega)} = +\infty. \tag{4.15}$$

*Proof.* Firstly, by using (4.2) with $u = \varphi$ we get

$$\begin{aligned}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} &= \frac{d}{dt}\int_\Omega u^2 dx = 2\int_\Omega uu_t dx \\ &= -2\left(\int_\Omega \langle(-\Delta_s)u, u\rangle - u^2 \log|u| dx\right) = -2I(u). \end{aligned} \tag{4.16}$$

So by using this, (4.2) and (4.6), we obtain

$$\begin{aligned}\frac{dI(u)}{dt} &= \frac{d}{dt}\left(2J(u) - \frac{1}{2}\int_\Omega u^2 dx\right) = 2\int_\Omega\int_\Omega \frac{(u(x,t)-u(y,t))(u_t(x,t)-u_t(y,t))}{|y^{-1}x|^{Q+2s}} dxdy \\ &- 2\int_\Omega uu_t \log|u| dx - \int_\Omega uu_t dx + \int_\Omega uu_t dx - \int_\Omega uu_t dx = -2\|u_t\|^2_{L^2(\Omega)} - \int_\Omega uu_t dx \\ &= -2\|u_t\|^2_{L^2(\Omega)} + I(u) \le I(u). \end{aligned} \tag{4.17}$$

By using Grönwall–Bellman's inequality and $I(u_0) < 0$ we have

$$I(u) \le I(u_0)e^t \le I(u_0) < 0, \quad \forall t \in (0, +\infty). \tag{4.18}$$

It means that $I(u(x,t))$ is decreasing functional with respect to the argument $t$. Let us set

$$A(t) = \int_0^t \|u\|^2_{L^2(\Omega)} dt, \quad A'(t) = \|u\|^2_{L^2(\Omega)}, \tag{4.19}$$

and by Definition 4.1 we have

$$A''(t) = 2\int_\Omega uu_t dx = -2[u]_s + 2\int_\Omega u^2 \log|u| dx = -2I(u). \tag{4.20}$$

A simple calculation gives

$$(\log A(t))' = \frac{A'(t)}{A(t)}, \quad (\log A(t))'' = \frac{A''(t)A(t) - (A'(t))^2}{A^2(t)}. \tag{4.21}$$

Now let us estimate $\frac{A''(t)A(t)-(A'(t))^2}{A^2(t)}$. By using (4.19), (4.6) and Lemma 4.3, we obtain

$$A''(t) = -2I(u) = -4J(u) + A'(t) = -4J(u_0) + 4\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + A'(t). \tag{4.22}$$

Similarly, from (4.19) we obtain

$$\begin{aligned}(A'(t))^2 &= \|u\|^4_{L^2(\Omega)} = \|u\|^4_{L^2(\Omega)} + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} \\ &+ \|u_0\|^4_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)} = \left(\int_\Omega \left(u^2 - u_0^2\right) dx\right)^2 + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)} \\ &= \left(\int_\Omega \int_0^t \frac{\partial u^2}{\partial \tau} d\tau dx\right)^2 + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)} = 4\left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2 \\ &\qquad + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)}. \end{aligned} \tag{4.23}$$

Finally, we have

$$(A'(t))^2 = 4\left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2 + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)}. \tag{4.24}$$

It follows that

$$\begin{aligned} A''(t)A(t) - (A'(t))^2 &= -4J(u_0)A(t) + 4\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau A(t) + A'(t)A(t) \\ &- 4\left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2 - 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} + \|u_0\|^4_{L^2(\Omega)} \\ &= 4\left(\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau \int_0^t \|u\|^2_{L^2(\Omega)} - \left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2\right) \\ &- 4J(u_0)A(t) + A'(t)A(t) - 2\|u_0\|^2_{L^2(\Omega)}A'(t) + \|u_0\|^4_{L^2(\Omega)}. \end{aligned} \tag{4.25}$$

By using the Cauchy-Bunyakovsky-Schwarz inequality, we get

$$\begin{aligned} A''(t)A(t) - (A'(t))^2 &= 4\left(\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau \int_0^t \|u\|^2_{L^2(\Omega)} - \left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2\right) \\ &- 4J(u_0)A(t) + A'(t)A(t) - 2\|u_0\|^2_{L^2(\Omega)}A'(t) + \|u_0\|^4_{L^2(\Omega)} \\ &\geq A'(t)\left(\frac{A(t)}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(\frac{A'(t)}{2} - 4J(u_0)\right). \end{aligned} \tag{4.26}$$

From (4.19), (4.20) and $I(u) \leq I(u_0) < 0$, we obtain

$$\begin{aligned} &A'(t) = A'(0) - 2\int_0^t I(u(x,\tau))d\tau = -2I(u_0)t \geq 0, \quad t \geq 0, \\ &A(t) = -I(u_0)t^2 \geq 0, \quad t \geq 0. \end{aligned} \tag{4.27}$$

By using (4.27) and (4.6) in (4.26), we calculate

$$\begin{aligned}
A''(t)A(t)-(A'(t))^2 &\geq A'(t)\left(\frac{A(t)}{2}-\|u_0\|^2_{L^2(\Omega)}\right)+A(t)\left(\frac{A'(t)}{2}-4J(u_0)\right)\\
&\geq A'(t)\left(\frac{-I(u_0)t^2}{2}-\|u_0\|^2_{L^2(\Omega)}\right)+A(t)\left(-I(u_0)t-4J(u_0)\right)\\
&\geq A'(t)\left(\frac{-I(u_0)t^2}{2}-\|u_0\|^2_{L^2(\Omega)}\right)+A(t)\left(-I(u_0)t-2I(u_0)-\|u_0\|^2_{L^2(\Omega)}\right)\\
&\geq A'(t)\left(\frac{-I(u_0)t^2}{2}-\|u_0\|^2_{L^2(\Omega)}\right)+A(t)\left(-I(u_0)(t+2)-\|u_0\|^2_{L^2(\Omega)}\right).
\end{aligned}\tag{4.28}$$

From Definition 4.1, we have that $u_0 \in H^s_0(\Omega)$ and let

$$t > t_0 = \max\left\{\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)}-2,\ \frac{\sqrt{2}\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}\right\} \geq 0. \tag{4.29}$$

Let us consider the case

$$t_0 = \max\left\{\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)}-2,\ \frac{\sqrt{2}\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}\right\} = \frac{\sqrt{2}\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}. \tag{4.30}$$

By using this fact in (4.28), we obtain

$$\begin{aligned}
&A''(t)A(t)-(A'(t))^2\\
&\geq A'(t)\left(\frac{-I(u_0)t^2}{2}-\|u_0\|^2_{L^2(\Omega)}\right)+A(t)\left(-I(u_0)(t+2)-\|u_0\|^2_{L^2(\Omega)}\right)\\
&\geq A'(t)\left(\frac{-I(u_0)t_0^2}{2}-\|u_0\|^2_{L^2(\Omega)}\right)+A(t)\left(-I(u_0)(t_0+2)-\|u_0\|^2_{L^2(\Omega)}\right)\\
&= A'(t)\left(\|u_0\|^2_{L^2(\Omega)}-\|u_0\|^2_{L^2(\Omega)}\right)+A(t)\left(-I(u_0)(t_0+2)-\|u_0\|^2_{L^2(\Omega)}\right)\\
&= A(t)\left(-I(u_0)(t_0+2)-\|u_0\|^2_{L^2(\Omega)}\right)\\
&\geq A(t)\left(-I(u_0)\left(\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)}-2+2\right)-\|u_0\|^2_{L^2(\Omega)}\right)\\
&= 0.
\end{aligned}\tag{4.31}$$

Thus, we get

$$A''(t)A(t)-(A'(t))^2 \geq 0. \tag{4.32}$$

Similarly, in the other case

$$t_0 = \max\left\{\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)}-2,\ \frac{\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}\right\} = \frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)}-2, \tag{4.33}$$

we get

$$A''(t)A(t)-(A'(t))^2 \geq 0. \tag{4.34}$$

So we have

$$(\log A(t))'' = \frac{A''(t)A(t) - (A'(t))^2}{A^2(t)}, \tag{4.35}$$

and integrating over $(t_0, t)$, we get

$$(\log A(t))' - (\log A(t))'|_{t=t_0} = \int_{t_0}^{t} \frac{A''(\tau)A(\tau) - (A'(\tau))^2}{A^2(\tau)} d\tau \geq 0. \tag{4.36}$$

Thus, we have

$$(\log A(t))' \geq (\log A(t))'|_{t=t_0}. \tag{4.37}$$

Similarly, we get

$$\frac{A'(t_0)}{A(t_0)}(t - t_0) = (\log A(t))'|_{t=t_0}(t - t_0) \leq \int_{t_0}^{t} \log(A(\tau))' d\tau = \log(A(t)) - \log(A(t_0)). \tag{4.38}$$

Finally, we arrive at

$$A(t_0)e^{\frac{A'(t_0)}{A(t)}(t-t_0)} \leq A(t). \tag{4.39}$$

By summarising above facts (4.37)-(4.39) with $t \geq t_0$, we compute

$$\begin{aligned} \|u\|^2_{L^2(\Omega)} &= A'(t) = (\log A(t))'A(t) \geq (\log A(t))'|_{t=t_0}A(t) = \frac{A'(t_0)}{A(t_0)}A(t) = \frac{A(t)}{A(t_0)}A'(t_0) \\ &\geq A'(t_0)e^{\frac{A'(t_0)}{A(t_0)}(t-t_0)} = \|u(\cdot, t_0)\|^2_{L^2(\Omega)}e^{\frac{A'(t_0)}{A(t_0)}(t-t_0)} \geq \|u_0\|^2_{L^2(\Omega)}e^{\frac{A'(t_0)}{A(t_0)}(t-t_0)}. \end{aligned} \tag{4.40}$$

That is,

$$\lim_{t\to+\infty} \|u(t, \cdot)\|^2_{L^2(\Omega)} = +\infty. \tag{4.41}$$

□

**Remark 4.5.** In the Abelian (Euclidean) case $\mathbb{G} = (\mathbb{R}^N, +)$, we have $Q = N$ and $|\cdot| = |\cdot|_E$ ($|\cdot|_E$ is the Euclidean distance), if $s \to 1^-$ we get blow-up result at infinity in [3] and [10].

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