

Scale-free unique continuation principle, eigenvalue lifting and Wegner estimates for random Schrödinger operators

Ivica Nakić¹, Matthias Täufer², Martin Tautenhahn², and Ivan Veselić²

¹*University of Zagreb, Department of Mathematics, Croatia*

²*Technische Universität Chemnitz, Fakultät für Mathematik, Germany*

Abstract

We prove a scale-free, quantitative unique continuation principle for functions in the range of the spectral projector $\chi_{(-\infty, E]}(H_L)$ of a Schrödinger operator H_L on a cube of side $L \in \mathbb{N}$, with bounded potential. Such estimates are also called, depending on the context, uncertainty principles, observability estimates, or spectral inequalities. We apply it to (i) prove a Wegner estimate for random Schrödinger operators with non-linear parameter-dependence and to (ii) exhibit the dependence of the control cost on geometric model parameters for the heat equation in a multi-scale domain.

Contents

1	Introduction	2
2	Results	3
2.1	Scale-free unique continuation and eigenvalue lifting	3
2.2	Application to random breather Schrödinger operators	5
2.3	More general non-linear models and localization	6
2.4	Application to control theory	7
3	Proof of scale-free unique continuation principle	9
3.1	Carleman inequalities	9
3.2	Extension to larger boxes	10
3.3	Ghost dimension	11
3.4	Interpolation inequalities	13
3.5	Proof of Theorem 2.2 and Corollary 2.5	19
4	Proof of Wegner and initial scale estimate	22
5	Proof of observability estimate	25
A	Sketch of proof of Proposition 3.2	25
B	Constants	26
B.1	Cutoff functions	26
B.1.1	The constants Θ_2 and Θ_3	27
B.1.2	The constant Θ_1	28

B.1.3	Distance of S_2 and $\mathbb{R}_+^{d+1} \setminus S_3$	29
B.2	The constant \tilde{C}_{sfuc}	31
C	On single-site potentials for the breather model	31
C.1	Our assumptions	31
C.2	Earlier assumptions	33
D	Proof of Corollary 2.3	35

1 Introduction

We prove a *quantitative unique continuation* inequality for functions in the range of the spectral projector $\chi_{(-\infty, E]}(H_L)$ of a Schrödinger operator H_L on a cube of side $L \in \mathbb{N}$. It has been announced in [36]. Depending on the area of mathematics and the context such estimates have various names: quantitative unique continuation principle (UCP), uncertainty principles, spectral inequalities, observability or sampling estimates, or bounds on the vanishing order. If the observability or sampling set respects in a certain way the underlying lattice structure, our estimate is independent of L ; for this reason we call it *scale-free*. For our applications it is crucial to exhibit explicitly the dependence of the quantitative unique continuation inequality on the model parameters.

A key motivation to study scale-free quantitative unique continuation estimates comes from the theory of random Schrödinger operators, in particular eigenvalue lifting estimates, Wegner bounds, and the continuity of the integrated density of states. (We defer precise definitions to §2.) In fact, there is quite a number of previous papers which have derived a scale-free UCP and eigenvalue lifting estimates under special assumptions. Naturally, the first situation to be considered was the case where the Schrödinger operator is the pure Laplacian $H = -\Delta$, i.e. the background potential V vanishes identically. For instance, [23] derives a UCP which is valid for energies in an interval at zero, i.e. the bottom of the spectrum, if one has a periodic arrangement of sampling sets. The proof uses detailed information about hitting probabilities of Brownian motion paths, and is in sense related to Harnack inequalities. A very elementary approach to eigenvalue lifting estimates is provided by the spatial averaging trick, used in [3] and [15] in periodic situations, and extended to non-periodic situations in [14]. It is applicable to energies near zero. A different approach for eigenvalue lifting was derived in [5]. In [4] it was shown how one can conclude an uncertainty principle at low energies based on an eigenvalue lifting estimate. Related results have been derived for energies near spectral edges in [24] and [9] using resolvent comparison. In one space dimension eigenvalue lifting results and Wegner estimates have been proven in [45], [25]. There a periodic arrangement of the sampling set is assumed. The proof carries over to the case of non-periodic arrangements verbatim, which has been used in the context of quantum graphs in [19]. In the case that both the deterministic background potential and the sampling set are periodic, an uncertainty principle and a Wegner estimate, which are valid for arbitrary bounded energy regions, have been proven in [6, 7]. These papers make use of Floquet theory, hence they are a priori restricted to periodic background potentials as well as periodic sampling sets. An alternative proof for the result in [7], with more explicit control of constants, has been worked out in [16]. The case where the background potential is periodic but the impurities need not be periodically arranged has been considered in [5] and [14] for low energies. Our main theorem unifies and generalizes all the results mentioned so far and makes the dependence on the model parameters explicit. Indeed, our scale-free

unique continuation principle answers positively a question asked in [40]. A partial answer was given already in [27]. While [40] concerns the case of a single eigenfunctions, [27] treats linear combinations of eigenfunctions corresponding to very close eigenvalues. For a broader discussion we refer to the summer school notes [43].

A second application of our scale-free UCP is in the control theory of the heat equation. Here one asks whether one can drive a given initial state to a desired state with a control function living in a specified subset, and what the minimal L^2 -norm of the control function (called control cost) is. Recently, the search for optimal placement of the control set and the dependence of the control cost on geometric features of this set has received much attention, see e.g. [39, 38]. Our scale-free UCP gives an explicit estimate of the control cost w.r.t. the model parameters in multi-scale domains.

Our proof of the scale-free unique continuation estimate uses two Carleman and nested interpolation estimates, an idea used before e.g. in [30, 21]. To obtain explicit estimates we need explicit weight functions. The first Carleman estimate includes a boundary term and uses a parabolic weight function as proposed in [21]. The second Carleman estimate is similar to the ones in [11, 3]. However, none of the two is quite sufficient for our purposes, so we use a variant developed in [35], see also Appendix §A. Moreover, typically the diameter of the ambient manifold enters in the Carleman estimate. In our case it grows unboundedly in L , hence the UCP constants would become worse and worse. Thus, to eliminate the L -dependence we have to use techniques developed in the context of random Schrödinger operators to accommodate for the multi-scale structure of the underlying domain and sampling set.

In the next section we state our main results, §3 is devoted to the proof of the scale-free unique continuation principle, §4 to proofs concerning random Schrödinger operators, §5 to the observability estimate of the control equation, while certain technical aspects are deferred to the appendix.

2 Results

2.1 Scale-free unique continuation and eigenvalue lifting

Let $d \in \mathbb{N}$. For $L > 0$ we denote by $\Lambda_L = (-L/2, L/2)^d \subset \mathbb{R}^d$ the cube with side length L , and by Δ_L the Laplace operator on $L^2(\Lambda_L)$ with Dirichlet, Neumann or periodic boundary conditions. Moreover, for a measurable and bounded $V : \mathbb{R}^d \rightarrow \mathbb{R}$ we denote by $V_L : \Lambda_L \rightarrow \mathbb{R}$ its restriction to Λ_L given by $V_L(x) = V(x)$ for $x \in \Lambda_L$, and by

$$H_L = -\Delta_L + V_L \quad \text{on} \quad L^2(\Lambda_L)$$

the corresponding Schrödinger operator. Note that H_L has purely discrete spectrum. For $x \in \mathbb{R}^d$ and $r > 0$ we denote by $B(x, r)$ the ball with center x and radius r with respect to Euclidean norm. If the ball is centered at zero we write $B(r) = B(0, r)$.

Definition 2.1. Let $G > 0$ and $\delta > 0$. We say that a sequence $z_j \in \mathbb{R}^d$, $j \in (G\mathbb{Z})^d$ is (G, δ) -*equidistributed*, if

$$\forall j \in (G\mathbb{Z})^d: \quad B(z_j, \delta) \subset \Lambda_G + j.$$

Corresponding to a (G, δ) -equidistributed sequence we define for $L \in G\mathbb{N}$ the set

$$W_\delta(L) = \bigcup_{j \in (G\mathbb{Z})^d} B(z_j, \delta) \cap \Lambda_L.$$

Theorem 2.2. *There is $N = N(d)$ such that for all $\delta \in (0, 1/2)$, all $(1, \delta)$ -equidistributed sequences, all measurable and bounded $V : \mathbb{R}^d \rightarrow \mathbb{R}$, all $L \in \mathbb{N}$, all $b \geq 0$ and all $\phi \in \text{Ran}(\chi_{(-\infty, b]}(H_L))$ we have*

$$\|\phi\|_{L^2(W_\delta(L))}^2 \geq C_{\text{sfuc}} \|\phi\|_{L^2(\Lambda_L)}^2 \quad (1)$$

where

$$C_{\text{sfuc}} = C_{\text{sfuc}}(d, \delta, b, \|V\|_\infty) := \delta^N (1 + \|V\|_\infty^{2/3} + \sqrt{b}).$$

For $t, L > 0$ and a measurable and bounded $V : \mathbb{R}^d \rightarrow \mathbb{R}$ we define the Schrödinger operator $H_{t,L} = -t\Delta_L + V_L$ on $L^2(\Lambda_L)$. By scaling we obtain the following corollary, see Appendix D.

Corollary 2.3. *Let $N = N(d)$ be the constant from Theorem 2.2. Then, for all $G, t > 0$, all $\delta \in (0, G/2)$, all (G, δ) -equidistributed sequences, all measurable and bounded $V : \mathbb{R}^d \rightarrow \mathbb{R}$, all $L \in G\mathbb{N}$, all $b \geq 0$ and all $\phi \in \text{Ran}(\chi_{(-\infty, b]}(H_{t,L}))$ we have*

$$\|\phi\|_{L^2(W_\delta(L))}^2 \geq C_{\text{sfuc}}^{G,t} \|\phi\|_{L^2(\Lambda_L)}^2$$

where

$$C_{\text{sfuc}}^{G,t} = C_{\text{sfuc}}^{G,t}(d, \delta, b, \|V\|_\infty) := \left(\frac{\delta}{G}\right)^{N(1+G^{4/3}\|V\|_\infty^{2/3}/t^{2/3}+G\sqrt{b/t})}.$$

Note that the set $W_\delta(L)$ depends on G and the choice of the (G, δ) -equidistributed sequence. In particular, there is a constant $M = M(d, G, t) \geq 1$ such that

$$C_{\text{sfuc}}^{G,t} \geq \delta^M (1 + \|V\|_\infty^{2/3} + \sqrt{b}). \quad (2)$$

Note that Theorem 2.2 and Corollary 2.3 also hold for $b < 0$, since

$$\text{Ran}(\chi_{(-\infty, b]}(H)) \subset \text{Ran}(\chi_{(-\infty, 0]}(H))$$

for any self-adjoint operator H .

Remark 2.4 (Previous results). If $L = G$ the result is closely related to doubling estimates and bounds on the vanishing order, cf. [30, 28, 21, 1]. These results, however, do not study the dependence of the bound on geometric data, e.g. the diameter of the domain or manifold. In the context of random Schrödinger operators results like (1) have been proven before under additional assumptions and using other methods: For $V \equiv 0$ and energies close to the minimum of the spectrum in [23] and [3]; near spectral edges of periodic Schrödinger operators in [24]; and for periodic geometries W_δ and potentials in [6]. More recently and using similar methods as we do, bounds like (1) have been established for individual eigenfunctions in [40]. This has then been extended in [27] to linear combinations of eigenfunctions of closeby eigenvalues. For more references and a broader discussion of the history see e.g. [40], [27], or [43].

As an application to spectral theory we have the following corollary. A proof is given at the end of Section 3.5.

Corollary 2.5. *Let $b, \alpha, G > 0$, $\delta \in (0, G/2)$, $L \in G\mathbb{N}$ and $A_L, B_L : \Lambda_L \rightarrow \mathbb{R}$ be measurable, bounded and assume that*

$$B_L \geq \alpha \chi_{W_\delta(L)}$$

for a (G, δ) -equidistributed sequence. Denote the eigenvalues of a self-adjoint operator H with discrete spectrum by $\lambda_i(H)$, enumerated increasingly and counting multiplicities. Then for all $i \in \mathbb{N}$ with $\lambda_i(-\Delta + A_L + B_L) \leq b$, we have

$$\lambda_i(-\Delta_L + A_L + B_L) \geq \lambda_i(-\Delta_L + A_L) + \alpha C_{\text{sfuc}}^{G,1}(d, \delta, b, \|A_L + B_L\|_\infty).$$

2.2 Application to random breather Schrödinger operators

An important application of our result is in the spectral theory of random Schrödinger operators. The above scale-free unique continuation estimate is the key for proving the Wegner estimate formulated below, which is a bound on the expected number of eigenvalues in a short energy interval of a finite box restriction of our random Hamiltonian. Together with a so-called initial scale estimate, Wegner estimates facilitate a proof of Anderson localization via multi-scale analysis. For more background on multi-scale analysis & localization and on Wegner estimates consult e.g. the monographs [42] and [47], respectively.

The main point is that the potentials we are dealing with here exhibit a *non-linear dependence* on the random parameters ω_j . Due to this challenge, previously established versions of (1), as discussed in Remark 2.4, are not sufficiently precise to be applied to such models. We emphasize that our scale-free unique continuation principle and Wegner estimate are valid for all bounded energy intervals, not only near the bottom of the spectrum.

Let us introduce a simple, but paradigmatic example of the models we are considering. (The general case will be studied in the next paragraph.)

Let \mathcal{D} be a countable set to be specified later. For $0 \leq \omega_- < \omega_+ < 1$ we define the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with

$$\Omega = \prod_{j \in \mathcal{D}} \mathbb{R}, \quad \mathcal{A} = \bigotimes_{j \in \mathcal{D}} \mathcal{B}(\mathbb{R}) \quad \text{and} \quad \mathbb{P} = \bigotimes_{j \in \mathcal{D}} \mu,$$

where $\mathcal{B}(\mathbb{R})$ is the Borel σ -algebra and μ is a probability measure with $\text{supp } \mu \subset [\omega_-, \omega_+]$ and a bounded density ν_μ . Hence, the projections $\omega \mapsto \omega_k$ give rise to a sequence of independent and identically distributed random variables ω_j , $j \in \mathcal{D}$. We denote by \mathbb{E} the expectation with respect to the measure \mathbb{P} .

The standard random breather model is defined as

$$H_\omega = -\Delta + V_\omega(x), \quad \text{with } V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \chi_{B_{\omega_j}}(x - j) \quad (3)$$

and the restriction of H_ω to the box Λ_L by $H_{\omega,L}$. Here obviously $\mathcal{D} = \mathbb{Z}^d$. Denote by $\chi_{[E-\varepsilon, E+\varepsilon]}$ the spectral projector of $H_{\omega,L}$. We formulate now a version of our general Theorem 2.8 applied to the standard random breather model.

Theorem 2.6 (Wegner estimate for the standard random breather model). *Assume that $[\omega_-, \omega_+] \subset [0, 1/4]$, fix $E_0 \in \mathbb{R}$, and set $\varepsilon_{\max} = \frac{1}{4} \cdot 8^{-N(2+|E_0+1|^{1/2})}$, where N is the constant from Theorem 2.2. Then there is $C = C(d, E_0) \in (0, \infty)$ such that for all $\varepsilon \in (0, \varepsilon_{\max})$ and $E \geq 0$ with $[E - \varepsilon, E + \varepsilon] \subset (-\infty, E_0]$, we have*

$$\mathbb{E} \left[\text{Tr} \left[\chi_{[E-\varepsilon, E+\varepsilon]}(H_{\omega,L}) \right] \right] \leq C \|\nu\|_\infty \varepsilon^{[N(2+|E_0+1|^{1/2})]^{-1}} |\ln \varepsilon|^d L^d.$$

Theorem 2.6 implies local Hölder continuity of the integrated density of states (IDS) and is sufficient for the multi-scale analysis proof of spectral localization, see the next paragraph.

Remark 2.7 (Previous results on the random breather model). The paper [8] introduced random breather potentials, while a Wegner estimate was proven in [9], however excluding any bounded and any continuous single site potential, cf. Appendix C. Lifshitz tails for random breather Schrödinger operators were proven in [26]. All of the papers mentioned so far approached

the breather model using techniques which have been developed for the alloy type model. Consequently, at some stage the non-linear dependence on the random variables was linearised, giving rise to certain differentiability conditions. As a result, characteristic functions of cubes or balls which would be the most basic example one can think of were excluded as single-site potentials. Only [46] considers a simple non-differentiable example, namely the standard random breather potential in one dimension, and proves a Lifshitz tail estimate.

2.3 More general non-linear models and localization

We formulate now a Wegner estimate for a general class of models, which includes the standard random breather potential, considered in the last paragraph as a special case. We state also an initial scale estimate which implies localization.

Here, in the general setting, we assume that $\mathcal{D} \subset \mathbb{R}^d$ is a Delone set, i.e. there are $0 < G_1 < G_2$ such that for any $x \in \mathbb{R}^d$, we have $|\{\mathcal{D} \cap (\Lambda_{G_1} + x)\}| \leq 1$ and $|\{\mathcal{D} \cap (\Lambda_{G_2} + x)\}| \geq 1$. Here, $|\cdot|$ stands for the cardinality. In other words, Delone sets are relatively dense and uniformly discrete subsets of \mathbb{R}^d . For more background about Delone sets, see, for example, the contributions in [22]. The reader unacquainted with the concept of a Delone set can always think of $\mathcal{D} = \mathbb{Z}^d$.

Furthermore, let $\{u_t : t \in [0, 1]\} \subset L_0^\infty(\mathbb{R}^d)$ be functions such that there are $G_u \in \mathbb{N}$, $u_{\max} \geq 0$, $\alpha_1, \beta_1 > 0$ and $\alpha_2, \beta_2 \geq 0$ with

$$\begin{cases} \forall t \in [0, 1] : \text{supp } u_t \subset \Lambda_{G_u}, \\ \forall t \in [0, 1] : \|u_t\|_\infty \leq u_{\max}, \\ \forall t \in [\omega_-, \omega_+], \delta \leq 1 - \omega_+ : \exists x_0 \in \Lambda_{G_u} : u_{t+\delta} - u_t \geq \alpha_1 \delta^{\alpha_2} \chi_{B(x_0, \beta_1 \delta^{\beta_2})}. \end{cases} \quad (4)$$

We define the family of Schrödinger operators H_ω , $\omega \in \Omega$, on $L^2(\mathbb{R}^d)$ given by

$$H_\omega := -\Delta + V_\omega \quad \text{where} \quad V_\omega(x) = \sum_{j \in \mathcal{D}} u_{\omega_j}(x - j).$$

Note that for all $\omega \in [0, 1]^{\mathcal{D}}$ we have $\|V_\omega\|_\infty \leq K_u := u_{\max} \lceil G_u/G_1 \rceil^d$, c.f. Lemma 4.1. Assumption (4) includes many prominent models of random Schrödinger operators - linear and non-linear. We give some examples.

Standard random breather model: Let μ be the uniform distribution on $[0, 1/4]$ and let $u_t(x) = \chi_{B(0,t)}$, $j \in \mathbb{Z}^d$. Then $V_\omega = \sum_{j \in \mathbb{Z}^d} \chi_{B(j, \omega_j)}$ is the characteristic function of a disjoint union of balls with random radii. Such models were introduced in §2.2.

General random breather models Let $0 \leq u \in L_0^\infty(\mathbb{R}^d)$ and define $u_t(x) := u(x/t)$ for $t > 0$ and $u_{j,0} := 0$, $j \in \mathbb{Z}^d$, and assume that the family $\{u_t : t \in [0, 1]\}$ satisfies (4). Natural examples are discussed in Appendix C. Then $V_\omega(x) = \sum_{j \in \mathbb{Z}^d} u_{\omega_j}(x - j)$ is a sum of random dilations of a single-site potential u at each lattice site $j \in \mathbb{Z}^d$.

Alloy type model Let $0 \leq u \in L_0^\infty(\mathbb{R}^d)$, $u \geq \alpha > 0$ on some open set and let $u_t(x) := tu(x)$. Then $V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \omega_j u(x - j)$ is a sum of copies of u at all lattice sites $j \in \mathbb{Z}^d$, multiplied with ω_j .

Delone-alloy type model Let $\mathcal{D} \subset \mathbb{R}^d$ be a Delone set, $0 \leq u \in L_0^\infty(\mathbb{R}^d)$, $u \geq \alpha > 0$ on some nonempty open set and let $u_t(x) := tu(x)$. Then $V_\omega(x) = \sum_{j \in \mathcal{D}} \omega_j u(x - j)$ is a sum of

copies of u at all lattice sites $j \in \mathcal{D}$, multiplied with ω_j . See [17] and the references therein for background on such models.

For $L > 0$ we denote by $H_{\omega,L}$ the restriction of H_ω to $L^2(\Lambda_L)$ with Dirichlet boundary conditions. Following the methods developed in [20], we obtain a Wegner estimate under our general assumption (4).

Theorem 2.8 (Wegner estimate). *For all $b \in \mathbb{R}$ there are constants $C, \kappa, \varepsilon_{\max} > 0$, depending only on $d, b, K_u, G_u, G_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \omega_+$ and $\|\nu_\mu\|_\infty$, such that for all $L \in (G_2 + G_u)\mathbb{N}$, all $E \in \mathbb{R}$ and all $\varepsilon \leq \varepsilon_{\max}$ with $[E - \varepsilon, E + \varepsilon] \subset (-\infty, b]$ we have*

$$\mathbb{E} [\text{Tr} [\chi_{(-\infty, b]}(H_{\omega, L})]] \leq C \varepsilon^{1/\kappa} |\ln \varepsilon|^d L^d. \quad (5)$$

Theorem 2.9 (Initial scale estimate). *Let κ be as in Theorem 2.8 for $b = d\pi^2 + K_u$. Assume that there are $t_0, C > 0$ such that*

$$0 \in \text{supp } \mu \quad \text{and} \quad \forall t \in [0, t_0]: \quad \mu([0, t]) \leq Ct^{d\kappa}.$$

Then there is $L_0 = L_0(t_0, \delta_{\max}, \kappa, G_u, G_1) \geq 1$ such that for all $L \in (G_2 + G_u)\mathbb{N}$, $L \geq L_0$ we have

$$\mathbb{P} \left(\left\{ \omega \in \Omega: \lambda_1(H_{\omega, L}) - \lambda_1(H_{0, L}) \geq \frac{1}{L^{3/2}} \right\} \right) \geq 1 - \frac{C}{L^{d/2}},$$

where $H_{0, L}$ is obtained from $H_{\omega, L}$ by setting ω_j to zero for all $j \in \mathcal{D}$.

Remark 2.10 (Discussion on initial scale estimate). Theorem 2.9 may serve as an initial scale estimate for a proof of localization via multi-scale analysis. More precisely, by using the Combes-Thomas estimate, an initial scale estimate in some neighbourhood of $a := \inf \sigma(H_0)$ follows. Note that the exponents $3/2$ and $d/2$ in Theorem 2.9 can be modified to some extent by adapting the proof and the assumption on the measure μ . Localization in a neighbourhood of a follows via multi-scale analysis, e.g., à la [42]. The question whether $\sigma(H_\omega) \cap I_a \neq \emptyset$ for almost all $\omega \in \Omega$ has to be settled. This is, however, satisfied for all examples mentioned above. In the special case of the standard random breather model one can get rid of the assumption on μ by proving and using the Lifshitz tail behaviour of the integrated density of states, cf. [46] for the one-dimensional case, and the forthcoming [41] for the multidimensional one.

2.4 Application to control theory

We consider the controlled heat equation

$$\begin{cases} \partial_t u - \Delta u + Vu = f\chi_\omega, & u \in L^2([0, T] \times \Omega), \\ u = 0, & \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0, & u_0 \in L^2(\Omega), \end{cases} \quad (6)$$

where ω is an open subset of the connected $\Omega \subset \mathbb{R}^d$, $T > 0$ and $V \in L^\infty(\Omega)$. In (6) u is the state and f is the control function which acts on the system through the control set ω .

Definition 2.11. For initial data $u_0 \in L^2(\Omega)$ and time $T > 0$, the set of reachable states $R(T, u_0)$ is

$$R(T, u_0) = \{u(T, \cdot): \text{there exists } f \in L^2([0, T] \times \omega) \text{ such that } u \text{ is solution of (6)}\}.$$

The system (6) is called null controllable at time T if $0 \in R(T; u_0)$ for all $u_0 \in L^2(\Omega)$. The controllability cost $\mathcal{C}(T, u_0)$ at time T for the initial state u_0 is

$$\mathcal{C}(T, u_0) = \inf \{ \|f\|_{L^2([0, T] \times \omega)} : u \text{ is solution of (6) and } u(T, \cdot) = 0 \}.$$

Since the system is linear, null controllability implies that the range of the semigroup generated by the heat equation is reachable too. It is well known that null controllability holds for any time $T > 0$, connected Ω and any nonempty and open set $\omega \subset \Omega$ on which the control acts, see [13].

It is also known, see for instance [48, Theorem 11.2.1], that null controllability of the system (6) at time T is equivalent to final state observability on the set ω at time T of the following system:

$$\begin{cases} \partial_t u - \Delta u + Vu = 0, & u \in L^2([0, T] \times \Omega), \\ u = 0, & \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0, & u_0 \in L^2(\Omega). \end{cases} \quad (7)$$

Definition 2.12. The system (7) is called final state observable on the set ω at time T if there exists $\kappa_T = \kappa_T(\Omega, \omega, V)$ such that for every initial state $u_0 \in L^2(\Omega)$ the solution $u \in L^2([0, T] \times \Omega)$ of (7) satisfies

$$\|u(T, \cdot)\|_{\Lambda_L}^2 \leq \kappa_T \|u\|_{L^2([0, T] \times \omega)}^2. \quad (8)$$

Moreover, the controllability cost $\mathcal{C}(T, u_0)$ of (6) coincides with the infimum over all observability costs $\sqrt{\kappa_T}$ in (8) times $\|u_0\|_{\Lambda_L}$ (see, for example, the proof of [48, Theorem 11.2.1]).

The problem of obtaining explicit bounds on $\mathcal{C}(T, u_0)$ received much consideration in the literature (see, for example, [18, 12, 37, 44, 33, 32, 34, 10, 31]), especially the case of small time, i.e. when T goes to zero. The dependencies of the controllability cost on T and $\|V\|_\infty$ are today well understood, see, for example [50]. However, the dependence on the geometry of the control set is less clear: in the known estimates the geometry enters only in terms of the distance to the boundary or in terms of the geometrical optics condition. To find an optimal control set is a very difficult problem, see for instance the recent articles [39, 38].

We are interested in the situation $\Omega = \Lambda_L \subset \mathbb{R}^d$ and $\omega = W_\delta(L)$ for a (G, δ) -equidistributed sequence with $L \in G\mathbb{N}$, $G > 0$ and $\delta < G/2$. In this specific setting we will give an estimate on the controllability cost. The novelty of our result is that the observability cost is independent of the scale L and the specific choice of the (G, δ) -equidistributed sequence. Moreover, the dependencies on $\|V\|_\infty$ and on the size of the control set via δ are known explicitly. As far as we are aware, this is the first time that such a scale-free estimate is obtained.

By the equivalence between null-controllability and final state observability, it is sufficient to construct an estimate of the form (8). In order to find such an estimate, we will combine Corollary 2.3 with results from [34] to obtain the following theorem.

Theorem 2.13. *For every $G > 0$, $\delta \in (0, G/2)$ and $K_V \geq 0$ there is $T' = T'(G, \delta, K_V) > 0$ such that for all $T \in (0, T')$, all (G, δ) -equidistributed sequences, all measurable and bounded $V : \mathbb{R}^d \rightarrow \mathbb{R}^d$ with $\|V\|_\infty \leq K_V$ and all $L \in G\mathbb{N}$, the system*

$$\begin{cases} \partial_t u - \Delta_L u + V_L u = 0, & u \in L^2([0, T] \times \Lambda_L), \\ u = 0, & \text{on } (0, T) \times \partial\Lambda_L, \\ u(0, \cdot) = u_0, & u_0 \in L^2(\Lambda_L). \end{cases}$$

is final state observable on the set $W_\delta(L)$ with cost κ_T satisfying

$$\kappa_T \leq 4a_0b_0e^{2c_*/T},$$

where $a_0 = (\delta/G)^{-N(1+G^{4/3}\|V\|_\infty^{2/3})}$, $b_0 = e^{2\|V\|_\infty}$, $c_* \leq \ln(G/\delta)^2 (NG + 4/\ln 2)^2$ and $N = N(d)$ is the constant from Theorem 2.2.

Remark 2.14. The same result holds also in the case of the controlled heat equation with periodic or Neumann boundary conditions with obvious modifications.

Remark 2.15. Null controllability of the heat equation implies a stronger type of controllability, so-called approximate controllability. Following [12], one can find an estimate for the cost of approximate controllability from the proof of Theorem 2.13. We will not pursue it in this paper.

3 Proof of scale-free unique continuation principle

3.1 Carleman inequalities

We denote by $\mathbb{R}_+^{d+1} := \{x \in \mathbb{R}^{d+1} : x_{d+1} \geq 0\}$ the $d+1$ -dimensional half-space and by $B_r^+ := \{x \in \mathbb{R}_+^{d+1} : |x| < r\}$ the $d+1$ -dimensional half-ball. For $x \in \mathbb{R}^{d+1}$ we denote by x' the projection on the first d coordinates, i.e. for $x = (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1}$ we use the notation $x' = (x_1, \dots, x_d) \in \mathbb{R}^d$. By $|x|$ and $|x'|$ we denote the Euclidean norms and by Δ the Laplacian on \mathbb{R}^{d+1} . For functions $f \in C^\infty(\mathbb{R}_+^{d+1})$ we use the notation $f_0 = f|_{x_{d+1}=0}$.

In the appendix of [30] Lebeau and Robbiano state a Carleman estimate for complex-valued functions with support in B_r^+ by using a real-valued weight function $\psi \in C^\infty(\mathbb{R}^{d+1})$ satisfying the two conditions

$$\forall x \in B_r^+ : \quad (\partial_{d+1}\psi)(x) \neq 0, \quad (9)$$

and for all $\xi \in \mathbb{R}^{d+1}$ and $x \in B_r^+$ there holds

$$\left. \begin{array}{l} 2\langle \xi, \nabla\psi \rangle = 0 \\ |\xi|^2 = |\nabla\psi|^2 \end{array} \right\} \Rightarrow \sum_{j,k=1}^{d+1} (\partial_{jk}\psi)(\xi_j\xi_k + (\partial_j\psi)(\partial_k\psi)) > 0. \quad (10)$$

As proposed in [21] we choose $r < 2 - \sqrt{2}$ and the special weight function $\psi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$,

$$\psi(x) = -x_{d+1} + \frac{x_{d+1}^2}{2} - \frac{|x'|^2}{4}. \quad (11)$$

Note that $\psi(x) \leq 0$ for all $x \in B_2^+$. This function ψ indeed satisfies the assumptions (9) and (10). Condition (9) is trivial for $r < 1$. In order to show the implication (10) we show

$$|\xi|^2 = |\nabla\psi|^2 \Rightarrow \sum_{j,k=1}^{d+1} \partial_{jk}\psi(\xi_j\xi_k + \partial_j\psi\partial_k\psi) > 0. \quad (12)$$

We use the hypothesis of (12) and calculate

$$\begin{aligned} \sum_{j,k=1}^{d+1} \partial_{jk}\psi(\xi_j\xi_k + \partial_j\psi\partial_k\psi) &= -\frac{1}{2} \sum_{i=1}^d \xi_i^2 + \xi_{d+1}^2 - \frac{1}{8}|x'|^2 + (x_{d+1} - 1)^2 \\ &= \frac{3}{2}\xi_{d+1}^2 - \frac{1}{4}|x'|^2 + \frac{1}{2}(x_{d+1} - 1)^2. \end{aligned}$$

Since $|x'|^2 \leq r^2$ and $(x_{d+1} - 1)^2 \geq (1 - r)^2$, assumption (12) is satisfied if $r < 2 - \sqrt{2}$. Now let

$$C_{c,0}^\infty(B_r^+) = \left\{ g : \mathbb{R}_+^{d+1} \rightarrow \mathbb{C} : g \equiv 0 \text{ on } \{x_{d+1} = 0\}, \right. \\ \left. \exists \psi \in C^\infty(\mathbb{R}^{d+1}) \text{ with } \text{supp } \psi \subset \{x \in \mathbb{R}^{d+1} : |x| < r\} \text{ and } g \equiv \psi \text{ on } \mathbb{R}_+^{d+1} \right\}.$$

Hence, as a corollary of Proposition 1 in the appendix of [30] we have

Proposition 3.1. *Let $\psi \in C^\infty(\mathbb{R}^{d+1}; \mathbb{R})$ be as in Eq. (11) and $\rho \in (0, 2 - \sqrt{2})$. Then there are constants $\beta_0, C_1 \geq 1$ such that for all $\beta \geq \beta_0$, and all $g \in C_{c,0}^\infty(B_\rho^+)$ we have*

$$\int_{\mathbb{R}^{d+1}} e^{2\beta\psi} (\beta|\nabla g|^2 + \beta^3|g|^2) \leq C_1 \left(\int_{\mathbb{R}^{d+1}} e^{2\beta\psi} |\Delta g|^2 + \beta \int_{\mathbb{R}^d} e^{2\beta\psi_0} |(\partial_{d+1}g)_0|^2 \right).$$

We will also need the following Carleman estimate.

Proposition 3.2. *Let $\rho > 0$. Then there are constants $\alpha_0, C_2 \geq 1$ depending only on the dimension and a function $w = \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying*

$$\forall x \in B(\rho) : \frac{|x|}{\rho e} \leq w(x) \leq \frac{|x|}{\rho},$$

such that for all $\alpha \geq \alpha_0$, and all $u \in W^{2,2}(\mathbb{R}^d)$ with support in $B(\rho) \setminus \{0\}$ we have

$$\int_{\mathbb{R}^d} (\alpha\rho^2 w^{1-2\alpha} |\nabla u|^2 + \alpha^3 w^{-1-2\alpha} |u|^2) dx \leq C_2 \rho^4 \int_{\mathbb{R}^d} w^{2-2\alpha} |\Delta u|^2 dx.$$

Proposition 3.2 is a special case of the result obtained in [35] where general second order elliptic partial differential operators with Lipschitz continuous coefficients are considered. The estimate has been previously obtained; (1) in [3], but there without the gradient term on the left hand side; (2) in [11], but there without a quantitative statement of the admissible functions u . These weaker versions are not sufficient for our purposes. In Appendix A we sketch for reader acquainted with the proof of [3] the difference between the two results.

3.2 Extension to larger boxes

For each measurable and bounded $V : \mathbb{R}^d \rightarrow \mathbb{R}$ and each $L \in \mathbb{N}$ we denote the eigenvalues of the corresponding operator H_L by E_k , $k \in \mathbb{N}$, enumerated in increasing order and counting multiplicities, and fix a corresponding sequence ϕ_k , $k \in \mathbb{N}$, of normalized eigenfunctions. Note that we suppress the dependence of E_k and ϕ_k on V and L .

Given V and L we define an extension of the potential V_L and the eigenfunctions ϕ_k to the set Λ_{RL} for some $R \in \mathbb{N}_{\text{odd}} = \{1, 3, 5, \dots\}$ to be chosen later on. The extension will depend on the type of boundary conditions we are considering for the Laplace operator.

Extension for periodic boundary conditions: We extend the potential V_L as well as the function ϕ_k , defined on the box Λ_L , periodically to $\tilde{V}, \tilde{\psi} : \mathbb{R}^d \rightarrow \mathbb{R}$ and then restrict them to Λ_{RL} . By the very definition of the operator domain of Δ_{Λ_L} with periodic boundary conditions the extension $\tilde{\psi}$ is locally in the Sobolev space $W^{2,2}(\mathbb{R}^d)$.

Extension for Dirichlet and Neumann boundary conditions: The potential V_L will be extended by symmetric reflections with respect to the hypersurfaces forming the boundaries of Λ_L . In the first step we extend $V_L : \Lambda_L \rightarrow \mathbb{R}$ to the set $H_L = \{x \in \Lambda_{3L} : x_i \in (-L/2, L/2), i \in \{2, \dots, d\}\}$ by

$$V_L(x) = \begin{cases} V_L(x) & \text{if } x \in \Lambda_L, \\ 0 & \text{if } x_1 \in \{-L/2, L/2\}, \\ V_L(L - x_1, x_2, \dots, x_d) & \text{if } x_1 > L/2, \\ V_L(-L - x_1, x_2, \dots, x_d) & \text{if } x_1 < -L/2. \end{cases}$$

Now we iteratively extend V_L in the remaining $d - 1$ directions using the same procedure and obtain a function $V_L : \Lambda_{3L} \rightarrow \mathbb{R}$. Iterating this procedure we obtain a function $V_L : \Lambda_{RL} \rightarrow \mathbb{R}$. The extensions of the eigenfunctions will depend on the boundary conditions. In the case of Dirichlet boundary conditions, we extend an eigenfunction similarly to the potential by antisymmetric reflections, while in the case of Neumann boundary conditions, we extend by symmetric reflections.

The extensions of the functions and V_L and ϕ_k , $k \in \mathbb{N}$, to the set Λ_{RL} will again be denoted by V_L and ϕ_k , $k \in \mathbb{N}$. The reader should be reminded that (the extended) $V_L : \Lambda_{RL} \rightarrow \mathbb{R}$ does in general not coincide with $V_{RL} : \Lambda_{RL} \rightarrow \mathbb{R}$. Note that for all three boundary conditions, $V_L : \Lambda_{RL} \rightarrow \mathbb{R}$ takes values in $[-\|V\|_\infty, \|V\|_\infty]$, the extended ϕ_k are elements of $W^{2,2}(\Lambda_{RL})$ with corresponding boundary conditions and they satisfy $\Delta\phi_k = (V_L - E_k)\phi_k$ on Λ_{RL} . Furthermore, the orthogonality relations remain valid.

3.3 Ghost dimension

For a measurable and bounded $V : \mathbb{R}^d \rightarrow \mathbb{R}$, $L \in \mathbb{N}$, $b \geq 0$ and $\phi \in \text{Ran}(\chi_{(-\infty, b]}(H_L))$ we have

$$\phi = \sum_{\substack{k \in \mathbb{N} \\ E_k \leq b}} \alpha_k \phi_k, \quad \text{with } \alpha_k = \langle \phi_k, \phi \rangle.$$

Since ϕ_k extend to Λ_{RL} as explained in Section 3.2, the function ϕ also extends to Λ_{RL} . We set $\omega_k := \sqrt{|E_k|}$ and define the function $F : \Lambda_{RL} \times \mathbb{R} \rightarrow \mathbb{C}$ by

$$F(x, x_{d+1}) = \sum_{\substack{k \in \mathbb{N} \\ E_k \leq b}} \alpha_k \phi_k(x) s_k(x_{d+1}),$$

where $s_k : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$s_k(t) = \begin{cases} \sinh(\omega_k t) / \omega_k, & E_k > 0, \\ x, & E_k = 0, \\ \sin(\omega_k t) / \omega_k, & E_k < 0. \end{cases}$$

Note that we suppress the dependence of ϕ and ϕ_k on V , L , b . Furthermore, the sums are finite since H_L is lower semibounded with purely discrete spectrum. The function F fulfills the handy relations

$$\Delta F = \sum_{i=1}^{d+1} \partial_i^2 F = V_L F \quad \text{on } \Lambda_{RL} \times \mathbb{R}$$

and

$$\partial_{d+1}F(x, 0) = \sum_{\substack{k \in \mathbb{N} \\ E_k \leq b}} \alpha_k \phi_k(x) \quad \text{for } x \in \Lambda_{RL}.$$

In particular, for all $x \in \Lambda_L$ we have $\partial_{d+1}F(x, 0) = \phi$. This way we recover the original function we are interested in.

Let us also fix the geometry. For $\delta \in (0, 1/2)$ we choose

$$\begin{aligned} \psi_1 &= -\delta^2/16, & \psi_2 &= -\delta^2/8, & \psi_3 &= -\delta^2/4, \\ r_1 &= \frac{1}{2} - \frac{1}{8}\sqrt{16 - \delta^2}, & r_2 &= 1, & r_3 &= 6e\sqrt{d}, \\ R_1 &= 1 - \frac{1}{4}\sqrt{16 - \delta^2}, & R_2 &= 3\sqrt{d}, & R_3 &= 9e\sqrt{d}, \end{aligned}$$

and define for $i \in \{1, 2, 3\}$ the sets

$$S_i := \{x \in \mathbb{R}^{d+1} : \psi(x) > \psi_i, x_{d+1} \in [0, 1]\} \subset \mathbb{R}_+^{d+1}$$

and

$$V_i := B(R_i) \setminus \overline{B(r_i)} \subset \mathbb{R}^{d+1}.$$

We also fix R to be the least odd integer larger than $2R_3 + 2$. For $i \in \{1, 2, 3\}$ and $x \in \mathbb{R}^d$ we denote by $S_i(x) = S_i + (x, 0)$ and $V_i(x) = V_i + (x, 0)$ the translates of the sets $S_i \subset \mathbb{R}^{d+1}$ and $V_i \subset \mathbb{R}^{d+1}$. Moreover, for $L \in \mathbb{N}$ and an $(1, \delta)$ -equidistributed sequence $z_j \in \mathbb{R}^d$, $j \in \mathbb{Z}^d$, we define $Q_L = \mathbb{Z}^d \cap \Lambda_L$, $U_i(L) = \cup_{j \in Q_L} S_i(z_j)$, $X_1 = \Lambda_L \times [-1, 1]$ and $\tilde{X}_{R_3} = \Lambda_{L+2R_3} \times [-R_3, R_3]$. Note that $W_\delta(L)$ is a disjoint union. In the following lemma we collect some consequences of our geometric setting. We will first restrict our attention to the case $L \in \mathbb{N}_{\text{odd}}$, and consider the case of even integers thereafter.

Lemma 3.3. (i) For all $\delta \in (0, 1/2)$ we have $S_1 \subset S_2 \subset S_3 \subset B_\delta^+ \subset \mathbb{R}_+^{d+1}$.

(ii) For all $L \in \mathbb{N}_{\text{odd}}$ with $L \geq 5$, all $\delta \in (0, 1/2)$ and all $(1, \delta)$ -equidistributed sequences z_j we have $\cup_{j \in Q_L} V_2(z_j) \supset X_1$.

(iii) There is a constant K_d , depending only on d , such that for all $L \in \mathbb{N}_{\text{odd}}$, all $\delta \in (0, 1/2)$, all $(1, \delta)$ -equidistributed sequences z_j , all measurable and bounded $V : \mathbb{R}^d \rightarrow \mathbb{R}$, all $b \geq 0$ and all $\phi \in \text{Ran}(\chi_{(-\infty, b]}(H_L))$ we have

$$\sum_{j \in Q_L} \|F\|_{H^1(V_3(z_j))}^2 \leq K_d \|F\|_{H^1(\cup_{j \in Q_L} V_3(z_j))}^2.$$

(iv) For all $L \in \mathbb{N}_{\text{odd}}$, $\delta \in (0, 1/2)$ and all $(1, \delta)$ -equidistributed sequences z_j we have $\cup_{j \in Q_L} V_3(z_j) \subset \tilde{X}_{R_3}$.

We note that part (ii) of Lemma 3.3 will be applied with L replaced by $5L$.

Proof. Parts (i) and (iv) are obvious.

To show (ii), we first prove that $[-1/2, 1/2]^d \times [-1, 1]$ can be covered by the sets $V_2(z_j)$. Let us take $j_1 = (-1, 0, \dots, 0)$, $j_2 = (-2, 0, \dots, 0)$, $j_1, j_2 \in Q_L$. Then

$$[-1/2, 1/2]^d \times [-1, 1] \subset V_2(z_{j_1}) \cup V_2(z_{j_2}), \quad (13)$$

cf. Fig. 1. Indeed, let $x = (x_1, \dots, x_{d+1})$ be an arbitrary point from $[-1/2, 1/2]^d \times [-1, 1]$. Then (13) is not satisfied only if $|(z_{j_1}, 0) - x|^2 < 1$ and $|(z_{j_2}, 0) - x|^2 > R_2^2$. Since $z_{j_1} \in (-3/2 + \delta, -1/2 - \delta) \times (-1/2 + \delta, 1/2 - \delta)^{d-1}$ and $z_{j_2} \in (-5/2 + \delta, -3/2 - \delta) \times (-1/2 + \delta, 1/2 - \delta)^{d-1}$, it follows

$$(-1/2 - \delta - x_1)^2 + x_{d+1}^2 < 1 \quad \text{and} \quad (-5/2 + \delta - x_1)^2 + (d-1)(1-\delta)^2 + x_{d+1}^2 > 9d.$$

Plugging the first relation into the second, we obtain

$$9d < (d-1)(1-\delta)^2 + 2(1-\delta)(3+2x_1) + 1 \leq (d-1)(1-\delta)^2 + 8(1-\delta) + 1.$$

But this relation is satisfied only for $d < 1$. Since $L \geq 5$ the same argument applies to cover every elementary cell $([-1/2, 1/2] + i) \times [-1, 1]$, $i \in Q_L$, by two neighboring sets $V_2(z_j)$.

Now we turn to the proof of (iii). Since $R \geq 2R_3 + 2$ the function F is defined on $V_3(z_j)$ for all $j \in Q_L$. For all $x \in \cup_{j \in Q_L} V_3(z_j)$, the number of indices $j \in Q_L$ such that $V_3(z_j) \ni x$ is bounded from above by $(2R_3 + 2)^d$. Hence,

$$\forall x \in \tilde{X}_{R_3}: \quad \sum_{j \in Q_L} \chi_{V_3(z_j)}(x) \leq (2R_3 + 2)^d \chi_{\cup_{j \in Q_L} V_3(z_j)}(x)$$

and thus

$$\begin{aligned} \sum_{j \in Q_L} \|F\|_{H^1(V_3(z_j))}^2 &= \int_{\tilde{X}_{R_3}} \left(\sum_{j \in Q_L} \chi_{V_3(z_j)}(x) \right) (|F(x)|^2 + |\nabla F(x)|^2) dx \\ &\leq (2R_3 + 2)^d \|F\|_{H^1(\cup_{j \in Q_L} V_3(z_j))}^2. \end{aligned}$$

Hence we can take $K_d = (2R_3 + 2)^d$. □

3.4 Interpolation inequalities

Proposition 3.4. *For all $\delta \in (0, 1/2)$, all $(1, \delta)$ -equidistributed sequences z_j , all measurable and bounded $V : \mathbb{R}^d \rightarrow \mathbb{R}$, all $L \in \mathbb{N}_{\text{odd}}$, all $b \geq 0$ and all $\phi \in \text{Ran}(\chi_{(-\infty, b]}(H_L))$*

(a) *there is $\beta_1 = \beta_1(d, \|V\|_\infty) \geq 1$ such that for all $\beta \geq \beta_1$ we have*

$$\|F\|_{H^1(U_1(L))}^2 \leq \tilde{D}_1(\beta) \|F\|_{H^1(U_3(L))}^2 + \hat{D}_1(\beta) \|(\partial_{d+1} F)_0\|_{L^2(W_\delta(L))}^2,$$

where β_1 is given in Eq. (15), and $\tilde{D}_1(\beta)$ and $\hat{D}_1(\beta)$ are given in Eq. (16).

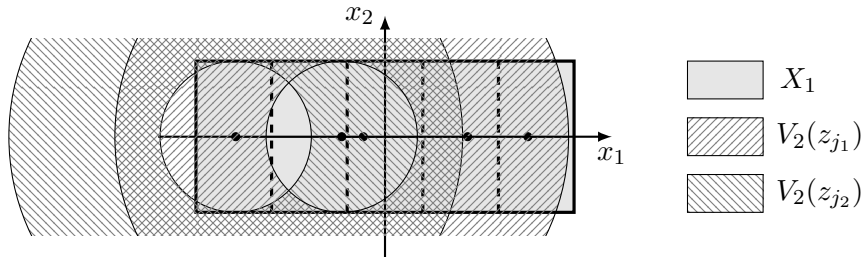


Figure 1: Illustration for (ii) in case $d = 1$, $L = 5$ and some configuration z_j , $j \in Q_L$. The set $[-1/2, 1/2] \times [-1, 1]$ is covered by $V_2(z_{j_1})$ and $V_2(z_{j_2})$.

(b) we have

$$\|F\|_{H^1(U_1(L))} \leq D_1 \|(\partial_{d+1}F)_0\|_{L^2(W_\delta(L))}^{1/2} \|F\|_{H^1(U_3(L))}^{1/2},$$

where D_1 is given in Eq. (20).

Proof. First we recall that $\Delta F = V_L F$, $\partial_{d+1}F(x', 0) = \phi(x')$ and $B_\delta^+ \supset S_3$. Now we choose a cutoff function $\chi \in C^\infty(\mathbb{R}^{d+1}; [0, 1])$ with $\text{supp } \chi \subset \overline{S_3}$, $\chi(x) = 1$ if $x \in S_2$ and

$$\max\{\|\Delta\chi\|_\infty, \|\nabla\chi\|_\infty\} \leq \frac{\tilde{\Theta}_1}{\delta^4} =: \Theta_1,$$

where $\tilde{\Theta}_1 = \tilde{\Theta}_1(d)$ depends only on the dimension, see Appendix B. Let φ be a non-negative function in $C_c^\infty(\mathbb{R}^d)$ with the properties that $\|\varphi\|_1 = 1$ and $\text{supp } \varphi \subset B(1)$. For $\varepsilon > 0$ we define $\varphi_\varepsilon : \mathbb{R}^d \rightarrow \mathbb{R}_0^+$ by $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$. The function φ_ε belongs to $C_c^\infty(\mathbb{R}^d)$ and satisfies $\text{supp } \varphi_\varepsilon \subset (\varepsilon)$. Now we continuously extend the eigenfunctions $\phi_k : \Lambda_{RL} \rightarrow \mathbb{R}$ to the set \mathbb{R}^d by zero and define for $\varepsilon > 0$ the function $F_\varepsilon : \mathbb{R}^d \times \mathbb{R}$ by

$$F_\varepsilon(x, x_{d+1}) = \sum_{\substack{k \in \mathbb{N} \\ E_k \leq b}} \alpha_k (\varphi_\varepsilon * \phi_k)(x) s_k(x_{d+1}).$$

By construction, the function $g = \chi F_\varepsilon$ is an element of $C_{c,0}^\infty(B_\delta^+)$. Hence, we can apply Proposition 3.1 with $g = \chi F_\varepsilon$ and $\rho = 1/2$ and obtain for all $\beta \geq \beta_0 \geq 1$

$$\int_{S_3} e^{2\beta\psi} (\beta|\nabla(\chi F_\varepsilon)|^2 + \beta^3|\chi F_\varepsilon|^2) \leq C_1 \int_{S_3} e^{2\beta\psi} |\Delta(\chi F_\varepsilon)|^2 + \beta C_1 \int_{B(\delta)} e^{2\beta\psi_0} |(\partial_{d+1}(\chi F_\varepsilon))_0|^2. \quad (14)$$

Note that β_0 and C_1 only depend on the dimension. By [49, Theorem 1.6.1 (iii)] we have $\varphi_\varepsilon * \phi_k \rightarrow \phi_k$, $\nabla(\varphi_\varepsilon * \phi_k) \rightarrow \nabla\phi_k$ and $\Delta(\varphi_\varepsilon * \phi_k) \rightarrow \Delta\phi_k$ in $L^2(S_3)$ as ε tends to zero. Consequently, the same holds for F_ε , ∇F_ε and ΔF_ε and thus we obtain Ineq. (14) with F_ε replaced by F . For the first summand on the right hand side we have the upper bound

$$\begin{aligned} \int_{S_3} e^{2\beta\psi} |\Delta(\chi F)|^2 &\leq 3 \int_{S_3} e^{2\beta\psi} (4|\nabla\chi|^2 |\nabla F|^2 + |\Delta\chi|^2 |F|^2 + |\Delta F|^2 |\chi|^2) \\ &\leq 3e^{2\beta\psi_2} \int_{S_3 \setminus S_2} (4\Theta_1^2 |\nabla F|^2 + \Theta_1^2 |F|^2) + \int_{S_3} 3e^{2\beta\psi} |V_L F \chi|^2 \\ &\leq 12\Theta_1^2 e^{2\beta\psi_2} \|F\|_{H^1(S_3)}^2 + 3\|V\|_\infty^2 \int_{S_3} e^{2\beta\psi} |\chi F|^2. \end{aligned}$$

The second summand is bounded from above by $\beta C_1 \int_{B(\delta)} |(\partial_{d+1}F)_0|^2$, since $F = 0$ and $\psi \leq 0$ on $\{x_{d+1} = 0\}$. Hence,

$$\begin{aligned} \beta \int_{S_3} e^{2\beta\psi} |\nabla(\chi F)|^2 + (\beta^3 - 3\|V\|_\infty^2 C_1) \int_{S_3} e^{2\beta\psi} |\chi F|^2 \\ \leq 12C_1 \Theta_1^2 e^{2\beta\psi_2} \|F\|_{H^1(S_3)}^2 + C_1 \beta \|(\partial_{d+1}F)_0\|_{L^2(B(\delta))}^2. \end{aligned}$$

Additionally to $\beta \geq \beta_0$ we choose $\beta \geq (6\|V\|_\infty^2 C_1)^{1/3} =: \tilde{\beta}_0$. This ensures that for all

$$\beta \geq \beta_1 := \max\{\beta_0, \tilde{\beta}_0\} \quad (15)$$

we have

$$\frac{1}{2} \int_{S_3} e^{2\beta\psi} (\beta |\nabla(\chi F)|^2 + \beta^3 |\chi F|^2) \leq 12C_1 \Theta_1^2 e^{2\beta\psi_2} \|F\|_{H^1(S_3)}^2 + C_1 \beta \|(\partial_{d+1} F)_0\|_{L^2(B(\delta))}^2.$$

Since $\beta \geq 1$, $S_3 \supset S_1$, $\chi = 1$ and $e^{2\beta\psi} \geq e^{2\beta\psi_1}$ on S_1 , we obtain

$$e^{2\beta\psi_1} \|F\|_{H^1(S_1)}^2 \leq 24C_1 \Theta_1^2 e^{2\beta\psi_2} \|F\|_{H^1(S_3)}^2 + 2C_1 \|(\partial_{d+1} F)_0\|_{L^2(B(\delta))}^2.$$

We apply this inequality for translates $S_i(z_j)$ and obtain by summing over $j \in Q_L = \mathbb{Z}^d \cap \Lambda_L$

$$e^{2\beta\psi_1} \sum_{j \in Q_L} \|F\|_{H^1(S_1(z_j))}^2 \leq 24C_1 \Theta_1^2 e^{2\beta\psi_2} \sum_{j \in Q_L} \|F\|_{H^1(S_3(z_j))}^2 + 2C_1 \sum_{j \in Q_L} \|(\partial_{d+1} F)_0\|_{L^2(B(z_j, \delta))}^2.$$

Recall that $U_i(L) = \cup_{j \in Q_L} S_i(z_j)$ and $W_\delta(L) = \cup_{j \in Q_L} B(z_j, \delta)$. Hence, for all $\beta \geq \beta_1$ we have

$$\|F\|_{H^1(U_1(L))}^2 \leq \tilde{D}_1 \|F\|_{H^1(U_3(L))}^2 + \hat{D}_1 \|(\partial_{d+1} F)_0\|_{L^2(W_\delta(L))}^2,$$

where

$$\tilde{D}_1(\beta) = 24C_1 \Theta_1^2 e^{2\beta(\psi_2 - \psi_1)} \quad \text{and} \quad \hat{D}_1(\beta) = 2C_1 e^{-2\beta\psi_1}. \quad (16)$$

We choose β such that

$$e^\beta = \left[\frac{1}{12\Theta_1^2} \frac{\|(\partial_{d+1} F)_0\|_{L^2(W_\delta(L))}^2}{\|F\|_{H^1(U_3(L))}^2} \right]^{\frac{1}{2\psi_2}}. \quad (17)$$

Now we distinguish two cases. If $\beta \geq \beta_1$ we obtain by using $\psi_1 = 2\psi_2$

$$\|F\|_{H^1(U_1(L))}^2 \leq 8\sqrt{3}C_1 \Theta_1 \|F\|_{H^1(U_3(L))} \|(\partial_{d+1} F)_0\|_{L^2(W_\delta(L))}. \quad (18)$$

If $\beta < \beta_1$ we use Lemma 5.2 of [29]. In particular, one concludes from Eq. (17) that

$$\|F\|_{H^1(U_3(L))}^2 < \frac{1}{12\Theta_1^2} e^{-2\beta_1\psi_2} \|(\partial_{d+1} F)_0\|_{L^2(W_\delta(L))}^2.$$

This gives us in the case $\beta < \beta_1$

$$\|F\|_{H^1(U_1(L))}^2 \leq \|F\|_{H^1(U_3(L))}^2 < \frac{e^{-\beta_1\psi_2}}{\sqrt{12}\Theta_1} \|F\|_{H^1(U_3(L))} \|(\partial_{d+1} F)_0\|_{L^2(W_\delta(L))}. \quad (19)$$

If we set

$$D_1^2 = \max \left\{ 8\sqrt{3}C_1 \Theta_1, \frac{e^{-\beta_1\psi_2}}{\Theta_1 \sqrt{12}} \right\}, \quad (20)$$

we conclude the statement of the proposition from Ineqs. (18) and (19). \square

Now we deduce from the second Carleman estimate, Proposition 3.2, another interpolation inequality.

Proposition 3.5. *For all $\delta \in (0, 1/2)$, all $(1, \delta)$ -equidistributed sequences z_j , all measurable and bounded $V : \mathbb{R}^d \rightarrow \mathbb{R}$, all $L \in \mathbb{N}_{\text{odd}}$, all $b \geq 0$ and all $\phi \in \text{Ran}(\chi_{(-\infty, b]}(H_L))$*

(a) there is $\alpha_1 = \alpha_1(d, \|V\|_\infty) \geq 1$ such that for all $\alpha \geq \alpha_1$ we have

$$\|F\|_{H^1(X_1)}^2 \leq \tilde{D}_2(\alpha) \|F\|_{H^1(U_1(L))}^2 + \hat{D}_2(\alpha) \|F\|_{H^1(\tilde{X}_{R_3})},$$

where α_1 is given in Eq. (22), and $\tilde{D}_2(\alpha)$ and $\hat{D}_2(\alpha)$ are given in Eq. (26);

(b) we have

$$\|F\|_{H^1(X_1)} \leq D_2 \|F\|_{H^1(U_1(L))}^\gamma \|F\|_{H^1(\tilde{X}_{R_3})}^{1-\gamma},$$

where γ and D_2 are given in Eq. (31) and (32).

Proof. We choose a cutoff function $\chi \in C_c^\infty(\mathbb{R}^{d+1}; [0, 1])$ with $\text{supp } \chi \subset B(R_3) \setminus \overline{B(r_1)}$, $\chi(x) = 1$ if $x \in B(r_3) \setminus \overline{B(R_1)}$,

$$\max\{\|\Delta\chi\|_{\infty, V_1}, \|\nabla\chi\|_{\infty, V_1}\} \leq \frac{\tilde{\Theta}_2}{\delta^4} =: \Theta_2$$

and

$$\max\{\|\Delta\chi\|_{\infty, V_3}, \|\nabla\chi\|_{\infty, V_3}\} \leq \Theta_3,$$

where $\tilde{\Theta}_2$ depends only on the dimension and Θ_3 is an absolute constant, see Appendix B. We set $u = \chi F$. We apply Proposition 3.2 with $\rho = R_3$ to the function u and obtain for all $\alpha \geq \alpha_0 \geq 1$

$$\int_{B(R_3)} (\alpha R_3^2 w^{1-2\alpha} |\nabla u|^2 + \alpha^3 w^{-1-2\alpha} |u|^2) dx \leq C_2 R_3^4 \int_{B(R_3)} w^{2-2\alpha} |\Delta u|^2 dx.$$

Since $w \leq 1$ on $B(R_3)$ we can replace the exponent of the weight function w at all three places by $2 - 2\alpha$, i.e.

$$\int_{B(R_3)} (\alpha R_3^2 w^{2-2\alpha} |\nabla u|^2 + \alpha^3 w^{2-2\alpha} |u|^2) dx \leq C_2 R_3^4 \int_{B(R_3)} w^{2-2\alpha} |\Delta u|^2 dx =: I. \quad (21)$$

For the right hand side we use

$$\Delta u = 2(\nabla\chi)(\nabla F) + (\Delta\chi)F + (\Delta F)\chi,$$

and $\Delta F = V_L F$, and obtain

$$I \leq 3C_2 R_3^4 \int_{B(R_3)} w^{2-2\alpha} (4|(\nabla\chi)(\nabla F)|^2 + |(\Delta\chi)F|^2 + \|V\|_\infty^2 |\chi F|^2) dx =: I_1 + I_2 + I_3.$$

If we choose α sufficiently large, i.e.

$$\alpha \geq (6C_2 R_3^4 \|V\|_\infty^2)^{1/3} =: \tilde{\alpha}_0,$$

we can subsume the term I_3 into the left hand side of Ineq. (21). We obtain for all

$$\alpha \geq \alpha_1 := \max\{\alpha_0, \tilde{\alpha}_0\} \quad (22)$$

the estimate

$$\int_{B(R_3)} \left(\alpha R_3^2 w^{2-2\alpha} |\nabla u|^2 + \frac{\alpha^3}{2} w^{2-2\alpha} |u|^2 \right) dx \leq I_1 + I_2.$$

For the “new” left hand side we have the lower bound

$$I_1 + I_2 \geq \int_{B(R_3)} \left(\alpha R_3^2 w^{2-2\alpha} |\nabla u|^2 + \frac{\alpha^3}{2} w^{2-2\alpha} |u|^2 \right) dx \geq \frac{1}{2} \left(\frac{R_3}{R_2} \right)^{2\alpha-2} \|F\|_{H^1(V_2)}^2.$$

For I_1 and I_2 we have the estimates

$$I_1 \leq 3C_2 R_3^4 \left[4\Theta_2^2 \left(\frac{eR_3}{r_1} \right)^{2\alpha-2} \int_{V_1} |\nabla F|^2 + 4\Theta_3^2 \left(\frac{eR_3}{r_3} \right)^{2\alpha-2} \int_{V_3} |\nabla F|^2 \right]$$

and

$$I_2 \leq 3C_2 R_3^4 \left[\Theta_2^2 \left(\frac{eR_3}{r_1} \right)^{2\alpha-2} \int_{V_1} |F|^2 + \Theta_3^2 \left(\frac{eR_3}{r_3} \right)^{2\alpha-2} \int_{V_3} |F|^2 \right].$$

Putting everything together, the Carleman estimate from Proposition 3.2 implies for $\alpha \geq \alpha_1$

$$\|F\|_{H^1(V_2)}^2 \leq 24C_2 R_3^4 \left[\Theta_2^2 \left(\frac{eR_2}{r_1} \right)^{2\alpha-2} \|F\|_{H^1(V_1)}^2 + \Theta_3^2 \left(\frac{eR_2}{r_3} \right)^{2\alpha-2} \|F\|_{H^1(V_3)}^2 \right]. \quad (23)$$

By translation, Ineq. (23) is still true if we replace V_1 , V_2 and V_3 by its translates $V_1(z_j)$, $V_2(z_j)$ and $V_3(z_j)$ for all $j \in Q_L$. Hence,

$$\sum_{j \in Q_L} \|F\|_{H^1(V_2(z_j))}^2 \leq 24C_2 R_3^4 \left[\Theta_2^2 \left(\frac{eR_2}{r_1} \right)^{2\alpha-2} \sum_{j \in Q_L} \|F\|_{H^1(V_1(z_j))}^2 + \Theta_3^2 \left(\frac{eR_2}{r_3} \right)^{2\alpha-2} \sum_{j \in Q_L} \|F\|_{H^1(V_3(z_j))}^2 \right]. \quad (24)$$

For all $L \in \mathbb{N}_{\text{odd}}$ Lemma 3.3 tells us that $\cup_{k \in Q_5} \cup_{j \in Q_L} V_2(z_j + kL) \supset X_1 = \Lambda_L \times [-1, 1]$ and the left hand side is bounded from below by

$$\sum_{j \in Q_L} \|F\|_{H^1(V_2(z_j))}^2 = \frac{1}{5^d} \sum_{k \in Q_5} \sum_{j \in Q_L} \|F\|_{H^1(V_2(z_j + kL))}^2 \geq \frac{1}{5^d} \|F\|_{H^1(X_1)}^2.$$

Since $V_1(z_j) \cap \mathbb{R}_+^{d+1} \subset S_1(z_j)$, $S_1(z_i) \cap S_1(z_j) = \emptyset$ for $i \neq j$, and since F is antisymmetric with respect to its last coordinate, we have

$$\sum_{j \in Q_L} \|F\|_{H^1(V_1(z_j))}^2 \leq 2 \sum_{j \in Q_L} \|F\|_{H^1(S_1(z_j))}^2 = 2 \|F\|_{H^1(U_1(L))}^2.$$

For the second summand on the right hand side of Ineq. (24), we find by Lemma 3.3 (iii) that there exists a constant K_d such that

$$\sum_{j \in Q_L} \|F\|_{H^1(V_3(z_j))}^2 \leq K_d \|F\|_{H^1(\cup_{j \in Q_L} V_3(z_j))}^2.$$

Moreover, since $\cup_{j \in Q_L} V_3(z_j) \subset \tilde{X}_{R_3} = \Lambda_{L+R_3} \times [-R_3, R_3]$, we have

$$\sum_{j \in Q_L} \|F\|_{H^1(V_3(z_j))}^2 \leq K_d \|F\|_{H^1(\tilde{X}_{R_3})}^2.$$

Putting everything together we obtain for all $\alpha \geq \alpha_1$

$$\frac{1}{5d} \|F\|_{H^1(X_1)}^2 \leq \tilde{D}_2(\alpha) \|F\|_{H^1(U_1(L))}^2 + \hat{D}_2(\alpha) \|F\|_{H^1(\tilde{X}_{R_3})}^2, \quad (25)$$

where

$$\tilde{D}_2(\alpha) = 48C_2R_3^4\Theta_2^2 \left(\frac{eR_2}{r_1}\right)^{2\alpha-2} \quad \text{and} \quad \hat{D}_2(\alpha) = 24C_2R_3^4\Theta_3^2K_d \left(\frac{eR_2}{r_3}\right)^{2\alpha-2}. \quad (26)$$

If we let $c_1 = 48C_2\Theta_2^2R_3^4r_1^2/(eR_2)^2$, $c_2 = 24C_2\Theta_3^2K_dR_3^4r_3^2/(eR_2)^2$,

$$p^+ = 2 \ln \left(\frac{eR_2}{r_1}\right) > 0 \quad \text{and} \quad p^- = 2 \ln \left(\frac{eR_2}{r_3}\right) < 0,$$

then Ineq. (25) reads

$$\frac{1}{5d} \|F\|_{H^1(X_1)}^2 \leq c_1 e^{p^+\alpha} \|F\|_{H^1(U_1(L))}^2 + c_2 e^{p^-\alpha} \|F\|_{H^1(\tilde{X}_{R_3})}^2. \quad (27)$$

We choose α such that

$$e^\alpha = \left(\frac{c_2 \|F\|_{H^1(\tilde{X}_{R_3})}^2}{c_1 \|F\|_{H^1(U_1(L))}^2} \right)^{\frac{1}{p^+ - p^-}}. \quad (28)$$

If $\alpha \geq \alpha_1$ we obtain from Ineq. (27) that

$$\frac{1}{5d} \|F\|_{H^1(X_1)}^2 \leq 2c_1^\gamma c_2^{1-\gamma} \|F\|_{H^1(U_1(L))}^{2\gamma} \|F\|_{H^1(\tilde{X}_{R_3})}^{2-2\gamma}, \quad \text{where} \quad \gamma = \frac{-p^-}{p^+ - p^-}. \quad (29)$$

If $\alpha < \alpha_1$, we proceed as in the last part of the proof of Proposition 3.4, i.e. we conclude from Eq. (28) that

$$\|F\|_{H^1(\tilde{X}_{R_3})}^2 < \frac{c_1}{c_2} e^{\alpha_1(p^+ - p^-)} \|F\|_{H^1(U_1(L))}^2$$

and thus

$$\|F\|_{H^1(X_1)}^2 \leq \|F\|_{H^1(\tilde{X}_{R_3})}^{\frac{2p^+ - p^-}{p^+ - p^-}} < \|F\|_{H^1(\tilde{X}_{R_3})}^{2(1-\gamma)} \left(\frac{c_1}{c_2} e^{\alpha_1(p^+ - p^-)} \right)^\gamma \|F\|_{H^1(U_1(L))}^{2\gamma}. \quad (30)$$

We calculate

$$\gamma = \frac{\ln 2}{\ln(r_3/r_1)}, \quad (31)$$

set

$$D_2^2 = \max \left\{ 5^d 192 \cdot 9^4 C_2 \Theta_3^2 K_d e^4 d^2 \left(\frac{2\Theta_2^2 r_1^2}{\Theta_3^2 K_d r_3^2} \right)^\gamma, \left(\frac{2\Theta_2^2}{\Theta_3^2 K_d} \left(\frac{r_3}{r_1} \right)^{2(\alpha_1-1)} \right)^\gamma \right\} \quad (32)$$

and conclude the statement of the proposition from Ineqs. (29) and (30). \square

3.5 Proof of Theorem 2.2 and Corollary 2.5

Proposition 3.6. *For all $T > 0$, all measurable and bounded $V : \mathbb{R}^d \rightarrow \mathbb{R}$, all $L \in \mathbb{N}_{\text{odd}}$, all $b \geq 0$ and all $\phi \in \text{Ran}(\chi_{(-\infty, b]}(H_L))$ we have*

$$\frac{T}{2} \sum_{\substack{k \in \mathbb{N} \\ E_k \leq b}} |\alpha_k|^2 \leq \frac{\|F\|_{H^1(\Lambda_{RL} \times [-T, T])}^2}{R^d} \leq 2T(1 + (1 + \|V\|_\infty)T^2) \sum_{\substack{k \in \mathbb{N} \\ E_k \leq b}} \beta_k(T) |\alpha_k|^2,$$

where

$$\beta_k(T) = \begin{cases} 1 & \text{if } E_k \leq 0, \\ e^{2T\sqrt{E_k}} & \text{if } E_k > 0. \end{cases}$$

Proof. For the function $F : \Lambda_{RL} \times \mathbb{R} \rightarrow \mathbb{C}$ we have for $T > 0$

$$\|F\|_{H^1(\Lambda_{RL} \times [-T, T])}^2 = \int_{-T}^T \int_{\Lambda_{RL}} (|\partial_{d+1} F|^2 + |\nabla' F|^2 + |F|^2) dx.$$

Note that $\|\phi_k\|_{L^2(\Lambda_{RL})} = R^d$. By Green's theorem we have

$$\int_{\Lambda_{RL}} |\nabla' F|^2 dx' = \int_{\Lambda_{RL}} \left(-\sum_{i=1}^d \partial_i^2 F\right) \bar{F} dx' = -\int_{\Lambda_{RL}} V|F|^2 dx' + \int_{\Lambda_{RL}} (\partial_{d+1}^2 F) \bar{F} dx'$$

for all $x_{d+1} \in \mathbb{R}$. First we estimate

$$\begin{aligned} \|F\|_{H^1(\Lambda_{RL} \times [-T, T])}^2 &= \int_{-T}^T \int_{\Lambda_{RL}} (|\partial_{d+1} F|^2 - V|F|^2 + (\partial_{d+1}^2 F) \bar{F} + |F|^2) dx \\ &\leq \int_{-T}^T \int_{\Lambda_{RL}} (|\partial_{d+1} F|^2 + (\partial_{d+1}^2 F) \bar{F} + (1 + \|V\|_\infty)|F|^2) dx \\ &= 2R^d \sum_{\substack{k \in \mathbb{N} \\ E_k \leq b}} |\alpha_k|^2 I_k, \end{aligned}$$

where

$$\begin{aligned} I_k &:= \int_0^T ((1 + \|V\|_\infty) s_k(x_{d+1})^2 + s_k'(x_{d+1})^2 + s_k''(x_{d+1}) s_k(x_{d+1})) dx_{d+1} \\ &= (1 + \|V\|_\infty) \int_0^T s_k(x_{d+1})^2 dx_{d+1} + s_k'(T) s_k(T). \end{aligned}$$

If $E_k \leq 0$, we estimate using $s_k(t) \leq t$ and $s_k'(t) s_k(t) \leq t$ for $t > 0$

$$I_k \leq (1 + \|V\|_\infty) T^3/3 + T \leq ((1 + \|V\|_\infty) T^3 + T) \beta_k(T).$$

For $E_k > 0$ we use $\sinh(\omega_k t)/\omega_k \leq t \cosh(\omega_k t)$ for $t > 0$ and $\cosh(\omega_k T)^2 \leq e^{2\omega_k T}$ to obtain

$$\begin{aligned} I_k &= (1 + \|V\|_\infty) \int_0^T \frac{\sinh^2(\omega_k x_{d+1})}{\omega_k^2} dx_{d+1} + \sinh(\omega_k T) \cosh(\omega_k T)/\omega_k \\ &\leq ((1 + \|V\|_\infty) T^3 \cosh^2(\omega_k T) + T \cosh^2(\omega_k T)) \leq ((1 + \|V\|_\infty) T^3 + T) \beta_k(T). \end{aligned}$$

This shows the upper bound. For the lower bound we drop the gradient term and obtain

$$\|F\|_{H^1(\Lambda_{RL} \times [-T, T])}^2 \geq \int_{-T}^T \int_{\Lambda_{RL}} (|\partial_{d+1} F|^2 + |F|^2) dx = 2 \cdot R^d \sum_{\substack{k \in \mathbb{N} \\ E_k \leq b}} |\alpha_k|^2 \tilde{I}_k,$$

where

$$\tilde{I}_k := \int_0^T [s_k(x_{d+1})^2 + s'_k(x_{d+1})^2] dx_{d+1}.$$

If $E_k = 0$, the lower bound $\tilde{I}_k \geq T$ follows immediately. Else, we have $s_k(t)^2 \geq \sin^2(\omega_k t)/\omega_k$ and $s'_k(t)^2 \geq \cos^2(\omega_k t)$ whence

$$\tilde{I}_k \geq \int_0^T \frac{\sin^2(\omega_k x_{d+1})}{\omega_k^2} + \cos^2(\omega_k x_{d+1}) dx_{d+1} \geq \int_0^T \cos^2(\omega_k x_{d+1}) dx_{d+1} = \frac{T}{2} + \frac{\sin(2\omega_k T)}{4\omega_k}.$$

Now, if $2\omega_k T < \pi$, the sinus term is positive and we drop it to find $\tilde{I}_k \geq T/2$. If $2\omega_k T \geq \pi$, we have $\sin(2\omega_k T) \geq -1$ and estimate

$$\tilde{I}_k \geq \frac{T}{2} - \frac{1}{4\omega_k} = \frac{T}{2} - \frac{\pi}{4\pi\omega_k} \geq \frac{T}{2} - \frac{T}{2\pi} \geq \frac{T}{4}. \quad \square$$

Proof of Theorem 2.2. First we consider the case $L \in \mathbb{N}_{\text{odd}}$. We note that Proposition 3.6 remains true if we replace Λ_{RL} by Λ_L and R^d by 1, i.e. for all $T > 0$ and $L \in \mathbb{N}_{\text{odd}}$ we have

$$\frac{T}{2} \sum_{\substack{k \in \mathbb{N} \\ E_k \leq b}} |\alpha_k|^2 \leq \|F\|_{H^1(\Lambda_L \times [-T, T])}^2 \leq 2T(1 + (1 + \|V\|_\infty)T^2) \sum_{\substack{k \in \mathbb{N} \\ E_k \leq b}} \beta_k(T) |\alpha_k|^2. \quad (33)$$

We have $\tilde{X}_{R_3} \subset \Lambda_{RL} \times [-R_3, R_3]$. By Ineq. (33) and Proposition 3.6 we have

$$\frac{\|F\|_{H^1(\tilde{X}_{R_3})}^2}{\|F\|_{H^1(X_1)}^2} \leq \frac{\|F\|_{H^1(\Lambda_{RL} \times [-R_3, R_3])}^2}{\|F\|_{H^1(X_1)}^2} \leq \tilde{D}_3^2 D_4^2$$

with

$$\tilde{D}_3^2 = \frac{\sum_{E_k \leq b} \theta_k |\alpha_k|^2}{\sum_{E_k \leq b} |\alpha_k|^2} \quad \text{and} \quad D_4^2 = 4 \cdot R^d R_3 (1 + (1 + \|V\|_\infty) R_3^2),$$

where $\theta_k = \beta_k(R_3)$. We use Propositions 3.4 and 3.5 and obtain

$$\|F\|_{H^1(\tilde{X}_{R_3})} \leq \tilde{D}_3 D_4 \|F\|_{H^1(X_1)} \leq D_1^\gamma D_2 \tilde{D}_3 D_4 \|F\|_{H^1(\tilde{X}_{R_3})}^{1-\gamma} \|(\partial_{d+1} F)_0\|_{L^2(W_\delta(L))}^{\gamma/2} \|F\|_{L^2(U_3(L))}^{\gamma/2}.$$

Since $U_3(L) \subset \tilde{X}_{R_3}$ we have

$$\|F\|_{H^1(\tilde{X}_{R_3})} \leq D_1^2 D_2^{2/\gamma} \tilde{D}_3^{2/\gamma} D_4^{2/\gamma} \|(\partial_{d+1} F)_0\|_{L^2(W_\delta(L))}.$$

By Ineq. (33), the square of the left hand side is bounded from below by

$$\|F\|_{H^1(\tilde{X}_{R_3})}^2 \geq \|F\|_{H^1(\Lambda_L \times [-R_3, R_3])}^2 \geq \frac{R_3}{2} \sum_{\substack{k \in \mathbb{N} \\ E_k \leq b}} |\alpha_k|^2.$$

Putting everything together we obtain by using $(\partial_{d+1}F)_0 = \phi$

$$\frac{R_3}{2} \sum_{\substack{k \in \mathbb{N} \\ E_k \leq b}} |\alpha_k|^2 \leq D_1^4 \left(D_2 \tilde{D}_3 D_4 \right)^{4/\gamma} \|\phi\|_{L^2(W_\delta(L))}^2.$$

In order to end the proof we will give an upper bound on \tilde{D}_3 which is independent of α_k , $k \in \mathbb{N}$. For this purpose, we recall that $\theta_k = \beta_k(R_3)$. Since $\theta_k \leq e^{2R_3\sqrt{b}}$ for all $k \in \mathbb{N}$ with $E_k \leq b$, we have

$$\tilde{D}_3^4 \leq D_3^4 := e^{4R_3\sqrt{b}}.$$

Hence, using $\sum_{E_k \leq b} |\alpha_k|^2 = \|\phi\|_{L^2(\Lambda_L)}^2$, we obtain for all $L \in \mathbb{N}_{\text{odd}}$ the estimate

$$\tilde{C}_{\text{sfuc}} \|\phi\|_{L^2(\Lambda_L)}^2 \leq \|\phi\|_{L^2(W_\delta(L))}^2$$

where $\tilde{C}_{\text{sfuc}} = \tilde{C}_{\text{sfuc}}(d, \delta, b, \|V\|_\infty) = D_1^{-4} (D_2 D_3 D_4)^{-4/\gamma}$. From the definitions of D_i , $i \in \{1, 2, 3, 4\}$, and γ one calculates that

$$\tilde{C}_{\text{sfuc}} \geq \delta^{\tilde{N}} (1 + \|V\|_\infty^{2/3} + \sqrt{b})$$

with some constant $\tilde{N} = \tilde{N}(d)$, see Appendix B. Now we treat the case of $L \in \mathbb{N}_{\text{even}} = \{2, 4, 6, \dots\}$. By a scaling argument as in Corollary 2.2 of [40], we immediately obtain that for all $G > 0$, $\delta \in (0, G/2)$, $L/G \in \mathbb{N}_{\text{odd}}$ and all (G, δ) -equidistributed sequences q_j we have

$$\|\phi\|_{L^2(W_\delta^q(L))}^2 \geq \tilde{C}_{\text{sfuc}}^G \|\phi\|_{L^2(\Lambda_L)}^2 \quad (34)$$

and $\tilde{C}_{\text{sfuc}}^G(d, \delta, b, \|V\|_\infty) = \tilde{C}_{\text{sfuc}}(d, \delta/G, bG^2, \|V\|_\infty G^2)$. Here $W_\delta^q(L)$ denotes the set $W_\delta(L)$ corresponding to the sequence q_j . Now we define

$$G = \begin{cases} \frac{L}{L/2-1} & \text{if } L \in 4\mathbb{N}, \\ 2 & \text{otherwise} \end{cases}$$

which satisfies $G \in [2, 4]$ and $L/G \in \mathbb{N}_{\text{odd}}$. Since $G \geq 2$, every elementary cell $\Lambda_G + j$, $j \in (G\mathbb{Z})^d$ contains at least one elementary cell $\Lambda_1 + j$, $j \in \mathbb{Z}^d$. Hence we can choose a (G, δ) -equidistributed subsequence q_j of z_j . We apply Ineq. (34) to this subsequence and obtain

$$\|\phi\|_{L^2(W_\delta(L))}^2 \geq \|\phi\|_{L^2(W_\delta^q(L))}^2 \geq \tilde{C}_{\text{sfuc}}^G \|\phi\|_{L^2(\Lambda_L)}^2.$$

Note that $W_\delta(L)$ corresponds to the sequence z_j . Putting everything together we obtain the statement of the theorem with

$$\min \left\{ \tilde{C}_{\text{sfuc}}, \inf_{G \in [2, 4]} \tilde{C}_{\text{sfuc}}^G \right\} \geq \delta^N (1 + \|V\|_\infty^{2/3} + \sqrt{b}) =: C_{\text{sfuc}}$$

and some constant $N = N(d)$. For the last inequality we use that $(1/4)^{\tilde{N}} \geq \delta^{2\tilde{N}}$. \square

Proof of Corollary 2.5. We denote the normalized eigenfunctions of $-\Delta_L + A_L + B_L$ corresponding to the eigenvalues $\lambda_i(-\Delta_L + A_L + B_L)$ by ϕ_i . Then we have

$$\begin{aligned} \lambda_i(-\Delta_L + A_L + B_L) &= \langle \phi_i, (-\Delta_L + A_L + B_L)\phi_i \rangle \\ &= \max_{\phi \in \text{Span}\{\phi_1, \dots, \phi_i\}, \|\phi\|=1} \langle \phi, (-\Delta_L + A_L)\phi \rangle + \langle \phi, B_L\phi \rangle \\ &\geq \max_{\phi \in \text{Span}\{\phi_1, \dots, \phi_i\}, \|\phi\|=1} \langle \phi, (-\Delta_L + A_L)\phi \rangle + \alpha \langle \phi, \chi_{W_\delta(L)}\phi \rangle. \end{aligned}$$

By Corollary 2.3, we conclude that for all $\phi \in \text{Span}\{\phi_1, \dots, \phi_i\}$, $\|\phi\| = 1$, we have

$$\langle \phi, \chi_{W_\delta(L)} \phi \rangle \geq C_{\text{sfuc}}^{G,1}(d, \delta, b, \|A_L + B_L\|_\infty)$$

and furthermore, by the variational characterization of eigenvalues, we find

$$\max_{\substack{\phi \in \text{Span}\{\phi_1, \dots, \phi_i\} \\ \|\phi\|=1}} \langle \phi, (-\Delta_L + A_L) \phi \rangle \geq \inf_{\dim \mathcal{D}=i} \max_{\substack{\phi \in \mathcal{D} \\ \|\phi\|=1}} \langle \phi, (-\Delta_L + A_L) \phi \rangle = \lambda_i(-\Delta_L + A_L).$$

Thus, we obtain the statement of the corollary. \square

4 Proof of Wegner and initial scale estimate

Recall that $0 < G_1 < G_2$ are the numbers from the Delone property such that $|\{\mathcal{D} \cap (\Lambda_{G_1} + x)\}| \leq 1$, $|\{\mathcal{D} \cap (\Lambda_{G_2} + x)\}| \geq 1$ for any $x \in \mathbb{R}^d$, and that for all $t \in [0, 1]$ we have $\text{supp } u_t \subset \Lambda_{G_u}$. Let $\delta_{\max} := 1 - \omega_+$ and $K_u := u_{\max}[G_u/G_1]^d$. For $\omega \in [\omega_-, \omega_+]^{\mathcal{D}}$ and $\delta \leq \delta_{\max}$, we use the notation $V_{\omega+\delta}$ for the potential V_ω , where every ω_j , $j \in \mathcal{D}$ has been replaced by $\omega_j + \delta$. The following lemma is a consequence of the properties of a Delone set, in particular $|\Lambda_L \cap \mathcal{D}| \leq \lceil L/G_1 \rceil^d$, and our assumption (4).

Lemma 4.1. (i) For all $\omega \in [\omega_-, \omega_+]^{\mathcal{D}}$, all $0 < \delta \leq \delta_{\max}$ and all $L \in (G_2 + G_u)\mathbb{N}$, the difference $V_{\omega+\delta} - V_\omega$ is on Λ_L bounded from below by $\alpha_1 \delta^{\alpha_2}$ times the characteristic function of $W_{\beta_1 \delta^{\beta_2}}(L)$ which corresponds to a $(G_2 + G_u, \beta_1 \delta^{\beta_2})$ -equidistributed sequence.

(ii) For all $\omega \in [0, 1]^{\mathcal{D}}$ we have $\|V_\omega\|_\infty \leq K_u$.

(iii) For all $L \in (G_2 + G_u)\mathbb{N}$, we have $|\{j \in \mathcal{D} : \exists t \in [0, 1] : \text{supp } u_t(\cdot - j) \cap \Lambda_L \neq \emptyset\}| \leq \lceil (L + G_u)/G_1 \rceil^d \leq (2L/G_1)^d$.

Proof of Theorem 2.8. Note that for all $b \in \mathbb{R}$, $\lambda_i(H_{\omega,L}) \leq b$ implies, by Lemma 4.1 part (ii), that $\lambda_i(H_{\omega+\delta,L}) \leq b + \|V_{\omega+\delta} - V_\omega\| \leq b + 2K_u$. Now we apply Corollary 2.5 with $A_L = V_\omega$ and $B_L = V_{\omega+\delta} - V_\omega$ (both restricted to Λ_L). Together with Lemma 4.1 part (i), we obtain for all $b \in \mathbb{R}$, all $L \in G_u\mathbb{N}$, all $\omega \in [\omega_-, \omega_+]^{\mathcal{D}}$, all $\delta \leq \delta_{\max}$ and all $i \in \mathbb{N}$ with $\lambda_i(H_{\omega,L}) \leq b$ the inequality

$$\lambda_i(H_{\omega+\delta,L}) \geq \lambda_i(H_{\omega,L}) + \alpha_1 \delta^{\alpha_2} C_{\text{sfuc}}^{G_2+G_u,1}(d, \beta_1 \delta^{\beta_2}, b + 2K_u, K_u).$$

In particular, there is $\kappa = \kappa(d, \omega_+, \alpha_1, \alpha_2, \beta_1, \beta_2, G_2, G_u, K_u, b) > 0$ such that

$$\lambda_i(H_{\omega+\delta,L}) \geq \lambda_i(H_{\omega,L}) + \delta^\kappa. \quad (35)$$

Now let $\varepsilon > 0$, satisfying $\varepsilon \leq \varepsilon_{\max} := \delta_{\max}^\kappa/4$. We choose $\delta := (4\varepsilon)^{1/\kappa}$, whence

$$\lambda_i(H_{\omega+\delta,L}) \geq \lambda_i(H_{\omega,L}) + 4\varepsilon. \quad (36)$$

Let $\rho \in C^\infty(\mathbb{R}, [-1, 0])$ be smooth, non-decreasing such that $\rho = -1$ on $(-\infty; -\varepsilon]$ and $\rho = 0$ on $[\varepsilon; \infty)$. We can assume $\|\rho'\|_\infty \leq 1/\varepsilon$. It holds that

$$\chi_{[E-\varepsilon; E+\varepsilon]}(x) \leq \rho(x - E + 2\varepsilon) - \rho(x - E - 2\varepsilon) = \rho(x - E - 2\varepsilon + 4\varepsilon) - \rho(x - E - 2\varepsilon)$$

for all $x \in \mathbb{R}$ and together with (36) this implies

$$\begin{aligned} \mathbb{E} [\text{Tr} [\chi_{[E-\varepsilon; E+\varepsilon]}(H_{\omega,L})]] &\leq \mathbb{E} [\text{Tr} [\rho(H_{\omega,L} - E - 2\varepsilon + 4\varepsilon) - \rho(H_{\omega,L} - E - 2\varepsilon)]] \\ &\leq \mathbb{E} [\text{Tr} [\rho(H_{\omega+\delta,L} - E - 2\varepsilon) - \rho(H_{\omega,L} - E - 2\varepsilon)]] \end{aligned} \quad (37)$$

Now let $\tilde{\Lambda}_L := \{j \in \mathcal{D} : \exists t \in [0, 1] : \text{supp } u_t(\cdot - j) \cap \Lambda_L \neq \emptyset\}$ be the set of lattice sites which can influence the potential within Λ_L . Note that $|\tilde{\Lambda}_L| \leq (2L/G_1)^d$. We enumerate the points in $\tilde{\Lambda}_L$ by $k : \{1, \dots, |\tilde{\Lambda}_L|\} \rightarrow \mathcal{D}$, $n \mapsto k(n)$. The upper bound in (37) will be expanded in a telescopic sum by changing the $|\tilde{\Lambda}_L|$ indices from ω_j to $\omega_j + \delta$ successively. In order to do that some notation is needed. Given $\omega \in [\omega_-, \omega_+]^{\mathcal{D}}$, $n \in \{1, \dots, |\tilde{\Lambda}_L|\}$, $\delta \in [0, \delta_{\max}]$ and $t \in [\omega_-, \omega_+]$, we define $\tilde{\omega}^{(n, \delta)}(t) \in [\omega_-, 1]^{\mathcal{D}}$ inductively via

$$\left(\tilde{\omega}^{(1, \delta)}(t)\right)_j := \begin{cases} t & \text{if } j = k(1), \\ \omega_j & \text{else,} \end{cases} \quad \text{and} \quad \left(\tilde{\omega}^{(n, \delta)}(t)\right)_j := \begin{cases} t & \text{if } j = k(n), \\ \left(\tilde{\omega}^{(n-1, \delta)}(\omega_j + \delta)\right)_j & \text{else.} \end{cases}$$

The function $\tilde{\omega}^{(n, \delta)} : [\omega_-, 1]^{\mathcal{D}} \rightarrow [\omega_-, 1]^{\mathcal{D}}$ is the rank-one perturbation of ω in the $k(n)$ -th coordinate with the additional requirement that all sites $k(1), \dots, k(n-1)$ have already been blown up by δ . We define

$$\Theta_n(t) := \text{Tr} \left[\rho \left(H_{\tilde{\omega}^{(n, \delta)}(t), L} - E - 2\varepsilon \right) \right], \quad \text{for } n = 1, \dots, |\tilde{\Lambda}_L|.$$

Note that

$$\begin{aligned} \Theta_1(\omega_{k(1)}) &= \text{Tr} [\rho (H_{\omega, L} - E - 2\varepsilon)], \\ \Theta_n(\omega_{k(n)}) &= \Theta_{n-1}(\omega_{k(n-1)} + \delta) \quad \text{for } n = 2, \dots, |\tilde{\Lambda}_L| \quad \text{and} \\ \Theta_{|\tilde{\Lambda}_L|}(\omega_{k(|\tilde{\Lambda}_L|)} + \delta) &= \text{Tr} [\rho (H_{\omega + \delta, L} - E - 2\varepsilon)]. \end{aligned}$$

Hence the upper bound in (37) is

$$\begin{aligned} &\mathbb{E} [\text{Tr} [\rho (H_{\omega + \delta, L} - E - 2\varepsilon)] - \text{Tr} [\rho (H_{\omega, L} - E - 2\varepsilon)]] \\ &= \mathbb{E} \left[\Theta_{|\tilde{\Lambda}_L|}(\omega_{k(|\tilde{\Lambda}_L|)} + \delta) - \Theta_1(\omega_{k(1)}) \right] = \sum_{n=1}^{|\tilde{\Lambda}_L|} \mathbb{E} [\Theta_n(\omega_{k(n)} + \delta) - \Theta_n(\omega_{k(n)})]. \end{aligned}$$

Due to the product structure of the probability space, we can apply Fubini's Theorem to each summand and obtain

$$\mathbb{E} [\Theta_n(\omega_{k(n)} + \delta) - \Theta_n(\omega_{k(n)})] = \mathbb{E} \left[\int_{\omega_-}^{\omega_+} \Theta_n(\omega_{k(n)} + \delta) - \Theta_n(\omega_{k(n)}) d\mu(\omega_{k(n)}) \right].$$

Note that $\Theta_n : [\omega_-, 1] \rightarrow \mathbb{R}$ is monotone and bounded. We will use the following Lemma.

Lemma 4.2. *Let $-\infty < \omega_- < \omega_+ \leq +\infty$. Assume that μ is a probability distribution with bounded density ν_μ and support in the interval $[\omega_-, \omega_+]$ and let Θ be a non-decreasing, bounded function. Then for all $\delta > 0$*

$$\int_{\mathbb{R}} [\Theta(\lambda + \delta) - \Theta(\lambda)] d\mu(\lambda) \leq \|\nu_\mu\|_\infty \cdot \delta [\Theta(\omega_+ + \delta) - \Theta(\omega_-)].$$

Proof of Lemma 4.2. We calculate

$$\begin{aligned} &\int_{\mathbb{R}} [\Theta(\lambda + \delta) - \Theta(\lambda)] d\mu(\lambda) \\ &\leq \|\nu_\mu\|_\infty \int_{\omega_-}^{\omega_+} [\Theta(\lambda + \delta) - \Theta(\lambda)] d\lambda = \|\nu_\mu\|_\infty \left[\int_{\omega_- + \delta}^{\omega_+ + \delta} \Theta(\lambda) d\lambda - \int_{\omega_-}^{\omega_+} \Theta(\lambda) d\lambda \right] \\ &= \|\nu_\mu\|_\infty \left[\int_{\omega_+}^{\omega_+ + \delta} \Theta(\lambda) d\lambda - \int_{\omega_-}^{\omega_- + \delta} \Theta(\lambda) d\lambda \right] \leq \|\nu_\mu\|_\infty \cdot \delta [\Theta(\omega_+ + \delta) - \Theta(\omega_-)]. \quad \square \end{aligned}$$

Thus, we find for all $n = 1, \dots, |\tilde{\Lambda}_L|$

$$\int_{\omega_-}^{\omega_+} [\Theta_n(\omega_{k(n)} + \delta) - \Theta_n(\omega_{k(n)})] d\mu(\omega_{k(n)}) \leq \|\nu_\mu\|_\infty \cdot \delta [\Theta_n(\omega_+ + \delta) - \Theta_n(\omega_-)].$$

We will also need the following result, see, e.g., Theorem 2 in [20].

Proposition 4.3. *Let $H_0 := -\Delta + A$ be a Schrödinger operator with a bounded potential $A \geq 0$, and let $H_1 := H_0 + B$ for some bounded $B \geq 0$ with compact support. Denote the corresponding Dirichlet restrictions to Λ by H_0^Λ and H_1^Λ , respectively. There are constants K_1, K_2 depending only on d and monotonously on $\text{diam supp } B$ such that for any smooth, bounded function $g : \mathbb{R} \rightarrow \mathbb{R}$ with compact support in $(-\infty, b]$ and the property that $g(H_1^\Lambda) - g(H_0^\Lambda)$ is trace class we have*

$$\text{Tr} [g(H_1^\Lambda) - g(H_0^\Lambda)] \leq K_1 e^b + K_2 \left(\ln(1 + \|g'\|_\infty)^d \right) \|g'\|_1.$$

Proposition 4.3 implies

Lemma 4.4. *Let $0 < \varepsilon \leq \varepsilon_{\max}$. Then $\Theta_n(\omega_+ + \delta) - \Theta_n(\omega_-) \leq (K_1 e^b + 2^d K_2) |\ln \varepsilon|^d$, where K_1, K_2 are as in Proposition 4.3 and thus only depend on d and on G_u .*

Proof of Lemma 4.4. Let $g(\cdot) := \rho(\cdot - E - 2\varepsilon)$. By our choice of ρ , g has support in $(-\infty, b]$, $\|g'\|_\infty \leq 1/\varepsilon$ and $\|g'\|_1 = 1$. We define the operators

$$H_0^\Lambda := H \left(\tilde{\omega}^{(n,\delta)}(\omega_-), L \right) \quad \text{and} \quad H_1^\Lambda := H \left(\tilde{\omega}^{(n,\delta)}(\omega_+ + \delta), L \right).$$

They are lower semibounded operators with purely discrete spectrum and since g has support in $(-\infty, b]$, the difference $g(H_1^\Lambda) - g(H_0^\Lambda)$ is trace class. By the previous proposition

$$\Theta_n(\omega_+ + \delta) - \Theta_n(\omega_-) = \text{Tr} [g(H_1^\Lambda) - g(H_0^\Lambda)] \leq K_1 e^b + K_2 (\ln(1 + 1/\varepsilon))^d.$$

To conclude, note that $\varepsilon \leq \varepsilon_{\max} < \frac{1}{2}$ and thus $\ln(1 + 1/\varepsilon) \leq 2|\ln \varepsilon|$ and $1 \leq |\ln \varepsilon| \leq |\ln \varepsilon|^d$. \square

Putting everything together and recalling $\delta = (4\varepsilon)^{1/\kappa}$ we find

$$\begin{aligned} \mathbb{E} [\text{Tr} [\chi_{[E-\varepsilon, E+\varepsilon]}(H_{\omega, L})]] &\leq \left(K_1 e^b + 2^d K_2 \right) \|\nu_\mu\|_\infty \cdot \delta |\ln \varepsilon|^d |\tilde{\Lambda}_L| \\ &\leq \left(K_1 e^b + 2^d K_2 \right) \|\nu_\mu\|_\infty \cdot (4\varepsilon)^{1/\kappa} |\ln \varepsilon|^d (2/G_1)^d L^d. \quad \square \end{aligned}$$

Proof of Theorem 2.9. We follow the ideas developed in [2, 24]. Let $t \leq \delta_{\max}$, $V_{t,L}$ be the restriction of V_ω to Λ_L obtained by setting all random variables to t , and $H_{t,L} = -\Delta_{\Lambda_L} + V_{t,L}$ on $L^2(\Lambda_L)$ with Dirichlet boundary conditions. Note that $H_{0,L} = -\Delta_{\Lambda_L} + V_{0,L}$ and that the first eigenvalue of $H_{t,L}$ is bounded from above by $d(\pi/L)^2 + K_u$. Ineq. (35) with $b = d\pi^2 + K_u$, $\omega_k = 0$, $k \in \mathcal{D}$, and $\delta = t$ yields that there is $\kappa = \kappa(d, \delta_{\max}, \alpha_1, \alpha_2, \beta_1, \beta_2, G_2, G_u, K_u)$ such that for all $t \leq \delta_{\max}$

$$\lambda_1(H_{t,L}) \geq \lambda_1(H_{0,L}) + t^\kappa.$$

We choose $t = L^{-7/(4\kappa)}$ and L sufficiently large such that $t < \min\{\delta_{\max}, t_0\}$. Then,

$$\lambda_1(H_{t,L}) - \lambda_1(H_{0,L}) \geq L^{-7/4}.$$

Let $\Omega_0 := \{\omega \in \Omega : \lambda_1(H_{\omega,L}) \geq \lambda_1(H_{t,L})\}$. Since the potential values in Λ_L only depend on ω_k , $k \in \Lambda_{L+G_u} \cap \mathcal{D}$, we calculate using $|\Lambda_{L+G_u} \cap \mathcal{D}| \leq [(L+G_u)/G_1]^d$ and our assumption on the measure μ that

$$\mathbb{P}(\Omega_0) \geq 1 - \mathbb{P}(\exists \gamma \in \Lambda_{L+G_u} \cap \mathcal{D} : \omega_\gamma \leq t) \geq 1 - \left[\frac{L+G_u}{G_1} \right]^d \mu([0, t]) \geq 1 - \left[\frac{L+G_u}{G_1} \right]^d \frac{C}{L^{7d/4}}.$$

Since $[(L+G_u)/G_1]^d \leq L^{5d/4}$ for L sufficiently large, we obtain the statement of the theorem. \square

5 Proof of observability estimate

We want to apply [34, Theorem 2.2] where we choose $A = \Delta_L - V_L$ on $L^2(\Lambda_L)$ with Dirichlet boundary conditions, $C = \chi_{W_\delta(L)}$ and $C_0 = \text{Id}$. Note that A is self-adjoint with spectrum contained in $(-\infty, \|V\|_\infty]$. For $\lambda > 0$ we define the increasing sequence of spectral subspaces $\mathcal{E}_\lambda := \text{Ran} \chi_{[-\lambda, \infty)}(\Delta_L - V_L)$.

We need to check [34, (5),(6),(7)]. By spectral calculus, we have for all $\lambda > 0$

$$\|e^{(\Delta_L - V_L)t} u\|_{\Lambda_L} \leq e^{-\lambda t} \|u\|_{\Lambda_L}, \quad u \in \mathcal{E}_\lambda^\perp = \text{Ran} \chi_{(-\infty, -\lambda)}(\Delta_L - V_L), \quad t > 0.$$

Furthermore, Corollary 2.3 implies for all $\lambda > 0$ and $u \in \mathcal{E}_\lambda$

$$\|u\|_{\Lambda_L}^2 \leq a_0 e^{-N \ln(\delta/G) G \sqrt{\lambda}} \|u\|_{W_\delta(L)}^2.$$

For $T \leq 1$ we have $e^{2T\|V\|_\infty}/T \leq e^{2\|V\|_\infty} e^{2/T}$ whence

$$\|e^{T(\Delta - V)} u\|_{\Lambda_L}^2 \leq \frac{e^{2T\|V\|_\infty}}{T} \int_0^T \|e^{t(\Delta - V)} u\|_{\Lambda_L}^2 dt \leq e^{2\|V\|_\infty} e^{2/T} \int_0^T \|e^{t(\Delta - V)} u\|_{\Lambda_L}^2 dt.$$

Thus we found [34, (5),(6),(7)] with $m_0 = 1$, $m = 0$, $\alpha = \nu = 1/2$, a_0 and b_0 as in the theorem, $a = -(N/2) \ln(\delta/G) G > 0$, $b = 1$ and $\beta = 1$. By [34, Theorem 2.2 and Corollary 1 (i)], there exists $T' > 0$ such that for all $T \leq T'$

$$\kappa_T \leq 4a_0 b_0 e^{2c_*/T}, \quad \text{where } c_* = 4(\sqrt{a+2} - \sqrt{a})^{-4}.$$

From the proof in [34], it can be inferred that T' only depends on m_0 , α , β , a , b , a_0 , b_0 and on our choice $T \leq 1$. Thus, in our case, T' only depends on G , δ and $\|V\|_\infty$. Using $\sqrt{a+2} - \sqrt{a} = \int_a^{a+2} (2\sqrt{x})^{-1} dx \geq (a+2)^{-1/2}$ and the fact that from $\delta \leq G/2$, it follows that $2 \leq 2a/a_{\min}$ where $a_{\min} := (N/2) \ln(2)G$, and we obtain

$$c_* \leq 4(a+2)^2 \leq 4a^2(1+2/a_{\min})^2 = \ln(G/\delta)^2 (NG + 4/\ln 2)^2.$$

A Sketch of proof of Proposition 3.2

We follow [3, 35] and consider the case $\rho = 1$ and $u \in C_c^\infty(B(1) \setminus \{0\}; \mathbb{R})$ only. The general case follows by regularization ($u \in W^{2,2}(\mathbb{R}^d)$ with support in $B(1) \setminus \{0\}$), scaling (to $\rho > 0$), and adding the two Carleman estimates for the real and imaginary parts of u . Let $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}$ be given by $\sigma(x) = |x|$, $\phi(s) = e^s$, $g = w^{-\alpha} u$, $w(x) = \psi(|x|)$,

$$\psi(s) = s \exp \left[- \int_0^s \frac{1 - e^{-t}}{t} dt \right], \quad \tilde{\nabla} g = \nabla g - \frac{\nabla \sigma^T \nabla g}{|\nabla \sigma|^2} \nabla \sigma = \nabla g - \frac{\nabla w^T \nabla g}{|\nabla w|^2} \nabla w,$$

$F_w := (w\Delta w - |\nabla w|^2)/|\nabla w|^2 : \mathbb{R}^d \rightarrow \mathbb{R}$ and $A(g) := (w\nabla w^T \nabla g)/|\nabla w|^2 + (1/2)gF_w : \mathbb{R}^d \rightarrow \mathbb{R}$. We follow the proof of [3, Lemma 3.15] until the estimate (8.2) in [3], i.e.

$$4\alpha^2 \int \frac{|\nabla w|^2}{w^2} A(g)^2 + 2\alpha \int \sigma\phi'(\sigma)|\tilde{\nabla}g|^2 + 2\alpha^3 \int \sigma\phi'(\sigma) \frac{|\nabla w|^2}{w^2} g^2 \leq \int \frac{w^2}{|\nabla w|^2} (w^{-\alpha}\Delta u)^2 + R_1, \quad (38)$$

where

$$R_1 = C \left(\alpha \int w^{1-\alpha}|g||\Delta u| + \alpha \int w^{-1}g^2 + \alpha \int w \frac{|\nabla \sigma^T \nabla g|^2}{|\nabla \sigma|^2} + \alpha^2 \int w^{-1}|A(g)||g| \right).$$

As explained in [3], one can drop the positive term $\int \sigma\phi'(\sigma)|\tilde{\nabla}g|^2$ in (38), and obtain for sufficiently large α the Carleman estimate

$$\alpha^3 \int_{\mathbb{R}^d} w^{-1-2\alpha} u^2 \leq \tilde{C}_2 \int_{\mathbb{R}^d} w^{2-2\alpha} (\Delta u)^2. \quad (39)$$

Following now [35] we do not drop the term $\int \sigma\phi'(\sigma)|\tilde{\nabla}g|^2$ and use instead

$$|\tilde{\nabla}g|^2 = w^{-2\alpha}|\nabla u|^2 - 2\alpha w^{-2}g|\nabla w|^2 A(g) + \alpha w^{-2}g^2 F_w |\nabla w|^2 - \alpha^2 w^{-2}g^2 |\nabla w|^2 - \frac{(\nabla w^T \nabla g)^2}{|\nabla w|^2}. \quad (40)$$

Combining Eq. (40) with Ineq. (38), and using the bounds $F_w \geq -C_F = \inf_{B_1^c} F_w$ and $w \leq \sigma\phi'(\sigma)$, we obtain

$$4\alpha^2 \int \frac{|\nabla w|^2}{w^2} A(g)^2 + 2\alpha \int w^{-2\alpha+1} |\nabla u|^2 - 2C_F \alpha^2 \int \sigma\phi'(\sigma) \frac{|\nabla w|^2}{w^2} g^2 \leq \int \frac{w^2}{|\nabla w|^2} (w^{-\alpha}\Delta u)^2 + R_2, \quad (41)$$

with some appropriate rest term R_2 . If we compare Ineqs. (38) and (41), we observe that the required gradient term is now included, while the g^2 -term, which corresponds to the lower bound of Ineq. (39), is now negative and goes with α^2 instead of α^3 ! In a similar way as Ineq. (38) implies Ineq. (39), one calculates that Ineq. (41) implies for sufficiently large α

$$\alpha \int_{\mathbb{R}^d} w^{1-2\alpha} |\nabla u|^2 - \alpha^2 \int_{\mathbb{R}^d} w^{-1-2\alpha} u^2 \leq \hat{C}_2 \int_{\mathbb{R}^d} w^{2-2\alpha} (\Delta u)^2 dx. \quad (42)$$

By adding the two estimates (39) and (42) we obtain the desired estimate by choosing α sufficiently large.

B Constants

B.1 Cutoff functions

Let $f, \psi : \mathbb{R} \rightarrow [0, 1]$ be given by

$$f(x) = \begin{cases} e^{-1/x} & x > 0, \\ 0 & x \leq 0, \end{cases} \quad \text{and} \quad \psi(x) = \frac{f(x)}{f(x) + f(1-x)}.$$

Note that the function ψ is $C^\infty(\mathbb{R})$ and satisfies

$$\sup_{x \in \mathbb{R}} \psi'(x) \leq 2 =: C', \quad \sup_{x \in \mathbb{R}} \psi''(x) \leq 10 =: C'', \quad \text{and} \quad \psi(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x \geq 1. \end{cases}$$

For $\varepsilon > 0$ we define $\psi_\varepsilon : \mathbb{R} \rightarrow [0, 1]$ by

$$\psi_\varepsilon(x) = \psi(x/\varepsilon).$$

Let now $M \subset \mathbb{R}^{d+1}$ and $h_M : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ with $h_M(x) \geq \text{dist}(x, M)$ if $x \notin M$ and $h_M(x) \leq 0$ if $x \in M$. For $\varepsilon > 0$ we define $\chi : \mathbb{R}^{d+1} \rightarrow [0, 1]$ by

$$\chi_{M,\varepsilon}(x) = \psi_\varepsilon(\varepsilon - h_M(x)).$$

Of course, $h_M(x) := \text{dist}(x, M)$ is a possible choice, but in applications we will require h_M to have certain additional properties. By construction we have (cf. Fig. 2)

$$\chi_{M,\varepsilon}(x) = \begin{cases} 1 & \text{if } x \in M, \\ 0 & \text{if } \text{dist}(x, M) \geq \varepsilon. \end{cases}$$

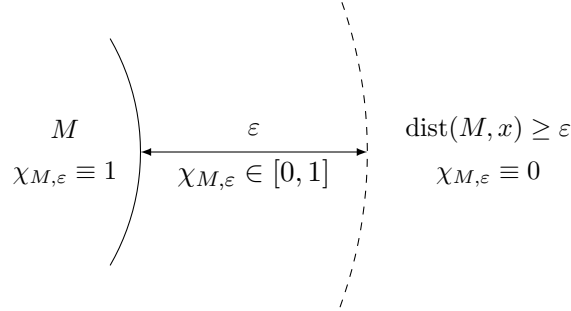


Figure 2: Cutoff function $\chi_{M,\varepsilon}$

B.1.1 The constants Θ_2 and Θ_3

We want to construct a cutoff function $\chi \in C_c^\infty(\mathbb{R}^{d+1}; [0, 1])$ with $\text{supp } \chi \subset B(R_3) \setminus \{0\}$ and $\chi(x) = 1$ if $x \in B(r_3) \setminus \overline{B(R_1)}$. We set $\tilde{M} = B(r_3)$, $2\tilde{\varepsilon} = R_3 - r_3$, $h_{\tilde{M}}(x) = |x| - r_3$ and define

$$\tilde{\chi}(x) = \chi_{\tilde{M},\tilde{\varepsilon}}(x).$$

Note that

$$\tilde{\chi}(x) = \begin{cases} 1 & \text{if } x \in B(r_3), \\ 0 & \text{if } x \notin B((r_3 + R_3)/2). \end{cases}$$

For the partial derivatives we calculate

$$\begin{aligned} (\partial_i \tilde{\chi})(x) &= -\frac{1}{\tilde{\varepsilon}} \psi'(1 - h_{\tilde{M}}(x)/\tilde{\varepsilon}) \frac{x_i}{|x|}, \\ (\partial_i^2 \tilde{\chi})(x) &= \frac{1}{\tilde{\varepsilon}^2} \psi''(1 - h_{\tilde{M}}(x)/\tilde{\varepsilon}) \frac{x_i^2}{|x|^2} - \frac{1}{\tilde{\varepsilon}} \psi'(1 - h_{\tilde{M}}(x)/\tilde{\varepsilon}) \left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3} \right). \end{aligned}$$

Hence, using $\Delta\tilde{\chi}(x) = 0$ if $x \notin B(R_3) \setminus B(r_3)$ and $2\tilde{\varepsilon} = R_3 - r_3 = 3e\sqrt{d}$, we obtain

$$\begin{aligned}\|\nabla\tilde{\chi}\|_\infty &\leq \frac{C'}{\tilde{\varepsilon}} = \frac{4}{R_3 - r_3} = \frac{4}{3e\sqrt{d}} \leq 1, \\ \|\Delta\tilde{\chi}\|_\infty &\leq \frac{C''}{\tilde{\varepsilon}^2} + \frac{C'}{\tilde{\varepsilon}} \frac{d}{r_3} \leq \frac{80 + 4d}{18e^2d} \leq \frac{84}{18e^2} \leq 1.\end{aligned}$$

Analogously we find a function $\hat{\chi}$ with values in $[0, 1]$, $\hat{\chi}(x) = 0$ if $x \in B(r_1)$, $\hat{\chi}(x) = 1$ if $x \notin B(R_1)$ and, using $R_1 - r_1 = r_1 \geq \delta^2/64$,

$$\begin{aligned}\|\nabla\hat{\chi}\|_\infty &\leq \frac{C'}{R_1 - r_1} \leq \frac{128}{\delta^2}, \\ \|\Delta\hat{\chi}\|_\infty &\leq \frac{C''}{(R_1 - r_1)^2} + \frac{C'}{(R_1 - r_1)} \frac{d}{r_1} \leq \frac{10 \cdot 64^2}{\delta^4} + \frac{2d64^2}{\delta^4} \leq \frac{12d64^2}{\delta^4}.\end{aligned}$$

Our cutoff function $\chi \in C_c^\infty(\mathbb{R}^{d+1}; [0, 1])$ with $\text{supp } \chi \subset B(R_3) \setminus \{0\}$ and $\chi(x) = 1$ if $x \in B(r_3) \setminus \overline{B(R_1)}$ can be defined by

$$\chi(x) = \begin{cases} \chi(x) = \hat{\chi}(x) & \text{if } x \in B(R_1) \setminus \overline{B(r_1)}, \\ \chi(x) = 1 & \text{if } x \in B(r_3) \setminus \overline{B(R_1)}, \\ \chi(x) = \tilde{\chi}(x) & \text{if } x \in B(R_3) \setminus \overline{B(r_3)}, \end{cases}$$

and has the properties (recall $V_i = B(R_i) \setminus \overline{B(r_i)}$)

$$\max\{\|\Delta\chi\|_{\infty, V_1}, \|\nabla\chi\|_{\infty, V_1}\} \leq \frac{12d64^2}{\delta^4} =: \frac{\tilde{\Theta}_2}{\delta^4} =: \Theta_2$$

and

$$\max\{\|\Delta\chi\|_{\infty, V_3}, \|\nabla\chi\|_{\infty, V_3}\} \leq \frac{4}{3e} =: \Theta_3.$$

B.1.2 The constant Θ_1

We choose $M = S_2$, $\varepsilon = \delta^2/16$ and

$$h_{S_2}(x) = x_{d+1} - 1 + \sqrt{a_2^2 + \frac{|x'|^2}{2}}.$$

Obviously, $h_{S_2}(x) \geq \text{dist}(x, S_2)$ if $x \notin S$ and $h_{S_2}(x) \leq 0$ if $x \in S_2$, cf. Fig. 3. Since the distance between the sets S_2 and $\mathbb{R}_+^{d+1} \setminus S_3$ is bounded from below by $\delta^2/16$, see Appendix B.1.3, we find that

$$\chi_{S, \varepsilon}(x) = \begin{cases} 1 & \text{if } x \in S_2, \\ 0 & \text{if } x \in \mathbb{R}_+^{d+1} \setminus S_3. \end{cases}$$

For the partial derivatives we calculate for $x \in S_3 \setminus S_2$

$$(\partial_i \chi)(x) = -\frac{1}{\varepsilon} \psi'(1 - h_{S_2}(x)/\varepsilon) \begin{cases} \frac{x_i}{2} \left(a_2^2 + \frac{|x'|^2}{2}\right)^{-1/2} & \text{if } i \in \{1, \dots, d\}, \\ 1 & \text{if } i = d + 1, \end{cases}$$

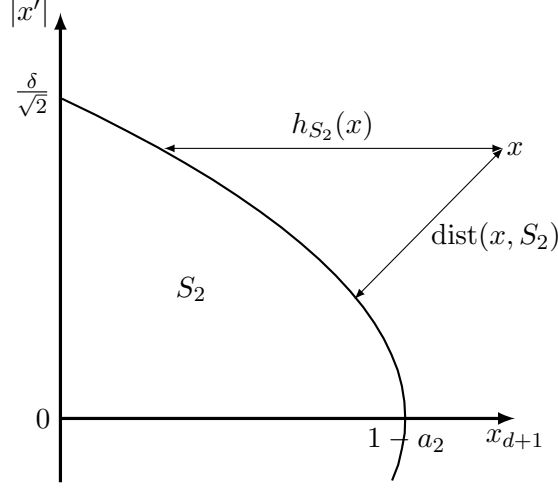


Figure 3: Illustration of the hyperbolas h_2 and h_3

and find by using $|x'|^2 \leq 1/4$ for $x \in S_3 \setminus S_2$ and $a_2^2 \in [15/16, 1]$

$$\|\nabla \chi_{S,\varepsilon}\|_\infty^2 \leq \frac{16}{466} \left(\frac{C'}{\varepsilon}\right)^2, \quad \text{hence,} \quad \|\nabla \chi_{S,\varepsilon}\|_\infty \leq \frac{6}{\delta^2}.$$

For the second partial derivatives we calculate for $i \in \{1, \dots, d\}$

$$\begin{aligned} (\partial_i^2 \chi)(x) &= \frac{1}{\varepsilon^2} \psi''(1 - h_S(x)/\varepsilon) \frac{x_i^2}{4} \left(a_2^2 + \frac{|x'|^2}{2}\right)^{-1} \\ &\quad - \frac{1}{\varepsilon} \psi'(1 - h_S(x)/\varepsilon) \left[\frac{1}{2} \left(a_2^2 + \frac{|x'|^2}{2}\right)^{-1/2} - \frac{x_i^2}{4} \left(a_2^2 + \frac{|x'|^2}{2}\right)^{-3/2} \right], \end{aligned}$$

and $\partial_{d+1}^2 \chi(x) = (1/\varepsilon^2) \psi''(1 - h_S(x)/\varepsilon)$. Hence, using $|x'|^2 \leq 1/4$ for $x \in S_3 \setminus S_2$ and $a_2^2 \in [15/16, 1]$

$$\|\Delta \chi\|_\infty \leq \frac{C''}{\varepsilon^2} \frac{237}{233} + \frac{C'}{2\varepsilon a_2} (d + 8/233) \leq \frac{16^2 \cdot 11d}{\delta^4} =: \frac{\tilde{\Theta}_1}{\delta^4} =: \Theta_1.$$

B.1.3 Distance of S_2 and $\mathbb{R}_+^{d+1} \setminus S_3$

The distance between the sets S_2 and $\mathbb{R}_+^{d+1} \setminus S_3$ is given by the distance between the two hyperbolas

$$h_i: \frac{(x-1)^2}{a_i^2} - \frac{y^2}{b_i^2} = 1, \quad i \in \{2, 3\}$$

in $\{(x, y) \in \mathbb{R}^2: x, y \geq 0\}$, where a_i and b_i are given by

$$a_2^2 = 1 - \frac{\delta^2}{4}, \quad a_3^2 = 1 - \frac{\delta^2}{2} \quad \text{and} \quad b_i^2 = 2a_i^2.$$

See Fig. 4 for an illustration. By symmetry we can consider the case $y \geq 0$ only. First we show that in order to estimate the distance between h_2 and h_3 from below, it is sufficient to consider

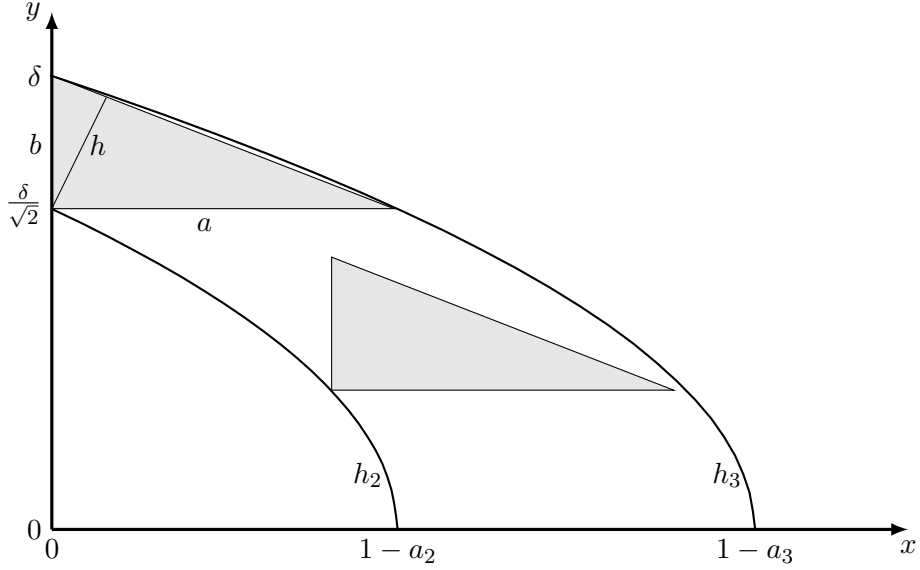


Figure 4: Illustration of the hyperbolas h_2 and h_3

the distance between the intersection point of h_2 with the x -axis and h_3 . For every point (x, y) on h_2 , we define the distance $a(y)$ between h_2 and h_3 in x -direction and the distance $b(x)$ in y -direction. This gives rise to a rectangular triangle with catheti of length a and b . Due to concavity and monotonicity of h_2 and h_3 , considered as functions of x , a lower bound for the distance of (x, y) to h_3 is given by the height of this rectangular triangle, given by

$$h(x) := \frac{a(x)b(x)}{\sqrt{a^2(x) + b^2(x)}}.$$

By a straightforward calculation, we see that $b(x)$ is strictly increasing as a function of x while $a(y)$ is strictly decreasing as a function of y . Thus, taking the triangle at the point $(0, \delta/\sqrt{2})$ and moving it along h_2 , the triangle will always stay below h_3 , see Fig. 4. Hence, h evaluated at the point $(0, \delta/\sqrt{2})$ is a lower bound for $\text{dist}(h_2, h_3)$. We have

$$a(\delta/\sqrt{2}) = 1 - \sqrt{1 - \frac{\delta^2}{4}} \quad \text{and} \quad b(0) = \left(1 - \frac{1}{\sqrt{2}}\right) \delta.$$

Hence,

$$\text{dist}(h_2, h_3) \geq \frac{\left(1 - \frac{1}{\sqrt{2}}\right) \delta \left(1 - \sqrt{1 - \frac{\delta^2}{4}}\right)}{\sqrt{\left(1 - \frac{1}{\sqrt{2}}\right)^2 \delta^2 + \left(1 - \sqrt{1 - \frac{\delta^2}{4}}\right)^2}}.$$

We use $\delta^2/8 \leq 1 - \sqrt{1 - \delta^2/4} \leq \delta/2$ and obtain the bound

$$\text{dist}(h_2, h_3) \geq \frac{\left(1 - \frac{1}{\sqrt{2}}\right) \delta^2}{8\sqrt{\left(1 - \frac{1}{\sqrt{2}}\right)^2 + 1/4}} > \frac{\delta^2}{16}.$$

B.2 The constant \tilde{C}_{sfuc}

We estimate $\tilde{C}_{\text{sfuc}} = D_1^{-4} (D_2 D_3 D_4)^{-4/\gamma}$. We start by estimating the constants D_i , $i \in \{1, \dots, 4\}$ separately. By K_i , $i \in \{1, \dots, 11\}$ we will denote positive constants which do not depend on δ , b and $\|V\|_\infty$, and will change from line to line. We will frequently use $\delta^2/64 \leq r_1 \leq \delta/8$, $K_1 \geq (1/2)^{K_2} \geq \delta^{K_2}$, and $a^{\ln b} = b^{\ln a}$ for $a, b > 0$. For D_1 and D_2 we calculate

$$D_1^{-4} \geq \delta^{K_1(1+\|V\|_\infty^{2/3})}, \quad \text{and} \quad D_2^{-4/\gamma} \geq \delta^{K_2(1+\|V\|_\infty^{2/3})}.$$

For the constant D_3 we have

$$D_3^{-4/\gamma} = \left(e^{4R_3\sqrt{b}} \right)^{-\ln(r_3/r_1)/\ln 2} = \left(\frac{r_1}{r_3} \right)^{\ln(e^{4R_3\sqrt{b}})/\ln 2} \geq \delta^{K_1(1+\sqrt{b})}.$$

For the constant D_4 we have $D_4^2 \leq K_1(1 + \|V\|_\infty)$ and hence

$$D_4^{-4/\gamma} \geq K_1^{-2/\gamma} (1 + \|V\|_\infty)^{-2/\gamma} \geq \delta^{K_2} \delta^{K_3 \ln(1+\|V\|_\infty)} \geq \delta^{K_4(1+\|V\|_\infty^{2/3})}.$$

Hence, we obtain the desired behaviour

$$\tilde{C}_{\text{sfuc}} \geq \delta^{K_1(1+\|V\|_\infty^{2/3} + \sqrt{b})}.$$

C On single-site potentials for the breather model

C.1 Our assumptions

In this section we discuss our conditions on the single-site potential in the random breather model. Recall that the ω_j were supported in $[\omega_-, \omega_+] \subset [0, 1)$ whence we consider $t \in [\omega_-, \omega_+]$ and $\delta \in [0, 1 - \omega_+]$.

Definition C.1. We say that a family $\{u_t\}_{t \in [0,1]}$ of measurable functions $u_t : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies condition

- (A) if the u_t are uniformly bounded, have uniform compact support and if there are $\alpha_1, \beta_1 > 0$ and $\alpha_2, \beta_2 \geq 0$ such that for all $t \in [\omega_-, \omega_+]$, $\delta \leq 1 - \omega_+$ there is $x_0 = x_0(t, \delta) \in \mathbb{R}^d$ with

$$u_{t+\delta} - u_t \geq \alpha_1 \delta^{\alpha_2} \chi_{B(x_0, \beta_1 \delta^{\beta_2})}. \quad (43)$$

- (B) if u_t is the dilation of a function u by t , defined as $u_t(x) := u(x/t)$ for $t > 0$ and $u_0 \equiv 0$, where u is the characteristic function of a bounded convex set K with $0 \in \bar{K}$.
- (C) if u_t is the dilation of a measurable function u which is positive, radially symmetric, compactly supported, bounded with decreasing radial part $r_u : [0, \infty) \rightarrow [0, \infty)$ and such there is a point $\tilde{x} > 0$ where r_u is differentiable, $r'_u(\tilde{x}) < 0$ and $r_u(\tilde{x}) > 0$.
- (D) if u_t is the dilation of a measurable function u which is positive, radially symmetric, radially decreasing, compactly supported, bounded and which has a discontinuity away from 0.

- (E) if u_t is the dilation of a measurable function which is non-positive, radially symmetric, radially increasing, compactly supported, bounded, and such there is a point $\tilde{x} > 0$ where the radial part r_u is differentiable, $r'_u(\tilde{x}) > 0$ and $r_u(\tilde{x}) < 0$.

Remark C.2. Condition (A) is the abstract assumption we used in the proof of the Wegner estimate for the random breather model. Conditions (B) to (E) are relatively easy to verify for specific examples of single-site potentials. In particular, (C) holds for many natural choices of single-site potentials such as the smooth function $\chi_{|x|<1} \exp(1/(|x|^2 - 1))$ or the hat-potential $\chi_{|x|<1}(1 - |x|)$. Furthermore, we note that if we have families $\{u_t\}_{t \in [0,1]}$ and $\{v_t\}_{t \in [0,1]}$ where u_t satisfies (A) and $v_{t+\delta} - v_t \geq 0$ for all $t \in [\omega_-, \omega_+]$ and $\delta \in (0, 1 - \omega_+]$, then the family $\{u_t + v_t\}_{t \in [0,1]}$ also satisfies (A).

Lemma C.3. *We have that each of the assumptions (B) to (E) implies (A).*

Proof. Assume (B). We will show (A) with $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_2 = 1$ and $\beta_1 = c$, and hence it is enough to show the existence of a $c\delta$ -ball in $K_{t+\delta} \setminus K_t$.

For $K \subset \mathbb{R}^d$ and $t > 0$ we define $K_t := \{x \in \mathbb{R}^d : x/t \in K\}$ and $K_0 := \emptyset$. Without loss of generality let $x := (1, 0, \dots, 0)$ be a point in \overline{K} which maximizes $|x|$ over \overline{K} . For $\lambda \in \mathbb{R}$ define the half-space $H_\lambda := \{x \in \mathbb{R}^d : x_1 \leq \lambda\}$, where x_1 stands for the first coordinate of x . By scaling, the existence of a $c\delta$ -ball in $K_{t+\delta} \setminus K_t$ is equivalent to the existence of a $c\delta/(t + \delta)$ -ball in $K \setminus K_{t/(t+\delta)}$. By maximality of $(1, 0, \dots, 0)$, we have $K \subset H_1$ and hence $K_{t/(t+\delta)} \subset H_{t/(t+\delta)}$. Thus, it is sufficient to find a $c \frac{\delta}{t+\delta}$ -ball in $K \setminus H_{t/(t+\delta)}$. By convexity of K , the set $\{z \in K : z_1 = 1/2\}$ is nonempty and since K is open, we find $z_0 \in K$ with $z_1 = 1/2$ and $0 < c < 1/2$ such that $B(z_0, c) \subset K$. We define for $\lambda \in [0, 1)$ the set $X(\lambda) \subset \mathbb{R}^d$ as $X(\lambda) := B(z_0 + \lambda((1, 0, \dots, 0) - z_0), c \cdot (1 - \lambda))$. By convexity and the fact that $(1, 0, \dots, 0) \in \overline{K}$, we have $X(\lambda) \subset K$. In fact, let $\{x_n\}_{n \in \mathbb{N}} \subset K$ be a sequence with $x_n \rightarrow (1, 0, \dots, 0)$. We define open sets $X_n(\lambda)$ by replacing $(1, 0, \dots, 0)$ by x_n in the definition of $X(\lambda)$. By convexity of K , every X_n is a subset of K whence $\bigcup_{n \in \mathbb{N}} X_n(\lambda) \subset K$. Furthermore we have $X(\lambda) \subset \bigcup_{n \in \mathbb{N}} X_n(\lambda)$. Thus $X(\lambda) \subset K$. We now choose $\lambda := \frac{t}{t+\delta}$. Then $X(\lambda) \cap H_\lambda = \emptyset$. Noting that $c(1 - \lambda) = c \frac{\delta}{t+\delta}$, we see that $X(\lambda)$ is the desired $c \frac{\delta}{t+\delta}$ -ball.

Now we assume (C). Let $r'_u(\tilde{x}) = -C_1$. Then there is $\tilde{\varepsilon} > 0$ such that

$$r_u(\tilde{x} + \varepsilon) - r_u(\tilde{x}) \in \left[-2\varepsilon C_1, \frac{-\varepsilon}{2} C_1 \right] \quad \text{for all } |\varepsilon| < \tilde{\varepsilon}. \quad (44)$$

It is sufficient to prove the following: There are $C_2, C_3 > 0$ such that for every $0 \leq t \leq \omega_+$ and every $0 < \delta \leq 1 - \omega_+$ there is $\hat{x} = \hat{x}(t, \delta)$ such that

$$r_u \left(\frac{\hat{x} + C_2 \delta}{t + \delta} \right) - r_u \left(\frac{\hat{x}}{t} \right) \geq C_3 \delta. \quad (45)$$

Indeed, by monotonicity of r_u , (45) implies that for every $x \in [\hat{x}, \hat{x} + C_2 \delta]$ we have

$$r_u \left(\frac{x}{t + \delta} \right) - r_u \left(\frac{x}{t} \right) \geq r_u \left(\frac{\hat{x} + C_2 \delta}{t + \delta} \right) - r_u \left(\frac{\hat{x}}{t} \right) \geq C_3 \delta$$

whence (A) holds with $x_0 := (\hat{x} + C_2 \delta / 2) e_1$, $\alpha_1 = C_3$, $\beta_1 = C_2 / 2$, $\alpha_2 = \beta_2 = 1$.

In order to see (45), let $\hat{x} = (t + \delta) \tilde{x}$. We choose $\kappa \in (0, 1/4)$ and assume that $\tilde{x} - 4\kappa \tilde{\varepsilon} > 0$ (this is no restriction since (44) also holds for smaller $\tilde{\varepsilon}$). Furthermore, we define $C_2 := \kappa \tilde{\varepsilon}$.

Now we distinguish two cases. If $\tilde{x}\delta/t \leq \tilde{\varepsilon}$, then (44) implies

$$\begin{aligned} r_u\left(\frac{\hat{x} + C_2\delta}{t + \delta}\right) - r_u\left(\frac{\hat{x}}{t}\right) &= r_u\left(\tilde{x} + \kappa\frac{\tilde{\varepsilon}\delta}{t + \delta}\right) - r_u(\tilde{x}) + r_u(\tilde{x}) - r_u\left(\tilde{x} + \tilde{x}\frac{\delta}{t}\right) \\ &\geq -2\kappa C_1\frac{\tilde{\varepsilon}\delta}{t + \delta} + C_1\frac{\tilde{x}\delta}{2t} \geq \delta\frac{C_1\tilde{x} - 4\kappa\tilde{\varepsilon}}{2(t + \delta)}. \end{aligned}$$

If $\tilde{x}\delta/t > \tilde{\varepsilon}$, we use $r_u(\tilde{x}) - r_u(\tilde{x} + \tilde{x}\delta/t) \geq r_u(\tilde{x}) - r_u(\tilde{x} + \tilde{\varepsilon})$ and (44) to obtain

$$r_u\left(\frac{\hat{x} + C_2\delta}{t + \delta}\right) - r_u\left(\frac{\hat{x}}{t}\right) \geq -2\kappa C_1\frac{\tilde{\varepsilon}\delta}{t + \delta} + C_1\frac{\tilde{\varepsilon}}{2} = \frac{C_1\tilde{\varepsilon}}{2}\left(1 - \frac{4\kappa\delta}{t + \delta}\right) \geq \frac{C_1\tilde{\varepsilon}}{2}(1 - 4\kappa).$$

Hence

$$r_u\left(\frac{\hat{x} + C_2\delta}{t + \delta}\right) - r_u\left(\frac{\hat{x}}{t}\right) \geq C_3\delta, \text{ where } C_3 := \min\left\{\frac{C_1(\tilde{x} - 4\kappa\tilde{\varepsilon})}{2}, \frac{C_1\tilde{\varepsilon}(1 - 4\kappa)}{2(1 - \omega_+)}\right\} > 0.$$

The fact that (D) implies (A) is a consequence of (B). In fact, a function u as in (D) can be decomposed $u = v + w$ where v is (a multiple of) a characteristic function of a ball, centered at the origin, and w is positive, radially symmetric and decreasing. Indeed, let x_0 be the point of discontinuity with the smallest norm. Then we can take $v = (u(x_0-) - u(x_0+))\chi_{B(0,|x_0|)}$, where χ_A denotes the characteristic function of the set A .

The function v satisfies (A) by (B) (since balls are convex) and we have $w_{t+\delta} - w_t \geq 0$. By Remark C.2, the family $\{u_t\}_{t \in [0,1]} = \{v_t + w_t\}_{t \in [0,1]}$ also satisfies (A). The case (E) is an adaptation of (C). \square

C.2 Earlier assumptions

For certain types of random breather potentials Wegner estimates have been given before, cf. [8] and [9]. As we will show below, none of these results covers the *standard breather model*. The methods of [8, 9] seem to be motivated by reducing, thanks to linearization, the random breather model to a model of alloy type and then applying methods designed for the latter one. They are not focused to take advantage of the inherent, albeit non-linear, monotonicity of the random breather model. The following assumptions on the single site potential are considered in [8] and [9], respectively.

Definition C.4. We say that a measurable function $u: \mathbb{R}^d \rightarrow [0, \infty)$ satisfies condition

(F) if u is compactly supported, in $C^2(\mathbb{R}^d)$, nonzero in a neighbourhood of the origin and for some $c_0 > 0$ we have the inequalities

$$-x \cdot \nabla u \geq 0 \text{ for all } x \in \mathbb{R}^d \quad \text{and} \quad \left| \frac{(x, \text{Hess}[u]x)}{x \cdot \nabla u} \right| \leq c_0 < \infty \text{ for all } x \in \mathbb{R}^d \setminus \{0\}. \quad (46)$$

(G) if $u \not\equiv 0$ is compactly supported, in $C^1(B_1 \setminus \{0\})$, and there is $\varepsilon_0 > 0$ such that

$$-x \cdot \nabla u - \varepsilon_0 u \geq 0 \text{ for all } x \in \mathbb{R}^d \setminus \{0\}. \quad (47)$$

We have the following Lemma.

Lemma C.5. *We have that*

- (F) never holds,
- (G) implies that u has a singularity at the origin.

Proof. We first show the statements in dimension 1. Assume (F) and let $x_0 := \min \text{supp } u$. Note that $x_0 < 0$. By the first inequality in (46) we have that $u' \geq 0$ for $x \in (x_0, 0)$. The second inequality in (46) implies

$$|u''(x)| \leq \frac{c_0 u'(x)}{|x|} \leq \frac{2c_0 u'(x)}{|x_0|} \text{ for all } x \in (x_0, x_0/2)$$

whence we have

$$u'(x) = \int_{x_0}^x u''(y) dy \leq \int_{x_0}^x |u''(y)| dy \leq \frac{2c_0}{|x_0|} \int_{x_0}^x u'(y) dy$$

and iteratively

$$\begin{aligned} u'(x) &\leq \frac{(2c_0)^n}{|x_0|^n} \int_{x_0}^x \int_{x_0}^{x^{(1)}} \dots \int_{x_0}^{x^{(n-1)}} u'(x^{(n)}) dx^{(n)} \dots dx^{(1)} \\ &\leq \|u'\|_\infty \cdot \frac{(2c_0)^n}{|x_0|^n} \int_{x_0}^x \int_{x_0}^{x^{(1)}} \dots \int_{x_0}^{x^{(n-1)}} dx^{(n)} \dots dx^{(1)} \\ &= \|u'\|_\infty \cdot \left(\frac{2c_0(x-x_0)}{|x_0|} \right)^n / n! \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

for all $x \in x_0, x_0/2$. We found $u' \equiv 0$ on $(x_0, x_0/2)$, which is a contradiction.

Now we assume (G). The function u cannot have its supremum at a point of differentiability for else it would have to be zero at its maximum which would imply $u \equiv 0$. Condition (47) implies that u is increasing on the negative half axis and decreasing on the positive half axis. We conclude that the supremum has to be the limit at the only possible non-differentiable point $x = 0$ and we will show that this limit is ∞ . By monotonicity of u and the assumption $u \not\equiv 0$, there is $\delta_0 > 0$ such that

$$u(x) \geq u(\delta_0) > 0 \text{ on } (0, \delta_0) \text{ or } u(x) \geq u(-\delta_0) > 0 \text{ on } (-\delta_0, 0).$$

Without loss of generality, we assume $u(x) \geq u(\delta_0) > 0$ on $(0, \delta_0)$. Furthermore, from (47) it follows that

$$-u'(x) \geq \varepsilon_0 \frac{u(x)}{x} \text{ for } x > 0.$$

Using this inequality we estimate for $0 < x < \delta_0$:

$$\begin{aligned} u(x) &\geq u(x) - u(\delta_0) = - \int_x^{\delta_0} u'(s) ds \geq \varepsilon_0 \int_x^{\delta_0} \frac{u(s)}{s} ds \\ &\geq \varepsilon_0 u(\delta_0) \int_x^{\delta_0} s^{-1} ds = \varepsilon_0 u(\delta_0) [\ln(\delta_0) - \ln(x)] \rightarrow \infty \text{ as } x \rightarrow 0. \end{aligned}$$

Now we show the claim in higher dimensions. If the single site potential $U : \mathbb{R}^d \rightarrow [0, \infty)$ does not vanish identically there is a point y such that $U(y) > 0$. Assume without loss of generality that y lies on the x_1 -axis and define $u : \mathbb{R} \rightarrow [0, \infty)$ by $u(x_1) = U(x_1, 0, \dots, 0)$. Note that if U satisfies the assumption (F) or (G), respectively then u satisfies (F) or (G) as well and the one-dimensional argument can be applied to u . Hence, the statement of the Lemma also holds for U . \square

In the light of the comments made at the beginning of this section, the occurrence of a singularity is not surprising since in the case of a single-site potential with a polynomial singularity, $u(x) = |x|^{-\alpha}$, we have

$$u(x/\omega_j) = |x/\omega_j|^{-\alpha} = \omega_j^\alpha |x|^{-\alpha} = \omega_j^\alpha u(x).$$

and thus the random breathing would correspond to a multiplication which would allow to reduce the breather model to the well-understood alloy type model $V_\omega(x) = \sum_j \omega_j u(x-j)$.

D Proof of Corollary 2.3

We fix

$$\phi = \sum_{k \in \mathbb{N}: E_k \leq b} \alpha_k \phi_k \in \text{Ran } \chi_{(-\infty, b]}(H_{t,L})$$

and define the map $g : \Lambda_{L/G} \rightarrow \Lambda_L$, $g(y) = G \cdot y$. For all ϕ_k the eigenvalue equation reads $-t\Delta_L \phi_k + V_L \phi_k = E_k \phi_k$ in Λ_L where $E_k \leq b$.

We want to transform this into an eigenvalue equation for $\phi_k \circ g$ in $\Lambda_{L/G}$. Therefore we compose with g and find

$$-t(\Delta_L \phi_k) \circ g + (V_L \circ g)(\phi_k \circ g) = E_k(\phi_k \circ g)$$

in $\Lambda_{L/G}$. The chain rule yields $(\Delta_L \phi_k) \circ g = (1/G^2)\Delta_{L/G}(\phi_k \circ g)$ which implies

$$-t/G^2 \Delta_{L/G}(\phi_k \circ g) + (V_L \circ g)(\phi_k \circ g) = E_k(\phi_k \circ g).$$

Thus the eigenvalue equation for $\phi_k \circ g$ is

$$-\Delta_{L/G}(\phi_k \circ g) + \left(\frac{G^2}{t} V_L \circ g\right)(\phi_k \circ g) = \left(\frac{G^2}{t} E_k\right)(\phi_k \circ g) \text{ on } \Lambda_{L/G}.$$

Hence,

$$\phi \circ g \in \text{Ran } \chi_{(-\infty, G^2 b/t]}(-\Delta_{L/G} + (G^2/t)(V_L \circ g)).$$

The set $W_\delta(L) \subset \Lambda_L$ arises from a (G, δ) -equidistributed sequence whence the set $W_\delta(L)/G := \{x \in \mathbb{R}^d : x \cdot G \in W_\delta(L)\} \subset \Lambda_{L/G}$ arises from a $(1, \delta/G)$ -equidistributed sequence. By a coordinate transformation and Theorem 2.2 we obtain

$$\|\phi\|_{W_\delta(L)}^2 = G^d \|\phi \circ g\|_{W_\delta(L)/G}^2 \geq G^d C_{\text{sfuc}}^{G,t} \|\phi \circ g\|_{\Lambda_{L/G}}^2 = C_{\text{sfuc}}^{G,t} \|\phi\|_{\Lambda_L}^2,$$

where $C_{\text{sfuc}}^{G,t} = C_{\text{sfuc}}(d, \delta/G, bG^2/t, \|V\|_\infty G^2/t)$.

Acknowledgement

This work has been partially supported by the DFG under grant *Unique continuation principles and equidistribution properties of eigenfunctions*. and by the binational German-Croatian DAAD-MZOS project *Scale-uniform controllability of partial differential equations*. I.N. was partially supported by HRZZ project grant 9345. M.T. thanks Constanza Rojas-Molina for pointing out that the initial length scale estimate follows from the unique continuation principle.

References

- [1] L. Bakri, *Carleman estimates for the Schrödinger operator. Applications to quantitative uniqueness*, Commun. Part. Diff. Eq. **38** (2013), no. 1, 69–91.
- [2] J.-M. Barbaroux, J.-M. Combes, and P. D. Hislop, *Localization near band edges for random Schrödinger operators*, Helv. Phys. Acta **70** (1997), no. 1-2, 16–43.
- [3] J. Bourgain and C. E. Kenig, *On localization in the continuous Anderson-Bernoulli model in higher dimension*, Invent. Math. **161** (2005), no. 2, 398–426.
- [4] A. Boutet de Monvel, D. Lenz, and P. Stollmann, *An uncertainty principle, Wegner estimates and localization near fluctuation boundaries*, Math. Z. **269** (2011), no. 1, 663–670.
- [5] A. Boutet de Monvel, S. Naboko, P. Stollmann, and G. Stolz, *Localization near fluctuation boundaries via fractional moments and applications*, J. Anal. Math. **100** (2006), no. 1, 83–116.
- [6] J.-M. Combes, P. D. Hislop, and F. Klopp, *Hölder continuity of the integrated density of states for some random operators at all energies*, Int. Math. Res. Notices (2003), no. 4, 179–209.
- [7] ———, *An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators*, Duke Math. J. **140** (2007), no. 3, 469–498.
- [8] J.-M. Combes, P. D. Hislop, and E. Mourre, *Spectral averaging, perturbation of singular spectra, and localization*, T. Am. Math. Soc. **348** (1996), 4883–4894.
- [9] J.-M. Combes, P. D. Hislop, and S. Nakamura, *The L^p -theory of the spectral shift function, the Wegner estimate, and the integrated density of states for some random Schrödinger operators*, Commun. Math. Phys. **70** (2001), no. 218, 113–130.
- [10] S. Ervedoza and E. Zuazua, *Sharp observability estimates for heat equations*, Arch. Ration. Mech. An. **202** (2011), no. 3, 975–1017.
- [11] L. Escauriaza and S. Vessella, *Optimal three cylinder inequalities for solutions to parabolic equations with Lipschitz leading coefficients*, Inverse Problems: Theory and Applications (G. Alessandrini and G. Uhlmann, eds.), Contemp. Math., vol. 333, American Mathematical Society, Providence, 2003, pp. 79–87.
- [12] E. Fernández-Cara and E. Zuazua, *The cost of approximate controllability for heat equations: The linear case*, Adv. Differential Equations **5** (2000), no. 4-6, 465–514.
- [13] A. V. Fursikov and O. Y. Imanuvilov, *Controllability of evolution equations*, Suhak kangüirok, vol. 34, Seoul National University, Seoul, 1996.
- [14] F. Germinet, *Recent advances about localization in continuum random Schrödinger operators with an extension to underlying Delone sets*, Mathematical results in quantum mechanics (I. Beltita, G. Nenciu, and R. Purice, eds.), World Scientific, Singapore, 2008, pp. 79–96.
- [15] F. Germinet, P. Hislop, and A. Klein, *Localization for Schrödinger operators with Poisson random potential*, J. Eur. Math. Soc. **9** (2007), no. 3, 577–607.
- [16] F. Germinet and A. Klein, *A comprehensive proof of localization for continuous Anderson models with singular random potentials*, J. Eur. Math. Soc. **15** (2013), no. 1, 53–143.
- [17] F. Germinet, P. Müller, and C. Rojas-Molina, *Ergodicity and dynamical localization for Delone-Anderson operators*, Rev. Math. Phys. **27** (2015), no. 9, 1550020, 36.
- [18] E. N. Güichal, *A lower bound of the norm of the control operator for the heat equation*, J. Math. Anal. Appl. **110** (1985), no. 2, 519–527.
- [19] M. Helm and I. Veselić, *Linear Wegner estimate for alloy-type Schrödinger operators on metric graphs*, J. Math. Phys. **48** (2007), no. 9, 092107.

- [20] D. Hundertmark, R. Killip, S. Nakamura, P. Stollmann, and I. Veselić, *Bounds on the spectral shift function and the density of states*, Commun. Math. Phys. **262** (2006), no. 2, 489–503.
- [21] D. Jerison and G. Lebeau, *Nodal sets of sums of eigenfunctions*, Harmonic analysis and partial differential equations (M. Christ, C. E. Kenig, and C. Sadosky, eds.), The University of Chicago Press, Chicago, 1999.
- [22] J. Kellendonk, D. Lenz, and J. Savinien (eds.), *Mathematics of aperiodic order*, Progress in Mathematics, vol. 309, Birkhäuser, Basel, 2015.
- [23] W. Kirsch, *Wegner estimates and Anderson localization for alloy-type potentials*, Math. Z. **221** (1996), no. 1, 507–512.
- [24] W. Kirsch, P. Stollmann, and G. Stolz, *Localization for random perturbations of periodic Schrödinger operators*, Random Oper. Stochastic Equations **6** (1998), no. 3, 241–268.
- [25] W. Kirsch and I. Veselić, *Existence of the density of states for one-dimensional alloy-type potentials with small support*, Mathematical Results in Quantum Mechanics (R. Weber, P. Exner, and B. Grébert, eds.), Contemp. Math., vol. 307, American Mathematical Society, 2002, pp. 171–176.
- [26] ———, *Lifshitz tails for a class of Schrödinger operators with random breather-type potential*, Lett. Math. Phys. **94** (2010), no. 1, 27–39.
- [27] A. Klein, *Unique continuation principle for spectral projections of Schrödinger operators and optimal Wegner estimates for non-ergodic random Schrödinger operators*, Commun. Math. Phys. **323** (2013), no. 3, 1229–1246.
- [28] I. Kukavica, *Quantitative uniqueness for second-order elliptic operators*, Duke Math. J. **91** (1998), no. 2, 225–240.
- [29] J. Le Rousseau and G. Lebeau, *On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations*, ESAIM Contr. Optim. Ca. **18** (2012), no. 3, 712–747.
- [30] G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur*, Commun. Part. Diff. Eq. **20** (1995), no. 1&2, 335–356.
- [31] P. Lissy, *A link between the cost of fast controls for the 1-d heat equation and the uniform controllability of a 1-d transport-diffusion equation*, C. R. Math. **350** (2012), no. 11, 591–595.
- [32] L. Miller, *Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time*, J. Differ. Equations **204** (2004), no. 1, 202–226.
- [33] ———, *The control transmutation method and the cost of fast controls*, SIAM J. Control Optim. **45** (2006), no. 2, 762–772.
- [34] ———, *A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups*, Discrete Cont. Dyn.-B **14** (2010), no. 4, 1465–1485.
- [35] I. Nakić, C. Rose, and M. Tautenhahn, *A quantitative Carleman estimate for second order elliptic operators*, arXiv:1502.07575 [math.AP], 2015.
- [36] I. Nakić, M. Täufer, M. Tautenhahn, and I. Veselić, *Scale-free uncertainty principles and Wegner estimates for random breather potentials*, C. R. Math. **353** (2015), no. 10, 919–923.
- [37] K.-D. Phung, *Note on the cost of the approximate controllability for the heat equation with potential*, J. Math. Anal. Appl. **295** (2004), no. 2, 527–538.
- [38] Y. Privat, E. Trélat, and E. Zuazua, *Complexity and regularity of maximal energy domains for the wave equation with fixed initial data*, Discrete Contin. Dyn. S. **35** (2015), no. 12, 6133–6153.
- [39] ———, *Optimal shape and location of sensors for parabolic equations with random initial data*, Arch. Ration. Mech. An. **216** (2015), no. 3, 921–981.

- [40] C. Rojas-Molina and I. Veselić, *Scale-free unique continuation estimates and applications to random Schrödinger operators*, Commun. Math. Phys. **320** (2013), no. 1, 245–274.
- [41] C. Schumacher and I. Veselić, In preparation.
- [42] P. Stollmann, *Caught by disorder*, Birkhäuser, Basel, 2001.
- [43] M. Täufer, M. Tautenhahn, and I. Veselić, *Harmonic analysis and random Schrödinger operators*, Spectral Theory and Mathematical Physics (M. Mantoiu, G. Raikov, and R. Tiedra de Aldecoa, eds.), Operator Theory: Advances and Applications, vol. 254, Birkhäuser, Basel, 2016, pp. 223–255.
- [44] G. Tenenbaum and M. Tucsnak, *New blow-up rates for fast controls of Schrödinger and heat equations*, J. Differ. Equations **243** (2007), no. 1, 70–100.
- [45] I. Veselić, *Lokalisierung bei zufällig gestörten periodischen Schrödingeroperatoren in Dimension Eins*, Diplomarbeit, Ruhr-Universität Bochum, 1996.
- [46] ———, *Lifshitz asymptotics for Hamiltonians monotone in the randomness*, Oberwolfach Rep. **4** (2007), no. 1, 380–382.
- [47] ———, *Existence and regularity properties of the integrated density of states of random Schrödinger operators*, Lecture Notes in Mathematics, vol. 1917, Springer, 2008.
- [48] G. Weiss and M. Tucsnak, *Observation and control for operator semigroups*, Birkhäuser, Basel, 2009.
- [49] P. W. Ziemer, *Weakly differentiable functions*, Springer, New York, 1989.
- [50] E. Zuazua, *Controllability and observability of partial differential equations: Some results and open problems*, Handbook of Differential Equations: Evolutionary Equations (C. M. Dafermos and M. Pokorný, eds.), vol. 3, Elsevier, Amsterdam, 2007, pp. 527–621.