ON THE CONTINUITY OF THE INTEGRATED DENSITY OF STATES IN THE DISORDER

MIRA SHAMIS

Abstract. Recently, Hislop and Marx studied the dependence of the integrated density of states on the underlying probability distribution for a class of discrete random Schrödinger operators, and established a quantitative form of continuity in weak* topology. We develop an alternative approach to the problem, based on Ky Fan inequalities, and establish a sharp version of the estimate of Hislop and Marx. We also consider a corresponding problem for continual random Schrödinger operators on $\mathbb{R}^d$.

1. Introduction

Recently, Hislop and Marx [5] studied the dependence of the integrated density of states (IDS) of random Schrödinger operators on the distribution of the potential.

Let $\{V(n)\}_{n\in\mathbb{Z}^d}$ be independent identically distributed random variables (i.i.d.r.v.) with the common probability distribution μ. Let H be the random Schrödinger operator acting on $\ell^2(\mathbb{Z}^d)$ by

$$H = -\Delta + V, \quad (H\psi)(n) = \sum_{m \text{ is adjacent to } n} (\psi(n) - \psi(m)) + V(n)\psi(n). \tag{1.1}$$

The IDS corresponding to the operator H is the function

$$N(E) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \#\{\text{eigenvalues of } H_\Lambda \text{ in } (-\infty, E]\}, \tag{1.2}$$

where H_Λ is the restriction of H to a finite box $\Lambda \subset \mathbb{Z}^d$, i.e. $H_\Lambda = P_\Lambda H P_\Lambda^*$, where $P_\Lambda : \ell^2(\mathbb{Z}^d) \to \ell^2(\Lambda)$ is the coordinate projection ((1.2) holds with probability 1). The measure with cumulative distribution function N is denoted by ρ.

To discuss the dependence of ρ on the distribution of the potential μ, we introduce two metrics on the space of Borel probability measures on $\mathbb{R}$.

The Kantorovich-Rubinstein (Wasserstein) metric is defined via

$$\mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}) = \sup\left\{ \left| \int f\, d\mu - \int f\, d\tilde{\mu} \right| \, : \, f : \mathbb{R} \to \mathbb{R} \text{ is 1-Lipschitz} \right\}. \tag{1.3}$$

By the Kantorovich-Rubinstein duality theorem

$$\mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}) = \inf\{\mathbb{E}|X - \tilde{X}|\}, \tag{1.4}$$

where the infimum is taken over $\mathbb{R}^2$-valued random variables $(X, \tilde{X})$, such that $X \sim \mu, \tilde{X} \sim \tilde{\mu}$. Following [5], we also consider the bounded Lipschitz metric, defined by

$$\mathrm{d}_{\mathrm{BL}}(\mu, \tilde{\mu}) = \sup\left\{\left|\int f\,d\mu - \int f\,d\tilde{\mu}\right| \,:\, f : \mathbb{R} \to \mathbb{R} \text{ is 1-Lip, } \|f\|_\infty \le 1\right\}. \tag{1.5}$$

Our definition differs from [5] by a multiplicative constant. Observe that

$$\mathrm{d}_{\mathrm{BL}}(\mu, \tilde{\mu}) \le \mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}),$$

and if $\operatorname{supp}\mu, \operatorname{supp}\tilde{\mu} \subset [-A, A]$, then

$$\mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}) \le \max(A, 1)\mathrm{d}_{\mathrm{BL}}(\mu, \tilde{\mu}).$$

In this notation Theorem 1.1 of [5] (formulated here in slightly less general setting than in the cited work) asserts the following.

Theorem (Hislop–Marx)**.** *Suppose $H, \tilde{H}$ are random Schrödinger operators of the form (1.1) with potentials $\{V(n)\}, \{\tilde{V}(n)\}$ sampled from a probability distributions $\mu, \tilde{\mu}$, respectively. Denote by $N, \tilde{N}$ the IDS corresponding to $H, \tilde{H}$, and let $\rho, \tilde{\rho}$ be the measures with cumulative distribution functions $N, \tilde{N}$, respectively. If $\operatorname{supp}\mu, \operatorname{supp}\tilde{\mu} \subset [-A, A]$, then*

$$\mathrm{d}_{\mathrm{BL}}(\rho, \tilde{\rho}) \le C_A \mathrm{d}_{\mathrm{BL}}(\mu, \tilde{\mu})^{1/(1+2d)}, \tag{1.6}$$

$$\sup |N(E) - \tilde{N}(E)| \le \frac{C_A}{\log_+ \frac{1}{\mathrm{d}_{\mathrm{BL}}(\mu, \tilde{\mu})}}, \tag{1.7}$$

where C_A depends only on A.

We refer to [5] for a discussion of earlier work on the subject, and only mention the result [4] on the continuity of the integrated density of states as a function of the coupling constant.

Hislop and Marx [5] presented several applications, particularly, to the continuity of the Lyapunov exponent of a one-dimensional operator as a function of the underlying distribution of the potential. The proof of the theorem in [5] is based on the approximation of the function f (in (1.5)) by polynomials.

We suggest a different approach to estimates of the form (1.6) using the Ky Fan inequalities. Our first result is the following theorem.

Theorem 1. *Suppose $H, \tilde{H}$ are random Schrödinger operators of the form (1.1), where $\{V(n)\}, \{\tilde{V}(n)\}$ are i.i.d.r.v. distributed accordingly to $\mu, \tilde{\mu}$ respectively. Let $N, \tilde{N}$ be the IDS corresponding to $H, \tilde{H}$, and let $\rho, \tilde{\rho}$ be the measures with cumulative distribution functions $N, \tilde{N}$ respectively. Then,*

$$\mathrm{d}_{\mathrm{KR}}(\rho, \tilde{\rho}) \le \mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}), \tag{1.8}$$

$$\sup_E |N(E) - \tilde{N}(E)| \le \frac{C}{\log_+ \frac{1}{\mathrm{d}_{\mathrm{KR}}(\mu,\tilde{\mu})}}, \tag{1.9}$$

where $C > 0$ is a numerical constant.

Remark 1.1. *The power* 1 *as well as the prefactor* 1 *in (1.8) are optimal in general.*

Remark 1.2. *This result can be extended to other models in which the potential is of the form*

$$\sum v_j P_j, \tag{1.10}$$

where v_j are i.i.d.r.v. with common Borel distribution supported on a finite interval and P_j are finite rank projections (see [5]).

Remark 1.3. *Theorem 1 can be extended to different underlying lattices, since the proof does not rely on the structure of $\mathbb{Z}^d$.*

In the follow up paper [6], Hislop and Marx prove a version of their results for the continual Anderson model, which is *not* of the form (1.10). A modification of our argument can be applied to the continual setting as well. We illustrate it by the following theorem.

Let H be a random Schrödinger operator acting on $L^2(\mathbb{R}^d)$, defined by

$$H = -\Delta + V, \tag{1.11}$$

where the potential V is of the form

$$V(x) = \sum_{j\in\mathbb{Z}^d} v_j u(x-j), \quad x \in \mathbb{R}^d, \tag{1.12}$$

where v_j are i.i.d.r.v. distributed accordingly to μ, and u is real-valued continuous compactly supported function: $u \in C_c(\mathbb{R})$. Denote by Λ the cube of side length L around the origin

$$\Lambda = \left[-\frac{L}{2}, \frac{L}{2}\right]^d.$$

Let H_Λ be the restriction of H to $L^2(\Lambda)$ with Dirichlet boundary conditions. Define the IDS corresponding to H similarly to (1.2)

$$N(E) = \lim_{L\to\infty} \frac{1}{L^d} \# \{\text{eigenvalues of}\, H_\Lambda \,\text{in}\, (-\infty, E]\}, \tag{1.13}$$

and let ρ be the measure with cumulative distribution function N.

Theorem 2. *Suppose $H, \tilde{H}$ are random Schrödinger operators of the form (1.11), and suppose that* $\operatorname{supp}\mu, \operatorname{supp}\tilde{\mu} \subset \mathbb{R}_+$ *and $u \ge 0$. Let $N, \tilde{N}$ be the*

IDS corresponding to $H, \tilde{H}$, *and let* $\rho, \tilde{\rho}$ *be the measures with cumulative distribution functions* $N, \tilde{N}$ *respectively. If* $\alpha > \frac{d}{2} - 1$, *then*

$$\left| \int f\left(\frac{1}{(1+E)^\alpha}\right) d\rho(E) - \int f\left(\frac{1}{(1+E)^\alpha}\right) d\tilde{\rho}(E) \right| \leq C(d,u,\alpha) \mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}), \tag{1.14}$$

for any 1*-Lipschitz function* f *for which* $\int f\left(\frac{1}{1+E}\right) d\rho(E)$ *converges.*

If $d = 1, 2, 3$ *and* $\operatorname{supp} \mu \subset [0, A]$ *then for any* $E_0 \in \mathbb{R}$

$$\sup_{E \leq E_0} |N(E) - \tilde{N}(E)| \leq \frac{C(d, E_0, A)}{\log_+^{\kappa_d} \frac{1}{\mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu})}}, \tag{1.15}$$

where $\kappa_1 = 1, \kappa_2 = 1/4, \kappa_3 = 1/8$.

Remark 1.4. *The following example shows that the condition* $\alpha > \frac{d}{2} - 1$ *is optimal in general, and in particular one can not expect a result of the same form as in the discrete case (which would correspond to* $\alpha = -1$*).*

Assume $\alpha \leq \frac{d}{2} - 1$. *Let* u *be such that* $\sum_{j \in \mathbb{Z}^d} u(x-j) \equiv 1$, *and let* $v_j \equiv 0, \tilde{v}_j \equiv \delta, f_\epsilon(x) = \max((x-\epsilon), 0)$. *The integrated density of states of the free Lapacian is given by*

$$d\rho(\lambda) = C_d \lambda^{\frac{d}{2}-1} d\lambda.$$

Therefore we have

$$\int f((1+\lambda)^{-\alpha}) d\rho(\lambda) = C_d \int_0^{\epsilon^{-1/\alpha}-1} ((1+\lambda)^{-\alpha} - \epsilon) \lambda^{\frac{d}{2}-1} d\lambda, \tag{1.16}$$

$$\int f((1+\lambda)^{-\alpha}) d\tilde{\rho}(\lambda) = C_d \int_0^{\epsilon^{-1/\alpha}-1-\delta} ((1+\lambda+\delta)^{-\alpha} - \epsilon) \lambda^{\frac{d}{2}-1} d\lambda, \tag{1.17}$$

and for any $\delta > 0$ *we obtain*

$$\begin{aligned} &\liminf_{\epsilon \to 0} \left| \int f((1+\lambda)^{-\alpha}) d\rho(\lambda) - \int f((1+\lambda)^{-\alpha}) d\tilde{\rho}(\lambda) \right| \\ &\geq \liminf_{\epsilon \to 0} C_d \int_0^{\epsilon^{-1/\alpha}-1-\delta} ((1+\lambda)^{-\alpha} - (1+\lambda+\delta)^{-\alpha}) \lambda^{\frac{d}{2}-1} d\lambda \\ &\geq \liminf_{\epsilon \to 0} C_d \alpha \delta \int_0^{\epsilon^{-1/\alpha}-1-\delta} (1+\lambda+\delta)^{-\alpha-1} \lambda^{\frac{d}{2}-1} d\lambda = \infty. \end{aligned}$$

Remark 1.5. *The restrictions on the dimension and on* V *in the second part of Theorem 2 come from the work of Bourgain and Klein [2] which we use to deduce (1.15) from (1.14).*

Remark 1.6. *The restriction* $\operatorname{supp} \mu \subset [0, A]$, *also coming from [2], can be relaxed using the work of Klein and Tsang [7, Theorem 1.3].*

Remark 1.7. *Theorem 2 formally implies a similar result for sign-indefinite* V *bounded from below.*

2. Preliminaries

2.1. **Discrete case.** The main ingredient of the proof of Theorem 1 is the Ky Fan inequality [8]:

Assume that A, B, and $\tilde{A} = A + B$ are linear self-adjoint operators that act on n-dimensional Euclidean space. Let $\lambda_n \le \lambda_{n-1} \le \cdots \le \lambda_1$, $e_n \le e_{n-1} \le \cdots \le e_1$, $\tilde{\lambda}_n \le \tilde{\lambda}_{n-1} \le \cdots \le \tilde{\lambda}_1$ be the eigenvalues of A, B, and $\tilde{A}$ respectively. Then, for any continuous convex function $\phi : \mathbb{R} \to \mathbb{R}$

$$\sum_{j=1}^{n} \phi(\tilde{\lambda}_j - \lambda_j) \le \sum_{j=1}^{n} \phi(e_j). \tag{2.1}$$

In particular,

$$\sum_{j=1}^{n} |\tilde{\lambda}_j - \lambda_j| \le \sum_{j=1}^{n} |e_j|. \tag{2.2}$$

To deduce (1.9) from (1.8) (similarly to [5]) we shall use the following result due to Craig and Simon [3]. Denote by

$$\omega(\delta) = \sup \{|\rho(E) - \rho(E')| \; : \; E' < E \le E + \delta\}, \tag{2.3}$$

the modulus of continuity of ρ. Then ([3]) the measure ρ with the cumulative distribution function N (the IDS) of any ergodic Schrödinger operator on $\ell^2(\mathbb{Z}^d)$ is log-Hölder continuous, namely, for any $\delta \in (0, \frac{1}{2}]$

$$\omega(\delta) \le \frac{C}{\log \frac{1}{\delta}}, \tag{2.4}$$

where $C > 0$ is a universal constant.

2.2. **Continual case.** First, recall that for $1 \le p < \infty$ the Schatten class S_p is the class of all compact operators in a given Hilbert space such that

$$\|A\|_p = \left(\sum_{n=1}^{\infty} s_n(A)^p \right)^{1/p} < \infty,$$

where $\{s_n(A)\}$ is the sequence of all singular values of the operator A enumerated with multiplicities taken into account. The class S_∞ consists of all compact operators.

The main ingredient in the proof of Theorem 2 is the following version of the Ky Fan inequality (see Markus [9]).

If $A \in S_1$, $B \in S_\infty$ that are self-adjoint, and $\tilde{A} = A + B$, $\lambda_1 \ge \lambda_2 \ge \cdots$, $e_1 \ge e_2 \ge \cdots$, $\tilde{\lambda}_1 \ge \tilde{\lambda}_2 \ge \cdots$, are the eigenvalues of A, B, and $\tilde{A}$ respectively, then, for any continuous convex function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$

$$\sum_{j=1}^{\infty} \phi(\tilde{\lambda}_j - \lambda_j) \le \sum_{j=1}^{\infty} \phi(e_j). \tag{2.5}$$

In particular,

$$\sum_{j=1}^{\infty} |\tilde{\lambda}_j - \lambda_j| \le \sum_{j=1}^{\infty} |e_j| = \|B\|_1. \tag{2.6}$$

To deduce (1.15) from (1.14) we will need the following result due to Bourgain and Klein [2].

Theorem (BK)**.** *Assume that H as in (1.11)–(1.12) on $L^2(\mathbb{R}^d)$, $d = 1, 2, 3$, with* $\operatorname{supp}\mu \subset [-A, A]$*. Let N be the corresponding IDS. Then, given $E_0 \in \mathbb{R}$, for all $E \le E_0$ and $\delta \le 1/2$*

$$|N(E) - N(E + \delta)| \le \frac{C(d, E_0, A)}{\log^{\kappa_d} \frac{1}{\delta}}, \tag{2.7}$$

where $C(d, E_0, A) > 0$, and $\kappa_1 = 1, \kappa_2 = 1/4, \kappa_3 = 1/8$.

3. Proof of Theorem 1 and Theorem 2

3.1. **Proof of Theorem 1.** Denote by $\Lambda \subset \mathbb{Z}^d$ a finite box and let (in the notation of Ky Fan's inequality)

$$A = H_\Lambda = (-\Delta + V)_\Lambda, \;\; \tilde{A} = \tilde{H}_\Lambda = (-\Delta + \tilde{V})_\Lambda,$$

be the restrictions of the operators H and $\tilde{H}$ to the box Λ. Then,

$$\begin{aligned} |\mathrm{tr} f(A) - \mathrm{tr} f(\tilde{A})| &= |\sum_{j=1}^{|\Lambda|} f(\lambda_j) - \sum_{j=1}^{|\Lambda|} f(\tilde{\lambda}_j)| \\ &\le \sum_{j=1}^{|\Lambda|} |f(\lambda_j) - f(\tilde{\lambda}_j)| \le \sum_{j=1}^{|\Lambda|} |\lambda_j - \tilde{\lambda}_j| \\ &\le \sum_{j=1}^{|\Lambda|} |e_j| = \sum_{x\in\Lambda} |V(x) - \tilde{V}(x)|, \end{aligned} \tag{3.1}$$

where the second inequality holds since f is 1-Lipschitz and the last inequality follows from (2.1).

By (1.4) there is a realization of V and $\tilde{V}$ on a common probability space such that

$$\mathbb{E}|V(x) - \tilde{V}(x)| \le \mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}).$$

Thus, using (3.1) for any 1-Lipschitz function f, we obtain

$$|\mathbb{E}\,\mathrm{tr}\, f(A) - \mathbb{E}\,\mathrm{tr}\, f(\tilde{A})| \le \mathbb{E} \sum_{x\in\Lambda} |V(x) - \tilde{V}(x)| \le |\Lambda| \mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}). \tag{3.2}$$

Since

$$\int f d\rho = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathbb{E}\,\mathrm{tr}\, f(A),$$

we obtain by passing to the limit $\Lambda \nearrow \mathbb{Z}^d$

$$\mathrm{d}_{\mathrm{KR}}(\rho, \tilde{\rho}) \leq \mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}), \tag{3.3}$$

thus concluding the proof of (1.8).

To deduce (1.9), we choose

$$f(x) = \begin{cases} \delta, & x \leq E \\ -x + E + \delta, & E \leq x \leq E + \delta \\ 0, & x \geq E + \delta, \end{cases} \tag{3.4}$$

for $\delta > 0$. Then, by definition of the IDS, we get for any $E \in \mathbb{R}$

$$\delta N(E) \leq \int f(E) d\rho(E) \leq \delta N(E + \delta), \tag{3.5}$$

$$\delta \tilde{N}(E) \leq \int f(E) d\tilde{\rho}(E) \leq \delta \tilde{N}(E + \delta). \tag{3.6}$$

Since

$$\int f(E) d\tilde{\rho}(E) = \int f(E) d\rho(E) + \int f(E) d(\tilde{\rho} - \rho)(E),$$

combining (3.3), (3.5), and (3.6), we obtain

$$\delta \tilde{N}(E) \leq \delta N(E + \delta) + \mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}),$$

namely

$$\tilde{N}(E) \leq N(E + \delta) + \frac{\mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu})}{\delta}. \tag{3.7}$$

In the same way we get

$$\tilde{N}(E) \geq N(E - \delta) - \frac{\mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu})}{\delta}. \tag{3.8}$$

Let ω be the modulus of continuity of N. Combining (2.4), (3.7), and (3.8), we obtain

$$\sup_E |N(E) - \tilde{N}(E)| \leq \inf_\delta \left(\omega(\delta) + \frac{\mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu})}{\delta} \right) \leq \frac{C}{\log_+ \frac{1}{\mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu})}}, \tag{3.9}$$

where $C > 0$ is a constant and we choose $\delta = \mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}) / \omega(\mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu}))$. This finishes the proof of (1.9).

□

Remark 3.1. *If the operator H is such that the modulus of continuity ω satisfies*

$$\omega(\delta) \leq C\delta^a,$$

for some $C, a > 0$, then (3.9) implies that

$$\sup_E |N(E) - \tilde{N}(E)| \leq \inf_\delta \left(C\delta^a + \frac{\mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu})}{\delta} \right) \leq \tilde{C}\, \mathrm{d}_{\mathrm{KR}}(\mu, \tilde{\mu})^{1/(1+a)}.$$

3.2. **Proof of Theorem 2.** Let

$$H_\Lambda = (-\Delta + V)_\Lambda,\ \tilde{H}_\Lambda = (-\Delta + \tilde{V})_\Lambda,$$

be the restrictions of the operators H and $\tilde{H}$ to a finite box $\Lambda \in \mathbb{R}^d$ with Dirichlet boundary conditions. Let (in the notation of Ky Fan's inequality)

$$A = (H_\Lambda + 1)^{-\alpha},\ \tilde{A} = (\tilde{H}_\Lambda + 1)^{-\alpha}.$$

We have the following

Claim 3.2. *If $\alpha > \frac{d}{2} - 1$, then*

$$\|A - \tilde{A}\|_1 \leq C(d, u, \alpha) \sum_{j \in 2\Lambda \cap \mathbb{Z}^d} |v_j - \tilde{v}_j| \ .$$

Proof. Let us number

$$2\Lambda \cap \mathbb{Z}^d = \{j_1, \ldots, j_n\},\ \ n \leq C|\Lambda| \ ,$$

and let $H_\Lambda^{(k)}$, $0 \leq k \leq n+1$, be the (restricted) operator corresponding to the potential

$$V(x) = \sum_{l<k} \tilde{v}_{j_l} u(x - j_l) + \sum_{l \geq k} v_{j_l} u(x - j_l) \ .$$

Observe that

(3.10)
$$\|(H_\Lambda^{(k)} + 1)^{-\alpha} - (H_\Lambda^{(k+1)} + 1)^{-\alpha}\|_1 \leq$$
$$|\alpha| \left(\left\| \frac{1}{(H_\Lambda^{(k)} + 1)^{\alpha+1}} (H_\Lambda^{(k)} - H_\Lambda^{(k+1)}) \right\|_1 + \left\| (H_\Lambda^{(k)} - H_\Lambda^{(k+1)}) \frac{1}{(H_\Lambda^{(k+1)} + 1)^{\alpha+1}} \right\|_1 \right) \ ,$$

as follows, for example, from the Birman-Solomyak formula [1, Theorem 8.1] (note that for $\alpha = 1$ it suffices to use the second resolvent identity). Then we have

$$\begin{aligned}\left\|\frac{1}{(H_\Lambda^{(k)}+1)^{\alpha+1}}u_{j_k}\right\|_1 &= \left\|\frac{1}{(H_\Lambda^{(k)}+1)^{\frac{\alpha+1}{2}}}\sqrt{u_{j_k}}\sqrt{u_{j_k}}\frac{1}{(H_\Lambda^{(k)}+1)^{\frac{\alpha+1}{2}}}\right\|_1 \\ &= \left\|\sqrt{u_{j_k}}\frac{1}{(H_\Lambda^{(k)}+1)^{\frac{\alpha+1}{2}}}\right\|_2^2 \\ &= \left\|{u_{j_k}}^{1/4}\frac{1}{(H_\Lambda^{(k)}+1)^{\frac{\alpha+1}{2}}}{u_{j_k}}^{1/4}\right\|_2^2 \\ &\le \left\|{u_{j_k}}^{1/4}\frac{1}{(-\Delta_\Lambda+1)^{\frac{\alpha+1}{2}}}{u_{j_k}}^{1/4}\right\|_2^2 \\ &\le \|u\|_\infty\left\|\mathbf{1}_{\mathrm{supp}\ u}\frac{1}{(-\Delta_\Lambda+1)^{\frac{\alpha+1}{2}}}\mathbf{1}_{\mathrm{supp}\ u}\right\|_2^2,\end{aligned}$$

where the first inequality follows from the positivity of the potential. A similar bound holds for the second term of (3.10). By Weyl's law the last norm is bounded by $C(d,u,\alpha)$ (uniformly in Λ) whenever $\alpha+1>\frac{d}{2}$. Thus, we obtain

$$\|(H_\Lambda^{(k)}+1)^{-\alpha}-(H_\Lambda^{(k+1)}+1)^{-\alpha}\|_1 \le C(d,u,\alpha)|v_{j_k}-\tilde{v}_{j_k}|.$$

□

By the Kantorovich-Rubinstein duality (1.4) there is a realization of v and $\tilde{v}$ on a common probability space such that

$$\mathbb{E}|v_j-\tilde{v}_j| \le \mathrm{d_{KR}}(\mu,\tilde{\mu}).$$

Thus, using Claim 3.2, we get for $\alpha>\frac{d}{2}-1$

$$\begin{aligned}\mathbb{E}\|A-\tilde{A}\|_1 &\le C(d,u,\alpha)\mathbb{E}\sum_{j\in 2\Lambda\cap\mathbb{Z}^d}|v_j-\tilde{v}_j| = C(d,u,\alpha)\sum_{j\in 2\Lambda\cap\mathbb{Z}^d}\mathbb{E}|\tilde{v}_j-v_j| \\ &\le C(d,u,\alpha)|\Lambda|\mathrm{d_{KR}}(\mu,\tilde{\mu}).\end{aligned}$$

The eigenvalues of A are exactly $\frac{1}{(1+\lambda_j)^\alpha}$, where λ_j are the eigenvalues of H_Λ, thus using (2.6), we obtain for $\alpha>\frac{d}{2}-1$ and for any 1-Lipschitz function f for which $\int f\left(\frac{1}{(1+E)^\alpha}\right)d\rho(E)$ converges (in particular, $f(0)=0$)

$$
\begin{aligned}
&\left|\sum_{j=1}^{\infty} f\left(\frac{1}{(1+\lambda_j)^\alpha}\right) - \sum_{j=1}^{\infty} f\left(\frac{1}{(1+\tilde\lambda_j)^\alpha}\right)\right| \\
&\le \sum_{j=1}^{\infty}\left| f\left(\frac{1}{(1+\lambda_j)^\alpha}\right) - f\left(\frac{1}{(1+\tilde\lambda_j)^\alpha}\right)\right| \\
&\le \sum_{j=1}^{\infty}\left| \frac{1}{(1+\lambda_j)^\alpha} - \frac{1}{(1+\tilde\lambda_j)^\alpha}\right| \le C(u,d,\alpha)\,|\Lambda|\,\mathrm{d}_{\mathrm{KR}}(\mu,\tilde\mu),
\end{aligned}
\tag{3.11}
$$

where the last step follows from the Ky Fan inequality. Using the definition (1.13) of ρ and passing to the limit $\Lambda \nearrow \mathbb{R}^d$, we conclude that if $\alpha > \frac{d}{2} - 1$, then

$$
\left|\int f\left(\frac{1}{(1+E)^\alpha}\right) d\rho(E) - \int f\left(\frac{1}{(1+E)^\alpha}\right) d\tilde\rho(E)\right| \le C(d,u,\alpha)\mathrm{d}_{\mathrm{KR}}(\mu,\tilde\mu)\ . \tag{3.12}
$$

Thus we complete the proof of (1.14).

To deduce (1.15), we define

$$
f(x) = \begin{cases} \frac{\delta}{2(1+E)^2}, & x \ge \frac{1}{1+E} \\ \frac{1+E+\delta}{2(1+E)}x - \frac{1}{2(1+E)}, & \frac{1}{1+E+\delta} \le x \le \frac{1}{1+E} \\ 0, & x \le \frac{1}{1+E+\delta}, \end{cases} \tag{3.13}
$$

for $\delta > 0$. Then, by the definition of the IDS (1.13) we get for a fixed $E_0 \in \mathbb{R}$ in the same way as in the proof of Theorem 1

$$
\tilde N(E_0) \le N(E_0+\delta) + \frac{2\mathrm{d}_{\mathrm{KR}}(\mu,\tilde\mu)\,(1+E_0)^2}{\delta}, \tag{3.14}
$$

$$
\tilde N(E_0) \ge N(E_0-\delta) - \frac{2\mathrm{d}_{\mathrm{KR}}(\mu,\tilde\mu)\,(1+E_0)^2}{\delta}. \tag{3.15}
$$

Let ω be the modulus of continuity of N. Then, by (2.7) for any $E \le E_0$ and $\delta \le 1/2$

$$
|N(E) - N(E+\delta)| \le \frac{C(d,E_0,A)}{\log_+^{\kappa_d} \frac{1}{\delta}}.
$$

Thus, choosing $\delta = \frac{C(d,E_0,A)\,\mathrm{d}_{\mathrm{KR}}(\mu,\tilde\mu)}{\omega(C(d,E_0,A)\,\mathrm{d}_{\mathrm{KR}}(\mu,\tilde\mu))}$, we obtain

$$
\sup_{E\le E_0} |\tilde N(E) - N(E)| \le \frac{\tilde C(d,E_0,A)}{\log_+^{\kappa_d} \frac{1}{\mathrm{d}_{\mathrm{KR}}(\mu,\tilde\mu)}},
$$

therefore completing the proof.

□

Acknowledgements: I am grateful to Peter Hislop and Christoph Marx for helpful correspondence. I would like to thank Abel Klein for helpful comments. I would also like to thank Sasha Sodin for pleasant and useful discussions pertaining to the Ky Fan inequalities.

References

[1] M. Sh. Birman, M. Solomyak, *Double operator integrals in a Hilbert space*, Integral equations and operator theory **47.2** 131–168 (2003)

[2] J. Bourgain, A. Klein, *Bounds on the density of states for Schrödinger operators*, Inventiones mathematicae **194.1** 41–72 (2013)

[3] W. Craig, B. Simon, *Log Hölder continuity of the integrated density of states for stochastic Jacobi matrices*, Comm. Math. Phys. **90**(**2**) 207–218 (1983)

[4] P. D. Hislop, F. Klopp, and J. H. Schenker, *Continuity with respect to disorder of the integrated density of states*, Illinois Journal of Mathematics **49.3** 893–904 (2005)

[5] P. D. Hislop, C. A. Marx, *Dependence of the density of states on the probability distribution for discrete random Schrödinger operators*, to appear in International Mathematics Research Notices (2018)

[6] P. D. Hislop, C. A. Marx, *Dependence of the density of states on the probability distribution - part II: Schrödinger operators on $\mathbb{R}^d$ and non-compactly supported probability measures*, arXiv:1904.01118

[7] A. Klein, CS. S. Tsang, *Local behavior of solutions of the stationary Schrödinger equation with singular potentials and bounds on the density of states of Schrödinger operators*, Communications in Partial Differential Equations **41.7** 1040–1055 (2016)

[8] V. B. Lidskii, *Inequalities for eigenvalues and singular values*, appendix to: F. R. Gantmaher. Theory of matrices. Second supplemented edition. Izdat. "Nauka", Moscow 1966 576 pp.

[9] A. S. Markus, *Eigenvalues and singular values of the sum and product of linear operators.* (Russian) Uspehi. Mat. Nauk **19**, no. 4 (118), 93–123 (1964)

School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, England

Email address: `m.shamis@qmul.ac.uk`