A perturbative approach to the construction of initial data on compact manifolds

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Abstract

We discuss the implementation, on compact manifolds, of the perturbative method of Friedrich-Butscher for the construction of solutions to the vacuum Einstein constraint equations. This method is of a perturbative nature and exploits the properties of the extended constraint equations —a larger system of equations whose solutions imply a solution to the Einstein constraints. The method is applied to the construction of nonlinear perturbations of constant mean curvature initial data of constant negative sectional curvature. We prove the existence of a neighbourhood of solutions to the constraint equations around such initial data, with particular components of the extrinsic curvature and electric/magnetic parts of the spacetime Weyl curvature prescribed as free data. The space of such free data is parametrised explicitly.

1 Introduction

The problem of constructing initial data for the Cauchy problem in General Relativity, with origins in the work of Lichnerowicz, has proven to be a rich and interesting problem both from the mathematical and the physical points of view. Recall that an initial data set for the Cauchy problem in General Relativity consists of a triple $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K})$, with $\mathcal{S}$ a 3-dimensional smooth orientable manifold (the *initial hypersurface*), $\boldsymbol{h}$ a Riemannian metric on $\mathcal{S}$, and $\boldsymbol{K}$ (the *extrinsic curvature*) a symmetric 2-tensor over $\mathcal{S}$, satisfying the *Einstein constraint equations*

$$r[\boldsymbol{h}] + K^2 - K_{ij}K^{ij} = 2\lambda, \tag{1a}$$

$$D^i K_{ij} - D_j K = 0. \tag{1b}$$

Here, $r[\boldsymbol{h}]$ denotes the Ricci scalar curvature of $\boldsymbol{h}$ and $K \equiv h^{ij}K_{ij}$, the *mean extrinsic curvature*. Given a solution to the Einstein constraints, the foundational result of Choquet-Bruhat (see [11]) guarantees the existence of a Cauchy development, $(\mathcal{M}, \boldsymbol{g})$, of $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K})$ —i.e. a solution $(\mathcal{M}, \boldsymbol{g})$ to the Einstein field equations with $\boldsymbol{h}$ and $\boldsymbol{K}$ equal to the first and second fundamental forms induced by $\mathcal{S} \hookrightarrow \mathcal{M}$. The *Hamiltonian* and *momentum* constraints (1a)–(1b) comprise a highly-coupled system of partial differential equations, and their analysis therefore presents a significant challenge. The challenge is, however, twofold: in addition to the mathematical difficulty of analysing such a system of equations, there is on the other hand the difficulty of ensuring that the solutions, however obtained, are *physically meaningful.* The latter problem is increasingly pertinent as we move into the age of gravitational wave astronomy.

To date, the most popular solution methods have been the so-called *conformal method* of Lichnerowicz and Choquet-Bruhat (see e.g. [11]), and the related *conformal thin sandwich* method.

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Additionally, there are various techniques based on "gluing" constructions, for example. For an overview of these methods, we refer the reader to [3, 11, 17, 26]. These techniques share in common the fact that they rely on reformulating the constraint equations (which are underdetermined elliptic) as a system of elliptic PDEs —requiring, in particular, the appropriate choice of *freely prescribed* and *determined fields*— to which the tools of the theory elliptic PDEs may then be applied. One of the features of the conformal method, in particular, is that the free data are *York-scaled*, so that one needs to solve the full system of (conformally formulated) constraint equations, solving in particular for the conformal factor, before one can obtain the corresponding physically meaningful counterparts of the free data via conformal rescaling. Recent work aiming at making the conformal method more physically relevant can be found in e.g. [23, 24].

The purpose of the present article is to explore an alternative *perturbative* approach (to be called the *Friedrich-Butscher* method), first considered in [8, 9] and implemented there to prove the existence of non-linear perturbative solutions of the constraint equations around flat initial data. The method was adapted in [13] to prove, in particular, the existence of constant scalar curvature manifolds as perturbations of hyperbolic space, and to hence construct hyperboloidal (umbilical) initial data sets that can be thought of as perturbations of the standard hyperboloid of Minkoswki space. Here we will be interested in applications to closed (i.e. compact, without boundary) initial hypersurfaces $\mathcal{S}$ —i.e. the construction of initial data for "cosmological spacetimes". In this approach, the central object of study is the system of so-called *extended constraint equations*. While the extended constraint equations are entirely equivalent to the Einstein constraint equations —see Section 2— their additional structure naturally lends itself to a choice of freely prescribed data and determined fields that differs from that of the conformal method. In particular, in this method certain components of the Weyl curvature (restricted to the initial hypersurface $\mathcal{S}$) of the development $(\mathcal{M}, \boldsymbol{g})$ have the natural interpretation of being freely prescribed data. Note that since the method is not based on a conformal reformulation of the constraints, the free data are physical in the sense of determining, *a priori*, physically relevant properties of the initial data set. This method, therefore, offers a new perspective on the classical problem of identifying the *gravitational degrees of freedom* of solutions to the Einstein field equations —the free data can be thought as parametrising the space of solutions of the constraints in a neighbourhood of the given background initial data set. Although local, in the sense that the free data is given with reference to a fixed background solution, this is perhaps a natural approach within the framework of the Cauchy problem, in particular in problems relating to Cauchy stability.

The extended constraint equations can also be seen as a particular case of the *conformal constraint equations* of Friedrich (see [16]), corresponding to a trivial conformal factor. The conformal constraint equations offer a promising alternative for the construction (on non-compact manifolds) of initial data with *controlled asymptotics*. A detailed understanding of the extended constraints is a necessary first step towards the study of the conformal constraint equations.

In restricting to the case of closed initial hypersurfaces, $\mathcal{S}$, we hope to bring to the foreground the more geometric aspects of the method, emphasising the key structural features of the extended constraints that enable such an approach. In the first half of the article —Sections 2 and 3— we discuss in fairly general terms the main aspects of the method, identifying structural features of the extended constraint equations, in addition to the potential restrictions imposed on the background initial data. In particular, we identify certain obstructions to the implementation of the method, at least in its present form —see Section 3.4. As proof of concept, the method is then implemented for a class of background initial data which we refer to as *conformally rigid hyperbolic* initial data. Here, the property of *conformal rigidity* is, roughly speaking, the requirement that there exist no perturbations of the metric that preserve *conformal flatness* to first order (except, of course, the pure-gauge perturbations) —in the case considered here, this is equivalent to the requirement that the metric admit no tracefree *Codazzi tensors*, see Section 3.4 for more details. Such a background solution may be thought of as constant extrinsic mean curvature (*CMC*) initial data for a spatially compact analogue of the $k = -1$ *Friedmann-Lemaître-Robertson-Walker* spacetime. We will see in Section 4.4 that this class of background initial data, being conformally flat, has the additional feature that it allows for an explicit construction and parametrisation of the free data.

So far, it is unclear whether the obstructions to the method associated to the existence of globally defined conformal Killing vectors and Codazzi tensors are an unavoidable deficiency of the method, or whether they can be overcome with some appropriate modifications. An analogy can be drawn here with the conformal method, in which the existence of a non-trivial conformal Killing vector for the seed metric is an obstruction to its implementation —see, for example, [3]. Similar obstructions also arise in the gluing methods. In the case of the conformal method, there have been recent attempts to remove the assumption of the non-existence of conformal Killing fields —see, for example [19]. It is plausible that the obstructions in the Friedrich–Butscher method, too, are not essential.

The main result of this article can be summarised as follows:

Theorem. *Let* $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$ *be a conformally rigid hyperbolic initial data set on a compact manifold* $\mathcal{S}$*. Then for each pair of sufficiently small tensor fields* $T_{ij}, \bar{T}_{ij}$ *over* $\mathcal{S}$*, transverse-tracefree with respect to* $\mathring{\boldsymbol{h}}$*, and each sufficiently small scalar field* ϕ *over* $\mathcal{S}$*, there exists a solution of the Einstein constraint equations* $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K})$ *with* $tr_{\mathring{\boldsymbol{h}}}(\boldsymbol{K} - \mathring{\boldsymbol{K}}) = \phi$ *and for which the electric and magnetic parts of the Weyl curvature (restricted to* $\mathcal{S}$*) of the resulting spacetime development take the form*

$$S_{ij} = \mathring{L}(\boldsymbol{X})_{ij} + T_{ij} - \tfrac{1}{3} tr_{\boldsymbol{h}}(\mathring{L}(\boldsymbol{X}) + \boldsymbol{T})\ h_{ij},$$
$$\bar{S}_{ij} = \mathring{L}(\bar{\boldsymbol{X}})_{ij} + \bar{T}_{ij} - \tfrac{1}{3} tr_{\boldsymbol{h}}(\mathring{L}(\bar{\boldsymbol{X}}) + \bar{\boldsymbol{T}})\ h_{ij},$$

for some covectors $\boldsymbol{X}$, $\bar{\boldsymbol{X}}$ *over* $\mathcal{S}$*, where* $\mathring{L}$ *denotes the* conformal Killing operator *with respect to* $\mathring{\boldsymbol{h}}$*.*

A precise statement of the above theorem is given in Section 4, Theorem 1.

Outline of the article

The structure of this article is as follows: in Section 2 we introduce the extended constraint equations and discuss their relationship to the Einstein constraint equations. In Section 3, we describe in general terms the Friedrich-Butscher method; in Section 3.2 we outline the general procedure for the reformulation of the extended constraint equations as an elliptic system; the potential obstructions to the implementation of the method are discussed in Section 3.4, motivating our subsequent restriction to *conformally rigid hyperbolic* background initial data. In Section 4 the method is carried out in this case, the main result being given in Theorem 1 of Section 4.1, and proved by means of Propositions 1 and 4 in Sections 4.2 and 4.3.

Notation and Conventions

In the following we will use $(\mathcal{S}, \boldsymbol{h})$ to denote a Riemannian manifold. The metric $\boldsymbol{h}$ is assumed to be positive definite. The Levi-Civita connection will be denoted by D, and the Latin indices $i, j, k, \ldots$ will denote abstract tensorial 3-dimensional indices. Where convenient we make use of *index-free* notation in which tensorial objects are written in boldface.

Our conventions for the Riemann curvature are fixed by

$$(D_i D_j - D_j D_i)v^k = r^k{}_{lij} v^l.$$

The Ricci curvature and scalar are $r_{ij} \equiv r^l{}_{ilj}$, $r \equiv h^{ij} r_{ij}$.

2 The extended Einstein constraint equations

The *extended Einstein constraint equations* (or *extended constraints* for short) on a spacelike hypersurface $\mathcal{S}$ of a 4-dimensional Lorentzian manifold $(\mathcal{M}, \boldsymbol{g})$ are given by the conditions

$$J_{ijk} = 0, \qquad \bar{\Lambda}_i = 0, \qquad \Lambda_i = 0, \qquad V_{ij} = 0, \tag{2}$$

in terms of the *zero-quantities*

$$J_{ijk} \equiv D_i K_{jk} - D_j K_{ik} - \epsilon^l{}_{ij} \bar{S}_{kl}, \tag{3a}$$
$$\Lambda_i \equiv D_j S_i{}^j - \epsilon_{ikl} K^{jk} \bar{S}_j{}^l, \tag{3b}$$
$$\bar{\Lambda}_l \equiv D^i \bar{S}_{il} - \epsilon_{ljk} K_i{}^k r^{ij}, \tag{3c}$$
$$V_{ij} \equiv r_{ij} - \tfrac{2}{3}\lambda h_{ij} - S_{ij} - K_i{}^k K_{jk} + K_k{}^k K_{ij}. \tag{3d}$$

They are to be read as equations for a Riemannian metric h_{ij}, a symmetric 2-tensor K_{ij} to be interpreted as the extrinsic curvature, and two symmetric $\boldsymbol{h}$-tracefree tensors S_{ij}, $\bar{S}_{ij}$.

The system (3a)-(3d) can be seen as a particular case of Friedrich's conformal constraint equations —namely, when the conformal rescaling is trivial, see [27]. The equations associated to the zero-quantities (3a) and (3d) are nothing other than the *Codazzi–Mainardi* and *Gauss–Codazzi equations* —recall that in three dimensions the essential components of the Riemann curvature tensor are contained in the Ricci tensor. The equations associated to the zero-quantities defined in (3b)-(3c) are the projections onto $\mathcal{S}$ of the second Bianchi identity of the ambient spacetime (assuming that the Einstein vacuum field equations hold):

$$\nabla_{[a} C_{bc]de} = 0,$$

where C_{abcd} denotes the Weyl tensor. Accordingly, the fields S_{ij} and $\bar{S}_{ij}$ can be interpreted, respectively, as the *electric* and *magnetic parts* of C_{abcd} with respect to the normal of $\mathcal{S}$ —the latter 3-manifold being thought of as a spacelike hypersurface of a spacetime $(\mathcal{M}, \boldsymbol{g})$.

Remark 1. The equations associated to the zero-quantities defined in (3b)-(3c) may also be interpreted as integrability conditions for the equations associated to (3a) and (3d). More specifically, the zero-quantities satisfy the relations

$$\bar{\Lambda}_l + \tfrac{1}{2}\epsilon_{ijk} D^k J^{ij}{}_l = 0, \tag{4a}$$
$$\Lambda_j + D_i V_j{}^i - \tfrac{1}{2} D_j V_i{}^i - K_{ik} J_j{}^{ik} + K_{jk} J^{ik}{}_i + K J_j{}^i{}_i = D^i r_{ij} - \tfrac{1}{2} D_j r = 0, \tag{4b}$$

where in the latter we are making use of the contracted Bianchi identity and K denotes the trace of K_{ij} with respect to h_{ij}. In particular, if $J_{ijk} = 0$ and $V_{ij} = 0$, then $\Lambda_i = \bar{\Lambda}_i = 0$ automatically.

Taking the appropriate traces of (3a) and (3d), one obtains the Einstein constraint equations

$$J_{ij}{}^i \equiv D^i K_{ij} - D_j K = 0, \tag{5a}$$
$$V_i{}^i \equiv r - 2\lambda - K_{ij} K^{ij} + K^2 = 0. \tag{5b}$$

It follows then that any solution to the equations associated to the zero-quantities (3a)-(3d) gives rise also to a solution of the Einstein constraints. The reverse is also true, since, having obtained a solution $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K})$ of the Einstein constraints, one simply *defines*

$$S_{ij} = r_{ij} - \tfrac{2}{3}\lambda h_{ij} - K_i{}^k K_{jk} + K K_{ij}, \tag{6a}$$
$$\bar{S}_{kl} = -\epsilon_{lij} D^j K_k{}^i. \tag{6b}$$

By construction then we have $J_{ijk} = 0$, $V_{ij} = 0$, whence the integrability conditions imply $\Lambda_i = \bar{\Lambda}_i = 0$. Hence, solutions of the extended constraints and of the Einstein constraint equations are in direct correspondence.

Remark 2. Note that, assuming $V_{ij} = 0$, if one substitutes (3d) into (3c) one obtains

$$\bar{\Lambda}_l \equiv D^i \bar{S}_{il} - S^{ij} \epsilon_{ljk} K_i{}^k, \tag{7}$$

which better exhibits the *electromagnetic duality* between the electric and magnetic parts of the Weyl tensor: namely, that under the transformation

$$S_{ij} \longrightarrow \bar{S}_{ij}, \qquad \bar{S}_{ij} \longrightarrow -S_{ij},$$

the corresponding zero quantities transform as

$$\Lambda_i \longrightarrow \bar{\Lambda}_i, \qquad \bar{\Lambda}_i \longrightarrow -\Lambda_i.$$

We choose, however, to work with the system (3a)–(3d), since the resulting integrability conditions (18a)–(18b) enjoy a particular semi-decoupling of the zero-quantities J_{ijk} and V_{ij} that is convenient for the subsequent analysis, and that is lost if one uses the alternative definition of the zero-quantity $\bar{\Lambda}_i$, given by (7).

3 The Friedrich–Butscher Method

In this section, we outline the general procedure introduced in [8, 9] to construct solutions to the Einstein constraint equations, in addition to describing some of the potential obstructions to its implementation. As mentioned in the introduction, the procedure is of a perturbative nature —that is, one proves the existence of nonlinear perturbations of some *background initial data set*, denoted $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$, through the use of the *implicit function theorem.* In order to apply the implicit function theorem, one first derives from the extended constraint equations a so-called *auxiliary* system of equations which, given the appropriate choice of free and determined data, has a linearisation which is manifestly elliptic. By construction, any solution of the extended constraint equations is also a solution of the auxiliary equations. Having found, via the inverse function theorem, an open neighbourhood of solutions to the auxiliary system around the given background initial data set one must then show that such *candidate initial data set* is indeed a solution to the extended constraints —we refer to the latter as the problem of *sufficiency of the auxiliary system.*

In short, the Friedrich–Butscher method may be divided into two stages:

(i) ***Construction of candidate solutions:*** derive a auxiliary system of equations, with elliptic linearisation, and apply the implicit function theorem to guarantee existence of solutions.

(ii) ***Sufficiency***: prove that the solutions to the auxiliary system constructed in *Step (i)* are also solutions to the extended constraint equations.

In Section 3.4 we discuss the potential obstructions to the implementation of the above procedure. The desire to avoid such obstructions motivates our restriction to *conformally rigid hyperbolic* manifolds in Section 4.

3.1 Preliminaries

In the following, it will be convenient to a adopt a slightly more index-free notation that emphasises the structure of the equations. Given the Riemannian 3-manifold $(\mathcal{S}, \boldsymbol{h})$, we introduce the following spaces of tensors:

- $\Lambda^1(\mathcal{S})$, the space of covectors over $\mathcal{S}$;
- $\mathscr{S}^2(\mathcal{S})$, the space of symmetric 2-tensors over $\mathcal{S}$;
- $\mathscr{S}^2_0(\mathcal{S}; \boldsymbol{h})$, the space of symmetric 2-tensors over $\mathcal{S}$ that are tracefree with respect to the metric $\boldsymbol{h}$;
- $\mathscr{S}_{TT}(\mathcal{S}; \boldsymbol{h})$, the space of transverse-tracefree tensors over $\mathcal{S}$ with respect to the metric $\boldsymbol{h}$;
- $\mathcal{J}(\mathcal{S})$, the space of *Jacobi* tensors —i.e. tensors J_{ijk} satisfying

$$J_{ijk} = -J_{jik}, \qquad J_{ijk} + J_{jki} + J_{kij} = 0.$$

Remark 3. It will be useful to note that

$$\mathcal{J}(\mathcal{S}) \simeq \Lambda^1(\mathcal{S}) \oplus \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h}).$$

More precisely, any $J_{ijk} \in \mathcal{J}(\mathcal{S})$ may be uniquely decomposed as

$$J_{ijk} = \tfrac{1}{2}\left(\epsilon_{ij}{}^{l}F_{lk} + A_i h_{jk} - A_j h_{ik}\right), \tag{8}$$

where

$$A_j \equiv J_{jk}{}^{k}, \qquad F_{km} \equiv \epsilon_{ij(m}J^{ij}{}_{k)},$$

the latter being tracefree. In the previous expressions and in the following ϵ_{ijk} denotes the *volume form* of the metric $\boldsymbol{h}$. We will refer to (8) as the *Jacobi decomposition, with respect to* $\boldsymbol{h}$ of J_{ijk}.

We also introduce the following operators:

- $\Pi_{\boldsymbol{h}} : \mathscr{S}^2(\mathcal{S}) \longrightarrow \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$, the *projection of symmetric 2-tensors into the space of symmetric tracefree 2-tensors*, given by

$$\Pi_{\boldsymbol{h}}(\eta)_{ij} \equiv \eta_{ij} - \tfrac{1}{3}\mathrm{tr}_{\boldsymbol{h}}(\boldsymbol{\eta})h_{ij};$$

- $\star : \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h}) \longrightarrow \mathcal{J}(\mathcal{S})$, given by

$$(\star\eta)_{ijk} \equiv \epsilon^{l}{}_{ij}\eta_{kl};$$

  where ϵ_{ijk} denotes the volume form;

- $\delta_{\boldsymbol{h}} : \mathscr{S}^2(\mathcal{S}) \longrightarrow \Lambda^1(\mathcal{S})$, the *divergence operator*,

$$\delta_{\boldsymbol{h}}(\eta)_j \equiv D^i\eta_{ij};$$

- $L_{\boldsymbol{h}} : \Lambda^1(\mathcal{S}) \longrightarrow \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$ the *conformal Killing operator*,

$$L_{\boldsymbol{h}}(X)_{ij} \equiv D_iX_j + D_jX_i - \tfrac{2}{3}D^kX_kh_{ij};$$

- $\mathcal{D}_{\boldsymbol{h}} : \mathscr{S}^2(\mathcal{S}) \longrightarrow \mathcal{J}(\mathcal{S})$ the *Codazzi operator*,

$$\mathcal{D}_{\boldsymbol{h}}(\eta)_{ijk} \equiv D_i\eta_{jk} - D_j\eta_{ik},$$

- $\mathcal{D}^*_{\boldsymbol{h}} : \mathcal{J}(\mathcal{S}) \longrightarrow \mathscr{S}^2(\mathcal{S})$, the *formal L^2-adjoint of $\mathcal{D}_{\boldsymbol{h}}$ restricted to $\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$*, and given by

$$\mathcal{D}^*_{\boldsymbol{h}}(\mu)_{ij} \equiv D^k\mu_{ikj} + D^k\mu_{jki} - \tfrac{2}{3}D^k\mu_{lk}{}^{l}h_{ij};$$

- $\Delta_L : \mathscr{S}^2(\mathcal{S}) \longrightarrow \mathscr{S}^2(\mathcal{S})$, the *Lichnerowicz Laplacian*, acting as

$$\Delta_L\eta_{ij} \equiv -\Delta_h\eta_{ij} + 2r_{(i}{}^{k}\eta_{j)k} - 2r_{ikjl}\eta^{kl},$$

  where $\Delta_{\boldsymbol{h}} \equiv h^{ij}D_iD_j$ is the *rough Laplacian.*

Notation. Often, for the sake of simplicity, the subscript $\boldsymbol{h}$ in the symbol of the above operators will be omitted. When the above operators are defined with respect to the background metric $\mathring{\boldsymbol{h}}$ they will be distinguished by the symbol $^\circ$.

Remark 4. Following the standard usage, covectors in the Kernel of the conformal Killing operator $L_{\boldsymbol{h}}$ will be called *conformal Killing vectors*, while symmetric tensors in the Kernel of the Codazzi operator $\mathcal{D}_{\boldsymbol{h}}$ will be called *Codazzi tensors.* If, in addition, the tensor is tracefree with respect to the metric $\boldsymbol{h}$ then we talk of a *tracefree Codazzi tensor.*

Remark 5. Since $\mathcal{D}_{\boldsymbol{h}} : \mathscr{S}^2(\mathcal{S}) \longrightarrow \mathcal{J}(\mathcal{S})$, the image of $\mathcal{D}_{\boldsymbol{h}}$ may be decomposed as in Remark 3. In particular, given $\eta_{ij} \in \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$, $\mathcal{D}_{\boldsymbol{h}}(\boldsymbol{\eta})_{ijk}$ may be decomposed as follows

$$\mathcal{D}_{\boldsymbol{h}}(\boldsymbol{\eta})_{ijk} = \tfrac{1}{2}(\epsilon_{ij}{}^{l}\mathrm{rot}_2(\boldsymbol{\eta})_{lk} - \delta_{\boldsymbol{h}}(\boldsymbol{\eta})_i h_{jk} + \delta_{\boldsymbol{h}}(\boldsymbol{\eta})_j h_{ik}), \tag{9}$$

where $\mathrm{rot}_2(\boldsymbol{\eta})_{ij} \equiv \epsilon_{kl(i}D^k\eta^l{}_{j)}$. It therefore follows that $\mathcal{D}_{\boldsymbol{h}}(\boldsymbol{\eta})_{ijk} = 0$ for $\eta_{ij} \in \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$ if and only if $\delta(\boldsymbol{\eta})_i = 0$ *and* $\mathrm{rot}_2(\boldsymbol{\eta})_{ij} = 0$.

We recall that the divergence operator is undetermined elliptic and (equivalently) the conformal Killing operator L is overdetermined elliptic. Moreover, as shown in [9], the operator $\mathcal{D}_{\boldsymbol{h}}$ is overdetermined elliptic when restricted to $\mathscr{S}_0^2(\mathcal{S};\boldsymbol{h})$. More precisely, one has the following:

Lemma 1. *Given a covector* $\boldsymbol{\xi}$ *let*

$$\sigma_{\boldsymbol{\xi}}[\mathcal{D}_{\boldsymbol{h}}] : \mathscr{S}^2(\mathcal{S}) \longrightarrow \mathcal{J}(\mathcal{S})$$

denote the symbol map of $\mathcal{D}_{\boldsymbol{h}}$*. For* $\boldsymbol{\xi} \neq 0$*, the kernel of* $\sigma_{\boldsymbol{\xi}}[\mathcal{D}_{\boldsymbol{h}}]$ *is one dimensional —it consists of elements of the form* $c\xi_i\xi_j$*. It follows that the operator* $\mathcal{D}_{\boldsymbol{h}}|_{\mathscr{S}_0^2(\mathcal{S};\boldsymbol{h})}$ *is overdetermined elliptic.*

The proof is straightforward; the details can be found in [9].

Remark 6. In terms of the above definitions, the extended constraints encoded in the zero-quantities (3a)–(3d) may be rewritten as

$$\mathcal{D}_{\boldsymbol{h}}(K)_{ijk} - (\star\bar{S})_{ijk} = 0, \tag{10a}$$

$$\delta_{\boldsymbol{h}}(S)_i + \epsilon^{jk}{}_i K_j{}^l \bar{S}_{kl} = 0, \tag{10b}$$

$$\delta_{\boldsymbol{h}}(\bar{S})_i - \epsilon_i{}^{jk} K_k{}^l r_{lj} = 0, \tag{10c}$$

$$r_{ij} - \tfrac{2}{3}\lambda h_{ij} - S_{ij} + KK_{ij} - K_i{}^k K_{jk} = 0. \tag{10d}$$

3.2 The auxiliary system

The Friedrich–Butscher method for the construction of solutions to the Einstein constraint equations relies on first using the extended constraint equations to obtain a auxiliary system of equations whose linearisation is elliptic. The existence of solutions is then established through an application of the implicit function theorem. In general, the linearised system is a highly coupled second order system of partial differential equations. In the case of background data with metric of constant sectional curvature (i.e. Einstein manifolds), the linearised equations decouple sufficiently so as to enable a straightforward analysis of its kernel and cokernel —this system will be given in Section 4.2. Here, we discuss the procedure in full generality, but for simplicity we restrict attention to the principal parts of the equations, since they suffice for the description of ellipticity.

3.2.1 The ansatz

First note that, given a background initial data set $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$, there exists (see (6a) and (6b)) a corresponding background solution to the extended constraints, denoted $(\mathcal{S}, \mathring{\boldsymbol{K}}, \mathring{\bar{\boldsymbol{S}}}, \mathring{\boldsymbol{S}}, \mathring{\boldsymbol{h}})$, and which may moreover be decomposed as follows

$$\mathring{K}_{ij} = \kappa_{ij} + \tfrac{1}{3}\mathring{K}\mathring{h}_{ij}, \tag{11a}$$

$$\mathring{S}_{ij} = \mathring{L}(\boldsymbol{v})_{ij} + \psi_{ij}, \tag{11b}$$

$$\mathring{\bar{S}}_{ij} = \mathring{L}(\bar{\boldsymbol{v}})_{ij} + \bar{\psi}_{ij}, \tag{11c}$$

with $\kappa_{ij} \in \mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}})$, $v_i, \bar{v}_i \in \Lambda^1(\mathcal{S})$ and $\psi_{ij}, \bar{\psi}_{ij} \in \mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}})$. Decompositions (11b) and (11c) are precisely the *York splits* (see [28, 10]) of the electric and magnetic parts; such a split is always possible, and is moreover unique up to the addition of conformal Killing fields to $v_i, \bar{v}_i$.

Remark 7. In Section 4, we will restrict to background initial data which is Einstein and umbilical, for which $\kappa_{ij} = 0$, $v_i = \bar{v}_i = 0$ and $\psi_{ij} = \bar{\psi}_{ij} = 0$.

We will seek solutions of the extended constraints of the form

$$K_{ij} = \kappa_{ij} + \chi_{ij} + \tfrac{1}{3}(\mathring{K} + \phi)\,\mathring{h}_{ij}, \tag{12a}$$

$$S_{ij} = \Pi_{\boldsymbol{h}}(\mathring{L}(\boldsymbol{v} + \boldsymbol{X}) + \boldsymbol{\psi} + \boldsymbol{T})_{ij}, \tag{12b}$$

$$\bar{S}_{ij} = \Pi_{\boldsymbol{h}}(\mathring{L}(\bar{\boldsymbol{v}} + \bar{\boldsymbol{X}}) + \bar{\boldsymbol{\psi}} + \bar{\boldsymbol{T}})_{ij}, \tag{12c}$$

where χ_{ij} is tracefree with respect to the background metric $\mathring{\boldsymbol{h}}$, $\mathring{K}+\phi$ being the trace part, and where T_{ij}, $\bar{T}_{ij}$ are taken to be transverse-tracefree with respect to the background metric. Recall that $\Pi_{\boldsymbol{h}}$ is the projection onto $\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$, so that S_{ij} and $\bar{S}_{ij}$ are $\boldsymbol{h}$-tracefree, as required. We will use $\boldsymbol{S}(\boldsymbol{X},\boldsymbol{T})$, $\bar{\boldsymbol{S}}(\bar{\boldsymbol{X}},\bar{\boldsymbol{T}})$ as shorthands for (12b) and (12c). The above ansatz is motivated by the fact that the operator $\delta_{\boldsymbol{h}}$ is undetermined elliptic, while $\mathcal{D}_{\boldsymbol{h}}|_{\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})}$ is overdetermined elliptic. Note that the background solution corresponds to taking

$$(\boldsymbol{\chi},\bar{\boldsymbol{X}},\boldsymbol{X},\boldsymbol{h})=(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0},\mathring{\boldsymbol{h}}) \qquad \text{and} \qquad (\phi,\bar{\boldsymbol{T}},\boldsymbol{T})=(0,\boldsymbol{0},\boldsymbol{0})$$

in (12a)–(12c).

Remark 8. Here we adopt a slightly different approach to that of [8, 9], which uses the ansatz

$$S_{ij}=L_{\boldsymbol{h}}(\boldsymbol{X})_{ij}+\Pi_{\boldsymbol{h}}T_{ij},$$

with T_{ij} a transverse-tracefree tensor with respect to $\mathring{\boldsymbol{h}}$. The reason for using (12b)–(12c) is that we will be able to use the orthogonality property of the York split (with respect to $\mathring{\boldsymbol{h}}$) — see [10]— to argue, in a straightforward way, that the solutions are uniquely determined by the freely-prescribed data $(\phi,\boldsymbol{T},\bar{\boldsymbol{T}})$.

3.2.2 The linearisation of the Ricci operator

Let us now consider equation (3d). As is well known, the linearised Ricci operator is not elliptic. The failure of the linearised Ricci operator to be elliptic is a consequence of diffeomorphism-invariance, as encoded by the contracted Bianchi identity —see, for instance, [12]. One method of breaking the gauge-invariance is via the use of a variation of the so-called *DeTurck trick*. Here we follow this approach.

Let $\mathring{D}$ denote the Levi-Civita connection associated to $\mathring{\boldsymbol{h}}$. The *linearisation of the Ricci operator*, $\breve{r}(\gamma)_{ij}$, about $\mathring{h}_{ij}$ acting on a symmetric tensor field γ_{ij} (the *metric perturbation*) is given by the following *Fréchet derivative*

$$\begin{aligned}\breve{r}(\gamma)_{ij} &\equiv \frac{d}{d\tau}r[\mathring{\boldsymbol{h}}+\tau\boldsymbol{\gamma}]_{ij}\bigg|_{\tau=0} &(13)\\ &= -\tfrac{1}{2}\mathring{\Delta}\gamma_{ij}+\tfrac{1}{2}\mathring{D}_k\mathring{D}_i\gamma_j{}^k+\tfrac{1}{2}\mathring{D}_k\mathring{D}_j\gamma_i{}^k-\tfrac{1}{2}\mathring{D}_i\mathring{D}_j\gamma\\ &= -\tfrac{1}{2}\mathring{\Delta}\gamma_{ij}+\tfrac{1}{2}\mathring{D}_i\mathring{D}_k\gamma_j{}^k+\tfrac{1}{2}\mathring{D}_j\mathring{D}_k\gamma_i{}^k-\tfrac{1}{2}\mathring{D}_i\mathring{D}_j\gamma+\mathring{r}_{(i}{}^k\gamma_{j)k}-\mathring{r}_{ikjl}\gamma^{kl}\\ &= \tfrac{1}{2}\Delta_L\gamma_{ij}+\mathring{D}_{(i}C(\boldsymbol{\gamma})_{j)}{}^k{}_k, &(14)\end{aligned}$$

where, here, τ is a real parameter describing a a one-parameter-family of metrics, $\boldsymbol{h}(\tau)=\mathring{\boldsymbol{h}}+\tau\boldsymbol{\gamma}$, and $C(\cdot)^i{}_{jk}$ is defined by

$$C(\boldsymbol{\gamma})^i{}_{jk}\equiv\tfrac{1}{2}(\mathring{D}_j\gamma_k{}^i+\mathring{D}_k\gamma_j{}^i-\mathring{D}^i\gamma_{jk}). \tag{15}$$

Here, and it what follows, index raising and lowering within a linearised covariant will be carried out with respect to the background metric, $\mathring{\boldsymbol{h}}$. The first term of (14), $\Delta_L\gamma_{ij}$, is manifestly elliptic, but the ellipticity is spoiled by the second-order term $\mathring{D}_{(i}C_{j)}{}^k{}_k$. Now, given an arbitrary local coordinate system, (x^α), define the following

$$Q(\tau)^\alpha\equiv\frac{1}{2}h(\tau)^{\beta\gamma}(\Gamma(\boldsymbol{h}(\tau))^\alpha_{\beta\gamma}-\mathring{\Gamma}^\alpha_{\beta\gamma}),$$

where $h(\tau)^{\beta\gamma}$ is the inverse of $h(\tau)_{\alpha\beta}$, and $\Gamma(\boldsymbol{h}(\tau))^\alpha_{\beta\gamma}$, $\mathring{\Gamma}^\alpha_{\beta\gamma}$ denote respectively the Christoffel symbols of the metrics $\boldsymbol{h}(\tau)$ and $\mathring{\boldsymbol{h}}$ in the local coordinates, (x^α).

Remark 9. Note that, though Q^α is defined with respect to a fixed local coordinate system, the expression is in fact covariant, being given by the trace of the difference of two connections (i.e the trace of the *transition tensor*, $S^k{}_{ij}$). Hence, $\boldsymbol{Q}$ represents a (globally-defined) vector field, which we will denote in the abstract index formalism by Q^i. The remaining calculations of the article will be carried out in the abstract index notation.

Consider now the Lie derivative of the metric along $Q(\tau)$, $\mathcal{L}_{Q(\tau)}h(\tau)_{ij}$, the linearisation of which is given by

$$\frac{d}{d\tau}(\mathcal{L}_{Q(\tau)}\boldsymbol{h}(\tau))_{ij}\bigg|_{\tau=0} = \mathring{D}_{(i}C_{j)}{}^{k}{}_{k},$$

which is precisely the term in (14) obstructing the ellipticity in the linearised Ricci operator. Accordingly, we define the *reduced Ricci operator*, $\mathrm{Ric}^Q(\cdot)$, as

$$\mathrm{Ric}^Q(\boldsymbol{h})_{ij} \equiv r_{ij} - (\mathcal{L}_Q h)_{ij}.$$

The linearisation of the reduced Ricci operator can then be seen to be proportional to the Lichnerowicz Laplacian of the background metric —that is,

$$D\mathrm{Ric}^Q(\mathring{\boldsymbol{h}})\cdot\gamma_{ij} = \tfrac{1}{2}\mathring{\Delta}_L\gamma_{ij},$$

which is manifestly elliptic —note that, modulo curvature terms, Δ_L is simply the rough Laplacian and, therefore, clearly elliptic —see e.g. also [14] for an alternative discussion of the above.

Remark 10. The reduced Ricci operator coincides with the Ricci operator when $Q^i = 0$. The linearisation $D\mathrm{Ric}^Q(\cdot)$ is formally identical to that obtained through the use of (generalised) harmonic coordinates.

3.2.3 The auxiliary extended constraint map

Following the discussion of the previous subsections, it is convenient to define the *auxiliary extended constraint map*

$$\Psi(\boldsymbol{\chi},\bar{\boldsymbol{X}},\boldsymbol{X},\boldsymbol{h};\phi,\bar{\boldsymbol{T}},\boldsymbol{T}) \equiv \begin{pmatrix} \mathring{\mathcal{D}}^*(\boldsymbol{J})_{ij} \\ \bar{\Lambda}_i \\ \Lambda_i \\ V_{ij} - \mathcal{L}_Q h_{ij} \end{pmatrix} = \begin{pmatrix} \mathring{\mathcal{D}}^*\left(\mathcal{D}_{\boldsymbol{h}}(\boldsymbol{K}) - \star\bar{\boldsymbol{S}}\right)_{ij} \\ \delta_{\boldsymbol{h}}(\bar{\boldsymbol{S}})_i - \epsilon^{jk}{}_i\chi_j{}^l S_{kl} \\ \delta_{\boldsymbol{h}}(\boldsymbol{S})_i + \epsilon^{jk}{}_i\chi_j{}^l \bar{S}_{kl} \\ \mathrm{Ric}^Q(\boldsymbol{h})_{ij} - \frac{2}{3}\lambda h_{ij} - S_{ij} + KK_{ij} - K_i{}^k K_{jk} \end{pmatrix}$$

with the understanding that the fields K_{ij}, S_{ij}, $\bar{S}_{ij}$ should be substituted by the ansatz (12a)–(12c). In terms of the latter, the *auxiliary system* is then given by

$$\Psi(\boldsymbol{\chi},\bar{\boldsymbol{X}},\boldsymbol{X},\boldsymbol{h};\phi,\bar{\boldsymbol{T}},\boldsymbol{T}) = 0, \tag{16}$$

which is to be read as a (second-order) system of partial differential equations for the fields $\boldsymbol{\chi},\bar{\boldsymbol{X}},\boldsymbol{X},\boldsymbol{h}$ while the fields $\phi,\bar{\boldsymbol{T}},\boldsymbol{T}$ are regarded as input —i.e. they are the freely specifiable data.

Remark 11. Note that the auxiliary system is defined always with reference to some fixed *background solution* $(\mathring{\boldsymbol{K}},\mathring{\bar{\boldsymbol{S}}},\mathring{\boldsymbol{S}},\mathring{\boldsymbol{h}})$ of the extended constraints, both through the ansatz (12a)-(12c) and through the definition of the reduced Ricci operator. It is straightforward to see that, for any given background solution, we have

$$\Psi(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0},\boldsymbol{0};0,\boldsymbol{0},\boldsymbol{0}) = 0$$

—that is to say, that the background solution (corresponding to trivial free and determined fields) itself solves the corresponding auxiliary equations.

In the following, we denote by $D\Psi[\mathring{\boldsymbol{K}},\mathring{\bar{\boldsymbol{X}}},\mathring{\boldsymbol{X}},\mathring{\boldsymbol{h}}]\cdot(\boldsymbol{\sigma},\bar{\boldsymbol{\xi}},\boldsymbol{\xi},\gamma)$ the linearisation of Ψ at $(\mathring{\boldsymbol{K}},\mathring{\bar{\boldsymbol{X}}},\mathring{\boldsymbol{X}},\mathring{\boldsymbol{h}})$ in the direction of the determined fields —that is to say, the following

$$D\Psi[\mathring{\boldsymbol{K}},\mathring{\bar{\boldsymbol{X}}},\mathring{\boldsymbol{X}},\mathring{\boldsymbol{h}}]\cdot(\gamma,\boldsymbol{\sigma},\boldsymbol{\xi},\bar{\boldsymbol{\xi}}) = \frac{d}{d\tau}\Psi(\mathring{\boldsymbol{h}}+\tau\gamma,\mathring{\boldsymbol{\chi}}+\tau\boldsymbol{\sigma},\mathring{\boldsymbol{X}}+\tau\boldsymbol{\xi},\mathring{\bar{\boldsymbol{X}}}+\tau\bar{\boldsymbol{\xi}};\ \phi,\bar{\boldsymbol{T}},\boldsymbol{T})\bigg|_{\tau=0},$$

where $\mathring{\boldsymbol{X}}_i$, $\mathring{\bar{\boldsymbol{X}}}$ are the covector fields appearing in the York decomposition of the background electric and magnetic Weyl curvatures, $\mathring{\boldsymbol{S}}$, $\mathring{\bar{\boldsymbol{S}}}$, and $\mathring{\boldsymbol{\chi}}$ is the tracefree part of $\mathring{\boldsymbol{K}}$ with respect to $\mathring{\boldsymbol{h}}$.

Notation. We will often denote $D\Psi[\mathring{\boldsymbol{K}}, \mathring{\bar{\boldsymbol{X}}}, \mathring{\boldsymbol{X}}, \mathring{\boldsymbol{h}}]$ by $D\Psi$ for notational convenience.

Note that, as they are held fixed, the free data $(\phi, \bar{\boldsymbol{T}}, \boldsymbol{T})$ are not an input for $D\Psi$. We will not give the expression for $D\Psi$ for a general background here. It will suffice for the purposes of this section to consider only the principal parts as a second-order system of partial differential equations —namely,

$$\begin{pmatrix} \mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}} & \mathring{\mathcal{D}}^*(\star \mathring{L}) & 0 & 0 \\ 0 & \mathring{\delta} \circ \mathring{L} & 0 & 0 \\ 0 & 0 & \mathring{\delta} \circ \mathring{L} & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\mathring{\Delta} \end{pmatrix} \begin{pmatrix} \sigma_{ij} \\ \bar{\xi}_i \\ \xi_i \\ \gamma_{ij} \end{pmatrix}.$$

Since the principal part is upper-triangular, to verify ellipticity of the full system we need consider only the diagonal entries, which are elliptic by construction —one proceeds from the bottom-right, verifying invertibility of the symbol of each row, and successively substituting into the row above where necessary. It follows then that $D\Psi$ is a Fredholm operator. The dimension of the Kernel of the operator and that of its adjoint can be conveniently analysed using the Atiyah-Singer Index theorem —see Remark 23.

3.3 The sufficiency argument

Let us now assume that *Step (i)* (see beginning of Section 3) has been carried out: that is to say, that we have established the existence of a small neighbourhood of solutions to the auxiliary system (16). In particular we have

$$\mathring{\mathcal{D}}^*(J)_{ij} = 0, \tag{17a}$$
$$V_{ij} = (\mathcal{L}_Q \boldsymbol{h})_{ij}, \tag{17b}$$
$$\Lambda_i = \bar{\Lambda}_i = 0. \tag{17c}$$

In order to conclude that such solutions of the auxiliary system indeed solve the extended constraint equations, there remains the task of showing:

(a) that $(\mathcal{L}_Q \boldsymbol{h})_{ij} = 0$ in order that $\mathrm{Ric}(\boldsymbol{h}) = \mathrm{Ric}^Q(\boldsymbol{h})$, implying (3d);

(b) that $J_{ijk} = 0$ so that (3a) is satisfied.

Remark 12. Item (a) can be thought of as the analogue of gauge propagation in the hyperbolic reduction of the Einstein field equations.

The tasks (a)-(b) will be established with the help of the integrability conditions (4a)-(4b), which in view of (17c), reduce to

$$\epsilon^{ijk} D_i J_{jkl} = 0, \tag{18a}$$
$$D^i (\mathcal{L}_Q \boldsymbol{h})_{ij} - \tfrac{1}{2} D_j (\mathcal{L}_Q \boldsymbol{h})_i{}^i = K_{ik} J_j{}^{ik} - K_{jk} J^{ik}{}_i - K J_j{}^i{}_i. \tag{18b}$$

The strategy will be to use (17a) and (18a) to first show that $J_{ijk} = 0$, and then to substitute into (18b), which will be used to show $Q_i = 0$.

3.3.1 Elliptic equations for Q_i and J_{ijk}

First, it will prove convenient to first define the operator

$$\mathcal{K}_{\boldsymbol{h}} : \mathcal{J}(\mathcal{S}) \longrightarrow \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}}) \oplus \Lambda^1(\mathcal{S})$$

acting as

$$\mathcal{K}_{\boldsymbol{h}}(\boldsymbol{J}) = \begin{pmatrix} \mathring{\mathcal{D}}^*(\boldsymbol{J})_{ij} \\ \epsilon^{ijk} D_i J_{jkl} \end{pmatrix}.$$

As remarked previously, a solution $(\boldsymbol{K}, \bar{\boldsymbol{S}}, \boldsymbol{S}, \boldsymbol{h})$ furnished in Step (i) gives rise to a zero quantity J_{ijk} satisfying equations (17a) and (18a), and which therefore lies in the kernel of the operator

$\mathcal{K}_{\boldsymbol{h}}$ —that is to say, $\mathcal{K}_{\boldsymbol{h}}(\boldsymbol{J}) = 0$. In order to establish that $J_{ijk} = 0$ (see point (b), above), it suffices to show that $\mathcal{K}_{\boldsymbol{h}}$ has a trivial kernel. To do so, we aim to first establish injectivity of the operator $\mathcal{K}_{\mathring{\boldsymbol{h}}}$, and then to show that injectivity is preserved provided the metric $\boldsymbol{h}$ is sufficiently close to $\mathring{\boldsymbol{h}}$, in the appropriate norm. This "stability" property of the kernel of $\mathcal{K}_{\boldsymbol{h}}$ relies crucially on the observation that the operator is, in fact, first-order elliptic —see Lemma 2 and Proposition 3 in Sections 4.3.1 and 4.3.2.

On the other hand, note that

$$
\begin{aligned}
D^i(\mathcal{L}_Q\boldsymbol{h})_{ij} - \tfrac{1}{2}D_j(\mathcal{L}_Q\boldsymbol{h})_i{}^i &= D^i\left(D_iQ_j + D_jQ_i - D^kQ_kh_{ij}\right) \\
&= \Delta_{\boldsymbol{h}}Q_j + D^iD_jQ_i - D_jD^kQ_k \\
&= \Delta_{\boldsymbol{h}}Q_j + \left(D_jD^iQ_i + r_{ij}Q^i\right) - D_jD^kQ_k \\
&= \Delta_{\boldsymbol{h}}Q_j + r_{ij}Q^i.
\end{aligned}
$$

Therefore, if $J_{ijk} = 0$, then (18b) implies the elliptic equation

$$
\Delta_{\boldsymbol{h}}Q_j + r_{ij}Q^i = 0,
$$

for the zero quantity Q_i. Integrating by parts over the closed manifold $\mathcal{S}$, it follows that

$$
\int_{\mathcal{S}} \left(\|D\boldsymbol{Q}\|^2_{\boldsymbol{h}} - r_{ij}Q^iQ^j\right)\, d\mu_{\boldsymbol{h}} = 0. \tag{19}
$$

Note that the above identity only follows once it has been established that $J_{ijk} = 0$. Fortunately, the equation $\mathcal{K}_{\boldsymbol{h}}(\boldsymbol{J}) = 0$ is decoupled from Q_i as a consequence of the semi-decoupling of (18b)–(18a), as described in Remark 2. This decoupling allows for a two step approach in which we first show $J_{ijk} = 0$ and then use (19) to show $Q_i = 0$. The full argument is given in Proposition 4 of Section 4.3.2.

3.4 Obstructions to the existence of solutions

In order to use the implicit function theorem (see Section 4.2) to establish existence of solutions to the auxiliary system

$$
\Psi = 0,
$$

one would like to show that the linearisation $D\Psi$ is an isomorphism between suitable Banach spaces. Accordingly, by an *obstruction to the existence of solutions*, we mean a non-trivial element of either $\ker(D\Psi)$ or $\operatorname{coker}(D\Psi)$ —recalling that $D\Psi$ is an elliptic (and hence Fredholm) operator, the existence of a non-trivial cokernel is precisely the obstruction to surjectivity of $D\Psi$ while the existence of a non-trivial kernel is the obstruction to injectivity.

As it will be seen, among the potential obstructions to the existence of solutions one has non-trivial conformal Killing vectors and tracefree Codazzi tensors of the background manifold. Precluding the existence of such obstructions is the fundamental motivation behind our choice of background data.

Remark 13. It is not clear whether the obstructions that will be identified in the sequel are essential, or may be circumvented. In [8, 9], for instance, the method follows through despite the existence of non-trivial conformal Killing vectors. There, in *Step (i)* the auxiliary system is solved only up to an *error term*, constrained to lie in a finite-dimensional space. In *Step (ii)*, it is then simultaneously shown that the error term must necessarily vanish and that the extended constraints are indeed satisfied, as a consequence of the non-linear integrability conditions (18a)-(18b). Whether such a procedure may be implemented in general is unclear. One might expect the method to be more rigid in the compact case —the non-existence of conformal Killing vectors, for instance, may be a prerequisite. An analogy may be drawn here with the problem of *linearisation stability* of the constraint equations, in which the obstructions to integrability are precisely the so-called *KID sets*, describing the projection onto $\mathcal{S}$ of a spacetime Killing vector. In the case of non-compact $\mathcal{S}$, a solution of the constraint equations may still be linearisation stable even when it admits a KID set, at least when the perturbations of the initial data are restricted to those of sufficiently fast decay at infinity (see for example [2]), while the compact case is more rigid.

3.4.1 Conformal Killing vectors

It is clear from the construction of the auxiliary system that the existence of a non-trivial conformal Killing vector in the background Riemannian manifold $(\mathcal{S}, \mathring{\boldsymbol{h}})$, η_i say, destroys the injectivity of $D\Psi$, because of the use of the ansatz (12b)-(12c). Indeed, $\ker(D\Psi)$ contains linear combinations of

$$(\sigma_{ij},\ \bar{\xi}_i,\ \xi_i,\ \gamma_{ij}) = (\mathbf{0},\ \eta_i,\ \mathbf{0},\ \mathbf{0}) \qquad \text{and} \qquad (\sigma_{ij},\ \bar{\xi}_i,\ \xi_i,\ \gamma_{ij}) = (\mathbf{0},\ \mathbf{0},\ \eta_i,\ \mathbf{0}).$$

Moreover, in the case of a constant mean curvature background, the second component of $D\Psi$ takes the form

$$\mathring{\delta}(\mathring{L}(\bar{\boldsymbol{\xi}})) = 0$$

and therefore in this case $\mathrm{coker}(D\Psi)$ also contains elements of the form

$$(\sigma_{ij},\ \bar{\xi}_i,\ \xi_i,\ \gamma_{ij})^* = (\mathbf{0},\ \eta_i,\ \mathbf{0},\ \mathbf{0}),$$

so that $D\Psi$ also fails to be surjective —here we are using the suffix * as a shorthand to denote an arbitrary element of the codomain of $D\Psi$. Similar difficulties arise in both the conformal method and the gluing methods, whenever there exist non-trivial conformal Killing vectors —see, for instance, [3].

Remark 14. From the previous discussion, it follows that the implementation of the Friedrich–Butscher method will be simplified if one restricts to *background initial data sets which do not admit a conformal Killing vector.* This condition holds, in particular, for manifolds of negative-definite Ricci curvature —the conformal Killing equation implies after contraction with $D^i\eta^j$ and integration by parts that

$$\int_{\mathcal{S}} \left(\|\mathring{D}\boldsymbol{\eta}\|^2_{\mathring{\boldsymbol{h}}} + \tfrac{1}{3}|\mathring{\delta}(\boldsymbol{\eta})|^2 - \mathring{r}_{ij}\eta^i\eta^j \right)\, d\mu_{\mathring{\boldsymbol{h}}} = 0.$$

Thus, if the Ricci tensor is negative-definite then $\eta_i = 0$ as a consequence of the positive-definiteness of the integrand. This is valid in particular for Einstein metrics of negative scalar curvature, despite them being locally maximally-symmetric —that is to say that, while there exists the maximal number of *local* Killing vector fields in a neighbourhood of each point, none may be extended globally to the whole manifold. A sufficient condition for the stronger requirement of non-existence of local conformal Killing vector fields is given in [6].

3.4.2 Non-trivial tracefree Codazzi tensors

Inspection of the auxiliary equation for the extrinsic curvature, equation (10a), readily shows that the existence of non-trivial tracefree *Codazzi* tensors in the background initial data set —i.e. elements of $\ker(\mathring{\mathcal{D}}) \cap \mathscr{S}^2_0(\mathcal{S}; \boldsymbol{h})$— also give rise to obstructions similar in nature to those arising from the existence of conformal Killing vectors. In this case, given a tracefree Codazzi tensor, η_{ij} say, $\ker(D\Psi)$ and $\mathrm{coker}(D\Psi)$ both contain elements of the form

$$(\eta_{ij},\ \mathbf{0},\ \mathbf{0},\ \mathbf{0})$$

which destroy both the injectivity and the surjectivity of $D\Psi$.

For examples of initial data sets which *do* admit tracefree Codazzi tensors, one needs only consider umbilical, conformally-flat initial data sets. Consider $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}} = \tfrac{1}{3}\mathring{K}\mathring{\boldsymbol{h}})$, $\mathring{K}$ a constant, which constitutes an *umbilical* initial data set provided

$$\mathring{r} = 2\lambda - \tfrac{2}{3}\mathring{K}^2.$$

If we restrict to those metrics $\mathring{\boldsymbol{h}}$ which are, in addition, conformally flat then it follows from the Weyl-Schouten Theorem (see Theorem 5.1 in [27]) that

$$0 = \mathcal{H}_{ij} \equiv \mathring{\epsilon}_{kl(i}\mathring{D}^k\mathring{r}_{j)}{}^l \equiv \mathring{\epsilon}_{kl(i}\mathring{D}^k\mathring{d}_{j)}{}^l,$$

where $\mathring{d}_{ij}$ denotes the tracefree part of the Ricci curvature. Moreover, it follows from the contracted second Bianchi identity that $\delta_{\mathring{\boldsymbol{h}}}(\mathring{\boldsymbol{d}})_i = 0$, again using the fact that $\mathring{r}$ is constant. Combining the above observations it follows (see Remark 5) that $\mathring{d}_{ij}$ is a tracefree Codazzi tensor —i.e. $\mathring{\mathcal{D}}(\mathring{\boldsymbol{d}})_{ijk} = 0$. *This Codazzi tensor is non-trivial (i.e. non-zero) if $\mathring{\boldsymbol{h}}$ is not an Einstein metric.*

Remark 15. The above observation is pertinent also to the case of non-compact $\mathcal{S}$. In particular, it suggests that the time-symmetric initial data set for the Schwarzschild spacetime, with metric

$$\mathring{\boldsymbol{h}} = \left(1 + \frac{m}{2r}\right)^4 \boldsymbol{\delta},$$

is potentially unsuitable (as background initial data) for the application of the Friedrich–Butscher method as $\mathring{\boldsymbol{h}}$ is not an Einstein metric.

3.4.3 Conformally rigid hyperbolic manifolds

From the previous two sections, we know that the existence of either a non-trivial conformal Killing vector or a non-trivial tracefree Codazzi tensor is undesirable for the application of the Friedrich–Butscher method on compact manifolds. Moreover, it was noted in Section 3.4.1 that a Riemannian manifold of negative-definite Ricci curvature cannot admit a globally-defined conformal Killing field, rendering such a manifold a natural first candidate for the background manifold $(\mathcal{S}, \mathring{\boldsymbol{h}})$.

Due to the highly-coupled nature of the auxiliary system of equations, $\Psi = 0$, the tractability of the required analysis is, of course, dependent on the specific properties of the background manifold, $(\mathcal{S}, \mathring{\boldsymbol{h}})$. In particular, if we consider a manifold $(\mathcal{S}, \mathring{\boldsymbol{h}})$ that is Einstein (or, equivalently, a *space form* since we are in dimension 3):

$$\mathring{r}_{ij} = \tfrac{1}{3}\mathring{r}\mathring{h}_{ij},$$

with $\mathring{r}$ (necessarily) constant, then $D\Psi$ simplifies significantly. The requirement that $\mathring{r}_{ij}$ be negative-definite is then simply that $\mathring{r}$ be negative.

Accordingly, let us restrict to an Einstein background manifold with negative Ricci scalar —we will refer to such a manifold as *hyperbolic*. Recall that, by the *Killing–Hopf Theorem* $(\mathcal{S}, \mathring{\boldsymbol{h}})$ is isometric to a quotient of the hyperbolic 3-space $\mathbb{H}^3$. We refer the reader to [7] for results concerning the admissible topologies of $\mathcal{S}$. Moreover, we would also like to exclude the possibility of a non-trivial tracefree Codazzi tensor —i.e. ensure that $\ker(\mathring{\mathcal{D}}) \cap \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}}) = \{0\}$. Now, in the case of hyperbolic manifolds —see [21] and also also [4]— the space of tracefree Codazzi tensors coincides with the space of *essential conformally flat deformations* —i.e. one has

$$\ker\{\mathring{\mathcal{D}} : \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}}) \to \mathcal{J}(\mathcal{S})\} = \ker\, \mathring{H} \cap \ker\, \mathring{\delta} \simeq \ker\, \mathring{H}/\mathring{L}(\Lambda^1(\mathcal{S})),$$

where $\mathring{H}$ denotes the *linearised Cotton map* —see Section 4.4 for more details. Consequently, we will refer to a hyperbolic manifold which admits no no-trivial tracefree Codazzi tensors as being *conformally rigid*. The requirement of conformal rigidity places additional restrictions on the topology of $\mathcal{S}$, but there remains a non-empty family of such manifolds —see [20].

4 Nonlinear perturbations of compact hyperbolic initial data

In the remainder of this article we restrict our attention to conformally rigid hyperbolic background initial data, since such manifolds admit neither conformal Killing fields nor tracefree Codazzi tensors.

The results here can be thought of spatially-closed analogues of those in [13], in which a version of the Friedrich–Butscher method was applied to non-compact hyperbolic background manifolds. We note however that here we solve the full extended constraint equations, rather than the reduced system corresponding to initial data sets of umbilic extrinsic curvature, as considered in [13] —i.e. we allow for non-trivial perturbations of the extrinsic curvature.

4.1 Statement of the main result

In the following, let $(\mathcal{S}, \mathring{\boldsymbol{h}})$ be a closed hyperbolic Einstein manifold with sectional curvature normalised to $k = -1$ (or, equivalently, with $\mathring{r} = -6$). Then, for any given constant $\mathring{K}$, the tensor fields

$$\mathring{h}_{ij}, \qquad \mathring{K}_{ij} = \tfrac{1}{3}\mathring{K}\mathring{h}_{ij}, \tag{20}$$

over $\mathcal{S}$ constitute a solution to the Einstein constraint equations with constant mean extrinsic curvature $\mathring{K}$ and with cosmological constant given by

$$\lambda = \tfrac{1}{3}(\mathring{K}^2 - 9),$$

as it can be readily seen from the Hamiltonian constraint (5b). Initial data of this type will be called *hyperbolic initial data*. The Cauchy stability of the development of initial data sets of this type, with $\lambda = 0$, was studied in [1].

Remark 16. Note that here we are choosing to normalise the intrinsic curvature, which in turn fixes the value of the cosmological constant, once the extrinsic curvature has been given. One could alternatively rescale the intrinsic and extrinsic curvatures appropriately so as to normalise the cosmological constant. The former option is chosen since, in the subsequent analysis, it is the intrinsic geometry of $(\mathcal{S}, \mathring{\boldsymbol{h}})$ that will be of primary importance.

Remark 17. The (unique) solution to the extended Einstein constraint equations associated to (20) is obtained by setting $\mathring{S}_{ij} = \mathring{\bar{S}}_{ij} = 0$ —see (6a)–(6b). Note that the sign of λ is dependent on the choice of $\mathring{K}$: $\lambda < 0$ for $|\mathring{K}| < 3$, $\lambda = 0$ for $\mathring{K} = \pm 3$ and $\lambda > 0$ for $|\mathring{K}| > 3$.

In the following it will prove convenient to define the constants

$$\alpha \equiv -4 + \frac{2}{9}\mathring{K}^2, \qquad \beta \equiv -4 + \frac{8}{9}\mathring{K}^2. \tag{21}$$

Define also for $s \geq 4$ the Banach spaces $\mathcal{X}^s, \mathcal{Y}^s, \mathcal{Z}^s$, as follows

$$\begin{aligned}
\mathcal{X}^s &\equiv H^{s-1}(\mathscr{C}(\mathcal{S})) \times H^{s-1}(\mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}})) \times H^{s-1}(\mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}})), \\
\mathcal{Y}^s &\equiv H^s(\mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})) \times H^s(\Lambda^1(\mathcal{S})) \times H^s(\Lambda^1(\mathcal{S})) \times H^s(\mathscr{S}^2(\mathcal{S})), \\
\mathcal{Z}^s &\equiv H^{s-2}(\mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})) \times H^{s-2}(\Lambda^1(\mathcal{S})) \times H^{s-2}(\Lambda^1(\mathcal{S})) \times H^{s-2}(\mathscr{S}^2(\mathcal{S})).
\end{aligned}$$

where $H^s(\cdot)$ denotes the Sobolev norm $W^{2,s}(\cdot)$ with the pointwise norms of tensor fields defined with respect to the background metric $\mathring{\boldsymbol{h}}$ —unless explicitly indicated otherwise, all H^s-norms from now on will be defined with respect to $\mathring{\boldsymbol{h}}$.

Remark 18. That the image of $\Psi : \mathcal{X}^s \times \mathcal{Y}^s$ is indeed contained in $\mathcal{Z}^s$ may be easily checked using the Schauder ring property: namely that $(\boldsymbol{u}, \boldsymbol{v}) \mapsto \boldsymbol{u} \otimes \boldsymbol{v}$ is continuous as a mapping from $H^{s_1} \times H^{s_2}$ to H^{s_3} provided $s_1 + s_2 > s_3 + n/2$ and $s_1, s_2 > s_3$ —see [11], for instance.

We are now in a position to state our main theorem:

Theorem 1. *Let $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$ be a smooth conformally rigid hyperbolic initial data set with constant mean extrinsic curvature $\mathring{K}$ satisfying*

$$\beta \notin \operatorname{Spec}\big(-\mathring{\Delta} : C^\infty(\mathcal{S}) \to C^\infty(\mathcal{S})\big). \tag{22}$$

Then, there exists an open neighbourhood $\mathcal{U} \subset \mathcal{X}$ of $(\mathbf{0}, \mathbf{0}, \mathbf{0})$, an open neighbourhood $\mathcal{W} \subset \mathcal{Y}$ of $(\mathring{\boldsymbol{h}}, \mathbf{0}, \mathbf{0}, \mathring{\boldsymbol{K}})$ and a smooth map $\nu : \mathcal{U} \to \mathcal{W}$ such that, defining

$$u \equiv (\phi, \boldsymbol{T}, \bar{\boldsymbol{T}}), \qquad \nu(u) \equiv \big(\boldsymbol{\chi}(u), \bar{\boldsymbol{X}}(u), \boldsymbol{X}(u), \boldsymbol{h}(u)\big),$$

the following assertions hold:

i) for each $(\phi, \boldsymbol{T}, \bar{\boldsymbol{T}}) \in \mathcal{U}$*,*

$$w(u) \equiv \big(\boldsymbol{\chi}(u) + \tfrac{1}{3}(\phi + \mathring{K})\mathring{\boldsymbol{h}},\ \bar{\boldsymbol{S}}(\bar{\boldsymbol{X}}(u), \bar{\boldsymbol{T}}),\ \boldsymbol{S}(\boldsymbol{X}(u), \boldsymbol{T}),\ \boldsymbol{h}(u)\big)$$

is a solution to the extended constraint equations (2) *with cosmological constant* $\lambda = (\mathring{K}^2 - 9)/3$*;*

ii) the map $u \mapsto w(u)$ *is injective for* $\mathring{K} \neq 0$*. Moreover, it is injective for* $\mathring{K} = 0$ *if we restrict the free datum* ϕ *to the sub-Banach space of functions which integrate to zero over* $\mathcal{S}$ *—that is to say that each such solution* w *corresponds to a unique choice of free data* $u = (\phi, \boldsymbol{T}, \bar{\boldsymbol{T}})$*.*

Remark 19. Notice that when $|\mathring{K}| \leq \sqrt{9/2}$ —and, in particular in the time-symmetric case, $\mathring{K} = 0$— condition (22) is satisfied trivially since $\beta < 0$ but $-\mathring{\Delta}$ is positive-semi-definite. Note that in this case the cosmological constant is negative ($\lambda < 0$). Moreover, since the spectrum of $-\mathring{\Delta}$ is discrete, condition (22) excludes only countably many values of $\mathring{K}$.

The theorem will be proven in two stages in the forthcoming sections, by means of Propositions 1 and 4. In Section 4.4 we describe a parametrisation of the free data through the use of the linearised Cotton map, based on the results of [4, 18], and summarised in Proposition 6.

4.2 Existence of solutions of the auxiliary system

The purpose of this section is to show the existence of perturbative solutions to the auxiliary system in the case of conformally rigid hyperbolic initial data sets.

4.2.1 Technical tools

The main tool used in establishing existence is the *Implicit Function Theorem* —see e.g. [15]— which we state here for completeness.

Theorem (***Implicit Function Theorem***)**.** *Let* $\mathcal{X}$*,* $\mathcal{Y}$*,* $\mathcal{Z}$ *be Banach spaces, and*

$$\Psi : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$$

a mapping with continuous Fréchet derivative. Suppose that $(x_0, y_0) \in \mathcal{X} \times \mathcal{Y}$ *satisfies* $\Psi(x_0, y_0) = 0$ *and that the map* $y \mapsto D\Psi(x_0, y_0)(0, y)$ *is a Banach space isomorphism from* $\mathcal{Y}$ *onto* $\mathcal{Z}$*. Then, there exist open neighbourhoods* $\mathcal{U}$ *of* x_0 *and* $\mathcal{V}$ *of* y_0 *and a Fréchet-differentiable mapping* $\nu : \mathcal{U} \to \mathcal{V}$ *such that* $\Psi(x, \nu(x)) = 0$ *for all* $x \in \mathcal{U}$*, and* $\Psi(x, y) = 0$ *for* $(x, y) \in \mathcal{U} \times \mathcal{V}$ *if and only if* $y = \nu(x)$*. Moreover, if the map* $x \mapsto D\Psi(x_0, y_0)(x, 0)$ *is injective, then* ν *is also injective.*

In order to establish that the various mappings of interest are isomorphisms, we will make use of the following *Splitting Lemma* —see e.g. [22].

Lemma (***Splitting Lemma***)**.** *Let* E *and* F *be vector bundles over* $\mathcal{S}$*, with fixed Riemannian metric* $\boldsymbol{h}$*. Let*

$$\mathscr{D} : C^\infty(E) \longrightarrow C^\infty(F)$$

be a differential operator of order k*, and* $\mathscr{D}^*$ *the corresponding formal* L^2*-adjoint. Suppose that* $\mathscr{D}$ *is overdetermined elliptic (equivalently,* $\mathscr{D}^*$ *is underdetermined elliptic), then for* $s \in [k, \infty)$

$$H^s(\mathcal{S}) = \mathrm{Im}\ \mathscr{D}^* \oplus \ker\ \mathscr{D},$$

where both factors are closed and are L^2*-orthogonal and* $\mathrm{Im}\ \mathscr{D}^* = \mathscr{D}^*(H^{s+k}(\mathcal{S}))$*. Moreover, if* $\mathscr{D}$ *is injective, then* $\mathscr{D}^*$ *is surjective, and the composition* $\mathscr{D}^* \circ \mathscr{D}$ *is an isomorphism.*

4.2.2 The application of the Implicit Function Theorem

Since the background solution admits no conformal Killing vectors and no non-trivial tracefree Codazzi tensors, the operators $\mathring{L}$ and $\mathring{\mathcal{D}}$ are both injective. Therefore, by the Splitting Lemma, the following are isomorphisms for $s \geq 4$:

$$\mathring{\delta} \circ \mathring{L} : H^s(\Lambda^1(\mathcal{S})) \to H^{s-2}(\Lambda^1(\mathcal{S})),$$
$$\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}} : H^s(\mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})) \to H^{s-2}(\mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})).$$

Since the background initial data, being hyperbolic, consists of an Einstein metric and umbilical extrinsic curvature, the linearisation of the auxiliary extended constraint map in the direction of the determined fields, $D\Psi$, takes the form

$$D\Psi \cdot (\boldsymbol{\sigma}, \bar{\boldsymbol{\xi}}, \boldsymbol{\xi}, \gamma; \phi, \bar{\boldsymbol{T}}, \boldsymbol{T}) = \begin{pmatrix} \mathring{\mathcal{D}}^*\big(\mathring{\mathcal{D}}(\boldsymbol{\sigma}) - \frac{1}{3}\mathring{K}\mathring{\mathcal{D}}(\gamma) - \mathring{\star}\mathring{L}(\bar{\boldsymbol{\xi}})\big)_{ij} \\ \mathring{\delta} \circ \mathring{L}(\bar{\boldsymbol{\xi}})_i \\ \mathring{\delta} \circ \mathring{L}(\boldsymbol{\xi})_i \\ \frac{1}{2}\mathring{\Delta}_L \gamma_{ij} - \frac{1}{2}\alpha\bar{\gamma}_{ij} - \frac{1}{6}\beta\gamma\mathring{h}_{ij} + \frac{1}{3}\mathring{K}\sigma_{ij} - \mathring{L}(\boldsymbol{\xi})_{ij} \end{pmatrix}.$$

Remark 20. Let $(A_{ij},\ \bar{B}_i,\ B_i,\ C_{ij}) \in \mathcal{Z}^s$ be arbitrary. Then in order to establish whether $D\Psi$ is an isomorphism, we are concerned with solving the system of equations

$$\mathring{\mathcal{D}}^*(\mathring{\mathcal{D}}(\boldsymbol{\sigma}) - \tfrac{1}{3}\mathring{K}\mathring{\mathcal{D}}(\boldsymbol{\gamma}) - \mathring{\star}\mathring{L}(\bar{\boldsymbol{\xi}}))_{ij} = A_{ij}, \tag{23a}$$
$$\mathring{\delta} \circ \mathring{L}(\bar{\boldsymbol{\xi}})_i = \bar{B}_i, \tag{23b}$$
$$\mathring{\delta} \circ \mathring{L}(\boldsymbol{\xi})_i = B_i, \tag{23c}$$
$$\mathring{\Delta}_L \gamma_{ij} - \alpha\bar{\gamma}_{ij} - \tfrac{1}{3}\beta\gamma\mathring{h}_{ij} + \tfrac{2}{3}\mathring{K}\sigma_{ij} - 2\mathring{L}(\boldsymbol{\xi})_{ij} = C_{ij}, \tag{23d}$$

where here γ and $\bar{\gamma}_{ij}$ denote the *trace and tracefree parts* of γ_{ij} with respect to $\mathring{\boldsymbol{h}}$, and the constants α, β are as defined in (21). Note the semi-decoupled form of the system: one can first solve (23b)-(23c), and then proceed to solve (23a) and (23d), in turn.

In order to address injectivity if the map ν, we also need to consider the linearisation of Ψ in the direction of the free data. For a general data set $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$ the linearisation is given by

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Psi(\boldsymbol{\chi}, \bar{\boldsymbol{X}}, \boldsymbol{X}, \boldsymbol{h};\ \mathring{K} + \tau\phi, \tau\bar{\boldsymbol{T}}, \tau\boldsymbol{T})\bigg|_{\tau=0} = \begin{pmatrix} -\frac{1}{6}\mathring{L}(d\phi)_{jk} - \frac{1}{2}\mathring{\epsilon}_{kil}\mathring{D}^l\bar{T}_j{}^i - \frac{1}{2}\mathring{\epsilon}_{jil}\mathring{D}^l\bar{T}_k{}^i \\ \mathring{\epsilon}_{ljk}\mathring{K}^{ij}T_i{}^k + \mathring{D}^i\bar{T}_{il} \\ -\mathring{\epsilon}_{ikl}\mathring{K}^{jk}\bar{T}_j{}^l + \mathring{D}^jT_{ij} \\ -T_{ij} + \frac{1}{3}(\mathring{K}_{ij} + \mathring{K}\mathring{h}_{ij})\phi \end{pmatrix}. \tag{24}$$

Remark 21. It is clear that if the above map is to be injective then we should at least require $T_{ij}, \bar{T}_{ij}$ to be tracefree with respect to $\mathring{\boldsymbol{h}}$ —it is easy to verify that pure trace T_{ij} and $\bar{T}_{ij}$ would be in the kernel. This further justifies the use of the ansatz (12b)-(12c).

The existence of solutions to the auxiliary system is established in the following proposition.

Proposition 1 (***existence of solutions to the auxiliary system***)**.** *Let $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$ be a smooth conformally rigid hyperbolic initial data set with (constant) mean extrinsic curvature $\mathring{K}$ satisfying condition* (22)*. Then $D\Psi : \mathcal{Y}^s \to \mathcal{Z}^s$ is a Banach space isomorphism for $s \geq 4$, and so (by the implicit function theorem) there exist open neighbourhoods $(\mathring{K}, \mathbf{0}, \mathbf{0}) \in \mathcal{V} \subset \mathcal{Y}^s$ and $(\mathring{\boldsymbol{K}}, \mathbf{0}, \mathbf{0}, \mathring{\boldsymbol{h}}) \in \mathcal{U} \subset \mathcal{X}^s$ and a Fréchet differentiable map $\nu : \mathcal{U} \to \mathcal{V}$ mapping free data to solutions of the auxiliary system $\Psi = 0$. Moreover the map ν is injective.*

Proof.

Injectivity of $\mathbf{D\Psi}$. Taking $A_{ij} = C_{ij} = 0,\ B_i = \bar{B}_i = 0$ in equations (23a)-(23d), we aim to show triviality of solutions $(\boldsymbol{\sigma}, \bar{\boldsymbol{\xi}}, \boldsymbol{\xi}, \boldsymbol{\gamma})$. Note that by elliptic regularity (see Appendix I of [7], for instance), it suffices to show restrict to smooth $(\boldsymbol{\sigma}, \bar{\boldsymbol{\xi}}, \boldsymbol{\xi}, \boldsymbol{\gamma})$. Equations (23b)-(23c) imply, firstly, that $\xi_i = \bar{\xi}_i = 0$ since the background metric admits no global conformal Killing vectors. Substituting into (23a) and (23d)

$$\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}}(\boldsymbol{\sigma} - \tfrac{1}{3}\mathring{K}\boldsymbol{\gamma})_{ij} = 0, \tag{25a}$$

$$\mathring{\Delta}_L \gamma_{ij} - \alpha\bar{\gamma}_{ij} - \tfrac{1}{3}\beta\gamma\mathring{h}_{ij} + \tfrac{2}{3}\mathring{K}\sigma_{ij} = 0. \tag{25b}$$

Tracing (25b) we obtain

$$-(\mathring{\Delta} + \beta)\gamma = 0.$$

By assumption $\beta \notin \mathrm{Spec}(-\mathring{\Delta})$ and therefore $\gamma = 0$. Substituting into (25a)

$$\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}}(\boldsymbol{\sigma} - \tfrac{1}{3}\mathring{K}\bar{\boldsymbol{\gamma}})_{ij} = 0. \tag{26}$$

Now, since $\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}} : \mathscr{S}_0^2(\mathcal{S}; \mathring{\boldsymbol{h}}) \to \mathscr{S}_0^2(\mathcal{S}; \mathring{\boldsymbol{h}})$ is an isomorphism, $\sigma_{ij} = \frac{1}{3}\mathring{K}\bar{\gamma}_{ij}$. Substituting into (25b) along with $\gamma = 0$ yields

$$\mathring{\Delta}_L\bar{\gamma}_{ij} + 4\bar{\gamma}_{ij} \equiv -\mathring{\Delta}\bar{\gamma}_{ij} - 2\bar{\gamma}_{ij} = 0. \tag{27}$$

We will now show that $(\mathring{\Delta}_L + 4) : \mathscr{S}_0^2(\mathcal{S}; \mathring{h}) \to \mathscr{S}_0^2(\mathcal{S}; \mathring{h})$ is injective (and hence, by self-adjointness, an isomorphism). First, taking the divergence of (27), commuting derivatives and using the fact that the background metric is Einstein (with $\mathring{r} = -6$), we find that

$$\begin{aligned} 0 &= -\mathring{D}^i(\mathring{\Delta}\bar{\gamma}_{ij} + 2\bar{\gamma}_{ij}) \\ &= -\mathring{\Delta}\mathring{\delta}(\bar{\boldsymbol{\gamma}})_j - \mathring{D}^k(\mathring{r}_k{}^l\bar{\gamma}_{lj} - \mathring{r}_j{}^l{}_k{}^i\bar{\gamma}_{il}) - \mathring{r}_j{}^{lik}\mathring{D}_k\bar{\gamma}_{il} - 2\mathring{\delta}(\bar{\boldsymbol{\gamma}})_j \\ &= -\mathring{\Delta}\mathring{\delta}(\bar{\boldsymbol{\gamma}})_j - \mathring{r}^{kl}\mathring{D}_k\bar{\gamma}_{lj} - 2\mathring{r}_j{}^{lik}\mathring{D}_k\bar{\gamma}_{il} - 2\mathring{\delta}(\bar{\boldsymbol{\gamma}})_j \\ &= (-\mathring{\Delta} + 2)\mathring{\delta}(\bar{\boldsymbol{\gamma}})_j, \end{aligned}$$

and hence we see that $\mathring{\delta}(\bar{\boldsymbol{\gamma}}) = 0$ by positivity of $(-\mathring{\Delta} + 2) : \Lambda^1(\mathcal{S}) \to \Lambda^1(\mathcal{S})$. Now,

$$\begin{aligned} \mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}}(\bar{\gamma})_{ij} &= \mathring{\Delta}\bar{\gamma}_{ij} - \tfrac{1}{2}\mathring{D}_k\mathring{D}_i\bar{\gamma}_j{}^k - \tfrac{1}{2}\mathring{D}_k\mathring{D}_j\bar{\gamma}_i{}^k + \tfrac{1}{3}\mathring{D}^k\mathring{D}^l\bar{\gamma}_{kl}\mathring{h}_{ij} \\ &= \mathring{\Delta}\bar{\gamma}_{ij} - \tfrac{1}{2}\mathring{D}_i\mathring{D}_k\bar{\gamma}_j{}^k - \tfrac{1}{2}\mathring{D}_j\mathring{D}_k\bar{\gamma}_i{}^k + \tfrac{1}{3}\mathring{D}^k\mathring{D}^l\bar{\gamma}_{kl}\mathring{h}_{ij} + 3\bar{\gamma}_{ij} \\ &= -(\mathring{\Delta}_L + 4)\bar{\gamma}_{ij} + \bar{\gamma}_{ij} \\ &= \bar{\gamma}_{ij}, \end{aligned}$$

where in the third line we are using $\mathring{\delta}(\bar{\boldsymbol{\gamma}}) = 0$ and in the fourth we are using (27). However, clearly $\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}}$ is negative-definite, and so we find that $\bar{\gamma}_{ij} = 0$ —that is to say, $(\mathring{\Delta}_L + 4)$ is injective. Collecting everything together, we have found that

$$\sigma_{ij} = \gamma_{ij} = 0, \qquad \xi_i = \bar{\xi}_i = 0,$$

—i.e. the map $D\Psi$ is injective.

Surjectivity of $\mathbf{D\Psi}$. The argument for surjectivity is similar. First, since $\mathring{\delta}\circ\mathring{L}$ is an isomorphism, equations (23b)-(23c) admit (unique) solutions $\bar{\xi}_i$, ξ_i, for any given $\bar{B}_i$, B_i. Substituting into equations (23a) and (23d) and rearranging one obtains

$$\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}}(\boldsymbol{\varsigma} - \tfrac{1}{9}\mathring{K}\gamma\mathring{\boldsymbol{h}})_{ij} = A_{ij} + \mathring{\mathcal{D}}^*(\mathring{\star}\mathring{L}(\bar{\boldsymbol{\xi}})), \tag{28a}$$

$$\mathring{\Delta}_L\gamma_{ij} + 4\bar{\gamma}_{ij} - \tfrac{1}{3}\beta\gamma\mathring{h}_{ij} + \tfrac{2}{3}\mathring{K}\varsigma_{ij} = C_{ij} + 2\mathring{L}(\boldsymbol{\xi})_{ij}, \tag{28b}$$

where, for simplicity, we have defined

$$\varsigma_{ij} \equiv \sigma_{ij} - \tfrac{1}{3}\mathring{K}\bar{\gamma}_{ij}.$$

Note that ς_{ij} is tracefree with respect to $\mathring{\boldsymbol{h}}$. Taking the trace of (28b) one obtains

$$-(\mathring{\Delta} + \beta)\gamma = C_k{}^k,$$

which admits a unique solution, since $\beta \notin \mathrm{Spec}(-\mathring{\Delta})$ implies that $-(\mathring{\Delta}+\beta)$ is invertible. Substituting into (28a) yields

$$\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}}(\boldsymbol{\varsigma})_{ij} = A_{ij} + \mathring{\mathcal{D}}^*(\mathring{\star}\mathring{L}(\bar{\boldsymbol{\xi}}))_{ij} + \tfrac{1}{9}\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}}(\gamma \mathring{h}_{ij})$$

where γ is as determined in the previous step, for which there exists a unique solution ς_{ij}, since $\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}} : \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}}) \to \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})$ is an isomorphism. Finally, substituting the γ and ς_{ij} so obtained into (28b), one obtains

$$\mathring{\Delta}_L \bar{\gamma}_{ij} + 4\bar{\gamma}_{ij} = C_{ij} + 2\mathring{L}(\xi)_{ij} + \tfrac{1}{3}\beta\gamma\mathring{h}_{ij} - \tfrac{2}{3}\mathring{K}\varsigma_{ij},$$

which admits a unique solution since $(\mathring{\Delta}_L + 4)$ is an isomorphism.

The previous two steps conclude the proof that $D\Psi$ is an isomorphism, and so by the Implicit Function Theorem there exists a map ν from the freely-prescribed data to the space of solutions of the auxiliary system $\Psi = 0$. It only remains to establish the injectivity of the map ν.

Injectivity of ν. To establish the injectivity of ν, we need to consider the linearisation of Ψ in the direction of the free data —namely

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Psi(\boldsymbol{\chi}, \bar{\boldsymbol{X}}, \boldsymbol{X}, \boldsymbol{h};\ \mathring{K} + \tau\phi, \tau\bar{\boldsymbol{T}}, \tau\boldsymbol{T})\bigg|_{\tau=0} = 0.$$

Since the background initial data, being hyperbolic, has umbilical extrinsic curvature, the expression (24) simplifies to

$$\mathring{L}(d\phi)_{jk} + 3\mathring{\epsilon}_{kil}\mathring{D}^l\bar{T}_j{}^i + 3\mathring{\epsilon}_{jil}\mathring{D}^l\bar{T}_k{}^i = 0, \tag{29a}$$
$$\mathring{D}^i\bar{T}_{il} = 0, \tag{29b}$$
$$\mathring{D}^j T_{ij} = 0, \tag{29c}$$
$$T_{ij} - \tfrac{4}{9}\mathring{K}\phi\mathring{h}_{ij} = 0. \tag{29d}$$

First consider the case $\mathring{K} \neq 0$: taking the trace of the algebraic equation (29d) one finds that $\phi = 0$, and so $T_{ij} = 0$. Combining (29a)–(29b) —see Remark 6— and using $\phi = 0$, one obtains

$$(\mathring{\mathcal{D}}\bar{T})_{ijk} \equiv \mathring{D}_i\bar{T}_{jk} - \mathring{D}_j\bar{T}_{ik} = 0.$$

Now, we have assumed the non-existence of non-trivial tracefree Codazzi tensors, so $\bar{T}_{ij} = 0$. Hence, in the non–time symmetric case $\mathring{K} \neq 0$, the map ν is injective.

Consider on the other hand the time-symmetric case $\mathring{K} = 0$. Clearly, the kernel of the system contains triples of the form

$$(T_{ij}, \bar{T}_{ij}, \phi) = (\mathbf{0},\ \mathbf{0},\ \text{const.}). \tag{30}$$

We show that these are indeed the only solutions. First, note that condition (29d) (setting $\mathring{K} = 0$) again implies $T_{ij} = 0$. Now, taking the divergence of (29a), one has that

$$\begin{aligned} 0 &= \mathring{\delta}\mathring{L}(d\phi)_k + 3\mathring{\epsilon}_{kil}\mathring{D}^j\mathring{D}^l\bar{T}_j{}^i + 3\mathring{\epsilon}_{jil}\mathring{D}^j\mathring{D}^l\bar{T}_k{}^i \\ &= \mathring{\delta}\mathring{L}(d\phi)_k + \tfrac{3}{2}\mathring{\epsilon}^{jlm}\bar{T}_k{}^i\mathring{r}_{ijlm} - \tfrac{3}{2}\mathring{\epsilon}_i{}^{lm}\bar{T}^{ij}\mathring{r}_{kjlm} + 3\mathring{\epsilon}_{kjl}\mathring{D}_i\mathring{D}^l\bar{T}^{ij} \\ &= \mathring{\delta}\mathring{L}(d\phi)_k + 6\mathring{\epsilon}_{kjl}\bar{T}^{ij}\mathring{r}_i{}^l + 3\mathring{\epsilon}_{kjl}\mathring{D}^l\mathring{D}_i\bar{T}^{ij} \\ &= \mathring{\delta}\mathring{L}(d\phi)_k, \end{aligned}$$

after commuting covariant derivatives and where in the last step we are using the fact that the background metric is Einstein, along with the fact that $\bar{T}_{ij}$ is divergence-free. Integrating by parts, one then finds that $\mathring{L}(d\phi) = 0$ —that is to say, $d\phi$ is a conformal Killing vector. Since $\mathring{\boldsymbol{h}}$ admits no non-trivial conformal Killing vectors, $d\phi = 0$ and so ϕ is constant. Proceeding as in the $\mathring{K} \neq 0$ case, we again see that $\bar{T}_{ij} = 0$, as a consequence of there being no non-trivial tracefree Codazzi tensors. By restricting the choice of ϕ to the sub-Banach space of functions integrating to zero, we clearly exclude from the kernel triples of the form (30), ensuring that ν is injective.

In order to show that $u \mapsto w(u)$ is injective, all that remains to be shown is that the map $u \equiv (\phi, \boldsymbol{T}, \bar{\boldsymbol{T}}) \mapsto \boldsymbol{S}(\boldsymbol{X}(u), \boldsymbol{T})$ is injective (and likewise for $\bar{\boldsymbol{X}}$). The injectivity of the map $u \mapsto \mathring{L}(\boldsymbol{X}(u)) + \boldsymbol{T}$ follows from injectivity of ν and uniqueness of the York split —using, once again, the non-existence of conformal Killing vectors for $\mathring{\boldsymbol{h}}$, see [10]. Finally, we need to show that $\Pi_{\boldsymbol{h}}$ is injective (for $\boldsymbol{h}$ sufficiently close to $\mathring{\boldsymbol{h}}$ in $\mathcal{B}_{\boldsymbol{h}}$). To see this, note that if $T_{ij} \in \ker(\Pi_{\boldsymbol{h}}) \cap \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})$, then

$$T_{ij} = \tfrac{1}{3} T h_{ij}$$

with $T = \mathrm{tr}_{\boldsymbol{h}}(\boldsymbol{T})$, and

$$0 = T \cdot \mathrm{tr}_{\mathring{\boldsymbol{h}}} \boldsymbol{h} = T \cdot (3 + \mathrm{tr}_{\mathring{\boldsymbol{h}}}(\boldsymbol{h} - \mathring{\boldsymbol{h}})).$$

Now, by Sobolev Embedding (see [22]) the C^0−norm of $(\boldsymbol{h} - \mathring{\boldsymbol{h}})$ is bounded above by the H^2−norm and hence, for $\boldsymbol{h}$ sufficiently close to $\mathring{\boldsymbol{h}}$ in $\mathcal{B}_{\boldsymbol{h}}$, it follows that $T = 0$ and hence $T_{ij} = 0$ —that is to say, $\Pi_{\boldsymbol{h}}$ is injective for such an $\boldsymbol{h}$.

□

Remark 22. Recall the notion of *total mean extrinsic curvature*

$$\int_{\mathcal{S}} \mathrm{tr}_{\mathring{\boldsymbol{h}}}(\boldsymbol{K}) \, d\mathring{\mu},$$

given here with respect to the background metric $\mathring{\boldsymbol{h}}$. The additional requirement that ϕ integrates to zero in the time-symmetric case $\mathring{K} = 0$ therefore ensures that the corresponding solutions furnished by Theorem 1 have zero total mean extrinsic curvature with respect to $\mathring{\boldsymbol{h}}$. While the proof guarantees a solution for any choice of (smooth, sufficiently small) ϕ, the injectivity of the map ν is only guaranteed if we further restrict to those ϕ that integrate to zero.

Remark 23. In the proof of Proposition 1, we could have instead used the vanishing of the index to establish surjectivity. Recall that the Atiyah–Singer index theorem (see [25], for example) relates the analytical and topological index of an elliptic operator over a compact manifold. For an odd-dimensional base manifold $\mathcal{S}$ the topological index vanishes —see the discussion in [25]— and so the index theorem guarantees that an injective elliptic operator defined over an odd-dimensional manifold must in fact be an isomorphism of the appropriate Banach spaces.

4.3 Sufficiency of the auxiliary system

In this section we establish *sufficiency* of auxiliary constraint system —that is, we show that the solutions of the auxiliary system established in the previous section are indeed solutions of the extended constraint equations.

4.3.1 Injectivity of $\mathcal{K}_{\boldsymbol{h}}$

Recall the operator $\mathcal{K}_{\boldsymbol{h}}$ (see Section 3.3.1) given by

$$\mathcal{K}_{\boldsymbol{h}}(\boldsymbol{J}) = \begin{pmatrix} \mathring{\mathcal{D}}^*(\boldsymbol{J})_{ij} \\ \epsilon^{ijk} D_i J_{jkl} \end{pmatrix}.$$

As described in Section 3.3, the sufficiency argument will involve establishing injectivity of the operator $\mathcal{K}_{\boldsymbol{h}}$. We first consider the operator evaluated at the background metric, $\mathring{\boldsymbol{h}}$:

Proposition 2. *Let $(\mathcal{S}, \mathring{\boldsymbol{h}})$ be a smooth conformally rigid hyperbolic manifold, then the operator $\mathring{\mathcal{K}} \equiv \mathcal{K}_{\mathring{\boldsymbol{h}}}$ is injective —i.e. the system of equations $\mathring{\mathcal{K}}(\boldsymbol{J}) = 0$ admits only the trivial solution $J_{ijk} = 0$.*

Proof. Suppose $J_{ijk} = 0$ is a Jacobi tensor satisfying $\mathring{\mathcal{K}}(\boldsymbol{J}) = 0$. Performing the Jacobi decomposition of J_{ijk} with respect to $\mathring{\boldsymbol{h}}$ we obtain

$$2\mathring{\mathrm{rot}}_2(\boldsymbol{F})_{ij} + \mathring{L}(\boldsymbol{A})_{ij} = 0, \tag{31a}$$

$$\mathring{\delta}(\boldsymbol{F})_i + \mathring{\text{curl}}(\boldsymbol{A})_i = 0, \tag{31b}$$

with $\mathring{\text{curl}}(\boldsymbol{A})_i \equiv \mathring{\epsilon}_{ijk}\mathring{D}^j A^k$, to be read as equations for $F_{ij} \in \mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}})$ and $A_i \in \Lambda^1(\mathcal{S})$. It then follows that

$$\begin{aligned}
0 &= \mathring{\delta}(\mathring{L}(\boldsymbol{A}) + 2\mathring{\text{rot}}_2(\boldsymbol{F}))_i \\
&= \mathring{\delta} \circ \mathring{L}(\boldsymbol{A})_i + 2\mathring{\delta} \circ \mathring{\text{rot}}_2(\boldsymbol{F})_i \\
&= \mathring{\delta} \circ \mathring{L}(\boldsymbol{A})_i + \mathring{\text{curl}} \circ \mathring{\delta}(\boldsymbol{F})_i - 2\mathring{\epsilon}_{iml}\mathring{r}_j{}^l F^{jm} \\
&= \mathring{\delta} \circ \mathring{L}(\boldsymbol{A})_i - \mathring{\text{curl}}^2(\boldsymbol{A})_i - 2\mathring{\epsilon}_{iml}\mathring{r}_j{}^l F^{jm},
\end{aligned}$$

where the first line follows from (31a), the third uses the identity

$$\mathring{\delta} \circ \mathring{\text{rot}}_2(\boldsymbol{F})_i = \tfrac{1}{2}\mathring{\text{curl}} \circ \mathring{\delta}(\boldsymbol{F})_i - \mathring{\epsilon}_{iml}\mathring{r}_j{}^l F^{jm},$$

and the fourth follows from substitution using (31b). Since $\mathring{\boldsymbol{h}}$ is Einstein, we find

$$\mathring{\delta} \circ \mathring{L}(\boldsymbol{A})_i - \mathring{\text{curl}}^2(\boldsymbol{A})_i = 0.$$

Contracting with A^i and integrating by parts:

$$0 = \int_{\mathcal{S}} \left(\tfrac{1}{2}\|\mathring{L}(\boldsymbol{A})\|^2 + \|\mathring{\text{curl}}(\boldsymbol{A})\|^2 \right) d\mu_{\mathring{\boldsymbol{h}}}, \tag{32}$$

where we are using the fact that $\mathring{\delta}^* = -\frac{1}{2}\mathring{L}$ and $\mathring{\text{curl}}^* = \mathring{\text{curl}}$. Hence, we find that $A_i = 0$, since $\mathring{\boldsymbol{h}}$ admits no conformal Killing vector fields. Substituting into (31a)–(31b), we see that $\mathring{\text{rot}}_2(\boldsymbol{F})_{ij} = \mathring{\delta}(\boldsymbol{F})_i = 0$ and hence $F_{ij} = 0$ since $\mathring{\boldsymbol{h}}$ admits no tracefree Codazzi tensors. It follows then that $J_{ijk} = 0$. □

In order to show that $\mathcal{K}_{\boldsymbol{h}}$ is injective for $\boldsymbol{h}$ sufficiently close to $\mathring{\boldsymbol{h}}$, we will first show that the operator $\mathcal{K}_{\boldsymbol{h}}$ is elliptic and then appeal to a particular stability property of the kernel of elliptic operators. Let us first establish ellipticity:

Lemma 2. *The operator $\mathcal{K}_{\boldsymbol{h}}$ is first-order elliptic for any Riemannian metric $\boldsymbol{h}$.*

Proof. Recall from Remark 3 that $\mathcal{J}(\mathcal{S})$ and $\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}}) \oplus \Lambda^1(\mathcal{S})$ are isomorphic as vector spaces. Therefore, in order to establish ellipticity it suffices to show that $\mathcal{K}_{\boldsymbol{h}}$ is overdetermined elliptic. Note that the second component of $\mathcal{K}_{\boldsymbol{h}} = 0$ is equivalent to

$$D_{[i}J_{jk]l} = 0.$$

Note also that a change of connection $D_i \to \mathring{D}_i$ only introduces lower-order (i.e. algebraic) terms involving J_{ijk}, so in order to show ellipticity it suffices to consider the operator $\mathring{\mathcal{K}}$, or equivalently an operator with principal part

$$\begin{pmatrix} \mathring{\mathcal{D}}^*(\boldsymbol{J})_{ij} \\ \mathring{D}_{[i}J_{jk]l}. \end{pmatrix}.$$

Accordingly, suppose $J_{ijk} \in \mathcal{J}(\mathcal{S})$ is in the kernel of the symbol map, $\sigma_\xi[\mathring{\mathcal{K}}]$, for a given fixed ξ_i, so that

$$\xi^k J_{ikj} + \xi^k J_{jki} - \tfrac{2}{3}\xi^k J_{lk}{}^l \mathring{h}_{ij} = 0, \tag{33a}$$

$$\xi_i J_{jkl} + \xi_j J_{kil} + \xi_k J_{ijl} = 0. \tag{33b}$$

Note that the latter is indeed equivalent to $\epsilon^{ijk}\xi_i J_{jkl} = 0$, taking into account the fact that $J_{ijk} = -J_{jik}$. Contracting indices i, l in equation (33b), we obtain

$$\xi^l J_{jkl} = -\xi_j J_{kl}{}^l + \xi_k J_{jl}{}^l. \tag{34}$$

On the other hand, contracting (33a) with ξ^j, we obtain

$$\begin{aligned}0 &= \xi^k \xi^j J_{ikj} + \xi^k \xi^j J_{jki} - \tfrac{2}{3}\xi^k \xi_i J_{lk}{}^l \\ &= \xi^k \xi^j J_{ikj} - \tfrac{2}{3}\xi^k \xi_i J_{lk}{}^l \\ &= \tfrac{1}{3}\xi_i \xi^k J_{kl}{}^l + |\boldsymbol{\xi}|^2 J_{il}{}^l \end{aligned} \tag{35}$$

where the second line follows from the fact that $J_{ijk} = -J_{jik}$ and the third line follows from substituting (34). Contracting (35) with ξ^i, we find that $\xi^i J_{il}{}^l = 0$, which when substituted back into (35) yields $J_{il}{}^l = 0$ for $|\boldsymbol{\xi}| \neq 0$. Substituting the latter into (33a) we see that

$$\xi^k J_{ikj} + \xi^k J_{jki} = 0. \tag{36}$$

Moreover, substitution of $J_{il}{}^l = 0$ into (34) yields

$$\xi^k J_{ijk} = 0. \tag{37}$$

Now, contracting the cyclic identity $J_{ijk} + J_{jki} + J_{kij} = 0$ with ξ^k one finds that

$$\begin{aligned}0 &= \xi^k J_{ijk} + \xi^k J_{jki} + \xi^k J_{kij} \\ &= \xi^k J_{jki} - \xi^k J_{ikj}, \end{aligned} \tag{38}$$

where to pass from the first to the second line we have used (37) and that $J_{kij} = -J_{ikj}$. Combining equations (36) and (38) one thus concludes that

$$\xi^k J_{ikj} = 0. \tag{39}$$

Finally, contracting (33b) with ξ^i, and using the relations (37) and (39) we obtain

$$0 = |\boldsymbol{\xi}|^2 J_{jkl} + \xi_j \xi^i J_{kil} + \xi_k \xi^i J_{ijl} = |\boldsymbol{\xi}|^2 J_{jkl}$$

Hence, for $|\boldsymbol{\xi}| \neq 0$, we see that the symbol map is injective —that is to say, $\mathcal{K}_{\boldsymbol{h}}$ is overdetermined elliptic and hence determined elliptic, since its domain and codomain are of equal dimension as vector spaces. □

In order to establish injectivity of $\mathcal{K}_{\boldsymbol{h}}$ we will make use of an elliptic estimate. Rather than working directly with the first-order operator $\mathcal{K}_{\boldsymbol{h}}$ we choose instead to work with the elliptic operator $\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}}$ to which the more standard results of second-order elliptic operators may be applied —note that the kernel of the latter operator agrees with the kernel of $\mathcal{K}_{\boldsymbol{h}}$, so it suffices to show injectivity of the second-order operator. Our starting point is the following elliptic estimate for $\mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}}$: there exists $C > 0$ such that, for all $\boldsymbol{\eta} \in H^2(\mathcal{J}(\mathcal{S}))$

$$\|\boldsymbol{\eta}\|_{H^2} \leq C \left(\|\mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}}(\boldsymbol{\eta})\|_{L^2} + \|\boldsymbol{\eta}\|_{H^1} \right) \tag{40}$$

—see Appendix II of [11], for instance. In fact, we will require a *uniform* version of the above elliptic estimate which allows for small perturbations of the metric:

Lemma 3. *There exists $\varepsilon > 0$ such that, for all $\boldsymbol{h}$ satisfying $\|\boldsymbol{h} - \mathring{\boldsymbol{h}}\|_{H^s} < \varepsilon$, $s \geq 4$, we have the estimate*

$$\|\boldsymbol{\eta}\|_{H^2} \leq 2C \left(\|\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}}(\boldsymbol{\eta})\|_{L^2} + \|\boldsymbol{\eta}\|_{H^1} \right) \tag{41}$$

for all $\boldsymbol{\eta} \in H^2(\mathcal{J}(\mathcal{S}))$, with C as in (40)*, depending only on $\mathring{\boldsymbol{h}}$.*

Proof. We first note that there exists some constant $\tilde{C}$ such that for any given $\boldsymbol{\eta} \in \mathcal{J}(\mathcal{S})$, we have

$$\|(\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}} - \mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}})\boldsymbol{\eta}\|_{L^2} \leq \tilde{C} \|\boldsymbol{h} - \mathring{\boldsymbol{h}}\|_{H^2} \|\boldsymbol{\eta}\|_{H^2} \tag{42}$$

—this follows from the fact that, schematically,

$$(\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}} - \mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}})\boldsymbol{\eta} \sim (\boldsymbol{h} - \mathring{\boldsymbol{h}})\partial\partial\boldsymbol{\eta} + \boldsymbol{S} \cdot \partial\boldsymbol{\eta} + (\partial\boldsymbol{S} + \boldsymbol{S} \cdot \boldsymbol{S})\boldsymbol{\eta}$$

with $\boldsymbol{S}$ the transition tensor covariant derivatives associated to the metrics $\mathring{\boldsymbol{h}}$ and $\boldsymbol{h}$, from which it is clear then that $(\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}} - \mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}})\boldsymbol{\eta}$ may be bounded above by $\|\boldsymbol{h} - \mathring{\boldsymbol{h}}\|_{H^2}\|\boldsymbol{\eta}\|_{H^2}$.

Now, using inequality (42) we find that for all $\boldsymbol{h}$ satisfying $\|\boldsymbol{h}-\mathring{\boldsymbol{h}}\|_{H^2} < \varepsilon$, and for all $\boldsymbol{\eta} \in \mathcal{J}(\mathcal{S})$,

$$
\begin{aligned}
\|\boldsymbol{\eta}\|_{H^2} &\leq C\left(\|\mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}}(\boldsymbol{\eta})\|_{L^2} + \|\boldsymbol{\eta}\|_{H^1}\right) \\
&\leq C\left(\|\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}}(\boldsymbol{\eta})\|_{L^2} + \|(\mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}} - \mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}})\boldsymbol{\eta}\|_{L^2} + \|\boldsymbol{\eta}\|_{H^1}\right) \\
&\leq C\left(\|\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}}(\boldsymbol{\eta})\|_{L^2} + \varepsilon\tilde{C}\|\boldsymbol{\eta}\|_{H^2} + \|\boldsymbol{\eta}\|_{H^1}\right),
\end{aligned}
$$

with C depending only on $\mathring{\boldsymbol{h}}$. Thus, taking $\varepsilon = 1/(2C\tilde{C})$ and rearranging we have that

$$
\|\boldsymbol{\eta}\|_{H^2} \leq 2C\left(\|\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}}(\boldsymbol{\eta})\|_{L^2} + \|\boldsymbol{\eta}\|_{H^1}\right) \tag{43}
$$

for all $\boldsymbol{\eta} \in H^2(\mathcal{J}(\mathcal{S}))$ and for all $\|\boldsymbol{h} - \mathring{\boldsymbol{h}}\|_{H^2} < \varepsilon$ as required. □

Remark 24. The content of inequality (42) may be summarised by the statement that the map

$$
\begin{array}{cccc}
M: & H^2(\mathscr{S}^2(\mathcal{S})) & \longrightarrow & B(H^2(\mathcal{J}(\mathcal{S})), L^2(\mathcal{J}(\mathcal{S})) \\
 & \boldsymbol{h} & \longmapsto & \mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}}
\end{array}
$$

is Lipschitz continuous at $\boldsymbol{h} = \mathring{\boldsymbol{h}}$ —here, $B(\cdot,\cdot)$ denotes the Banach space of bounded linear maps between the indicated Banach spaces, endowed with the operator norm— with $\tilde{C}$ the Lipschitz constant, which depends on the precise structure of $\mathcal{K}^* \circ \mathcal{K}$ and may be computed explicitly.

4.3.2 The main argument

Assume now that the procedure described in Section 4.2 has been carried out —that is to say, we have established the existence of a neighbourhood of solutions to the auxiliary system. For each such solution, the corresponding zero quantities $Q_i,\ J_{ijk}$ necessarily satisfy

$$
\mathcal{K}_{\boldsymbol{h}}(\boldsymbol{J}) = 0, \tag{44a}
$$

$$
D^i(\mathcal{L}_Q\boldsymbol{h})_{ij} - \tfrac{1}{2}D_j(\mathcal{L}_Q\boldsymbol{h})_i{}^i = K_{ik}J_j{}^{ik} - K_{jk}J^{ik}{}_i - KJ_j{}^i{}_i. \tag{44b}
$$

The first equation collects together (17a) and (18a), while the latter is the remaining integrability condition — see Section 3.3. We regard the above as equations for a pair of tensor fields $\boldsymbol{Q} \in \Lambda^1(\mathcal{S}),\ \boldsymbol{J} \in \mathcal{J}(\mathcal{S})$, which we aim to prove are necessarily vanishing —at this point we forget about the definitions of the zero quantities $Q_i,\ J_{ijk}$ in terms of the unknown tensor fields.

We first use the results of the previous section to show that injectivity of the operator $\mathcal{K}_{\boldsymbol{h}}$ is stable under H^s-perturbations, $s \geq 4$, of the metric. Note that, in the following, all Sobolev norms are taken with respect to the background metric, $\mathring{\boldsymbol{h}}$.

Proposition 3. *There exists $\varepsilon > 0$ such that for any metric $\boldsymbol{h}$ satisfying $\|\boldsymbol{h} - \mathring{\boldsymbol{h}}\|_{H^s} < \varepsilon$, the corresponding operator $\mathcal{K}_{\boldsymbol{h}}$ is injective in H^2.*

Proof. Suppose not. Then there exists a *failure sequence* $\{(\boldsymbol{h}^{(n)},\ \boldsymbol{\eta}^{(n)})\}$, $n \in \mathbb{N}$ —i.e. a sequence of Riemannian metrics $\boldsymbol{h}^{(n)}$ converging to $\mathring{\boldsymbol{h}}$ in H^2 and corresponding non-zero Jacobi tensors $\boldsymbol{\eta}^{(n)} \in \mathcal{J}(\mathcal{S})$ for which

$$
\mathcal{K}_{(n)}(\boldsymbol{\eta}^{(n)}) = 0
$$

for each $n \in \mathbb{N}$ —here, $\mathcal{K}_{(n)} \equiv \mathcal{K}_{\boldsymbol{h}^{(n)}}$. Since $\mathcal{K}_{(n)}$ is linear, we may take each $\boldsymbol{\eta}^{(n)}$ to be of unit H^2-norm. Hence, by the Rellich-Kondrakov Theorem, since the sequence $\{\boldsymbol{\eta}^{(n)}\}$ is bounded in H^2, there is a subsequence that is Cauchy in H^1 —let us assume without loss of generality that $\{\boldsymbol{\eta}^{(n)}\}$ is Cauchy— converging to some limit $\boldsymbol{\eta}^\bullet \in \mathcal{J}(\mathcal{S})$. We now aim to show using the inequality (43) that the sequence is in fact Cauchy in H^2. Let us restrict to a the tail of the subsequence (relabelling, if necessary) for which $\|\boldsymbol{h}^{(n)} - \mathring{\boldsymbol{h}}\| < \varepsilon$ with ε as given in Proposition 3. Applying the inequality (43) to $\boldsymbol{\eta}^{(m,n)} \equiv \bar{\boldsymbol{\eta}}^{(n)} - \bar{\boldsymbol{\eta}}^{(m)}$, with $\boldsymbol{h} = \boldsymbol{h}^{(n)}$, we have

$$
\begin{aligned}
&\|\boldsymbol{\eta}^{(m,n)}\|_{H^2} \\
&\quad \leq 2C\left(\|\mathcal{K}^*_{(n)} \circ \mathcal{K}_{(n)}(\boldsymbol{\eta}^{(m,n)})\|_{L^2} + \|\boldsymbol{\eta}^{(m,n)}\|_{H^1}\right) \\
&\quad = 2C\left(\|\mathcal{K}^*_{(n)} \circ \mathcal{K}_{(n)}(\boldsymbol{\eta}^{(m)})\|_{L^2} + \|\boldsymbol{\eta}^{(m,n)}\|_{H^1}\right) \\
&\quad = 2C\left(\|(\mathcal{K}^*_{(n)} \circ \mathcal{K}_{(n)} - \mathcal{K}^*_{(m)} \circ \mathcal{K}_{(m)})\boldsymbol{\eta}^{(m)}\|_{L^2} + \|\boldsymbol{\eta}^{(m,n)}\|_{H^1}\right). \qquad (45)
\end{aligned}
$$

The second line follows from by substituting for $\boldsymbol{\eta}^{(m,n)}$ in the first term and using the fact that, by assumption, $\mathcal{K}_{(n)}(\bar{\boldsymbol{\eta}}^{(n)}) = 0$; the third line follows similarly. Now,

$$
\begin{aligned}
\|(\mathcal{K}^*_{(n)} \circ \mathcal{K}_{(n)} - \mathcal{K}^*_{(m)} \circ \mathcal{K}_{(m)})\boldsymbol{\eta}^{(m)}\|_{L^2} \leq{}& \|(\mathcal{K}^*_{(n)} \circ \mathcal{K}_{(n)} - \mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}})\boldsymbol{\eta}^{(m)}\|_{L^2} \\
&+ \|(\mathcal{K}^*_{(m)} \circ \mathcal{K}_{(m)} - \mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}})\boldsymbol{\eta}^{(m)}\|_{L^2},
\end{aligned}
$$

which goes to zero in the limit $m, n \longrightarrow \infty$, again using the Lipschitz property of M and the fact that $\boldsymbol{\eta}^{(m)}$ is bounded in H^2. Collecting together the above observations, we see from (45) that as $m, n \longrightarrow \infty$, $\boldsymbol{\eta}^{(m,n)} \longrightarrow 0$ in H^2 —i.e. the sequence $\bar{\boldsymbol{\eta}}^{(n)}$ is Cauchy in H^2, and therefore the limit $\boldsymbol{\eta}^\bullet \in \mathcal{J}(\mathcal{S})$ is in H^2. Clearly $\boldsymbol{\eta}^\bullet$ is non-zero —in fact, one has that $\|\boldsymbol{\eta}^\bullet\|_{H^2} = 1$.

Using the Lipschitz property of M once more, along with the fact that $\boldsymbol{\eta}^{(n)}$ converges to $\boldsymbol{\eta}^\bullet$ in H^2, one finds that

$$
\|\mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}}(\boldsymbol{\eta}^\bullet)\|_{L^2} = \lim_{n\to\infty} \|\mathcal{K}^*_{(n)} \circ \mathcal{K}_{(n)}(\boldsymbol{\eta}^{(n)})\|_{L^2} = 0.
$$

Hence, $\mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}}(\boldsymbol{\eta}^\bullet) = 0$, and it follows via integration by parts that $\mathring{\mathcal{K}}(\boldsymbol{\eta}^\bullet) = 0$. However, $\boldsymbol{\eta}^\bullet \in \mathcal{J}(\mathcal{S}) \setminus \{0\}$ and so we obtain a contradiction, since $\mathring{\mathcal{K}}$ is injective, as shown in Proposition 2. □

We are now in a position to prove the main result of this section:

Proposition 4 (***Sufficiency***). *There exists an open neighbourhood $\mathcal{V}$ of $\mathring{\boldsymbol{h}} \in \mathcal{B}_{\boldsymbol{h}}$, such that for each $\boldsymbol{h} \in \mathcal{V}$, $(J_{ijk}, Q_i) = (\mathbf{0}, \mathbf{0})$ is the unique H^2 solution of* (44a)–(44b).

Proof. We begin by showing that $J_{ijk} = 0$. This follows immediately from the previous proposition provided we choose $\mathcal{V}$ to be a suitably small neighbourhood.

Having established that $J_{ijk} = 0$, (44b) implies that Q_i satisfies the integral identity (19). Hence, it follows that

$$
0 = \int_{\mathcal{S}} \left(\|D\boldsymbol{Q}\|^2_{\boldsymbol{h}} - r_{ij}Q^iQ^j\right)\, d\mu_{\boldsymbol{h}} \geq \int_{\mathcal{S}} -r_{ij}Q^iQ^j\, d\mu_{\boldsymbol{h}} \longrightarrow \int_{\mathcal{S}} 2\|\boldsymbol{Q}\|^2_{\mathring{\boldsymbol{h}}}\, d\mu_{\mathring{\boldsymbol{h}}},
$$

where convergence follows from the fact that, since $\boldsymbol{h} \to \mathring{\boldsymbol{h}}$ in H^4, we have $r[\boldsymbol{h}]_{ij} \to \mathring{r}_{ij} = -2\mathring{h}_{ij}$ in C^0 —convergence of the latter in H^2 is immediate, and an application of the Sobolev Embedding Theorem establishes convergence in C^0. Hence, provided we take $\mathcal{V}$ to be a suitably-small neighbourhood, it follows that for any $\boldsymbol{h} \in \mathcal{V}$ we necessarily have $\boldsymbol{Q} = 0$. □

Hence, it follows that for solutions $(K_{ij}, S_{ij}, \bar{S}_{ij}, h_{ij})$ of the auxiliary system sufficiently close to the background data, the corresponding zero quantities Q_i, J_{ijk} must necessarily vanish, implying $(K_{ij}, S_{ij}, \bar{S}_{ij}, h_{ij})$ indeed solves the extended constraint equations. This concludes the proof of sufficiency. Collecting together Propositions 1 and 4, one obtains Theorem 1.

Remark 25. Alternatively, we could also have shown $Q_i = 0$ by using identity (19) to first establish injectivity of the operator $Q_i \mapsto \mathring{\Delta} Q_i + \mathring{r}_{ij}Q^j$, and again appealing to the stability property of kernels of elliptic operators.

4.4 Parametrising the space of freely-prescribed data

We have seen that, according to Theorem 1, there exist solutions of the extended constraints corresponding to freely-prescribed data $(\phi, \boldsymbol{T}, \bar{\boldsymbol{T}})$ sufficiently close to $(\mathbf{0}, \mathbf{0}, \mathbf{0})$, where $\boldsymbol{T}, \bar{\boldsymbol{T}} \in \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}})$. In this last subsection we aim to give an explicit parametrisation of the space of freely-prescribed data, using the ideas of [4] for the construction of transverse-tracefree tensors on conformally flat manifolds, which have previously been applied to the construction of *generalised* Bowen-York data —see [5]. We first review the basic ideas.

4.4.1 The Gasqui–Goldschmidt complex

Let $\mathcal{H}(\boldsymbol{h})_{ij}$ denote the *Cotton–York tensor* associated to a metric $\boldsymbol{h}$ —namely

$$\mathcal{H}_{ij} \equiv \epsilon_{kl(i} D^k r_{j)}{}^l.$$

The Cotton tensor $\mathcal{H}_{ij}$ is symmetric and tracefree. Moreover, by the *third Bianchi identity* it is also divergence-free. Recall also that, in dimension 3, the vanishing of the Cotton-York tensor is equivalent to local conformal-flatness —see e.g. [27]. Now consider the linearisation, $\mathring{H}(\boldsymbol{\eta})_{ij}$, about a background metric $\mathring{\boldsymbol{h}}$, given by the Fréchet derivative

$$\begin{aligned}\mathring{H}(\boldsymbol{\eta})_{ij} &\equiv \frac{d}{d\tau}\mathcal{H}(\mathring{\boldsymbol{h}} + \tau\boldsymbol{\eta})_{ij}\bigg|_{\tau=0} \\ &= \mathring{\epsilon}^{kl}{}_i(\mathring{D}_k \check{r}(\eta)_{lj} - C(\boldsymbol{\eta})^m{}_{kj}\mathring{r}_{lm}) + \eta_{(i}{}^k \mathring{\mathcal{H}}_{j)k} - \tfrac{1}{2}\eta\mathring{\mathcal{H}}_{ij}\end{aligned}$$

with indices raised using $\mathring{\boldsymbol{h}}$. Here, $\eta \equiv \mathrm{tr}_{\mathring{\boldsymbol{h}}}(\boldsymbol{\eta})$, the operator $C(\cdot)^i{}_{jk}$is as defined in (15) and $\check{r}(\eta)_{ij}$ is the linearised Ricci operator acting on the metric perturbation η_{ij}, and given by equation (14).

According to the above observations, if $\mathring{\boldsymbol{h}}$ is *conformally flat*, then $\mathring{H}(\boldsymbol{\eta}) \in \mathscr{S}_0^2(\mathcal{S}; \mathring{\boldsymbol{h}})$. Moreover, in the case of conformally-flat data, $\mathring{H}(\boldsymbol{\eta})_{ij}$ is also divergence-free since the linearisation of the third Bianchi identity gives

$$\begin{aligned}0 &= \frac{d}{d\tau}\delta_{\boldsymbol{h}}(\mathcal{H}(\boldsymbol{h}))_i\bigg|_{\tau=0} \\ &= \mathring{\delta}(\mathring{H}(\boldsymbol{\eta}))_i - \eta^{kj}\mathring{D}_k\mathring{\mathcal{H}}_{ij} - \tfrac{1}{2}\mathring{\mathcal{H}}^{jk}\mathring{D}_i\eta_{jk} - \mathring{\mathcal{H}}_i{}^k\mathring{D}^j\eta_{jk} + \tfrac{1}{2}\mathring{\mathcal{H}}_i{}^k\mathring{D}_k\eta \\ &= \mathring{\delta}(\mathring{H}(\boldsymbol{\eta}))_i\end{aligned}$$

where to pass from the second to the third line it has been used that $\mathring{\mathcal{H}}_{ij} = 0$ for a conformally flat background. Hence, $\mathring{H}(\boldsymbol{\eta})_{ij} \in \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}})$. The above features are expressed succinctly in the *Gasqui-Goldschmidt* elliptic complex —see [18, 4]:

$$0 \to \Gamma(\Lambda^1(\mathcal{S})) \xrightarrow{\mathring{L}} \Gamma(\mathscr{S}_0^2(\mathcal{S}; \mathring{\boldsymbol{h}})) \xrightarrow{\mathring{H}} \Gamma(\mathscr{S}_0^2(\mathcal{S}; \mathring{\boldsymbol{h}})) \xrightarrow{\mathring{\delta}} \Gamma(\Lambda^1(\mathcal{S})) \to 0,$$

which holds for any conformally flat manifold $(\mathcal{S}, \mathring{\boldsymbol{h}})$. Here, we are using $\Gamma(\cdot)$ to denote smooth sections of the indicated tensor bundle. Another consequence of the elliptic complex is that the linear sixth-order operator $P \equiv \mathring{H}^2 + (\mathring{L} \circ \mathring{\delta})^3$ is elliptic —see [4]. It is straightforward to see that $\ker P = \ker \mathring{H} \cap \ker \mathring{\delta}$, and hence that P is injective for a conformally rigid manifold $(\mathcal{S}, \mathring{\boldsymbol{h}})$.

For compact $\mathcal{S}$, the above elliptic complex admits the following expression of *Poincaré duality*:

$$\ker \mathring{\delta}/\mathring{H}(\Gamma(\mathscr{S}_0^2(\mathcal{S}; \mathring{\boldsymbol{h}}))) \simeq \ker \mathring{H}/\mathring{L}(\Gamma(\Lambda^1(\mathcal{S}))).$$

Hence, *given our assumption of conformal rigidity*, it follows that the map

$$\mathring{H} : \Gamma(\mathscr{S}_0^2(\mathcal{S}; \mathring{\boldsymbol{h}})) \to \Gamma(\mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}}))$$

is, in fact, surjective —any *smooth* TT tensor may be constructed as the image under H of some smooth tracefree $2-$tensor. This result is generalised in the following Proposition:

Proposition 5. *Let* $(\mathcal{S},\mathring{\boldsymbol{h}})$ *be a smooth conformally-rigid (not necessarily hyperbolic) manifold, then the map*

$$\mathring{H}: H^{s+2}(\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}})) \to H^{s-1}(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}})),$$

is surjective for $s \geq 4$.

Proof. Given $T_{ij} \in H^{s-1}(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}}))$, then since $\Gamma(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}})) \cap H^{s-1}(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}}))$ is dense in $H^{s-1}(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}}))$ we can approximate T_{ij} by a Cauchy sequence $T^{(n)}_{ij} \in \Gamma(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}}))$. Since $\mathring{\boldsymbol{h}}$ is conformally rigid there exists, for each $n \in \mathbb{N}$, an element $\eta^{(n)}_{ij} \in \Gamma(\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}}))$ for which $\mathring{H}(\boldsymbol{\eta}^{(n)})_{ij} = T^{(n)}_{ij}$. Without loss of generality, we may assume that $\eta^{(n)}_{ij} \in \Gamma(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}}))$ for each $n \in \mathbb{N}$ —one takes the TT part of the York split of a given $\eta^{(n)}_{ij}$, if necessary, and uses the fact that Im $\mathring{L} \subset$ ker $\mathring{H}$. Now since the elliptic operator $P \equiv \mathring{H}^2 + (\mathring{L}\circ\mathring{\delta})^3$ is injective, it follows from standard results of elliptic PDE theory (see Appendix H of [7], for instance) that there exists some constant $C > 0$ for which the elliptic estimate

$$\|\boldsymbol{\eta}\|_{H^{s+2}} \leq C\|P(\boldsymbol{\eta})\|_{H^{s-4}}$$

holds for all $\eta_{ij} \in H^{s+2}(\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}}))$. In particular, it follows that

$$\begin{aligned}
\|\boldsymbol{\eta}^{(m)} - \boldsymbol{\eta}^{(n)}\|_{H^{s+2}} &\leq C\|P(\boldsymbol{\eta}^{(m)} - \boldsymbol{\eta}^{(n)})\|_{H^{s-4}}\\
&\leq C\|\mathring{H}\circ\mathring{H}(\boldsymbol{\eta}^{(m)} - \boldsymbol{\eta}^{(n)})\|_{H^{s-4}}\\
&\leq C\|\mathring{H}(\boldsymbol{T}^{(m)} - \boldsymbol{T}^{(n)})\|_{H^{s-4}}\\
&\leq C\|\boldsymbol{T}^{(m)} - \boldsymbol{T}^{(n)}\|_{H^{s-1}},
\end{aligned}$$

where the second line follows from the fact that, by assumption, $\eta^{(n)}_{ij}$ are divergence-free, and the fourth follows by continuity of $\mathring{H}$ as a map from H^{s-1} to H^{s-4}. It follows that the sequence $\{\boldsymbol{\eta}^{(n)}\}$, $n \in \mathbb{N}$, is Cauchy in the H^{s+2}-norm and therefore converges to some $\eta_{ij} \in H^{s+2}(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}}))$. By continuity we then have that $\mathring{H}(\boldsymbol{\eta})_{ij} = T_{ij}$, as required. □

4.4.2 The parametrisation

The above ideas can now be applied to obtain the *parametrisation* of the free data T_{ij}, $\bar{T}_{ij}$:

Proposition 6. *Let* $(\mathcal{S},\mathring{\boldsymbol{h}},\mathring{\boldsymbol{K}})$ *satisfy the conditions of Theorem 1, and let* $\mathcal{U}$ *be the neighbourhood of the freely specifiable data as given there. There exists an open subset*

$$\tilde{\mathcal{U}} \subset \mathcal{B}_{\boldsymbol{\eta}} \equiv H^{s-1}(\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}}))),$$

such that:

i) for each $\boldsymbol{\eta},\bar{\boldsymbol{\eta}} \in \tilde{\mathcal{U}}$ *there exists a solution to the extended constraint equations with free data*

$$T_{ij} = \mathring{H}(\boldsymbol{\eta})_{ij}, \qquad \bar{T}_{ij} = \mathring{H}(\bar{\boldsymbol{\eta}})_{ij}; \tag{46}$$

ii) all admissible free data (i.e. $\boldsymbol{T},\bar{\boldsymbol{T}} \in \mathcal{U}$*) may be obtained in the form* (46)*, for some* $\boldsymbol{\eta},\bar{\boldsymbol{\eta}} \in \tilde{\mathcal{U}}$.

For a given T_{ij} $\bar{T}_{ij}$*, the choice of* $\eta_{ij},\bar{\eta}_{ij}$ *in* (46) *is unique up to the addition of elements in Im*$(\mathring{L})$.

Proof. Take $\tilde{\mathcal{U}} \equiv \mathring{H}^{-1}(\mathcal{U}\cap \mathrm{Im}(\mathring{H}))$. The map

$$\mathring{H}: \mathcal{B}_{\boldsymbol{\eta}} \to \mathcal{B}_T$$

is continuous, so $\tilde{\mathcal{U}}$ is open in $\mathcal{B}_T$. Applying Theorem 1 with free data (46) establishes *(i)*. By assumption of conformal rigidity and using Proposition 5 it follows that

$$\mathring{H}: H^{s+2}(\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}})) \to H^{s-1}(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}}))$$

is surjective, so $\mathring{H}(\tilde{\mathcal{U}}) = \mathcal{U}$, establishing *(ii)*. Uniqueness (up to addition of elements in Im$(\mathring{L})$) follows immediately from the assumption of conformal rigidity. □

5 Conclusions and Outlook

The Friedrich-Butscher method originally applied in [8, 9] to the asymptotically flat case, was implemented here to the case of hyperbolic background initial data. This method provides a promising alternative to the standard conformal method for the construction of initial data; in particular, it allows for the possibility of generating solutions to the constraint equations that are tailored in the sense of having certain components of the Weyl curvature (restricted to $\mathcal{S}$) prescribed from the outset.

Work is currently under progress to extend the present results to a broader class of background initial data, in addition to extending the analysis to the full conformal constraint equations. It would be interesting to see whether the method can be implemented numerically through an iterative convergence scheme.

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