

Quantum statistics in Network Geometry with Fractional Flavor

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Abstract. Growing network models have been shown to display emergent quantum statistics when nodes are associated to a fitness value describing the intrinsic ability of a node to acquire new links. Recently it has been shown that quantum statistics emerge also in a growing simplicial complex model called Network Geometry with Flavor which allows for the description of many-body interactions between the nodes. This model depends on an external parameter called *flavor* that is responsible for the underlying topology of the simplicial complex. When the flavor takes the value $s = -1$ the d -dimensional simplicial complex is a manifold in which every $(d - 1)$ -dimensional face can only have an incidence number $n_\alpha \in \{0, 1\}$. In this case the faces of the simplicial complex are naturally described by the Bose-Einstein, Boltzmann and Fermi-Dirac distribution depending on their dimension. In this paper we extend the study of Network Geometry with Flavor to fractional values of the flavor $s = -1/m$ in which every $(d - 1)$ -dimensional face can only have incidence number $n_\alpha \in \{0, 1, 2, \dots, m\}$. We show that in this case the statistical properties of the faces of the simplicial complex are described by the Bose-Einstein or the Fermi-Dirac distribution only. Finally we comment on the spectral properties of the networks constituting the underlying structure of the considered simplicial complexes.

1. Introduction

Quantum statistics have been shown to emerge spontaneously in the description of growing network models with fitness of the nodes [1–7]. In particular the Bianconi-Barabási model [1,2] is a textbook [8] example of this phenomenon in which the non-equilibrium dynamics of a classical network is mathematically described by quantum statistics. The implications of this mapping are profound. In particular the mapping of the Bianconi-Barabási model with a Bose gas is able to predict a topological phase transition [1,9] in the network in which the dynamics of the networks is not stationary anymore but instead it is dominated by the sequence of nodes with high fitness that arrive in the network and eventually become super-hubs. Interestingly this model has a large variety of applications ranging from the Internet [10] and the WWW [11] and explains the basic mechanism beyond the winner-takes-all phenomena observed in such structures (like the emergence of super-hubs as Google and Facebook). A symmetric generation of a growing Cayley tree with fitness of nodes is instead described by the Fermi-Dirac distribution [3] and leads to the analytical description of Invasion Percolation on these structures.

Recently these classical results of network theory have been related to the properties of growing simplicial complexes [12–16]. A simplicial complex [17–21] is a generalized network structure that allows the description of many-body interactions between a set of nodes. In particular simplicial complexes are not only formed by nodes and links like networks but they are instead also formed by triangles, tetrahedra and so on. Given that a simplicial complex is build by geometrical building blocks, simplicial complexes are natural structures to study network geometry. As such simplicial complexes have been widely used in quantum gravity to describe the discrete (or discretised) structure of space-time [23–26]. In the last five years simplicial complexes are becoming increasingly popular to describe complex systems as well including collaboration networks, social networks, financial networks, nano-structures, and brain networks [20,21,27–36]. The Network Geometry with Flavor (NGF) [12–16] is a non-equilibrium model of growing simplicial complexes with fitness that has been proposed to study emergent network geometry. In fact the NGFs evolve thanks to purely combinatorial rules that make no use of any embedding space, but when the same length is attributed to each link of the simplicial complex they are able to generate structures with an emergent hyperbolic geometry [14].

The *flavor* s of the NGF is a parameter that can change the topological nature of the simplicial complex and their evolution. For $s = -1$ the NGF is a manifold; for $s = 0$ the network grows by uniform attachment of d -dimensional simplices on $(d - 1)$ -dimensional faces; finally for $s = 1$ the network evolves according to a generalized preferential attachment rule of d -dimensional simplices on $(d - 1)$ -dimensional faces.

Interestingly NGFs have a stochastic topology that is described by quantum statistics [12,13]. In particular for $s = -1$ where we associate to the $(d - 1)$ -dimensional faces an incidence number $n_\alpha \in \{0, 1\}$ we obtain that the $(d - 1)$ -dimensional faces

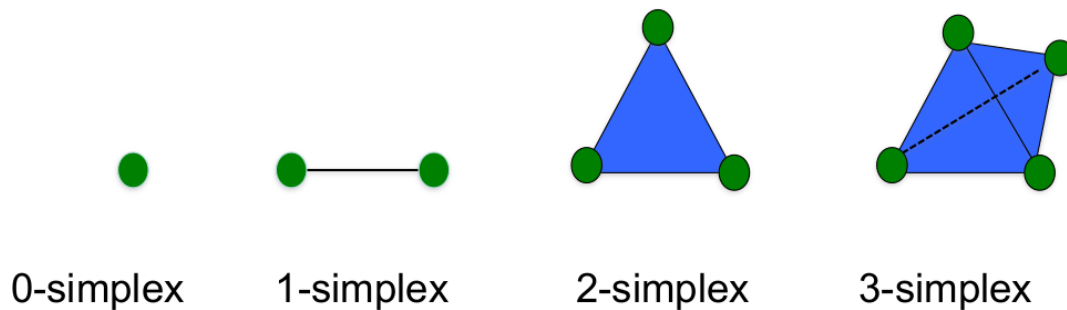


Figure 1. Examples of δ -dimensional simplices with $\delta = 0, 1, 2, 3$

are described by the Fermi-Dirac statistics. Moreover the lower dimensional faces are described by either the Boltzmann or the Bose-Einstein statistics. For instance in a NGF with flavor $s = -1$ and dimension $d = 3$ the statistical properties of the triangles, links and nodes of the simplicial complexes are described by the Fermi-Dirac, the Boltzmann and the Bose-Einstein statistics respectively.

In this paper we extend the study of this model to Network Geometry with Fractional Flavor $s = -1/m$ and $m > 1$. In principle, since for these networks the incidence number of the $(d-1)$ -faces is allowed to take only values $n_\alpha \in \{0, 1, 2, \dots, m\}$ we might expect to find that $(d-1)$ -faces are described by fractional statistics [37–39]. Contrary to this naive expectation here we show that also in this case $(d-1)$ -dimensional faces are described by the Fermi-Dirac statistics and that instead the main difference with the NGF with integer flavor $s = -1$ is that we do not find any face described by the Boltzmann statistics. This result sheds light on the effect that dimensionality and flavor have on the emergence of quantum statistics in NGFs. In particular while for integer flavor we must require $s = -1$ and $d \geq 3$ to observe the co-existence of the Fermi-Dirac and Bose-Einstein distribution describing the statistical properties of faces of different dimension δ , for fractional flavor $s = -1/m$ (with $m > 1$) we can have the coexistence of these two statistics already for dimension $d = 2$.

2. Simplicial complexes and their generalized degrees

2.1. Simplicial complexes

A simplicial complex describes the many body interactions between a set of N nodes. In particular a simplicial complex is formed by simplices glued along their faces. A δ -dimensional simplex is a set of $\delta + 1$ nodes. Therefore a 0-dimensional simplex is a node, a 1-dimensional simplex is a link, a 2-dimensional simplex is a triangle, and so on (see Figure 1). A δ' -dimensional *face* α' of a δ -dimensional simplex α , is a simplex formed by a subset of $\delta' + 1$ nodes of α , i.e. $\alpha' \subset \alpha$ (see Figure 2). A d -dimensional *simplicial complex* \mathcal{K} (see Figure 3 for examples) is formed by a set of simplices of dimensions $0 \leq \delta \leq d$ (including at least a d -dimensional simplex) that obey the following two conditions:

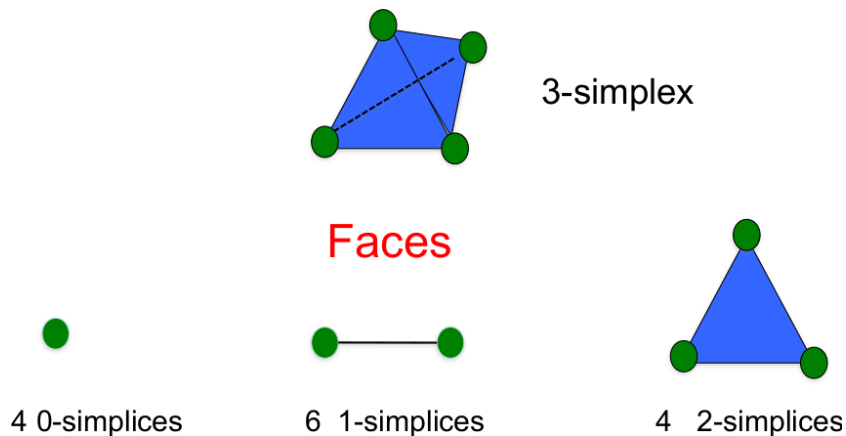


Figure 2. The faces of a $\delta = 3$ dimensional simplex (a tetrahedron) are shown and their relative number is indicated correspondingly.

- (a) if a simplex α belongs to the simplicial complex, i.e. $\alpha \in \mathcal{K}$ then also all its faces $\alpha' \subset \alpha$ belong to the simplicial complex, i.e. $\alpha' \in \mathcal{K}$;
- (b) if two simplices α and α' belong to the simplicial complex, i.e. $\alpha \in \mathcal{K}$ and $\alpha' \in \mathcal{K}$, then either their intersection is the null set $\alpha \cap \alpha' = \emptyset$ or their intersection belongs to the simplicial complex, i.e. $\alpha \cap \alpha' \in \mathcal{K}$.

A d -dimensional simplicial complex is called *pure* if it is only formed by d -dimensional simplices and their faces (see Figure 3 for examples).

From a simplicial complex it is always possible to extract a network called the *1-skeleton* by considering only the nodes and links of the simplicial complex.

In this paper we will focus on pure d -dimensional simplicial complexes \mathcal{K} . In the following we will indicate with $\mathcal{Q}_\delta(N)$ the set of all possible δ simplices in a simplicial complex of N nodes and with $\mathcal{S}_{d,\delta}$ the set of all the δ -dimensional faces of the pure d -dimensional simplicial complex \mathcal{K} .

Note that here we take a purely topological [22] rather than a geometrical point of view. Therefore the nodes belonging to each simplex are not assigned a priori a position in an embedded space and the links of the simplices do not have an a priori defined length. This is the ideal starting point for studying the problem of emergent hyperbolic network geometry [17] as one is interested on the minimal assumptions on the link lengths that allow the embedding on a space with a given sign of the curvature.

2.2. Generalized degrees

The topology of a pure d -dimensional simplicial complex \mathcal{K} is fully specified by the adjacency tensor \mathbf{a} of elements a_α with $\alpha \in \mathcal{Q}_d(N)$ given by

$$a_\alpha = \begin{cases} 1 & \text{if } \alpha \in \mathcal{K}, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

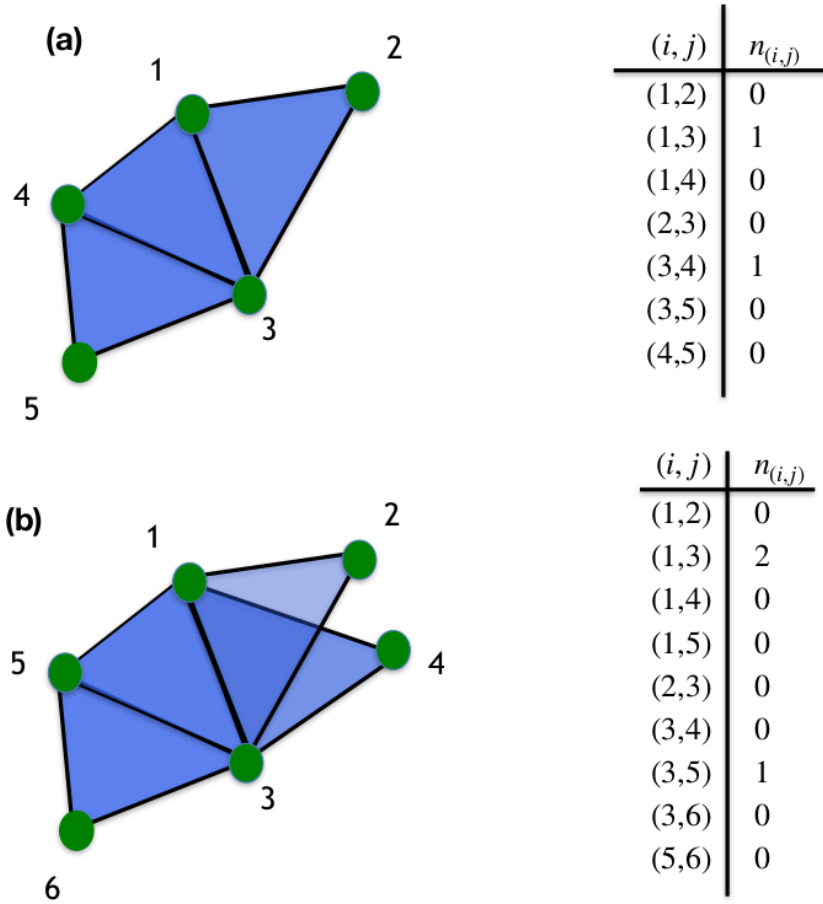


Figure 3. Two examples of pure 2-dimensional simplicial complexes are shown together with the list of the incidence number of their links. The simplicial complex in panel (a) is a manifold as $n_\alpha \in \{0, 1\}$ for all links α of the simplicial complex. The simplicial complex in panel (b) is not a manifold as $n_{(1,3)} = 2$.

The *generalized degree* $k_{d,\delta}(\alpha)$ [13, 19] of the δ -face α is defined as the number of d -dimensional simplices incident to it. Using the adjacency tensor we can evaluate $k_{d,\delta}$ of a δ -face α as

$$k_{d,\delta}(\alpha) = \sum_{\alpha' \in Q_d(N) | \alpha' \supset \alpha} a_{\alpha'}. \quad (2)$$

Therefore, in $d = 2$, the generalized degree $k_{2,1}(\alpha)$ is the number of triangles incident to a link α while the generalized degree $k_{2,0}(\alpha)$ indicates the number of triangles incident to a node α . Similarly in a pure $d = 3$ dimensional simplicial complex, the generalized degrees $k_{3,2}$, $k_{3,1}$ and $k_{3,0}$ indicate the number of tetrahedra incident respectively to a triangular face, a link or a node. The generalized degrees of faces are not independent of the generalized degree of the simplices to which they belong [19]. In fact the generalized degree of a δ -face α is related to the generalized degree of the δ' -dimensional faces

incident to it, with $\delta' > \delta$, by the simple combinatorial relation

$$k_{d,\delta}(\alpha) = \frac{1}{\binom{d-\delta}{\delta'-\delta}} \sum_{\alpha' \in \mathcal{Q}_d(N) | \alpha' \supset \alpha} k_{d,\delta'}(\alpha'). \quad (3)$$

Moreover, since every d -dimensional simplex belongs to $\binom{d+1}{\delta+1}$ δ -dimensional faces, in a simplicial complex with M d -dimensional simplices we have

$$\sum_{\alpha \in \mathcal{S}_{d,\delta}} k_{d,\delta}(\alpha) = \binom{d+1}{\delta+1} M. \quad (4)$$

2.3. Incidence number

The $(d-1)$ -dimensional faces of a pure d -dimensional simplicial complex deserve some special attention. In particular to each $(d-1)$ -dimensional face α we associate an *incidence number* n_α given by the number of incident d -dimensional simplices minus one, i.e.

$$n_\alpha = k_{d,d-1}(\alpha) - 1. \quad (5)$$

Interestingly a simplicial complex can define a discrete d -dimensional manifold only if $n_\alpha \in \{0, 1\}$, i.e. a discrete d -dimensional manifold must have all its $(d-1)$ -dimensional faces incident at most to two d -dimensional simplices. Therefore if $n_\alpha > 1$ at least for one face $\alpha \in \mathcal{S}_{d,d-1}$ then the simplicial complex is not a discrete manifold. In Figure 3 we show two examples of 2-dimensional simplicial complexes (a manifold and a simplicial complex that is not a manifold) and the corresponding list of the incidence numbers of their links.

3. Network Geometry with Flavor

3.1. Energy and fitness of the simplices

In the Network Geometry with Flavor each simplex $\alpha \in \mathcal{K}$ is associated to an energy ϵ_α that does not change in time. The energy of a face describes its intrinsic and heterogeneous properties and has an important effect on the simplicial complex evolution.

The energy ϵ_i of node i is drawn randomly from a given distribution $g(\epsilon)$. To every δ -face $\alpha \in \mathcal{S}_{d,\delta}$ with $0 < \delta \leq d$ we associate an *energy* ϵ_α given by the sum of the energy of the nodes that belong to the face α ,

$$\epsilon_\alpha = \sum_{i \in \alpha} \epsilon_i. \quad (6)$$

Therefore, the energy of a link is given by the sum of energies of the two nodes that belong to it, the energy of a triangular face is given by the sum of the energy of the three nodes belonging to it and so on. The energy $\epsilon_{(i,j)}$ of the generic link $\alpha = (i, j)$

belonging to any given triangle of the NGF formed by the nodes i , j and r satisfy the triangular inequality

$$|\epsilon_{(i,r)} - \epsilon_{(j,r)}| \leq \epsilon_{(i,j)} \leq \epsilon_{(i,r)} + \epsilon_{(j,r)}. \quad (7)$$

This result remains valid for any permutation of the order of the nodes i , j and r belonging to the triangle. Therefore it is possible to consider the energy of the link as a possible candidate for the length of the link.

Finally to each simplex $\alpha \in \mathcal{K}$ we associate a *fitness* η_α given by

$$\eta_\alpha = e^{-\beta\epsilon_\alpha}, \quad (8)$$

where $\beta \leq 0$ is an external parameter of the model called *inverse temperature*. If $\beta = 0$ we have that $\eta_\alpha = 1$ for every simplex $\alpha \in \mathcal{K}$, therefore every simplex has the same fitness independently of their differences in energy. On the contrary when β is large small differences in energy lead to large differences in the fitness of different simplices.

3.2. Evolution of the Network Geometry with Flavor

Network Geometry with Flavor (NGF) is a growing model generating pure d -dimensional simplicial complexes. The stochastic evolution of NGF is determined by a parameter s called the *flavor* and by the fitness of the simplices of the simplicial complex. The evolution of the NGF obeys a simple iterative algorithm.

Initially at time $t = 1$ the simplicial complex is formed by a single d -dimensional simplex. At each time $t > 1$ we glue a d -dimensional simplex to a $(d - 1)$ -face α chosen with probability

$$\Pi_{d,d-1}(\alpha) = \frac{\eta_\alpha(1 + sn_\alpha)}{Z^{[s]}}, \quad (9)$$

where $Z^{[s]}$ is called the *partition function* of the NGF and is given by

$$Z^{[s]}(t) = \sum_{\alpha' \in S_{d,d-1}} \eta_{\alpha'}(1 + sn_{\alpha'}). \quad (10)$$

3.3. Possible values of the flavor and their topological implications

The Network Geometry with Flavor describes a growing simplicial complex that depends on the value of the flavor s . Let us consider the attachment probability $\Pi_{d,d-1}(\alpha)$ for $\beta = 0$ and the integer flavors $s \in \{-1, 0, 1\}$. In this case we have that the attachment probability satisfies

$$\Pi_{d,d-1}(\alpha) \propto (1 + sn_\alpha) = \begin{cases} 1 - n_\alpha & \text{for } s = -1, \\ 1 & \text{for } s = 0, \\ k_{d,d-1}(\alpha) & \text{for } s = 1. \end{cases} \quad (11)$$

Therefore the flavor $s = -1$ enforces the generation of a manifold. In fact we have $\Pi_{d,d-1}(\alpha) > 0$ if $n_\alpha = 0$ and $\Pi_{d,d-1}(\alpha) = 0$ if $n_\alpha = 1$. Therefore $n_\alpha \in \{0, 1\}$ for every $(d - 1)$ -dimensional face α of the simplicial complex. However in both cases $s = 0$ and $s = 1$ the incidence number can take any integer value $n_\alpha \geq 0$. The flavor $s = 0$

Table 1. Distribution of generalized degrees of faces of dimension δ in a d -dimensional NGF of flavor s at $\beta = 0$. For $d \geq 2\delta + 2 - s$ the power-law distributions are scale-free, i.e. the second moment of the distribution diverges.

flavor	$s = -1$	$s = 0$	$s = 1$
$\delta = d - 1$	Bimodal	Exponential	Power-law
$\delta = d - 2$	Exponential	Power-law	Power-law
$\delta \leq d - 3$	Power-law	Power-law	Power-law

corresponds to a uniform attachment of d -dimensional simplices of $(d - 1)$ -dimensional faces, while $s = 1$ corresponds to a higher dimensional preferential attachment of d -dimensional simplices of $(d - 1)$ -dimensional faces.

The NGF with integer flavor reduces to several known models for different values of the parameters s, d and β . For $d = 1, s = 1, \beta = 0$ the NGF reduces to the Barabási-Albert [40] model while for $d = 1, s = 1, \beta > 0$ it reduces to the Bianconi-Barabási model [1, 2]. For $d = 2, s = 0, \beta = 0$ it reduces to the model proposed in Ref. [41] Finally for $d = 3, s = -1, \beta = 0$ it reduces to a random Apollonian network [42].

Values of the flavor s different from the values $\{-1, 0, 1\}$ are also allowed as long as they lead to a suitable probability $\Pi_{d,d-1}(\alpha) \in [0, 1]$ for every face $\alpha \in \mathcal{K}$. Therefore positive values of the flavor $s \in \mathbb{R}^+$ are always allowed. In this case via a rescaling of the attachment probability it is easy to show that NGFs display a stochastic topology with statistical properties equivalent to NGF with flavor $s = 1$.

For negative values of the flavor s the requirement of observing a well defined attachment probability $\Pi_{d,d-1}(\alpha) \in [0, 1]$ implies instead some restriction on the possible values of s . In particular if $s < 0$, then s should be of the form

$$s = -\frac{1}{m}, \quad (12)$$

with $m \in \mathbb{N}$. For such values of the flavor s the incidence number of any $(d - 1)$ -dimensional face α can only take $m + 1$ values, i.e.

$$n_\alpha \in \{0, 1, 2, \dots, m\}. \quad (13)$$

Therefore as long as $m > 1$ NGF with $s = -1/m$ are not anymore manifolds, but they have a generalized degree of the $(d - 1)$ -dimensional faces bounded by $m + 1$, i.e.

$$k_{d,d-1}(\alpha) \in \{0, 1, 2, \dots, m + 1\}. \quad (14)$$

This case, that we will call NGF with Fractional Flavor, is therefore expected to display statistical properties that are not equivalent to the ones observed for any of the integer flavors $s \in \{-1, 0, 1\}$.

4. Network Geometry with Integer Flavor

The distribution $P_{d,\delta}(k)$ of generalized degrees $k_{d,\delta} = k$ of δ -dimensional faces on the d -dimensional NGF has been derived for integer flavors $s \in \{-1, 0, 1\}$ in Ref. [13]. It has been found that for $\beta = 0$ the generalized degree distribution $P_{d,\delta}(k)$ can follow a

Table 2. The average $\langle k_{d,\delta} - 1 | \epsilon \rangle$ of the generalized degrees $k_{d,\delta} - 1$ of δ -faces with energy ϵ in a d -dimensional NGF of flavor s follows either the Fermi-Dirac, the Boltzmann or the Bose-Einstein statistics depending on the values of the dimensions d and δ .

flavor	$s = -1$	$s = 0$	$s = 1$
$\delta = d - 1$	Fermi-Dirac	Boltzmann	Bose-Einstein
$\delta = d - 2$	Boltzmann	Bose-Einstein	Bose-Einstein
$\delta \leq d - 3$	Bose-Einstein	Bose-Einstein	Bose-Einstein

Table 3. Distribution of generalized degrees of faces of dimension δ in a d -dimensional NGF of flavor s at $\beta = 0$. Only for $d - 2\delta \geq 2 + \frac{3}{m}$ the power-law distributions are scale-free, i.e. the second moment of the distribution diverges.

flavor	$s = -1/m$
$\delta = d - 1$	Bounded $k \leq m + 1$
$\delta \leq d - 2$	Power-law

bimodal, exponential or power-law distribution (see Table 1) depending on the dimension δ and the flavor s of the NGF.

For $\beta > 0$ emergent quantum statistics describe the statistical properties of NGFs as long as the NGF has a stationary generalized degree distribution, i.e for sufficiently low value of $\beta \leq \beta_c$. Specifically it has been found in Ref. [12, 13] that the average of the generalized degree minus one, $k_{d,\delta} - 1$, over δ -dimensional faces of energy ϵ can follow the Fermi-Dirac, the Boltzmann or the Bose-Einstein distribution depending on the dimension δ and the flavor s of the NGF (see Table 2). For instance in a NGF with $s = -1$ and $d = 3$ the average of the generalized degree distribution minus one performed over faces of energy ϵ follows the Fermi-Dirac, the Boltzmann or the Bose-Einstein distribution for triangular faces, links and nodes respectively.

In the next section we will show how these statistical properties change for NGF with Fractional Flavor.

5. Network Geometry with Fractional Flavor and $\beta = 0$

5.1. Main results

In this section we will evaluate the generalized degree distribution $P_{d,\delta}(k)$ of NGF with Fractional flavor $s = -1/m$ and $m > 1$ for $\beta = 0$. In particular we will show that differently from the cases $s = -1$ and $s = 0$ the generalized degree distributions are never exponential. In fact for $s = -1/m$ and $m > 1$ we obtain that the $(d - 1)$ -faces have a generalized degree distribution with bounded support with $k \leq m + 1$ and the δ -dimensional faces with $0 \leq \delta < d - 1$ have a generalized degree distribution which is power-law (see Table 3). In order to proof these results, in the following paragraph we first derive the generalized attachment probability. Subsequently we derive the generalized degree distribution first using the mean-field approximation and finally using the master equation approach providing exact asymptotic results.

5.2. Attachment probability

For fractional flavor $s = -\frac{1}{m}$ the attachment probability for $\beta = 0$, given by Eq. (9) can also be expressed as

$$\Pi_{d,d-1}(\alpha) = \frac{m - n_\alpha}{\tilde{Z}} = \frac{m + 1 - k_{d,d-1}(\alpha)}{\tilde{Z}}, \quad (15)$$

where \tilde{Z} is given by

$$\tilde{Z} = \sum_{\alpha \in \mathcal{S}_{d,d-1}} (m - n_\alpha) = \sum_{\alpha \in \mathcal{S}_{d,d-1}} (m + 1 - k_{d,d-1}(\alpha)).$$

Therefore the normalization constant \tilde{Z} counts each $(d-1)$ -dimensional face α with a degeneracy $m - n_\alpha \in \{0, 1, 2, \dots, m\}$.

Since at time $t = 1$ we have $d + 1$ $(d-1)$ -dimensional faces with degeneracy m we have that $\tilde{Z}(t = 1) = m(d + 1)$. Moreover at each time we add d new $(d-1)$ -dimensional faces with degeneracy m and we reduce the degeneracy of the $(d-1)$ -faces α to which we add the new d -dimensional simplex by 1. Therefore at each time \tilde{Z} increases by $md - 1$. It follows that the normalization constant \tilde{Z} is given by

$$\tilde{Z} = (md - 1)t + 1 + m \simeq (md - 1)t, \quad (16)$$

where the last expression is valid for $t \gg 1$. The probability $\Pi_{d,\delta}(\alpha)$ that a new d -dimensional simplex is attached to a $\delta \leq d - 2$ dimensional face α is given by

$$\Pi_{d,\delta}(\alpha) = \sum_{\alpha' \in \mathcal{S}_{d,d-1} | \alpha' \supset \alpha} \frac{m - n_{\alpha'}}{\tilde{Z}}. \quad (17)$$

In order to calculate the numerator of this expression we make the following considerations. If we assume that the face α has incidence number $n_\alpha = 0$, then the numerator of Eq. (17) is given by $d - \delta$. In fact every δ -dimensional face α with $\delta < d - 1$ and generalized degree $k_{d,\delta}(\alpha) = 1$ is incident to

$$\binom{d - \delta}{d - \delta - 1} = d - \delta \quad (18)$$

$(d-1)$ -dimensional faces with degeneracy m . This follows from the fact that its incident $(d-1)$ -dimensional faces must contain $d - \delta - 1$ nodes that do not belong to the face α . These nodes should be chosen among the $(d - \delta)$ nodes of the single d -dimensional simplex that contains the face α and are external to face α . Following a similar argument it is easy to check that at each time we add a d -dimensional simplex to the δ -dimensional face the number of $(d-1)$ -dimensional faces with degeneracy m increases by

$$\binom{d - \delta - 1}{d - \delta - 2} = d - \delta - 1 \quad (19)$$

and additionally we reduce the degeneracy of the $(d-1)$ -dimensional face to which we attach the new d -dimensional simplex by one. Therefore for $t \gg 1$ we have

$$\Pi_{d,\delta}(\alpha) \simeq \begin{cases} \frac{m+1-k_{d,\delta}(\alpha)}{(md-1)t} & \text{for } \delta = d - 1, \\ \frac{[m(d-\delta-1)-1]k_{d,\delta}(\alpha)+m+1}{(md-1)t} & \text{for } \delta \leq d - 2. \end{cases} \quad (20)$$

Finally given the above expression we can express the probability $\tilde{\Pi}_{d,\delta}(k)$ that a new d -dimensional simplex is attached to a δ -dimensional face α with generalized degree $k_{d,\delta}(\alpha) = k$ as

$$\tilde{\Pi}_{d,\delta}(k) \simeq \begin{cases} \frac{m+1-k}{(md-1)t} & \text{for } \delta = d-1, \\ \frac{[m(d-\delta-1)-1]k+m+1}{(md-1)t} & \text{for } \delta \leq d-2. \end{cases} \quad (21)$$

5.3. Mean-field approach

In order to find an approximated generalized degree distribution we can consider the popular mean field approach [43]. In this case we assume that the generalized degrees can be approximated by their average over different network realizations and that they evolve by a deterministic equation

$$\frac{dk_{d,\delta}(\alpha)}{dt} = \Pi_{d,\delta}(\alpha). \quad (22)$$

Let us consider separately the case in which $\delta = d-1$ and the case $\delta \leq d-2$. The mean-field equation for the generalized degree of $\delta = d-1$ dimensional faces is given by

$$\frac{dk_{d,d-1}}{dt} = \frac{m+1-k_{d,d-1}}{(md-1)t}, \quad (23)$$

with initial condition $k_{d,d-1}(t_\alpha) = 1$. It follows that in the mean-field approximation the generalized degree of a $(d-1)$ -face added at time t_α evolves in time as

$$k_{d,d-1}(t) = m+1 - m \left(\frac{t_\alpha}{t} \right)^{1/(md-1)}. \quad (24)$$

It follows that, as expected, the generalized degrees of the $(d-1)$ -dimensional faces are bounded and asymptotically in time saturate to the value $k_{d,d-1} = m+1$. In order to derive the generalized degree distribution in the mean-field approximation we calculate the probability $\mathcal{P}(k_{d,\delta} > k)$. This is given by

$$\begin{aligned} \mathcal{P}(k_{d,d-1} > k) &= \mathcal{P} \left(t_\alpha < \left(\frac{m+1-k}{m} \right)^{md-1} t \right) \\ &= \left(\frac{m+1-k}{m} \right)^{md-1}. \end{aligned} \quad (25)$$

Therefore in the mean-field approximation the generalized degree distribution is given by

$$\tilde{P}_{d,d-1}(k) = -\frac{\mathcal{P}(k_{d,d-1} > k)}{dk} = \frac{md-1}{m} \left(\frac{m+1-k}{m} \right)^{md-2}, \quad (26)$$

valid for $1 \leq k \leq m$. Let us now consider the mean-field equation for the $\delta \leq d-2$ dimensional faces. This equation reads

$$\frac{dk_{d,d-1}}{dt} = \frac{[m(d-\delta-1)-1]k_{d,\delta} + m+1}{(md-1)t} \quad (27)$$

with initial condition $k_{d,d-1}(t_\alpha) = 1$. Therefore in the mean field approximation the generalized degree of a δ -face added at time t_α grows in time as

$$k_{d,d-1}(t) = (1 + 1) \left(\frac{t}{t_\alpha} \right)^{\frac{m(d-\delta-1)-1}{md-1}} - a \quad (28)$$

where for convenience we have defined the constant a as

$$a = \frac{m + 1}{m(d - \delta - 1) - 1}. \quad (29)$$

Given this definition we can express the generalized degree distribution of $\delta \leq d - 2$ faces in the mean field approximation. In fact we have that the cumulative generalized degree distribution is given by

$$\begin{aligned} \mathcal{P}(k_{d,\delta} > k) &= \mathcal{P} \left(t_\alpha < \left(\frac{1 + a}{k + a} \right)^{\frac{md-1}{m(d-\delta-1)-1}} t \right) \\ &= \left(\frac{1 + a}{k + a} \right)^{\frac{md-1}{m(d-\delta-1)-1}}. \end{aligned} \quad (30)$$

By differentiating this expression we find the generalized degree distribution of $\delta \leq d - 2$ faces in the mean field approximation is given by

$$\begin{aligned} \tilde{P}_{d,\delta}(k) &= - \frac{\mathcal{P}(k_{d,\delta} > k)}{dk} \\ &= \frac{md - 1}{m(d - \delta)} \left(\frac{1 + a}{k + a} \right)^{\frac{md-1}{m(d-\delta-1)-1} + 1}. \end{aligned} \quad (31)$$

Therefore we find in this approximation that generalized degrees have a bounded distribution for $\delta \leq d - 1$ faces and a power-law distribution for $\delta \leq d - 2$ faces.

5.4. Master equation approach

The mean-field approach gives only approximate results for the generalized degree distribution. In order to get the exact asymptotic results we need to consider the master equation approach [43]. The master equation describes the evolution of the average number $N_{d,\delta}^t(k)$ of δ -dimensional faces that at time t have generalized degree k in a d -dimensional NGF with flavor $s = -\frac{1}{m}$. We notice that at each time we add

$$m_{d,\delta} = \binom{d}{\delta}$$

number of δ -faces of generalized degree $k_{d,\delta} = 1$ and the average number of δ -faces of generalized degree $k_{d,\delta} = k$ increases by

$$\Pi_{d,\delta}(k - 1) N_{d,\delta}^t(k - 1)$$

if $k > 1$ and decreases by

$$\Pi_{d,\delta}(k) N_{d,\delta}^t(k).$$

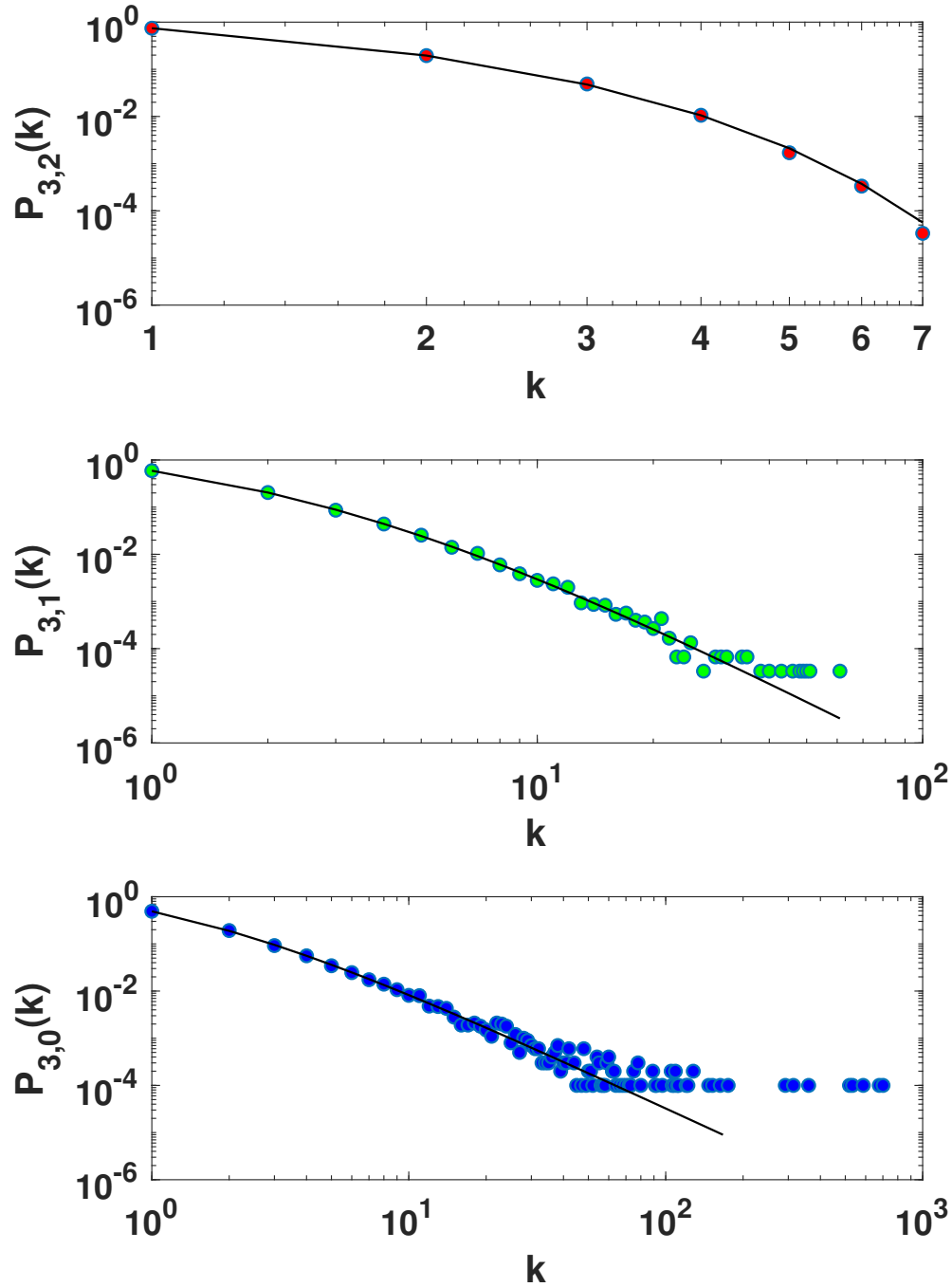


Figure 4. Generalized degree distribution $P_{d,\delta}(k)$ of nodes ($\delta = 0$), links ($\delta = 1$) and triangles ($\delta = 2$) of a NGF with $N = 5000$ nodes, flavor $s = -1/6$, dimension $d = 3$ and inverse temperature $\beta = 0$. The symbols indicate the results of simulations, the solid lines indicate the theoretical predictions obtained using the master equation approach.

Therefore the master equation reads

$$\begin{aligned} N_{d,\delta}^{t+1}(k) &= N_{d,\delta}^t(k) + \Pi_{d,\delta}(k-1)N_{d,\delta}^t(k-1)(1-\delta_{k,1}) \\ &\quad - \Pi_{d,\delta}(k)N_{d,\delta}^t(k) + m_{d,\delta}\delta_{k,1}. \end{aligned} \quad (32)$$

For large network sizes when $t \gg 1$ the average number of δ -dimensional faces is given by

$$N_{d,\delta}^t(k) \simeq m_{d,\delta}tP_{d,\delta}(k) \quad (33)$$

where $P_{d,\delta}(k)$ is the generalized degree distribution of the δ -dimensional faces. Inserting this asymptotic expression in the Eq. (32) we can derive the generalized degree distribution as explained in the following by distinguishing between the case in which $\delta = d - 1$ and the case in which $\delta \leq d - 2$.

For the $\delta = d - 1$ dimensional faces, by using the expression

$$\Pi_{d,d-1}(k) = \frac{m+1-k}{(md-1)t}, \quad (34)$$

and the asymptotic scaling of $N_{d,\delta}^t(k)$ given by Eq. (33) the master equation can be re-written in terms of the generalized degree distribution obtaining

$$\begin{aligned} P_{d,d-1}(k) &= \frac{m+2-k}{(md-1)}P_{d,d-1}(k-1)(1-\delta_{k,1}) + \\ &\quad - \frac{m+1-k}{(md-1)}P_{d,d-1}(k) + \delta_{k,1}. \end{aligned} \quad (35)$$

obtaining

$$P_{d,d-1}(k) = \frac{md-1}{m(d+1)-1} \frac{\Gamma(m+1)}{\Gamma(md+m-1)} \frac{\Gamma(md+m-k)}{\Gamma(m-k+2)}, \quad (36)$$

valid for $1 \leq k \leq m$.

For the $\delta \leq d - 2$ dimensional faces by using the expression

$$\Pi_{d,\delta}(k) = \frac{[m(d-\delta-1)-1]k+m+1}{(md-1)t}, \quad (37)$$

and the asymptotic scaling of $N_{d,\delta}^t(k)$ given by Eq. (33) the master equation can be re-written in terms of the generalized degree distribution obtaining

$$\begin{aligned} P_{d,\delta}(k) &= \frac{[m(d-\delta-1)-1]k+m+1}{md-1}P_{d,\delta}(k-1)(1-\delta_{k,1}) \\ &\quad - \frac{[m(d-\delta-1)-1]k+m+1}{md-1}P_{d,\delta}(k) + \delta_{k,1}. \end{aligned}$$

This latter recursive equation has explicit solution

$$\begin{aligned} P_{d,\delta}(k) &= \frac{md-1}{m(2d-\delta)-1} \frac{\Gamma\left(2 + \frac{m(d+1)}{m(d-\delta-1)-1}\right)}{\Gamma\left(1 + \frac{m+1}{m(d-\delta-1)-1}\right)} \\ &\quad \times \frac{\Gamma\left(k + \frac{m+1}{m(d-\delta-1)-1}\right)}{\Gamma\left(k+1 + \frac{m(d+1)}{m(d-\delta-1)-1}\right)}. \end{aligned} \quad (38)$$

This distribution for large $k \gg 1$ decays as a power-law

$$P_{d,\delta}(k) \simeq k^{-\gamma_{d,\delta}} \quad (39)$$

with

$$\gamma_{d,\delta} = 1 + \frac{md - 1}{m(d - \delta - 1) - 1}. \quad (40)$$

These distributions are therefore scale-free, i.e. $\gamma \leq 3$ if and only if

$$d - 2\delta \geq 2 + \frac{3}{m}. \quad (41)$$

These theoretical predictions show that the generalized degree distribution is indeed bounded for $\delta = d - 1$ dimensional faces and power-law for $\delta \leq d - 2$ dimensions. As expected these results perfectly match the simulation results providing exact asymptotic expression for the generalized degree distribution $P_{d,\delta}(k)$ for NGF with fractional flavor $s = -\frac{1}{m}$ (see Figure 4).

6. Network Geometry with Fractional Flavor and $\beta > 0$

6.1. Main results

Quantum statistics have been shown to characterize the statistical properties of NGF with integer flavor $s \in \{-1, 0, 1\}$. In particular d -dimensional NGFs with flavor $s = -1$ have an average degree of δ faces with energy ϵ described by the Fermi-Dirac (for $\delta = d - 1$), the Boltzmann (for $\delta = d - 2$) and the Bose-Einstein distribution (for $\delta \leq d - 3$). On the contrary on NGF with flavor $s = 0$ the average degree of δ faces with energy ϵ can be only described by the Boltzmann and the Bose-Einstein distribution. Finally in NGF with flavor $s = 1$ all the faces, independently of their dimension δ , have an average degree described only by the Bose-Einstein distribution. Interestingly if we consider integer flavors $s \in \{-1, 0, 1\}$ the Fermi-Dirac distribution emerges as the natural distribution characterizing the statistical properties of $\delta = d - 1$ faces only if the flavor is given by $s = -1$, which corresponds to the case in which the incidence number n_α for the $\delta = d - 1$ faces can only take the values $n_\alpha \in \{0, 1\}$. This suggests a relation between the emergence of the Fermi-Dirac statistics and the constraint imposed by the flavor $s = -1$ on the possible values of the incidence number. It is therefore interesting to investigate the properties of NGF with fractional flavor $s = -\frac{1}{m}$ in which the only allowed values of the incidence number are $n_\alpha \in \{0, 1, \dots, m\}$. In principle one could expect that in this case the statistical properties of the generalized degree of the NGF would be described by generalized quantum statistics such as the Gentile statistics [37] or the anyons statistics [38, 39]. To our surprise instead the result of our calculations has revealed that in NGF with fractional flavor $s = -\frac{1}{m}$ and $m > 1$ does not display any fractional statistics (see Table 4). If we compare the results obtained for to the NGFs with fractional flavor $s = -\frac{1}{m}$ and $m > 1$ to the results obtained for NGF with $s = -1$ we observe that

Table 4. The average $\langle k_{d,\delta} - 1 | \epsilon \rangle$ of the generalized degrees $k_{d,\delta}$ of δ -faces with energy ϵ minus one in a d -dimensional NGF of flavor $s = -\frac{1}{m}$ follows either the Fermi-Dirac or the Bose-Einstein statistics depending on the values of the dimensions d and δ .

$s = -1/m$	
$\delta = d - 1$	Fermi-Dirac
$\delta \leq d - 2$	Bose-Einstein

- the average degree of $(d - 1)$ -dimensional faces with energy ϵ still remain described by the Fermi-Dirac statistics also if the incidence number of $\delta = d - 1$ faces can take values $n_\alpha \in \{0, 1, \dots, m\}$ with $m > 1$;
- the average degree of $\delta = (d - 2)$ dimensional faces with energy ϵ is already characterized by the Bose-Einstein statistics and not by the Boltzmann statistics;
- the average degree of $\delta < d - 2$ -dimensional faces with energy ϵ is characterized by the Bose-Einstein statistics like in the case $s = -1$.

In particular these results imply that when we consider the fractional flavor $s = -\frac{1}{m}$ and $m > 1$ we have that already for $d = 2$ dimensional NGF we can observe the coexistence of faces with statistical properties described respectively by the Fermi-Dirac and Bose-Einstein distribution while for observing the co-existence of these two statistics in NGF with flavor $s = -1$ we should have dimension $d \geq 3$.

6.2. Attachment probability and chemical potentials

When $\beta > 0$ the generalized degree distribution of the NGF can be solved by extending the self-consistent approach proposed for solving the Bianconi-Barabási model [1, 2] which constitutes the NGF model for $s = 1$ and $d = 1$. In this approach it is assumed that the statistical properties of the NGF reach a steady state and that it is possible to define suitable parameters $\mu_{d,\delta}$ called *chemical potentials*. In particular the chemical potential $\mu_{d,d-1}$ of the $\delta = d - 1$ faces is defined as

$$e^{\beta\mu_{d,d-1}} = \lim_{t \rightarrow \infty} \frac{t}{mZ^{[s]}}, \quad (42)$$

while the chemical potential $\mu_{d,\delta}$ of the $\delta < d - 1$ faces is defined as

$$e^{\beta\mu_{d,\delta}} = \lim_{t \rightarrow \infty} \left\langle \frac{\sum_{\alpha' \in S_{d,d-1} | \alpha \subset \alpha'} e^{-\beta(\epsilon_{\alpha'} - \epsilon_\alpha)} (1 + sn_{\alpha'}) t}{Z^{[s]}(a + k_{d,\delta}(\alpha))} \right\rangle,$$

where here the average is done over δ -dimensional faces $\alpha \in S_{d,\delta}$. In both cases it is assumed that if the network evolution reaches a stationary state, then the chemical potential is self-averaging, i.e. it does not depend on the specific network realization of the NGF over which the limit $t \rightarrow \infty$ is performed. As long as the chemical potentials are well determined and self-averaging quantities the attachment probabilities can be expressed in terms of the chemical potentials, and it can be easily shown that the probability $\Pi_{d,\delta}(\alpha)$ that a new d -dimensional simplex is attached to a new δ -dimensional

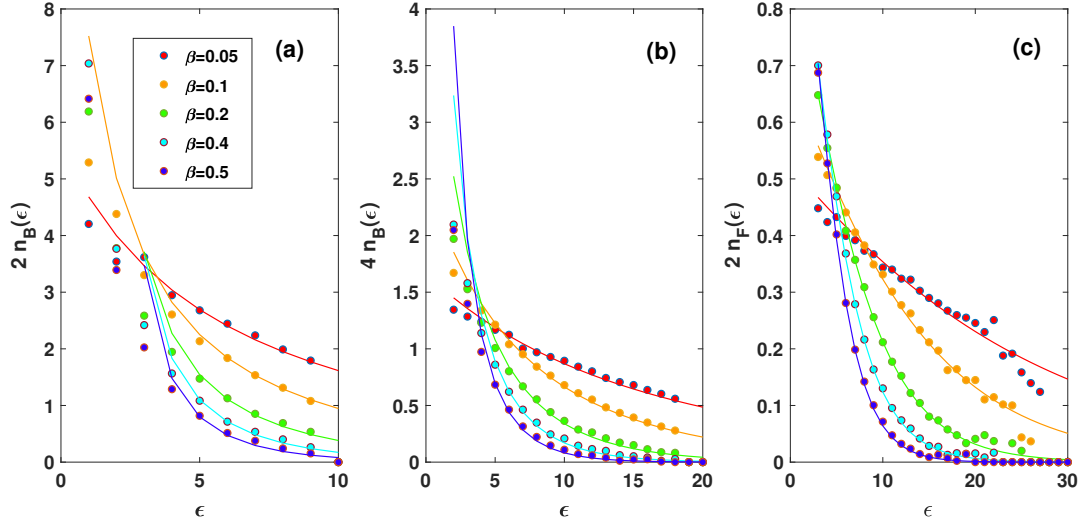


Figure 5. Average generalized degree minus one, over faces of energy ϵ and dimension $\delta = 0$ (panel a) $\delta = 1$ (panel b) or $\delta = 2$ (panel c) for $d = 3$ dimensional NGF with fractional flavor $s = -1/2$ are plotted for different values of β (symbols) and compared to the theoretical expectations (Fermi-Dirac and Bose-Einstein statistics). The simulations are performed for NGF with $N = 3000$ nodes. The data are averaged over 50 NGF realizations.

face α is given by

$$\Pi_{d,\delta}(\alpha) \simeq \begin{cases} e^{-\beta(\epsilon_\alpha - \mu_{d,d-1}) \frac{m+1-k_{d,\delta}(\alpha)}{t}} & \text{for } \delta = d-1, \\ e^{-\beta(\epsilon_\alpha - \mu_{d,d-1}) \frac{k_{d,\delta}(\alpha)+a}{t}} & \text{for } \delta \leq d-2, \end{cases} \quad (43)$$

where a is given by Eq. (29). From this it follows that the probability $\tilde{\Pi}_{d,\delta}(k)$ that a new d -dimensional simplex is attached to a δ -face with generalized degree $k_{d,\delta}(\alpha) = k$ is given by

$$\tilde{\Pi}_{d,\delta}(k) \simeq \begin{cases} e^{-\beta(\epsilon_\alpha - \mu_{d,d-1}) \frac{m+1-k}{t}} & \text{for } \delta \leq d-1, \\ e^{-\beta(\epsilon_\alpha - \mu_{d,d-1}) \frac{k+a}{t}} & \text{for } \delta \leq d-2. \end{cases} \quad (44)$$

6.3. Mean-field approach

Let us consider first the results that can be obtained within the mean field approximation. As mentioned before for the case $\beta = 0$ in the mean-field approximation we neglect the fluctuations and we consider a deterministic evolution of the generalized degree that is assumed to be equal to the average generalized degree over different NGF realizations. Let us consider separately the case in which the cases $\delta = d-1$ and $\delta \leq d-2$. By assuming that the chemical potential $\mu_{d,d-1}$ is well defined, and using Eq. (44) for the attachment probability, the mean-field equation (Eq. (22)) for generalized degree $k_{d,d-1}(\alpha) = k_{d,d-1}$ of the generic $(d-1)$ -face α with energy $\epsilon_\alpha = \epsilon$ can be written

explicitly as

$$\frac{dk_{d,d-1}}{dt} = \frac{e^{-\beta(\epsilon - \mu_{d,d-1})}(m + 1 - k_{d,d-1})}{t}, \quad (45)$$

with initial condition $k_{d,d-1}(t_\alpha) = 1$. This equation has solution

$$k_{d,d-1}(t|\epsilon) = m + 1 - m \left(\frac{t_\alpha}{t} \right)^{e^{-\beta(\epsilon - \mu_{d,d-1})}}, \quad (46)$$

which like in the case $\beta = 0$ clearly implies that the generalized degree of the $(d - 1)$ -dimensional faces is bounded. Interestingly in the mean-field approximation we can evaluate the average of the generalized degrees minus one over faces with energy ϵ getting

$$\langle k_{d,d-1} - 1 | \epsilon \rangle = \frac{m}{e^{\beta(\epsilon_\alpha - \mu_{d,d-1})} + 1} = mn_F(\epsilon). \quad (47)$$

Therefore this quantity is proportional to the Fermi-Dirac distribution $n_F(\epsilon)$ with chemical potential $\mu_{d,d-1}$. Interestingly, as we will show in the next paragraph this result is exact, in fact it is a result that concerns the average of the generalized degrees and therefore is not affected by the mean-field approximation. However the generalized degree distribution of $d - 1$ faces that can be derived from the mean-field approach is instead an approximation. By proceeding similarly to the case $\beta = 0$ we obtain that in the mean field approximation the probability $\tilde{P}_{d,d-1}(k|\epsilon)$ that a $(d - 1)$ -dimensional face with energy ϵ has generalized degree $k_{d,d-1}(\alpha) = k$ is given by

$$\tilde{P}_{d,d-1}(k|\epsilon) = \frac{1}{m} e^{\beta(\epsilon - \mu_{d,d-1})} \left(\frac{m + 1 - k}{m} \right)^{e^{\beta(\epsilon - \mu_{d,d-1})} - 1}. \quad (48)$$

We can proceed similarly for the $\delta \leq d - 2$ dimensional faces. In particular in this case the mean-field equations read

$$\frac{dk_{d,\delta}}{dt} = \frac{e^{-\beta(\epsilon - \mu_{d,\delta})}(k_{d,\delta} + a)}{t}, \quad (49)$$

with initial condition $k_{d,\delta}(t_\alpha) = 1$. Here we have assumed that the chemical potential $\mu_{d,\delta}$ is well defined and we have used Eq. (44) for the attachment probability $\Pi_{d,\delta}$. The above mean-field equations have the solution

$$k_{d,\delta}(t|\epsilon) = (1 + a) \left(\frac{t}{t_\alpha} \right)^{e^{-\beta(\epsilon_\alpha - \mu_{d,\delta})}} - a. \quad (50)$$

By using this expression it is possible to calculate the average of the generalized degree minus one over faces of energy ϵ finding

$$\langle k_{d,\delta} - 1 | \epsilon \rangle = A_\delta \frac{1}{e^{\beta(\epsilon_\alpha - \mu_{d,\delta})} - 1} = A_\delta n_B(\epsilon), \quad (51)$$

where

$$A_\delta = 1 + a = \frac{m(d - \delta)}{m(d - \delta - 1) - 1}. \quad (52)$$

Therefore we find that $\delta \leq d - 2$ dimensional faces of energy ϵ have statistical properties described by the Bose-Einstein distribution $n_B(\epsilon)$ with chemical potential $\mu_{d,\delta}$. However

in the mean-field approximation the derived generalized degree distribution is not exact but approximated. Proceeding as in the previous case we find that in the mean field approximation the probability $\tilde{P}_{d,d-1}(k|\epsilon)$ that a $(d-1)$ -dimensional face with energy ϵ has generalized degree $k_{d,d-1}(\alpha) = k$ is given by

$$\tilde{P}_{d,\delta}(k|\epsilon) = e^{\beta(\epsilon-\mu_{d,\delta})}(1+a) \left(\frac{1+a}{k+a} \right)^{e^{\beta(\epsilon-\mu_{d,\delta})}+1} \quad (53)$$

6.4. Master equation approach

For $\beta > 0$ we can find the exact asymptotic result for the generalized degree distribution of faces of given energy ϵ . The master equation from which we start is written for the number $N_{d,\delta}^t(k|\epsilon)$ of δ -dimensional faces with energy ϵ and reads

$$\begin{aligned} N_{d,\delta}^{t+1}(k|\epsilon) &= N_{d,\delta}^t(k|\epsilon) + \Pi_{d,\delta}(k-1)N_{d,\delta}^t(k-1|\epsilon)(1-\delta_{k,1}) \\ &\quad - \Pi_{d,\delta}(k)N_{d,\delta}^t(k|\epsilon) + \rho_{d,\delta}(\epsilon)\delta_{k,1}. \end{aligned}$$

where $\Pi_{d,\delta}(k-1)$ is given by Eq. (44) and where $\rho_{d,\delta}(\epsilon)$ indicates the density of new faces with energy ϵ that we add at time t . For large network sizes when $t \gg 1$ the average number of δ -dimensional faces with energy ϵ is given by

$$N_{d,\delta}^t(k|\epsilon) \simeq \rho_{d,\delta}(\epsilon)tP_{d,\delta}(k|\epsilon). \quad (54)$$

Inserting this asymptotic expression we get the exact asymptotic result for the generalized degree distribution $P_{d,\delta}(k|\epsilon)$ of δ -dimensional faces with energy ϵ . Specifically in the case $\delta = d-1$ we obtain the bounded distribution

$$\begin{aligned} P_{d,d-1}(k|\epsilon) &= \frac{e^{\beta(\epsilon-\mu_{d,d-1})}}{(e^{\beta(\epsilon-\mu_{d,d-1})}+m)} \frac{\Gamma(m+1)}{\Gamma(m+e^{\beta(\epsilon-\mu_{d,d-1})})} \\ &\quad \times \frac{\Gamma(m-k+1+e^{\beta(\epsilon-\mu_{d,d-1})})}{\Gamma(m-k+2)}, \end{aligned} \quad (55)$$

for $1 \leq k \leq m+1$. For $\delta \leq d-2$ we obtain instead the power-law distribution

$$\begin{aligned} P_{d,\delta}(k|\epsilon) &= \frac{e^{\beta(\epsilon-\mu_{d,\delta})}[m(d-\delta-1)-1]}{e^{\beta(\epsilon-\mu_{d,\delta})}[m(d-\delta-1)-1]+m(d-\delta)} \\ &\quad \times \frac{\Gamma(2+a+e^{\beta(\epsilon-\mu_{d,\delta})})}{\Gamma(1+a)} \\ &\quad \times \frac{\Gamma(k+a)}{\Gamma(k+1+a+e^{\beta(\epsilon-\mu_{d,\delta})})}. \end{aligned} \quad (56)$$

Therefore for $k \gg 1$ the generalized degree distribution of δ -dimensional faces with energy ϵ decays as a power-law with an energy dependent power-law exponent $\gamma(\epsilon)$, i.e.

$$P_{d,\delta}(k|\epsilon) \simeq k^{-\gamma(\epsilon)} \quad (57)$$

with

$$\gamma(\epsilon) = 1 + e^{\beta(\epsilon-\mu_{d,\delta})}. \quad (58)$$

Having the exact asymptotic results of the generalised degree distribution $P_{d,\delta}(k|\epsilon)$ valid as long as the chemical potentials $\mu_{d,\delta}$ are well defined, we can perform the average over all δ -faces with energy ϵ , i.e.

$$\langle k_{d,\delta} - 1 | \epsilon \rangle = \sum_k (k - 1) P_{k,\delta}(k|\epsilon). \quad (59)$$

In this result, we obtain total agreement with the mean-field results, i.e.

$$\langle k_{d,d-1} - 1 | \epsilon \rangle = mn_F(\epsilon) \text{ for } \delta = d - 1, \quad (60)$$

$$\langle k_{d,\delta} - 1 | \epsilon \rangle = A_\delta n_B(\epsilon) \text{ for } \delta \leq d - 2. \quad (61)$$

Therefore the generalized degree minus one averaged over faces of energy ϵ is proportional to the Fermi-Dirac distribution for $\delta = d - 1$ while it is proportional to the Bose-Einstein distribution for faces of dimension $\delta \leq d - 2$. In Figure 5 we compare the simulation results with the theoretical predictions showing very good agreement as long as the inverse temperature β is sufficiently low. For higher values of β the system does not reach a stationary state and the description of this phase transition is beyond the scope of this work.

7. Spectral properties of the NGF with Fractional Flavor

The spectral dimension [44] determines the properties of a diffusion process defined on the 1-skeleton of the NGF, i.e. the network constructed by starting from the simplicial complex by considering exclusively its nodes and links. Given the Laplacian matrix \mathbf{L} with elements defined as

$$L_{ij} = K_i \delta_{ij} - a_{ij} \quad (62)$$

where \mathbf{a} is the adjacency matrix of the network, K_i indicates the degree of the generic node i , and the density of eigenvalues $\rho(\lambda)$ for $\lambda \ll 1$ obeys the power-law scaling

$$\rho(\lambda) \simeq \lambda^{d_S/2-1} \quad (63)$$

we say that the network has *spectral dimension* d_S . Note that in this case the cumulative density of eigenvalues $\rho_c(\lambda)$ obeys the scaling

$$\rho_c(\lambda) \simeq \lambda^{d_S/2} \quad (64)$$

for $\lambda \ll 1$. The NGFs with integer flavor $s \in \{-1, 0, 1\}$ have been shown to display a finite spectral dimension [16, 33, 34]. Therefore it is interesting to investigate here how the spectral dimension changes for NGF with fractional flavor. By calculating numerically the spectrum of large NGF we found that for $\beta = 0$ the spectral dimension d_S of the NGF with flavor $s = -\frac{1}{m}$ is an increasing function of m . Therefore it achieves its smallest value for $s = -1$ and increases as m increases (see Figure 6a). Moreover the spectral properties of the NGF changes also with β . In particular numerical results indicate that the spectral dimension d_S decreases as the inverse temperature β increases (see Figure 6b).

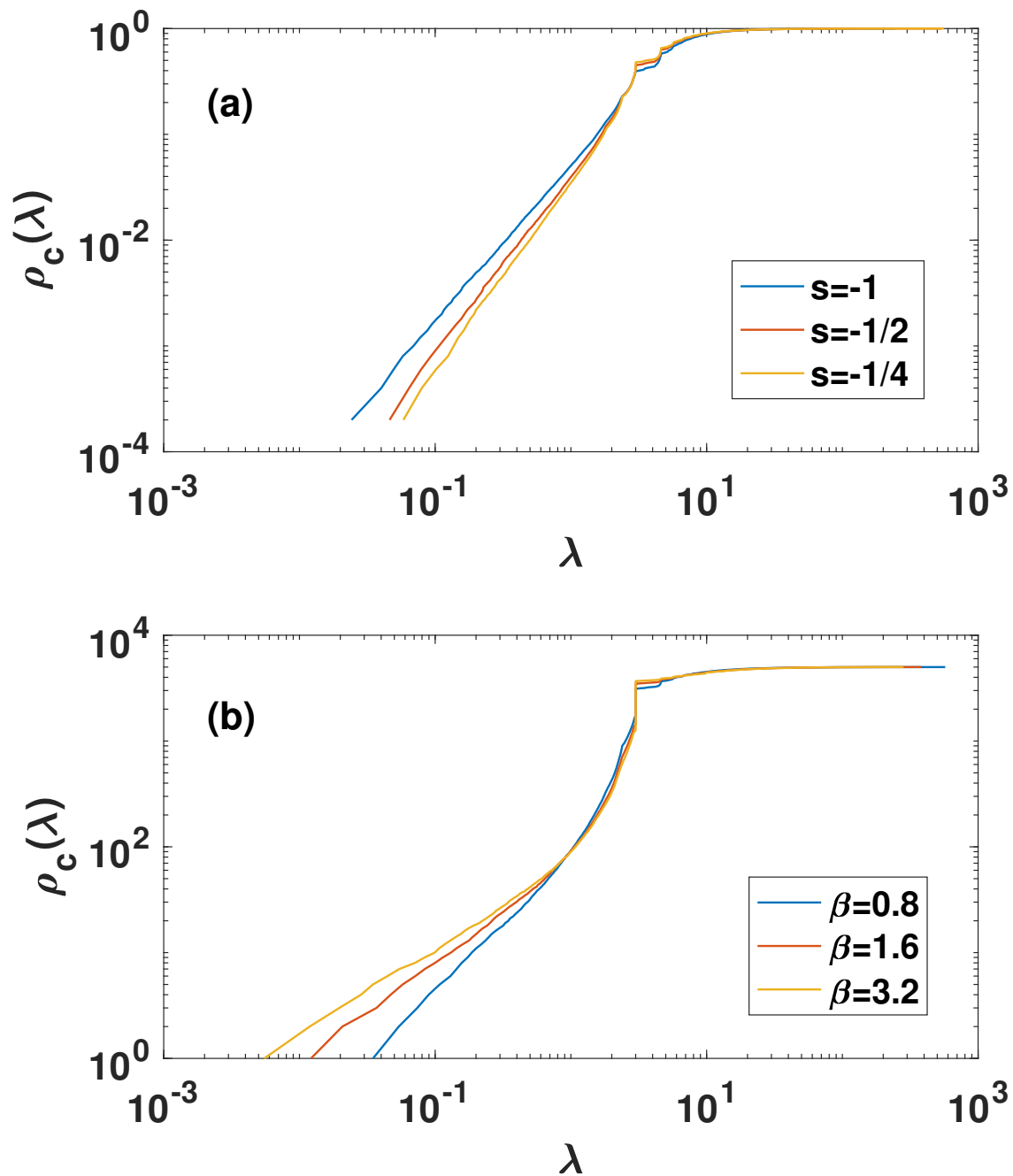


Figure 6. Cumulative distribution $\rho_c(\lambda)$ of the eigenvalues of the Laplacian for NGFs of dimension $d = 3$ formed by $N = 5000$ nodes. In panel (a) we show the cumulative distribution $\rho_c(\lambda)$ as a function of s for $\beta = 0$. In panel (b) we show the cumulative distribution $\rho_c(\lambda)$ as a function of β for $s = -1/2$. Every spectrum is averaged over 10 realizations of the NGFs.

8. Conclusions

In conclusion here we have extended the model Network Geometry with Flavor to fractional negative values of the flavor $s = -\frac{1}{m}$. This choice of parameters enforces

the condition that each $(d - 1)$ -dimensional face of the pure d -dimensional simplicial complexes generated by the model is incident at most to $m + 1$ d -dimensional simplices. For the limiting case $m = 1$ this model generates discrete manifolds where $(d - 1)$ -dimensional faces have incidence numbers $n_\alpha \in \{0, 1\}$. For $m > 1$ instead the simplicial complexes generated by the model are not anymore manifolds and have incidence number $n_\alpha \in \{0, 1, 2, \dots, m\}$. In previous studies it has been shown that NGF displays emergent quantum statistics. In particular the generalized degrees of δ faces for energy ϵ in the NGFs with $s = -1$ can be simply related to the Fermi-Dirac, Boltzmann and Bose-Einstein statistics depending on the dimensionality δ . This result implies that for a NGF in $d = 3$ with $s = -1$ the triangles, the links, and the nodes of given energy ϵ have an average generalized degree minus one given by the Fermi-Dirac, the Boltzmann and the Bose-Einstein statistics respectively. Here we show that when we consider NGF with flavor $s = -\frac{1}{m}$ we still observe different statistics as a function of the dimensionality of the faces but the only two types of statistics emerging are the Fermi-Dirac and Bose-Einstein statistics as long as the NGF evolution reaches a steady state. This implies that for $d = 2$ we observe links and nodes of energy ϵ whose average generalized degree follows the Fermi-Dirac and Bose-Einstein distribution respectively. Therefore already in $d = 2$ we observe the co-existence of the two quantum statistics determining properties of faces of different dimension. Finally in this paper we have also numerically analysed the spectral properties of NGF with fractional flavor showing how the spectral dimension of the 1-skeleton of NGF changes as a function of m and β .

The proposed NGF model with fractional flavor can be used to model real simplicial complexes in which nodes have some intrinsic features that can be associated with their fitness. As generalized network structures with metadata are increasingly available we believe that this is a very promising possible application of our modeling framework. Moreover the NGF with fractional flavor can be used as well controlled artificial models in which to test dynamical processes defined on simplicial complexes, such as topological percolation [32], synchronization [33–35] or social contagion [36].

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