K-THEORY FOR GENERALIZED LAMPLIGHTER GROUPS

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Abstract. We compute K-theory for the reduced group C*-algebras of generalized Lamplighter groups.

1. Introduction

The classical Lamplighter group is given by the semidirect product $\left(\bigoplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})\right) \rtimes \mathbb{Z}$, where the $\mathbb{Z}$-action on $\bigoplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ is induced by the canonical translation action of $\mathbb{Z}$ on itself. This construction can be generalized by replacing $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}$ by other groups. The classical Lamplighter group and its generalizations are important examples in group theory which led to solutions of several open problems (see for instance [7, 6, 10]).

The goal of these notes is to derive a K-theory formula for group C*-algebras of generalized Lamplighter groups of the form $\left(\bigoplus_{\Gamma}\Sigma\right) \rtimes \Gamma$, where Σ is an arbitrary finite group and Γ is an arbitrary countable group. As in the classical setting, the Γ-action on $\bigoplus_{\Gamma}\Sigma$ is induced by the canonical left translation action of Γ on itself. Our computations are inspired by [9, 13], which treat the special case of free groups Γ ([9] deals with the case $\Gamma = \mathbb{Z}$). Our method, however, is completely different from the ones adopted in [9, 13].

Our main result reads as follows. Let Σ be a finite group and Γ a countable group. Let $\mathrm{con}\,\Sigma$ be the set of conjugacy classes in Σ, and $\mathrm{con}^{\times}\Sigma := \mathrm{con}\,\Sigma \setminus \{\{1\}\}$ the set of non-trivial conjugacy classes. Let $\mathcal{C}$ be the set of conjugacy classes of finite subgroups of Γ. For a finite subgroup C of Γ, let $F(C)$ be the set of non-empty finite subsets of the right coset space $C\backslash\Gamma$ which are not of the form $\pi^{-1}(Y)$ for a finite subgroup $D \subseteq \Gamma$ with $C \subsetneq D$ and $Y \subseteq D\backslash\Gamma$, where $\pi: C\backslash\Gamma \twoheadrightarrow D\backslash\Gamma$ is the canonical projection. The normalizer $N_C := \left\{\gamma \in \Gamma\colon \gamma C \gamma^{-1} = C\right\}$ acts on $F(C)$ by left multiplication, and we denote the set of orbits by $N_C\backslash F(C)$. Given $X \in F(C)$, we write $C \cdot X := \bigsqcup_{x \in X} C \cdot x$ and let $(\mathrm{con}^{\times}\Sigma)^{C\cdot X}$ be the set of functions $C \cdot X \to \mathrm{con}^{\times}\Sigma$. $\gamma \in C$ acts on $\varphi \in (\mathrm{con}^{\times}\Sigma)^{C\cdot X}$ via $(\gamma.\varphi)(x) = \varphi(\gamma^{-1}x)$, and we set $\mathrm{Stab}_C(\varphi) = \{\gamma \in C\colon \gamma.\varphi = \varphi\}$ for $\varphi \in (\mathrm{con}^{\times}\Sigma)^{C\cdot X}$.

Theorem 1.1. *If Γ satisfies the Baum-Connes conjecture with coefficients, then the K-theory of $C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)$ is given by*

$$K_*(C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)) \cong K_*(C^*_\lambda(\Gamma)) \oplus \left(\bigoplus_{[C]\in\mathcal{C}} \ \bigoplus_{[X]\in N_C\backslash F(C)} \ \bigoplus_{[\varphi]\in C\backslash\left((\mathrm{con}^{\times}\Sigma)^{C\cdot X}\right)} K_*(C^*_\lambda(\mathrm{Stab}_C(\varphi))) \right).$$

Here we take one representative C out of each class in $\mathcal{C}$, one representative X out of each class in $N_C\backslash F(C)$, and one representative φ out of each class in $C\backslash\left((\mathrm{con}^{\times}\Sigma)^{C\cdot X}\right)$.

We refer the reader to [1, 14, 5] and the references therein for more information about the Baum-Connes conjecture. For instance, Theorem 1.1 applies to all groups with the Haagerup property [11] and all hyperbolic groups [12].

Note that Σ enters our formula only in the form of $\mathrm{con}^{\times}\Sigma$. What is more, if Γ is infinite, then for each $[C] \in \mathcal{C}$, we simply get a free abelian group of countably infinite rank, so that $K_*(C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma))$ does not depend on Σ at all. This becomes particularly evident in K_1, where Theorem 1.1 yields the following

Corollary 1.2. *Let Σ be a finite group and Γ a countable group. If Γ satisfies the Baum-Connes conjecture with coefficients, then the canonical inclusion $\Gamma \hookrightarrow \Sigma \rtimes \Gamma$ induces an isomorphism*

$$K_1(C^*_\lambda(\Gamma)) \cong K_1(C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)).$$

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Moreover, for torsion-free Γ, our formula becomes particularly simple.

Corollary 1.3. *Let Σ and Γ be as in Theorem 1.1. Assume that Γ is torsion-free. Write $\mathrm{FIN}^{\times}$ for the set of non-empty finite subsets of Γ. Then, under the same assumptions as in Theorem 1.1, we have*

$$K_*(C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)) \cong K_*(C^*_\lambda(\Gamma)) \oplus \left(\bigoplus_{[X] \in \Gamma \backslash \mathrm{FIN}^{\times}} \bigoplus_{(\mathrm{con}^{\times} \Sigma)^X} K_*(\mathbb{C}) \right).$$

The proof of our main theorem proceeds in two steps. First, using the Going-Down principle from [2, 8] (see also [5, § 3]), we show that $C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)$ has the same K-theory as the crossed product $C((\mathrm{con}\,\Sigma)^\Gamma) \rtimes_r \Gamma$ for the topological full shift $\Gamma \curvearrowright (\mathrm{con}\,\Sigma)^\Gamma$. Here we view $\mathrm{con}\,\Sigma$ as a finite alphabet. Secondly, we compute K-theory for $C((\mathrm{con}\,\Sigma)^\Gamma) \rtimes_r \Gamma$ using [3, 4]. As a by-product, we obtain a general K-theory formula for crossed products of topological full shifts (see Proposition 2.4). Both steps require our assumption that Γ satisfies the Baum-Connes conjecture with coefficients.

We point out that it is not possible to apply the results in [3, 4] directly because [3, 4] only deal with crossed products attached to actions on commutative C*-algebras.

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2. K-theory for certain crossed products and generalized Lamplighter groups

We first discuss the following abstract situation: Let $A = \bigoplus_{i=0}^n M_{k_i}$ be a finite dimensional C*-algebra, where M_k is the algebra of $k \times k$-matrices. We assume that $k_0 = 1$, i.e., $A = \mathbb{C} \oplus M_{k_1} \oplus \ldots \oplus M_{k_n}$. Let Γ be a countable group. We form the tensor product $\bigotimes_\Gamma A$ as follows: For every finite subset $F \subseteq \Gamma$, we form the ordinary tensor product $\bigotimes_F A$, and for $F_1 \subseteq F_2$, we have the canonical embedding $\bigotimes_{F_1} A \hookrightarrow \bigotimes_{F_2} A$, $x \mapsto x \otimes 1$ (here 1 denotes the unit of $\bigotimes_{F_2 \setminus F_1} A$, and we used the canonical isomorphism $\bigotimes_{F_2} A \cong \left(\bigotimes_{F_1} A\right) \otimes \left(\bigotimes_{F_2 \setminus F_1} A\right)$). Then set $\bigotimes_\Gamma A := \varinjlim_F \bigotimes_F A$. The left Γ-action on itself by translations induces an action $\Gamma \curvearrowright \bigotimes_\Gamma A$. Our goal is to compute the K-theory of $\left(\bigotimes_\Gamma A\right) \rtimes_r \Gamma$. The special case $A = C^*_\lambda(\Sigma)$ will lead to Theorem 1.1.

Let e_i be a minimal projection in $M_{k_i} \subseteq A$. In particular, $e_0 = 1 \in \mathbb{C} \subseteq A$. For $F \subseteq \Gamma$ finite, let $\varphi \in \{1, \ldots, n\}^F$, i.e., φ is a function $\varphi: F \to \{1, \ldots, n\}$. Define $e_\varphi := \bigotimes_{f \in F} e_{\varphi(f)} \in \bigotimes_F A \subseteq \bigotimes_\Gamma A$. If $F = \emptyset$, then for $\varphi: \emptyset \to \{1, \ldots, n\}$, we set $e_\varphi := 1$ (where 1 denotes the unit of $\bigotimes_\Gamma A$). The set

$$\left\{ e_\varphi \colon \varphi \in \{1, \ldots, n\}^F,\ F \subseteq \Gamma \text{ finite} \right\} \tag{1}$$

is a Γ-invariant family of commuting non-zero projections, which is closed under multiplication up to zero (i.e., the product of two projections in the family is either zero or again a projection in the family). We do not need it now, but the family is also linearly independent (see Lemma 2.3 and the proof of (2)). Let D be the C*-subalgebra of $\bigotimes_\Gamma A$ generated by the projections in (1). Let $\iota: D \hookrightarrow \bigotimes_\Gamma A$ be the canonical embedding. Note that ι is Γ-equivariant.

Proposition 2.1. *If Γ satisfies the Baum-Connes conjecture with coefficients, then $\iota \rtimes_r \Gamma$ induces an isomorphism $K_*(D \rtimes_r \Gamma) \cong K_*(\left(\bigotimes_\Gamma A\right) \rtimes_r \Gamma)$.*

Proof. By the Going-Down principle (see [5, § 3]), it suffices to show that for every finite subgroup $H \subseteq \Gamma$, $\iota \rtimes_r H$ induces an isomorphism $K_*(D \rtimes_r H) \cong K_*(\left(\bigotimes_\Gamma A\right) \rtimes_r H)$.

Let us first treat the case of the trivial subgroup, $H = \{1\}$. For a fixed finite subset $F \subseteq \Gamma$, let

$$D_F = C^*(\left\{ e_\varphi \colon \varphi \in \{1, \ldots, n\}^{F'} \text{ for } F' \subseteq F \right\}).$$

Then $D = \varinjlim_F D_F$. We also have $\bigotimes_\Gamma A = \varinjlim_F \bigotimes_F A$. As K-theory is continuous, i.e., preserves direct limits, it suffices to show that $\iota_F := \iota|_{D_F}: D_F \to \bigotimes_F A$ induces an isomorphism in K_*. Let $[\iota_F] \in KK(D_F, \bigotimes_F A)$

be the KK-element determined by ι_F. Consider the projection $e = \sum_{i=0}^n e_i$ in A. e is a full projection in A, and we have $eAe = \bigoplus_{i=0}^n \mathbb{C}e_i$. The $\bigotimes_F A$–$\bigotimes_F eAe$-imprimitivity bimodule $\bigotimes_F Ae$ gives rise to a KK-element $\mathbf{j}_F \in KK(\bigotimes_F A, \bigotimes_F eAe)$. $\mathbf{j}_F$ is invertible, and its inverse is the KK-element induced by the inclusion $\bigotimes_F eAe \hookrightarrow \bigotimes_F A$. Hence it suffices to show that the Kasparov product $[\iota_F] \cdot \mathbf{j}_F \in KK(D_F, \bigotimes_F eAe)$ induces an isomorphism $K_*(D_F) \to K_*(\bigotimes_F eAe)$.

First, consider the special case of a single element subset, $F = \{f\}$ for some $f \in \Gamma$. Let us write $D_f := D_{\{f\}}$, $\iota_f := \iota_{\{f\}}$ and $\mathbf{j}_f := \mathbf{j}_{\{f\}}$. Since $D_f = \mathbb{C} \cdot 1 + \mathbb{C}e_1 + \ldots + \mathbb{C}e_n$ (where 1 denotes the unit of $\bigotimes_\Gamma A$) and $eAe = \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \ldots \oplus \mathbb{C}e_n$, we can describe the map $K_*(D_F) \to K_*(\bigotimes_F eAe)$ induced by $[\iota_f] \cdot \mathbf{j}_f$ by the commutative diagram

$$\begin{array}{ccc} K_*(D_f) & \longrightarrow & K_*(eAe) \\ \| & & \| \\ \mathbb{Z}[1] \oplus \bigoplus_{i=1}^n \mathbb{Z}[e_i] & \xrightarrow{M_f} & \bigoplus_{i=0}^n \mathbb{Z}[e_i] \end{array}$$

where the upper horizontal map is the map we want to describe, and M_f is the $(n+1) \times (n+1)$-matrix

$$M_f = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ k_1 & 1 & & 0 \\ \vdots & & \ddots & \\ k_n & 0 & & 1 \end{pmatrix}.$$

Obviously, M_f is invertible. Note that everything is independent of f.

Now consider the case of a general finite subset $F \subseteq \Gamma$. Since $D_F = \bigotimes_{f \in F} D_f$, we have $K_*(D_F) \cong \bigotimes_{f \in F} K_*(D_f)$, and we also have $K_*(\bigotimes_F eAe) \cong \bigotimes_{f \in F} K_*(eAe)$. The homomorphism $K_*(D_F) \to K_*(\bigotimes_F eAe)$ induced by $[\iota_F] \cdot \mathbf{j}_F$ respects this tensor product decomposition, in the sense that we have a commutative diagram

$$\begin{array}{ccccccc} \bigotimes_{f \in F} K_*(D_f) & \overset{\sim}{=\!=\!=} & K_*(D_F) & \longrightarrow & K_*(\bigotimes_F eAe) & \overset{\sim}{=\!=\!=} & \bigotimes_{f \in F} K_*(eAe) \\ \| & & & & & & \| \\ \bigotimes_{f \in F} \left(\mathbb{Z}[1] \oplus \bigoplus_{i=1}^n \mathbb{Z}[e_i]\right) & & & \xrightarrow{M_F = \otimes_{f \in F} M_f} & & & \bigotimes_{f \in F} \left(\bigoplus_{i=0}^n \mathbb{Z}[e_i]\right) \end{array}$$

Again, we see that M_F is invertible because all the M_f, $f \in F$, are.

Now let us deal with the case of an arbitrary finite subgroup $H \subseteq \Gamma$. If we choose an increasing sequence of H-invariant finite subsets $F \subseteq \Gamma$ whose union is Γ, we obtain H-equivariant inductive limit decompositions $D = \varinjlim_F D_F$ and $\bigotimes_\Gamma A = \varinjlim_F \bigotimes_F A$. Hence, again by continuity of K-theory, it suffices to show that, for every F, $\iota_F \rtimes_r H : D_F \rtimes_r H \to \left(\bigotimes_F A\right) \rtimes_r H$ induces an isomorphism in K_*. Let $\mathbf{j}_F \in KK(\bigotimes_F A, \bigotimes_F eAe)$ be as before. Since the full projection $\bigotimes_F e \in \bigotimes_F A$ giving rise to $\mathbf{j}_F$ is H-invariant, $\mathbf{j}_F$ is a KK^H-equivalence (see [5, Remark 3.3.16]). Thus, to show that $\iota_F \rtimes_r H : D_F \rtimes_r H \to \left(\bigotimes_F A\right) \rtimes_r H$ induces an isomorphism in K_*, it suffices to show that $[\iota_F] \cdot \mathbf{j}_F \in KK^H(D_F, \bigotimes_F eAe)$ induces an isomorphism $K_*(D_F \rtimes_r H) \to K_*(\left(\bigotimes_F eAe\right) \rtimes_r H)$, for which in turn it is enough to prove that $[\iota_F] \cdot \mathbf{j}_F$ is a KK^H-equivalence.

Now both D_F and $\bigotimes_F eAe$ are finite dimensional commutative C*-algebras with an H-action, so that we are exactly in the setting of [4, Appendix]. It is straightforward to check that $[\iota_F] \cdot \mathbf{j}_F = x^H_{M_F}$, where $x^H_{M_F}$ is the element in $KK^H(D_F, \bigotimes_F eAe)$ corresponding to the matrix M_F, as constructed in [4, Appendix]. By [4, Lemma A.2], $x^H_{M_F}$ is a KK^H-equivalence because M_F is an invertible matrix. The inverse of $x^H_{M_F}$ is given by $x^H_{M_F^{-1}}$. □

Remark 2.2. Note that our assumption on A that $\mathbb{C}$ appears as a direct summand is really necessary. For instance, if $A = M_2$, then $\bigotimes_\Gamma A$ would be the UHF algebra M_{2^∞} (as soon as Γ is infinite). But we have $K_0(M_{2^\infty}) \cong \mathbb{Z}[\frac{1}{2}]$, while our method would always yield a free abelian group for K_0. Hence our method fails.

Let us now compare with the topological full shift $\Gamma \curvearrowright \{0,\dots,n\}^{\Gamma}$. For a finite subset $F \subseteq \Gamma$, let π_F be the canonical projection $\{0,\dots,n\}^{\Gamma} \twoheadrightarrow \{0,\dots,n\}^{F}$. Given $\varphi \in \{0,\dots,n\}^{F}$, we have the cylinder set $\pi_F^{-1}(\varphi)$ and its characteristic function $1_{\pi_F^{-1}(\varphi)} \in C(\{0,\dots,n\}^{\Gamma})$. The following is now easy to see.

Lemma 2.3. *The Γ-equivariant isomorphism $D \cong C(\{0,\dots,n\}^{\Gamma})$, $e_\varphi \mapsto 1_{\pi_F^{-1}(\varphi)}$ induces an isomorphism $D \rtimes_r \Gamma \cong C(\{0,\dots,n\}^{\Gamma}) \rtimes_r \Gamma$.*

We now compute K-theory for $C(\{0,\dots,n\}^{\Gamma}) \rtimes_r \Gamma$.

Proposition 2.4. *If Γ satisfies the Baum-Connes conjecture with coefficients, then*

$$K_*(C(\{0,\dots,n\}^{\Gamma}) \rtimes_r \Gamma) \cong K_*(C^*_\lambda(\Gamma)) \oplus \left(\bigoplus_{[C]\in \mathcal{C}} \bigoplus_{[X]\in N_C \backslash \mathcal{F}(C)} \bigoplus_{[\varphi]\in C \backslash \left(\{1,\dots,n\}^{C\cdot X}\right)} K_*(C^*_\lambda(\mathrm{Stab}_C(\varphi))) \right).$$

Here we use the same notation as in Theorem 1.1.

Proof. First of all, the same arguments as for [4, Examples 2.13 & 3.1] show that the family

$$\left\{\pi_F^{-1}(\varphi) \colon \varphi \in \{1,\dots,n\}^F,\ F \subseteq \Gamma \text{ finite}\right\}$$

is a Γ-invariant regular basis for the compact open sets in $\{0,\dots,n\}^{\Gamma}$. Here Γ acts via $\gamma.\pi_F^{-1}(\varphi) = \pi_{\gamma\cdot F}^{-1}(\gamma.\varphi)$, where $\gamma.\varphi \in \{1,\dots,n\}^{\gamma\cdot F}$ is given by $(\gamma.\varphi)(x) = \varphi(\gamma^{-1}x)$. Therefore, using the bijection

$$\bigsqcup_{[F]\in\Gamma\backslash\mathrm{FIN}} \mathrm{Stab}_\Gamma(F) \backslash \left(\{1,\dots,n\}^F\right) \cong \Gamma \backslash \left\{\pi_F^{-1}(\varphi) \colon \varphi \in \{1,\dots,n\}^F,\ F \subseteq \Gamma \text{ finite}\right\},\ [\varphi] \mapsto [\varphi],$$

and the observation that for $\gamma \in \Gamma$ and $\varphi \in \{1,\dots,n\}^F$, we have $\gamma.\pi_F^{-1}(\varphi) = \pi_F^{-1}(\varphi)$ if and only if $\gamma \cdot F = F$ and $\gamma.\varphi = \varphi$, we may apply [4, Corollary 3.14], and obtain

$$K_*\left(C\left(\{0,\dots,n\}^{\Gamma}\right) \rtimes_r \Gamma\right) \cong \bigoplus_{[F]\in\Gamma\backslash\mathrm{FIN}} \bigoplus_{[\varphi]\in \mathrm{Stab}_\Gamma(F)\backslash\left(\{1,\dots,n\}^F\right)} K_*(C^*_\lambda(\mathrm{Stab}_\Gamma(\varphi))). \tag{2}$$

Here FIN is the set of all finite subsets of Γ, and $\Gamma\backslash\mathrm{FIN}$ is the set of orbits of the left translation action $\Gamma \curvearrowright \mathrm{FIN}$. Moreover, $\mathrm{Stab}_\Gamma(F)$ and $\mathrm{Stab}_\Gamma(\varphi)$ denote the stabilizer groups $\mathrm{Stab}_\Gamma(F) = \{\gamma \in \Gamma \colon \gamma \cdot F = F\}$ and $\mathrm{Stab}_\Gamma(\varphi) = \{\gamma \in \Gamma \colon \gamma.\varphi = \varphi\}$, and $C^*_\lambda(\mathrm{Stab}_\Gamma(\varphi))$ is the reduced group C*-algebra of $\mathrm{Stab}_\Gamma(\varphi)$.

Now we analyse $\Gamma\backslash\mathrm{FIN}$ and $\mathrm{Stab}_\Gamma(F)$ for $[F] \in \Gamma\backslash\mathrm{FIN}$. For $F = \emptyset$, we have $\mathrm{Stab}_\Gamma(\varphi) = \Gamma$. This yields $K_*(C^*_\lambda(\Gamma))$ as one direct summand on the right-hand side of (2). We set $\mathrm{FIN}^\times := \mathrm{FIN} \setminus \{\emptyset\}$. Let us describe $\Gamma\backslash\mathrm{FIN}^\times$. Let $\mathcal{C}$, $\mathcal{F}(C)$ and N_C be as in Theorem 1.1. Then we claim that

$$\bigsqcup_{[C]\in\mathcal{C}} N_C\backslash\mathcal{F}(C) \to \Gamma\backslash\mathrm{FIN}^\times,\ [X] \mapsto [C \cdot X] \tag{3}$$

is a bijection, and that for every $[C] \in \mathcal{C}$, $[X] \in N_C\backslash\mathcal{F}(C)$, we have

$$\mathrm{Stab}_\Gamma(C \cdot X) = C. \tag{4}$$

First note that the map (3) is well-defined. Moreover, this map is surjective because every $F \in \mathrm{FIN}^\times$ with $\mathrm{Stab}_\Gamma(F) = C$ is of the form $F = C \cdot X$ for some finite, non-empty subset $X \subseteq C\backslash\Gamma$. Now, X must lie in $\mathcal{F}(C)$. Suppose not, i.e., $X = \pi^{-1}(Y)$ for a finite subgroup $D \subseteq \Gamma$ with $C \subsetneq D$ and $Y \subseteq D\backslash\Gamma$, where $\pi \colon C\backslash\Gamma \twoheadrightarrow D\backslash\Gamma$ is the canonical projection. Then $F = C \cdot X = D \cdot Y$, so that $D \subseteq \mathrm{Stab}_\Gamma(F)$, in contradiction to $\mathrm{Stab}_\Gamma(F) = C$. Not only does this show surjectivity, but it also proves (4). To see injectivity of (3), assume that $X \in \mathcal{F}(C)$ and $X' \in \mathcal{F}(C')$ satisfy $[C \cdot X] = [C' \cdot X']$, say $C' \cdot X' = \gamma \cdot C \cdot X$. It follows that $C' = \mathrm{Stab}_\Gamma(C' \cdot X') = \gamma \mathrm{Stab}_\Gamma(C \cdot X)\gamma^{-1} = \gamma C \gamma^{-1}$. Hence $[C] = [C']$, and since we are taking one representative out of each class, we must actually have $C = C'$. But then γ must lie in N_C, and we must have $C \cdot X' = \gamma \cdot C \cdot X = C \cdot \gamma \cdot X$, so that $X' = \gamma \cdot X$, i.e., $[X'] = [X]$ in $N_C\backslash\mathcal{F}(C)$. This shows injectivity.

We now complete the proof by plugging the bijections (3), (4) into (2) and observing that for $X \in \mathcal{F}(C)$ and $\varphi \in \{1,\dots,n\}^{C\cdot X}$, we have $\mathrm{Stab}_\Gamma(\varphi) \subseteq \mathrm{Stab}_\Gamma(C \cdot X) = C$. □

Combining Proposition 2.1, Lemma 2.3 and Proposition 2.4, and using the concrete construction in [4, § 3] for our following assertion on K_1, we obtain

Corollary 2.5. *In the situation of Proposition 2.1, if* Γ *satisfies the Baum-Connes conjecture with coefficients, then we have*

$$K_*((\bigotimes_\Gamma A) \rtimes_r \Gamma) \cong K_*(C^*_\lambda(\Gamma)) \oplus \left(\bigoplus_{[C] \in \mathcal{C}} \bigoplus_{[X] \in N_C \backslash F(C)} \bigoplus_{[\varphi] \in C \backslash \left(\{1,\dots,n\}^{C \cdot X}\right)} K_*(C^*_\lambda(\mathrm{Stab}_C(\varphi))) \right).$$

In K_1*, the canonical map* $C^*_\lambda(\Gamma) \hookrightarrow (\bigotimes_\Gamma A) \rtimes_r \Gamma$ *induces an isomorphism*

$$K_1(C^*_\lambda(\Gamma)) \cong K_1((\bigotimes_\Gamma A) \rtimes_r \Gamma).$$

If Γ *is in addition torsion-free, then we obtain*

$$K_*((\bigotimes_\Gamma A) \rtimes_r \Gamma) \cong K_*(C^*_\lambda(\Gamma)) \oplus \left(\bigoplus_{[X] \in \Gamma \backslash \mathrm{FIN}^\times} \bigoplus_{\{1,\dots,n\}^X} K_*(\mathbb{C}) \right).$$

Now let us apply our K-theory formula to generalized Lamplighter groups. Consider the case $A = C^*_\lambda(\Sigma)$ for a finite group Σ. Our assumption on A that $\mathbb{C}$ appears as a direct summand is satisfied because the trivial representation gives rise to a direct summand $\mathbb{C}$ in $C^*_\lambda(\Sigma)$. The remaining direct summands of A are in one-to-one correspondence with $\mathrm{con}^\times \Sigma$. Hence we obtain

Corollary 2.6. *Let* Σ *be a finite group. If* Γ *satisfies the Baum-Connes conjecture with coefficients, then we have*

$$K_*(C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)) \cong K_*(C^*_\lambda(\Gamma)) \oplus \left(\bigoplus_{[C] \in \mathcal{C}} \bigoplus_{[X] \in N_C \backslash F(C)} \bigoplus_{[\varphi] \in C \backslash \left((\mathrm{con}^\times \Sigma)^{C \cdot X}\right)} K_*(C^*_\lambda(\mathrm{Stab}_C(\varphi))) \right).$$

In K_1*, the canonical inclusion* $\Gamma \hookrightarrow \Sigma \rtimes \Gamma$ *induces an isomorphism*

$$K_1(C^*_\lambda(\Gamma)) \cong K_1(C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)).$$

If Γ *is in addition torsion-free, then we obtain*

$$K_*(C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)) \cong K_*(C^*_\lambda(\Gamma)) \oplus \left(\bigoplus_{[X] \in \Gamma \backslash \mathrm{FIN}^\times} \bigoplus_{(\mathrm{con}^\times \Sigma)^X} K_*(\mathbb{C}) \right).$$

This completes the proofs of Theorem 1.1, Corollary 1.2 and Corollary 1.3.

Remark 2.7. As in [4, Corollary 3.14], we get KK-equivalences in Proposition 2.4, Corollary 2.5 and Corollary 2.6 if Γ satisfies the strong Baum-Connes conjecture.

Remark 2.8. Moreover, as in [4, Corollary 3.14], we could allow coefficients in Proposition 2.4, Corollary 2.5 and Corollary 2.6. However, in Corollary 2.6, we would only get a K-theory formula for crossed products $B \rtimes_r \left((\bigoplus_\Gamma \Sigma) \rtimes \Gamma\right)$ where the $(\bigoplus_\Gamma \Sigma)$-action on the C*-algebra B is trivial.

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