On unified crack propagation laws

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Abstract

The anomalous propagation of short cracks shows generally exponential fatigue crack growth but the dependence on stress range at high stress levels is not compatible with Paris' law with exponent $m = 2$. Indeed, some authors have shown that the standard uncracked SN curve is obtained mostly from short crack propagation, assuming that the crack size a increases with the number of cycles N as $\frac{da}{dN} = H\Delta\sigma^h a$ where h is close to the exponent of the Basquin's power law SN curve. We therefore propose a general equation for crack growth which for short cracks has the latter form, and for long cracks returns to the Paris' law. We show generalized SN curves, generalized Kitagawa-Takahashi diagrams, and discuss the application to some experimental data. The problem of short cracks remains however controversial, as we discuss with reference to some examples.

Keywords: fatigue, SN curves, crack growth, damage tolerance

Nomenclature	
a	crack length
a_i	initial crack size
a_f	final crack size
a_0	critical or additional crack size
A	constant of the NASGRO Paris' law
A_1, A_2	constants
C	constant of the Paris' law
d_3	microstructural length scale
D	constant of the NASGRO Paris' law
${}_2F_1$	Gauss' hypergeometric function
h	exponent of the exponential crack growth law
H	constant of the exponential crack growth law
K	stress-intensity factor
K_{eff}	stress-intensity factor of the NASGRO Paris' law
$K_{eff,th}$	threshold of the stress-intensity factor of the NASGRO Paris' law
K_f	fatigue knock-down stress-intensity factor
K_{max}	maximum stress-intensity factor of the NASGRO Paris' law
K_{th}	threshold of the stress-intensity factor
l	macroscopic length scale
m	exponent of the Paris' law
N	number of cycles
N_f	final number of cycles
p	exponent of the NASGRO Paris' law
q	"fracture quantum"
Y	geometrical factor
Δa	"fracture quantum" in the original QFM formulation
σ	stress
σ_L	fatigue stress limit
σ_{min}	minimum stress in a cycle
σ_{max}	maximum stress in a cycle
σ_R	ultimate material strength
σ_∞	infinite-life stress limit
ASTM	American Society for Testing and Materials
DT	Damage Tolerance
EPFM	Elasto-Plastic Fracture Mechanics
HSC	highly stressed cracks
LEFM	Linear Elastic Fracture Mechanics
MSC	microstructurally short cracks
PSC	physically small cracks
QFM	Quantized Fracture Mechanics
R	stress ratio
SN	stress vs number of cycles
USAF	United States Air Force

1 Introduction

The problem of propagation of cracks is central to fatigue, and despite the large efforts, a universal picture of crack propagation is still elusive, particularly for the problem of short cracks. Paris *et al.* [1] suggest that 'a specific accumulation damage model for the computation of damage growth under a wide variety of service loads is still lacking'. Short cracks are obviously more difficult to observe experimentally due to their size, and tend to show a number of deviations from "long crack" growth. This is partly due to the fact that long cracks have a size which is much larger than the material length scales, and therefore are more naturally amenable to a continuum mechanics treatment. In particular, when long cracks are loaded under "small scale yielding", Linear Elastic Fracture Mechanics (LEFM) becomes effective for growth rate prediction. Instead, for short cracks, LEFM limits are invalidated as the size of the plastic zone at the crack-tip is equal or greater than the length of the crack.

While all the fatigue models remain fundamentally empirical, there are two main approaches which remain unfortunately separated: a "stress-life" approach, which was developed by Wöhler and others and for almost a century was the only viable route to design, and the "crack propagation" approach, which originated the Damage Tolerance (DT) philosophy. The two approaches are generally applied to different types of components, with the DT approach having a great success in aeronautical fields, particularly for metals. The idea that it should be possible to take the best of the two worlds and develop a "unified" approach, and use both sets of material constants, has been put forward in earlier studies [2]. In previous works, generalized Paris' and Wöhler's laws have been proposed [3, 4]. Recently, we have taken into consideration the effect of the notch and crack sizes [5], as well as the effect of the initial crack size distribution [6]. However, a generally accepted unified theoretical framework capable of describing crack propagation, fatigue life of uncracked specimens, with all the "thresholds" and transitions clearly identified and modelled, is still not available. The present paper briefly discusses one significant difficulty in dealing with the "short crack" problem.

Pugno *et al.* [3] used a Quantized Fracture Mechanics (QFM) approach [7] to obtain generalized crack propagation equations which, in the limit of short cracks, permit to obtain a standard SN curve, independent on the crack size. This early proposal will be revisited here, in view of obtaining a different limit for short cracks. To this end, we first briefly review the literature on short crack propagation laws, then on long cracks and the most well known laws used in DT calculations. Finally, we make our proposal of a new unified approach.

1.1 Short cracks

Some authors have suggested that the crack propagation rate for a short crack should be described by some power law of the stress range [8–13], and by the crack length. Frost and Dugdale [14] were the first to propose a law of the type:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = H\Delta\sigma^h a \tag{1}$$

where $\mathrm{d}a/\mathrm{d}N$ is the advancement of fatigue crack per cycle, $\Delta\sigma$ the stress range and h and H constants, with h being originally equal to 3 in the Frost-Dugdale proposal (which perhaps was an approximation to the later proposal by Paris for long cracks). In other cases, mostly discussed by Japanese investigators [8–13], a much higher h was found, of the order of $h \simeq 10$. For steels, h decreases with an increase of the ultimate strength of the material: Goto and Nisitani [15] report data for S45C, SCr 440, SCM 435 and SNCM 439 steels showing, respectively $h = 9.3, 7.0, 7.0$ and 7.0.

Equation (1) is generally known as the "exponential crack growth" and it was proposed even earlier by Head [16] to justify a derivation of the strength versus number of cycles to failure (SN) curves, but not with reference to actual crack sizes (which were not observed in those early times). Indeed, this is a crucial point: the integration of Equation (1) leads obviously to a SN curve of the type

$$\Delta\sigma^h N = \mathrm{const} = \frac{1}{H}\log a_f/a_i \tag{2}$$

and the typical value of h introduced above (i.e. $h \simeq 10$) is consistent with the Basquin exponent of the SN curves of metals [17], whereas Frost-Dugdale's cubic rule would not. Of course, in the classical "stress-life" approach to fatigue, the constant usually assumed on the right hand side of Equation (2) does not recognize the dependence on initial and final crack sizes, a_f and a_i, respectively. This could have occurred because of the logarihtmic weak dependence on the ratio a_f/a_i, which means that the variation of the constant (close to 3-6 in practice) may well have been confused with the well known large scatter in fatigue SN curves.

In earlier studies by some of the present authors [3], however, we did not include the short crack limit in generalizing crack propagation laws, and considered the limit to be that given by the standard SN Basquin's law.

More recently, the exponential crack growth has been suggested in more general contexts, and in crack growth of actual structures when fatigue originates from material discontinuities, and even under spectrum loading [18, 19]. However in the latter cases the exponent h probably returns to smaller values, and not much reference is made on SN curves and how they are obtained from crack growth rates. Notice however that Berens *et al.* refer to "exponential fits" either for short crack sizes, i.e. $a < 0.005$ in, or as an approximation in small increments of propagation [18]. Also Molent *et al.* make use of the exponential crack growth [19], but an in-depth discussion about the content of these previous works is beyond the scope of the present effort. Anyway, we believe that it is not correct in general to conclude that an exponential crack growth can be assumed for the entire lifetime.

1.2 Long cracks and Damage Tolerance

For long cracks, the fatigue crack growth can be obtained from the celebrated Paris' law [20], which is written in terms of the amplitude of the stress-intensity factor $\Delta K = Y \Delta\sigma\sqrt{\pi a}$ where Y is a geometrical factor, as:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C\Delta K^m \tag{3}$$

where C and m are experimentally determined "material parameters" (although they depend also on geometry and other parameters, see Ref. [21]) and, in the following, we assume $Y = 1$, like in the case of a central crack in a infinite plate under tension.

Figure 1 shows crack propagation curves in steel from [10] within smooth specimen, or specimen with very small holes (diameter of 100 μm), which result in crack propagation law of the type (1) when cracks are small (less than about 1 mm), and a Paris-like regime above this value. Intermediate crack sizes, as can be expected, seem to follow an intermediate power law regime with a slope nearly equal to 2. For long cracks, a Paris regime is found, for which the dependence on the stress range is collapsed into the ΔK parameter, with constants $C = 4.95 \cdot 10^{-13}\ \frac{\mathrm{m}^{1-m/2}}{\mathrm{cycle}\cdot\mathrm{MPa}^m}$ and $m = 3.7$. *Vice versa*, for short cracks, a strong dependence on the stress range is found, namely higher growth rates are obtained with higher $\Delta\sigma$, and the data depart from the Paris' curve, including propagation at much lower values than the long crack threshold (which, for this material, is expected to be about $\Delta K_{th} \approx 7$ MPa$\sqrt{\mathrm{m}}$).

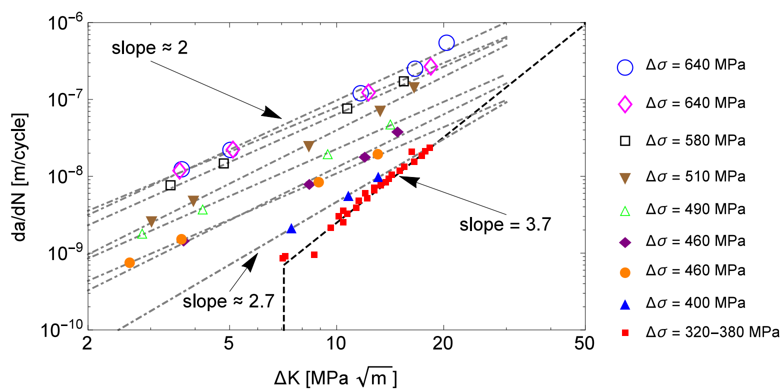


Fig.1. The crack growth curves from Nisitani and Goto [10] clearly show an exponential crack growth and a transition towards a Paris' law regime when the cracks are larger than about 1 mm (filled red squares). Experimental data have been extracted from Fig 3 (b) and Fig. 6 (a) in Ref. [10]. For all the data stress ratio $R = \sigma_{min}/\sigma_{max} = -1$ and, for each marker, the stress range $\Delta\sigma$ is reported in the legend.

A very important successful framework using Paris' law concept is the well known United States Air Force (USAF) DT methodology [22]. This approach effectively avoids many sources of scatter by removing the uncertainty of short crack growth problems. In fact, it considers that cracks of conservative deterministic dimensions are present after pre-service (in the order of 1 mm) or in-service inspection techniques (up to 12 mm) at critical conditions. The DT requirement consists in prescribing that crack propagation should not make the structure at risk before the subsequent inspection. Few codes (NASGRO, AFGROW, NASTRAN) exist that utilize various forms of the long-crack Paris' type of crack growth equations, taking into account various phenomena to some degree of confidence, such as crack closure, crack retardation during spectrum loading, etc. However, techniques also able to take into account of the short crack effects may be relevant in some specific cases, particularly in military contexts[1].

1.3 Transition from short to long cracks

Some authors suggest that existing long crack equations can include short crack effects without much modification. For example, Jones [23] suggests that the NASGRO (specifically, Hartman–Schijve variant) Paris' equation is quite general and includes short crack effects:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = D\frac{(\Delta K_{eff} - \Delta K_{eff,th})^p}{(1 - K_{\max}/A)^{p/2}} \tag{4}$$

where A, D and p are deterministic constants, while ΔK_{eff} corrects ΔK for crack closure, and $\Delta K_{eff,th}$ is a threshold in terms of effective stress intensity factor range ΔK_{eff}. In particular, the author suggests that usually $p \approx 2$ (at least for some alloys and steels used in aeronautical, railways and civil applications, see [24]), meaning that the crack propagation would be exponential if $\Delta K_{eff,th} \approx 0$ so that the higher values of m typically recorded should be just an effect of this threshold effect, in turn due to crack tip plasticity and closure shielding.

[1] Jones [23] argues this is because the requirement 'that aircraft cannot be flown once the aircraft has exceeded half of the number of cycles seen in the associated full-scale fatigue test, does not hold for military aircraft'.

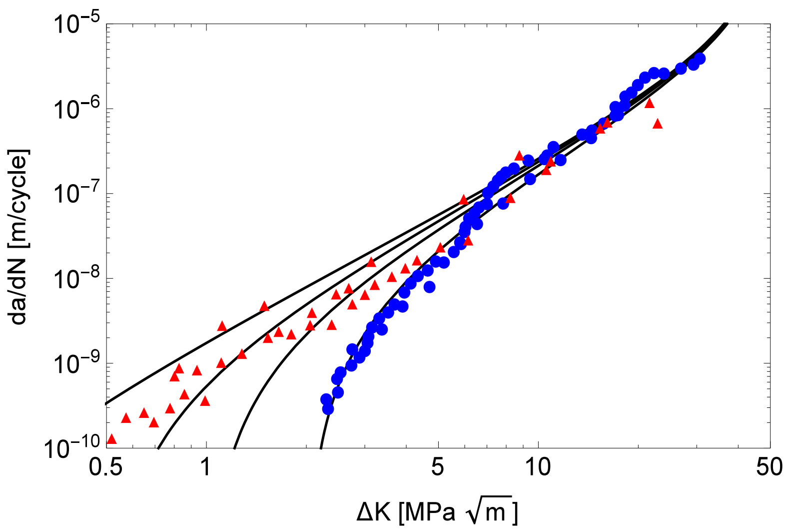


Fig.2. Example of the NASGRO Hartman-Schijve crack propagation equation fitting both long and short crack data on 7050-T7451 (from [23], Fig. 1). Blue circles for long cracks ($R = 0.33$), red triangles for short cracks (different R ratio), and the fit is done with Equation (4) using $D = 2.1 \cdot 10^{-9}$ $\frac{\mathrm{m}}{\mathrm{cycle}\cdot\left(\mathrm{MPa}\sqrt{\mathrm{m}}\right)^p}$, $A = 50$ MPa$\sqrt{\mathrm{m}}$, $p = 2$, $\Delta K_{eff} = \Delta K$ and $\Delta K_{eff,th}$ from 0.1 to 2 MPa$\sqrt{\mathrm{m}}$

Jones [23] suggests that short cracks see very little crack closure and Equation (4) can be further simplified by taking $\Delta K_{eff} \approx \Delta K$. Thus, for short cracks an exponential law seems to be obtained, as introduced through Equation (1). The author concludes that 'small crack curves generally have a Paris like shape with no clear threshold' as NASGRO Hartman-Schijve variant reduces basically to the standard power-law Paris' form (**??**). We report in Figure 2 some data for long and short cracks, on 7050-T7451 Al alloy (data extracted from [23]). It is clearly visible that short crack data present a higher grow rate with respect to long cracks for the same ΔK, without showing a clear threshold. Even above the long crack threshold, short cracks (within the experimental scatter) do not show a clear threshold while they seem to suggest a power of $m = 2$, and no R-ratio effects. There seems however not to be a dependence on a high power of the stress range, as we have discussed before.

There remains therefore a controversy over the important issue of the stress range power for short cracks: if it is 2 in the form suggested by Jones [23], this in turn requires some additional explanation on how to obtain SN curves as virtually, obviously, no material shows a SN law with Basquin slope equal or close to 2. If instead it is a higher power like $h \simeq 10$ as in the references cited above [8–13], we can conclude, as indeed these reference do, that SN curves are entirely explained with short crack propagation, without much need to recur to initiation life and long crack propagation. In the discussion we shall return to this point.

A quite successful distinction from short (i.e. fatigue-limit dominated) to long (i.e. fatigue-threshold dominated) cracks, was introduced by various authors to occur at crack sizes in the order of [25–29]:

$$a_0 = \frac{1}{\pi}\left(\frac{\Delta K_{th}}{\Delta\sigma_L}\right)^2 \tag{5}$$

where $\Delta\sigma_L$ is the fatigue limit range at a given load R-ratio and ΔK_{th} the threshold at the same R-ratio. Kitagawa and Takahashi [25] introduced the $\Delta\sigma - a$ diagram to show the transition which El Haddad *et al.* [26] formulated in an empirical equation, simply adding a_0 as an "intrinsic" additional crack size for the threshold equation:

$$\Delta\sigma_\infty = \frac{\Delta K_{th}}{\sqrt{\pi\left(a+a_0\right)}} \tag{6}$$

and a_0 turned out to be linearly proportional to the grain size.

The El Haddad idea is one aspect of the "Critical Distance Method" which in turn originates in the very old days of fatigue, when Neuber [30] and Peterson [31] introduced the fatigue knock-down factor K_f by averaging the stress over a certain length scale, as summarized in a recent book by Taylor [29]. El Haddad *et al.* [26] formulated also the extension of this idea to crack propagation laws, suggesting however that a_0 should be added to the Irwin stress intensity range in Paris' law, independently on stress level, i.e.:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C\left(Y\Delta\sigma\sqrt{\pi\left(a+a_0\right)}\right)^m \tag{7}$$

Note that this law leads, in the limit of very short cracks, to a constant crack propagation rate. However, the accuracy of this assumption was not validated extensively (some tests are used from a single reference which are very likely under strain control, and the use of J-integral complicates the interpretation). Again, it certainly suffers from the drawback that the SN curve of the uncracked specimen is not obtained from integrating the crack propagation curve, because the power on the stress range is not compatible with the Basquin typical values.

2 Model

El Haddad *et al.* [26]'s idea is a special case of what in previous works [3,4] has been considered as generalized Paris' laws in the context of the QFM approach [7], and which we shall use again here, for different purpose of obtaining the general exponential crack growth law (1) in the limit of short cracks. Let us consider the expression of the stress-intensity factor in the Griffith's case (i.e. for an infinite elastic plate with a symmetric crack of size $2a$). According to QFM, we have [3,4]:

$$\Delta K^* = \Delta\sigma\sqrt{\pi\left(a+\frac{\Delta a}{2}\right)} = \Delta\sigma\sqrt{\pi\left(a+q\right)} \tag{8}$$

where we use $q = \Delta a/2$ for convenience, being Δa the "fracture quantum" in the original QFM formulation. When $q \gg a$, Equation (8) becomes:

$$\Delta K^* \approx \Delta\sigma\sqrt{\pi q} \tag{9}$$

Considering the crack propagation rate $\mathrm{d}a/\mathrm{d}N$ according to the exponential crack growth and the Paris' law, from Equations (1) and (3) respectively, we can write the equality:

$$H\Delta\sigma^h a \equiv C\Delta K^{*m} \tag{10}$$

We can now substitute Equation (9) into Equation (10), obtaining:

$$q^{\frac{m}{2}} \approx \frac{H}{C\pi^{\frac{m}{2}}}\Delta\sigma^{h-m}a \tag{11}$$

hence the crack propagation rate is:

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}N} &\approx C\Delta\sigma^m\pi^{\frac{m}{2}}\left[a+\left(\frac{H}{C\pi^{\frac{m}{2}}}\Delta\sigma^{h-m}a\right)^{\frac{2}{m}}\right]^{\frac{m}{2}} = \\ &= C\Delta K^m\left[1+\left(\frac{H}{C\pi}\Delta\sigma^{h-2}\Delta K^{2-m}\right)^{\frac{2}{m}}\right]^{\frac{m}{2}}\end{aligned} \tag{12}$$

Note that for $m > 2$ and $\Delta K \to +\infty$, Equation (12) tends exactly to the Paris' law (3). Instead, for $\Delta K \to 0$, we get

$$\frac{\mathrm{d}a}{\mathrm{d}N} \approx \frac{H}{\pi}\Delta\sigma^{h-2}\Delta K^2 \tag{13}$$

which corresponds to the exponential crack growth law (1), written in terms of the amplitude of the stress-intensity factor.
We have thus derived a generalized crack propagation law valid both for short and for long cracks. Equation (12) can be integrated from the initial crack size a_i to the final crack size a_f, obtaining, for $m > 2$:

$$N_f \approx \int_{a_i}^{a_f} \frac{da}{A_1 a\left(A_2 + a^{1-2/m}\right)^{m/2}} = \tag{14}$$

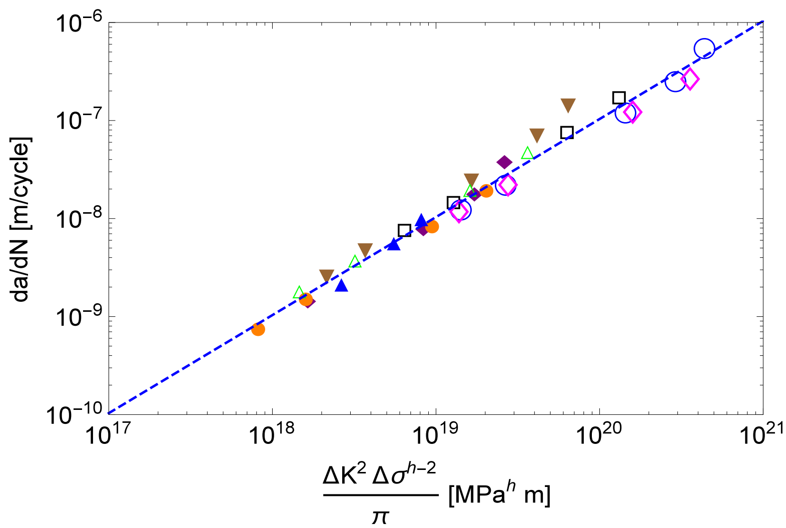
$$\begin{aligned}&= \frac{2}{A_1(m-2)}a_i^{1-m/2}\left[{}_2F_1\left[\frac{m}{2},\frac{m}{2},\frac{2+m}{2},-a_i^{\frac{2}{m}-1}A_2\right] + \right. \\ &\left. -\left(\frac{a_i}{a_f}\right)^{\frac{m}{2}-1} {}_2F_1\left[\frac{m}{2},\frac{m}{2},\frac{2+m}{2},-a_f^{\frac{2}{m}-1}A_2\right]\right]\end{aligned} \tag{15}$$

where ${}_2F_1[a,b,c,z] = \Sigma_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$ is the Gauss' hypergeometric function (special results are obtained for $m = 2, 3, 4$), and for clarity we have introduced the constants $A_1 = C\Delta\sigma^m\pi^{m/2}$ and $A_2 = \left(\frac{H}{C\pi^{m/2}}\Delta\sigma^{h-m}\right)^{2/m}$. Note that, under the typical assumption $\frac{a_i}{a_f} \ll 1$ Equation (15) reduces to:

$$N_f \approx \frac{2}{C\Delta\sigma^m \pi^{m/2}(m-2)} a_i^{1-m/2} \; {}_2F_1\left[\frac{m}{2}, \frac{m}{2}, \frac{2+m}{2}, -a_i^{\frac{2}{m}-1}\left(\frac{H}{C\pi^{m/2}}\Delta\sigma^{h-m}\right)^{2/m}\right] \tag{16}$$

3 Examples

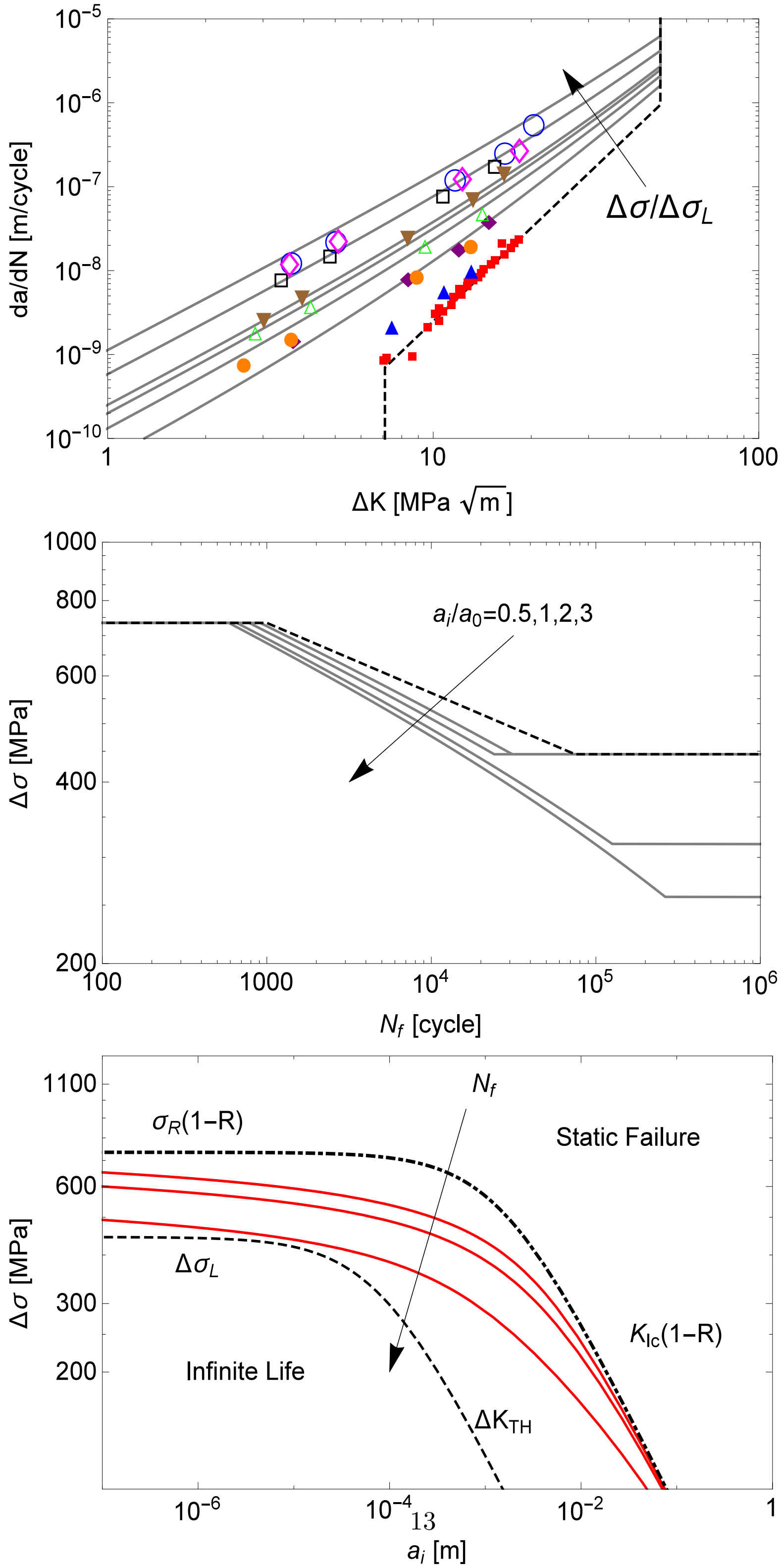
Going back to the experimental results by Nisitani and Goto [10], we show in Figure 3 that a reasonable fit is obtained using Equation (13) with $h = 8.6$ and $H = 1.04 \cdot 10^{-27}\ \frac{1}{\text{cycle}\cdot\text{MPa}^h}$.



(17)

Fig. 3 - Experimental data from Nisitani and Goto [10] (data and symbols as in Fig. 1, $R = -1$) and fit (dashed line) with Equation (13), with $h = 8.6$ and $H = 1.04 \cdot 10^{-27}\ \frac{1}{\text{cycle}\cdot\text{MPa}^h}$.

In Figure 4 Equation (12) is plotted using the constants obtained from the data in [10]. In particular, considering $\log \frac{a_f}{a_i} = \log 10^2$, we obtain $H = 1.04 \cdot 10^{-27}\ \frac{1}{\text{cycle}\cdot\text{MPa}^h}$, $h = 8.6$, $C = 4.95 \cdot 10^{-13} \frac{\text{m}^{1-m/2}}{\text{cycle}*\text{MPa}^m}$ and $m = 3.7$. Figure 4a shows the crack growth rate curves for the levels of stress range in the experiments $\Delta\sigma/\Delta\sigma_L = [0.9, 1.03, 1.10, 1.14, 1.30, 1.44]$, being $\Delta\sigma_L = 445$ MPa the fatigue range limit. As discussed above, the two limit cases are retrieved, while the complete expression of Equation (12) describes the behaviour for intermediate values of ΔK. Note that we have plotted the equation only for stress ranges higher than fatigue limit range, as we expect essentially this is the condition where short cracks can propagate. As a result of the transition the equation shows a crack growth rate, which is significantly higher than long crack growth even above the "long crack threshold", which here would be near 7 MPa$\sqrt{\text{m}}$. However, the crack growth equation should be changed to the long crack growth form, perhaps in the form of Equation (4), showing the threshold, for values of the stress range below the fatigue limit. The quality of the fit, when seen in the form of crack-growth rate of Figure 4a, is reasonable but clearly some trends remain qualitative rather than quantitative. Note also that we obtained the constants associated to the short-crack law independently from the original data and from the original tables reported in [10], which seem to fit the SN curve in Figure 9 of the same paper. Anyway, it is not clear if they represent the same cases, given that in the paper there are different specimens, for different loading conditions (rotating bending vs tension-compression) and different materials, thus some confusion may have arisen. We therefore could not check independently if the SN curves do show only short crack propagation lives, and almost no initiation life, as the authors say.



da/dN [m/cycle]
ΔK [MPa √m]
Δσ/Δσ_L
Δσ [MPa]
N_f [cycle]
a_i/a_0=0.5,1,2,3
σ_R(1−R)
Static Failure
N_f
Δσ_L
K_Ic(1−R)
Infinite Life
ΔK_TH
a_i [m]

(a)

(b)

(c)

Fig.4. Example of the proposed Equation (12), plotted with the data from Nisitani and Goto [10] (data and symbols as in Fig. 1, $R = -1$), in terms of crack growth rate (a), SN curve (b), and extended Kitagawa diagram (c) (σ_R is the ultimate material strength).

In Figure 4b we plot Equation (16) as a generalized SN curve for the same parameters used in Figure 4a and four different a_i/a_0 ratios (i.e. $0.5, 1, 2, 3$). The "fatigue limit" is added independently to our short crack law (which contains neither of the two thresholds) and is imposed either at $\Delta\sigma = \Delta\sigma_L$ or at $\Delta K = \Delta K_{th}$. For comparison, we have plotted the pure power law SN curve for the example case material (dashed black line), i.e. Equation (2).

Finally, in Figure 4c, an extension of the celebrated Kitagawa-Takahashi diagram is presented. The infinite life region is bounded from above by the El Haddad curve, while the finite life region is bounded from below by the El Haddad curve and superiorly by its extension (from [2]) that accounts approximately for the static resistance of the material:

$$\Delta\sigma_{CM} = \frac{K_{Ic}}{\sqrt{\pi\left(a + a_0^S\right)}} \tag{18}$$

being $a_0^S = \frac{1}{\pi}\left(\frac{K_{Ic}}{\sigma_R}\right)^2$. The solid red curves in the finite life region have been drawn using Equation (15) (thus accounting for a_f) for $N_f = \left[10^3, 10^4, 10^5\right]$ cycles. Some uncertainty remains over the region near the threshold, which comes from the fact that the size effect in the Paris' law is intrinsically different from that of the threshold.

4 Discussion

The different forms of short crack propagation laws that we discussed suggest that probably there are various regimes of "short cracks", which require different approaches. An illuminating map on the problem of short cracks perhaps worth citing here is given by Miller [32], reproduced schematically in Figure 5. It is clearly shown that short cracks can be divided into three categories: microstructurally short cracks (MSC), for $a < d_3$ in the Figure where d_3 identifies microstructural length scales associated with inclusions or grain sizes; physically small cracks (PSC) below the size where LEFM can be safely applied; and finally highly stressed cracks (HSC), i.e. those larger than l but for which elasto-plastic corrections are needed. Note that even for PSC and HSC, if the stress levels are sufficiently high, in principle Elasto-Plastic Fracture Mechanics (EPFM) should be applied.

We clearly see some possible motivations as to why there is not a single model for short cracks, or why various models seem conflicting. While Jones [23] refers probably to low stress levels, where short crack effects are limited to the removal of crack tip closure effects, resulting in a "clean" form of the Paris' law for long cracks, the Japanese authors short crack type of law, and our paper, is rather mainly an attempt to model the EPFM effects in a simple way.

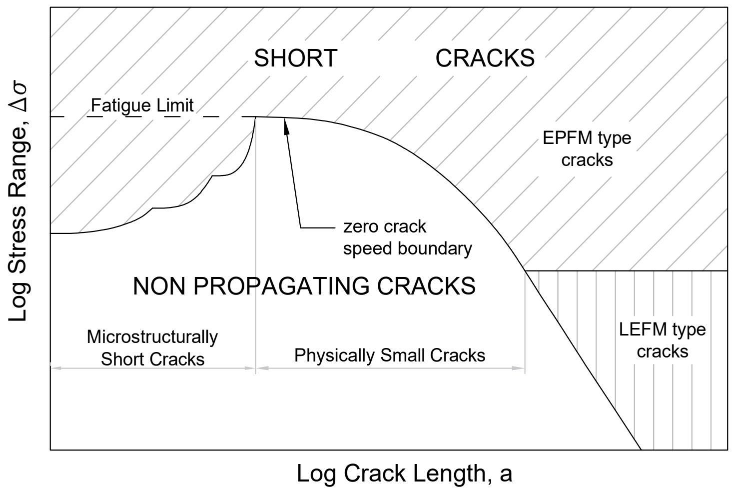


Fig. 5. Miller's version of the extended Kitagawa-Takahashi diagram [32].

In these respects, it may be worth citing McEvily *et al.* [33] who, in a similar attempt to model elasto-plastic loading of short cracks, introduced a correction to the crack size which is in some respects similar to our attempt, although more precisely connected to the size of the plastic zone ahead of the crack tip.

5 Conclusions

There is some controversy in the literature about what should be the correct model for the propagation of short cracks. El Haddad *et al.* [34] proposed the addition of a fictitious crack length, which was obtained from fatigue limit and fatigue threshold, and was independent on the stress level. Other authors used a NASGRO equation in the Hartman-Schijve variant which, for short cracks, returns very closely to a simple Paris' law for $m = 2$ and ideal crack growth behaviour (no crack closure, R-ratio effects, etc.).

We have proposed a new unified formulation for crack growth rate, which leads in the limit of short cracks, to the form suggested by Frost and Dugdale, and in the limit of long cracks to Paris' simple type of equation. This form is compatible with obtaining the SN curve of uncracked materials from the simple integration of the crack growth equation. We have therefore provided a unified equation for crack growth. Comparison with experimental data from the literature is still difficult, and different authors seem to report different behaviours.

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