

Polyhomogeneous expansions from time symmetric initial data

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Abstract

We make use of Friedrich's construction of the cylinder at spatial infinity to relate the logarithmic terms appearing in asymptotic expansions of components of the Weyl tensor to the freely specifiable parts of time symmetric initial data sets for the Einstein field equations. Our analysis is based on the assumption that a particular type of formal expansions near the cylinder at spatial infinity corresponds to the leading terms of actual solutions to the Einstein field equations. In particular, we show that if the Bach tensor of the initial conformal metric does not vanish at the point at infinity then the most singular component of the Weyl tensor decays near null infinity as $O(\tilde{r}^{-3} \ln \tilde{r})$ so that spacetime will not peel. We also provide necessary conditions on the initial data which should lead to a peeling spacetime. Finally, we show how to construct global spacetimes which are candidates for non-peeling (polyhomogeneous) asymptotics.

Keywords: Conformal methods, spinors, spatial infinity, peeling

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1 Introduction

The *Peeling theorem* has played a very important role in the development of the modern notion of gravitational radiation. It is usually formulated within the context of *asymptotically simple spacetimes* —see [43], Section 10.2 for a definition of this class of spacetimes. The Peeling theorem can be formulated as:

Theorem 1. *Let $(\tilde{\mathcal{M}}, \tilde{g})$ denote a vacuum asymptotically simple spacetime with vanishing Cosmological constant. Then the components of the Weyl tensor with respect to a frame adapted to a foliation of outgoing light cones satisfy*

$$\tilde{\psi}_0 = O\left(\frac{1}{\tilde{r}^5}\right), \quad \tilde{\psi}_1 = O\left(\frac{1}{\tilde{r}^4}\right), \quad \tilde{\psi}_2 = O\left(\frac{1}{\tilde{r}^3}\right), \quad \tilde{\psi}_3 = O\left(\frac{1}{\tilde{r}^2}\right), \quad \tilde{\psi}_4 = O\left(\frac{1}{\tilde{r}}\right),$$

where \tilde{r} is a suitable parameter along the generators of the light cones.

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The definition of asymptotically simple spacetimes involves an assumption on the existence of a smooth (i.e. C^∞) conformal extension of the spacetime (\mathcal{M}, \tilde{g}) . An inspection of the proof of the Peeling theorem reveals that, in fact, it is only necessary to assume that the conformal extension is C^4 . In view of the latter, the question of the existence and genericity of spacetimes satisfying the *peeling behaviour* can be rephrased in terms of the construction of asymptotically flat spacetimes with, at least, this minimum of differentiability.

There exists a vast body of work aimed at the construction of spacetimes satisfying the peeling behaviour and at understanding the genericity of this property —see e.g. [16, 21, 43] for an entry point to the literature on this subject. The seminal work in [30] already singles out a key feature of the problem —namely, that whereas *Penrose’s compactification procedure* applied to the Minkowski spacetime renders a fully smooth conformal extension, for spacetimes with a non-vanishing mass (e.g. the Schwarzschild spacetime) the conformal structure degenerates at *spatial infinity*. As spatial infinity can be regarded as the (past/future) endpoint of the generators of (future/past) null infinity, it is natural to expect that the behaviour of the gravitational field near spatial infinity will, somehow, reflect on the peeling properties of the spacetime —particularly, if one tries to analyse these from the point of view of a Cauchy initial value problem.

The first systematic attempt to understand the *generic* properties of the Einstein field equations near spatial infinity from the point of view of an initial value problem are due to Beig & Schmidt [8, 5] who integrate the equations along the so-called *hyperboloid* at spatial infinity. Further insight on the relation between the peeling property and spatial infinity was provided by Friedrich’s proof of the semiglobal existence and stability of perturbations of the Minkowski spacetime from *hyperboloidal initial data* —see[18]. This result ensures the existence of (semiglobal) developments with a smooth null infinity (and thus peeling) if suitable hyperboloidal initial data is provided. A posterior analysis of the solutions to the Einstein constraint equations in the hyperboloidal setting by Andersson, Chrusciel & Friedrich [4] and later Andersson & Chrusciel [2, 3] revealed that the initial data sets considered in Friedrich’s semiglobal results are non-generic. More precisely, their results reveal the existence of certain obstructions to the smoothness null infinity and suggest that a consistent framework for the analysis of the asymptotics of the gravitational field of isolated bodies is that of *polyhomogeneous expansions* —i.e expansions involving powers of $1/\tilde{r}$ and $\ln \tilde{r}$. The formal properties of spacetimes possessing polyhomogeneous asymptotic expansions were explored in [15] —similar types of expansions had been considered earlier in e.g. [29, 44]; further properties have been later discussed in [36, 38, 37].

The proof of the non-linear stability of the Minkowski spacetime by Christodoulou & Klainerman [14] provides a further body of evidence of the non-generic character of spacetimes satisfying the peeling behaviour. Indeed, their analysis provides spacetimes, which, in the notation of Theorem 1 satisfy:

$$\tilde{\psi}_0 = O\left(\frac{1}{\tilde{r}^{7/2}}\right), \quad \tilde{\psi}_1 = O\left(\frac{1}{\tilde{r}^{7/2}}\right), \quad \tilde{\psi}_2 = O\left(\frac{1}{\tilde{r}^3}\right), \quad \tilde{\psi}_3 = O\left(\frac{1}{\tilde{r}^2}\right), \quad \tilde{\psi}_4 = O\left(\frac{1}{\tilde{r}}\right), \quad (1)$$

—see [19]. Christodoulou & Klainerman’s original results left open the question of whether these decay estimates are sharp or could actually be improved through further analysis. The relation between non-peeling expansions and a *no incoming radiation condition* expressed in terms of the constancy of the Bondi mass on \mathcal{I}^- in Christodoulou & Klainerman’s construction has been explored in [13] —remarkably, it turns out that the no incoming radiation condition is not enough to preclude the development of logarithmic terms in the expansions. Later, by making further assumptions on the initial data Klainerman & Nicolò have been able to ensure the existence of spacetimes with the peeling property —see [26]¹.

¹Notice, however, that the conditions on the initial data considered in this reference exclude the Kerr spacetime.

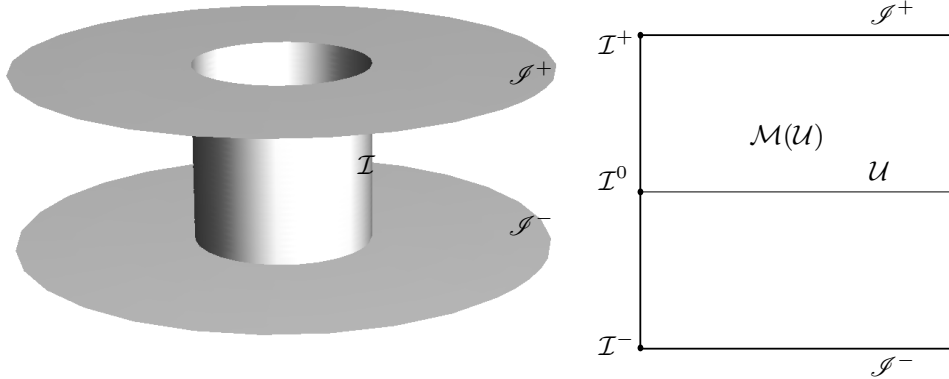


Figure 1: Left: schematic representation of the cylinder at spatial infinity in the so-called *horizontal representation* where null infinity corresponds to the locus of points with $\tau = \pm 1$. The cylinder \mathcal{I} is a total characteristic of the conformal evolution equations —see Section 3.3 for further details. Right: longitudinal section in which the angular dependence has been suppressed. Here \mathcal{U} denotes an open set in a neighbourhood of i and $\mathcal{M}(\mathcal{U})$ its development; \mathcal{I}^\pm are the critical sets where the cylinder meets spatial infinity and \mathcal{I}^0 is the intersection of the cylinder with the initial hypersurface. These figures are coordinate rather than conformal diagrams—in particular, conformal geodesics correspond to vertical lines. Figure taken from Figure 20.3 in page 543 of [43]. ©Cambridge University Press. Reprinted with permission.

The cylinder at spatial infinity

Arguably, the most systematic approach to the analysis of the properties of the Einstein field equation near spatial infinity is based on Friedrich’s construction of the *cylinder at spatial infinity* [20]. This framework makes use of the so-called *extended conformal Einstein field equations* and a gauge based on the properties of conformal geodesics to formulate a regular initial value problem at spatial infinity. Crucial in this framework is a careful description of the singular behaviour of the Weyl tensor and the degeneracy of the conformal field equations near spatial infinity —both at the level of data and solutions. In particular, the use of a gauge based on conformal invariants supports the intuition that the singularities identified in this construction are truly geometric —as opposed to singularities due to a deficiency of the gauge. This point is further supported by the analysis of static and stationary solutions given in [23, 1].

Friedrich’s framework represents spatial infinity through an extended set obtained first by blowing-up the point of spatial infinity to a 2-sphere and then by considering the timelike conformal geodesics passing through this sphere and orthogonal to some fiducial initial hypersurface \mathcal{S} . These curves rule the cylinder at spatial infinity \mathcal{I} and allow to transport information from past null infinity to future null infinity. The cylinder \mathcal{I} intersects null infinity transversally at so-called *critical sets* \mathcal{I}^\pm —see Figure 1.

From the point of view of the conformal Einstein field equations the set \mathcal{I} is very special as it turns out to be a *total characteristic* —that is, the full conformal evolution equations reduce to a system of transport equations on the cylinder. Thus, it is not possible to prescribe boundary data on \mathcal{I} . Rather, the value of the various conformal fields is obtained from an integration of the transport equations using the data at the intersection of the \mathcal{I} with \mathcal{S} . The integration of these transport equations reveals that, in fact, the conformal evolution equations degenerate at the critical sets —in the sense that the matrix associated to the time derivatives in the evolution equations loses rank.

Crucially, the total characteristic nature of the cylinder at spatial infinity implies the existence of a hierarchy of transport equations at \mathcal{I} . This, in turn, allows to construct formal expansions of

the solutions to the extended conformal Einstein field equations —so-called *F-expansions*. These expansions provide valuable information about the type of singular behaviour one can expect at the *intersection of null and spatial infinity*. In particular, in [20] it has been shown that for *time symmetric initial data sets*, the solutions to the hierarchy of transport equations at \mathcal{I} develop a particular type of logarithmic singularities at the critical sets. This class of singularities are a manifestation of the degeneracy of the conformal evolution equations at \mathcal{I}^\pm and, roughly speaking, are associated to the *linear part* of the evolution equations. It is worth noticing that the structural properties leading to this type of singular behaviour is shared by a large class of equations —the Maxwell, Yang-Mills, scalar field, etc.

The key result in [20] is that the class of logarithmic singularities alluded to in the previous paragraph do not arise if the *regularity condition*

$$D_{\{i_p D_{i_{p-1}} \cdots D_{i_1} b_{jk}\}}(i) = 0, \quad p = 0, 1, \dots, p \quad (2)$$

on the Bach tensor b_{jk} of the *initial conformal metric* \mathbf{h} and its derivatives at i is satisfied. In condition (2) D is the Levi-Civita connection of \mathbf{h} and $\{ab\dots c\}$ denotes the operation of taking the symmetric, \mathbf{h} -tracefree part. If condition (2) is not satisfied at some order p_i then logarithmic singularities arise in the solutions to the transport equations at order $p_i + 2$. The regularity condition (2) is a condition on the conformal class $[\mathbf{h}]$ of the metric \mathbf{h} and thus can be verified to be conformally invariant. It is satisfied by static initial data sets —see [6, 17, 23]. It is also (trivially) satisfied by conformally flat time symmetric initial data sets —e.g. the Brill-Lindquist and Misner initial data sets [12, 28]. Another class of *obstructions to the smoothness of null infinity*, has been identified in [40, 39]. The logarithmic singularities associated to these obstructions are related to the specific nonlinear structure of the Einstein field equations.

As already emphasised, the asymptotic expansions obtained through the framework of the cylinder at spatial infinity are obtained in a very specific gauge (the *F-gauge*) based on the properties of conformal geodesics. Accordingly, a natural question to be asked is the following:

Q: *what type of asymptotic expansions near null infinity is implied by the formal F-expansions near the cylinder at spatial infinity?*

As the discussion of asymptotic properties near null infinity (cf. e.g. Theorem 1) is usually expressed in terms of a gauge hinged on null infinity (the *Newman-Penrose (NP) gauge*), in order to address **Q** one needs to relate the F and NP gauges. The required analysis has been discussed in [25].

Main results

In this article we address **Q** for time symmetric initial data sets admitting a conformal metric that is analytic near spatial infinity —this is the class of initial data sets considered in [20]. For this class of initial data there exist detailed computations of the solution to the transport equations near spatial infinity —see e.g. [40, 39, 41]. Under the assumption *that the formal expansions correspond to actual solutions to the conformal Einstein field equations* (Assumption 2 in Section 3) our main result is that generic time symmetric initial data (see Assumption 1 in Section 3) gives rise to developments with expansions near null infinity of the form

$$\tilde{\psi}_0 = O\left(\frac{\ln \tilde{r}}{\tilde{r}^3}\right), \quad \tilde{\psi}_1 = O\left(\frac{\ln \tilde{r}}{\tilde{r}^3}\right), \quad \tilde{\psi}_2 = O\left(\frac{\ln \tilde{r}}{\tilde{r}^3}\right), \quad \tilde{\psi}_3 = O\left(\frac{1}{\tilde{r}^2}\right), \quad \tilde{\psi}_4 = O\left(\frac{1}{\tilde{r}}\right), \quad (3)$$

thus, suggesting *polyhomogeneous asymptotics*. As

$$\left(\frac{\ln \tilde{r}}{\tilde{r}^3}\right) / \left(\frac{1}{\tilde{r}^{7/2}}\right) = \tilde{r}^{1/2} \ln \tilde{r} \longrightarrow \infty \quad \text{as } \tilde{r} \longrightarrow \infty, \quad (4)$$

the expansions in (3) imply a decay of the components of the Weyl tensor which is even slower than the one given by Christodoulou & Klainerman —cf. (1). Moreover, our analysis shows how

the non-peeling decay in (3) is related to Friedrich’s regularity condition (2). More precisely, the decay (3) is obtained if the regularity condition (2) is violated at order $p = 0$. We also show that, under our assumptions, a necessary condition to obtain peeling decay is that (2) holds up to order $p = 3$.

Once the part of the initial data responsible for the decay in (3) has been identified, it is natural to ask whether it is possible to construct a candidate (global) spacetime to have such asymptotic behaviour. We show that this is indeed possible by first constructing a family of initial data sets which violate the regularity condition (2) to order $p = 0$ and which can be regarded as perturbations of time symmetric Minkowski initial data. Consistent with the limit in (4), it turns out that Christodoulou & Klainerman’s result is not applicable to initial data sets that violate Friedrich’s regularity condition to order $p = 0$ —the quantity measuring global smallness is not well defined. Fortunately, Bieri’s generalisation of Christodoulou & Klainerman’s global existence result [9] can, nevertheless, be used to construct the maximal global hyperbolic development of the initial data. This construction render the desired candidate spacetime.

Remark 1. The various pieces of information required to obtain the expressions (3) have been available in the literature for sometime. However, they have remained scattered as, for historical reasons, the main focus has for a while been to understand the conditions leading to a smooth conformal extension. The main task in this article is to put these pieces together.

Remark 2. The analysis in this article can be generalised, to the expense of lengthier computations, to classes of initial with a non-vanishing extrinsic curvature. As the decays in (3) are almost borderline with what is allowed by the Einstein field equations, it is conjectured that the inclusion of a non-vanishing extrinsic curvature without linear momentum will not modify our main result.

Remark 3. It should be stressed that our main result is formal —i.e. it assumes the existence of a spacetime with the given asymptotics —cf. Assumption 2. The main open problem in the subject of the asymptotics of the gravitational field of isolated systems can be described as showing that there indeed exist solutions to the Einstein field equations with the given asymptotics —in other words, one needs to analyse the relation between the expansions used to obtain (3) and actual solutions. This is a challenging and deep problem which deserves careful consideration. A possible line of attack of this question is to obtain a suitable generalisation of the estimates for the spin-2 equation in [22] to the full conformal Einstein equations.

Outline of the article

The article is structured as follows. Section 2 provides a brief review of the main technical tool in this article, the extended conformal Einstein field equations —including a discussion of gauge issues and the construction of initial data. Section 3 provides an overview of the construction of the cylinder at spatial infinity. The purpose of this section is not so much to describe the construction but to set notation and describe the features relevant for the purposes of the present article —in particular, Subsection 3.4 provides a description of the asymptotic expansions of the solutions to the conformal field equations near the cylinder for time symmetric initial data. Section 4 provides a discussion of the relation between the F-gauge used in the construction of the cylinder at spatial infinity and the NP-gauge used in the discussion of peeling properties. Our main results are provided in Section 5. Section 6 discusses the construction of global spacetimes which are candidates to have the non-peeling asymptotics of our main results. We provide some concluding remarks and outlooks in Section 7. Finally, in the Appendix we provide detailed expressions of the solutions to the transport equations on the cylinder at spatial infinity.

Notation and conventions

In this article $\{a, b, c, \dots\}$ denote abstract tensor indices and $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots\}$ will be used as spacetime frame indices taking the values $0, \dots, 3$. In this way, given a basis $\{\mathbf{e}_a\}$ a generic tensor is denoted by T_{ab} while its components in the given basis are denoted by $T_{\mathbf{ab}} \equiv T_{ab} \mathbf{e}_a^a \mathbf{e}_b^b$. Additionally, spatial frame indices respect to an adapted frame will be denoted by i, j, k, \dots and will take the values $1, 2, 3$. Since part of the analysis will require the use of spinors the notation and conventions of Penrose & Rindler [31] will be followed. In particular, capital Latin indices $\{A, B, C, \dots\}$ will denote abstract spinor indices while boldface capital Latin indices $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots\}$ will denote frame spinorial indices with respect to some specified spin dyad $\{\epsilon_{\mathbf{A}}^A\}$.

2 The conformal Einstein field equations

This section provides a brief discussion of the basic technical tools used in this article. For a more detailed discussion on the structural properties of the conformal Einstein field equations, the reader is referred to [43].

2.1 The extended conformal Einstein field equations

The *extended conformal Einstein field equations* are a conformal representation of the Einstein field equations expressed in terms of a Weyl connection. In what follows let $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ denote a spacetime satisfying the vacuum Einstein field equations

$$\tilde{R}_{ab} = 0, \quad (5)$$

and let $(\mathcal{M}, \mathbf{g})$ be a conformal extension thereof with

$$g_{ab} = \Theta^2 \tilde{g}_{ab},$$

where Θ is some suitable conformal factor. Furthermore, let $\{\mathbf{e}_a\}$ denote a \mathbf{g} -orthonormal frame. In terms of the above, the extended conformal Einstein field equations are given by the concise expressions

$$[\mathbf{e}_a, \mathbf{e}_a] = (\hat{\Gamma}_a^c{}_b - \hat{\Gamma}_b^c{}_a) \mathbf{e}_c, \quad (6a)$$

$$\hat{P}^c{}_{dab} = \hat{\rho}^c{}_{dab}, \quad (6b)$$

$$\hat{\nabla}_c \hat{L}_{db} - \hat{\nabla}_d \hat{L}_{cb} = d_a d^a{}_{bcd}, \quad (6c)$$

$$\hat{\nabla}_a d^a{}_{bcd} = f_a d^a{}_{bcd}. \quad (6d)$$

In the previous equations $\hat{\nabla}$ denotes the Weyl connection defined through the relation

$$\hat{\nabla}_a g_{bc} = -2f_a g_{bc},$$

with f_a a smooth covector and $\hat{\Gamma}_a^b{}_c$ the associated connection coefficients. Moreover,

$$d_a = \Theta f_a + \nabla_a \Theta,$$

while

$$d^a{}_{bcd} \equiv \Theta^{-1} C^a{}_{bcd}$$

are the components of the *rescaled Weyl tensor* and \hat{L}_{ab} denote components of the Schouten tensor of the connection $\hat{\nabla}$. Finally, $\hat{P}^c{}_{dab}$ and $\hat{\rho}^c{}_{dab}$ denote, respectively, the *geometric* and *algebraic curvatures* —i.e. the classical expression of the curvature in terms of the connection coefficients and its irreducible decomposition in terms of $d^a{}_{bcd}$ and \hat{L}_{ab} .

Remark 4. Equations (6a) and (6b) are a rewriting of the first and second Cartan structure equations. In particular, equation (6b) provides a differential condition for the components $f_{\mathbf{a}}$.

Remark 5. A key property of the system (6a)-(6d) is that a solution thereof implies, whenever $\Theta \neq 0$, a solution to the Einstein field equations (5) —see e.g. [43], Proposition 8.3.

Remark 6. Most applications of the extended conformal Einstein field equations make use of a spinorial frame version of thereof. The latter is readily obtained by contraction with the constant Infeld-van der Waerden symbols $\sigma^{\mathbf{a}}_{AA'}$. In particular, exploiting the symmetries of the rescaled Weyl tensor one has that its spinorial counterpart is defined as

$$d_{AA'BB'CC'DD'} \equiv \sigma^{\mathbf{a}}_{AA'} \sigma^{\mathbf{b}}_{BB'} \sigma^{\mathbf{c}}_{CC'} \sigma^{\mathbf{d}}_{DD'} d_{\mathbf{abcd}},$$

so that one has the decomposition

$$d_{AA'BB'CC'DD'} = -\phi_{ABCD} \epsilon_{A'B'} \epsilon_{CC'} - \bar{\phi}_{A'B'C'D'} \epsilon_{AB} \epsilon_{CD},$$

with $\phi_{ABCD} = \phi_{(ABCD)}$ the components of the *Weyl spinor* ϕ_{ABCD} . Equation (6d) then takes the form

$$\hat{\nabla}^A{}_{A'} \phi_{ABCD} = f^A{}_{A'} \phi_{ABCD}.$$

2.2 Gauge considerations

A conformal geodesic on $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ can be viewed as a curve $x(\tau)$, $\tau \in I$ with I some interval, for which there exists a Weyl connection along the curve $\hat{\nabla}$ such that

$$\hat{\nabla}_{\dot{x}} \dot{x} = 0,$$

with \dot{x} the tangent vector to the curve —in other words, $x(\tau)$ is an affine geodesic of the curve, see [43] Section 5.5.2. This property can be used to construct *Gaussian gauge systems* if a given region of spacetime is covered by a non-intersecting congruence of conformal geodesics. In this type of gauge systems an orthonormal frame $\{\mathbf{e}_{\mathbf{a}}\}$ is Weyl propagated and spatial coordinates \underline{x}_* are extended off an initial hypersurface by requiring them to be constant along the curve. These *spatial* coordinates are completed by including the parameter τ of the curve so as to obtain the spacetime coordinates (τ, \underline{x}_*) . Moreover, conformal geodesics single out a *canonical representative* of the conformal class $[\tilde{\mathbf{g}}]$ by imposing the condition

$$\mathbf{g}(\dot{x}, \dot{x}) = \Theta^2 \tilde{\mathbf{g}}(\dot{x}, \dot{x}) = 1. \quad (7)$$

Remarkably, the above condition leads to an explicit expression for the conformal factor Θ in terms of the parameter of the curve —namely

$$\Theta(\tau) = \Theta_* + \dot{\Theta}_*(\tau - \tau_*) + \frac{1}{2} \ddot{\Theta}_*(\tau - \tau_*)^2, \quad (8)$$

where the coefficients Θ_* , $\dot{\Theta}_*$ and $\ddot{\Theta}_*$ are constant along a given conformal geodesic and are expressible in terms of initial data for the congruence —see [43], Proposition 5.1. In view of (7) one can set $\mathbf{e}_0 = \dot{x}$. It follows then that

$$\hat{\Gamma}_0{}^{\mathbf{a}}{}_{\mathbf{b}} = 0, \quad f_0 = 0, \quad \hat{L}_{0\mathbf{a}} = 0.$$

The extended conformal Einstein field equations (6a)-(6d) expressed in terms of a conformal Gaussian gauge system as described in the previous paragraph take the form

$$\partial_{\tau} \mathbf{e} = \mathbf{Q}_1(\Gamma, \mathbf{e}) + \mathbf{K}_1(\mathbf{e}), \quad (9a)$$

$$\partial_\tau \Gamma = \mathbf{Q}_2(\Gamma, \Gamma) + \mathbf{K}_2(\Phi) + \mathbf{L}_2\phi, \quad (9b)$$

$$\partial_\tau \Phi = \mathbf{Q}_3(\Gamma, \Phi) + \mathbf{L}_3\phi, \quad (9c)$$

$$(\mathbf{I} + \mathbf{A}^0(\mathbf{e}))\partial_\tau \phi + \mathbf{A}^\alpha \partial_\alpha \phi = \mathbf{B}(\Gamma, \phi), \quad (9d)$$

where \mathbf{e} , Γ , Φ and ϕ denote the independent components of the frame, connection, Schouten tensor and Weyl tensor. In equations (9a)-(9c) one has that \mathbf{Q}_1 , \mathbf{Q}_2 and \mathbf{Q}_3 denote a quadratic expression with constant coefficients of the arguments, \mathbf{K}_1 and \mathbf{K}_2 denote linear functions with constant coefficients of the argument and $\mathbf{L}_2(x)$ and $\mathbf{L}_3(x)$ denote smooth matrix-valued functions of the coordinates. Moreover, in equation (9d), \mathbf{I} is the 5×5 unit matrix, \mathbf{A}^μ are Hermitian matrices depending smoothly on the frame coefficients \mathbf{e} and \mathbf{B} is a quadratic expression with constant coefficients of its arguments.

2.3 Initial data for the evolution equations

Initial data for the conformal evolution equations (9a)-(9d) can be constructed if a conformal metric \mathbf{h} and a conformal factor Ω are provided such that

$$r[\Omega^{-2}\mathbf{h}] = 0$$

so that the time symmetric Hamiltonian constraint is satisfied. In particular, for the components d_{ij} of the electric part of the Weyl tensor (the magnetic part vanishes due to the time symmetry) one has that

$$d_{ij} = \frac{1}{\Omega^2} (D_{\{i}D_{j\}}\Omega + \Omega s_{ij}) \quad (10)$$

with s_{ij} the components of the tracefree Ricci tensor of the metric \mathbf{h} . Further details on the construction of initial data for the conformal evolution equations can be found in [43], Section 11.4.3.

The Bach tensor. The Bach tensor of the conformal metric \mathbf{h} will play a key role in the subsequent discussion. Given the Schouten tensor l_{ij} of \mathbf{h} , the *Cotton tensor* is given by

$$b_{kij} \equiv D_i l_{jk} - D_j l_{ik}.$$

Its Hodge dual b_{ij} , is called the *Bach tensor*, and is given by

$$b_{ij} \equiv \frac{1}{2} \epsilon_j{}^{kl} b_{ikl},$$

so that

$$b_i{}^i = 0, \quad b_{ij} = b_{ji}, \quad D^i b_{ij} = 0.$$

The spinorial counterpart of b_{ij} is a totally symmetric spinor b_{ABCD} —the *Bach spinor*. Finally it is observed that under the rescaling $\mathbf{h} \mapsto \mathbf{h}' = \phi^2 \mathbf{h}$ one has that

$$b'_{ijk} = b_{ijk}, \quad b'_{ij} = \phi^{-1} b_{ij}, \quad b'_{ABCD} = \phi^{-1} b_{ABCD}.$$

3 The cylinder at spatial infinity

In this section we provide a brief discussion of the construction of the cylinder at spatial infinity and the asymptotic expansions (*F-expansions*) it gives rise to. The basic references on the construction of the cylinder at spatial infinity are [20, 23]—see also Chapter 21 in [43].

3.1 Boundary conditions

In the present discussion we will restrict our attention to time symmetric initial data sets for the Einstein field equations which are asymptotically Euclidean and admit a point compactification yielding an analytic conformal metric. More precisely, we assume that there exists a 3-dimensional Riemannian manifold $(\mathcal{S}, \mathbf{h})$ with a point $i \in \mathcal{S}$ and a function $\Omega \in C^2$ such that

$$\Omega(i) = 0, \quad \mathbf{d}\Omega(i) = 0, \quad \mathbf{Hess}\Omega(i) = -2\mathbf{h}(i), \quad (11)$$

with $\Omega > 0$ away from i and \mathbf{h} analytic at least in a neighbourhood of i —cf. [43], Definition 11.2.

Remark 7. In the following, our discussion will be restricted to a suitably small neighbourhood of i on \mathcal{S} and the Cauchy development thereof.

Remark 8. There is some conformal gauge freedom left in Conditions (11). A replacement of the form $\mathbf{h} \mapsto \vartheta^4 \mathbf{h}$, $\Omega \mapsto \vartheta^2 \Omega$ with $\vartheta(i) = 1$ gives rise to the same physical metric $\tilde{\mathbf{h}} = \Omega^{-4} \mathbf{h}$. This gauge freedom has been used in [20] to construct a *conformal normal gauge* for which there exist coordinates $\underline{x} = (x^\alpha)$ with $x^\alpha(i) = 0$ such that

$$h_{\alpha\beta} = -\delta_{\alpha\beta} + O(|x|^3). \quad (12)$$

In particular, the curvature of \mathbf{h} vanishes at i .

In what follows, we make the following assumption:

Assumption 1. *The metric \mathbf{h} satisfies the boundary conditions (11) with a conformal factor $\Omega \in C^2(\mathcal{S}) \cap C^\infty(\mathcal{S} \setminus \{i\})$. Moreover, it is analytic in a neighbourhood of i and there exists coordinates $\underline{x} = (x^\alpha)$ for which the components of \mathbf{h} satisfy (12).*

Remark 9. The assumption of analyticity has been made for convenience and easy reference with the analysis in [20]. As pointed out in that reference, this assumption is not essential and for particular computations (like the ones considered here) a finite degree of differentiability suffices—see also similar remarks in [24].

Under Assumption 1 it follows that the conformal factor Ω admits, in a suitable neighbourhood $\mathcal{B}_a(i)$ of i , the parametrisation

$$\Omega = \frac{U}{|x|} + W$$

where U and W are analytic functions on $\mathcal{B}_a(i)$

$$U = 1 + O(|x|^4), \quad W(i) = \frac{m}{2},$$

where m is the ADM mass of the initial data. The function $U/|x|$ is the Green's function of the Yamabe equation implied by the time symmetric Hamiltonian constraint and contains information about the local geometry around i while W encodes global information—in particular the mass. The function W can be expanded as

$$W = \frac{m}{2} + W_1|x| + \frac{1}{2}W_2|x|^2 + O(|x|^3),$$

where W_1, W_2 are smooth functions of the *angular coordinates*.

Singular behaviour of the rescaled Weyl tensor. A direct computation combining the expansions in the previous paragraphs with equation (10) leads to

$$d_{ij} = -\frac{3mx_{\{i}x_{j\}}}{|x|^5} + O(|x|^{-2}),$$

where $x_{\mathbf{i}} \equiv \delta_{\mathbf{i}^\beta} \delta_{\alpha\beta} x^\alpha$ is the position vector centred at i . Consequently, one has that the Weyl tensor blows up at i as $|x|^{-3}$.

Remark 10. As pointed out in the introduction the above singularity is the main technical difficulty in formulating an initial value problem for the conformal Einstein field equations in a neighbourhood of spatial infinity.

3.2 The manifold \mathcal{C}_a

As it is well known, the standard approach to the discussion of the conformal structure of asymptotically flat spacetimes makes use of representations in which spatial infinity is a point i^0 . By contrast, the representation introduced in [20] describes spatial infinity in terms of an extended set—the *cylinder at spatial infinity*. Friedrich’s representation makes use of a blow up of the point at infinity, i , on the initial hypersurface \mathcal{S} to a 2-sphere. The construction makes use of a particular bundle of spin-frames (*bundle space*) over $\mathcal{B}_a(i)$. As we will not require the full details of this construction, in the following we provide a brief description in terms of (vector) frames.

The bundle space. Given an \mathbf{h} -orthonormal frame $\{\mathbf{e}_i\}$ at i , any other frame at i can be obtained by means of a rotation—that is, any other frame is of the form $e_i(s) = s^j_i \mathbf{e}_j$ with $s = (s^j_i) \in SO(3)$. In particular $\mathbf{e}_3(s)$ sweeps all possible directions at i as one lets s exhaust $SO(3)$. For a given value of s , one distinguishes $\mathbf{e}_3(s)$ as the radial vector at i . Keeping s fixed, one then constructs the \mathbf{h} -geodesic starting i that has tangent vector $\mathbf{e}_3(s)$ and denote by ρ the affine parameter along the curve that vanishes at i . One then parallelly propagates the rest of the frame $\{\mathbf{e}_i\}$ along this curve. For a particular value of the parameter ρ , let denote by $\{\mathbf{e}_i(\rho, s)\}$ the frame thus obtained. The subsequent discussion will be then restricted to a suitable small metric ball $\mathcal{B}_a(i)$ on which this construction can be carried out. Let $SO(\mathcal{B}_a(i))$ denote the bundle of oriented orthonormal frames over $\mathcal{B}_a(i)$. It follows then that the map from the set $(-a, a) \times SO(3)$ into $SO(3)$ given by $(\rho, s) \mapsto \mathbf{e}_i(\rho, s)$ as described above defines a smooth embedding of a 4-dimensional manifold into $SO(\mathcal{B}_a(i))$. In the following, only non-negative values of ρ will be considered.

The blow-up of i . Denote by \mathcal{C}_a the image of the set $[0, a) \times SO(3)$ and define

$$\mathcal{I}^0 \equiv \{(\rho, s) \in \mathcal{C}_a \mid \rho = 0\} \approx SO(3).$$

Finally, denote by π the projection of $SO(\mathcal{C}_a(i))$ onto $\mathcal{C}_a(i)$. One then restricts the attention to the subgroup $SO(2)$ of $SO(3)$ which leaves \mathbf{e}_3 invariant—i.e. $SO(2) \equiv \{s' \in SO(3) \mid s'^j_3 \mathbf{e}_j = \mathbf{e}_3\}$. Accordingly, if $s \in SO(3)$ and $s' \in SO(2)$ then $\mathbf{e}_i(s)$ and $\mathbf{e}_i(ss')$ are parallelly transported along the same geodesic. It then follows that $\pi(\mathbf{e}_i(\rho, s)) = \pi(\mathbf{e}_i(\rho, ss'))$, so that the projection factorises as

$$\mathcal{C}_a \xrightarrow{\pi'} \mathcal{C}'_a \equiv \mathcal{C}_a / SO(2) \xrightarrow{\pi''} \mathcal{B}_a(i).$$

It can be then verified that the projection π'' maps $\pi'(\mathcal{I}^0) \approx \mathbb{S}^2$ onto i and induces an isomorphism between $\mathcal{C}'_a \setminus \pi'(\mathcal{I}^0)$ onto the punctured ball $\mathcal{B}_a(i) \setminus \{i\}$. This diffeomorphism allows to identify these sets. Notwithstanding the above factorisation, in [20] the data for the conformal evolution equations, e.g. formula (10), on $\mathcal{B}_a(i)$ is lifted to \mathcal{C}_a using π . In this manner \mathcal{C}_a becomes the initial manifold, (ρ, s) are used as coordinates and the boundary \mathcal{I}^0 becomes a blow-up of i . The manifold \mathcal{C}_a has one dimension more than $\mathcal{B}_a(i)$ —this has to do with the action of $SO(2)$.

A frame formalism. It can be verified that all the fields appearing in the extended conformal Einstein field equations have a well-defined transformation behaviour (spin-weight) under such action. Thus, it is possible to define vector fields $\{\mathbf{X}, \mathbf{c}_i(\rho, s)\}$ such that \mathbf{X} is generated by the action of $SO(2)$ and the vector fields $\{\mathbf{c}_i(\rho, s)\}$ project to $\mathbf{e}_a(\rho, s)$ via π . These vector fields allow the introduction of a frame formalism on \mathcal{C}_a in which the frame is expanded by ∂_ρ and vectors \mathbf{X}_\pm which are a basis of the Lie algebra $\mathfrak{su}(2)$ —the vectors \mathbf{X}_\pm can be shown to be related to the $\bar{\partial}$ and ∂ operators, see [25].

Fixing the conformal gauge. The initial data for the conformal factor in the conformal Gaussian gauge system is set by requiring

$$\Theta_\star \equiv \kappa^{-1}\Omega,$$

where the function κ is chosen so that

$$\kappa = \rho\kappa', \quad \kappa' \in C^\infty(\mathcal{C}_a), \quad \kappa' > 0, \quad \mathbf{X}\kappa' = 0, \quad \kappa'|_{\mathcal{I}^0} = 1.$$

The function κ induces a conformal rescaling of the frame $\mathbf{e}_a \mapsto \kappa\mathbf{e}_a$. This rescaling maps bijectively the set $\mathcal{C}_{a,\kappa} \setminus \mathcal{I}^0$ onto a smooth submanifold of the bundle of frame fields over \mathcal{B}_a . This submanifold is denoted by $\mathcal{C}_{a,\kappa}$. In order to complete the construction of the conformal Gaussian system, one needs to provide initial data for the 1-form f_a . This is done setting

$$f_0 = 0, \quad f_i = \kappa^{-1}D_i\kappa.$$

3.3 The manifold $\mathcal{M}_{a,\kappa}$

Following the previous discussion, the conformal evolution equations (9a)-(9d) can be regarded as equations on the development of $\mathcal{C}_{a,\kappa}$. This development will be denoted as $\mathcal{M}_{a,\kappa}$ and is a 5-dimensional manifold embedded in the bundle of frame fields over the unphysical spacetime \mathcal{M} . The manifold $\mathcal{M}_{a,\kappa}$ is a $SO(2)$ -bundle over the spacetime. The projection sending $\mathcal{M}_{a,\kappa}$ to \mathcal{M} will be denoted, in a slight abuse of notation, by π . The coordinates (ρ, s) and the vector fields \mathbf{X}, \mathbf{c}_i are extended from $\mathcal{C}_{a,\kappa}$ into $\mathcal{M}_{a,\kappa}$ by the flow of conformal geodesics ruling $\mathcal{M}_{a,\kappa}$ in such a way that the vectors do not pick up a component in the direction of the fibres —i.e. in the direction of \mathbf{X} . Further, we make use of the parameter τ of the conformal geodesics as a further coordinate on $\mathcal{M}_{a,\kappa}$ —that is, we set $x^0 = \tau$.

From the conformal gauge choice it follows that the conformal factor (8) takes the form

$$\Theta = \Theta_\star \left(1 - \frac{\kappa_\star^2}{\omega_\star^2} \tau^2 \right) \quad \text{on} \quad \mathcal{M}_{a,\kappa}$$

where

$$\omega \equiv \frac{2\Omega}{\sqrt{|D_i\Omega D^i\Omega|}}.$$

The subscript \star indicates that the relevant functions are constant along a given conformal geodesic.

Remark 11. A key property in the previous construction is that if the initial data for the conformal evolution equations has a smooth limit as $\rho \rightarrow 0$, then it can be smoothly extended into the coordinate range $\rho \leq 0$. Similarly for the fields Θ and d_a . Thus, the initial value problem for the conformal evolution equations can be extended smoothly into a range where $\rho \leq 0$ in such a way that the reduced equations are still symmetric hyperbolic.

Assuming that the development of the initial value problem for the conformal evolution equations (9a)-(9d) extends far enough then one can write

$$\begin{aligned} \mathcal{M}_{a,\kappa} &= \left\{ |\tau| \leq \frac{\omega}{\kappa}, s \in SO(3) \right\}, \\ \tilde{\mathcal{M}}_{a,\kappa} &= \left\{ |\tau| < \frac{\omega}{\kappa}, s \in SO(3) \right\}. \end{aligned}$$

In addition, it is natural to define

$$\mathcal{I}^\pm \equiv \left\{ \tau = \pm \frac{\omega}{\kappa}, s \in SO(3) \right\},$$

$$\mathcal{I} \equiv \{|\tau| < 1, \rho = 0, s \in SO(3)\}$$

and

$$\mathcal{I}^\pm \equiv \{\tau = \pm 1, \rho = 0, s \in SO(3)\},$$

$$\mathcal{I}^0 \equiv \{\tau = 0, \rho = 0, s \in SO(3)\}.$$

Observing that $\kappa/\rho \rightarrow 1$ as $\rho \rightarrow 0$ it can be readily checked that

$$\begin{aligned} \Theta &> 0 && \text{on } \tilde{\mathcal{M}}_{a,\kappa}, \\ \Theta &= 0, \quad \mathbf{d}\Theta \neq 0 && \text{on } \mathcal{I}^- \cup \mathcal{I}^+ \cup \mathcal{I}, \\ \Theta &= 0, \quad \mathbf{d}\Theta = 0 && \text{on } \mathcal{I}^+ \cup \mathcal{I}^-, \end{aligned}$$

justifying the names of the various pieces of the conformal boundary. Moreover, the initial hypersurface corresponds to

$$\mathcal{C}_{a,\kappa} \equiv \{\tau = 0, 0 \leq \rho < a, s \in SO(3)\},$$

so that its boundary is given by \mathcal{I}^0 .

Remark 12. By assumption, the initial data extend smoothly, and in a unique manner to \mathcal{I}^0 . The solution on $\mathcal{M}_{a,\kappa}$ depends only on the data given in $\mathcal{C}_{a,\kappa}$ as the set \mathcal{I} is a total characteristic of the conformal evolution equations (9a)-(9d) so that the solution on \mathcal{I} depends only on the data on \mathcal{I}^0 —that is, no boundary data can be prescribed. The set \mathcal{I} is known as the *cylinder at spatial infinity*. It can be regarded as a blow-up of i^0 . Of particular relevance in the following discussion are the sets \mathcal{I}^\pm , the *critical sets*, where the conformal evolution equations degenerate. Dealing with the consequences of this degeneracy is the key issue of the *problem of spatial infinity*.

3.4 The solutions to the transport equations

In [20] it has been shown how the construction of the cylinder at spatial infinity can be used to construct Taylor like expansions of the solutions to the conformal evolution equations (9a)-(9d)—to be called *F-expansions*. In order to do this, one differentiates the conformal evolutions with respect to the radial direction ρ an arbitrary number of times, say p , and then evaluates at the ρ . In what follows, for conciseness we restrict our attention to the Bianchi equations (9d). In particular, for the Bianchi evolution equations (9d), letting

$$\phi^{[p]} \equiv \partial_\rho^p \phi|_{\rho=0}, \quad p = 0, 1, 2, \dots,$$

a calculation yields that

$$\begin{aligned} &(\mathbf{I} + \mathbf{A}^0(e^{[0]}))\partial_\tau \phi^{[p]} + \mathbf{A}^+(e^{[0]})\mathbf{X}_+ \phi^{[p]} + \mathbf{A}^-(e^{[0]})\mathbf{X}_- \phi^{[p]} \\ &= \mathbf{B}(\hat{\Gamma}^{[0]}, \phi^{[p]}) + \sum_{j=1}^p \binom{p}{j} (\mathbf{B}(\hat{\Gamma}^{[j]}, \phi^{[p-j]}) - \mathbf{A}^+(e^{[j]})\partial_+ \phi^{[p-j]} - \mathbf{A}^-(e^{[j]})\partial_- \phi^{[p-j]}). \end{aligned} \tag{13}$$

The above equation will be called the *Bianchi transport equation of order p* . Similar equations can be obtained for

$$e^{[p]} \equiv \partial_\rho^p e|_{\rho=0}, \quad \Gamma^{[p]} \equiv \partial_\rho^p \Gamma|_{\rho=0}, \quad \Phi^{[p]} \equiv \partial_\rho^p \Phi|_{\rho=0} \quad p = 0, 1, 2, \dots$$

Remark 13. Observe the absence of ρ -derivatives in equation (13). This is a consequence of the fact that \mathcal{I} is a total characteristic of the conformal evolution equations (9a)-(9d). A similar

property is satisfied by the equations for $e^{[p]}$, $\Gamma^{[p]}$, $\Phi^{[p]}$. Collecting all these transport equations for $p = 0, 1, 2, \dots, p$ one obtains a hierarchy of equations which can be solved recursively starting from $p = 0$. This is a process completely amenable to an implementation in a computer algebra system —see [40, 39]. Thus, equation (13) constitutes a system of linear evolution equations intrinsic to \mathcal{I} for $\phi^{[p]}$.

If the lower terms

$$e^{[j]}, \quad \Gamma^{[j]}, \quad j = 0, 1, \dots, p$$

and

$$\phi^{[j]}, \quad j = 0, 1, \dots, p-1,$$

are known then one can use the restriction of the initial data at \mathcal{I} , $\phi_*^{[p]} = \phi^{[p]}(0, s)$ to solve for $\phi^{[p]} = \phi^{[p]}(\tau, s)$. The result can be collected in an expansion

$$\phi \simeq \sum_{j=0}^p \frac{1}{j!} \phi^{[j]} \rho^j,$$

where the symbol \simeq has been used to indicate the formal character of the expansion. The relation of these expansion to actual solutions to the conformal Einstein field equations is a challenging open problem.

3.4.1 Solutions in the $\kappa = \omega$ gauge

The solutions of the transport equations to (13) for the orders $p = 0, 1, 2$ with initial data satisfying Assumption 1 have been studied in [20] in a conformal gauge for which $\kappa = \rho$. Solutions for the orders $p = 3, 4, 5$ have been computed in [39]. The conformal gauge for which $\kappa = \rho$ was used in these references as it renders simpler expressions. In the present article we are interested in analysing expansions near null infinity. For this, a conformal gauge for which

$$\kappa = \omega$$

is more convenient as the location of future null infinity is given by the simple condition $\tau = 1$. Calculations in this gauge have been reported in [39] —see also [41]. In the rest of this section we provide a summary of the properties of these expansions which will be required in the remainder of the article.

Proposition 1. *For initial data satisfying Assumption 1, the solutions to the transport equation for the components of the rescaled Weyl tensor, equation (13), satisfy:*

- (i) *At orders $p = 0$ and $p = 1$ the solutions have polynomial dependence in τ .*
- (ii) *At order $p = 2$ the solutions have polynomial dependence in τ if and only if the condition*

$$b_{ij}(i) = 0$$

is satisfied by the Bach tensor of the conformal metric \mathbf{h} . If $b_{ij}(i) \neq 0$ the solution develops logarithmic singularities at $\tau = \pm 1$.

- (iii) *At order $p = 3$ the solutions have polynomial dependence if and only if*

$$b_{ij}(i) = 0, \quad D_{\{i} b_{jk\}}(i) = 0.$$

If $D_{\{i} b_{jk\}}(i) \neq 0$ then the solution develops logarithmic singularities at $\tau = \pm 1$.

(iv) At order $p = 4$ the solutions have polynomial dependence if and only if

$$b_{ij}(i) = 0, \quad D_{\{i}b_{jk\}}(i) = 0, \quad D_{\{i}D_j b_{kl\}}(i) = 0.$$

If $D_{\{i}D_j b_{kl\}}(i) \neq 0$ then the solution develops logarithmic singularities at $\tau = \pm 1$.

(v) At order $p = 5$ if

$$b_{ij}(i) = 0, \quad D_{\{i}b_{jk\}}(i) = 0, \quad D_{\{i_2}D_{i_1}b_{jk\}}(i) = 0, \quad D_{\{i_3}D_{i_2}D_{i_1}b_{jk\}}(i) = 0,$$

then the solution develops logarithmic singularities at $\tau = \pm 1$.

These qualitative properties of the solutions to the transport equations are independent of whether one uses the $\kappa = \rho$ or the $\kappa = \omega$ gauge.

Detailed expressions for the independent components of the spinor ϕ_{ABCD} up to order $p = 2$ in the gauge $\kappa = \omega$ are given in Appendix A. We notice, however, that

$$\phi_0^{[2]} = \mathfrak{b}_0(1 + \tau)^4 \left(\ln(1 - \tau) - \ln(1 + \tau) \right) + \check{\phi}_0^{[2]}, \quad (14a)$$

$$\phi_1^{[2]} = \mathfrak{b}_1(1 - \tau)(1 + \tau)^3 \left(\ln(1 - \tau) - \ln(1 + \tau) \right) + \check{\phi}_1^{[2]}, \quad (14b)$$

$$\phi_2^{[2]} = \mathfrak{b}_2(1 - \tau)^2(1 + \tau)^2 \left(\ln(1 - \tau) - \ln(1 + \tau) \right) + \check{\phi}_2^{[2]}, \quad (14c)$$

$$\phi_3^{[2]} = \mathfrak{b}_3(1 - \tau)^3(1 + \tau) \left(\ln(1 - \tau) - \ln(1 + \tau) \right) + \check{\phi}_3^{[2]}, \quad (14d)$$

$$\phi_4^{[2]} = \mathfrak{b}_4(1 - \tau)^4 \left(\ln(1 - \tau) - \ln(1 + \tau) \right) + \check{\phi}_4^{[2]}, \quad (14e)$$

where

$$\mathfrak{b}_k(i) \equiv b_{(ABCD)_k}(i)$$

denotes the independent components of the Bach spinor (i.e. spinorial counterpart of the 3-dimensional tensor b_{ij}) evaluated at the point at infinity and $\check{\phi}_0^{[2]}, \dots, \check{\phi}_4^{[2]}$ denote expressions with polynomial dependence (hence smooth) on τ .

Remark 14. Observe that the coefficient $\phi_0^{[2]}$ is the most singular one at $\tau = 1$ (i.e. \mathcal{I}^+), not being even continuous, while $\phi_4^{[2]}$ is the most regular having 4 derivatives at $\tau = 1$. The role at $\tau = -1$ is reversed.

Remark 15. That the structure of the logarithmic singularities in (14a)-(14e) suggest that, generically, the development of initial data satisfying Assumption 1 does not admit a C^2 conformal extension. This observation is likely to hold also for more general classes of initial data.

In order to be able to relate the formal F-gauge expansions discussed in the previous paragraphs we need to make the following assumption:

Assumption 2. *The formal F-gauge expansions correspond to the leading orders of an actual solution to the extended conformal Einstein field equations. In particular, for the components of the rescaled Weyl tensor, one has that*

$$\phi = \sum_{j=0}^2 \frac{1}{j!} \phi^{[j]} \rho^j + \mathbf{R}_3$$

where the remainder satisfies $\mathbf{R}_3 \in C^\infty(\tilde{\mathcal{M}}_{a,\kappa}) \cap C^0(\mathcal{M}_{a,\kappa})$ and $\mathbf{R}_3 = O(\rho^3)$. Similarly, for the components of the frame one has

$$e = \sum_{j=0}^2 \frac{1}{j!} e^{[j]} \rho^j + \mathbf{R}_3.$$

Remark 16. As already mentioned in the introduction, controlling the residue of the F-expansions by means of estimates obtained from the conformal evolution equations is the major outstanding issue in the so-called *problem of spatial infinity*.

4 The NP gauge

While the F-gauge expressions discussed in the previous section provide a great deal of information about the singular behaviour of the Weyl tensor at the conformal boundary of the development of the class of initial data of Assumption 1, they are given in a gauge which does not directly lead to assertions about the peeling (or lack thereof) behaviour of the spacetime. For this, one has to transform into a gauge hinged on (future) null infinity —the so-called Newman-Penrose (NP) gauge.

In the following we use the notation \simeq to denote equality at \mathcal{I}^+ .

4.1 Construction of the gauge

In what follows consider a conformal extension (\mathcal{M}, g', Ξ) of a *suitably* asymptotically spacetime satisfying the vacuum Einstein equations containing at least a piece of future null infinity. A frame $\{e'_{AA'}\}$ satisfying $g'(e'_{AA'}, e'_{BB'}) = \epsilon_{AB}\epsilon_{A'B'}$ defined in a neighbourhood \mathcal{U} of \mathcal{I}^+ is said to be *adapted to \mathcal{I}^+* if:

- (i) The vector $e'_{11'}$ is tangent to \mathcal{I}^+ and is parallelly propagated along its generators.
- (ii) On \mathcal{U} there exists a function u (a *retarded time*) which is an affine parameter of the generators of \mathcal{I}^+ satisfying $e'_{11'}(u) \simeq 1$. The function u is propagated off the conformal boundary by requiring it to be constant on null hypersurfaces transverse to \mathcal{I}^+ and satisfies $e'_{00'} = g'^{\sharp}(\mathbf{d}u, \cdot)$ —thus, $e'_{00'}$ is tangent to the hypersurfaces \mathcal{N}_{u_\bullet} defined by the condition $u = u_\bullet$ where u_\bullet is a constant.
- (iii) The fields are tangent to the cuts $\mathcal{C}_{u_\bullet} \equiv \mathcal{N}_{u_\bullet} \cap \mathcal{I}^+$ and parallelly propagated along the direction of $e'_{00'}$.

The conditions (i)-(iii) imply restrictions on the form of several components of the spin connection coefficients. Exploiting the conformal freedom in the choice of the conformal factor Ξ , the frame can be refined even further to fix the values of certain components of the tracefree Ricci tensor of the metric g' . More precisely, one has the following:

Proposition 2. *Given a suitably asymptotically simple spacetime, locally, it is always possible to find a conformal extension (\mathcal{M}', g', Ξ) for which*

$$R[g'] \simeq 0,$$

and an adapted frame $\{e'_{AA'}\}$ such that the associated spin connection coefficients $\Gamma'_{AA'BC}$ satisfy

$$\begin{aligned} \Gamma'_{00'BC} &\simeq 0, & \Gamma'_{11'BC} &\simeq 0, \\ \Gamma'_{01'11} &\simeq 0, & \Gamma'_{10'00} &\simeq 0, & \Gamma'_{1011} &\simeq 0, \\ \bar{\Gamma}'_{1'00'1'} + \Gamma'_{01'01} &\simeq 0. \end{aligned}$$

In addition, one has that

$$\Phi'_{12} \simeq 0, \quad \Phi'_{22} \simeq 0,$$

and $e'_{00'}(\Xi)$ is constant on \mathcal{I}^+ .

The construction summarised in the previous proposition is supplemented by adapted coordinates. On a fiduciary cut $\mathcal{C}_\star \approx \mathbb{S}^2$ one chooses some coordinates $\theta = (\theta^A)$, $A = 2, 3$ and extends them along \mathcal{I}^+ by requiring them to be constant along the null generators. On the hypersurfaces \mathcal{N}_u transverse to \mathcal{I}^+ one identifies an affine parameter r of the null generators of these hypersurfaces in such a way that $e'_{00'}(r) = 1$ and $r \simeq 0$. Finally, the coordinates $\theta = (\theta^A)$ are propagated off \mathcal{I}^+ so that they are constant along the generators of \mathcal{N}_u . From this construction one obtains *unphysical Bondi coordinates* $x = (u, r, \theta^A)$ in a neighbourhood of \mathcal{I}^+ . *Physical Bondi coordinates* are obtained with the inversion $\tilde{r} = 1/r$.

4.2 Basic transformation formulae

The relation between the F-gauge and the NP-gauge has been analysed in [25] —the reader is referred to this article for full details on this construction. The translation from the F-gauge to the NP gauge involves:

- (i) a conformal rescaling of the metric,

$$g' = \varkappa^2 g \tag{15}$$

where \varkappa is a suitable conformal factor;

- (ii) a Lorentz transformation and rescaling of the frame $\{e_{AA'}\}$ of the form

$$e'_{AA'} = \varkappa^{-1} \Lambda^B{}_A \bar{\Lambda}^{B'}{}_{A'} e_{BB'}, \tag{16}$$

where $(\Lambda^B{}_A) \in SL(2, \mathbb{C})$;

- (iii) a change of coordinates

$$u = u(\tau, \rho, s), \quad r = r(\tau, \rho, s)$$

where u is a suitable retarded time on \mathcal{I}^+ and r is an affine parameter the generators of outgoing light cones such that $r|_{\mathcal{I}^+} = 0$.

As a consequence of relation (16), the spin dyads $\{\epsilon_A{}^A\}$ and $\{\epsilon'_{A'}{}^{A'}\}$, associated to the frames $\{e_{AA'}\}$ and $\{e'_{AA'}\}$, respectively, are related to each other via

$$\epsilon'_{A'}{}^{A'} = \varkappa^{-1/2} \Lambda^B{}_A \epsilon_B{}^A. \tag{17}$$

From the discussion in [25] it readily follows that:

Proposition 3. *For the development of initial data satisfying Assumption 1 and under Assumption 2 it follows that:*

$$\Lambda^0{}_1 = \rho^{1/2} + O(\rho^{7/2}, (1 - \tau)), \tag{18a}$$

$$\Lambda^1{}_1 = O(\rho^{5/2}, (1 - \tau)), \tag{18b}$$

$$\Lambda^0{}_0 = O(\rho^{3/2}, (1 - \tau)) \tag{18c}$$

$$\Lambda^1{}_0 = -\rho^{-1/2} + O(\rho^{3/2}, (1 - \tau)), \tag{18d}$$

and

$$\varkappa = 1 + O(\rho, (1 - \tau)).$$

Moreover, one has that

$$u = \sqrt{2} \left(-\frac{1}{\rho} + 4m \ln \rho + u_\star \right) + O(\rho, (1 - \tau)),$$

$$r = (1 - \tau) + O(\rho, (1 - \tau)^2).$$

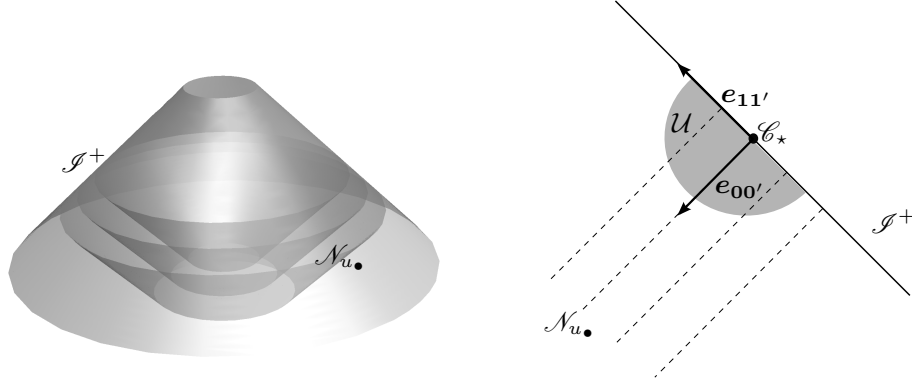


Figure 2: Schematic depiction of the NP gauge. Left: a foliation of outgoing null hypersurfaces near future null infinity. Right: the vectors of the adapted frame. Figure taken from Figure 10.2 in page 232 of [43]. ©Cambridge University Press. Reprinted with permission.

Remark 17. Observe that $\varkappa \neq 0$ on \mathcal{I}^+ , thus, the NP gauge used in the previous discussion is expressed in the unphysical spacetime.

4.3 Transformation formulae for the rescaled Weyl spinor

Consistent with equation (15) one has that, the rescaled Weyl spinor associated with the \mathbf{g}' and \mathbf{g} representation denoted respectively as ϕ'_{ABCD} and ϕ_{ABCD} , are related to each other via

$$\phi'_{ABCD} = \varkappa^{-1} \phi_{ABCD}.$$

Transvecting this last equation with $\epsilon_{\mathbf{A}}^A$ and using equation (17) one concludes that the relation between the components of the rescaled Weyl spinors, respect to the NP and F-frames, is given by

$$\phi'_{ABCD} = \varkappa^{-3} \Lambda^F_A \Lambda^H_B \Lambda^P_C \Lambda^Q_D \phi_{FHPQ}. \quad (19)$$

The components of the rescaled Weyl spinor ϕ_{ABCD} in the F-gauge will be expressed through contractions with the spin dyad $\epsilon_{\mathbf{A}}^A$. Following the standard conventions of [32, 35] one defines:

$$\begin{aligned} \phi_0 &\equiv o^A o^B o^C o^D \phi_{ABCD}, \\ \phi_1 &\equiv \iota^A o^B o^C o^D \phi_{ABCD}, \\ \phi_2 &\equiv \iota^A \iota^B o^C o^D \phi_{ABCD}, \\ \phi_3 &\equiv \iota^A \iota^B \iota^C o^D \phi_{ABCD}, \\ \phi_4 &\equiv \iota^A \iota^B \iota^C \iota^D \phi_{ABCD}. \end{aligned}$$

so that ϕ_{ABCD} can be expressed as

$$\phi_{ABCD} = \phi_0 \iota^A \iota^B \iota^C \iota^D - 4\phi_1 \iota^A \iota^B \iota^C o^D + 6\phi_2 \iota^A \iota^B o^C o^D - 4\phi_3 \iota^A o^B o^C o^D + \phi_4 o^A o^B o^C o^D.$$

Expanding equation (19) and taking into the above introduced notation one obtains

$$\begin{aligned} \phi'_0 &= \varkappa^{-3} \left((\Lambda^0_{\mathbf{0}})^4 \phi_0 + 4(\Lambda^0_{\mathbf{0}})^3 (\Lambda^1_{\mathbf{0}}) \phi_1 + 6(\Lambda^0_{\mathbf{0}})^2 (\Lambda^1_{\mathbf{0}})^2 \phi_2 + 4(\Lambda^0_{\mathbf{0}}) (\Lambda^1_{\mathbf{0}})^3 \phi_3 + (\Lambda^1_{\mathbf{0}})^4 \phi_4 \right), \\ \phi'_1 &= \varkappa^{-3} \left((\Lambda^0_{\mathbf{0}})^3 (\Lambda^0_{\mathbf{1}}) \phi_0 + 3(\Lambda^0_{\mathbf{0}})^2 (\Lambda^0_{\mathbf{1}}) (\Lambda^1_{\mathbf{0}}) \phi_1 + (\Lambda^0_{\mathbf{0}})^3 (\Lambda^1_{\mathbf{1}}) \phi_1 \right) \end{aligned}$$

$$\begin{aligned}
& +3(\Lambda^0_{\mathbf{0}})(\Lambda^0_{\mathbf{1}})(\Lambda^1_{\mathbf{0}})^2\phi_2 + 3(\Lambda^0_{\mathbf{0}})^2(\Lambda^1_{\mathbf{0}})(\Lambda^1_{\mathbf{1}})\phi_2 + (\Lambda^0_{\mathbf{1}})(\Lambda^1_{\mathbf{0}})^3\phi_3 \\
& +3(\Lambda^0_{\mathbf{0}})(\Lambda^1_{\mathbf{0}})^2(\Lambda^1_{\mathbf{1}})\phi_3 + (\Lambda^1_{\mathbf{0}})^3(\Lambda^1_{\mathbf{1}})\phi_4), \\
\phi'_2 = \varkappa^{-3} & \left((\Lambda^0_{\mathbf{0}})^2(\Lambda^0_{\mathbf{1}})^2\phi_0 + 2(\Lambda^0_{\mathbf{0}})(\Lambda^0_{\mathbf{1}})^2(\Lambda^1_{\mathbf{0}})\phi_1 + 2(\Lambda^0_{\mathbf{0}})^2(\Lambda^0_{\mathbf{1}})(\Lambda^1_{\mathbf{1}})\phi_1 \right. \\
& + (\Lambda^0_{\mathbf{1}})^2(\Lambda^1_{\mathbf{0}})^2\phi_2 + 4(\Lambda^0_{\mathbf{0}})(\Lambda^0_{\mathbf{1}})(\Lambda^1_{\mathbf{0}})(\Lambda^1_{\mathbf{1}})\phi_2 + (\Lambda^0_{\mathbf{0}})^2(\Lambda^1_{\mathbf{1}})^2\phi_2 \\
& \left. + 2(\Lambda^0_{\mathbf{1}})(\Lambda^1_{\mathbf{0}})^2(\Lambda^1_{\mathbf{1}})\phi_3 + 2(\Lambda^0_{\mathbf{0}})(\Lambda^1_{\mathbf{0}})(\Lambda^1_{\mathbf{1}})^2\phi_3 + (\Lambda^1_{\mathbf{0}})^2(\Lambda^1_{\mathbf{1}})^2\phi_4 \right), \\
\phi'_3 = \varkappa^{-3} & \left((\Lambda^0_{\mathbf{0}})(\Lambda^0_{\mathbf{1}})^3\phi_0 + (\Lambda^0_{\mathbf{1}})^3(\Lambda^1_{\mathbf{0}})\phi_1 + 3(\Lambda^0_{\mathbf{0}})(\Lambda^0_{\mathbf{1}})^2(\Lambda^1_{\mathbf{1}})\phi_1 \right. \\
& + 3(\Lambda^0_{\mathbf{1}})^2(\Lambda^1_{\mathbf{0}})(\Lambda^1_{\mathbf{1}})\phi_2 + 3(\Lambda^0_{\mathbf{0}})(\Lambda^0_{\mathbf{1}})(\Lambda^1_{\mathbf{1}})^2\phi_2 + 3(\Lambda^0_{\mathbf{1}})(\Lambda^1_{\mathbf{0}})(\Lambda^1_{\mathbf{1}})^2\phi_3 \\
& \left. + (\Lambda^0_{\mathbf{0}})(\Lambda^1_{\mathbf{1}})^3\phi_3 + (\Lambda^1_{\mathbf{0}})(\Lambda^1_{\mathbf{1}})^3\phi_4 \right), \\
\phi'_4 = \varkappa^{-3} & \left((\Lambda^0_{\mathbf{1}})^4\phi_0 + 4(\Lambda^0_{\mathbf{1}})^3(\Lambda^1_{\mathbf{1}})\phi_1 + 6(\Lambda^0_{\mathbf{1}})^2(\Lambda^1_{\mathbf{1}})^2\phi_2 + 4(\Lambda^0_{\mathbf{1}})(\Lambda^1_{\mathbf{1}})^3\phi_3 + (\Lambda^1_{\mathbf{1}})^4\phi_4 \right).
\end{aligned}$$

Finally, to obtain to compute the (physical) components of the Weyl tensor we observe that

$$\mathbf{g}' = r^2 \tilde{\mathbf{g}} \quad (21)$$

where r is the (unphysical) Bondi radial coordinate. By definition one has then that

$$\phi'_{ABCD} = r \Psi_{ABCD}$$

where Ψ_{ABCD} is the (conformally invariant) Weyl spinor. Consistent with the rescaling (21) one sets

$$o'^A = r \tilde{o}^A, \quad \iota'^A = \tilde{\iota}^A. \quad (22)$$

The physical Bondi radial coordinate \tilde{r} is obtained from r via the inversion

$$\tilde{r} = 1/r.$$

It follows from the above that the components $\tilde{\psi}_0, \dots, \tilde{\psi}_4$ of Ψ_{ABCD} with respect to the spin dyad $\{\tilde{o}^A, \tilde{\iota}^A\}$ are given by

$$\tilde{\psi}_0 = \frac{1}{\tilde{r}^3} \phi'_0, \quad \tilde{\psi}_1 = \frac{1}{\tilde{r}^2} \phi'_1, \quad \tilde{\psi}_2 = \frac{1}{\tilde{r}} \phi'_2, \quad \tilde{\psi}_3 = \phi'_3, \quad \tilde{\psi}_4 = \tilde{r} \phi'_4.$$

Remark 18. Observe that the rescaling of the spin dyad given in (22) is asymmetric in o^A and ι^A —this choice is customary in the discussion of the peeling behaviour.

5 Main results

In this section we provide the main results of our analysis: expressions for the asymptotic decay of the components of the Weyl tensor dictated by the F-expansions under the premise that these expansions are related to actual solutions to the conformal Einstein field equations as stated in Assumption 2. These results bring together the discussion in Sections 3 and 4. The statements in this section are obtained via a direct computation involving the transformation formulae of Section 4.3, the explicit expansions of Proposition 3 and the solutions to the transport equations as given by Proposition 1.

5.1 Decay near null infinity

The main result in this article is the following:

Theorem 2. *Under Assumption 2, given time symmetric initial data satisfying Assumption 1, the F -expansions imply that*

$$\begin{aligned}\tilde{\psi}_0 &= O(\tilde{r}^{-3} \ln \tilde{r}), \\ \tilde{\psi}_1 &= O(\tilde{r}^{-3} \ln \tilde{r}), \\ \tilde{\psi}_2 &= O(\tilde{r}^{-3} \ln \tilde{r}), \\ \tilde{\psi}_3 &= O(\tilde{r}^{-2}), \\ \tilde{\psi}_4 &= O(\tilde{r}^{-1}).\end{aligned}$$

Remark 19. A general framework for the discussion of polyhomogeneous spacetimes with the above asymptotics is given in [15]. Polyhomogeneous spacetimes satisfying the above decay have been studied in [38] where a number of their properties are discussed—in particular the existence of so-called *logarithmic Newman-Penrose constants*.

Remark 20. It is worth recalling that, for the purposes of the discussion of peeling properties, the key component is $\tilde{\psi}_0$ —i.e. the most singular one. Making use of an asymptotic characteristic initial value problem it is possible to deduce the decay of the other components if that of $\tilde{\psi}_0$ is prescribed on a fiduciary outgoing light cone. Using these methods it is possible to show that the Einstein field equations are, in fact, consistent with a decay of the form $\tilde{\psi}_0 = O(\tilde{r}^{-3} \ln^N \tilde{r})$ with N some positive integer—see [36].

For more restricted classes of initial data one has the following:

Theorem 3. *Under Assumption 2, given time symmetric initial data satisfying Assumption 1, the F -expansions are such that:*

(i) *If*

$$b_{ij}(i) = 0,$$

then

$$\begin{aligned}\tilde{\psi}_0 &= O(\tilde{r}^{-4} \ln \tilde{r}), \\ \tilde{\psi}_1 &= O(\tilde{r}^{-4} \ln \tilde{r}), \\ \tilde{\psi}_2 &= O(\tilde{r}^{-3}), \\ \tilde{\psi}_3 &= O(\tilde{r}^{-2}), \\ \tilde{\psi}_4 &= O(\tilde{r}^{-1}).\end{aligned}$$

(ii) *If*

$$b_{ij}(i) = 0, \quad D_{\{k} b_{ij\}}(i) = 0,$$

then

$$\begin{aligned}\tilde{\psi}_0 &= O(\tilde{r}^{-5} \ln \tilde{r}), \\ \tilde{\psi}_1 &= O(\tilde{r}^{-4}), \\ \tilde{\psi}_2 &= O(\tilde{r}^{-3}), \\ \tilde{\psi}_3 &= O(\tilde{r}^{-2}), \\ \tilde{\psi}_4 &= O(\tilde{r}^{-1}).\end{aligned}$$

(iii) *The classical peeling behaviour is obtained if*

$$b_{ij}(i) = 0, \quad D_{\{k} b_{ij\}}(i) = 0, \quad D_{\{k} D_l b_{ij\}}(i) = 0.$$

Remark 21. Polyhomogeneous spacetimes with asymptotics of the form (i) and (ii) in the previous result have been discussed in [36]. The relation to the *outgoing radiation condition* of Bondi et al. [11, 33] has been analysed in [37]. The decay in (i) includes the *minimal* polyhomogeneous spacetimes of [15].

Remark 22. The peeling spacetimes of (iii) are not, generically, smooth at the conformal boundary. In fact, they will exhibit the logarithmic singularities first observed in [40, 39] —see also (iv) in Proposition 1. These will appear at order $O(\tilde{r}^{-6})$. The developments of explicit time symmetric initial data sets like those of Brill-Lindquist and Misner data should have this type of asymptotics —i.e. peeling but with a conformal boundary of finite differentiability.

5.2 Behaviour on null infinity

The methods discussed in the previous sections also allow us to analyse the behaviour of the rescaled Weyl tensor on \mathcal{I}^+ near spatial infinity. Observe that the restriction of ϕ'_{ABCD} corresponds, essentially, to the leading terms of $\tilde{\psi}_{ABCD}$.

Theorem 4. *Under Assumption 2, given time symmetric initial data satisfying Assumption 1, the F -expansions imply that:*

- (i) *If $b_{ij}(i) \neq 0$ then $\phi'_{ABCD}|_{\mathcal{I}^+}$ are singular.*
- (ii) *If $b_{ij}(i) = 0$ then one has that*

$$\phi'_0 \simeq O(1), \quad \phi'_1 \simeq O(1) \quad \phi'_2 \simeq O(1) \quad \phi'_3 \simeq O(\rho^3) \quad \phi'_4 \simeq O(\rho^3).$$

6 Construction of candidate spacetimes with non-peeling behaviour

The purpose of this section is to construct an explicit family of global spacetimes which are candidates for not satisfying the peeling behaviour. In a first step we construct a family of asymptotically Euclidean time symmetric initial data sets which can be regarded as perturbations of data for the Minkowski spacetime and such that $\mathfrak{b}_k \neq 0$. In a second step, we make use of Bieri's generalisation of Christodoulou & Klainerman's global existence and stability result [9] to guarantee the existence of a geodesically complete maximal hyperbolic development of the members of the family of initial data sets.

6.1 Time symmetric initial data with prescribed asymptotics

In this subsection we construct a family of time symmetric, asymptotically Euclidean vacuum initial data sets such that $\mathfrak{b}_k \neq 0$ —hence the regularity condition (2) is violated at order zero. For simplicity, in what follows we consider time symmetric initial data sets with one asymptotic end.

6.1.1 Construction of a suitable conformal metric

We begin by recalling the connection between the Bach tensor and normal expansions. As it is well known (see e.g. [43] Section 11.6.2), it is always possible to find, locally, a conformal gauge such that the conformal metric \mathbf{h} admits in a neighbourhood of i , in terms of normal coordinates $\underline{x} = (x^\alpha)$, the expansion

$$h_{\alpha\beta} = \delta_{\alpha\beta} - \frac{1}{6} \partial_\eta r_{\alpha\gamma\beta\delta}(i) x^\gamma x^\delta x^\eta + O(|x|^4). \quad (23)$$

The relation between the value of the derivative of the Riemann tensor of the metric \mathbf{h} at i and the value of the Bach tensor is best discussed using spinors. Let $r_{AGBHCDEF}$ denote the (space) spinor counterpart of the 3-dimensional Riemann tensor $r_{\alpha\gamma\beta\delta}$. As a consequence of the symmetries of the Riemann tensor, its spinorial counterpart admits the decomposition

$$r_{AGBHCDEF} = -r_{ABCDEF}\epsilon_{GH} - r_{GHCDEF}\epsilon_{AB}$$

where

$$r_{ABCDEF} = \left(\frac{1}{2}s_{ABCE} - \frac{1}{12}rh_{ABCE}\right)\epsilon_{DF} + \left(\frac{1}{2}s_{ABDF} - \frac{1}{12}h_{ABDF}\right)\epsilon_{CE},$$

and $s_{ABCD} = s_{(ABCD)}$ denotes the spinorial counterpart of the tracefree Ricci tensor and r is the Ricci scalar. The spinor s_{ABCD} is related to the Bach spinor b_{ABCD} via

$$b_{ABCD} = D^Q{}_{(A} s_{BCD)Q}.$$

Now, from the expansion (23) it follows that

$$r(i) = 0, \quad s_{ABCD}(i) = 0.$$

A computation then shows that

$$D_{AB} s_{CDEF} = -\frac{5}{6}(\epsilon_A(Cb_{BDEF}) + \epsilon_B(Cb_{ADEF})) \quad \text{at } i.$$

Thus, if there is at least one non-vanishing essential component of $b_{ABCD}(i)$ —say \mathbf{b}_0 , then $(D_{AB} s_{CDEF})(i) \neq 0$ and accordingly, $(\partial_\eta r_{\alpha\gamma\beta\delta})(i) \neq 0$.

Let now, $\mathfrak{z}_{\eta\alpha\gamma\beta\delta}$ denote real constants, not all of them vanishing, satisfying the same symmetries with respect to the indices $\alpha, \beta, \gamma, \delta, \eta$ as the covariant derivative of the Riemann tensor. Moreover, let $\check{\mathbf{h}}$ denote the standard metric of \mathbb{S}^3 . The previous discussion motivates the following:

Lemma 1. *Let \mathcal{S} denote a compact 3-manifold with $\mathcal{S} \approx \mathbb{S}^3$. Given constants $\mathfrak{z}_{\eta\alpha\gamma\beta\delta}$ as above, there exists a smooth Riemannian metric $\check{\mathbf{h}}$ together with a smooth function ω over \mathcal{S} and an $\varepsilon > 0$ such that:*

- (i) *there exists a point $i \in \mathcal{S}$ such that in terms of normal coordinates $\underline{x} = (x^\alpha)$ centred at i the metric $\check{\mathbf{h}}$ takes the form*

$$\check{h}_{\alpha\beta} = \delta_{\alpha\beta} + \mathfrak{z}_{\eta\alpha\gamma\beta\delta} x^\gamma x^\delta x^\eta \quad (24)$$

in a $\check{\mathbf{h}}$ -open ball of radius ε , $\mathcal{B}_\varepsilon(i)$, centred at i ;

- (ii) *$\check{\mathbf{h}} = \omega^2 \mathbf{h}$ on $\mathcal{S} \setminus \mathcal{B}_{2\varepsilon}(i)$;*

- (iii) *the metric $\check{\mathbf{h}}$ has positive Yamabe number $Y(\check{\mathbf{h}})$;*

- (iv) *one has that $\check{\mathbf{h}} \rightarrow \omega^2 \mathbf{h}$ as $\varepsilon \rightarrow 0$ in the C^2 topology of smooth metrics.*

Proof. The result is a direct application of Lemma 2.1 in [20]. □

Remark 23. From the expansion (23) one has that for $\check{\mathbf{h}}$ as given above,

$$\partial_\eta \check{r}_{\alpha\gamma\beta\delta}(i) = -6\mathfrak{z}_{\eta\alpha\gamma\beta\delta}.$$

Thus, as a consequence of (24) the metric $\check{\mathbf{h}}$ is not conformally flat in a small enough neighbourhood of i . Moreover, it is clear that the constants $\mathfrak{z}_{\eta\alpha\gamma\beta\delta}$ can be chosen so that $\mathbf{b}_0 \neq 0$ to that the component of the Bach tensor giving rise to the most singular behaviour of the rescaled Weyl tensor at the critical set \mathcal{I}^+ is non-vanishing.

6.1.2 Solving the time symmetric Hamiltonian constraint

The family of metrics given by Lemma 1 will now be used to construct time symmetric initial data sets for the Einstein field equations via the *method of punctures* —see [20, 7]. Accordingly, we look for a conformal factor Ω such that $\Omega \in C^2(\mathcal{S}) \cap C^\infty(\mathcal{S} \setminus \{i\})$ satisfying the properties

$$\Omega = 0, \quad \mathbf{d}\Omega = 0, \quad \mathbf{Hess}\Omega = -2\check{\mathbf{h}} \quad \text{at } i, \quad (25a)$$

$$\Omega > 0 \quad \text{on } \mathcal{S} \setminus \{i\}, \quad (25b)$$

and

$$r[\Omega^{-2}\check{\mathbf{h}}] = 0. \quad (26)$$

Setting

$$\Omega = \vartheta^{-2},$$

the time symmetric Hamiltonian constraint (26) yields the Yamabe equation

$$\left(\Delta_{\check{\mathbf{h}}} - \frac{1}{8}r[\check{\mathbf{h}}]\right)\vartheta = 0. \quad (27)$$

Consistent with conditions (25a)-(25b) we impose the boundary condition

$$\vartheta|x| \rightarrow 1 \quad \text{as } x \rightarrow 0. \quad (28)$$

One then has the following:

Proposition 4. *Let $\check{\mathbf{h}}$ and $\varepsilon > 0$ be as given by Lemma 1. There exists a unique solution ϑ to the Yamabe equation (27) with boundary conditions (28). In a suitably neighbourhood of i the function ϑ can be written as*

$$\vartheta = \frac{U}{|x|} + W$$

where U and W are analytic at i and satisfy

$$U = 1 + O(|x|^4), \quad W(i) = \frac{m}{2}.$$

The function ϑ depends smoothly on $\check{\mathbf{h}}$ and, in particular,

$$\vartheta \rightarrow \frac{1}{|x|} \quad \text{as } \varepsilon \rightarrow 0,$$

so that

$$\vartheta^4\check{\mathbf{h}} \rightarrow \delta \quad \text{as } \varepsilon \rightarrow 0,$$

where δ denotes the 3-dimensional Euclidean metric.

Proof. The positivity of the Yamabe invariant of $\check{\mathbf{h}}$ ensures the solvability of equation (27) subject to the condition (28). A detailed discussion of the construction of the solution ϑ can be found in [20], Section 2.2. Observe that in the limit $\varepsilon \rightarrow 0$ one has that $m = 0$ so from the rigidity part of the mass positivity theorem [34] the limiting initial data must be data for the Minkowski spacetime. \square

Remark 24. The limit $\varepsilon \rightarrow 0$ is independent of the choice of constants $\mathfrak{z}_{\eta\alpha\gamma\beta\delta}$.

Remark 25. The decompactification obtained via the conformal factor ϑ gives rise to a metric $\tilde{\mathbf{h}} \equiv \vartheta^4\check{\mathbf{h}}$ defined over $\tilde{\mathcal{S}} \approx \mathbb{R}^3$ solving the time symmetric Hamiltonian constraint. By choosing ε small enough one can make $\tilde{\mathbf{h}}$ suitably close to the 3-dimensional Euclidean metric.

Remark 26. The procedure describe in the previous paragraphs can be readily adapted to construct initial data sets for which $b_{ABCD}(i) = 0$ but $D_{(EF}b_{ABCD)}(i) \neq 0$.

Remark 27. A tedious calculation shows that if $b_{ABCD}(i) = 0$ then the Bach tensor of the metric $\tilde{\mathbf{h}} = \vartheta^4 \check{\mathbf{h}}$, expressed in terms of asymptotically Cartesian coordinates $\underline{y} = (y^\alpha)$ in the asymptotic end of $\tilde{\mathcal{S}}$ satisfies

$$b_{\alpha\beta} = \frac{\mathbf{b}_{\alpha\beta}}{|y|^4} + O(|y|^5), \quad (29)$$

where $\mathbf{b}_{\alpha\beta}$ are constants, not all of them zero.

6.2 Global existence of a candidate non-peeling spacetimes

In a next step we make use, for suitably small $\varepsilon > 0$, of Bieri's global existence and stability result of [9] to construct the maximal globally hyperbolic development of the metric $\tilde{\mathbf{h}}$ of Proposition 4. The resulting spacetime will be a candidate spacetime for exhibiting non-peeling behaviour.

The analysis in [9] considers initial data sets $(\tilde{\mathcal{S}}, \tilde{\mathbf{h}}, \tilde{\mathbf{K}})$, $\tilde{\mathcal{S}} \approx \mathbb{R}^3$, which outside a compact set admit asymptotically Euclidean coordinates $\underline{x} = (x^\alpha)$ such that:

$$\tilde{h}_{\alpha\beta} = \delta_{\alpha\beta} + O_{H^3}(|x|^{-1/2}), \quad (30a)$$

$$\tilde{K}_{\alpha\beta} = O_{H^2}(|x|^{-3/2}). \quad (30b)$$

² The metric $\tilde{\mathbf{h}}$ obtained in Proposition 4 can be readily seen to satisfy the above asymptotic conditions —in particular $\tilde{K}_{\alpha\beta} = 0$.

In what follows $x_\odot \in \tilde{\mathcal{S}}$ is an arbitrary origin in the (physical) hypersurface $\tilde{\mathcal{S}}$. Let d_\odot denote the distance function from the origin x_\odot . The global smallness assumption in Bieri's global existence result makes use of the quantity

$$Q(x_\odot, a) \equiv a^{-1} \int_{\mathbb{R}^3} \left(|K|^2 + (a^2 + d_\odot^2) |DK|^2 + (a^2 + d_\odot^2)^2 |D^2K|^2 \right. \\ \left. + (a^2 + d_\odot^2) |\text{Ric}[\tilde{\mathbf{h}}]|^2 + (a^2 + d_\odot^2)^2 |DRic[\tilde{\mathbf{h}}]|^2 \right) d\mu.$$

with $a \in \mathbb{R}^+$ and where $\text{Ric}[\tilde{\mathbf{h}}]$ denotes the Ricci tensor of the metric $\tilde{\mathbf{h}}$. The global existence and stability result in [9] can be phrased as —see [10]:

Theorem 5. *There exists $\epsilon > 0$ such that for all smooth vacuum initial data sets with (30a)-(30b), $K_\alpha^\alpha = 0$ and satisfying*

$$\inf_{x_\odot \in \mathbb{R}^3, a > 0} Q(x_\odot, a) < \epsilon,$$

the associated maximal globally hyperbolic vacuum development is geodesically complete, with the metric asymptotically approaching the Minkowski metric in all directions.

Now, as $\tilde{\mathbf{h}} \rightarrow \boldsymbol{\delta}$ for $\varepsilon \rightarrow 0$, and $\inf_{x_\odot \in \mathbb{R}^3, a > 0} Q(x_\odot, a) = 0$ for time symmetric Minkowski data, one can choose the $\varepsilon > 0$ in Proposition 4 so that $\inf_{x_\odot \in \mathbb{R}^3, a > 0} Q(x_\odot, a)$ is small enough for Theorem 5 to apply. Summarising, one has the following:

Proposition 5. *For suitably small $\varepsilon > 0$, the maximal globally hyperbolic development, $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$, of the time symmetric initial data set $(\tilde{\mathcal{S}}, \tilde{\mathbf{h}})$ is geodesically complete with $\tilde{\mathbf{g}}$ asymptotically approaching the Minkowski metric in all directions.*

Based on the results of Section 5 we have the following:

²One says that $u = O_{H^k}(r^\alpha)$ if the weighted Sobolev norm $\|u\|_{H_k^\alpha}$ is finite —see Appendix A in [10] for detailed definition. In particular, if $k > l + 3/2$ then $|\partial^l u| = o(r^\alpha)$.

Conjecture 1. *The spacetime $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ obtained in Proposition 5 has polyhomogeneous (non-peeling) asymptotics as in Theorem 2.*

Remark 28. The construction in the previous paragraphs can be readily adapted to obtain global spacetimes with polyhomogeneous (non-peeling) asymptotics as in Theorem 3 part (i) or the peeling asymptotics of part (ii).

Remark 29. It should be pointed out that Christodoulou & Klainerman’s global existence result in [14] cannot be applied to the time symmetric initial data $(\tilde{\mathcal{S}}, \tilde{\mathbf{h}})$ as the quantity measuring the global smallness in this case requires $d_{\odot}^6 |\text{Bach}[\tilde{\mathbf{h}}]|^2$ to be integrable over \mathbb{R}^3 —this is not satisfied by a Bach tensor of the form given by (29).

7 Concluding remarks

The main analysis of this article concerning the potential non-peeling behaviour of the development of time symmetric initial data sets can, in principle, be extended to more general classes of data for which the extrinsic curvature is non-vanishing. Preliminary computations and the general structure of the conformal Einstein field equations suggest that the main picture will remain the same —in particular, the generic non-peeling behaviour seems to be that of Theorem 2. It is of interest to remark that an analysis of the peeling properties of the development of (conformally flat) Bowen-York initial data sets has been given in [41] —notice however, that because of the conformal flatness of this family of initial data, the regularity condition 2 is automatically satisfied and the spacetimes have much better peeling properties.

As already mentioned in the introduction, the key open problem in the analysis of the *problem of spatial infinity* through the cylinder at spatial infinity is to establish the relation between the F-expansions obtained by solving the transport equations along \mathcal{I} and actual solutions to the conformal Einstein field equations. More precisely, to assert that the F-expansions correspond to a Taylor-like polynomial of a solution one needs to be able to control the remainder of the expansion. A type of estimates that, in principle, should provide the desired control have been constructed in [22] for the (linear) spin-2 massless field on the Minkowski spacetime. Adapting this construction to the case of the full conformal Einstein field equations is a challenging endeavour which requires a much bigger insight into the structure of the equations than the one it is currently available —however, some preliminary investigations have been carried out in [42]. Further investigations along these lines will be pursued elsewhere. Needless to say that the development of techniques to transform, say, Conjecture 1 into rigorous statement would constitute a major milestone in mathematical General Relativity and the conclusion of a research programme started in the early 1960’s.

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A Expressions in the gauge $\kappa = \omega$

In this appendix we present the explicit solutions to the transport equations at \mathcal{I} for the orders $p = 0, 1, 2$. In view of the applications in this article we restrict the attention to the coefficients of the frame and the components of the rescaled Weyl spinor.

A.1 Expansions for the frame coefficients

In what follows let

$$(c_{AB}^\mu)^{[p]} \equiv (\partial_\rho^p c_{AB}^\mu)|_{\mathcal{I}},$$

and

$$x_{AB} \equiv \sqrt{2}\delta_{(A}^0\delta_{B)}^1, \quad y_{AB} \equiv -\frac{1}{\sqrt{2}}\delta_A^1\delta_B^1, \quad z_{AB} \equiv \frac{1}{\sqrt{2}}\delta_A^0\delta_B^1.$$

A.1.1 Order $p = 0$

$$\begin{aligned} (c_{AB}^0)^0 &= -\tau x_{AB}, \\ (c_{AB}^1)^0 &= 0, \\ (c_{AB}^+)^0 &= z_{AB}, \\ (c_{AB}^-)^0 &= y_{AB}. \end{aligned}$$

A.1.2 Order $p = 1$

$$\begin{aligned} (c_{AB}^0)^1 &= m(-\tau + \frac{4}{3}\tau^3 - \frac{1}{3}\tau^5)x_{AB}, \\ (c_{AB}^1)^1 &= x_{AB}, \\ (c_{AB}^+)^1 &= m(\frac{1}{2} + \tau^2 - \frac{1}{6}\tau^4)z_{AB}, \\ (c_{AB}^-)^1 &= m(\frac{1}{2} + \tau^2 - \frac{1}{6}\tau^4)y_{AB}. \end{aligned}$$

A.1.3 Order $p = 2$

$$\begin{aligned} (c_{AB}^0)^2 &= (m^2(4\tau^3 - 5\tau^5 + \frac{8}{7}\tau^7 - \frac{1}{7}\tau^9) + W_1(-12\tau + 16\tau^3 - \frac{26}{5}\tau^5 + \frac{6}{5}\tau^7))x_{AB} \\ &\quad + (4\tau - 8\tau^3 + \frac{7}{5}\tau^5 + \frac{3}{5}\tau^7)(X_-W_1y_{AB} + X_+W_1z_{AB}), \\ (c_{AB}^1)^2 &= m(1 - 4\tau^2 + \frac{2}{3}\tau^4)x_{AB}, \\ (c_{AB}^+)^2 &= (m^2(\tau^2 + \frac{13}{6}\tau^4 - \frac{8}{9}\tau^6 + \frac{1}{14}\tau^8) + W_1(4 + 12\tau^2 - 3\tau^4 + \frac{3}{5}\tau^6))z_{AB} \\ &\quad - (6\tau^2 + \frac{1}{2}\tau^4 - \frac{3}{10}\tau^6)X_-W_1x_{AB}, \\ (c_{AB}^-)^2 &= (m^2(\tau^2 + \frac{13}{6}\tau^4 - \frac{8}{9}\tau^6 + \frac{1}{14}\tau^8) + W_1(4 + 12\tau^2 - 3\tau^4 + \frac{3}{5}\tau^6))z_{AB} \\ &\quad - (6\tau^2 + \frac{1}{2}\tau^4 - \frac{3}{10}\tau^6)X_+W_1x_{AB}. \end{aligned}$$

A.2 Expansions of the rescaled Weyl tensor

Consistent with the notation introduced above let

$$(\phi_{ABCD})^{[p]} \equiv (\partial_\rho^p \phi_{ABCD})|_{\mathcal{I}}.$$

Taking into account that

$$\epsilon_A^0 = -\iota_A, \quad \epsilon_A^1 = o_A,$$

and

$$\epsilon_A^A \epsilon^B_A = \delta_A^B.$$

the components ϕ_{ABCD} can be expressed as

$$\begin{aligned} \phi_{ABCD} &= \phi_0\delta_A^0\delta_B^0\delta_C^0\delta_D^0 + 4\phi_1\delta_{(A}^0\delta_{B)}^0\delta_C^0\delta_D^1 \\ &\quad + 6\phi_2\delta_{(A}^0\delta_{B)}^0\delta_C^1\delta_D^1 + 4\phi_3\delta_{(A}^0\delta_{B)}^1\delta_C^1\delta_D^1 + \phi_4\delta_A^1\delta_B^1\delta_C^1\delta_D^1. \end{aligned}$$

where

$$\begin{aligned}
\phi_0 &\equiv o^A o^B o^C o^D \phi_{ABCD}, \\
\phi_1 &\equiv \iota^A o^B o^C o^D \phi_{ABCD}, \\
\phi_2 &\equiv \iota^A \iota^B o^C o^D \phi_{ABCD}, \\
\phi_3 &\equiv \iota^A \iota^B \iota^C o^D \phi_{ABCD}, \\
\phi_4 &\equiv \iota^A \iota^B \iota^C \iota^D \phi_{ABCD}.
\end{aligned}$$

A.2.1 Order $p = 0$

$$\phi_{ABCD}^{(0)} = -m \delta_{(A}{}^0 \delta_B{}^0 \delta_C{}^1 \delta_D){}^1.$$

A.2.2 Order $p = 1$

$$\begin{aligned}
\phi_{ABCD}^{(1)} &= -\frac{1}{6} (36W_1(1-\tau^2) + m^2(9+18\tau^2-3\tau^4)) \delta_{(A}{}^0 \delta_B{}^0 \delta_C{}^1 \delta_D){}^1 \\
&\quad - 3(1-\tau)^2 X_+ W_1 \delta_{(A}{}^0 \delta_B{}^0 \delta_C{}^0 \delta_D){}^1 + 3(1+\tau)^2 X_- W_1 \delta_{(A}{}^0 \delta_B{}^1 \delta_C{}^1 \delta_D){}^1.
\end{aligned}$$

A.2.3 Order $p = 2$

$$\begin{aligned}
\phi_{ABCD}^{(2)} &= \phi_0^{(2)} \delta_A{}^0 \delta_B{}^0 \delta_C{}^0 \delta_D{}^0 + 4\phi_1^{(2)} \delta_{(A}{}^0 \delta_B{}^0 \delta_C{}^0 \delta_D){}^1 \\
&\quad + 6\phi_2^{(2)} \delta_{(A}{}^0 \delta_B{}^0 \delta_C{}^1 \delta_D){}^1 + 4\phi_3^{(2)} \delta_{(A}{}^0 \delta_B{}^1 \delta_C{}^1 \delta_D){}^1 + \phi_4^{(2)} \delta_A{}^1 \delta_B{}^1 \delta_C{}^1 \delta_D{}^1
\end{aligned}$$

where

$$\begin{aligned}
\phi_0^{(2)} &= -2X_+^2 W_2 (1-\tau)^4 + \frac{\sqrt{2}}{3} b_0 f_0(\tau), \\
\phi_1^{(2)} &= \frac{1}{4} a_1(\tau) m X_+ W_1 - 2X_+ W_2 (1+\tau)(1-\tau)^3 + \frac{\sqrt{2}}{6} b_1 f_1(\tau), \\
\phi_2^{(2)} &= \frac{1}{6} c_2(\tau) m^3 + \frac{1}{6} a_2(\tau) m W_1 - 4W_2 (1+\tau)^2 (1-\tau)^2 + \frac{\sqrt{3}}{9} b_2 f_2(\tau), \\
\phi_3^{(2)} &= \frac{1}{4} a_3(\tau) m X_- W_1 + 2X_- W_2 (1+\tau)^3 (1-\tau) + \frac{\sqrt{2}}{6} b_3 f_3(\tau), \\
\phi_4^{(2)} &= -2X_-^2 W_2 (1+\tau)^4 + \frac{\sqrt{2}}{3} b_4 f_4(\tau),
\end{aligned}$$

where

$$\begin{aligned}
c_2(\tau) &= -9 - 54\tau^2 - 69\tau^4 + 28\tau^6 - \frac{16}{7}\tau^8, \\
a_1(\tau) &= -36 + 96\tau - 132\tau^2 + 184\tau^3 - 82\tau^4 - \frac{164}{5}\tau^5 + \frac{74}{5}\tau^6, \\
a_2(\tau) &= -180 - 252\tau^2 + 372\tau^4 - \frac{276}{5}\tau^6, \\
a_3(\tau) &= -36 - 96\tau - 132\tau^2 - 184\tau^3 - 82\tau^4 + \frac{164}{5}\tau^5 + \frac{74}{5}\tau^6, \\
f_0(\tau) &= 2(1-\tau)^4 K(-\tau), \\
f_1(\tau) &= 4(1-\tau)^3 (1+\tau) K(-\tau) - \frac{3}{1-\tau},
\end{aligned}$$

$$\begin{aligned}
f_2(\tau) &= \sqrt{6} \left(\frac{2-\tau}{(1+\tau)^2} - 2(1-\tau)^2(1+\tau)^2 K(\tau) \right), \\
f_3(\tau) &= -4(1+\tau)(1-\tau)^3 K(\tau) + \frac{3}{1+\tau}, \\
f_4(\tau) &= -2(1+\tau)^4 K(\tau),
\end{aligned}$$

and

$$\begin{aligned}
K(\tau) &= 1 - 3 \int_0^\tau \frac{ds}{(1-s)(1+s)^5} \\
&= \frac{1}{32} \left(3 \ln(1-\tau) - 3 \ln(1+\tau) + \frac{32 + 38\tau + 24\tau^2 + 6\tau^3}{(1+\tau)^4} \right)
\end{aligned}$$

while

$$b_k \equiv b_{(ABCD)_k}, \quad k = 0, 1, \dots, 4,$$

are the independent components of the Bach spinor evaluated at i .

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