Spinorial characterisations of rotating black hole spacetimes

by

Michael Cole

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School of Mathematical Sciences
Queen Mary, University of London
United Kingdom

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Statement of Originality

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The contents of this thesis are based on the following papers:

- Michael J. Cole and Juan A. Valiente Kroon. Killing spinors as a characterisation of rotating black hole spacetimes. *Classical and Quantum Gravity*, 33(12):125019, 2016.

- Michael J. Cole and Juan A. Valiente Kroon. A geometric invariant characterising initial data for the KerrNewman spacetime. *Annales Henri Poincaré*, 18(11):36513693, Aug 2017.

- Michael J. Cole, Juan A. Valiente Kroon and István Rácz. Killing spinor data on non-expanding horizons and the uniqueness of vacuum stationary black holes, *currently in preparation.*

Abstract

In this thesis, the implications of the existence of Killing spinors in a spacetime are investigated. In particular, it is shown that in vacuum and electrovacuum spacetimes a Killing spinor, along with some assumptions on the associated Killing vector in an asymptotic region, guarantees that the spacetime is locally isometric to a member of the Kerr or Kerr-Newman family. It is shown that the characterisation of these spacetimes in terms of Killing spinors is an alternative expression of characterisation results of Mars (Kerr) and Wong (Kerr-Newman) involving restrictions on the Weyl curvature and matter content.

In the next section, the construction of a geometric invariant characterising initial data for the Kerr-Newman spacetime is described. This geometric invariant vanishes if and only if the initial data set corresponds to exact Kerr-Newman initial data, and so characterises this type of data. First, the characterisation of the Kerr-Newman spacetime in terms of Killing spinors is illustrated. The space spinor formalism is then used to obtain a set of four independent conditions on an initial Cauchy hypersurface that guarantee the existence of a Killing spinor on the development of the initial data. Following a similar analysis in the vacuum case, the properties of solutions to the approximate Killing spinor equation are studied, and used to construct the geometric invariant.

Finally, the problem of Killing spinor initial data in the characteristic problem is investigated. It is shown that data need only be specified on the bifurcation surface of the two intersecting null hypersurfaces in order to guarantee the existence of a Killing spinor in a neighbourhood of the bifurcation surface. This characterises the class of spacetimes known as distorted black holes, which include but is strictly larger than the Kerr family of spacetimes.

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Chapter 1

Background

1.1 The Einstein equations

The theory of general relativity is a mathematical model of gravity, where the system of consideration is represented by a *spacetime*. This consists of a pair $(\mathcal{M}, g)$, containing:

1. A differentiable manifold $\mathcal{M}$:

   A *differentiable manifold* is a Hausdorff, paracompact topological space $\mathcal{M}$, together with a collection of *charts* $\{\mathcal{U}_i, \psi_i\}$ containing open sets $\mathcal{U}_i$ of $\mathcal{M}$, which satisfy:

   (a) $\mathcal{M} = \bigcup_i \mathcal{U}_i$ (the sets cover $\mathcal{M}$).

   (b) For all i, there exists a bijection $\psi_i : \mathcal{U}_i \to \mathcal{V}_i$, where $\mathcal{V}_i$ is some open subset of $\mathbb{R}^n$.

   (c) If $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$, then the transition maps $\psi_j \circ \psi_i^{-1}$ from $\psi_i(\mathcal{U}_i \cap \mathcal{U}_j)$ to $\psi_j(\mathcal{U}_i \cap \mathcal{U}_j)$ are continuously differentiable.

   The manifold is k-differentiable if the transition maps are k-times continuously differentiable, and smooth if the transition maps are infinitely differentiable. The manifolds considered in this thesis will be assumed to be smooth and 4-dimensional.

2. A Lorentzian metric field g:

   Consider the space of tangent vectors to smooth curves $\gamma : \mathcal{M} \to \mathbb{R}$ passing through $p \in \mathcal{M}$. These are defined as linear maps from the space of smooth functions on $\mathcal{M}$ to $\mathbb{R}$, given by:

   $$X_p(f) := \frac{\mathrm{d}}{\mathrm{d}t}\left(f(\gamma(t))\right)|_{t=0}.$$

   It can be shown that this forms a vector space at p, called the *tangent space* $T_p\mathcal{M}$. Then, the metric at p is a multilinear map:

   $$g : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$$

   that is symmetric ($g(X,Y) = g(Y,X)$) and non-degenerate ($g(X,Y) = 0$ for all $Y \in T_p\mathcal{M}$ if and only if $X = 0$). It is often represented in component form as a real n by n matrix. As a real symmetric matrix, it is diagonalisable, with the eigenvalues of g on the diagonal; non-degeneracy ensures all of these eigenvalues are non-zero. The *signature* of g is defined by the number of positive and negative eigenvalues; in 4-dimensional general relativity, we consider *Lorentzian* metrics, with one positive and 3 negative eigenvalues, denoted $(+---)$ (it is common for the signature to be defined to be $(-+++)$, but we will keep the 'mostly-minus' convention in order to be consistent with the spinor formalism discussed later). A Lorentzian metric field is a smooth choice of Lorentzian metric g_p at every point $p \in \mathcal{M}$ (i.e. such that the map $p \to g_p(X|_p, Y|_p)$ is a smooth function of p for all smooth vector fields $X, Y \in T\mathcal{M}$, the tangent bundle of $\mathcal{M}$).

Furthermore, a spacetime metric is required to satisfy the *Einstein equations*:

$$\mathrm{Ric}[g] - \frac{1}{2}\mathrm{R}[g]\, g = 8\pi T$$

where $\mathrm{Ric}[g]$ is the Ricci curvature tensor of g, $\mathrm{R}[g]$ is the Ricci scalar (the trace of the Ricci tensor) and T is the *energy-momentum* tensor, describing the distribution of matter and energy in the spacetime.

Remark 1. Using index notation, this equation can be written

$$R_{ab} - \frac{1}{2} R\, g_{ab} = 8\pi T_{ab}$$

We will use the Einstein summation convention (for both tensor and spinor indices) throughout this thesis.

Remark 2. A symmetry of the Riemann curvature tensor known as the Bianchi identity gives rise to the conservation of energy-momentum:

$$\nabla^a T_{ab} = 0$$

This can also be interpreted as providing the equations of motion for the matter content of the spacetime.

In this thesis, we will be considering two choices of energy-momentum tensor: first the vacuum choice, $T_{ab} = 0$, in which case the Einstein equations reduce to:

$$R_{ab} = 0$$

and secondly the electrovacuum choice, describing a spacetime containing only electromagnetic fields, where the energy momentum tensor takes the form

$$T_{ab} = \frac{1}{4\pi}(F_{ac}F^b{}_c - \frac{1}{4} g_{ab} F_{cd} F^{cd}).$$

Here, F_{ab} is the Faraday tensor, completely determined by the electric and magnetic vector fields $\underline{E}$ and $\underline{B}$ (upon choice of a timelike vector u^a). The conservation of energy-momentum gives rise to the Maxwell equations, and the Einstein equations restricted to this choice of energy-momentum tensor are referred to as the *Einstein-Maxwell* equations, and reduce to the vacuum Einstein equations when the electromagnetic fields vanish.

1.2 The Kerr-Newman solution

1.2.1 The Kerr-Newman metric and its properties

Solutions to the full Einstein-Maxwell equations are in general hard to find; they consist of a system of multi-dimensional, coupled partial differential equations, and so currently known exact solutions in general assume restrictions such as symmetries or algebraic speciality. For example, the Schwarzschild solution was one of the first non-trivial exact solutions to the vacuum Einstein equations to be found, under the assumption of spherical symmetry, in [52] (and is the unique such spacetime due to Birkhoff's theorem, see [13]).

The Schwarzschild solution is now understood to represent the gravitational field of a spherically symmetric and stationary black hole; it possesses a curvature singularity at area radius $r = 0$, and an event horizon at $r = 2M$, where M is the mass parameter of the solution. However, it is not believed to be an accurate description of a physical black hole, which is expected to have a non-zero angular momentum J which breaks the assumption of spherical symmetry. Thus, after discovery of the Schwarzschild solution in 1916, there was significant effort to find a generalisation possessing angular momentum. The search took considerably longer than expected, but such a solution was finally found by Kerr in 1963 [36], now called the Kerr solution and expressed here in Boyer-Lindquist coordinates (t, r, θ, ϕ):

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{\Delta - a^2\sin^2\theta}{\Sigma}\mathrm{d}t^2 - 2a\sin^2\theta\frac{r^2+a^2-\Delta}{\Sigma}\mathrm{d}t\mathrm{d}\phi \\ &+\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta\mathrm{d}\phi^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 \end{aligned} \tag{1.1}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. The solution possesses a number of properties desirable for describing a rotating black hole:

- The solution is stationary and axisymmetric (admits the two Killing vectors $\partial_t, \partial_\phi$).
- The solution depends on 2 parameters M, a, determining the mass (M) and angular

momentum ($J = aM$) of the black hole. Therefore, (1.1) actually represents a *family* of vacuum solutions, referred to as the Kerr family; however, for simplicity one often uses the *Kerr solution* to refer to a member of this family, or as referring to the family as a whole.

- In the limit $r \to \infty$, the metric reduces to the Minkowski metric, i.e. the solution is asymptotically flat.

- When the rotation parameter a is set to zero, the metric reduces to the Schwarzschild metric.

- There is a curvature singularity at $r = 0$, and an event horizon at $r = M + \sqrt{M^2 - a^2}$.

- Like the Schwarzschild solution, the Kerr solution has 2 asymptotically flat ends (see section 2.3 for the full definition), but it can also be extended to asymptotically flat regions to the past and future of the singularity. However, generically rotating black holes are expected to exhibit singular behaviour along the Cauchy horizon (the boundary of the globally hyperbolic region of the spacetime – see section 1.3), providing confirmation of the strong cosmic censorship hypothesis. Justification for the inextendibility of the metric past the Cauchy horizon in a sufficiently regular way is given in [21]; there, it is shown that although the metric can be extended across the Cauchy horizon as a C^0 field, it generically exhibits a "weak null singularity" (for example, preventing extendibility as a C^2 metric). Thus, a modified version of the strong cosmic censorship hypothesis is preserved.

This vacuum solution can be extended to a solution to the Einstein-Maxwell equations in a straightforward way, called the Kerr-Newman solution:

$$\begin{aligned} \mathrm{d}s^2 =& -\frac{\Delta - a^2\sin^2\theta}{\Sigma}\mathrm{d}t^2 - 2a\sin^2\theta\frac{r^2+a^2-\Delta}{\Sigma}\mathrm{d}t\mathrm{d}\phi \\ &+ \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta\mathrm{d}\phi^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 \\ A =& -\frac{Qr(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi)}{\Sigma} \end{aligned} \tag{1.2}$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2 + Q^2$, and the Faraday tensor F of the solution is the exterior derivative of the electromagnetic 4-potential, $F = \mathrm{d}A$. The solution now depends on 3 parameters (M, a, Q), with the new parameter Q having the interpretation of electric charge. The expression for the 4-potential A is not unique – the gauge transformation $A \to A + \mathrm{d}\chi$ for an arbitrary scalar field χ preserves the value of the Faraday tensor, and so is a solution to the Einstein-Maxwell equations with the same metric; It is clear that the Kerr-Newman solution reduces to the Kerr solution when $Q = 0$, and it possesses many of the same physical properties as the Kerr solution. Although the Kerr-Newman solution is a solution to a more general set of equations, in reality we expect to only observe Kerr black holes in the universe - charged black holes would attract matter of the opposite charge, thereby reducing the charge of the black hole to zero, over a short time frame.

1.2.2 The Carter constant and hidden symmetries

As mentioned above, each member of the Kerr-Newman family admits two Killing vectors, representing two isometries of the spacetime. These vectors are ∂_t, representing stationarity, and ∂_ϕ, representing axisymmetry. A consequence of the existence of these two symmetries is the existence of conserved quantities for observers moving along geodesics. Explicitly, if X^a is the tangent vector to a geodesic of a Kerr-Newman spacetime, then the quantities

$$E = g_{ab} X^a (\partial_t)^b$$
$$L = g_{ab} X^a (\partial_\phi)^b$$

are conserved along the geodesic. Along with the Hamiltonian $H = g_{ab} X^a X^b$ of the particle (determined by the particle's mass), this provides 3 conserved quantities for an observer moving along a geodesic.

This falls short of the 4 conserved quantities needed, in the special case of a 4-dimensional spacetime, to allow the geodesic equation to be completely integrated. However, an interesting property of the Kerr-Newman metric is the existence of a further conserved quantity, known as the Carter constant (found by Carter in [14]). The existence of this constant of motion is a direct consequence of the existence of a *Killing tensor field* K_{ab} in the Kerr-Newman spacetime, satisfying a modification of the Killing vector equations:

$$\nabla_{(a} K_{bc)} = 0. \tag{1.3}$$

The Carter constant is then constructed using the Killing tensor and the geodesic tangent vector X^a:

$$C = K_{ab} X^a X^b.$$

It is straightforward to show that C is conserved along geodesics. Furthermore, when combined with the 3 conserved quantities described earlier, this allows all geodesics to be parametrised uniquely by the values of (H, E, L, C), thereby allowing the geodesic equation to be integrated completely.

The existence of a Killing tensor is a highly non-trivial property of a spacetime, so it can be considered fortunate that the Kerr-Newman family, one of the most physically important class of spacetimes, admits such a tensor; in this thesis, the implications of this fact will be investigated and used to study the ways in which the Kerr-Newman solution can be characterised.

1.2.3 Uniqueness and stability of the Kerr-Newman family

The current family of uniqueness results regarding the Kerr-Newman solution contain assumptions on the spacetime that are often considered too restrictive, such as analyticity – see e.g. [17] for a review on the subject. There exist a variety of results removing this assumption – for example, showing that stationary spacetimes with sufficiently small Mars-Simon tensor must be isometric to a member of the Kerr family, at least in the exterior region [33]. The latter is of particular relevance here, as the Mars-Simon tensor is integral to the characterisation of the Kerr spacetime due to Mars [40, 41], which will be examined in Chapter 2. Although there has been significant progress on proving the linear stability of the Kerr-Newman solution (for example, by investigating the behaviour of the Teukolsky equation on a Kerr background [20]), the question of non-linear stability has been far more stubborn – see e.g. [22] for a discussion on this topic. In particular, although there exist results for spacetimes with a higher degree of symmetry (such as the non-linear stability of Schwarzschild under axially symmetric perturbations [37]), the full non-linear stability of the Kerr-Newman family under arbitrary perturbations is still an open problem.

1.3 The Cauchy problem

In order to investigate the behaviour of perturbations to Kerr-Newman black holes, it is useful to be able to specify an 'initial state' of the spacetime, where initial data specifying the perturbation is prescribed and evolved to future times. The set-up and theoretical motivation for this process was first outlined by Fourés-Bruhat in [25], with further results extending the argument in collaboration with Geroch in [15].

To construct this formalism, we assume that the spacetime can be foliated by a family

$\mathcal{S}_t$ of spacelike partial Cauchy surfaces (surfaces containing points which are not causally related); otherwise, the spacetime can be restricted to regions where this is true. Such spacetimes are called *globally hyperbolic*; the implications of global hyperbolicity are discussed in [29]. The spacetime metric g_{ab} induces an $(n-1)$-dimensional Riemannian metric h_{ij} on each $\mathcal{S}_t$, and each $\mathcal{S}_t$ also admits an extrinsic curvature tensor K_{ij} describing the embedding of the $\mathcal{S}_t$ in the larger manifold $\mathcal{M}$. Singling out the $t = 0$ surface, together the collection $(\mathcal{S}_0, h_{ij}, K_{ij})$ constitutes an *initial data set* for the spacetime.

As the spacetime metric g_{ab} is constrained by the Einstein equations, we expect the induced metric and extrinsic curvature to also be constrained; in fact, by projecting the Einstein equations along the normal direction to the surface $\mathcal{S}$, we obtain the Hamiltonian and momentum constraints:

$$r - K^{ij}K_{ij} + tr(K)^2 = 16\pi\rho$$
$$D_j K^j{}_i - D_i K = 8\pi p_i$$

where r is the Ricci scalar of h_{ij}, D_i is the Levi-Civita connection associated to h_{ij}, and ρ and p_i are the matter energy and matter momentum densities respectively, obtained from the energy momentum tensor T_{ab}. By projecting the Einstein equations fully onto $\mathcal{S}$, one obtains a set of evolution equations for the data h_{ij}, K_{ij}.

Reversing the perspective, one can ask whether these constraints are sufficient to reconstruct the full spacetime – in other words, whether an initial data set satisfying these constraints gives rise to a unique spacetime upon evolution. The following result due to Choquet-Bruhat and Geroch [15] answers this (in the vacuum case):

Theorem 1. *Let $(\mathcal{S}, h_{ab}, K_{ab})$ be an* initial data set *satisfying the constraint equations in vacuum, consisting of a spacelike partial Cauchy surface $\mathcal{S}$, a Riemannian metric h_{ij} on $\mathcal{S}$, and the extrinsic curvature tensor K_{ij} of $\mathcal{S}$. Then, there exists a unique spacetime $(\mathcal{M}, g_{ab})$ (up to diffeomorphisms) such that:*

1. *$(\mathcal{M}, g_{ab})$ satisfies the vacuum Einstein equations;*

2. *$(\mathcal{M}, g_{ab})$ is globally hyperbolic, with $\mathcal{S}$ a member of the foliation of Cauchy surfaces;*

3. *h_{ij} is the metric on $\mathcal{S}$ induced by g_{ab};*

4. *K_{ij} is the extrinsic curvature of $\mathcal{S}$;*

5. *$(\mathcal{M}, g_{ab})$ is an extension of any other spacetime satisfying the above conditions.*

There exist generalisations of this result to spacetimes with non-trivial matter contents – see e.g. the review in [49]. Note that the obtained spacetime is only unique up to diffeomorphisms; in particular, if $(\mathcal{M}, g)$ and $(\tilde{\mathcal{M}}, \tilde{g})$ are spacetimes satisfying the conditions of the theorem, then there exists a smooth bijective map $\phi : \mathcal{M} \to \tilde{\mathcal{M}}$ with a smooth inverse, and $\tilde{g} = \phi_*(g)$, the pushforward of g with respect to ϕ. Two diffeomorphic spacetimes have equivalent physical properties (such as giving rise to the same dynamics), and so diffeomorphisms can be thought of as a gauge symmetry of general relativity.

One can also ask how properties of the larger spacetime are encoded in the initial data, particularly symmetries. Representing symmetries of a spacetime in terms of conditions on an initial hypersurface is not a new idea; the *Killing initial data (KID) equations* – see e.g. [10] – are conditions on a spacelike Cauchy surface $\mathcal{S}$ which guarantee the existence of a Killing vector in the resulting evolution of the initial data. In this way, isometries of the whole spacetime can be encoded at the level of initial data. The resulting conditions form a system of overdetermined equations, so do not necessarily admit a solution for an arbitrary initial data set. In fact, it has been shown that solutions to the KID equations are non-generic, in the sense that generic solutions of the vacuum constraint equations do not possess any global or local spacetime Killing vectors – see [11].

Recalling that the Kerr-Newman spacetime admits a Killing tensor which satisfies a generalised form of the Killing vector equation, one can now ask whether a similar procedure can be performed here; explicitly, can one obtain an overdetermined system on an initial data hypersurface which, when a solution exists, guarantees the existence of a Killing tensor in the resulting spacetime? Chapter 3 will address this question.

1.4 Spinors

Throughout this thesis, spinorial methods will be used to simplify the analysis and illustrate some of the ideas in a more intuitive way. There are conflicting conventions and notation used across the literature for these methods, so in this section the chosen conventions to be used in this thesis will be set out. We will for the most part be using the conventions set out in Stewart [55] and Penrose & Rindler [45, 46].

The curvature spinors $\Psi_{ABCD}, \Phi_{ABA'B'}$ (spinorial counterparts of the Weyl and trace-free Ricci tensors respectively) and Λ (proportional to the Ricci scalar) are defined by the relations

$$\Box_{AB}\xi_C = \Psi_{ABCD}\xi^D - 2\Lambda\xi_{(A}\epsilon_{B)C}, \qquad \Box_{A'B'}\xi_C = \xi^D\Phi_{CDA'B'} \tag{1.4}$$

where $\Box_{AB} \equiv \nabla_{A'(A}\nabla_{B)}{}^{A'}$. In spinorial notation the Einstein-Maxwell equations read

$$\Phi_{ABA'B'} = 2\phi_{AB}\bar{\phi}_{A'B'}, \qquad \Lambda = 0 \tag{1.5}$$

where $\phi_{AB} = \phi_{(AB)}$ is the Maxwell spinor satisfying

$$\nabla^A{}_{A'}\phi_{AB} = 0. \tag{1.6}$$

The Bianchi identity in electrovacuum spacetimes takes the form

$$\nabla^A{}_{A'}\Psi_{ABCD} = 2\bar{\phi}_{A'B'}\nabla_B{}^{B'}\phi_{CD}. \tag{1.7}$$

We systematically use of the following expression for the (once contracted) second derivative of a spinor:

$$\nabla_{AQ'}\nabla_B{}^{Q'} = \frac{1}{2}\epsilon_{AB}\square + \square_{AB}. \tag{1.8}$$

In particular, from the Maxwell equation (1.6) it follows that

$$\nabla_{A'B}\phi_{CD} = \nabla_{A'(B}\phi_{CD)}. \tag{1.9}$$

Our conventions for the curvature are that

$$\nabla_c\nabla_d u^b - \nabla_d\nabla_c u^b = R_{dca}{}^b u^a.$$

Given an antisymmetric rank 2 tensor F_{ab}, the Hodge dual of F_{ab} is defined by

$$F^{\star}_{ab} \equiv \frac{1}{2}\epsilon_{ab}{}^{cd}F_{cd}. \tag{1.10}$$

The self-dual version of F_{ab} is then defined by

$$\mathcal{F}_{ab} \equiv F_{ab} + \mathrm{i}F^{\star}{}_{ab}. \tag{1.11}$$

1.5 The space-spinor formalism

In what follows assume that the spacetime $(\mathcal{M}, g)$ obtained as the development of Cauchy initial data $(\mathcal{S}, h_{ij}, K_{ij})$ can be covered by a congruence of smooth timelike curves with tangent vector τ^a satisfying the normalisation condition $\tau_a\tau^a = 2$. The reason for normalisation will be clarified in the following – see equation (1.15). Associated to the vector τ^a one has the projector

$$h_a{}^b \equiv \delta_a{}^b - \frac{1}{2}\tau_a\tau^b \tag{1.12}$$

projecting tensors into the distribution $\langle\boldsymbol{\tau}\rangle^{\perp}$ of hyperplanes orthogonal to τ^a.

Remark 3. The congruence of curves does not need to be hypersurface orthogonal – however, for convenience it will be assumed that the vector field τ^a is orthogonal to the Cauchy hypersurface $\mathcal{S}$.

Now, let $\tau^{AA'}$ denote the spinorial counterpart of the vector τ^a – by definition one has that

$$\tau_{AA'}\tau^{AA'} = 2. \tag{1.13}$$

Let $\{o^A, \iota^A\}$ denote a normalised spin-dyad satisfying $o_A\iota^A = 1$. In the following we restrict the attention to spin-dyads such that

$$\tau^{AA'} = o^A\bar{o}^{A'} + \iota^A\bar{\iota}^{A'}. \tag{1.14}$$

It follows then that

$$\tau_{AA'}\tau^{BA'} = \delta_A{}^B, \tag{1.15}$$

consistent with the normalisation condition (1.13). As a consequence of this relation, the spinor $\tau^{AA'}$ can be used to introduce a formalism in which all primed indices in spinors and spinorial equations are replaced by unprimed indices by suitable contractions with $\tau_A{}^{A'}$.

Remark 4. The set of transformations on the dyad $\{o^A, \iota^A\}$ preserving the expansion (1.14) is given by the group $SU(2,\mathbb{C})$. In particular, a general linear transformation of a spinor dyad $\{o^A, \iota^A\}$ of the form $o^A \mapsto \alpha o^A + \beta\iota^A$, $\iota^A \mapsto \gamma o^A + \delta\iota^A$ must be an element of $SL(2,\mathbb{C})$ to preserve the normalisation condition $o_A\iota^A = 1$; it is an easy exercise to show that unitarity is a necessary and sufficient condition to preserve the form of (1.14).

1.5.1 The Sen connection

The *space-spinor* counterpart of the spinorial covariant derivative $\nabla_{AA'}$ is defined as

$$\nabla_{AB} \equiv \tau_B{}^{A'}\nabla_{AA'}. \tag{1.16}$$

The derivative operator ∇_{AB} can be decomposed in irreducible terms as

$$\nabla_{AB} = \frac{1}{2}\epsilon_{AB}\mathcal{P} + \mathcal{D}_{AB} \tag{1.17}$$

where

$$\mathcal{P} \equiv \tau^{AA'}\nabla_{AA'} = \nabla_Q{}^Q, \qquad \mathcal{D}_{AB} \equiv \tau_{(A}{}^{A'}\nabla_{B)A'} = \nabla_{(AB)}.$$

The operator $\mathcal{P}$ is the directional derivative of $\nabla_{AA'}$ in the direction of $\tau^{AA'}$ while $\mathcal{D}_{AB}$ corresponds to the so-called *Sen connection of the covariant derivative* $\nabla_{AA'}$ *implied by* $\tau^{AA'}$.

1.5.2 The acceleration and the extrinsic curvature

Of particular relevance in the subsequent discussion is the decomposition of the covariant derivative of the spinor $\tau_{BB'}$, namely $\nabla_{AA'}\tau_{BB'}$. A calculation readily shows that the content of this derivative is encoded in the spinors

$$K_{AB} \equiv \tau_B{}^{A'}\mathcal{P}\tau_{AA'}, \qquad K_{ABCD} \equiv \tau_D{}^{C'}\mathcal{D}_{AB}\tau_{CC'} \tag{1.18}$$

corresponding, respectively, to the spinorial counterparts of the acceleration and the Weingarten tensor, expressed in tensorial terms as

$$K_a \equiv -\frac{1}{2}\tau^b\nabla_b\tau_a, \qquad K_{ab} \equiv -h_a{}^c h_b{}^d\nabla_c\tau_d.$$

It can be readily verified that

$$K_{AB} = K_{(AB)}, \qquad K_{ABCD} = K_{(AB)(CD)}. \tag{1.19}$$

In the sequel it will be convenient to express K_{ABCD} in terms of its irreducible components. To this end define

$$\Omega_{ABCD} \equiv K_{(ABCD)}, \qquad \Omega_{AB} \equiv K_{(A}{}^{Q}{}_{B)Q}, \qquad K \equiv K_{AB}{}^{CD}, \tag{1.20}$$

so that one can define

$$K_{ABCD} = \Omega_{ABCD} - \frac{1}{2}\epsilon_{A(C}\Omega_{D)B} - \frac{1}{2}\epsilon_{B(C}\Omega_{D)A} - \frac{1}{3}\epsilon_{A(C}\epsilon_{D)B}K. \tag{1.21}$$

If the vector field τ^a is hypersurface orthogonal, then one has that $\Omega_{AB} = 0$, and thus the Weingarten tensor satisfies the symmetry $K_{ab} = K_{(ab)}$ so that it can be regarded as the extrinsic curvature of the leaves of a foliation of the spacetime $(\mathcal{M}, g)$. If this is the case, in addition to the second symmetry in (1.19) one has that

$$K_{ABCD} = K_{CDAB}.$$

In particular, K_{ABCD} restricted to the hypersurface $\mathcal{S}$ satisfies the above symmetry and one has $\Omega_{AB} = 0$ – cfr. Remark 3.

In what follows denote by $D_{AB} = D_{(AB)}$ the spinorial counterpart of the Levi-Civita connection of the metric h_{ij} on $\mathcal{S}$. The Sen connection $\mathcal{D}_{AB}$ and the Levi-Civita connection D_{AB} are related to each other through the spinor K_{ABCD}. For example, for a valence 1 spinor π_A one has that

$$\mathcal{D}_{AB}\pi_C = D_{AB}\pi_C + \frac{1}{2}K_{ABC}{}^{Q}\pi_Q,$$

with the obvious generalisations for higher order spinors. A consequence of this relationship is that, even when τ is hypersurface-orthogonal, the Sen connection does not coincide with the Levi-Civita connection - for example, the Sen connection will in general have a non-zero torsion. The definition of the Sen connection as an irreducible component of the projection of the covariant derivative operator means that it is a natural choice of differential operator in this setting; if needed, equations using the Sen connection can be transformed to ones involving the induced Levi-Civita connection via the above relation.

1.5.3 Hermitian conjugation

Given a spinor π_A, its *Hermitian conjugate* is defined as

$$\widehat{\pi}_A \equiv \tau_A{}^{Q'}\bar{\pi}_{Q'}. \tag{1.22}$$

This operation can be extended in the obvious way to higher valence pairwise symmetric spinors. The operation of Hermitian conjugation allows to introduce a notion of *reality*. Given spinors $\nu_{AB} = \nu_{(AB)}$ and $\xi_{ABCD} = \xi_{(AB)(CD)}$, we say that they are *real* if and only if

$$\widehat{\nu}_{AB} = -\nu_{AB}, \qquad \widehat{\xi}_{ABCD} = \xi_{ABCD}.$$

If the spinors are real then it can be shown that there exist real spatial 3-dimensional tensors ν_i and ξ_{ij} such that ν_{AB} and ξ_{ABCD} are their spinorial counterparts. We also note that

$$\nu_{AB}\widehat{\nu}^{AB} \geq 0, \qquad \xi_{ABCD}\widehat{\xi}^{ABCD} \geq 0$$

independently of whether ν_{AB} and ξ_{ABCD} are real or not.

Finally, it is observed that while the Levi-Civita covariant derivative D_{AB} is real in the sense that

$$\widehat{D_{AB}\pi_C} = -D_{AB}\hat{\pi}_C,$$

the Sen connection $\mathcal{D}_{AB}$ is not. More precisely, one has that

$$\widehat{\mathcal{D}_{AB}\pi_C} = -\mathcal{D}_{AB}\widehat{\pi}_C + \frac{1}{2}K_{ABC}{}^{Q}\hat{\pi}_Q. \tag{1.23}$$

1.5.4 Commutators

The main analysis of this section will require a systematic use of the commutators of the covariant derivatives $\mathcal{P}$ and $\mathcal{D}_{AB}$. In order to discuss these in a convenient manner it is convenient to define the Hermitian conjugate of the Penrose box operator $\square_{AB} \equiv \nabla_{C'(A}\nabla_{B)}{}^{C'}$ in the natural manner as

$$\widehat{\square}_{AB} \equiv \tau_A{}^{A'}\tau_B{}^{B'}\square_{A'B'}.$$

From the definition of $\square_{A'B'}$ it follows that

$$\widehat{\square}_{AB}\pi_C = \tau_A{}^{A'}\tau_B{}^{B'}\Phi_{FCA'B'}\pi^F.$$

In terms of $\square_{AB}$ and $\widehat{\square}_{AB}$, the commutators of $\mathcal{P}$ and $\mathcal{D}_{AB}$ read

$$[\mathcal{P}, \mathcal{D}_{AB}] = \widehat{\square}_{AB} - \square_{AB} - \frac{1}{2}K_{AB}\mathcal{P} + K^D{}_{(A}\mathcal{D}_{B)D} - K_{ABCD}\mathcal{D}^{CD}, \tag{1.24a}$$

$$[\mathcal{D}_{AB}, \mathcal{D}_{CD}] = \frac{1}{2}\big(\epsilon_{A(C}\square_{D)B} + \epsilon_{B(C}\square_{D)A}\big) + \frac{1}{2}\big(\epsilon_{A(C}\widehat{\square}_{D)B} + \epsilon_{B(C}\widehat{\square}_{D)A}\big) + \frac{1}{2}\big(K_{CDAB}\mathcal{P} - K_{ABCD}\mathcal{P}\big) + K_{CDF(A}\mathcal{D}_{B)}{}^F - K_{ABF(C}\mathcal{D}_{D)}{}^F. \tag{1.24b}$$

Remark 5. Observe that on the hypersurface $\mathcal{S}$ the commutator (1.24b) involves only objects intrinsic to $\mathcal{S}$. Notice, also, that the Sen connection $\mathcal{D}_{AB}$ has torsion. Namely, for a scalar ϕ one has that

$$[\mathcal{D}_{AB}, \mathcal{D}_{CD}]\phi = K_{CDF(A}\mathcal{D}_{B)}{}^F\phi - K_{ABF(C}\mathcal{D}_{D)}{}^F\phi.$$

1.6 Outline of the thesis

Chapter 2 investigates the implications of the existence of Killing spinors in a spacetime. In particular, it is shown that in electrovacuum spacetimes the existence of a Killing spinor, along with some assumptions on the associated Killing vector in an asymptotic region, guarantees that the spacetime is locally isometric to the Kerr-Newman solution. This extends work by Bäckdahl and Valiente Kroon [6], which proved the vacuum case; in electrovacuum spacetimes, a further assumption linking the electromagnetic content to the Killing spinor is necessary. It is shown that the characterisation of these spacetimes in terms of Killing spinors is an alternative expression of characterisation results of Mars (Kerr) and Wong (Kerr-Newman) involving restrictions on the Weyl curvature and matter content; in particular, the existence of a Killing spinor gives rise to a set of constants linking the Ernst potential, Killing form and Faraday tensor, which when set to certain values single out the exact Kerr or Kerr-Newman solutions. It is shown that the additional assumption of asymptotic flatness sets these constants to the required values automatically.

Chapter 3 describes the construction of a geometric invariant characterising initial data for the Kerr-Newman spacetime. This geometric invariant vanishes if and only if the initial data set corresponds to exact Kerr-Newman initial data, and so characterises this type of data. Making use of the characterisation of the Kerr-Newman solution in terms of Killing spinors given in Chapter 2, the space spinor formalism is then used to obtain a set of four independent conditions on an initial Cauchy hypersurface that guarantee the existence of a Killing spinor on the development of the initial data. Following an analysis similar to that of the vacuum case given in [6], the properties of solutions to the approximate Killing spinor equation are used to construct the geometric invariant.

Chapter 4 investigates the problem of Killing spinor initial data in the characteristic problem. The motivation for investigating this comes from the fact that a spacetime admitting a bifurcate Killing horizon can be uniquely determined (at least in the domain of dependence of the horizon structure) once initial data is provided on the bifurcation surface. In a similar way, it is shown that data for a Killing spinor candidate field need only be specified on the bifurcation surface of the bifurcate horizon in order to guarantee the existence of a Killing spinor in a neighbourhood of the bifurcation surface. This characterises the class of spacetimes known as distorted black holes, which includes but is strictly larger than the Kerr family of spacetimes.

Finally, **Chapter 5** will provide a brief summary of the contents of each chapter, as well as some observations regarding interpretations as well as limitations of the obtained results.

Chapter 2

Killing spinors as a characterisation of rotating black hole spacetimes

The contents of this chapter reproduces the arguments and results given in the paper [18].

2.1 Introduction

The *Kerr spacetime*, describing a rotating black hole in vacuum, is one of the most interesting exact solutions to the Einstein field equations. As well as having physical relevance, the existence of various incarnations of uniqueness theorems (see e.g. [17] and references within for a survey of this vast topic) has cemented its place as one of the most important vacuum solutions mathematically and physically. There also exist generalisations to spacetimes containing restricted forms of matter – for example, the Kerr-Newman solution to the Einstein-Maxwell equations. Although less physically relevant than the vacuum case, these solutions still retain many interesting features of the

Kerr solution, including uniqueness under further assumptions on the matter content. Thus, these generalisations still retain a mathematical importance.

The remarks in the previous paragraph justify the attention given to finding *characterisations of the Kerr spacetime* and its relations – see e.g. [24, 40]. Such characterisations can be used to study various open questions about these black hole spacetimes. For example, they can be used to reformulate uniqueness theorems and clarify relations between them; study the stability of the solutions, by indicating the behaviour of perturbations; and illustrate the special characteristics of these particular solutions, in particular through the use of symmetries – see e.g. [3] for a recent discussion on these and related ideas. The last of these is elegantly achieved through the use of *Killing spinors*. Closely related to *Killing-Yano tensors*, these spinorial objects represent "hidden symmetries" of the spacetime, which cannot be represented using Killing vectors. It has been shown previously (see [4, 6, 7]) that a vacuum spacetime admitting a Killing spinor, along with conditions on the Weyl curvature and an asymptotic condition, must be isometric to the Kerr spacetime. This result crucially depends on a result of Mars (see [41]) which uses the structure of the Weyl tensor, and its relation to the Killing vectors of the spacetime, to characterise the Kerr solution in a way that exploits to the maximum possible extent the asymptotic flatness of the spacetime – more precisely, it is required that the self-dual Killing form of the stationary Killing vector is an eigenform of the self-dual Weyl tensor.

The characterisation of the Kerr spacetime by Mars given in [41] relies on a previous characterisation of this solution to the vacuum Einstein field equations in terms of the vanishing of the so-called *Mars-Simon tensor* – see [40]. Interestingly, the latter characterisation has been generalised to the electrovacuum case by Wong [56] assuming some restrictions on the matter content. This characterisation is not optimal, in the sense that it assumes the existence of certain relations among the relevant geometric objects; by contrast, in [40], the existence of the vacuum counterpart of these relations is a consequence of the characterisation. Nevertheless, as a consequence of the analysis in [56], one may expect that the Kerr-Newman solution can be characterised by the use of Killing spinors in a similar way to the vacuum case. The characterisations in both [40] and [56] come in both a *local version* (in which certain constraints arising in the characterisation are fixed by evaluating them at finite points of the manifold) and a *global version* (in which asymptotic flatness is used to fix the value of the constants). Remarkably, the generalisation of the characterisation in [41] to the electrovacuum case has, so far, not been obtained.

The purpose of this chapter is to revisit the characterisation of the Kerr spacetime using Killing spinors and then generalise to the electrovacuum case using Wong's result in [56]. The analysis suggests that Wong's result can be strengthened to obtain a characterisation of the Kerr-Newman spacetime more in the spirit of Mars's original result in [40] and, in turn, be used to obtain a generalisation of the analysis of [41] in which the Kerr-Newman spacetime is characterised in an optimal way by a combination of local and global assumptions.

This chapter is organised as follows. Section 2.2 gives an introduction to Killing spinors, their relation to Killing vectors and investigates the implications on the curvature of the spacetime. Some time will be spent defining 1-forms and potentials which are useful for the characterisations later on. In section 2.3, the asymptotic conditions required for the characterisation theorems are defined. Then, in section 2.4, it is shown that the conditions of the characterisation result of Mars [41] are satisfied when the spacetime admits an appropriate Killing spinor. Finally, section 2.5 shows the same for Wong's characterisation of the Kerr-Newman spacetime – i.e. the existence of an appropriate Killing spinor on a electrovacuum spacetime guarantees that the solution is Kerr-Newman up to an isometry.

Conventions

In what follows, $(\mathcal{M}, \boldsymbol{g})$ will denote an electrovacuum spacetime satisfying the Einstein equations with vanishing cosmological constant. The signature of the metric throughout this thesis will be $(+,-,-,-)$, to be consistent with most of the existing literature using spinors. We use the spinorial conventions of [45], outlined in Chapter 1. The lowercase Latin letters a, b, $c, \ldots$ are used as abstract spacetime tensor indices while the uppercase letters A, B, $C, \ldots$ will serve as abstract spinor indices. The Greek letters μ, ν, $\lambda, \ldots$ will be used as spacetime coordinate indices while α, β, $\gamma, \ldots$ will serve as spatial coordinate indices.

2.2 Killing spinors

The purpose of this section is to provide a summary of the basic theory of Killing spinors in electrovacuum spacetimes – see [30–32]. Throughout the chapter, $(\mathcal{M}, \boldsymbol{g})$ will denote an electrovacuum spacetime. Recall that in spinorial notation the Einstein-Maxwell equations read

$$\Phi_{ABA'B'} = 2\phi_{AB}\bar{\phi}_{A'B'}, \qquad \Lambda = 0$$

where $\phi_{AB} = \phi_{(AB)}$ is the Maxwell spinor satisfying the Maxwell equation (1.6).

2.2.1 Basic equations

A *Killing spinor* is a valence-2 symmetric spinor κ_{AB} satisfying the equation

$$\nabla_{A'(A}\kappa_{BC)} = 0. \tag{2.1}$$

By taking a further contracted derivative of this equation, it can be shown that a solution to equation (2.1) must also satisfy the integrability condition

$$\kappa_{(A}{}^{F}\Psi_{BCD)F} = 0 \tag{2.2}$$

where Ψ_{ABCD} is the Weyl spinor, a completely symmetric spinor which is the spinorial equivalent of the Weyl tensor. This condition restricts the form of the Weyl spinor as it requires that

$$\Psi_{ABCD} \propto \kappa_{(AB}\kappa_{CD)}.$$

This proportionality condition forces the spacetime to be of Petrov type D, N or O (i.e. conformally flat). In particular, if a non-vanishing Killing spinor has a repeated principal spinor α_A so that $\kappa_{AB} = \alpha_{(A}\alpha_{B)}$, then the Weyl spinor has four repeated null directions, and so it is of Petrov type N. If the Killing spinor is algebraically general, i.e. there exist α_A and β_B such that $\kappa_{AB} = \alpha_{(A}\beta_{B)}$, then the Weyl spinor has two pairs of repeated null directions, and so it is of Petrov type D.

Algebraically general Killing spinors

In the case that the Killing spinor κ_{AB} is algebraically general, the principal spinors α_A and β_B can be used to form a normalised spin dyad which we will denote by $\{o^A, \iota^B\}$ and such that $o_A\iota^A = 1$. The Killing spinor κ_{AB} is then expanded in terms of the basis as

$$\kappa_{AB} = \varkappa o_{(A}\iota_{B)} \tag{2.3}$$

for some factor of proportionality $\varkappa$. Due to equation (2.2), the Weyl spinor can be expanded in a similar way as

$$\Psi_{ABCD} = \psi o_{(A}o_{B}\iota_{C}\iota_{D)} \tag{2.4}$$

for some factor of proportionality ψ.

The substitution of expression (2.3) in the Killing spinor equation (2.1) implies restrictions on the Newman-Penrose (NP) spin connection coefficients. Namely, one has that

$$\kappa = \lambda = \nu = \sigma = 0,$$

consistent with the fact that the spacetime is, at least, of Petrov type D.

2.2.2 The Killing vector associated to a Killing spinor

A Killing spinor κ_{AB} can be used to define the spinorial counterpart $\xi_{AA'}$ of a (possibly complex) vector via the relation

$$\xi_{AA'} \equiv \nabla^{C}{}_{A'}\kappa_{AC}. \tag{2.5}$$

It can be shown, using the Killing spinor equation (2.1) and commuting covariant derivatives, that $\xi_{AA'}$ satisfies the equation

$$\nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'} = -6\kappa_{(A}{}^{C}\Phi_{B)CA'B'}.$$

Therefore, if

$$\kappa_{(A}{}^{C}\Phi_{B)CA'B'} = 0 \tag{2.6}$$

then $\xi_{AA'}$ is the spinorial counterpart of a (possibly complex) Killing vector in the spacetime. In what follows, condition (2.6) will be referred to as the *matter alignment condition.* In the particular case of an electrovacuum spacetime the matter alignment condition takes the form

$$\kappa_{(A}{}^{C}\phi_{B)C} = 0 \tag{2.7}$$

implying that the spinors κ_{AB} and ϕ_{AB} are proportional to each other. Thus, in terms of the basis dyad $\{o, \iota\}$ used to express equation (2.3) one can write

$$\phi_{AB} = \varphi o_{(A}\iota_{B)} \tag{2.8}$$

with φ a proportionality factor.

As discussed in [46], the notion of a Lie derivative is, in general, not well defined for spinors. However, in the case of a Hermitian spinor $\xi^{AA'}$ associated to a real Killing vector, and recalling that the Maxwell spinor ϕ_{AB} is the spinorial counterpart of the Faraday tensor F_{ab}, there exists a consistent expression which can be used to obtain the spinorial counterpart of $\mathcal{L}_\xi F_{ab} = 0$, the derivative of the Faraday tensor along the integral curves of the real vector field ξ:

$$\mathcal{L}_\xi \phi_{AB} \equiv \xi^{CC'}\nabla_{CC'}\phi_{AB} + \phi_{C(A}\nabla_{B)C'}\xi^{CC'}. \tag{2.9}$$

The Maxwell spinor will be said to *inherit the symmetry generated by the Killing vector* ξ^a if $\mathcal{L}_\xi \phi_{AB} = 0$. Explicitly, the Maxwell spinor ϕ_{AB} and Faraday tensor F_{ab} are related via the relation

$$\mathcal{F}_{AA'BB'} = 2\phi_{AB}\epsilon_{A'B'}$$

where $\mathcal{F}_{AA'BB'}$ denotes the spinorial counterpart of the self-dual Faraday tensor $\mathcal{F}_{ab} = F_{ab} + \mathrm{i}F^{\star}_{ab}$.

Remark. In Section 2.2.5.3 it will be shown that in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ endowed with a Killing spinor κ_{AB} such that $\xi_{AA'}$ is Hermitian and ϕ_{AB} and κ_{AB} satisfy the alignment condition (2.7) then ϕ_{AB} inherits the symmetry of the spacetime.

2.2.3 Relation to Killing-Yano tensors

If a spacetime $(\mathcal{M}, \boldsymbol{g})$ admits a Killing spinor κ_{AB}, and the vector $\xi^{AA'}$ defined by (2.5) satisfies $\xi^{AA'} = \bar{\xi}^{AA'}$ (i.e. is a real vector), then one can construct a real, valence-2 antisymmetric tensor Y_{ab} as the tensorial counterpart of the Killing spinor:

$$Y_{AA'BB'} \equiv \mathrm{i}\big(\kappa_{AB}\epsilon_{A'B'} - \bar{\kappa}_{A'B'}\epsilon_{AB}\big)$$

which, as a consequence of (2.1), satisfies the Killing-Yano equation

$$\nabla_{(a}Y_{b)c} = 0.$$

Such a tensor is called a *Killing-Yano tensor*. Conversely, if a spacetime admits a Killing-Yano tensor Y_{ab}, one can construct a valence-2 symmetric spinor κ_{AB} from the relation

$$\kappa_{AB} \equiv -\frac{\mathrm{i}}{4}\epsilon^{A'B'}\left(Y_{AA'BB'} + \mathrm{i}Y^{*}_{AA'BB'}\right)$$

which satisfies the Killing spinor equation (2.1) – see e.g. [46], Section 6.7 page 107; also [44]. Furthermore, if a spacetime admits a Killing-Yano tensor Y_{ab}, then it is possible to construct a new tensor:

$$K_{ab} \equiv Y_a{}^bY_{cb}$$

that is a Killing tensor satisfying equation (1.3); accordingly, the spacetime will admit a Carter-like constant of motion along geodesics.

Remark. The existence of a Killing-Yano tensor for the Kerr-Newman spacetime is a key ingredient to showing the integrability of the Hamilton-Jacobi equations for geodesic motion, and the separability of the Maxwell equations and the Dirac equation on the Kerr-Newman spacetime – see e.g. [34] or [54] for further details.

2.2.4 The Killing form

In the reminder of this section assume that the matter alignment condition (2.6) is satisfied, so that $\xi_{AA'}$ is the spinorial counterpart of a Killing vector. Moreover, assume that $\xi_{AA'}$ is a Hermitian spinor so that, in fact, it is the spinorial counterpart of a real vector. Then, define the spinorial counterpart of the *Killing form* of ξ^a, namely

$$H_{ab} \equiv \nabla_{[a}\xi_{b]} = \nabla_a \xi_b \qquad (2.10)$$

by

$$H_{AA'BB'} \equiv \nabla_{AA'}\xi_{BB'}.$$

As a consequence of the antisymmetry in the pairs $_{AA'}$ and $_{BB'}$, $H_{AA'BB'}$ can be decomposed into irreducible parts as

$$H_{AA'BB'} = \eta_{AB}\epsilon_{A'B'} + \bar{\eta}_{A'B'}\epsilon_{AB} \qquad (2.11)$$

where η_{AB} is a symmetric spinor – the *Killing form spinor*. In the sequel, we will require the self-dual part of $H_{AA'BB'}$, denoted by $\mathcal{H}_{AA'BB'}$, and defined by

$$\mathcal{H}_{AA'BB'} \equiv H_{AA'BB'} + \mathrm{i}H^{\star}_{AA'BB'}.$$

A direct calculation then yields

$$\mathcal{H}_{AA'BB'} = 2\eta_{AB}\epsilon_{A'B'}. \qquad (2.12)$$

Using equation (2.11), the spinor η_{AB} can be expressed in terms of the Killing vector as

$$\eta_{AB} = \frac{1}{2}\nabla_{AA'}\xi_B{}^{A'}. \qquad (2.13)$$

Then, by using (2.5), this can be expanded in terms of the Killing spinor:

$$\eta_{AB} = -\frac{3}{4}\Psi_{ABCD}\kappa^{CD}. \tag{2.14}$$

Expansions for the algebraically general case

Assuming that κ_{AB} is algebraically general, the basis expansions of κ_{AB} and Ψ_{ABCD} in (2.3) and (2.4) can be used to find the basis expansion of η_{AB}:

$$\eta_{AB} = \frac{1}{4}\varkappa\psi o_{(A}\iota_{B)} = \eta o_{(A}\iota_{B)} \tag{2.15}$$

where

$$\eta \equiv \frac{1}{4}\varkappa\psi. \tag{2.16}$$

2.2.5 The Ernst forms and potentials

Throughout this section let ξ^a denote a real Killing vector on the electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g})$. A well-known consequence of the Killing equation

$$\nabla_a\xi_b + \nabla_b\xi_a = 0$$

and the definition of the Riemann tensor in terms of commutators of covariant derivatives is that

$$\nabla_a\nabla_b\xi_c = R_{cba}{}^d\xi_d. \tag{2.17}$$

The *Ernst form* of the Killing vector ξ^a is defined as

$$\chi_a = 2\xi^b\mathcal{H}_{ba}. \tag{2.18}$$

The Ernst form first arose in [23], in which the Einstein equations are reduced to a two-dimensional non-linear equation (the Ernst equation) under the assumptions of axisymmetry and asymptotic stationarity; an explicit solution to this equation allows the metric to be obtained by solving a system of ODEs. Several properties of the Ernst form follow from the identity (2.17) recast as

$$\nabla_a \mathcal{H}_{bc} = \mathcal{R}_{cba}{}^d \xi_d \tag{2.19}$$

where $\mathcal{R}_{abcd}$ denotes the *self-dual* Riemann tensor. From expression (2.19) it follows, using the identity

$$^*R_{[abc]d} = \frac{1}{3}\epsilon_{abce}R^e{}_d,$$

that

$$\nabla_{[a}\mathcal{H}_{bc]} = \frac{1}{3}\epsilon_{cbae}R^e{}_d\xi^d, \qquad \nabla^a\mathcal{H}_{ab} = -R_{ba}\xi^a.$$

A further computation using the above identities and the definition of the Ernst form, equation (2.18), yields

$$\nabla_a\chi_b - \nabla_b\chi_a = -2\epsilon_{cbae}\xi^c R^e{}_d\xi^d. \tag{2.20}$$

2.2.5.1 The vacuum case

In vacuum $\mathcal{R}_{abcd} = \mathcal{C}_{abcd}$, where $\mathcal{C}_{abcd}$ denotes the self-dual Weyl tensor, and so from the symmetries of the Weyl tensor one concludes that

$$\nabla_a\chi_b - \nabla_b\chi_a = 0.$$

Consequently, in vacuum the Ernst form is closed and thus locally exact. This means that there exists a scalar, the *Ernst potential* χ, satisfying

$$\chi_a = \nabla_a\chi.$$

Now let $\xi_{AA'}$ denote the (Hermitian) spinorial counterpart of the real Killing vector ξ^a. If $\xi_{AA'}$ arises from a Killing spinor through the relation (2.5), it follows from the spinor decomposition of $\mathcal{H}_{AA'BB'}$ that the spinorial counterpart $\chi_{AA'}$ of the Ernst form χ_a is given by

$$\begin{aligned}\chi_{AA'} &= 4\eta_{AB}\xi^{B}{}_{A'} \\ &= 3\kappa^{CF}\Psi_{ABCF}\nabla_{DA'}\kappa^{DB}.\end{aligned}$$

2.2.5.2 The electrovacuum case

In the electrovacuum case the Ernst form is no longer exact – cf. equation (2.20). However, if the Faraday tensor inherits the symmetry of the spacetime – i.e. $\mathcal{L}_\xi F_{ab} = 0$ – then it is possible to construct a further 1-form, the so-called *electromagnetic Ernst form*, which can be shown to be closed. In analogy to the definition in (2.18), define

$$\varsigma_a \equiv 2\xi^b \mathcal{F}_{ba}. \tag{2.21}$$

A calculation then shows that

$$\nabla_a \varsigma_b - \nabla_b \varsigma_a = 2\mathcal{L}_\xi \mathcal{F}_{ab}.$$

If $\mathcal{L}_\xi \mathcal{F}_{ab} = 0$ then ς_a is closed, and therefore locally exact – this means that there exists a scalar, the *electromagnetic Ernst potential* ς, which satisfies

$$\varsigma_a = \nabla_a \varsigma.$$

The spinorial version of equation (2.21) can be readily be found to be

$$\varsigma_{AA'} = 4\phi_{AB}\nabla^{Q}{}_{A'}\kappa^{B}{}_{Q}.$$

2.2.5.3 Expansions in the algebraically general case

Consider the case of an algebraically general spinor κ_{AB} such that $\xi_{AA'}$ as given by equation (2.5) is Hermitian. In order to find the full basis expansions of $\chi_{AA'}$ and $\varsigma_{AA'}$, the derivative of the proportionality factor $\varkappa$ needs to be calculated. First, note the expressions for the derivatives of the spin basis vectors in terms of the spin coefficients from the Newman-Penrose formalism:

$$\begin{aligned}\nabla_{AA'}o_B = &-\alpha o_A o_B \bar{\iota}_{A'} - \beta \iota_A o_B \bar{o}_{A'} + \gamma o_A o_B \bar{o}_{A'} + \epsilon \iota_A o_B \bar{\iota}_{A'} \\ &- \kappa \iota_A \iota_B \bar{\iota}_{A'} + \rho o_A \iota_B \bar{\iota}_{A'} + \sigma \iota_A \iota_B \bar{o}_{A'} - \tau o_A \iota_B \bar{o}_{A'}, \end{aligned} \tag{2.22a}$$

$$\begin{aligned}\nabla_{AA'}\iota_B = &\alpha o_A \iota_B \bar{\iota}_{A'} + \beta \iota_A \iota_B \bar{o}_{A'} - \gamma o_A \iota_B \bar{o}_{A'} - \epsilon \iota_A \iota_B \bar{\iota}_{A'} \\ &- \lambda o_A o_B \bar{\iota}_{A'} - \mu \iota_A o_B \bar{o}_{A'} + \nu o_A o_B \bar{o}_{A'} + \pi \iota_A o_B \bar{\iota}_{A'}. \end{aligned} \tag{2.22b}$$

Substituting the basis expansion for the Killing spinor into the Killing spinor equation, using expressions (2.22a)-(2.22b) and the relation $\epsilon_{AB} = o_A \iota_B - \iota_A o_B$, we find that

$$\nabla_{AA'}\varkappa = \varkappa \left(\mu o_A \bar{o}_{A'} - \pi o_A \bar{\iota}_{A'} + \tau \iota_A \bar{o}_{A'} - \rho \iota_A \bar{\iota}_{A'} \right). \tag{2.23}$$

The expressions obtained in the previous paragraphs allow one to obtain an expression of the Killing spinor in terms of the spin basis. A calculation starting from the definition (2.5) readily yields the expression

$$\xi_{AA'} = -\frac{3}{2}\varkappa\big(\mu o_A \bar{o}_{A'} - \pi o_A \bar{\iota}_{A'} - \tau \iota_A \bar{o}_{A'} + \rho \iota_A \bar{\iota}_{A'} \big).$$

If $\xi_{AA'}$ is a Hermitian spinor, i.e. $\xi_{AA'} = \bar{\xi}_{AA'}$, then the previous expression implies

$$\bar{\mu}\bar{\varkappa} = \mu\varkappa, \qquad \bar{\tau}\bar{\varkappa} = \varkappa\pi, \qquad \bar{\rho}\bar{\varkappa} = \varkappa\rho. \tag{2.24}$$

The vacuum case. Using the previous expression along with the basis expansions for

κ_{AB} and Ψ_{ABCD}, *in vacuum*, the Ernst form can be expanded as

$$\chi_{AA'} = \frac{3}{4}\varkappa^2\psi\left(\mu o_A\bar{o}_{A'} - \pi o_A\bar{\iota}_{A'} + \tau\iota_A\bar{o}_{A'} - \rho\iota_A\bar{\iota}_{A'}\right). \tag{2.25}$$

Intuitively, one would expect it should be possible to express the Ernst form χ in terms of the scalars $\varkappa$ and ψ. As it will be seen in Section 2.4, the characterisation of the Kerr spacetime given by Theorem 2 suggests that a combination of the form $\mathfrak{c} + \frac{3}{4}\varkappa^2\psi$ with $\mathfrak{c}$ a constant is a suitable candidate. In order to compute the derivative of this expression one needs an expression for $\nabla_{AA'}\psi$. This can be obtained from the vacuum Bianchi identity

$$\nabla^A{}_{A'}\Psi_{ABCD} = 0.$$

Substituting the basis expansion for the Weyl spinor into the above relation, using equations (2.22a) and (2.22b), collecting terms and finally making use of $\epsilon_{AB} = o_A\iota_B - \iota_A o_B$ one obtains

$$\nabla_{AA'}\psi = -3\psi\left(\mu o_A\bar{o}_{A'} - \pi o_A\bar{\iota}_{A'} + \tau\iota_A\bar{o}_{A'} - \rho\iota_A\bar{\iota}_{A'}\right). \tag{2.26}$$

Combining this with expression (2.23) for $\nabla_{AA'}\varkappa$ we find that

$$\nabla_{AA'}\left(\mathfrak{c} - \frac{3}{4}\varkappa^2\psi\right) = \chi_{AA'}$$

so that the Ernst potential can be written as

$$\chi = \mathfrak{c} - \frac{3}{4}\varkappa^2\psi \qquad \text{for some} \quad \mathfrak{c}\in\mathbb{C}.$$

This expression can be simplified using the following observation: combining expressions for $\nabla_{AA'}\varkappa$ and $\nabla_{AA'}\psi$ given by equations (2.23) and (2.26), respectively, it can be shown that

$$\nabla_{AA'}\left(\varkappa^3\psi\right) = 0;$$

therefore, we have that

$$\varkappa^3\psi = \mathfrak{M} \tag{2.27}$$

with $\mathfrak{M}$ a (possibly complex) constant, and furthermore

$$\chi = \mathfrak{c} - \frac{3\mathfrak{M}}{4\varkappa}. \tag{2.28}$$

The electrovacuum case. From the electrovacuum Bianchi identity given by (1.7), a calculation yields

$$\nabla_{AA'}\psi = -3\left(\psi + 2\varphi\bar{\varphi}\right)\mu o_A \bar{o}_{A'} + 3\left(\psi - 2\varphi\bar{\varphi}\right)\pi o_A \bar{\iota}_{A'} \\ -3\left(\psi - 2\varphi\bar{\varphi}\right)\tau \iota_A \bar{o}_{A'} + 3\left(\psi + 2\varphi\bar{\varphi}\right)\rho \iota_A \bar{\iota}_{A'}.$$

Similarly, using the Maxwell equations (1.6) and the derivatives of the basis vectors given by equations (2.22a) and (2.22b), the derivative of the Maxwell proportionality factor φ is given by

$$\nabla_{AA'}\varphi = -2\varphi\left(\mu o_A \bar{o}_{A'} - \pi o_A \bar{\iota}_{A'} + \tau \iota_A \bar{o}_{A'} - \rho \iota_A \bar{\iota}_{A'}\right). \tag{2.29}$$

Thus, a further calculation using the previous expressions yields the following explicit expression for the electromagnetic Ernst potential:

$$\varsigma_{AA'} = 3\varkappa\varphi\big(\mu o_A \bar{o}_{A'} - \pi o_A \bar{\iota}_{A'} + \tau \iota_A \bar{o}_{A'} - \rho \iota_A \bar{\iota}_{A'}\big).$$

In the electrovacuum case, assuming an algebraically general Killing spinor and that the Maxwell spinor and the Killing spinor satisfy the matter alignment condition (2.7), the characterisation of the Kerr-Newman spacetime given in Theorem 4 suggests an expression for ς in terms of the scalars $\varkappa$, ψ and φ – namely $\mathfrak{c}' - \bar{\varkappa}\bar{\psi}/2\bar{\varphi}$ with $\mathfrak{c}'$ a constant. Combining the above expressions we can conclude that

$$\nabla_{AA'}\left(\mathfrak{c}' - \frac{\bar{\varkappa}\bar{\psi}}{2\bar{\varphi}}\right) = \varsigma_{AA'} \tag{2.30}$$

so that the potential can be set to be

$$\varsigma = \mathfrak{c}' - \frac{1}{2}\frac{\bar{\varkappa}\bar{\psi}}{\bar{\varphi}} \qquad \text{for some} \quad \mathfrak{c}' \in \mathbb{C}. \tag{2.31}$$

Moreover, combining expression (2.23) for $\nabla_{AA'}\varkappa$ with (2.29) it is straightforward to verify that

$$\nabla_{AA'}\left(\varkappa^2\varphi\right) = 0;$$

therefore, there exists a (possibly complex) constant $\mathfrak{Q}$ such that

$$\varkappa^2\varphi = \mathfrak{Q}. \tag{2.32}$$

In the electrovacuum case the relation between the scalars $\varkappa$ and ψ takes a more complicated form than in vacuum – cf. equation (2.27). Given a complex constant $\mathfrak{C}'$, a calculation using expressions (2.23), (2.29) and relation (2.32) shows that

$$\begin{aligned}\nabla_{AA'}\left(\frac{\mathfrak{C}}{\bar{\varkappa}} + \varkappa^3\psi\right) = &-\left(\frac{6|\mathfrak{Q}|^2\varkappa\mu}{\bar{\varkappa}^2} + \frac{\mathfrak{C}\bar{\mu}}{\bar{\varkappa}}\right)o_A\bar{o}_{A'} - \left(\frac{6|\mathfrak{Q}|^2\varkappa\pi}{\bar{\varkappa}^2} + \frac{\mathfrak{C}\bar{\tau}}{\bar{\varkappa}}\right)o_A\bar{\iota}_{A'} \\ &+ \left(\frac{6|\mathfrak{Q}|^2\varkappa\tau}{\bar{\varkappa}^2} + \frac{\mathfrak{C}\bar{\pi}}{\bar{\varkappa}}\right)\iota_A\bar{o}_{A'} + \left(\frac{6|\mathfrak{Q}|^2\varkappa\rho}{\bar{\varkappa}^2} + \frac{\mathfrak{C}\bar{\rho}}{\bar{\varkappa}}\right)\iota_A\bar{\iota}_{A'}.\end{aligned}$$

If the spinor $\xi_{AA'}$ is assumed to be Hermitian, then the previous expression reduces to

$$\nabla_{AA'}\left(\frac{\mathfrak{C}}{\bar{\varkappa}} + \varkappa^3\psi\right) = -\frac{\varkappa(\mathfrak{C} + 6|\mathfrak{Q}|^2)}{\bar{\varkappa}^2}\big(\mu o_A\bar{o}_{A'} + \pi o_A\bar{\iota}_{A'} - \tau\iota_A\bar{o}_{A'} - \rho\iota_A\bar{\iota}_{A'}\big).$$

Thus, by choosing $\mathfrak{C} = -6|\mathfrak{Q}|^2$, then the combination $\mathfrak{C}/\bar{\varkappa} + \varkappa^3\psi$ is a constant – that is, there exists $\mathfrak{M}' \in \mathbb{C}$ such that

$$\varkappa^3\psi - \frac{6|\mathfrak{Q}|^2}{\bar{\varkappa}} = \mathfrak{M}'. \tag{2.33}$$

Therefore, the scalar ψ can be expressed solely in terms of $\varkappa$ as

$$\psi = \frac{1}{\varkappa^3}\left(\mathfrak{M}' + \frac{6|\mathfrak{Q}|^2}{\bar{\varkappa}}\right). \tag{2.34}$$

Note that when the Maxwell field vanishes, then the constant $\mathfrak{Q}$ also vanishes and this equation reduces to the vacuum case given by (2.27).

Finally, it is observed that expanding expression (2.9) in terms of the spinor basis $\{o, \iota\}$ and using expressions (2.15) and (2.29) one concludes, after a calculation, that

$$\mathcal{L}_\xi \phi_{AB} = 0$$

– so that ϕ_{AB} inherits the symmetry generated by the Killing spinor κ_{AB}.

2.2.6 Spacetimes with an algebraically special Killing spinor

So far, the Killing spinor has been assumed to be algebraically general; in this section, this assumption is justified by briefly considering electrovacuum spacetimes with an algebraically special Killing spinor. These spacetimes will not play a role in the remainder of this chapter. The reason for this is the following result:

Lemma 1. *Let $(\mathcal{M}, \boldsymbol{g})$ be a smooth electrovacuum spacetime with a matter content satisfying the matter alignment condition and admitting a valence-2 Killing spinor κ_{AB} such that the associated field $\xi^{AA'}$ is a Hermitian spinor. If κ_{AB} is algebraically special (i.e. $\kappa_{AB} = \alpha_A\alpha_B$ for some non-vanishing spinor α_A) then $\xi^a = 0$.*

Proof. It follows directly from the existence of a non-vanishing algebraically special Killing spinor that the spacetime $(\mathcal{M}, \boldsymbol{g})$ must be of Petrov type N – see equation (2.4). That is, the basis expansion of the Weyl spinor has the form:

$$\Psi_{ABCD} = \psi\alpha_A\alpha_B\alpha_C\alpha_D \tag{2.35}$$

for some function ψ. As the matter alignment condition holds by assumption, the Hermitian spinor $\xi_{AA'}$ is the spinorial counterpart of a real Killing vector ξ^a. The content of the Killing form of ξ^a is encoded in the symmetric spinor η_{AB}. Substituting the expansions (2.35) and $\kappa_{AB} = \alpha_A\alpha_B$ into equation (2.14), it follows directly that $\eta_{AB} = 0$. Thus, the Killing form H_{ab} of ξ^a vanishes. Accordingly, ξ^a is a covariantly constant vector on $(\mathcal{M}, \boldsymbol{g})$:

$$\nabla_a \xi^b = 0. \tag{2.36}$$

In order to further investigate the consequences of this observation, introduce a normalised spin dyad $\{o^A,\, \iota^A\}$ with $o_A = \alpha_A$ and $o_A\iota^A = 1$. The Killing and Maxwell spinors have the basis expansions

$$\kappa_{AB} = o_A o_B, \qquad \phi_{AB} = \varphi o_A o_B.$$

Substituting the first of the above expressions into the Killing spinor equation $\nabla_{A'(A}\kappa_{BC)} = 0$ immediately implies that

$$\gamma = \alpha = \sigma = \kappa = 0, \qquad \rho + \epsilon = 0, \qquad \tau + \beta = 0. \tag{2.37}$$

Moreover, the Hermitian spinor $\xi_{AA'}$ can be expressed as

$$\xi_{AA'} = -3\beta o_A \bar{o}_{A'} + 3\epsilon o_A \bar{\iota}_{A'}.$$

The spinorial version of equation (2.36) implies $D\xi_{AA'} = 0$, $\Delta\xi_{AA'} = 0$, $\delta\xi_{AA'} = 0$ and $\bar{\delta}\xi_{AA'} = 0$. In particular, from $\Delta\xi_{AA'} = 0$ and $\bar{\delta}\xi_{AA'} = 0$, expanding in terms of the basis one finds that $\beta\tau = 0$ and $\epsilon\rho = 0$. Combining this expression with the third and fourth conditions in (2.37) produces the conclusion that

$$\tau = \beta = \epsilon = \rho = 0.$$

It follows then that

$$\xi_{AA'} = 0.$$

□

As we want to use the asymptotics of the Killing vector $\xi_{AA'}$ in the characterisation of the Kerr and Kerr-Newman spacetime, we will rule out the algebraically special case and assume that the Killing spinor is algebraically general – i.e. $\kappa_{AB}\kappa^{AB} \neq 0$.

Remark. Note that because $\Psi_{ABCD} \propto \kappa_{(AB}\kappa_{CD)}$, the conditions $\Psi_{ABCD}\Psi^{ABCD} \neq 0, \Psi_{ABCD} \neq 0$ imply that the Killing spinor is algebraically general and non-zero, i.e. $\kappa_{AB}\kappa^{AB} \neq 0$, $\kappa_{AB} \neq 0$. These two conditions on the curvature are precisely the ones assumed in Theorem 6 of [6], and so the characterisation of Kerr in terms of Killing spinors presented in that article is essentially the same as the one presented here. Despite this, we reproduce the result here for completeness and ease of comparison with the electrovacuum case.Here, this is done using the local result of Mars given in [40], whereas the proof in [6] uses the global result from [41]. In the absence of a generalisation to the electrovacuum case of the characterisation of [41], the analysis of the Kerr-Newman spacetime must make use of the local result by Wong [56].

2.3 Boundary conditions

This section provides a brief discussion of the boundary conditions which will be used in conjunction with the properties of Killing spinors to characterise the Kerr and Kerr-Newman spacetimes.

2.3.1 Stationary asymptotically flat ends

The remainder of this chapter will be concerned with spacetimes admitting a *stationary asymptotically flat 4-end* – see e.g. [56].

Definition 1. *A stationary asymptotically flat 4-end in an electrovacuum spacetime* $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ *is an open submanifold* $\mathcal{M}_\infty \subset \mathcal{M}$ *diffeomorphic to* $I \times (\mathbb{R}^3 \setminus \mathcal{B}_R)$ *where* $I \subset \mathbb{R}$ *is an open interval and* $\mathcal{B}_R$ *is a closed ball of radius* R. *In the local coordinates* (t, x^α) *defined by the diffeomorphism the components* $g_{\mu\nu}$ *and* $F_{\mu\nu}$ *of the metric* $\boldsymbol{g}$ *and the Faraday tensor* $\boldsymbol{F}$ *satisfy*

$$|g_{\mu\nu} - \eta_{\mu\nu}| + |r\partial_\alpha g_{\mu\nu}| \leq Cr^{-1}, \tag{2.38a}$$

$$|F_{\mu\nu}| + |r\partial_\alpha F_{\mu\nu}| \leq C' r^{-2}, \tag{2.38b}$$

$$\partial_t g_{\mu\nu} = 0, \tag{2.38c}$$

$$\partial_t F_{\mu\nu} = 0, \tag{2.38d}$$

where C *and* C' *are positive constants,* $r \equiv (x^1)^2 + (x^2)^2 + (x^3)^2$, *and* $\eta_{\mu\nu}$ *denote the components of the Minkowski metric in Cartesian coordinates.*

Remark 1. It follows from condition (2.38c) in Definition 1 that the stationary asymptotically flat end $\mathcal{M}_\infty$ is endowed with a Killing vector ξ^a which takes the form $\boldsymbol{\partial}_t$ – a so-called *time translation.* Condition (2.38d) implies that the electromagnetic field inherits the symmetry of the spacetime – that is $\mathcal{L}_\xi \boldsymbol{F} = 0$, with $\mathcal{L}_\xi$ the Lie derivative along ξ^a.

Of particular interest will be those stationary asymptotically flat ends *generated by a Killing spinor*:

Definition 2. *A stationary asymptotically flat end* $\mathcal{M}_\infty \subset \mathcal{M}$ *in an electrovacuum spacetime* $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ *endowed with a Killing spinor* κ_{AB} *is said to be generated by a Killing spinor if the spinor* $\xi_{AA'} \equiv \nabla^B{}_{A'}\kappa_{AB}$ *is the spinorial counterpart of the Killing vector* ξ^a.

Remark 2. Stationary spacetimes have a natural definition of mass in terms of the Killing vector ξ^a that generates the isometry – the so-called *Komar mass* m defined as

$$m \equiv -\frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} \epsilon_{abcd} \nabla^c \xi^d \mathrm{d}S^{ab}$$

where S_r is the sphere of radius r centred at $r = 0$ and $\mathrm{d}S^{ab}$ is the bi-normal surface element to S_r. Similarly, the *total electromagnetic charge* of the spacetime is defined via the integral

$$q = -\frac{1}{4\pi} \lim_{r\to\infty} \int_{S_r} F_{ab} \mathrm{d}S^{ab}.$$

Remark 3. In the asymptotic region the components of the metric can be written in the form

$$\begin{aligned} g_{00} &= 1 - \frac{2m}{r} + O(r^{-2}), \\ g_{0\alpha} &= \frac{4\epsilon_{\alpha\beta\gamma} S^\beta x^\gamma}{r^3} + O(r^{-3}), \\ g_{\alpha\beta} &= -\delta_{\alpha\beta} + O(r^{-1}), \end{aligned}$$

where m is the Komar mass of ξ^a in the end $\mathcal{M}_\infty$, $\epsilon_{\alpha\beta\gamma}$ is the flat rank 3 totally anti-symmetric tensor and S^β denotes a 3-dimensional tensor with constant entries. The components of the Faraday tensor are

$$\begin{aligned} F_{0\alpha} &= \frac{q}{r^2} + O(r^{-3}), \\ F_{\alpha\beta} &= O(r^{-3}) \end{aligned}$$

– see e.g. [53]. Therefore, to leading order any stationary asymptotically flat electrovacuum spacetime is asymptotically a Kerr-Newman spacetime.

2.3.2 Killing spinor and Killing vector asymptotics

In general, the spinor $\xi_{AA'}$ obtained from a Killing spinor κ_{AB} using formula (2.5) is not Hermitian. It is, however, well known that for the Kerr-Newman spacetime $\xi_{AA'}$ is indeed the spinorial counterpart of a real Killing vector ξ^a – see e.g. [3]. More generally, this observation applies to any electrovacuum spacetime with a stationary asymptotically flat end. To see this, note the following:

Lemma 2. *Let $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ be a smooth electrovacuum spacetime with a stationary asymptotically flat end $\mathcal{M}_\infty$, admitting a complex Killing vector field ξ^a. If ξ^a tends to a time translation at infinity in $\mathcal{M}_\infty$, then ξ^a is in fact a real vector in $\mathcal{M}_\infty$.*

Proof. The complex Killing vector can be written $\xi^a = \xi^a_1 + \mathrm{i}\xi^a_2$ for two real vectors ξ^a_1, ξ^a_2, which are also Killing vectors by linearity of the Killing vector equation. As a time translation $(\partial_t)^a$ is a real vector, we have $\xi^a_1 \to (\partial_t)^a$ and $\xi^a_2 \to 0$ as $r \to \infty$ in the asymptotically flat end $\mathcal{M}_\infty$. However, it is well known that there are no non-trivial real Killing vectors which vanish at infinity – see e.g. [12, 16]. Therefore, $\xi^a_2 = 0$ on $\mathcal{M}_\infty$, and $\xi^a = \xi^a_1$ is a real Killing vector. □

Therefore, by assuming that the Killing vector ξ^a is asymptotically a time translation, then the assumption requiring its spinorial equivalent $\xi_{AA'}$ to be a Hermitian spinor can be dropped. In fact, it is possible to replace this condition on the Killing vector with an asymptotic condition on the Killing spinor, as described in the following proposition:

Proposition 1. *Let $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ denote an electrovacuum spacetime with a stationary asymptotically flat end $\mathcal{M}_\infty$ generated by a Killing spinor κ_{AB}. Then the functions $\varkappa$, φ and ψ as defined by equations* (2.3), (2.4) *and* (2.8) *satisfy*

$$\begin{aligned}
\varkappa &= \frac{2}{3}r + O(1), \\
\varphi &= \frac{q}{r^2} + O(r^{-3}), \\
\psi &= -\frac{6m}{r^3} + O(r^{-4}).
\end{aligned}$$

Moreover, one has that

$$\xi^2 \equiv \xi_{AA'}\xi^{AA'} = 1 + O(r^{-1}).$$

Proof. The analysis in [53] shows that to leading order the electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ coincides on $\mathcal{M}_\infty$ with the Kerr-Newman spacetime. Thus, the expansions for the fields $\varkappa$, φ and φ must coincide to leading order with their expressions for the Kerr-Newman spacetime – see [3].

□

2.4 Characterisations of the Kerr spacetime

The motivation behind the analysis in this chapter is the following theorem by M. Mars – see [41]:

Theorem 2. *Let $(\mathcal{M}, \boldsymbol{g})$ be a smooth, vacuum spacetime admitting a Killing vector ξ^a with self-dual Killing form $\mathcal{H}_{ab}$. Let $\mathcal{M}$ satisfy:*

(i) there exists a non-empty region $\mathcal{M}_\bullet \subset \mathcal{M}$ where

$$\mathcal{H}^2 \equiv \mathcal{H}_{ab}\mathcal{H}^{ab} \neq 0;$$

(ii) The self-dual Killing form and the Weyl tensor are related by

$$\mathcal{C}_{abcd} = H\left(\mathcal{H}_{ab}\mathcal{H}_{cd} - \frac{1}{3}\mathcal{H}^2\mathcal{I}_{abcd}\right) \qquad (2.39)$$

where

$$\mathcal{I}_{abcd} \equiv \frac{1}{4}(g_{ac}g_{bd} - g_{ad}g_{bc} + \mathrm{i}\epsilon_{abcd})$$

and H is a complex scalar function.

Then there exist two complex constants $\mathfrak{l}$ *and* $\mathfrak{c}$ *such that*

$$H = \frac{6}{\mathfrak{c} - \chi}, \qquad \mathcal{H}^2 = -\mathfrak{l}(\mathfrak{c} - \chi)^4.$$

If, in addition, $\mathfrak{c} = 1$ *and* $\mathfrak{l}$ *is real positive, then* $(\mathcal{M}, \boldsymbol{g})$ *is locally isometric to the Kerr spacetime.*

Remark 1. It is important to emphasise that in the above Theorem the existence of the constants $\mathfrak{c}$ and $\mathfrak{l}$ and the functional dependence of H and $\mathcal{H}^2$ with respect to χ are a consequence of the hypotheses of the theorem – this should be contrasted with the electrovacuum case in which the existence of the analogous constants needs to be assumed.

Remark 2. A particular case of Theorem 2 occurs when $(\mathcal{M}, \boldsymbol{g})$ is *a priori* assumed to have an stationary asymptotically flat end $\mathcal{M}_\infty$ with the Killing vector ξ^a tending asymptotically to a time translation at infinity and such that the Komar mass associated to ξ^a is non-zero. These assumptions ensure that $\mathcal{H}^2 \neq 0$ in a region of the spacetime – namely, in $\mathcal{M}_\infty$. Therefore, only condition (2.39) needs to be verified to conclude that

$$H = \frac{6}{1 - \chi}$$

and that the spacetime is locally isometric to the Kerr spacetime – see Theorem 2 in [40].

Remark 3. The subsequent discussion will make use of the spinorial counterparts of the conditions in the previous Theorem. In particular, notice that the content of the combination $\mathcal{H}_{ab}\mathcal{H}_{cd} - \frac{1}{3}\mathcal{H}^2\mathcal{I}_{abcd}$ can be encoded in terms of the spinor η_{AB} as defined in equation (2.13) as

$$\left(4\eta_{AB}\eta_{CD} - \frac{2}{3}\eta_{EF}\eta^{EF}(\epsilon_{AD}\epsilon_{BC} + \epsilon_{AC}\epsilon_{BD})\right)\epsilon_{A'B'}\epsilon_{C'D'} = 4\eta_{(AB}\eta_{CD)}\epsilon_{A'B'}\epsilon_{C'D'}$$

where the last expression follows from a decomposition in irreducible terms. Thus, condition (2.39) can be concisely re-expressed in terms of spinors as

$$\Psi_{ABCD} = 2H\eta_{(AB}\eta_{CD)}. \tag{2.40}$$

Finally, it is noticed that the condition $\mathcal{H}^2 \neq 0$ can be re-expressed as

$$\eta_{AB}\eta^{AB} \neq 0.$$

2.4.1 Killing spinors and the Mars characterisation

This section will analyse the extent to which existence of a Killing spinor on a vacuum spacetime implies the hypotheses of the characterisation of Kerr given in Theorem 2. The assumptions to be made in the remainder of this section shall here be explicitly stated for completeness:

Assumption 1. *Let $(\mathcal{M}, \boldsymbol{g})$ be a smooth vacuum spacetime and let $\mathcal{K} \subset \mathcal{M}$ such that:*

(i) on $\mathcal{K}$ there exists an algebraically general Killing spinor κ_{AB};

(ii) the spinor $\xi_{AA'} \equiv \nabla^B{}_{A'}\kappa_{AB}$ is on $\mathcal{K}$ the spinorial counterpart of a real Killing spinor ξ^a – i.e. $\xi_{AA'}$ is Hermitian.

Under the above assumptions, it follows from combining the basis expansion for Ψ_{ABCD} and η_{AB}, equations (2.4) and (2.15), respectively, that

$$\Psi_{ABCD} = \frac{16}{\varkappa^2\psi}\eta_{(AB}\eta_{CD)}.$$

Thus, hypothesis *(ii)* of Theorem 2 is satisfied with

$$H = \frac{8}{\varkappa^2\psi}$$

– cf. equation (2.40). Using the expression for the Ernst potential predicted by the theory of Killing spinors, equation (2.28), it follows that

$$H = \frac{6}{\mathfrak{c} - \chi}$$

which is precisely the form for H predicted by Theorem 2. From this expression one can further conclude that

$$\mathcal{H}^2 = -\frac{\mathfrak{M}}{3}\left(\frac{4}{3\mathfrak{M}}\right)^3 (\mathfrak{c} - \chi)^4. \tag{2.41}$$

This, again, is the form predicted by Theorem 2.

The above observations allow the formulation of the following *Killing spinor version* of Theorem 2:

Proposition 2. *Let $(\mathcal{M}, \boldsymbol{g})$ denote a smooth vacuum spacetime endowed with a Killing spinor κ_{AB} with $\kappa_{AB}\kappa^{AB} \neq 0$ such that the spinor $\xi_{AA'} \equiv \nabla^B{}_{A'}\kappa_{AB}$ is Hermitian. Then there exist two complex constants $\mathfrak{l}$ and $\mathfrak{c}$ such that*

$$\mathcal{H}^2 = -\mathfrak{l}(\mathfrak{c} - \chi)^4. \tag{2.42}$$

If, in addition, $\mathfrak{c} = 1$ and $\mathfrak{l}$ is real positive, then $(\mathcal{M}, \boldsymbol{g})$ is locally isometric to the Kerr spacetime.

2.4.1.1 A characterisation using asymptotic flatness

Proposition 2 requires the setting of two complex constants by hand, in order to recover the Kerr spacetime. It is possible to avoid this by introducing a further, physically reasonable assumption - that the set $\mathcal{K} \subset \mathcal{M}$ contains a stationary asymptotically flat end with the Killing spinor κ_{AB} generating the time translation Killing vector.

From Proposition 1 it readily follows that

$$(\mathfrak{c} - \chi)^4 = \frac{16m^4}{r^4} + O(r^{-5}).$$

Similarly, one has, using equation (2.15), that

$$\mathcal{H}^2 = -4\eta^2 = -\frac{4m^2}{r^4} + O(r^{-5}).$$

Thus, by consistency with the required asymptotic behaviour of the Ernst potential, one has to set $\mathfrak{c} = 1$ and the constant $\mathfrak{l}$ in Proposition 2 is given by $\mathfrak{l} = 1/4m^2$.

We can summarise the discussion of the previous section in the following:

Theorem 3. *Let $(\mathcal{M}, \boldsymbol{g})$ be a smooth vacuum spacetime containing a stationary asymptotically flat end $\mathcal{M}_\infty$ generated by a Killing spinor κ_{AB}. Let the Komar mass associated to the Killing vector $\xi_{AA'} = \nabla^B{}_{A'}\kappa_{AB}$ in $\mathcal{M}_\infty$ be non-zero. Then, $(\mathcal{M}, \boldsymbol{g})$ is locally isometric to the Kerr spacetime.*

Remark. As observed in [3] the requirement on the non-vanishing of the Komar mass can be replaced by an assumption on the existence of a horizon.

2.5 Characterisations of the Kerr-Newman spacetime

We now move on to discuss characterisations of the Kerr-Newman spacetime through Killing spinors. Our starting point is the following result – see [56]:

Theorem 4. *Let $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ be a smooth, electrovacuum spacetime admitting a real Killing vector ξ^a. Assume further that ξ^a is timelike somewhere in $\mathcal{M}$ and that F_{ab} is non-null on $\mathcal{M}$ (i.e. $\mathcal{F}^2 \equiv \mathcal{F}_{ab}\mathcal{F}^{ab} \neq 0$) and that it inherits the symmetry of the spacetime – i.e.*

$$\mathcal{L}_\xi \mathcal{F}_{ab} = 0. \tag{2.43}$$

Assume, furthermore, that there exists a complex scalar P, a normalisation for ς and a *complex constant* $\mathfrak{c}_1$ *such that:*

$$P^{-4} = -\mathfrak{c}_1^2 \mathcal{F}^2, \tag{2.44a}$$

$$\mathcal{H}_{ab} = -\frac{1}{2}\bar{\varsigma}\mathcal{F}_{ab}, \tag{2.44b}$$

$$\mathcal{C}_{abcd} = 3P\left(\frac{1}{2}\mathcal{F}_{ab}\mathcal{H}_{cd} + \frac{1}{2}\mathcal{F}_{ab}\mathcal{H}_{cd} - \frac{1}{3}\mathcal{I}_{abcd}\mathcal{F}_{ef}\mathcal{H}^{ef}\right). \tag{2.44c}$$

Then there exist complex constants $\mathfrak{c}_2$ *and* $\mathfrak{c}_3$ *such that:*

$$P = \frac{2}{\mathfrak{c}_2 - \varsigma}, \tag{2.45a}$$

$$4\xi^2 - |\varsigma|^2 = \mathfrak{c}_3. \tag{2.45b}$$

If, further, $\mathfrak{c}_2$ *is such that* $\mathfrak{c}_1\bar{\mathfrak{c}}_2$ *is real and* $\mathfrak{c}_3$ *is such that* $|\mathfrak{c}_2|^2 + \mathfrak{c}_3 = 4$*, then* $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ *is locally isometric to a Kerr-Newman spacetime.*

Remark 1. As in Section 2.4, this section will make use of a reformulation of the conditions in Theorem 4 in spinorial formalism. A direct computation shows that (2.44a) can be rewritten as

$$P^{-4} = -8\mathfrak{c}_1^2 \phi_{AB}\phi^{AB}.$$

Similarly, condition (2.44b) can be expressed in terms of the spinors η_{AB} and φ_{AB} as

$$\eta_{AB} = -\frac{1}{2}\bar{\varsigma}\phi_{AB},$$

while, finally, equation (2.44c) is equivalent to

$$\Psi_{ABCD} = 6P\eta_{(AB}\phi_{CD)}.$$

2.5.1 Killing spinors and Wong's characterisation

This section will investigate some further consequences of the existence of Killing spinors on electrovacuum spacetimes. Again, the assumptions to be made in the remainder of this section shall be explicitly stated here for completeness:

Assumption 2. *Let $(\mathcal{M}, \boldsymbol{g})$ be a smooth electrovacuum spacetime and let $\mathcal{K} \subset \mathcal{M}$ such that:*

(i) on $\mathcal{K}$ there exists an algebraically general Killing spinor κ_{AB};

(ii) the spinor $\xi_{AA'} \equiv \nabla^{B}{}_{A'}\kappa_{AB}$ is on $\mathcal{K}$ the spinorial counterpart of a real Killing spinor ξ^a – i.e. $\xi_{AA'}$ is Hermitian;

(iii) the Killing spinor κ_{AB} and the Maxwell spinor ϕ_{AB} satisfy the alignment condition $\kappa_{(A}{}^{Q}\phi_{B)Q} = 0$ – that is, they are proportional.

As already discussed in Section 2.2.5.3, under the above assumptions it follows that $\mathcal{L}_\xi \phi_{AB} = 0$ which, in turn, implies that $\mathcal{L}_\xi F_{ab} = 0$. Thus, the electromagnetic field inherits the symmetry generated by the Killing spinor κ_{AB}.

From the discussion in Sections 2.2.1 and 2.2.4 it follows that

$$\Psi_{ABCD} = \psi o_{(A}o_B\iota_C\iota_{D)}, \qquad \eta_{(AB}\phi_{CD)} = \eta\varphi o_{(A}o_B\iota_C\iota_{D)}.$$

Thus, the spinorial version of condition (2.44c) in Theorem 4 is satisfied with a proportionality function P given by

$$P = \frac{2}{3\varkappa\varphi}.$$

Now, making use of expressions (2.31), (2.32) and (2.34) to rewrite P in terms of the electromagnetic Ernst potential, it follows that

$$P = \frac{2}{\mathfrak{c}_2 - \varsigma}, \qquad \mathfrak{c}_2 \equiv \mathfrak{c}' - \frac{\bar{\mathfrak{M}}'}{2\bar{\mathfrak{Q}}}.$$

Thus, under the Assumptions 2, hypothesis (2.44c) and conclusion (2.45a) in Theorem 4 are satisfied.

Moreover, from the discussion in Section 2.2.5.3 it follows that the spinors η_{AB} and

ϕ_{AB} are proportional to each other with a proportionality function ς given by

$$\bar{\varsigma} = -\frac{\varkappa\psi}{2\varphi}.$$

The calculations of Section 2.2.5, cf. equation (2.30) in particular, show that ς satisfies the properties to be expected from the electromagnetic Ernst potential. Therefore, by setting the constant $\mathfrak{c}'$ in the definition of ς given by (2.31) to zero (and thereby fixing the normalisation of the potential), condition (2.44b) is satisfied. A similar remark holds for condition (2.44a) with the constant $\mathfrak{c}_1$ given by

$$\mathfrak{c}_1^2 = \frac{81}{64}\mathfrak{Q}^2.$$

In the presence of a Killing spinor, the norm $\xi^2 \equiv \xi_a\xi^a$ of the associated Killing vector is related to the electromagnetic form ς. To see this consider

$$\begin{aligned}\nabla_{AA'}\xi^2 &= 2\xi^{BB'}\nabla_{AA'}\xi_{BB'}\\ &= -2\eta_{AB}\xi^{B}{}_{A'} - 2\bar{\eta}_{A'B'}\xi_A{}^{B'}\end{aligned}$$

where in the second line it has been used that

$$\nabla_{AA'}\xi_{BB'} = \eta_{AB}\epsilon_{A'B'} + \bar{\eta}_{A'B'}\epsilon_{AB}.$$

As the spinors η_{AB} and ϕ_{AB} are proportional to each other, one can write

$$\begin{aligned}\nabla_{AA'}\xi^2 &= \bar{\varsigma}\xi^{A}{}_{B'}\phi_{AB} + \varsigma\xi_B{}^{A'}\bar{\phi}_{A'B'}\\ &= \frac{1}{4}\left(\bar{\varsigma}\nabla_{BB'}\varsigma + \varsigma\nabla_{BB'}\bar{\varsigma}\right)\\ &= \frac{1}{4}\nabla_{BB'}|\varsigma|^2.\end{aligned}$$

Therefore, locally there exists a constant $\mathfrak{c}_3$ such that

$$4\xi^2 - |\varsigma|^2 = \mathfrak{c}_3.$$

Thus, conclusion (2.45b) in Theorem 4 is also a consequence of the existence of a Killing spinor.

The discussion of this section can be summarised with the following Killing spinor version of Theorem 4:

Proposition 3. *Let* $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ *denote a smooth electrovacuum spacetime satisfying the matter alignment condition, endowed with a Killing spinor* κ_{AB} *with* $\kappa_{AB}\kappa^{AB} \neq 0$ *such that the spinor* $\xi_{AA'} \equiv \nabla^B{}_{A'}\kappa_{AB}$ *is Hermitian. Then there exist two constants* $\mathfrak{c}_2$ *and* $\mathfrak{c}_3$ *such that*

$$(\mathfrak{c}_2 - \varsigma)^4 = -\left(\frac{9}{8}\mathfrak{Q}\right)^2 \mathcal{F}^2, \qquad 4\xi^2 - |\varsigma|^2 = \mathfrak{c}_3.$$

If, further, $\mathfrak{c}_2$ *is such that* $\bar{\mathfrak{c}}_2\mathfrak{Q}$ *is real and* $\mathfrak{c}_3$ *is such that* $|\mathfrak{c}_2|^2 + \mathfrak{c}_3 = 4$*, then* $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ *is locally isometric to a Kerr-Newman spacetime.*

2.5.1.1 A characterisation using asymptotic flatness

As in Proposition 2, the above result relies on the setting of complex constants by hand; to avoid this, assume that the domain $\mathcal{K} \subset \mathcal{M}$ considered in the Assumptions 3 contains an stationary asymptotic flat end with the Killing spinor κ_{AB} generating the time translation Killing vector. We use this further assumption to determine the values of the constants in Proposition 3.

Combining the asymptotic expansions of Proposition 1 with the relation (2.32) gives

$$\mathfrak{Q} = \frac{4}{9}q \in \mathbb{R}.$$

Similarly, using equation (2.33) it follows that

$$\mathfrak{M}' = -\frac{16}{9}m.$$

A further computation using equation (2.31) and (2.32), respectively, shows that

$$(\mathfrak{c}_2 - \varsigma)^4 = \left(\mathfrak{c}_2 - \frac{2m}{q} + O(r^{-1})\right)^4, \qquad \left(\frac{9}{8}\mathfrak{Q}\right)^2 \mathcal{F}^2 = -\frac{q^4}{r^4} + O(r^{-5}).$$

Therefore, for consistency, one has to set

$$\mathfrak{c}_2 = \frac{2m}{q}$$

and we must have $\bar{\mathfrak{c}}_2\mathfrak{Q} \in \mathbb{R}$. From the previous discussion it follows that $\varsigma = 2m/q + O(r^{-1})$ so that, together with $\xi^2 = 1 + O(r^{-1})$, we can conclude that $\mathfrak{c}_3$ as defined by equation (2.45b) is given by

$$\mathfrak{c}_3 = 4\left(1 - \frac{m^2}{q^2}\right).$$

Accordingly one has that $|\mathfrak{c}_2|^2 + \mathfrak{c}_3 = 4$ as required.

The discussion of the previous paragraphs can be summarised in the following:

Theorem 5. *Let $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ be a smooth, electrovacuum spacetime satisfying the matter alignment condition, with a stationary asymptotically flat end $\mathcal{M}_\infty$ generated by a Killing spinor κ_{AB}. Let both the Komar mass associated to the Killing vector $\xi_{AA'} = \nabla^B{}_{A'}\kappa_{AB}$ and the total electromagnetic charge in $\mathcal{M}_\infty$ be non-zero. Then $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ is locally isometric to the Kerr-Newman spacetime.*

2.6 Applications

The advantage of the Killing spinor characterisation of the Kerr and Kerr-Newman solutions is that the existence of such a spinor is a geometric condition, with only reasonable asymptotic conditions needing to be further assumed for the results presented above.

This geometric condition can be converted into initial data for a spacelike Cauchy surface, in a way compatible with the constraint equations. This can then be exploited to construct a geometric invariant for the initial data set, which parametrises the deviation of the resulting global development of the initial data set from the exact Kerr or Kerr-Newman solution. Various versions of this construction analysis have been considered in [4–7] for the vacuum case. A generalisation of these constructions to the electrovacuum case will be given in the next chapter.

In this chapter, we have considered only vacuum and electrovacuum spacetimes; it may be possible to extend these results to spacetimes with other matter contents. For example, in [42] the authors extend the results of [41] to spacetimes with non-zero cosmological constant; it is possible that the necessary conditions of these generalised results can be replaced with the existence of a Killing spinor, or a conformal Killing spinor.

Finally, the results of this chapter suggest that the characterisations of the Kerr-Newman spacetime given by Wong in [56] can be improved to an optimal theorem in which condition (2.44a) in Theorem 4 is a consequence of the other hypotheses. An optimal result of this type is desirable if one is to attempt to use this type of characterisation to construct an alternative approach to the uniqueness of black holes.

Chapter 3

A geometric invariant characterising Kerr-Newman initial data

The contents of this chapter reproduce the arguments given in the paper [19].

3.1 Introduction

As we have seen, the Kerr-Newman solution can be characterised using the existence of a solution to the Killing spinor equation in a natural way. A key condition in the generalisation of the vacuum result to the electrovacuum case was the matter alignment condition - in other words, the assumption that the Maxwell spinor and Killing spinor are proportional. Although this condition restricts the form of the matter content of the spacetime, it has the reasonable physical interpretation of requiring the matter fields to 'inherit' the symmetry described by the Killing spinor. The most attractive feature of Theorem 5 is that all of the assumptions are either physically motivated asymptotic conditions, or requirements that the spacetime respect this 'hidden' symmetry.

Once the motivation for a characterisation of the Kerr-Newman spacetime in terms of Killing spinors has been established, it is useful to investigate how the existence of such a spinor can be expressed in terms of initial data. The initial value problem in General Relativity has played a crucial role in the systematic analysis of the properties of generic solutions to the Einstein field equations – see e.g. [25, 50, 51]. It also provides the framework necessary for numerical simulations of spacetimes to be performed – see e.g. [1, 9].

As described in Chapter 1, symmetries of a spacetime can be represented as conditions on an initial hypersurface via the KID equations – see [10]. These equations form a overdetermined system, so for arbitrary initial data sets satisfying the vacuum constraint equations, solutions will not necessarily exist. This corresponds to the observation that generic solutions to the vacuum Einstein equations do not admit any spacetime Killing vectors (see [11]). An analogous construction can, in principle, be performed for Killing spinors. This analysis has been performed for the vacuum case giving explicitly the conditions relating the Killing spinor candidate and the Weyl curvature of the spacetime – see [28] and also [6]. These conditions are, like the KID equations, an overdetermined system and so do not necessarily admit a solution for an arbitrary initial surface. However, in [5, 6] it has been shown that given an asymptotically Euclidean hypersurface it is always possible to construct a *Killing spinor candidate* which, whenever there exists a Killing spinor in the development, coincides with the restriction of the Killing spinor to the initial hypersurface. This approximate Killing spinor is obtained by solving a linear second order elliptic equation which is the Euler-Lagrange equation of a certain functional over $\mathcal{S}$. The approximate Killing spinor can be used to construct a geometric invariant which in some way parametrises the deviation of the initial data set from Kerr initial data. Variants of the basic construction in [6] have been given in [4, 7].

The purpose of this chapter is to extend the analysis of [6] to the electrovacuum case. In doing so, we rely on the characterisation of the Kerr-Newman spacetime given in [18] which, in turn, builds upon the characterisation provided in [41] for the vacuum case and [56] for the electrovacuum case. As a result of our analysis we find that the Killing spinor initial data equations remain largely unchanged, with extra conditions ensuring that the electromagnetic content of the spacetime inherits the symmetry of the Killing spinor. These electrovacuum Killing spinor equations, together with an appropriate approximate Killing spinor, are used to construct an invariant expressed in terms of suitable integrals over the hypersurface $\mathcal{S}$ whose vanishing characterises in a necessary and sufficient manner initial data for the Kerr-Newman spacetime. Our main result, in this respect, is given in Theorem 10.

Overview of the chapter

Section 3.2 provides a brief recap of the theory of Killing spinors in electrovacuum spacetimes, and defines some of the relevant quantities for this chapter. Section 3.3 discusses the evolution equations governing the propagation of the Killing spinor equation in an electrovacuum spacetime. The main conclusion from this analysis is that the resulting second-order system is linear and homogeneous in a certain set of zero-quantities and their first derivatives. The trivial initial data for these equations, sufficient to guarantee the existence of a unique solution, give rise to conditions implying the existence of a Killing spinor in the development of an initial hypersurface. In Section 3.4 the space-spinor formalism is used to re-express these conditions in terms of quantities defined on the initial hypersurface. In addition, in this section the interdependence between the various conditions is analysed and a minimal set of *Killing spinor data equations* is obtained. Section 3.5 introduces the notion of approximate Killing spinors for electrovacuum initial data sets and discusses some basic ellipticity properties of the associated approximate Killing spinor equation. Section 3.6 discusses the construction of a solution to the approximate Killing spinor equation for a class of asymptotically Euclidean manifolds. Finally, Section 3.7 brings together the analyses in the various section to construct a geometric invariant characterising initial data for the Kerr-Newman spacetime. The main result of this chapter is given in Theorem 10.

Recap of notation and conventions

Let $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ denote an electrovacuum spacetime, i.e. a solution to the Einstein-Maxwell field equations. The signature of the metric in this chapter will again be $(+,-,-,-)$, and we will continue to use the spinorial conventions of [45]. The lowercase Latin letters a, b, $c, \dots$ are used as abstract spacetime tensor indices while the uppercase letters A, B, $C, \dots$ will serve as abstract spinor indices. The Greek letters μ, ν, $\lambda, \dots$ will be used as spacetime coordinate indices while α, β, $\gamma, \dots$ will serve as spatial coordinate indices. Finally $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}, \dots$ will be used as spinorial frame indices.

The conventions for the spinorial curvature tensors will be as described in (1.4), and the expression for the once-contracted second derivative of a spinor given in (1.8) will be used systematically.

3.2 Killing spinors in electrovacuum spacetimes

3.2.1 The Einstein-Maxwell equations

To recap from a previous chapter, the Einstein-Maxwell equations are given by

$$\Phi_{ABA'B'} = 2\phi_{AB}\bar{\phi}_{A'B'}, \qquad \Lambda = 0,$$
$$\nabla^{A}{}_{A'}\phi_{AB} = 0.$$

Given a Maxwell spinor in an electrovacuum spacetime, applying the derivative $\nabla^{A'}{}_{C}$ to the Maxwell equation in the form $\nabla^{A}{}_{A'}\phi_{AB} = 0$ gives, after some standard manipulations, the following wave equation for the Maxwell spinor:

$$\square\phi_{AB} = 2\Psi_{ABCD}\phi^{CD}. \tag{3.1}$$

3.2.2 Killing spinors

Recall that a *Killing spinor* $\kappa_{AB} = \kappa_{(AB)}$ in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ is a solution to the Killing spinor equation

$$\nabla_{A'(A}\kappa_{BC)} = 0. \tag{3.2}$$

In this chapter, a prominent role will played by the integrability conditions implied by the Killing spinor equation. More precisely, one has the following:

Lemma 3. *Let* $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ *denote an electrovacuum spacetime endowed with a Killing spinor* κ_{AB}. *Then* κ_{AB} *satisfies the integrability conditions:*

$$\kappa_{(A}{}^{Q}\Psi_{BCD)Q} = 0, \tag{3.3a}$$

$$\square\kappa_{AB} + \Psi_{ABCD}\kappa^{CD} = 0. \tag{3.3b}$$

Proof. The integrability conditions follow from applying the derivative $\nabla_D{}^{A'}$ to the Killing spinor equation (3.2), then using the identity (1.8) together with the box commutators (1.4) and finally decomposing the resulting expression into its irreducible terms – the only non-trivial trace yields equation (3.3b) while the completely symmetric part gives equation (3.3a). □

Remark 6. Observe that although every solution to the Killing spinor equation (3.2) satisfies the wave equation (3.3b), the converse is not true. In what follows, a symmetric spinor satisfying equation (3.3b), but not necessarily equation (3.2), will be called a *Killing spinor candidate.* This notion will play a central role in our subsequent analysis – in particular, we will be concerned with the question of the further conditions that need to be imposed on a Killing spinor candidate to be a true Killing spinor.

In the previous chapter, the Killing spinor was used to construct a (complex) vector $\xi_{AA'}$ via equation (2.5). In vacuum, this vector is by construction a Killing vector;

however, this is only true in electrovacuum spacetimes if the matter alignment condition (given in equation (2.7)) holds. If this condition is satisfied, then the Killing spinor κ_{AB} and Maxwell spinor ϕ_{AB} are necessarily proportional - in other words, the electromagnetic fields in the spacetime are 'aligned' with the symmetry represented by the Killing spinor. Of course, the vector $\xi_{AA'}$ can be constructed from any symmetric candidate spinor κ_{AB} via (2.5), even if κ_{AB} is not a Killing spinor; in this case, $\xi_{AA'}$ will be referred to as the *Killing vector candidate* associated to κ_{AB}.

3.2.3 Zero-quantities

From here onwards, the calculations performed will make use of *xAct*, a suite of tensor computer algebra packages for Mathematica. Although the calculations could be straightforwardly carried out by hand, the size and complexity of some expressions makes this unwieldy; therefore, computer algebra packages provide significant time savings, and allow one to focus on the structure of and relationships between expressions rather than on their explicit composition. Some common operations which can be performed include:

- Commutation of derivatives (including covariant, Sen and normal)
- Decomposition of spinorial expressions into irreducible parts; this operation is one of the most useful ways of simplifying a long expression, as many of the irreducible parts trivially or can easily be shown to vanish
- Substitution of one equation into another - for example, the elimination of $\square\kappa_{AB}$ from wave equation computations using equation (3.3b)
- Reduction of second-order derivative terms to curvature, for example via the box commutators (1.4).

Some of the code used to perform these operations could be recycled from the analysis of the vacuum case, given in [6]; however, these rules had to updated and complemented with additional rules to take into account the non-vanishing matter content of electrovacuum spacetimes. The precise order in which these rules are applied is usually motivated by the form of the final expression that is wished to be obtained, mirroring the order that would be used were the calculations being done by hand. The website for the suite, including downloads, documentation and updates, can be found at `www.xact.es` [43].

In order to investigate the consequences of the Killing spinor equation (3.2) in a more systematic manner it is convenient to introduce the following *zero-quantities*:

$$H_{A'ABC} \equiv 3\nabla_{A'(A}\kappa_{BC)}, \tag{3.4a}$$

$$S_{AA'BB'} \equiv \nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'}, \tag{3.4b}$$

$$\Theta_{AB} \equiv 2\kappa_{(A}{}^{Q}\phi_{B)Q}. \tag{3.4c}$$

Observe that if $H_{A'ABC} = 0$ then κ_{AB} is a Killing spinor. Similarly, if $S_{AA'BB'} = 0$ then $\xi_{AA'}$ is the spinor counterpart of a Killing vector, while if $\Theta_{AB} = 0$ then the matter alignment condition (2.7) holds.

The decomposition in irreducible components of $\nabla_{AA'}\kappa_{BC}$ can be expressed in terms of $H_{A'ABC}$ and $\xi_{AA'}$ as

$$\nabla_{AA'}\kappa_{BC} = \frac{1}{3}H_{A'ABC} - \frac{2}{3}\epsilon_{A(B}\xi_{C)A'}. \tag{3.5}$$

Similarly, a further computation shows that for $\xi_{AA'}$ as given by equation (2.5) one has the decomposition

$$\nabla_{AA'}\xi_{BB'} = \bar{\eta}_{A'B'}\epsilon_{AB} + \eta_{AB}\epsilon_{A'B'} + \frac{1}{2}S_{AA'BB'} \tag{3.6}$$

where

$$\eta_{AB} \equiv \frac{1}{2}\nabla_{AQ'}\xi_{B}{}^{Q'}$$

is the spinorial equivalent of the self-dual Killing form defined in (2.10).

Remark 7. From equation (2.5) it readily follows by contraction that

$$\nabla^{AA'}\xi_{AA'} = 0$$

independently of whether the alignment condition (2.7) holds or not – i.e. the Killing vector candidate $\xi_{AA'}$ defined by equation (2.5) is always divergence free. This observation, in turn, implies that

$$S_{AA'}{}^{AA'} = 0,$$

so that one has the symmetry

$$S_{AA'BB'} = S_{(AB)(A'B')}. \tag{3.7}$$

Remark 8. The zero-quantities introduced in equations (3.4a)-(3.4c) are a helpful bookkeeping device. In particular, calculations analogous to that of the proof of Lemma 3 show that

$$\nabla_{(A}{}^{A'}H_{|A'|BCD)} = -6\Psi_{Q(ABC}\kappa_{D)}{}^{Q},$$
$$\nabla^{AA'}H_{A'ABC} = 2\big(\Box\kappa_{BC} + \Psi_{BCPQ}\kappa^{PQ}\big).$$

Therefore, the integrability conditions of Lemma 3 can be written, alternatively, as

$$\nabla_{(A}{}^{A'}H_{|A'|BCD)} = 0, \qquad \nabla^{AA'}H_{A'ABC} = 0.$$

In particular, observe that if κ_{AB} is a Killing spinor candidate, then the zero quantity $H_{A'ABC}$ constructed from κ_{AB} is divergence free.

3.3 The Killing spinor evolution system in electrovacuum spacetimes

In this section we systematically investigate the interrelations between the zero-quantities $H_{A'ABC}$, $S_{AA'BB'}$ and Θ_{AB}. The ultimate objective of this analysis is to obtain a system of linear, homogeneous wave equations for the zero-quantities; it will follow that the global solution to this system with vanishing zero and first order derivatives on an initial hypersurface $\mathcal{S}$ is unique and also vanishing, giving rise to a Killing spinor on the development of the initial data.

3.3.1 A wave equation for $\xi_{AA'}$

Given a Killing spinor candidate κ_{AB}, the wave equation (3.3b) naturally implies a wave equation for the Killing vector candidate $\xi_{AA'}$. First, note the following alternative expression for the field $S_{AA'BB'}$:

Lemma 4. *Let κ_{AB} denote a symmetric spinor field in an electrovacuum $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Then, one has that*

$$S_{AA'BB'} = 6\bar{\phi}_{A'B'}\Theta_{AB} - \frac{1}{2}\nabla_{PA'}H_{B'AB}{}^{P} - \frac{1}{2}\nabla_{PB'}H_{A'AB}{}^{P}. \qquad (3.8)$$

Proof. To obtain the identity, start by substituting the expression $\xi_{AA'} = \nabla^{Q}{}_{A'}\kappa_{QA}$ into the definition of $S_{AA'BB'}$, equation (3.4b). Then, commute covariant derivatives using the commutators (1.4) and make use of the decompositions of $\nabla_{AA'}\kappa_{BC}$, $\nabla_{AA'}\xi_{BB'}$ and $S_{AA'BB'}$ given by equations (3.5), (3.6) and (3.7), respectively, to simplify. □

Remark 9. Observe that in the above result the spinor κ_{AB} is not assumed to be a Killing spinor candidate.

The latter is used, in turn, to obtain a wave equation for the Killing vector candidate:

Lemma 5. *Let κ_{AB} denote a Killing spinor candidate in an electrovacuum spacetime*

$(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. *Then the Killing vector candidate* $\xi_{AA'} \equiv \nabla^{Q}{}_{A'}\kappa_{AQ}$ *satisfies the wave equation*

$$\square\xi_{AA'} = -2\xi^{PP'}\Phi_{APA'P'} + \Phi^{PQ}{}_{A'}{}^{P'}H_{P'APQ} - \Psi_{APQD}H_{A'}{}^{PQD} + 6\bar{\phi}_{A'}{}^{P'}\nabla_{PP'}\Theta_{A}{}^{P}. \quad (3.9)$$

Proof. One makes use of the definition of $S_{AA'BB'}$ and the identity (3.8) to write

$$\begin{aligned}\nabla^{AA'}\nabla_{AA'}\xi_{BB'} + \nabla^{AA'}\nabla_{BB'}\xi_{AA'} &= 6\nabla^{AA'}\left(\Theta_{AB}\bar{\phi}_{A'B'}\right) - \frac{1}{2}\nabla^{AA'}\nabla_{CA'}H_{B'AB}{}^{C} \\ &\quad -\frac{1}{2}\nabla^{AA'}\nabla_{CB'}H_{A'AB}{}^{C}.\end{aligned}$$

The above expression can be simplified using the Maxwell equations. Moreover, commuting covariant derivatives in the terms $\nabla^{AA'}\nabla_{CA'}H_{B'AB}{}^{C}$ and $\nabla^{AA'}\nabla_{CB'}H_{A'AB}{}^{C}$ gives:

$$\begin{aligned}\square\xi_{AA'} = -2\xi^{PP'}\Phi_{APA'P'} + \Phi^{PQ}{}_{A'}{}^{P'}H_{P'APQ} - \Psi_{APQD}H_{A'}{}^{PQD} + 6\bar{\phi}_{A'}{}^{P'}\nabla_{PP'}\Theta_{A}{}^{P} \\ -\nabla_{AA'}\nabla_{PP'}\xi^{PP'} - \frac{1}{2}\nabla_{QA'}\nabla_{PP'}H^{P'}{}_{A}{}^{PQ}.\end{aligned}$$

Finally, using $\nabla^{AA'}H_{A'ABC} = 0$ (see Remark 8) and the fact that $\xi_{AA'}$ is a Killing vector candidate (see Remark 7), the result follows. □

Remark 10. Important for the subsequent discussion is that the wave equation (3.9) takes, in tensorial terms, the form

$$\square\xi_{a} = -2\Phi_{ab}\xi^{b} + J_{a} \quad (3.10)$$

where J_a is defined in spinorial terms by

$$J_{AA'} \equiv \Phi^{PQ}{}_{A'}{}^{P'}H_{P'APQ} - \Psi_{APQD}H_{A'}{}^{PQD} + 6\bar{\phi}_{A'}{}^{P'}\nabla_{PP'}\Theta_{A}{}^{P}.$$

In terms of the zero-quantity $\zeta_{AA'}$ to be introduced in equation (3.12) one has

$$J_{AA'} \equiv \Phi^{PQ}{}_{A'}{}^{P'}H_{P'APQ} - \Psi_{APQD}H_{A'}{}^{PQD} - 6\bar{\phi}_{A'}{}^{P'}\zeta_{AP'}.$$

Therefore, $J_{AA'}$ is a linear and homogeneous expression of zero-quantities and does not involve their derivatives. This is confirmed by the fact that setting $J_a = 0$ in the above equation yields the familiar wave equation for a Killing vector in a spacetime with non-trivial matter content. Therefore, J_a can be thought of as a vector measuring the obstruction of ξ_a to being a Killing vector.

3.3.2 A wave equation for $H_{A'ABC}$

It is possible to construct a wave equation for the zero quantity $H_{A'ABC}$:

Lemma 6. *Let κ_{AB} denote a Killing spinor candidate in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Then the zero-quantity $H_{A'ABC}$ satisfies the wave equation*

$$\begin{aligned}\square H_{A'BCD} = {} & 2\Psi_{CDAF}H_{A'B}{}^{AF} + 2\Psi_{BDAF}H_{A'C}{}^{AF} + 4\phi_D{}^A\bar{\phi}_{A'}{}^{B'}H_{B'BCA} \\ & - 12\bar{\phi}_{A'}{}^{B'}\nabla_{DB'}\Theta_{BC} - 2\nabla_D{}^{B'}S_{(BC)(A'B')}. \end{aligned} \tag{3.11}$$

Proof. Consider, again, the identity (3.8) in the form

$$\nabla_{AB'}H_{A'BC}{}^A = 6\bar{\phi}_{A'B'}\Theta_{BC} - S_{(BC)(A'B')}.$$

Applying the derivative $\nabla_D{}^{B'}$ to the above expression one readily finds that

$$\nabla_D{}^{B'}\nabla_{AB'}H_{A'BC}{}^A = 6(\Theta_{BC}\nabla_D{}^{B'}\bar{\phi}_{A'B'} + \bar{\phi}_{A'B'}\nabla_D{}^{B'}\Theta_{BC}) - \nabla_D{}^{B'}S_{(BC)(A'B')}).$$

Using the identity (1.8) and the box commutators (1.4) one obtains, after simplifying using the Maxwell equations, the desired equation. □

Remark 11. Observe that the right hand side of the wave equation (3.11) is a linear and homogeneous expression in the zero-quantity $H_{A'ABC}$ and the first order derivatives of Θ_{AB} and $S_{AA'BB'}$.

3.3.3 A wave equation for Θ_{AB}

In order to compute a wave equation for the zero-quantity associated to the matter alignment condition it is convenient to introduce a further zero-quantity:

$$\zeta_{AA'} \equiv \nabla^{Q}{}_{A'}\Theta_{AQ}. \tag{3.12}$$

Clearly, if the matter alignment condition (2.7) is satisfied, then $\zeta_{AA'} = 0$. The reason for introducing this further field will become clear later. Using the above definition, it follows that:

Lemma 7. *Let κ_{AB} denote a symmetric spinor field in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Then, one has that*

$$\square\Theta_{AB} = 2\Psi_{ABPQ}\Theta^{PQ} - 2\nabla_{B}{}^{A'}\zeta_{AA'}. \tag{3.13}$$

Proof. The wave equation follows from applying the derivative $\nabla_{B}{}^{A'}$ to the definition of $\zeta_{AA'}$ and using the identity (1.8) together with the box commutators (1.4). □

Remark 12. A direct computation using the definitions of Θ_{AB} and $\zeta_{AA'}$ together with the expression for the irreducible decomposition of $\nabla_{AA'}\kappa_{BC}$ given by equation (3.5) and the Maxwell equations gives that

$$\zeta_{AA'} = -\nabla_{A'(A}\phi_{BC)}\kappa^{BC} + \frac{4}{3}\xi^{B}{}_{A'}\phi_{AB} + \frac{1}{3}H_{A'ABC}\phi^{BC}. \tag{3.14}$$

Remark 13. It follows directly from equation (3.13) that

$$\nabla^{AA'}\zeta_{AA'} = 0.$$

Alternatively, this property can be verified through a direct computation using the identity (3.14).

As the right hand side of equation (3.13) is a linear and homogeneous expression in

Θ_{AB} and a first order derivative of $\zeta_{AA'}$, to construct a closed system of wave equations one needs to construct a wave equation for $\zeta_{AA'}$. The required expression follows from an involved computation – as it can be seen from the proof of the following lemma:

Lemma 8. *Let κ_{AB} denote a symmetric spinor field in an electrovacuum $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Then, one has that*

$$\begin{aligned}\square\zeta_{AA'} = {} & 4\zeta^{DB'}\phi_{AD}\bar{\phi}_{A'B'} + \frac{2}{3}\phi^{DB}\Psi_{DBCF}H_{A'A}{}^{CF} - \frac{2}{3}\phi^{DB}\Psi_{ABCF}H_{A'D}{}^{CF} \\ & - \frac{4}{3}\phi_A{}^D\phi^{BC}\bar{\phi}_{A'}{}^{B'}H_{B'DBC} - \frac{2}{3}H^{B'DBC}\nabla_{AB'}\phi_{DA'BC} \\ & - \frac{2}{3}H_{A'}{}^{DBC}\nabla_{AB'}\phi_D{}^{B'}{}_{BC} + \frac{2}{3}\phi^{DB'BC}\nabla_{AB'}H_{A'DBC} \\ & + \frac{2}{3}\phi^D{}_{A'}{}^{BC}\nabla_{AB'}H^{B'}{}_{DBC} - \frac{4}{3}\phi^{DB}\nabla_A{}^{B'}S_{(BD)(A'B')} \\ & - 4\phi^{DB}\bar{\phi}_{A'}{}^{B'}\nabla_{BB'}\Theta_{AD} - \frac{2}{3}\phi^{DB}\nabla_B{}^{B'}S_{(AD)(A'B')} \\ & + \frac{2}{3}\nabla_A{}^{B'}\phi^{DB}\nabla_{CB'}H_{A'DB}{}^C - \frac{4}{3}\nabla_A{}^{B'}\phi^{DB}S_{(BD)(A'B')}. \end{aligned} \tag{3.15}$$

where $\phi_{AA'BC} \equiv \nabla_{AA'}\phi_{BC}$.

Proof. Consider the identity (3.14) and apply the derivative $\nabla^A{}_{B'}$ to obtain

$$\begin{aligned}\nabla^A{}_{B'}\zeta_{AA'} = {} & -\kappa^{BC}\nabla^A{}_{B'}\nabla_{AA'}\phi_{BC} + \frac{1}{3}(H_{A'ABC}\nabla^A{}_{B'}\phi^{BC} + \phi^{BC}\nabla^A{}_{B'}H_{A'ABC}) \\ & - \nabla_{AA'}\phi_{BC}\nabla^A{}_{B'}\kappa^{BC} + \frac{4}{3}(\phi_{AB}\nabla^A{}_{B'}\xi^B{}_{A'} + \xi^B{}_{A'}\nabla^A{}_{B'}\phi_{AB}).\end{aligned}$$

Some further simplifications give

$$\begin{aligned}\nabla^A{}_{B'}\zeta_{AA'} = {} & \frac{1}{3}\nabla^A{}_{B'}\phi^{BC}H_{A'ABC} + \frac{1}{3}\nabla^A{}_{A'}\phi^{BC}H_{B'ABC} - \frac{1}{3}\phi^{AB}\nabla_{CB'}H_{A'AB}{}^C \\ & + \frac{2}{3}\phi^{AB}S_{(AB)(A'B')}.\end{aligned}$$

To obtain the required wave equation, apply $\nabla_D{}^{B'}$ to the above expression and make use of the decomposition (1.8) on the terms

$$\frac{1}{3}\nabla_D{}^{B'}\nabla^A{}_{B'}\phi^{BC}H_{A'ABC}, \qquad \nabla_D{}^{B'}\nabla^A{}_{B'}\zeta_{AA'}, \qquad -\frac{1}{3}\phi^{AB}\nabla_D{}^{B'}\nabla_{CB'}H_{A'AB}{}^C.$$

Finally, substitution of the wave equations for ϕ_{AB} and $H_{A'ABCD}$, equations (3.1) and (3.11), yields the required expression homogeneous in zero-quantities. □

3.3.4 A wave equation for $S_{AA'BB'}$

The discussion of the wave equation for the spinorial field $S_{AA'BB'}$ is best carried out in tensorial notation. Accordingly, let S_{ab} denote the tensorial counterpart of the (not necessarily Hermitian) spinor $S_{AA'BB'}$. Key to this computation is the wave equation for the Killing vector candidate ξ^a, equation (3.10).

Lemma 9. *Let κ_{AB} denote a Killing spinor candidate in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Then the zero-quantity S_{ab} satisfies the wave equation*

$$\begin{aligned} \square S_{ab} = & -2\mathcal{L}_\xi T_{ab} + 2T_b{}^c S_{ac} + 2T_a{}^c S_{bc} - T^{cd} S_{cd} g_{ab} - T_{ab} S^c{}_c \\ & - 2C_{acbd} S^{cd} + \nabla_a J_b + \nabla_b J_a \end{aligned} \tag{3.16}$$

where $\mathcal{L}_\xi T_{ab}$ denotes the Lie derivative of the energy-momentum of the Faraday tensor.

Proof. The required expression follows from applying $\square = \nabla_a \nabla^a$ to

$$S_{ab} = \nabla_a \xi_b + \nabla_b \xi_a,$$

commuting covariant derivatives, using the wave equation (3.10), the Einstein equation

$$R_{ab} = T_{ab},$$

the contracted Bianchi identity

$$\nabla^a C_{abcd} = \nabla_{[c} T_{d]b}$$

and the relation

$$\nabla_a \xi_b = \frac{1}{2} S_{ab} + \nabla_{[a}\xi_{b]}.$$

□

A straightforward computation shows that the Lie derivative of the electromagnetic energy-momentum tensor can be expressed in terms of the Lie derivative of the Faraday tensor and the zero-quantity S_{ab} in the following way:

$$\mathcal{L}_\xi T_{ab} = -\frac{1}{4} F_{cd} F^{cd} S_{ab} - F_a{}^c F_b{}^d S_{cd} + \frac{1}{2} F_c{}^f F^{cd} g_{ab} S_{df} \\ + F_b{}^c \mathcal{L}_\xi F_{ac} + F_a{}^c \mathcal{L}_\xi F_{bc} - \frac{1}{2} F^{cd} g_{ab} \mathcal{L}_\xi F_{cd}.$$

Furthermore, the Lie derivative of the Faraday tensor can be expressed in terms of the Lie derivative of the Maxwell spinor as

$$\mathcal{L}_\xi F_{AA'BB'} = \left(\mathcal{L}_\xi \phi_{AB} - \frac{1}{2} S_{AC'BD'} \bar{\phi}^{C'D'} \right) \epsilon_{A'B'} + \text{complex conjugate},$$

where the *Lie derivative of the Maxwell spinor* is defined by

$$\mathcal{L}_\xi \phi_{AB} \equiv \xi^{CC'} \nabla_{CC'} \phi_{AB} + \phi_{C(A} \nabla_{B)C'} \xi^{CC'} \tag{3.17}$$

– see Section 6.6 in [46]. This expression can be written in terms of zero quantities by using the wave equations for the Killing and Maxwell spinors, the Maxwell equations and the identity

$$\kappa^D{}_{(A} \Psi_{B)DEF} \phi^{EF} = \frac{1}{2} \Psi_{ABCD} \Theta^{CD} + \frac{1}{3} \phi^{EF} \nabla_{(A|}{}^{A'} H_{A'|BEF)}$$

along with the wave equations for the Killing and Maxwell spinors and the Maxwell equations, (3.3b) and (3.9), so as to obtain

$$\mathcal{L}_\xi \phi_{AB} = -\frac{3}{2} \nabla_{(A}{}^{A'} \zeta_{B)A'} + H_{A'CD(A} \nabla_{B)}{}^{A'} \phi^{CD} - \phi^{CD} \nabla_{(A|}{}^{A'} H_{A'|BCD)}.$$

From this discussion, the result follows:

Lemma 10. *Let κ_{AB} denote a Killing spinor candidate in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Then the Lie derivative $\mathcal{L}_\xi T_{ab}$ can be expressed as a linear and homogeneous expression in the zero-quantities*

$$S_{AA'BB'}, \quad \zeta_{AA'}, \quad H_{A'ABC}$$

and their first order derivatives.

Remark 14. In the context of the present discussion the object $\mathcal{L}_\xi \phi_{AB}$, as defined in (3.17), must be regarded as a convenient shorthand for a complicated expression. It is only consistent with the usual notion of Lie derivative of tensor fields if $\xi^{AA'}$ is the spinorial counterpart of a conformal Killing vector ξ^a – see [46], Section 6.6, for further discussion on this point.

3.3.5 Summary

Collecting the results of the current section gives the following result:

Proposition 4. *Let κ_{AB} denote a Killing spinor candidate in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Then the zero-quantities*

$$H_{A'ABC}, \quad \Theta_{AB}, \quad \zeta_{AA'}, \quad S_{AA'BB'}$$

satisfy a system of wave equations, consisting of equations (3.11)*,* (3.13)*,* (3.15) *and* (3.16)*, which are linear and homogeneous in the above zero-quantities and their first order derivatives.*

The above proposition ensures that the system of equations given in (3.11), (3.13), (3.15) and (3.16) satisfy the conditions of Theorem 1 in [28], which guarantees the existence of a unique solution in a neighbourhood of the hypersurface $\mathcal{S}$, given arbitrary initial data. For the remainder of this chapter, when the existence of a unique solution to a system of wave equations is claimed, it is this result which is being used, with the understanding that the specific system satisfies the necessary assumptions. In particular, applying this theorem to the system obtained here gives rise to the following:

Theorem 6. *Let κ_{AB} denote a Killing spinor candidate in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ and let $\mathcal{S}$ denote a Cauchy hypersurface of $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. The spinor κ_{AB} is an actual Killing spinor if and only if on $\mathcal{S}$ one has that*

$$H_{A'ABC}|_{\mathcal{S}} = 0, \qquad \nabla_{EE'}H_{A'ABC}|_{\mathcal{S}} = 0 \tag{3.18a}$$

$$S_{AA'BB'}|_{\mathcal{S}} = 0, \qquad \nabla_{EE'}S_{AA'BB'}|_{\mathcal{S}} = 0 \tag{3.18b}$$

$$\Theta_{AB}|_{\mathcal{S}} = 0, \qquad \nabla_{EE'}\Theta_{AB}|_{\mathcal{S}} = 0 \tag{3.18c}$$

$$\zeta_{AA'}|_{\mathcal{S}} = 0, \qquad \nabla_{EE'}\zeta_{AA'}|_{\mathcal{S}} = 0. \tag{3.18d}$$

Proof. The initial data for the homogeneous system of wave equations for the fields $H_{A'ABC}$, Θ_{AB}, $\zeta_{AA'}$ and $S_{AA'BB'}$ given by equations (3.11), (3.13), (3.15) and (3.16) consists of the values of these fields and their normal derivatives on the Cauchy surface $\mathcal{S}$. As this system of wave equations satisfies the conditions of Theorem 1 in [28], the unique solution to these equations with vanishing initial data is given by

$$H_{A'ABC} = 0, \quad \Theta_{AB} = 0, \quad \zeta_{AA'} = 0, \quad S_{AA'BB'} = 0.$$

Thus, if this is the case, the spinor κ_{AB} satisfies the Killing equation on $\mathcal{M}$ and, accordingly, it is a Killing spinor. Conversely, given a Killing spinor κ_{AB} over $\mathcal{M}$, its restriction to $\mathcal{S}$ satisfies the conditions (3.18a)-(3.18d). □

Remark 15. As the spinorial zero-quantities $H_{A'ABC}$, Θ_{AB}, $\zeta_{AA'}$ and $S_{AA'BB'}$ can be expressed in terms of the spinor κ_{AB}, it follows that the conditions (3.18a)-(3.18d) are, in fact, conditions on κ_{AB}, and its (spacetime) covariant derivatives up to third order. In the next section it will be shown how these conditions can be expressed in terms of objects defined on the hypersurface $\mathcal{S}$.

3.4 The Killing spinor data equations

The purpose of this section is to show how the conditions (3.18a)-(3.18d) of Theorem 6 can be re-expressed as conditions which are defined on the hypersurface $\mathcal{S}$. To this end, the space-spinor formalism outlined in Chapter 1 will be used.

3.4.1 Basic decompositions

First, the irreducible decompositions of the various spinorial fields and equations required for the subsequent analysis will be investigated.

3.4.1.1 Decomposition of the Killing spinor and Maxwell equations

Contracting the Killing spinor equation (3.2) in the form $\nabla_{(A|A'|}\kappa_{CD)} = 0$ with $\tau_B{}^{A'}$ one obtains

$$\nabla_{(A|B|}\kappa_{CD)} = 0$$

where ∇_{AB} is the differential operator defined in equation (1.16). Using the decomposition (1.17) gives

$$\frac{1}{2}\epsilon_{(A|B|}\mathcal{P}\kappa_{CD)} + \mathcal{D}_{(A|B|}\kappa_{CD)} = 0.$$

Taking, respectively, the trace and the totally symmetric part of the above expression one readily obtains the equations

$$\mathcal{P}\kappa_{AB} + \mathcal{D}_{(A}{}^{Q}\kappa_{B)Q} = 0, \tag{3.19a}$$

$$\mathcal{D}_{(AB}\kappa_{CD)} = 0. \tag{3.19b}$$

Equation (3.19a) can be naturally interpreted as an evolution equation for the spinor κ_{AB} while equation (3.19b) plays the role of a constraint.

A similar calculation applied to the Maxwell equation, equation (1.6), in the form

$\nabla^{A}{}_{A'}\phi_{AC} = 0$ yields the equations

$$\mathcal{P}\phi_{AB} - 2\mathcal{D}^{Q}{}_{(A}\phi_{B)Q} = 0, \tag{3.20a}$$

$$\mathcal{D}^{AB}\phi_{AB} = 0. \tag{3.20b}$$

Again, equation (3.20a) is an evolution equation for the Maxwell spinor ϕ_{AB} while (3.20b) is the spinorial version of the electromagnetic Gauss constraint.

Remark 16. The operation of Hermitian conjugation can be used to define, respectively, the *electric* and *magnetic* parts of the Maxwell spinor:

$$E_{AB} \equiv \frac{1}{2}\big(\widehat{\phi}_{AB} - \phi_{AB}\big), \qquad B_{AB} \equiv \frac{\mathrm{i}}{2}\big(\phi_{AB} + \widehat{\phi}_{AB}\big).$$

It is straightforward to verify that

$$\widehat{E}_{AB} = -E_{AB}, \qquad \widehat{B}_{AB} = -B_{AB}.$$

Thus, E_{AB} and B_{AB} are the spinorial counterparts of 3-dimensional tensors E_i and B_i – the electric and magnetic parts of the Faraday tensor with respect to the normal to the hypersurface $\mathcal{S}$.

3.4.1.2 The decomposition of the components of the curvature

Crucial for the subsequent argument will be the fact that the restriction of the Weyl spinor Ψ_{ABCD} to an hypersurface $\mathcal{S}$ can be expressed in terms of quantities defined on the hypersurface. In analogy to the case of the Maxwell spinor ϕ_{AB}, the Hermitian conjugation operation can be used to decompose the Weyl spinor Ψ_{ABCD} into its electric and magnetic parts with respect to the normal to $\mathcal{S}$:

$$E_{ABCD} \equiv \frac{1}{2}\big(\Psi_{ABCD} + \hat{\Psi}_{ABCD}\big), \qquad B_{ABCD} \equiv \frac{\mathrm{i}}{2}\big(\hat{\Psi}_{ABCD} - \Psi_{ABCD}\big)$$

so that

$$\Psi_{ABCD} = E_{ABCD} + \mathrm{i}B_{ABCD}.$$

The electrovacuum Bianchi identity (1.7) implies on $\mathcal{S}$ the constraint

$$\mathcal{D}^{AB}\Psi_{ABCD} = -2\widehat{\phi}^{AB}\mathcal{D}_{AB}\phi_{CD}.$$

Finally, using the Gauss-Codazzi and Codazzi-Mainardi equations one finds that

$$\begin{aligned} E_{ABCD} &= -r_{(ABCD)} + \frac{1}{2}\Omega_{(AB}{}^{PQ}\Omega_{CD)PQ} - \frac{1}{6}\Omega_{ABCD}K + E_{(AB}E_{CD)}, \\ B_{ABCD} &= -\mathrm{i}D^{Q}{}_{(A}\Omega_{BCD)Q}, \end{aligned}$$

where r_{ABCD} is the spinorial counterpart of the Ricci tensor of the intrinsic metric of the hypersurface $\mathcal{S}$.

3.4.1.3 Decomposition of the derivatives of the Killing spinor candidate

Once again, the calculations in this section will utilise the computer algebra packages contained in the *xAct* suite (see [43]).

Given a spinor κ_{AB} defined on the Cauchy hypersurface $\mathcal{S}$, it will prove convenient to define:

$$\xi \equiv \mathcal{D}^{AB}\kappa_{AB}, \tag{3.21a}$$

$$\xi_{AB} \equiv \frac{3}{2}\mathcal{D}_{(A}{}^{C}\kappa_{B)C}, \tag{3.21b}$$

$$\xi_{ABCD} \equiv \mathcal{D}_{(AB}\kappa_{CD)}. \tag{3.21c}$$

These spinors correspond to the irreducible components of the Sen derivative of κ_{AB}, as follows:

$$\mathcal{D}_{AB}\kappa_{CD} = \xi_{ABCD} - \frac{1}{3}\epsilon_{A(C}\xi_{D)B} - \frac{1}{3}\epsilon_{B(C}\xi_{D)A} - \frac{1}{3}\epsilon_{A(C}\epsilon_{D)B}\xi.$$

Using the commutation relation for the Sen derivatives given in equation (1.24b), the derivatives of ξ and ξ_{AB} can also be calculated. The irreducible components of $\mathcal{D}_{AB}\xi_{CD}$ are given on $\mathcal{S}$ (where we can assume that $\Omega_{AB} = 0$, equivalent to assuming that the vector τ is normal to $\mathcal{S}$) by

$$\mathcal{D}_{AB}\xi^{AB} = -\frac{1}{2}K\xi + \frac{3}{4}\Omega^{ABCD}\xi_{ABCD} + \frac{3}{2}\Theta_{AB}\widehat{\phi}^{AB}, \tag{3.22a}$$

$$\begin{aligned}\mathcal{D}_{A(B}\xi_{C)}{}^{A} = &-\mathcal{D}_{BC}\xi - \frac{3}{2}\Psi_{BCAD}\kappa^{AD} + \frac{2}{3}K\xi_{BC} + \frac{1}{2}\Omega_{BCAD}\xi^{AD}\\ &- \frac{3}{2}\Omega_{(B}{}^{ADF}\xi_{C)ADF} + \frac{3}{2}\mathcal{D}_{AD}\xi_{BC}{}^{AD} - 3\Theta_{A(B}\hat{\phi}_{C)}{}^{A},\end{aligned} \tag{3.22b}$$

$$\begin{aligned}\mathcal{D}_{(AB}\xi_{CD)} = &\,3\Psi_{F(ABC}\kappa_{D)}{}^{F} + K\xi_{ABCD} - \frac{1}{2}\xi\Omega_{ABCD} + \Omega_{(ABC}{}^{F}\xi_{D)F}\\ &- \frac{3}{2}\Omega_{(AB}{}^{PQ}\xi_{CD)PQ} + 3\mathcal{D}^{F}{}_{(A}\xi_{BCD)F} - 3\Theta_{(AB}\widehat{\phi}_{CD)},\end{aligned} \tag{3.22c}$$

where the Hermitian conjugate of the Maxwell spinor $\widehat{\phi}_{AB}$ is defined by

$$\widehat{\phi}_{AB} \equiv \tau_{A}{}^{A'}\tau_{B}{}^{B'}\bar{\phi}_{A'B'}.$$

Note that in (3.22b), the term $\mathcal{D}_{AB}\xi$ appears – there is no independent equation for the Sen derivative of ξ.

The second order derivatives of ξ can also be calculated. On the hypersurface $\mathcal{S}$ these take the form:

$$\begin{aligned}\mathcal{D}_{AB}\mathcal{D}^{AB}\xi = &-\frac{1}{6}K^2\xi + \frac{1}{2}K\widehat{\phi}^{AB}\Theta_{AB} - 2\widehat{\phi}^{AB}\phi_{AB}\xi + \frac{2}{3}\xi^{AB}\mathcal{D}_{AB}K\\ &- 4\widehat{\phi}^{AB}\phi_{A}{}^{C}\xi_{BC} - \frac{3}{2}\Psi^{ABCD}\xi_{ABCD} + \frac{3}{2}\xi^{ABCD}\mathcal{D}_{DF}\Omega_{ABC}{}^{F}\\ &- 3\widehat{\phi}^{AB}\Theta^{CD}\Omega_{ABCD} - \frac{1}{2}\Omega_{ABCD}\Omega^{ABCD}\xi + \frac{5}{4}K\Omega^{ABCD}\xi_{ABCD}\\ &+ 3\kappa^{AB}\Psi_{A}{}^{CDF}\Omega_{BCDF} - \frac{3}{2}\Omega_{AB}{}^{FG}\Omega^{ABCD}\xi_{CDFG}\\ &- 3\kappa^{AB}\widehat{\phi}^{CD}\mathcal{D}_{BD}\phi_{AC} + 3\kappa^{AB}\widehat{\phi}_{A}{}^{C}\mathcal{D}_{CD}\phi_{B}{}^{D} - \frac{3}{2}\kappa^{AB}\mathcal{D}_{CD}\Psi_{AB}{}^{CD}\\ &+ \frac{1}{2}\xi^{AB}\mathcal{D}_{CD}\Omega_{AB}{}^{CD} + \frac{3}{2}\mathcal{D}_{CD}\mathcal{D}_{AB}\xi^{ABCD} + 3\widehat{\phi}^{AB}\phi^{CD}\xi_{ABCD}\\ &- \frac{9}{2}\Omega^{ABCD}\mathcal{D}_{DF}\xi_{ABC}{}^{F} + 3\Theta^{AB}\mathcal{D}_{BC}\widehat{\phi}_{A}{}^{C},\end{aligned} \tag{3.23a}$$

$$\mathcal{D}^{C}{}_{(A}\mathcal{D}_{B)C}\xi = \frac{1}{2}\Omega_{ABCD}\mathcal{D}^{CD}\xi - \frac{1}{3}K\mathcal{D}_{AB}\xi, \tag{3.23b}$$

$$\begin{aligned}
\mathcal{D}_{(AB}\mathcal{D}_{CD)}\xi = {}& \frac{1}{3}\widehat{\phi}^{EF}\Theta_{EF}\Omega_{ABCD} - \Psi_{ABCD}\xi - \frac{5}{9}K\Omega_{ABCD}\xi \\
& + \frac{1}{6}\Omega^{EFPQ}\Omega_{ABCD}\xi_{EFPQ} + \frac{8}{9}K^2\xi_{ABCD} + \frac{1}{3}\xi\mathcal{D}_{E(A}\Omega_{BCD)}{}^{E} \\
& - \frac{10}{3}K\mathcal{D}_{E(A}\xi_{BCD)}{}^{E} + \frac{3}{2}\mathcal{D}_{(AB}\mathcal{D}_{|EF|}\xi_{CD)}{}^{EF} + \frac{3}{2}\mathcal{D}_{F(A}\mathcal{D}_{B|E|}\xi_{CD)}{}^{EF} \\
& + \frac{1}{2}\mathcal{D}_{(A|F}\mathcal{D}_{E|}{}^{F}\xi_{BCD)}{}^{E} + \frac{8}{3}K\kappa_{(A}{}^{E}\Psi_{BCD)E} - \frac{3}{2}\kappa_{(A}{}^{E}\mathcal{D}_{B|F|}\Psi_{CD)E}{}^{F} \\
& - \frac{3}{2}\kappa^{EF}\mathcal{D}_{(AB}\Psi_{CD)EF} - \frac{1}{2}\kappa^{EF}\mathcal{D}_{F(A}\Psi_{BCD)E} + 2\xi\widehat{\phi}_{(AB}\phi_{CD)} \\
& - \frac{8}{3}K\widehat{\phi}_{(AB}\Theta_{CD)} + \Theta_{(AB}\mathcal{D}_{C|E|}\widehat{\phi}_{D)}{}^{E} + 3\Theta_{(A}{}^{E}\mathcal{D}_{BC}\widehat{\phi}_{D)E} \\
& + \Theta_{(A}{}^{E}\mathcal{D}_{B|E|}\widehat{\phi}_{CD)} + 2\Psi_{E(ABC}\xi_{D)}{}^{E} + \frac{1}{6}\xi\Omega_{(AB}{}^{EF}\Omega_{CD)EF} \\
& - \frac{14}{9}K\Omega_{E(ABC}\xi_{D)}{}^{E} - \frac{5}{3}K\Omega_{(AB}{}^{EF}\xi_{CD)EF} + \frac{2}{3}\Omega_{E(ABC}\mathcal{D}_{D)}{}^{E}\xi \\
& + \frac{3}{2}\Omega_{(ABC}{}^{E}\mathcal{D}^{FP}\xi_{D)EFP} - \Omega_{(AB}{}^{EF}\mathcal{D}_{C}{}^{P}\xi_{D)EFP} \\
& + \frac{1}{2}\Omega_{(AB}{}^{EF}\mathcal{D}_{|FP|}\xi_{CD)E}{}^{P} - \frac{3}{2}\Omega_{(A}{}^{EFP}\mathcal{D}_{BC}\xi_{D)EFP} \\
& + \frac{1}{2}\Omega_{(A}{}^{EFP}\mathcal{D}_{B|P|}\xi_{CD)EF} + \frac{2}{3}\xi_{(AB}\mathcal{D}_{CD)}K + \frac{1}{2}\xi_{(A}{}^{E}\mathcal{D}_{B|F|}\Omega_{CD)E}{}^{F} \\
& + \frac{1}{2}\xi^{EF}\mathcal{D}_{(AB}\Omega_{CD)EF} + \frac{1}{6}\xi^{EF}\mathcal{D}_{F(A}\Omega_{BCD)E} + \frac{2}{3}\xi_{E(ABC}\mathcal{D}_{D)}{}^{E}K \\
& + \frac{1}{2}\xi_{(AB}{}^{EF}\mathcal{D}_{C|P|}\Omega_{D)EF}{}^{P} + \frac{3}{2}\xi_{(A}{}^{EFP}\mathcal{D}_{BC}\Omega_{D)EFP} \\
& + \frac{1}{2}\xi_{(A}{}^{EFP}\mathcal{D}_{B|P|}\Omega_{CD)EF} + \kappa_{(A}{}^{E}\widehat{\phi}_{BC}\mathcal{D}_{D)F}\phi_{E}{}^{F} \\
& - 3\kappa_{(A}{}^{E}\widehat{\phi}_{B}{}^{F}\mathcal{D}_{CD)}\phi_{EF} + 2\kappa_{(A}{}^{E}\widehat{\phi}_{B}{}^{F}\mathcal{D}_{C|F|}\phi_{D)E} + \kappa^{EF}\widehat{\phi}_{(AB}\mathcal{D}_{C|F|}\phi_{D)E} \\
& + 3\kappa^{EF}\widehat{\phi}_{E(A}\mathcal{D}_{BC}\phi_{D)F} + \frac{1}{2}\kappa_{E(A}\Psi_{B}{}^{EFP}\Omega_{CD)FP} \\
& - \frac{1}{2}\kappa^{EF}\Psi^{P}{}_{E(AB}\Omega_{CD)FP} + \frac{3}{2}\kappa^{EF}\Psi^{P}{}_{EF(A}\Omega_{BCD)P} \\
& + \frac{10}{3}\widehat{\phi}_{(AB}\phi_{C}{}^{E}\xi_{D)E} + \frac{2}{3}\widehat{\phi}_{(A}{}^{E}\phi_{BC}\xi_{D)E} + \frac{2}{3}\phi_{E(A}\widehat{\phi}_{B}{}^{E}\xi_{CD)} \\
& - \widehat{\phi}_{(AB}\phi^{EF}\xi_{CD)EF} + \widehat{\phi}_{(A}{}^{E}\phi_{B}{}^{F}\xi_{CD)EF} + 3\phi_{E}{}^{F}\widehat{\phi}_{(A}{}^{E}\xi_{BCD)F} \\
& + \frac{1}{6}\widehat{\phi}_{(AB}\Theta^{EF}\Omega_{CD)EF} + \frac{2}{3}\widehat{\phi}_{(A}{}^{E}\Theta_{B}{}^{F}\Omega_{CD)EF} + \frac{3}{2}\Theta_{E}{}^{F}\widehat{\phi}_{(A}{}^{E}\Omega_{BCD)F} \\
& + \frac{1}{6}\widehat{\phi}^{EF}\Theta_{(AB}\Omega_{CD)EF} + \frac{3}{2}\widehat{\phi}^{EF}\Theta_{E(A}\Omega_{BCD)F} + \frac{1}{2}\Omega_{(ABC}{}^{P}\Omega_{D)EFP}\xi^{EF} \\
& + \frac{1}{6}\Omega_{EFP(A}\Omega_{BC}{}^{FP}\xi_{D)}{}^{E} - \frac{3}{4}\Omega_{(ABC}{}^{E}\Omega_{D)}{}^{FPQ}\xi_{EFPQ} \\
& - \frac{3}{4}\Omega_{E}{}^{FPQ}\Omega_{(ABC}{}^{E}\xi_{D)FPQ} + \frac{1}{12}\Omega_{(AB}{}^{EF}\Omega_{CD)}{}^{PQ}\xi_{EFPQ}
\end{aligned}$$

$$+\frac{1}{3}\Omega_{E(A}{}^{PQ}\Omega_{BC}{}^{EF}\xi_{D)FPQ}+\frac{1}{12}\Omega_{EF}{}^{PQ}\Omega_{(AB}{}^{EF}\xi_{CD)PQ}. \tag{3.23c}$$

Remark 17. It is of interest to remark that equation (3.23b) is just the statement that the Sen connection has torsion – cf. Remark 5.

An important and direct consequence of the above expressions is the following:

Lemma 11. *Assume that* $\Omega_{AB}=0$ *and* $\mathcal{D}_{(AB}\kappa_{CD)}=0$ *on* $\mathcal{S}$*. Then*

$$\mathcal{D}_{AB}\mathcal{D}_{CD}\mathcal{D}_{EF}\kappa_{GH}=H_{ABCDEFGH}$$

on $\mathcal{S}$*, where* $H_{ABCDEFGH}$ *is a linear combination of* κ_{AB}*,* $\mathcal{D}_{AB}\kappa_{CD}$ *and* $\mathcal{D}_{AB}\mathcal{D}_{CD}\kappa_{EF}$ *with coefficients depending on* Ψ_{ABCD}*,* K_{ABCD}*,* ϕ_{AB}*,* $\widehat{\phi}_{AB}$ *and* $\mathcal{D}_{AB}\phi_{CD}$*.*

Proof. The proof follows from direct inspection of equations (3.22a)-(3.22c) and (3.23a)-(3.23c). □

Remark 18. The above result is strictly not true if $\xi_{ABCD}=\mathcal{D}_{(AB}\kappa_{CD)}\neq 0$.

3.4.2 The decomposition of the Killing spinor data equations

This section will provide a systematic discussion of the decomposition of the *Killing initial data conditions* in Theorem 6. The main purpose of this decomposition is to untangle the interrelations between the various conditions and to obtain a *minimal* set of equations which is intrinsic to the Cauchy hypersurface $\mathcal{S}$.

For the ease of the discussion, the assumptions assumed to hold throughout this section will be stated explicitly here:

Assumption 3. *Given a Cauchy hypersurface* $\mathcal{S}$ *of an electrovacuum spacetime* $(\mathcal{M},\boldsymbol{g})$*, we assume that the hypothesis and conclusions of Theorem 6 hold.*

Assumption 4. *The spinor* $\tau^{AA'}$ *which on* $\mathcal{S}$ *is normal to* $\mathcal{S}$ *is extended off the initial hypersurface in such a way that it is the spinorial counterpart of the tangent vector to a* *congruence of $\boldsymbol{g}$-geodesics. Accordingly one has that $K_{AB} = 0$ – that is, the acceleration vanishes.*

The second of these holds without loss of generality.

3.4.2.1 Decomposing $H_{A'ABC} = 0$

Splitting the expression $\tau_D{}^{A'} H_{A'ABC}$ into irreducible parts and using the definitions (3.21a)-(3.21c) gives that the condition $H_{A'ABC} = 0$ is equivalent to

$$\xi_{ABCD} = 0, \tag{3.24a}$$

$$\mathcal{P}\kappa_{AB} = -\frac{2}{3}\xi_{AB}. \tag{3.24b}$$

Equation (3.24a) is a condition intrinsic to the hypersurface while (3.24b) is extrinsic – i.e. it involves derivatives in the direction normal to $\mathcal{S}$. Also, observe that the conditions (3.24a) and (3.24b) are essentially the equations (3.19a) and (3.19b).

3.4.2.2 Decomposing $\nabla_{EE'} H_{A'ABC} = 0$

If $H_{A'ABC} = 0$ on $\mathcal{S}$, it readily follows that $\mathcal{D}_{EF} H_{A'ABC} = 0$ on $\mathcal{S}$. Thus, in order investigate the consequences of the second condition in (3.18a) it is only necessary to consider the transverse derivative $\mathcal{P}H_{A'ABC}$. It follows that

$$\tau_D{}^{A'}\mathcal{P}H_{A'ABC} = \mathcal{P}\big(\tau_D{}^{A'}H_{A'ABC}\big) - H_{A'ABC}\mathcal{P}\tau_D{}^{A'}$$

and so as $H_{A'ABC}|_{\mathcal{S}} = 0$, the irreducible parts of $\tau_D{}^{A'}\mathcal{P}H_{A'ABC} = 0$ are given by

$$\mathcal{P}\xi_{ABCD} = 0, \tag{3.25a}$$

$$\mathcal{P}^2\kappa_{AB} = -\frac{2}{3}\mathcal{P}\xi_{AB}. \tag{3.25b}$$

Taking equation (3.25a) and commuting the $\mathcal{D}_{AB}$ and $\mathcal{P}$ derivatives, and using equations (3.24a) and (3.24b), gives

$$\begin{aligned}\mathcal{P}\xi_{ABCD} &= \mathcal{P}\mathcal{D}_{(AB}\kappa_{CD)}\\ &= 2\Psi^F{}_{(ABC}\kappa_{D)F} - \frac{1}{3}\xi\Omega_{ABCD} + \frac{2}{3}\Omega^F{}_{(ABC}\xi_{D)F}\\ &\quad - \frac{2}{3}\mathcal{D}_{(AB}\xi_{CD)} - 2\Theta_{(AB}\widehat{\phi}_{CD)}.\end{aligned}$$

Substituting for the derivative of ξ_{AB} using (3.22c), and using equations (3.24a) and (3.24b) again, gives

$$\mathcal{P}\xi_{ABCD} = 4\Psi^F{}_{(ABC}\kappa_{D)F} = 0. \tag{3.26}$$

To re-express condition (3.25b), the following result obtained by commuting the $\mathcal{P}$ and $\mathcal{D}_{AB}$ derivatives will be useful:

$$\begin{aligned}\mathcal{P}\xi_{AB} &= \frac{3}{2}\kappa^{CD}\Psi_{ABCD} - 3\Theta_{C(A}\widehat{\phi}_{B)}{}^C - \frac{1}{3}K\xi_{AB}\\ &\quad + \frac{1}{2}\Omega_{ABCD}\xi^{CD} - \frac{3}{2}\mathcal{D}_{C(A}\mathcal{P}\kappa_{B)}{}^C.\end{aligned} \tag{3.27}$$

Recall that the Killing spinor candidate κ_{AB} satisfies the homogeneous wave equation (3.3b). Using the space-spinor decomposition relations (1.16) and (1.17), the wave operator can be split into Sen and normal derivative operators. The result is:

$$\begin{aligned}\mathcal{P}^2\kappa_{AB} =& -2\kappa^{CD}\Psi_{ABCD} + \frac{1}{3}K_{AB}\xi + \frac{2}{3}\Omega_{AB}\xi - \tfrac{2}{3}K_{(A}{}^C\xi_{B)C}\\ & - \frac{4}{3}\Omega_{(A}{}^C\xi_{B)C} + K^{CD}\xi_{ABCD} + 2\Omega^{CD}\xi_{ABCD}\\ & - K\mathcal{P}\kappa_{AB} - \frac{2}{3}\mathcal{D}_{AB}\xi + \frac{4}{3}\mathcal{D}_{(A}{}^C\xi_{B)C} - 2\mathcal{D}_{CD}\xi_{AB}{}^{CD}\end{aligned}$$

Applying conditions (3.24a) and (3.24b) to the right hand side of the latter, evaluating at $\mathcal{S}$ (where $\Omega_{AB} = 0$) and setting $K_{AB} = 0$ gives

$$\mathcal{P}^2\kappa_{AB} = -2\kappa^{CD}\Psi_{ABCD} + \frac{2}{3}K\xi_{AB} - \frac{2}{3}\mathcal{D}_{AB}\xi + \frac{4}{3}\mathcal{D}_{(A}{}^C\xi_{B)C}.$$

Then, using equations (3.27) and (3.22b), as well as (3.24a) and (3.24b) as needed, it can be shown that

$$\mathcal{P}^2\kappa_{AB} = -\frac{2}{3}\mathcal{P}\xi_{AB} \tag{3.28}$$

which is exactly the required condition. Thus, we have shown that the condition (3.25b) is purely a consequence of the evolution equation for the Killing spinor candidate, along with the conditions arising from $H_{A'ABC}|_{\mathcal{S}} = 0$.

In summary, if κ_{AB} satisfies $\square\kappa_{AB} + \Psi_{ABCD}\kappa^{CD} = 0$, then the following are equivalent:

(i) $H_{A'ABC}|_{\mathcal{S}} = \mathcal{P}H_{A'ABC}|_{\mathcal{S}} = 0$

(ii) $\xi_{ABCD} = 0, \quad \mathcal{P}\kappa_{AB} + \frac{2}{3}\xi_{AB} = 0, \quad \Psi^F{}_{(ABC}\kappa_{D)F} = 0.$

3.4.2.3 Decomposing $\Theta_{AB} = 0$

As Θ_{AB} has no unprimed indices, it is already in a space-spinor compatible form:

$$\Theta_{AB} = \kappa_{(A}{}^C\phi_{B)C} = 0. \tag{3.29}$$

3.4.2.4 Decomposing $\nabla_{EE'}\Theta_{AB} = 0$

If $\Theta_{AB}|_{\mathcal{S}} = 0$, only the normal derivative $\mathcal{P}\Theta_{AB}$ need be considered. Using the evolution equation for the spinor ϕ_{AB} implied by Maxwell equations, equation (3.20a), along with (3.24b) in the condition $\mathcal{P}\Theta_{AB} = 0$ gives the spatially intrinsic condition

$$\kappa_{(A|}{}^C\mathcal{D}_{CD}\phi_{|B)}{}^D = \frac{1}{3}\phi_{(A}{}^C\xi_{B)C} \tag{3.30}$$

In summary, assuming (3.24b) holds, then the following are equivalent:

(i) $\Theta_{AB}|_{\mathcal{S}} = \mathcal{P}\Theta_{AB}|_{\mathcal{S}} = 0$

(ii) $\kappa_{(A}{}^{C}\phi_{B)C} = 0, \quad \kappa_{(A|}{}^{C}\mathcal{D}_{CD}\phi_{|B)}{}^{D} = \frac{1}{3}\phi_{(A}{}^{C}\xi_{B)C}.$

3.4.2.5 Decomposing $S_{AA'BB'} = 0$

A point of departure for decomposing the condition $S_{AA'BB'}|_{\mathcal{S}} = 0$ is the relation linking $S_{AA'BB'}$ to Θ_{AB} and the derivative of $H_{A'ABC}$ given by equation (3.8). Splitting the derivative of $H_{A'ABC}$ into normal and tangential parts gives

$$S_{AA'BB'} = -6\bar{\phi}_{A'B'}\Theta_{AB} + \frac{1}{2}\tau^{C}{}_{(A'}\mathcal{P}H_{B')ABC} + \tau_{D(A'}\mathcal{D}^{DC}H_{B')ABC}. \tag{3.31}$$

We already have conditions ensuring that $\Theta_{AB}|_{\mathcal{S}} = H_{A'ABC}|_{\mathcal{S}} = \mathcal{P}H_{A'ABC}|_{\mathcal{S}} = 0$, and so as a consequence we automatically have that $S_{AA'BB'}|_{\mathcal{S}} = 0$.

3.4.2.6 Decomposing $\nabla_{EE'}S_{AA'BB'} = 0$

Again as $S_{AA'BB'}|_{\mathcal{S}} = 0$, one only needs to consider the normal derivative $\mathcal{P}S_{AA'BB'}$. Taking the normal derivative of equation (3.31) and using the Gaussian gauge condition ($K_{AB} = 0$) gives that on $\mathcal{S}$:

$$\begin{aligned}\mathcal{P}S_{AA'BB'} =& -6\mathcal{P}\bar{\phi}_{A'B'}\Theta_{AB} - 6\bar{\phi}_{A'B'}\mathcal{P}\Theta_{AB} + \tau^{C}{}_{(A'}\mathcal{P}^{2}H_{B')ABC} \\ &+ \tau_{D(A'}\mathcal{P}\mathcal{D}^{DC}H_{B')ABC}.\end{aligned}$$

The first and second terms on the right hand side are zero as a consequence of conditions (3.29) and (3.30). The last term can be also shown to be zero by commuting the derivatives and using (3.24a), (3.24b) and (3.26). This leaves

$$0 = \mathcal{P}S_{AA'BB'} = \tau^{C}{}_{(A'}\mathcal{P}^{2}H_{B')ABC}. \tag{3.32}$$

Eliminating the primed indices by multiplying by factors of $\tau_{AA'}$ gives

$$\tau_{(C|}{}^{A'}\mathcal{P}^2 H_{A'AB|D)} = 0$$

Thus, if this condition is satisfied on $\mathcal{S}$, then it follows that $\mathcal{P}S_{AA'BB'}|_{\mathcal{S}} = 0$. In the following we investigate further the consequences of this condition. As in a Gaussian gauge $\mathcal{P}\tau_{AA'} = 0$ it readily follows that

$$\mathcal{P}^2\left(\tau_{(C|}{}^{A'}H_{A'AB|D)}\right) = 0.$$

Splitting into irreducible parts, one obtains two necessary conditions:

$$\mathcal{P}^2\xi_{ABCD} = 0, \tag{3.33a}$$

$$\mathcal{P}^2\left(\mathcal{P}\kappa_{AB} + \frac{2}{3}\xi_{AB}\right) = 0. \tag{3.33b}$$

Consider first condition (3.33a). Commuting the Sen derivative with one of the normal derivatives produces

$$\begin{aligned}\mathcal{P}\big(\mathcal{P}\xi_{ABCD}\big) &= \mathcal{P}\big(\mathcal{P}\mathcal{D}_{(AB}\kappa_{CD)}\big)\\ &= \mathcal{P}\bigg(2\Psi_{(ABC}{}^{F}\kappa_{D)F} - 2\Theta_{(AB}\widehat{\phi}_{CD)} - \frac{1}{3}\Omega_{ABCD}\xi - \frac{2}{3}\Omega_{F(ABC}\xi^{F}{}_{D)}\\ &\quad - \frac{1}{3}\Omega_{(AB}\xi_{CD)} - \frac{1}{3}K\xi_{ABCD} + \Omega^{F}{}_{(A}\xi_{BCD)F} - \Omega_{(AB}{}^{EF}\xi_{CD)EF}\\ &\quad + \mathcal{D}_{(AB}\mathcal{P}\kappa_{CD)}\bigg).\end{aligned}$$

The previous conditions on $\mathcal{S}$ can now be used to eliminate terms. For example, the second term in the bracket is zero from conditions (3.29) and (3.30). The fifth, sixth and seventh terms vanish from (3.24a) and (3.26). Equations (3.24b) and (3.28) ca be used to replace the last term – alternatively, one can commute the derivatives, use the substitution and then commute back; the result is the same. From this substitution one obtains a factor $\mathcal{D}_{(AB}\xi_{CD)}$ inside the normal derivative, which can be replaced using (3.22c) –

this equation is satisfied on the whole spacetime rather than just the hypersurface, so taking normal derivatives of it is valid.

Proceeding as above, condition (3.33a) can be reduced to

$$\mathcal{P}^2\xi_{ABCD} = \mathcal{P}\left(4\Psi_{(ABC}{}^{F}\kappa_{D)F}\right) = 0. \tag{3.34}$$

Splitting the covariant derivatives in the Bianchi identity (1.7) into normal and tangential components gives the following space-spinor version:

$$\mathcal{P}\Psi_{ABCD} = -4\widehat{\phi}_{F(A}\mathcal{D}^{F}{}_{B}\phi_{CD)} - 4\widehat{\phi}_{(AB}\mathcal{D}^{F}{}_{C}\phi_{D)F} - 2\mathcal{D}_{F(A}\Psi_{BCD)}{}^{F}.$$

One can use this expression to further reduce condition (3.34) to

$$\begin{aligned}\Psi_{F(ABC}\xi_{D)}{}^{F} &+ 6\widehat{\phi}_{F(A}\kappa^{E}{}_{B}\mathcal{D}^{F}{}_{C}\phi_{D)E}\\ &+ 6\widehat{\phi}_{(AB}\kappa^{E}{}_{C}\mathcal{D}^{F}{}_{D)}\phi_{EF} + 3\kappa_{(A}{}^{F}\mathcal{D}_{B|E}\Psi_{F|CD)}{}^{E} = 0.\end{aligned} \tag{3.35}$$

This is an intrinsic condition on $\mathcal{S}$.

Consider now the condition (3.33b). In order to obtain insight into this condition, we will make use, again, of the wave equation (3.3b) for the spinor κ_{AB}. Taking a normal derivative of this equation, one obtains

$$\mathcal{P}\left(\Box\kappa_{AB} + \Psi_{ABCD}\kappa^{CD}\right) = 0.$$

Splitting the spacetime derivatives into normal and tangential parts and rearranging gives

$$\begin{aligned}\mathcal{P}\left(\mathcal{P}^2\kappa_{AB}\right) = \mathcal{P}\big(&-2\kappa^{CD}\Psi_{ABCD} + \frac{2}{3}\Omega_{AB}\xi - \frac{4}{3}\Omega_{(A}{}^{C}\xi_{B)C} + 2\Omega^{CD}\xi_{ABCD}\\ &- K\mathcal{P}\kappa_{AB} - \frac{2}{3}\mathcal{D}_{AB}\xi - \frac{4}{3}\mathcal{D}_{C(A}\xi_{B)}{}^{C} - 2\mathcal{D}_{CD}\xi_{AB}{}^{CD}\big).\end{aligned}$$

As before, previous conditions can be used to eliminate terms. The fourth and eight terms on the right hand side vanish due to (3.24a) and (3.26). Also, equation (3.22b) can be used to replace the seventh term – this is because the relation (3.22b) holds on the whole spacetime, and so one can take normal derivatives of it freely. These steps give

$$\mathcal{P}\big(\mathcal{P}^2\kappa_{AB}\big) = \mathcal{P}\bigg(\frac{2}{3}\Omega_{(A}{}^{C}\xi_{B)C} - \frac{2}{9}K\xi_{AB} - \frac{2}{3}\Omega_{ABCD}\xi^{CD} + \frac{2}{3}\mathcal{D}_{AB}\xi\bigg).$$

Alternatively, consider the second derivative of ξ_{AB}, given by applying a normal derivative to equation (3.27) – note that equation (3.27) applies on the whole spacetime, so one can take the normal derivative. This yields

$$\begin{aligned}\mathcal{P}^2\xi_{AB} = \mathcal{P}\bigg(&\frac{3}{2}\kappa^{CD}\Psi_{ABCD} - 3\Theta_{C(A}\widehat{\phi}_{B)}{}^{C} - \frac{1}{2}\Omega_{AB}\xi - \frac{1}{3}K\xi_{AB} + \frac{1}{2}\Omega_{(A}{}^{C}\xi_{B)C}\\ &+ \frac{1}{2}\Omega_{ABCD}\xi^{CD} + \frac{3}{4}\Omega^{CD}\xi_{ABCD} - \frac{3}{2}\Omega_{(A}{}^{CDF}\xi_{B)CDF} - \frac{3}{2}\mathcal{D}_{C(A}\mathcal{P}\kappa_{B)}{}^{C}\bigg).\end{aligned}$$

As before, we can use the conditions (3.24a), (3.24b), (3.26) and (3.28), and the identity (3.22b) to reduce this to

$$\mathcal{P}^2\xi_{AB} = \mathcal{P}\bigg(\frac{1}{3}K\xi_{AB} - \Omega_{(A}{}^{C}\xi_{B)C} + \Omega_{ABCD}\xi^{CD} - \mathcal{D}_{AB}\xi\bigg).$$

By comparing terms, it follows that

$$\mathcal{P}^3\kappa_{AB} = -\frac{2}{3}\mathcal{P}^2\xi_{AB}$$

which is exactly the second condition (3.33b). So, no further conditions are needed to be prescribed on the hypersurface – this condition arises naturally from the evolution equation for the Killing spinor.

3.4.2.7 Decomposing $\zeta_{AA'} = 0$

Recalling the definition of $\zeta_{AA'}$, equation (3.12), and the decomposition of the spacetime covariant derivative given by (1.16) and (1.17), one obtains

$$\begin{aligned}\zeta_{AA'} &= \nabla^B{}_{A'}\Theta_{AB} \\ &= \frac{1}{2}\tau^B{}_{A'}\mathcal{P}\Theta_{AB} - \tau^C{}_{A'}\mathcal{D}_C{}^B\Theta_{AB}.\end{aligned}$$

From conditions (3.29) and (3.30) it then follows that $\zeta_{AA'}|_{\mathcal{S}} = 0$.

3.4.2.8 Decomposing $\nabla_{EE'}\zeta_{AA'} = 0$

Again, if $\zeta_{AA'}|_{\mathcal{S}} = 0$ then one one only needs to consider the transverse derivative $\mathcal{P}\zeta_{AA'}$. By definition one has that

$$\begin{aligned}\mathcal{P}\zeta_{AA'} &= \mathcal{P}\nabla^B{}_{A'}\Theta_{AB} \\ &= \mathcal{P}\Big(- \tau^C{}_{A'}\mathcal{D}^B{}_C + \frac{1}{2}\tau^B{}_{A'}\mathcal{P}\Big)\Theta_{AB} \\ &= \frac{1}{2}\tau^B{}_{A'}\mathcal{P}^2\Theta_{AB}\end{aligned}$$

where the last equation has been obtained by commuting the Sen and normal derivatives, and using (3.30). Therefore, the vanishing of the derivative of $\zeta_{AA'}$ is equivalent to

$$\mathcal{P}^2\Theta_{AB} = 0.$$

under the assumption that $\zeta_{AA'}|_{\mathcal{S}} = 0$. Now, recalling the wave equation for Θ_{AB} (equation (3.13)), notice that the right hand side vanishes on $\mathcal{S}$ as a consequence of (3.24a), (3.24b) and (3.26), so that one is left with

$$\square\Theta_{AB}|_{\mathcal{S}} = 0.$$

Finally, expanding the wave operator on the left hand side gives that on $\mathcal{S}$,

$$\begin{aligned}\square\Theta_{AB} &= \nabla^{CC'}\nabla_{CC'}\Theta_{AB} \\ &= \left(-\tau^{BC'}\mathcal{D}^{C}{}_{B} + \frac{1}{2}\tau^{CC'}\mathcal{P}\right)\left(-\tau^{B}{}_{C'}\mathcal{D}_{BC} + \frac{1}{2}\tau_{CC'}\mathcal{P}\right)\Theta_{AB} \\ &= \frac{1}{4}\tau^{CC'}\tau_{CC'}\mathcal{P}^2\Theta_{AB}\end{aligned}$$

where the last line follows by commuting the derivatives where appropriate and using conditions (3.29) and (3.30). As $\tau^{CC'}\tau_{CC'} = 2$ by definition, we find that $\mathcal{P}^2\Theta_{AB} = 0$ as a consequence of the evolution equation for Θ_{AB}.

3.4.3 Eliminating redundant conditions

The discussion of the previous subsections can be summarised in the following:

Theorem 7. *Let κ_{AB} denote a Killing spinor candidate on an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. If κ_{AB} satisfies on a Cauchy hypersurface $\mathcal{S}$ the intrinsic conditions*

$$\xi_{ABCD} = 0, \tag{3.36a}$$

$$\Psi_{F(ABC}\kappa_{D)}{}^{F} = 0, \tag{3.36b}$$

$$\kappa_{(A}{}^{C}\phi_{B)C} = 0, \tag{3.36c}$$

$$\kappa_{(A|}{}^{C}\mathcal{D}_{CD}\phi_{|B)}{}^{D} = \frac{1}{3}\phi_{(A}{}^{C}\xi_{B)C}, \tag{3.36d}$$

$$\begin{aligned}3\kappa_{(A}{}^{F}\mathcal{D}_{B}{}^{E}\Psi_{CD)EF} + \Psi_{(ABC}{}^{F}\xi_{D)F} = {}& 6\widehat{\phi}_{F(A}\kappa^{E}{}_{B}\mathcal{D}^{F}{}_{C}\phi_{D)E} \\ &+ 6\widehat{\phi}_{(AB}\kappa^{E}{}_{C}\mathcal{D}^{F}{}_{D)}\phi_{EF},\end{aligned} \tag{3.36e}$$

and its normal derivative at $\mathcal{S}$ is given by

$$\mathcal{P}\kappa_{AB} = -\frac{2}{3}\xi_{AB},$$

then κ_{AB} is, in fact, a Killing spinor.

Remark 19. Note that

$$\Theta_{AB} = \kappa_{(A}{}^{C}\phi_{B)C} = 0 \quad \Rightarrow \quad \phi_{AB} \propto \kappa_{AB}$$

Using this fact, it is possible to express (3.36d) and (3.36e) as a condition on the proportionality factor relating the Killing spinor κ_{AB} and the Maxwell spinor ϕ_{AB}.

In order to simplify the conditions in Theorem 7 and to analyse their various inter-relations, we proceed by looking at the different algebraic types that the Killing spinor can have. First, consider the algebraically general case:

Lemma 12. *Assume that a symmetric spinor κ_{AB} satisfies the conditions*

$$\kappa_{AB}\kappa^{AB} \neq 0, \qquad \xi_{ABCD} = \Psi_{F(ABC}\kappa_{D)}{}^{F} = \kappa_{(A}{}^{C}\phi_{B)C} = 0$$

on an open subset $\mathcal{U} \subset \mathcal{S}$. Then, there exists a spin basis $\{o^A, \iota^A\}$ with $o_A\iota^A = 1$ such that the spinors κ_{AB} and ϕ_{AB} can be expanded as

$$\kappa_{AB} = e^{\varkappa} o_{(A}\iota_{B)}, \qquad \phi_{AB} = \varphi o_{(A}\iota_{B)}.$$

Furthermore, if $\mathfrak{Q} \equiv \varphi e^{2\varkappa}$ is a constant on $\mathcal{U}$, then conditions (3.36d) *and* (3.36e) *are satisfied on $\mathcal{U}$.*

Proof. The first part of the lemma follows directly from $\kappa_{AB}\kappa^{AB} \neq 0$, and the fact that $\kappa_{(A}{}^{C}\phi_{B)C} = 0$ implies that $\phi_{AB} \propto \kappa_{AB}$. The condition $\Psi_{F(ABC}\kappa_{D)}{}^{F} = 0$ also allows the Weyl spinor to be expanded in the same basis:

$$\Psi_{ABCD} = \psi o_{(A}o_B\iota_C\iota_{D)}.$$

To show the redundancy of (3.36d) and (3.36e), the equation $\mathcal{D}_{(AB}\kappa_{CD)} = 0$ will be decomposed into irreducible components. To simplify the notation, use the D, Δ, δ

symbols from the Newman-Penrose formalism to represent directional derivatives:

$$D \equiv o^A o^B \mathcal{D}_{AB}, \qquad \Delta \equiv \iota^A \iota^B \mathcal{D}_{AB}, \qquad \delta \equiv o^A \iota^B \mathcal{D}_{AB}.$$

The irreducible components of $\mathcal{D}_{(AB}\kappa_{CD)} = 0$ then become:

$$o^C D o_C = 0, \tag{3.37a}$$

$$o^C \delta o_C = -\frac{1}{2} D\varkappa, \tag{3.37b}$$

$$\iota^C D\iota_C - o^C \Delta o_C = 2\delta\varkappa, \tag{3.37c}$$

$$\iota^C \delta\iota_C = \frac{1}{2}\Delta\varkappa, \tag{3.37d}$$

$$\iota^C \Delta\iota_C = 0. \tag{3.37e}$$

Using these, one can show that

$$e^{-\varkappa}\xi_{AB} = -3o_A o_B \iota^F \delta\iota_F - 3\iota_A\iota_B o^F \delta o_F + \frac{3}{2} o_{(A}\iota_{B)} \left(\iota^F D\iota_F + o^F \Delta o_F\right).$$

In a similar way, using the electromagnetic Gauss constraint, equation (3.20b), together with the basis expansion for ϕ_{AB}, one obtains

$$\delta\varphi + 2\varphi\delta\varkappa = 0 \tag{3.38}$$

on $\mathcal{S}$.

The spacetime Bianchi identity (1.7) implies the constraint

$$\mathcal{D}^{CD}\Psi_{ABCD} = -2\hat{\phi}^{CD}\mathcal{D}_{CD}\phi_{AB} \tag{3.39}$$

on $\mathcal{S}$, of which we now wish to find the irreducible components. To find the basis expansion of the Hermitian conjugate $\widehat{\phi}_{AB}$, note that:

$$o_A \widehat{o}^A \equiv o_A \tau^{AA'} \bar{o}_{A'} = \tau_{AA'} o^A \bar{o}^{A'} = \tau_a k^a$$

where $k_a \equiv o_A \bar{o}_{A'}$. As τ_a is timelike and k_a is null, this scalar product is non-zero, and so the pair $\{o_A, \widehat{o}_A\}$ forms a basis. We expand the spinor ι^A in this basis as

$$\iota^A = a\widehat{o}^A + bo^A.$$

for some $a, b \in \mathbb{C}$. Contracting this with o_A gives $1/a = o_A\widehat{o}^A \geq 0$, and so the basis coefficient a is non-negative. Taking the Hermitian conjugate gives:

$$\widehat{\iota}^A = -\bar{a}o^A + \bar{b}\widehat{o}^A.$$

Using the above expressions we can find the basis expansion of $\hat{\phi}_{AB}$. Namely, one has that:

$$\begin{aligned}
\widehat{\phi}_{AB} &= \frac{1}{2}\bar{\varphi}(\widehat{o}_A\widehat{\iota}_B + \widehat{\iota}_A\widehat{o}_B) \\
&= \frac{1}{2}\bar{\varphi}(-\bar{a}o_A\widehat{o}_B - \bar{a}\widehat{o}_A o_B + 2\bar{b}\widehat{o}_A\widehat{o}_B) \\
&= \frac{\bar{\varphi}\bar{b}}{a^2}\iota_A\iota_B + \frac{\bar{\varphi}b}{a^2}(|a|^2 + |b|^2)o_A o_B - \frac{\bar{\varphi}}{a^2}(|a|^2 + 2|b|^2)o_{(A}\iota_{B)}.
\end{aligned}$$

Now, using the basis expansion for the Weyl spinor, contracting with combinations of o^A and ι^A and using the relations given in (3.37a)-(3.37e) and (3.38), the components of (3.39) become

$$\begin{aligned}
D\psi + 3\psi D\varkappa &= \frac{6|\varphi|^2}{a^2}(|a|^2 + 2|b|^2)D\varkappa + \frac{12\bar{b}|\varphi|^2}{a^2}o^A\Delta o_A, \\
\Delta\psi + 3\psi\Delta\varkappa &= \frac{6|\varphi|^2}{a^2}(|a|^2 + 2|b|^2)\Delta\varkappa - \frac{12b|\varphi|^2}{a^2}(|a|^2 + |b|^2)\iota^A D\iota_A, \\
\delta\psi + 3\psi\delta\varkappa &= -\frac{6|\varphi|^2}{a^2}(|a|^2 + 2|b|^2)\delta\varkappa - \frac{3b\bar{\varphi}}{a^2}(|a|^2 + |b|^2)D\varphi - \frac{3\bar{b}\bar{\varphi}}{a^2}\Delta\varphi.
\end{aligned}$$

Exploiting the conditions (3.37a)-(3.37e) and the expansions of the Maxwell and the Bianchi constraints, condition (3.36e) can be decomposed into the following non-trivial irreducible parts:

$$\begin{aligned}
\frac{\bar{b}\bar{\varphi}}{a^2}\left(D\varphi + 2\varphi D\varkappa\right) &= 0,\\
\frac{\bar{\varphi}}{a^2}(|a|^2 + 2|b|^2)\left(D\varphi + 2\varphi D\varkappa\right) &= 0,\\
\frac{\bar{\varphi}}{a^2}(|a|^2 + 2|b|^2)\left(\Delta\varphi + 2\varphi \Delta\varkappa\right) &= 0,\\
\frac{b\bar{\varphi}}{a^2}(|a|^2 + |b|^2)\left(\Delta\varphi + 2\varphi \Delta\varkappa\right) &= 0.
\end{aligned}$$

Assuming $\varphi \neq 0$, these conditions along with the Maxwell constraint (3.38) are equivalent to the following basis-independent expression, also independent of the value of a and b:

$$\mathcal{D}_{AB}\varphi + 2\varphi\mathcal{D}_{AB}\varkappa = 0.$$

The latter can be written as

$$\mathcal{D}_{AB}\left(\varphi e^{2\varkappa}\right) = \mathcal{D}_{AB}\mathfrak{Q} = 0.$$

Therefore, under the hypotheses of the present lemma, equation (3.36e) is equivalent to the requirement of $\mathfrak{Q}$ being constant in a domain $\mathcal{U} \subset \mathcal{S}$. In a similar way, substituting the above relations in equation (3.36d) and splitting into irreducible parts gives the following set of equivalent conditions:

$$\begin{aligned}
e^{\varkappa}\left(D\varphi + 2\phi D\varkappa\right) &= 0,\\
e^{\varkappa}\left(\Delta\varphi + 2\varphi \Delta\varkappa\right) &= 0,\\
e^{\varkappa}\left(\delta\varphi + 2\varphi \delta\varkappa\right) &= 0.
\end{aligned}$$

As $e^{\varkappa}$ is non-zero, this set of conditions is again equivalent to the constancy of $\mathfrak{Q}$ in $\mathcal{U} \subset \mathcal{S}$. □

Next, consider the case when the Killing spinor is algebraically special:

Lemma 13. *Assume the symmetric spinor κ_{AB} satisfies the conditions*

$$\kappa_{AB}\kappa^{AB} = 0, \qquad \kappa_{AB}\hat{\kappa}^{AB} \neq 0, \qquad \xi_{ABCD} = \Psi_{F(ABC}\kappa_{D)}{}^{F} = \kappa_{(A}{}^{C}\phi_{B)C} = 0$$

on an open subset $\mathcal{U} \subset \mathcal{S}$. Then, there exists a normalised spin basis $\{o^A, \iota^A\}$ such that the spinors κ_{AB} and ϕ_{AB} can be expanded as

$$\kappa_{AB} = e^{\varkappa} o_A o_B, \qquad \phi_{AB} = \varphi o_A o_B.$$

Furthermore, the equations (3.36d) *and* (3.36e) *are satisfied on $\mathcal{U} \subset \mathcal{S}$.*

Proof. The first part of the lemma follows directly from the hypothesis $\kappa_{AB}\kappa^{AB} = 0$, $\kappa_{AB}\widehat{\kappa}^{AB} \neq 0$, and the fact that $\kappa_{(A}{}^{C}\phi_{B)C} = 0$ implies $\phi_{AB} \propto \kappa_{AB}$. The condition $\Psi_{F(ABC}\kappa_{D)}{}^{F} = 0$ also allows the Weyl spinor to be expanded in the same basis as

$$\Psi_{ABCD} = \psi o_A o_B o_C o_D.$$

In this basis, the components of the equation $\mathcal{D}_{(AB}\kappa_{CD)} = 0$ become

$$\begin{aligned}
&o^A D o_A = 0,\\
&D\varkappa + 4o^A \delta o_A + 2\iota^A D o_A = 0,\\
&\delta\varkappa + o^A \Delta o_A + 2\iota^A \delta o_A = 0,\\
&\Delta\varkappa + 2\iota^A \Delta o_A = 0.
\end{aligned}$$

Using these relations one can show that

$$e^{-\varkappa}\xi_{AB} = 3o_A o_B o^C \Delta o_C - 6 o_{(A}\iota_{B)} o^C \delta o_C.$$

The Maxwell constraint, equation (3.20b), on $\mathcal{S}$ is equivalent to

$$D\phi - \phi D\varkappa - 6\phi o^A \delta o_A = 0,$$

and the $o_{(A}\iota_{B)}$ component of the Bianchi constraint

$$\mathcal{D}^{CD}\Psi_{ABCD} = -2\widehat{\phi}^{CD}\mathcal{D}_{CD}\phi_{AB}$$

on $\mathcal{S}$, as a consequence of the previous relations, is equivalent to the following condition:

$$|\varphi|^2 o^A \Delta o_A - 2b|\varphi|^2 o^A \delta o_A = 0.$$

Then, by substituting all the relevant basis expansions into (3.36d) and (3.36e), and splitting the equations into irreducible parts, one finds that both conditions are automatically satisfied as a result of the above relations. □

We round up the discussion of this section with the following electrovacuum analogue of Theorem 4 in [4]:

Lemma 14. *Assume that one has a symmetric spinor κ_{AB} satisfying the conditions*

$$\mathcal{D}_{(AB}\kappa_{CD)} = \Psi_{F(ABC}\kappa_{D)}{}^F = \kappa_{(A}{}^C\phi_{B)C} = 0$$

on the Cauchy hypersurface $\mathcal{S}$ and that the complex function

$$\mathfrak{Q}^2 \equiv \left(\kappa_{AB}\kappa^{AB}\right)^2 \phi_{AB}\phi^{AB}$$

is constant on $\mathcal{S}$. Then the domain of dependence, $D^+(\mathcal{S})$, of the initial data set $(\mathcal{S}, \boldsymbol{g}, \boldsymbol{K}, \boldsymbol{F})$ will admit a Killing spinor.

Proof. Let $\mathcal{U}_1$ be the set of all points in $\mathcal{S}$ where $\kappa_{AB}\kappa^{AB} \neq 0$ and $\mathcal{U}_2$ be the set of all points in $\mathcal{S}$ where $\kappa_{AB}\widehat{\kappa}^{AB} \neq 0$. The scalar functions $\kappa_{AB}\kappa^{AB} : \mathcal{S} \to \mathbb{C}$ and $\kappa_{AB}\widehat{\kappa}^{AB} : \mathcal{S} \to \mathbb{R}$ are continuous. Therefore, $\mathcal{U}_1$ and $\mathcal{U}_2$ are open sets. Now, let $\mathcal{V}_1$ and $\mathcal{V}_2$ denote, respectively, the interiors of $\mathcal{S} \setminus \mathcal{U}_1$ and $\mathcal{V}_1 \setminus \mathcal{U}_2$. On the open set $\mathcal{V}_1 \cap \mathcal{U}_2$, we have that $\kappa_{AB}\kappa^{AB} = 0$ and $\kappa_{AB}\widehat{\kappa}^{AB} \neq 0$. Hence, by Lemma 13, the conditions (3.36d) and (3.36e) are satisfied on $\mathcal{V}_1 \cap \mathcal{U}_2$. Similarly, by Lemma 12, conditions (3.36d) and (3.36e) are satisfied on $\mathcal{U}_1$. On the open set $\mathcal{V}_2$, we have that $\kappa_{AB} = 0$ and therefore (3.36d) and (3.36e) are trivially satisfied on $\mathcal{V}_2$. Using the above sets, the 3-manifold $\mathcal{S}$ can be split as

$$\mathrm{int}\mathcal{S} = \mathcal{U}_1 \cup (\mathcal{V}_1 \cap \mathcal{U}_2) \cup \mathcal{V}_2 \cup \partial\mathcal{U}_1 \cup \partial\mathcal{V}_2.$$

By hypothesis, all terms in conditions (3.36d) and (3.36e) are continuous, and the conditions themselves are satisfied on the open sets $\mathcal{U}_1$, $\mathcal{V}_2$ and $\mathcal{V}_1 \cap \mathcal{U}_2$. By continuity, the conditions are also satisfied on the boundaries $\partial\mathcal{U}_1$ and $\partial\mathcal{V}_2$. Therefore, (3.36d) and (3.36e) are satisfied on $\mathrm{int}\,\mathcal{S}$, and by continuity this extends to the whole of $\mathcal{S}$. □

3.4.4 Summary

The calculations from the current section can be summarised in the following theorem:

Theorem 8. *Let $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \boldsymbol{F})$ be an initial data set for the Einstein-Maxwell field equations where $\mathcal{S}$ is a Cauchy hypersurface. If the conditions*

$$\xi_{ABCD} = 0, \tag{3.40a}$$

$$\Psi_{F(ABC}\kappa_{D)}{}^{F} = 0, \tag{3.40b}$$

$$\kappa_{(A}{}^{C}\phi_{B)C} = 0, \tag{3.40c}$$

$$\mathfrak{Q}^2 \equiv \left(\kappa_{AB}\kappa^{AB}\right)^2 \phi_{AB}\phi^{AB} = constant, \tag{3.40d}$$

are satisfied on $\mathcal{S}$, then the development of the initial data set will admit a Killing spinor in the domain of dependence of $\mathcal{S}$. The Killing spinor is obtained by evolving (3.3b) *with initial data on $\mathcal{S}$ satisfying the above conditions and*

$$\mathcal{P}\kappa_{AB} = -\frac{2}{3}\xi_{AB}.$$

3.5 The approximate Killing spinor equation

In the previous section, conditions on an initial data set for the Einstein-Maxwell equations were identified that guarantee the existence of a Killing spinor on the resultant spacetime – see Theorem 8. Together with the characterisation of the Kerr-Newman spacetime given by Theorem 5, this provides a way of characterising initial data for the Kerr-Newman spacetime. The key equation in this characterisation is the *spatial Killing spinor equation*

$$\mathcal{D}_{(AB}\kappa_{CD)} = 0.$$

As it will be seen in the following, this equation is overdetermined and thus, admits no solution for a generic initial data set $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \boldsymbol{F})$. Following the discussion of Section 5 in [6], this section will show how to construct an elliptic equation for a spinor κ_{AB} over $\mathcal{S}$ which can always be solved and which provides, in some sense, a best fit to a spatial Killing spinor. This *approximate Killing spinor* will be used, in turn, to measure the deviation of the electrovacuum initial data set under consideration from initial data for the Kerr-Newman spacetime.

3.5.1 Basic identities

First, we will briefly discuss the basic ellipticity properties of the spatial Killing equation. In what follows, let $\mathfrak{S}_{(AB)}(\mathcal{S})$ and $\mathfrak{S}_{(ABCD)}(\mathcal{S})$ denote, respectively, the space of totally symmetric valence 2 and 4 spinor fields over the 3-manifold $\mathcal{S}$. Given μ_{AB}, $\nu_{AB} \in \mathfrak{S}_{(AB)}(\mathcal{S})$, ζ_{ABCD}, $\chi_{ABCD} \in \mathfrak{S}_{(ABCD)}(\mathcal{S})$ one can use the Hermitian structure induced on $\mathcal{S}$ by $\tau^{AA'}$ to define an inner product in $\mathfrak{S}_{(AB)}(\mathcal{S})$ and $\mathfrak{S}_{(ABCD)}(\mathcal{S})$, respectively, via

$$\langle \boldsymbol{\mu}, \boldsymbol{\nu} \rangle_2 \equiv \int_{\mathcal{S}} \mu_{AB} \widehat{\nu}^{AB} \mathrm{d}\mu, \qquad \langle \boldsymbol{\zeta}, \boldsymbol{\chi} \rangle_2 \equiv \int_{\mathcal{S}} \zeta_{ABCD} \widehat{\chi}^{ABCD} \mathrm{d}\mu \tag{3.41}$$

where $\mathrm{d}\mu$ denotes volume form of the 3-metric $\boldsymbol{h}$.

Let now Φ denote the *spatial Killing spinor operator*

$$\Phi : \mathfrak{S}_{(AB)}(\mathcal{S}) \longrightarrow \mathfrak{S}_{(ABCD)}(\mathcal{S}), \qquad \Phi(\boldsymbol{\kappa}) \equiv \mathcal{D}_{(AB}\kappa_{CD)}.$$

The inner product (3.41) allows one to define $\Phi^* : \mathfrak{S}_{(ABCD)}(\mathcal{S}) \longrightarrow \mathfrak{S}_{(AB)}(\mathcal{S})$, the *formal adjoint of* Φ, through the condition

$$\langle \Phi(\boldsymbol{\kappa}), \boldsymbol{\zeta} \rangle_2 = \langle \boldsymbol{\kappa}, \Phi^*(\boldsymbol{\zeta}) \rangle_2.$$

In order to evaluate the above condition one makes use of the identity (obtained using integration by parts)

$$\begin{aligned} \int_{\partial\mathcal{U}} n^{AB}\kappa^{CD}\widehat{\zeta}_{ABCD}\mathrm{d}S = \int_{\mathcal{U}} \mathcal{D}^{AB}\kappa^{CD}\widehat{\zeta}_{ABCD}\mathrm{d}\mu - \int_{\mathcal{U}} \kappa^{AB}\mathcal{D}^{CD}\widehat{\zeta_{ABCD}}\mathrm{d}\mu \\ + \int_{\mathcal{U}} 2\kappa^{AB}\Omega^{CDF}{}_{A}\widehat{\zeta}_{BCDF}\mathrm{d}\mu \end{aligned} \tag{3.42}$$

with $\mathcal{U} \subset \mathcal{S}$ and where $\mathrm{d}S$ denotes the area element of $\partial\mathcal{U}$, n_{AB} is the spinorial counterpart of its outward pointing normal and ζ_{ABCD} is a totally symmetric spinorial field. Now, observing that

$$\begin{aligned} \langle \Phi(\boldsymbol{\kappa}), \boldsymbol{\zeta} \rangle_2 &= \int_{\mathcal{S}} \mathcal{D}_{(AB}\kappa_{CD)}\widehat{\zeta}^{ABCD}\mathrm{d}\mu \\ &= \int_{\mathcal{S}} \mathcal{D}_{AB}\kappa_{CD}\widehat{\zeta}^{ABCD}\mathrm{d}\mu, \end{aligned}$$

it follows then from the identity (3.42) that

$$\Phi^*(\boldsymbol{\zeta}) = \mathcal{D}^{AB}\zeta_{ABCD} - 2\Omega^{ABF}{}_{(C}\zeta_{D)ABF}.$$

Definition 3. The composition operator $L \equiv \Phi^* \circ \Phi : \mathfrak{S}_{(AB)}(\mathcal{S}) \longrightarrow \mathfrak{S}_{(AB)}(\mathcal{S})$ given by

$$L(\boldsymbol{\kappa}) \equiv \mathcal{D}^{AB}\mathcal{D}_{(AB}\kappa_{CD)} - \Omega^{ABF}{}_{(A}\mathcal{D}_{|DF|}\kappa_{B)C} - \Omega^{ABF}{}_{(A}\mathcal{D}_{B)F}\kappa_{CD} \tag{3.43}$$

will be called the *approximate Killing spinor operator* and the equation

$$L(\boldsymbol{\kappa}) = 0$$

the *approximate Killing spinor equation.*

Remark 20. A direct computation shows that the approximate Killing spinor equation (3.43) is, in fact, the Euler-Lagrange equation of the functional

$$J \equiv \int_{\mathcal{S}} \mathcal{D}_{(AB}\kappa_{CD)} \widehat{\mathcal{D}^{AB}\kappa^{CD}}\, d\mu.$$

3.5.2 Ellipticity of the approximate Killing spinor equation

The key observation concerning the approximate Killing spinor operator is given in the following:

Lemma 15. *The operator L is a formally self-adjoint elliptic operator.*

Proof. It is sufficient to look at the principal part of the operator L given by

$$P(L)(\boldsymbol{\kappa}) = \mathcal{D}^{AB}\mathcal{D}_{(AB}\kappa_{CD)}.$$

The symbol for this operator is given by

$$\sigma_L(\boldsymbol{\xi}) \equiv \xi^{AB}\xi_{(AB}\kappa_{CD)}$$

where the argument ξ_{AB} satisfies $\xi_{AB} = \xi_{(AB)}$ and $\widehat{\xi}_{AB} = -\xi_{AB}$ – i.e. ξ is a real symmetric spinor. Also, define an inner product $\langle\, , \rangle$ on the space of symmetric valence-2 spinors by

$$\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle \equiv \widehat{\xi}^{AB}\eta_{AB}.$$

The operator L is elliptic if the map

$$\sigma_L(\boldsymbol{\xi}) : \kappa_{AB} \mapsto \xi^{CD}\xi_{(CD}\kappa_{AB)}$$

is an isomorphism when $|\boldsymbol{\xi}|^2 \equiv \langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle \neq 0$. As the above mapping is linear and between vector spaces of the same dimension, one only needs to verify injectivity – in other words, that if $\xi^{AB}\xi_{(AB}\kappa_{CD)} = 0$, then $\kappa_{AB} = 0$. To show this, first expand the symmetrisation to obtain

$$-\kappa_{CD}|\boldsymbol{\xi}|^2 - \langle \boldsymbol{\xi}, \boldsymbol{\kappa} \rangle \xi_{CD} + 2\xi^{AB}\xi_{CB}\kappa_{AD} + 2\xi^{AB}\xi_{DB}\kappa_{AC} = 0,$$

where the reality condition $\widehat{\xi}_{AB} = -\xi_{AB}$ has been used. Note also that the spinorial Jacobi identity implies that

$$\xi^{AB}\xi_{CB} = -\frac{1}{2}\delta_C{}^A|\boldsymbol{\xi}|^2$$

which reduces the above equation to

$$3\kappa_{CD}|\boldsymbol{\xi}|^2 + \xi_{CD}\langle \boldsymbol{\xi}, \boldsymbol{\kappa} \rangle = 0.$$

Contracting this with $\widehat{\kappa}^{CD}$, and using the conjugate symmetry of the inner product, we obtain

$$3|\boldsymbol{\kappa}|^2|\boldsymbol{\xi}|^2 + |\langle \boldsymbol{\xi}, \boldsymbol{\kappa} \rangle|^2 = 0.$$

Both of these terms are positive, and so the equality can only hold if each term vanishes individually. Taking the first of these, one sees that when $|\boldsymbol{\xi}|^2 \neq 0$, we must have $|\boldsymbol{\kappa}|^2 = 0$. This is equivalent to $\kappa_{AB} = 0$, completing the proof of injectivity and establishing the ellipticity of L. □

3.6 The approximate Killing spinor equation in asymptotically Euclidean manifolds

Now that the elliptic character of the approximate Killing spinor equation has been established, we can now move on to discuss the construction of a solution of the approximate Killing spinor equation, equation (3.43), in asymptotically Euclidean manifolds. The main conclusion of this section is that for this type of initial data set for the Einstein-Maxwell equations it is always possible to construct an approximate Killing spinor.

3.6.1 Weighted Sobolev norms

The discussion of asymptotic boundary conditions for the approximate Killing equation on asymptotically flat manifolds makes use of *weighted Sobolev norms* and *spaces*; here, the necessary terminology and conventions to follow the discussion will be established, following those laid out in [8].

For a point $p \in \mathcal{S}$, define the function $\sigma : \mathcal{S} \to \mathbb{R}$ by

$$\sigma(x) \equiv \left(1 + d(p,x)^2\right)^{\frac{1}{2}}$$

where d is the Riemannian distance function on $\mathcal{S}$. Using this, define the following weighted L^2 norm, for $\delta \in \mathbb{R}$:

$$\parallel u \parallel_\delta \equiv \left(\int_{\mathcal{S}} |u|^2 \sigma^{-2\delta-3} \mathrm{d}x\right)^{\frac{1}{2}}$$

For example, the choice $\delta = -\frac{3}{2}$ gives the usual L^2 norm. Similarly, let H^s_δ with s a non-negative index denote the weighted Sobolev space of functions for which the norm

$$\parallel u \parallel_{s,\delta} \equiv \sum_{0 \leq |\alpha| \leq s} \parallel D^\alpha u \parallel_{\delta - |\alpha|}$$

is finite, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index and $|\alpha| \equiv \alpha_1 + \alpha_2 + \alpha_3$. We say that the function $u \in H^\infty_\delta$ if $u \in H^s_\delta$ for all s. Furthermore, a spinor or a tensor is said to belong to a function space if its norm does – so, for instance $\zeta_{AB} \in H^s_\delta$ is a shorthand for $(\zeta_{AB}\hat{\zeta}^{AB} + \zeta_A{}^A\hat{\zeta}_B{}^B)^{1/2} \in H^s_\delta$. A property of the weighted Sobolev spaces that will be used repeatedly is the following: if $u \in H^\infty_\delta$, then u is smooth (i.e. C^∞ over $\mathcal{S}$) and has a fall off at infinity such that $D^\alpha u = o(r^{\delta-|\alpha|})$[1]. In a slight abuse of notation, if $u \in H^\infty_\delta$ then we will often say that $u = o_\infty(r^\delta)$ in a given asymptotic end.

3.6.2 Asymptotically Euclidean manifolds

The remainder of this chapter will concern Einstein-Maxwell initial data sets with a specific asymptotic behaviour, imposing a restriction on the class of such initial data sets. The Einstein-Maxwell constraint equations are given by

$$\begin{aligned} r - K^2 + K_{ij}K^{ij} &= 2\rho, \\ D^j K_{ij} - D_i K &= j_i, \\ D^i E_i &= 0, \\ D^i B_i &= 0, \end{aligned}$$

where D_i denotes the Levi-Civita connection of the 3-metric $\boldsymbol{h}$, r is the associated Ricci scalar, K_{ij} is the extrinsic curvature, $K \equiv K_i{}^i$, ρ is the energy-density of the electromagnetic field, j_i is the associated Poynting vector and E_i and B_i denote the electric and magnetic parts of the Faraday tensor with respect to the unit normal of $\mathcal{S}$. The following restriction on this data will be assumed:

Assumption 5. *The initial data set $(\boldsymbol{h}, \boldsymbol{K}, \boldsymbol{E}, \boldsymbol{B})$ for the Einstein-Maxwell equations is asymptotically Reissner-Nordström in the sense that in each asymptotic end of $\mathcal{S}$ there*

[1] Recall that the *small o* indicates that if $f(x) = o(x^n)$, then $f(x)/x^n \to 0$ as $x \to 0$.

exist asymptotically Cartesian coordinates (x^α) *and two constants* m, q *for which*

$$h_{\alpha\beta} = -\left(1 + \frac{2m}{r}\right)\delta_{\alpha\beta} + o_\infty(r^{-3/2}), \tag{3.44a}$$

$$K_{\alpha\beta} = o_\infty(r^{-5/2}), \tag{3.44b}$$

$$E_\alpha = \frac{qx_\alpha}{r^3} + o_\infty(r^{-5/2}), \tag{3.44c}$$

$$B_\alpha = o_\infty(r^{-5/2}). \tag{3.44d}$$

Remark 21. The asymptotic conditions spelled in Assumption 5 ensure that the total electric charge of the initial data is non-vanishing. In particular, it contains standard initial data for the Kerr-Newman spacetime in, say, Boyer-Lindquist coordinates as an example. More generally, the assumptions are consistent with the notion of *stationary asymptotically flat end* provided in Definition 1.

Remark 22. The above class of initial data is not the most general one could consider. In particular, conditions (3.44a)-(3.44d) exclude boosted initial data. In order to do so one would require that

$$K_{\alpha\beta} = o_\infty(r^{-3/2}).$$

The Einstein-Maxwell constraint equations would then require one to modify the leading behaviour of the 3-metric $h_{\alpha\beta}$. The required modifications for this extension of the present analysis are discussed in [6].

3.6.3 Asymptotic behaviour of the approximate Killing spinor

We can now discuss the asymptotic behaviour of solutions to the spatial Killing spinor equation on asymptotically Euclidean manifolds of the type described in Assumption 5. The strategy for doing this will be to first consider the behaviour of the Killing spinor in the exact Kerr-Newman spacetime; then, impose the same asymptotics on solutions to the approximate Killing spinor equation on slices of a more general spacetime. In what follows, the analysis will be focussed on the asymptotic end of the spacetime.

3.6.3.1 Asymptotic behaviour in the exact Kerr-Newman spacetime

For the exact Kerr-Newman spacetime with mass m, angular momentum a and charge q it is possible to introduce a NP frame $\{l^a,\, n^a,\, m^a,\, \bar{m}^a\}$ with associated spin dyad $\{o^A,\, \iota^A\}$ such that the spinors κ_{AB}, ϕ_{AB} and Ψ_{ABCD} admit the expansion

$$\kappa_{AB} = \varkappa o_{(A}\iota_{B)}, \qquad \phi_{AB} = \varphi o_{(A}\iota_{B)}, \qquad \Psi_{ABCD} = \psi o_{(A}o_B\iota_C\iota_{D)},$$

with

$$\begin{aligned} \varkappa &= \frac{2}{3}(r - \mathrm{i}a\cos\theta), \\ \varphi &= \frac{q}{(r-\mathrm{i}a\cos\theta)^2}, \\ \psi &= \frac{6}{(r-\mathrm{i}a\cos\theta)^3}\left(\frac{q^2}{r+\mathrm{i}a\cos\theta} - m\right), \end{aligned}$$

where r denotes the standard *Boyer-Lindquist* radial coordinate – see [3] for more details. A further computation shows that the spinorial counterpart, $\xi^{AA'}$, of the Killing vector ξ^a takes the form

$$\xi_{AA'} = -\frac{3}{2}\varkappa(\mu o_A\bar{o}_{A'} - \pi o_A\bar{\iota}_{A'} + \tau\iota_A\bar{o}_{A'} - \rho\iota_A\bar{\iota}_{A'}) \tag{3.45}$$

where the NP spin connection coefficients μ, π, τ and ρ satisfy the conditions

$$\bar{\mu}\bar{\varkappa} = \mu\varkappa, \qquad \bar{\tau}\bar{\varkappa} = \varkappa\pi, \qquad \bar{\rho}\bar{\varkappa} = \varkappa\rho$$

which ensure that $\xi_{AA'}$ is a Hermitian spinor – i.e. $\xi_{AA'} = \bar{\xi}_{AA'}$. Despite the conciseness of the above expressions, the basis of *principal spinors* given by $\{o^A, \iota^A\}$ is not well adapted to the discussion of asymptotics on a stationary end of the Kerr-Newman spacetime.

From the point of view of asymptotics, a better representation of the Kerr-Newman spacetime is obtained using a NP frame $\{l'^a,\, n'^a,\, m'^a,\, \bar{m}'^a\}$ with associated spin dyad

$\{o'^A,\, \iota'^A\}$ such that

$$\tau^a = l'^a + n'^a = \sqrt{2}(\boldsymbol{\partial}_t)^a,$$

where the vector τ^a is the tensorial counterpart of the spinor $\tau^{AA'}$. Writing this in the spinorial basis gives

$$\tau^{AA'} = o'^A \bar{o}'^{A'} + \iota'^A \bar{\iota}'^{A'}. \tag{3.46}$$

Notice, in particular, that from the above expression it follows that $\iota'_A = \widehat{o'}_A$. As $\tau_{AA'} = \sqrt{2}\xi_{AA'}$, one can use the expressions (3.45) and (3.46) to compute the leading terms of the Lorentz transformation relating the NP frames $\{l^a,\, n^a,\, m^a,\, \bar{m}^a\}$ and $\{l'^a,\, n'^a,\, m'^a,\, \bar{m}'^a\}$.

In what follows it will be convenient to denote the spinors of the basis $\{o'^A,\, \iota'^A\}$ in the form $\{\epsilon_{\boldsymbol{A}}{}^A\}$ where

$$\epsilon_{\boldsymbol{0}}{}^A = o'^A, \qquad \epsilon_{\boldsymbol{1}}{}^A = \iota'^A.$$

Moreover, let $\kappa_{\boldsymbol{AB}} \equiv \epsilon_{\boldsymbol{A}}{}^A \epsilon_{\boldsymbol{B}}{}^B \kappa_{AB}$ denote the components of κ_{AB} with respect to the basis $\{\epsilon_{\boldsymbol{A}}{}^A\}$. It can then be shown (through a long computation) that for Kerr-Newman initial data satisfying the asymptotic conditions (3.44a)-(3.44d) one can choose asymptotically Cartesian coordinates $(x^\alpha) = (x^1, x^2, x^3)$ and orthonormal frames on the asymptotic ends such that

$$\kappa_{\boldsymbol{AB}} = \mp\frac{\sqrt{2}}{3} x_{\boldsymbol{AB}} \mp \frac{2\sqrt{2}m}{3r} x_{\boldsymbol{AB}} + o_\infty(r^{-1/2}), \tag{3.47}$$

with

$$x_{\boldsymbol{AB}} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -x^1 + \mathrm{i}x^2 & x^3 \\ x^3 & x^1 + \mathrm{i}x^2 \end{pmatrix}.$$

From the above expressions one finds that on the asymptotic ends

$$\xi = \pm\sqrt{2} + o_\infty(r^{-1/2}),$$
$$\xi_{\boldsymbol{AB}} = o_\infty(r^{-1/2}),$$

where $\xi_{\boldsymbol{AB}} \equiv \epsilon_{\boldsymbol{A}}{}^A \epsilon_{\boldsymbol{B}}{}^B \xi_{AB}$. Moreover, for any electrovacuum initial data set satisfying the conditions (3.44a)-(3.44d) a spinor of the form (3.47) satisfies

$$\mathcal{D}_{(\boldsymbol{AB}}\kappa_{\boldsymbol{CD})} = o_\infty(r^{-3/2}).$$

3.6.3.2 Asymptotic behaviour for non-Kerr data

Not unsurprisingly, given electrovacuum initial data satisfying the conditions (3.44a)-(3.44d), it is always possible to find a spinor κ_{AB} consistent with the expansion (3.47) in the asymptotic region. More precisely, one has:

Lemma 16. *For any asymptotic end of an electrovacuum initial data set satisfying* (3.44a)-(3.44d) *there exists a spinor* κ_{AB} *such that*

$$\kappa_{\boldsymbol{AB}} = \mp\frac{\sqrt{2}}{3}x_{\boldsymbol{AB}} \mp \frac{2\sqrt{2}m}{3r}x_{\boldsymbol{AB}} + o_\infty(r^{-1/2})$$

with

$$\xi = \pm\sqrt{2} + o_\infty(r^{-1/2}), \tag{3.48a}$$

$$\xi_{\boldsymbol{AB}} = o_\infty(r^{-1/2}), \tag{3.48b}$$

$$\xi_{\boldsymbol{ABCD}} = o_\infty(r^{-3/2}). \tag{3.48c}$$

The spinor $\kappa_{\boldsymbol{AB}}$ *is unique up to order* $o_\infty(r^{-1/2})$*, up to the addition of a constant term.*

Proof. The proof follows the same structure of Theorem 17 in [6], where the vacuum case is considered. The argument goes as follows: first, substitution of the expansion for $\kappa_{\boldsymbol{AB}}$ given yields the asymptotic behaviours in (3.48a)-(3.48c), and so all one is required to show is that this expansion is unique. To do this, let $\tilde{\kappa}_{\boldsymbol{AB}}$ be a spinor satisfying the conditions of the lemma, define

$$\mathring{\kappa}_{\boldsymbol{AB}} \equiv \mp\frac{\sqrt{2}}{3}x_{\boldsymbol{AB}} \mp \frac{2\sqrt{2}m}{3r}x_{\boldsymbol{AB}},$$

and consider $\kappa_{\boldsymbol{AB}} \equiv \tilde{\kappa}_{\boldsymbol{AB}} - \mathring{\kappa}_{\boldsymbol{AB}}$. To obtain the desired result, one need only show that $\kappa_{\boldsymbol{AB}} = C_{\boldsymbol{AB}} + o_\infty(r^{-1/2})$, for $C_{\boldsymbol{AB}}$ a constant spinor. This is equivalent to showing that $\mathcal{D}_{AB}\kappa_{CD} = o_\infty(r^{-3/2})$, a coordinate independent statement. This is done by evaluating the asymptotic behaviour of the right hand sides of equations (3.23a)-(3.23c) (using the additional fact not present in the vacuum case that the Maxwell spinor $\phi_{AB} = o_\infty(r^{-2})$, a consequence of (3.44c)-(3.44d)) and integrating to obtain stronger asymptotic decay behaviour for ξ and ξ_{AB}. As these are the irreducible components of $\mathcal{D}_{AB}\kappa_{CD}$, this gives the required result (these calculations are identical to the those in Theorem 17 in [6]). □

In the analysis of the construction of a solution to the approximate Killing spinor equation, it is crucial that there exist no nontrivial spatial Killing spinors that go to zero at infinity. More precisely, one has the following:

Lemma 17. *Let $\nu_{AB} \in H^\infty_{-1/2}$ be a solution to $\mathcal{D}_{(AB}\nu_{CD)} = 0$ on an electrovacuum initial data set satisfying the asymptotic conditions* (3.44a)-(3.44d)*. Then $\nu_{AB} = 0$ on $\mathcal{S}$.*

Proof. From Lemma 11 one can write $\mathcal{D}_{AB}\mathcal{D}_{CD}\mathcal{D}_{EF}\kappa_{GH}$ as a linear combination of lower order derivatives, with smooth coefficients. Direct inspection shows that the coefficients in this linear combination have the decay conditions to make use of Theorem 20 from [6] with $m = 2$. It then follows that ν_{AB} must vanish on $\mathcal{S}$. □

3.6.4 Solving the approximate Killing spinor equation

Consider now solutions to the approximate Killing spinor equation of the form:

$$\kappa_{AB} = \mathring{\kappa}_{AB} + \theta_{AB}, \qquad \theta_{AB} \in H^\infty_{-1/2} \tag{3.49}$$

with $\mathring{\kappa}_{AB}$ the spinor discussed in Lemma 16. For this ansatz one has the following:

Theorem 9. *Given an electrovacuum asymptotically Euclidean initial data set* $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \boldsymbol{E}, \boldsymbol{B})$ *satisfying the asymptotic conditions* (3.44a)-(3.44d)*, there exists a smooth unique solution to the approximate Killing spinor equation* (3.43) *of the form* (3.49)*.*

Proof. The proof is analogous to that of Theorem 25 in [6]; it is presented for completeness, as it is important for the main result of this chapter.

Substitution of the Ansatz (3.49) into equation (3.43) yields the equation

$$L(\theta_{AB}) = -L(\mathring{\kappa}_{AB}) \tag{3.50}$$

for the spinor θ_{AB}. Due to elliptic regularity, any solution to the above equation of class $H^2_{-1/2}$ is, in fact, a solution of class $H^\infty_{-1/2}$. Thus, if a solution θ_{AB} exists then it must be smooth. By construction – see Lemma 16 – it follows that $\mathcal{D}_{(AB}\kappa_{CD)} \in H^\infty_{-3/2}$ so that

$$F_{AB} \equiv -L(\mathring{\kappa}_{AB}) \in H^\infty_{-5/2}.$$

In order to discuss the existence of solutions we make use of the *Fredholm alternative* for weighted Sobolev spaces. In the particular case of equation (3.50) there exists a unique solution of class $H^2_{-1/2}$ if

$$\int_{\mathcal{S}} F_{AB}\widehat{\nu}^{AB}\mathrm{d}\mu = 0$$

for all $\nu_{AB} \in H^2_{-1/2}$ satisfying

$$L^*(\nu_{CD}) = L(\nu_{CD}) = 0.$$

It will now be shown that a spinor ν_{AB} satisfying the above must be trivial. Using the identity (3.42) with $\zeta_{ABCD} = \mathcal{D}_{(AB}\nu_{CD)}$ and assuming that $L(\nu_{CD}) = 0$ one obtains

$$\int_{\mathcal{S}} \mathcal{D}^{AB}\nu^{CD}\widehat{\mathcal{D}_{(AB}\nu_{CD)}}\mathrm{d}\mu = \int_{\partial\mathcal{S}_\infty} n^{AB}\nu^{CD}\widehat{\mathcal{D}_{(AB}\nu_{CD)}}\mathrm{d}S$$

where $\partial\mathcal{S}_\infty$ denotes the sphere at infinity. Now, as $\nu_{AB} \in H^2_{-1/2}$ by assumption, it follows that $\mathcal{D}_{(AB}\nu_{CD)} \in H^{\infty}_{-3/2}$ and that

$$n^{AB}\nu^{CD}\widehat{\mathcal{D}_{(AB}\nu_{CD)}} = o(r^{-2}).$$

The integration of the latter over a finite sphere is of order $o(1)$. Accordingly, the integral over the sphere at infinity $\partial\mathcal{S}_{\infty}$ vanishes and, moreover,

$$\int_{\mathcal{S}} \mathcal{D}^{AB}\nu^{CD}\widehat{\mathcal{D}_{(AB}\nu_{CD)}}\mathrm{d}\mu = 0.$$

Thus, one concludes that

$$\mathcal{D}_{(AB}\nu_{CD)} = 0 \qquad \text{over} \qquad \mathcal{S}$$

so that ν_{AB} is a Killing spinor candidate. Lemma 17 shows that there are no non-trivial Killing spinor candidates that go to zero at infinity.

It follows from the discussion in the previous paragraph that the kernel of the approximate Killing spinor operator is trivial and that the Fredholm alternative imposes no obstruction to the existence of solutions to (3.50). Thus, one obtains a unique solution to the approximate Killing spinor equation with the prescribed asymptotic behaviour at infinity. □

3.7 The geometric invariant

Now that a process for constructing an approximate Killing spinor on an initial data set with asymptotic behaviour given by (3.44a)-(3.44d) has been outlined, we can use this spinor to construct an invariant measuring the deviation of the initial data set from initial data for the exact Kerr-Newman spacetime.

In the following let κ_{AB} denote the approximate Killing spinor obtained from Theorem 9, and let

$$\begin{aligned}
J &\equiv \int_{\mathcal{S}} \mathcal{D}_{(AB}\kappa_{CD)}\widehat{\mathcal{D}^{AB}\kappa^{CD}}\mathrm{d}\mu, \\
I_1 &\equiv \int_{\mathcal{S}} \Psi_{(ABC}{}^{F}\kappa_{D)F}\widehat{\Psi^{ABCG}\kappa^{D}{}_{G}}\mathrm{d}\mu, \\
I_2 &\equiv \int_{\mathcal{S}} \Theta_{AB}\widehat{\Theta^{AB}}\mathrm{d}\mu, \\
I_3 &\equiv \int_{\mathcal{S}} \mathcal{D}_{AB}\mathfrak{Q}^2\widehat{\mathcal{D}^{AB}\mathfrak{Q}^2}\mathrm{d}\mu,
\end{aligned}$$

where following the notation of Section 3.4 one has

$$\Theta_{AB} \equiv 2\kappa_{(A}{}^{Q}\phi_{B)Q}, \qquad \mathfrak{Q}^2 \equiv \left(\kappa_{AB}\kappa^{AB}\right)^2 \phi_{AB}\phi^{AB}.$$

The above integrals are well-defined due to the following result:

Lemma 18. *Given the approximate Killing spinor κ_{AB} obtained from Theorem 9, one has that*

$$J,\, I_1,\, I_2,\, I_3 < \infty.$$

Proof. By construction one has that the spinor κ_{AB} obtained from Theorem 9 satisfies $\mathcal{D}_{(AB}\kappa_{CD)} \in H^0_{-3/2}$. It follows then from the definition of the weighted Sobolev norm that

$$\| \nabla_{(AB}\kappa_{CD)} \|_{H^0_{-3/2}} = \| \nabla_{(AB}\kappa_{CD)} \|_{L^2} = J < \infty.$$

To verify the boundedness of I_1, notice that by assumption $\Psi_{ABCD} \in H^\infty_{-3+\varepsilon}$ and $\kappa_{AB} \in H^\infty_{1+\varepsilon}$; it follows by the multiplication properties of weighted Sobolev spaces (see e.g. Lemma 14 in [6]) that

$$\Psi_{(ABC}{}^{F}\kappa_{D)F} \in H^\infty_{-3/2},$$

so that, in fact, $I_1 < \infty$.

We now look at the boundedness of I_2. By construction and due to the asymptotic conditions (3.44a)-(3.44d), one can choose asymptotically Cartesian coordinates and orthonormal frames on the asymptotic ends such that the approximate Killing spinor and Maxwell spinor satisfy

$$\kappa_{\boldsymbol{AB}} = \mp\frac{\sqrt{2}}{3}x_{\boldsymbol{AB}} + o_\infty\left(r^{1/2}\right)$$
$$\phi_{\boldsymbol{AB}} = \frac{q}{\sqrt{2}r^3}x_{\boldsymbol{AB}} + o_\infty\left(r^{-5/2}\right)$$

Therefore,

$$\begin{aligned}\Theta_{\boldsymbol{AB}} &= \kappa_{(\boldsymbol{A}}{}^{\boldsymbol{Q}}\phi_{\boldsymbol{B})\boldsymbol{Q}} \\ &= \mp\frac{q}{3r^3}x_{(\boldsymbol{A}}{}^{\boldsymbol{Q}}x_{\boldsymbol{B})\boldsymbol{Q}} + o_\infty\left(r^{-3/2}\right) \\ &= o_\infty\left(r^{-3/2}\right)\end{aligned}$$

and so $\Theta_{AB} \in H^\infty_{-3/2}$, and $I_2 < \infty$.

Finally, to show the boundedness of I_3, note that in the asymptotically Cartesian coordinates and orthonormal frames used above, we have

$$\left(\kappa_{AB}\kappa^{AB}\right)^2 = \frac{4}{81}r^4 + o_\infty\left(r^{-7/2}\right)$$
$$\phi_{AB}\phi^{AB} = \frac{q^2}{2r^4} + o_\infty\left(r^{-9/2}\right)$$

and so the quantity $\mathfrak{Q}$ satisfies

$$\mathfrak{Q}^2 = \frac{2}{81}q^2 + o_\infty\left(r^{-1/2}\right)$$

Taking a derivative, one obtains

$$\mathcal{D}_{AB}\mathfrak{Q}^2 = o_\infty\left(r^{-3/2}\right)$$

and therefore $\mathcal{D}_{AB}\mathfrak{Q}^2 \in H^\infty_{-3/2}$ and $I_3 < \infty$. □

Remark 23. As coordinate-invariant functions on the hypersurface $\mathcal{S}$, dependent on the curvature of $\mathcal{S}$, they may be related to topological invariants of the sub-manifold - for example, Chern classes or other characteristic classes.

The integrals J, I_1, I_2 and I_3 are then used to define the following geometric invariant:

$$I = J + I_1 + I_2 + I_3. \tag{3.51}$$

Combining all of the results obtained so far gives the main result of this chapter:

Theorem 10. *Let* $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \boldsymbol{E}, \boldsymbol{B})$ *denote a smooth asymptotically Euclidean initial data set for the Einstein-Maxwell equations satisfying the on each of its two asymptotic ends the decay conditions* (3.44a)-(3.44d) *with non-vanishing mass and electromagnetic charge. Let* I *be the invariant defined by equation* (3.51) *where* κ_{AB} *is the unique solution to equation* (3.43) *with asymptotic behaviour at each end given by* (3.47). *The invariant* I *vanishes if and only if* $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \boldsymbol{E}, \boldsymbol{B})$ *is locally an initial data set for the Kerr-Newman spacetime.*

Proof. The proof follows the same strategy of Theorem 28 in [6]. It follows from the assumptions that if $I = 0$ then the electrovacuum Killing spinor data equations (3.40a)-(3.40d) are satisfied on the whole of the hypersurface $\mathcal{S}$. Thus, from Theorem 8 the development of the electrovacuum initial data $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \boldsymbol{E}, \boldsymbol{B})$ will have a Killing spinor, at least on a slab.

Now, the idea is to make use of Theorem 5 to conclude that the development will be the Kerr-Newman spacetime. For this, one has to conclude that the spinor $\xi_{AA'} \equiv \nabla^{Q}{}_{A}\kappa_{BQ}$ is Hermitian so that it corresponds to the spinorial counterpart of a real Killing vector. By assumption, it follows from the expansions (3.48a)-(3.48c) that

$$\xi - \hat{\xi} = o_\infty(r^{-1/2}), \qquad \xi_{AB} + \hat{\xi}_{AB} = o_\infty(r^{-1/2}).$$

Together, the last two expressions correspond to the Killing initial data for the imaginary part of $\xi_{AA'}$ – thus, the imaginary part of $\xi_{AA'}$ goes to zero at infinity. It is well known that for electrovacuum spacetimes there exist no non-trivial Killing vectors of this type [12, 16]. Thus, $\xi_{AA'}$ is the spinorial counterpart of a real Killing vector. By construction, $\xi_{AA'}$ tends, asymptotically, to a time translation at infinity. Accordingly, the development of the electrovacuum initial data $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \boldsymbol{E}, \boldsymbol{B})$ contains two asymptotically stationary flat ends $\mathcal{M}_\infty$ and $\mathcal{M}'_\infty$ generated by the Killing spinor κ_{AB}. As the Komar mass and the electromagnetic charge of each end is, by assumption, non-zero, one concludes from Theorem 5 that the development $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ is locally isometric to the Kerr-Newman spacetime. □

3.8 Conclusions

As a natural extension to the vacuum case described by Bäckdahl and Valiente Kroon [6], the formalism presented above for the electrovacuum case has similar applications and possible modifications. For example, the use of asymptotically hyperboloidal rather than asymptotically flat slices can now be analysed for the full electrovacuum case, applying to the more general Kerr-Newman solution. In the next chapter, the arguments made here will be modified to obtain necessary and sufficient conditions for the existence of a Killing spinor in the future development of a pair of intersecting null hypersurfaces, as opposed to a asymptotically flat spacelike hypersurface.

A motivation for the above analysis was also to provide a way of tracking the deviation of initial data from exact Kerr-Newman data in numerical simulations. However, in order to be a useful tool, one would still have to show that the geometric invariant is suitably behaved under time evolution (such as monotonicity). As highlighted in [6], a major problem is that it is hard to find a evolution equation for κ_{AB} such that the elliptic equations (3.43) is satisfied on each leaf in the foliation. If these issues can be resolved, then this formalism may be of some use in the study of non-linear perturbations of the Kerr-Newman solution and the black hole stability problem.

Chapter 4

Killing spinor data on non-expanding horizons

4.1 Introduction

In a paper by Rácz [47], it is shown that a spacetime admitting a pair of non-expanding, shear-free null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ (the union of which is shown to form a bifurcate Killing horizon in Corollary 6.1) can be uniquely determined in the domain of dependence of $\mathcal{H}_1 \cup \mathcal{H}_2$, once data has been prescribed on the intersection surface $\mathcal{Z} = \mathcal{H}_1 \cap \mathcal{H}_2$. This set-up provides the basis for the *characteristic initial value problem*, and is useful for investigating the behaviour of a black hole spacetime given data only on the horizons. In fact, the set-up described in [47] can be considered to describe a general class of *stationary distorted electrovacuum black hole spacetimes* – within the class of solutions to the Einstein-Maxwell equations. Of course, the Kerr-Newman family of solutions is an example of a family of exact solutions to the Einstein-Maxwell equations satisfying these conditions, and so belongs to this class of solutions. One can ask what further conditions are necessary to impose on the horizons in order to single out the Kerr-Newman family from the more general class, and how restrictive these conditions are.

Furthermore, as the only restriction on spacetimes in the class of distorted black hole spacetimes is the presence of a single one-parameter group is isometries (generated by the Killing vector associated to the bifurcate Killing horizon), the class is expected to contain not just the Kerr-Newman solution but a large number of 'nearby' and similar solutions. In particular, no assumption of made on the asymptotic behaviour of the spacetime - only the geodesic completeness of the generators of the null hypersurfaces is necessary. Therefore, this class is expected to contain the asymptotically flat stationary electrovacuum spacetimes established by black hole uniqueness theorems, along with spacetimes with other asymptotic properties.

In order to investigate possible conditions on $\mathcal{H}_1 \cup \mathcal{H}_2$, we can rely on the previously established characterisation of the Kerr solution by Killing spinors. In Chapter 3, this characterisation was used to identify the Kerr-Newman family of solutions exactly from a larger class. Here, it is shown that it is possible to guarantee the existence of a Killing spinor in the domain of dependence of the non-expanding horizons $\mathcal{H}_1 \cup \mathcal{H}_2$ by prescribing data for the Killing spinor, and this data need only be given on the intersection surface $\mathcal{Z}$. The only restriction on the background spacetime is the prescription of the curvature component Ψ_2 in terms of this initial data.

In this chapter, the analysis will be restricted to the vacuum case, attempting to identify the Kerr family of solutions to the Einstein equations from the general class of *stationary* distorted vacuum black hole spacetimes. A set of conditions will be found which must be satisfied on the intersection surface $\mathcal{Z}$ to ensure the existence of a Killing spinor on a neighbourhood of $\mathcal{Z}$ in the interior of the black hole, and then investigate further conditions which must be given there to single out the Kerr solution. A key obstacle is the fact that the natural asymptotic flatness conditions used in results like Theorem 5 and Theorem 10 cannot be used in the characteristic problem; constants arising from local results must either be determined by hand, or by some other criterion.

The main result of this chapter (given in Theorem 14) can be formulated as:

Theorem. *Let $(\mathcal{M}, g)$ be a vacuum distorted black hole. Given a spin basis $\{o^A, \iota^A\}$ on the bifurcation surface $\mathcal{Z}$, assume that there exist constants c, $\mathfrak{M} \in \mathbb{C}$ such that the following relations hold on $\mathcal{Z}$:*

$$\begin{aligned}
&\kappa_0 = \kappa_2 = 0, \\
&\eth^2\kappa_1 = \bar{\eth}^2\kappa_1 = 0, \\
&\kappa_1^3\Psi_2 = \mathfrak{M}, \\
&\kappa_1 + \bar{\kappa}_1 = c, \\
&\eth\bar{\eth}\kappa_1 + 2\Psi_2\kappa_1 \in \mathbb{R}
\end{aligned}$$

where $\kappa_0, \kappa_1, \kappa_2$ are the basis components of a spinor κ_{AB} with respect to the spin basis $\{o^A\iota_A\}$. Then, there exist two complex constants $\mathfrak{c}$ and $\mathfrak{l}$ such that

$$\mathcal{H}^2 = -\mathfrak{l}(\mathfrak{c} - \chi)^4$$

in a neighbourhood $\mathcal{O}$ of the bifurcation surface, where $\mathcal{H}^2 = \mathcal{H}_{ab}\mathcal{H}^{ab}$ is the contraction of the self-dual Killing form with itself (see section 2.2.4 for the full definition) and χ is the Ernst potential (see section 2.2.5.1). Furthermore, if $\mathfrak{c} = 1$ and $\mathfrak{l}$ is real and positive, then $(\mathcal{O}, g)$ is locally isometric to a member of the Kerr family of spacetimes.

Also, as in the previous chapter, the calculations in this chapter will make extensive use of the *xAct* suite of tensor computer algebra packages, in order to speed up the computation of large and unwieldy expressions. All of these calculations could be done by hand; the software merely allows them to be done in an efficient and intuitive manner. The details of the software are given in [43].

Overview

This chapter is structured as follows: first, in Section 4.2, the construction of the characteristic problem given in [47] is summarised. This construction will be used to define the distorted black holes considered in this chapter. In Section 4.3, the wave equation for the Killing spinor is decomposed into equations intrinsic to the horizons, providing a system of transport equations for the components of the Killing spinor. Furthermore, by finding a system of homogeneous wave equations for a collection of zero-quantity fields and imposing appropriate initial data for the system, further conditions (differential and algebraic constraints) can be found for the components of the Killing spinor and their first derivatives on the bifurcate horizon $\mathcal{H}_1 \cup \mathcal{H}_2$. In Section 4.4, these conditions are investigated further; it is shown that the conditions intrinsic to the bifurcation surface $\mathcal{Z}$ imply a specific form for the Killing spinor components. Furthermore, the constraints intrinsic to $\mathcal{H}_1$ or $\mathcal{H}_2$ are shown to satisfy ordinary differential equations along the generators of the relevant horizons, and so can be replaced with conditions on the bifurcation surface, if not becoming redundant. In this way, conditions on the extended horizon construction are reduced to conditions only on the bifurcation surface $\mathcal{Z}$. In Section 4.5, the additional conditions required to fulfil the assumptions of Proposition 2 are investigated; in particular, it is shown that the requirement that the Killing vector $\xi_{AA'}$ is Hermitian can be encoded as initial data on $\mathcal{H}_1 \cup \mathcal{H}_2$. These conditions can be reduced to conditions only on the bifurcation surface; this puts further restrictions on the form of the Killing spinor components on $\mathcal{Z}$. A detailed version of the main result is given in subsection 4.5.3. In Section 4.6, an explicit expression for the only non-trivial Killing spinor component is given, satisfying the required conditions on $\mathcal{Z}$; in doing so, a restriction on the geometry of the bifurcation surface is obtained. Lastly, Section 4.7 illustrates how the previously obtained conditions on $\mathcal{Z}$ are insufficient to completely isolate the Kerr family from the larger class of 'distorted' black holes. Explicitly, it is shown that 'distorted' black holes with metrically spherical bifurcation surfaces include spacetimes other than the Schwarzschild solution, thereby showing that the Kerr family is a strict subset of the class of 'distorted' black hole spacetimes.

Recap of notation and conventions

In what follows $(\mathcal{M}, \boldsymbol{g})$ will denote a vacuum spacetime. The metric $\boldsymbol{g}$ is assumed to have signature $(+,-,-,-)$. The Latin letters a, $b, \ldots$ are used as abstract tensorial spacetime indices while the Greek letters μ, $\nu, \ldots$ denote spacetime indices. The script letters $\mathcal{A}$, $\mathcal{B}, \ldots$ are used to denote *angular coordinates*. The Latin capital letters A, $B, \ldots$ are used as abstract spinorial indices.

Systematic use of the standard NP and GHP formalisms as discussed in [45, 55] will be used in this chapter, along with standard NP and GHP notation and conventions. In particular, if η is a smooth scalar on a 2-surface $\mathcal{Z}$ with spin-weight s, the action of the $\eth$ and $\bar{\eth}$ operators on η is defined by

$$\eth\eta = \delta\eta + s\,(\overline{\alpha} - \beta)\,\eta\,, \qquad \bar{\eth}\eta = \bar{\delta}\,\eta - s\,(\alpha - \overline{\beta})\,\eta\,. \tag{4.2}$$

One also has that

$$(\bar{\eth}\eth - \eth\bar{\eth})\,\eta = s\,K_{\mathcal{G}}\,\eta\,, \tag{4.3}$$

where $K_{\mathcal{G}}$ denotes the Gaussian curvature of $\mathcal{Z}$.

An alternative representation of the $\eth$ and $\bar{\eth}$ operators is given via the construction in Section 4.14 of [45]. In particular, by choosing an arbitrary holomorphic function z the 2-metric $\boldsymbol{\sigma}$ on $\mathcal{Z}$ can be given as

$$\boldsymbol{\sigma} = -\frac{1}{P\overline{P}}\big(\mathbf{d}z \otimes \mathbf{d}\bar{z} + \mathbf{d}\bar{z} \otimes \mathbf{d}z\big), \tag{4.4}$$

where P is a complex function on $\mathcal{Z}$. For example, if $\mathcal{Z}$ is isometric to the unit sphere, then $P = \frac{1}{2}(1 + z\bar{z})$; this is used later in section 4.7, when the implications of $\mathcal{Z}$ being isometric to S^2 with the round metric is investigated. More general 2-metrics that are conformally related to the round metric (i.e. $\boldsymbol{\sigma} = \Omega^2\boldsymbol{\sigma}_{S^2}$) will have their corresponding P functions rescaled by the conformal factor Ω.

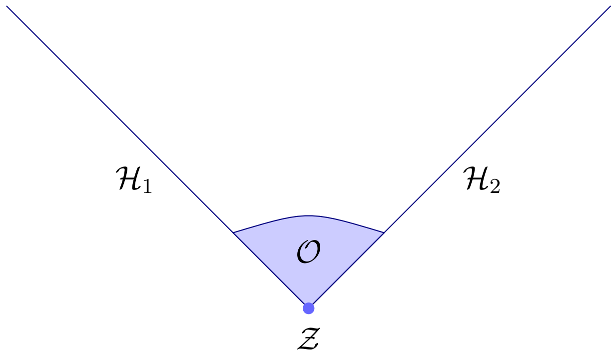

Figure 4.1: The set up for the characteristic initial value problem. The two non-expanding, shear-free null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ form a bifurcate Killing horizon, intersecting at the spacelike 2-surface $\mathcal{Z}$. $\mathcal{O}$ represents a neighbourhood of $\mathcal{Z}$ in $D(\mathcal{H}_1 \cup \mathcal{H}_2)$.

In terms of the holomorphic coordinate function z on $\mathcal{Z}$, the operators $\eth$ and $\bar{\eth}$ – acting on a scalar η of spin-weight s – are defined as (see (4.14.3)-(4.14.4) in [45])

$$\eth\eta \equiv P\overline{P}^{-s}\frac{\partial}{\partial z}\left(\overline{P}^{s}\,\eta\right), \qquad \bar{\eth}\eta \equiv \overline{P}P^{s}\frac{\partial}{\partial \overline{z}}\left(\eta P^{-s}\right). \tag{4.5}$$

As the complex coordinates z and $\bar{z}$ have zero spin-weight, it is easy to verify that

$$\eth z = P, \qquad \eth\overline{z} = 0\,,$$

and that

$$\eth P = 0\,, \qquad \bar{\eth} P = 0\,.$$

4.2 The characteristic initial value problem on expansion and shear-free hypersurfaces

In [47], by adopting and slightly generalising results of Friedrich in [26], a systematic analysis of the null characteristic initial value problem for the Einstein-Maxwell equations in terms of the Newman-Penrose formalism was performed. In particular, a procedure to obtain a system of reduced evolution equations forming a first order symmetric hyperbolic system was outlined. Moreover, it was shown that the solutions to these evolution equations imply, in turn, a solution to the full Einstein-Maxwell system provided that the inner (constraint) equations on the initial null hypersurfaces hold. For this setting, the theory for the characteristic initial value problem developed in [48] applies and ensures the local existence and uniqueness of a solution of the reduced evolution equations.

These general results were then used to investigate electrovacuum spacetimes $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ possessing a pair of null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ generated by expansion and shear-free geodesically complete null congruences, with intersection on a two dimensional spacelike hypersurface $\mathcal{Z} \equiv \mathcal{H}_1 \cap \mathcal{H}_2$. The configuration formed by $\mathcal{H}_1$ and $\mathcal{H}_2$ constitute a *bifurcate horizon*. In general, the freely specifiable data on $\mathcal{Z}$ does not possess any symmetry in addition to the horizon Killing vector (implied by the non-expanding character of the horizons). Thus, these spacetimes constitute the generic class of *stationary* distorted electrovacuum spacetimes. The key observation resulting from the analysis in [47] is, for the *vacuum case*, summarised in the following:

Theorem 11. *Suppose that $(\mathcal{M}, \boldsymbol{g})$ is a vacuum spacetime with a vanishing Cosmological constant possessing a pair of null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ generated by expansion and shear-free geodesically complete null congruences, intersecting on a 2-dimensional spacelike hypersurface $\mathcal{Z} \equiv \mathcal{H}_1 \cap \mathcal{H}_2$. Then, the metric $\boldsymbol{g}$ is uniquely determined (up to diffeomorphisms) on a neighbourhood $\mathcal{O}$ of $\mathcal{Z}$ contained in the domain of dependence $D(\mathcal{H}_1 \cap \mathcal{H}_2)$ of $\mathcal{H}_1$ and $\mathcal{H}_2$, once a complex vector field $\zeta^{\mathcal{A}}$ (determining the induced metric $\boldsymbol{\sigma}$ on $\mathcal{Z}$) and the spin connection coefficient τ are specified on $\mathcal{Z}$.*

4.2.1 Summary of the construction

Further information regarding the construction of the characteristic setup in [47] will be required for the analysis in this chapter. Throughout, let $(\mathcal{M}, \boldsymbol{g})$ denote a vacuum spacetime and let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote two null hypersurfaces in $(\mathcal{M}, \boldsymbol{g})$ intersecting on a spacelike 2-surface $\mathcal{Z}$.

Remark 24. In the rest of this section, the topology of $\mathcal{Z}$ will not be relevant for the discussion. The situation will, however, change when attempting to single out the Kerr spacetime.

Let n^a denote a smooth future-directed null vector on $\mathcal{Z}$ tangent to $\mathcal{H}_2$, which is extended to $\mathcal{H}_2$ by requiring it to satisfy $n^b\nabla_b n_a = 0$ on $\mathcal{H}_2$. Moreover, let u be an affine parameter along the null generators of $\mathcal{H}_2$, so that $u = 0$ on $\mathcal{Z}$ and $\mathcal{Z}_u$ are the associated 1-parameter family of smooth cross sections of $\mathcal{H}_2$. Choose a further null vector l^a as the unique future-directed null vector field on $\mathcal{H}_2$ which is orthogonal to the 2-dimensional cross sections $\mathcal{Z}_u$ and satisfies the normalisation condition $n_a l^a = 1$. Consider now the null geodesics starting on $\mathcal{H}_2$ with tangent l^a. Since $\mathcal{H}_2$ is assumed to be smooth and the vector fields n^a and l^a are smooth on $\mathcal{H}_2$ by construction, these geodesics do not intersect in a sufficiently small open neighbourhood $\mathcal{O} \subset \mathcal{M}$ of $\mathcal{H}_2$. Let r denote the affine parameter along the null geodesics starting on $\mathcal{H}_2$ with tangent l^a, chosen such that $r = 0$ on $\mathcal{H}_2$. By construction one has that $l^a = (\partial/\partial r)^a$, and the affine parameter defines a smooth function $r : \mathcal{O} \to \mathbb{R}$. The function $\mathcal{H}_2 \to \mathbb{R}$ defined by the affine parameter of the integral curves of n^a can be extended to a smooth function $u : \mathcal{O} \to \mathbb{R}$ by requiring it to be constant along the null geodesics with tangent l^a.

This construction is complemented by choosing suitable coordinates $(x^{\mathcal{A}})$ on patches of $\mathcal{Z}$ and extending them to $\mathcal{O}$ by requiring them to be constant along the integral curves of the vectors l^a and n^a. In this manner one obtains a system of *Gaussian null coordinates* $(x^\mu) = (u, r, x^{\mathcal{A}})$ on patches of $\mathcal{O}$. In each of these patches the spacetime metric $\boldsymbol{g}$ takes the form

$$
\begin{aligned}
\boldsymbol{g} = g_{00}\mathbf{d}u \otimes \mathbf{d}u + (\mathbf{d}u \otimes \mathbf{d}r + \mathbf{d}r \otimes \mathbf{d}u) \\
+ g_{0\mathcal{A}}(\mathbf{d}u \otimes \mathbf{d}x^{\mathcal{A}} + \mathbf{d}x^{\mathcal{A}} \otimes \mathbf{d}u) + g_{\mathcal{AB}}\mathbf{d}x^{\mathcal{A}} \otimes \mathbf{d}x^{\mathcal{B}},
\end{aligned}
$$

where g_{00}, $g_{0\mathcal{A}}$, $g_{\mathcal{AB}}$ are smooth functions of the coordinates (x^μ) such that

$$g_{00} = g_{0\mathcal{A}} = 0, \quad \text{on} \quad \mathcal{H}_2,$$

and $g_{\mathcal{AB}}$ is a negative definite 2×2 matrix. Observe that by construction

$$\mathcal{H}_1 \cap \mathcal{O} = \{(x^\mu) \in \mathcal{O} \mid u = 0\},$$
$$\mathcal{H}_2 \cap \mathcal{O} = \{(x^\mu) \in \mathcal{O} \mid r = 0\}.$$

In the following analysis, it will be convenient to consider the components of the contravariant form of the metric associated to the line element given above. A calculation shows that

$$(g^{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & g^{11} & g^{1\mathcal{B}} \\ 0 & g^{\mathcal{A}1} & g^{\mathcal{AB}} \end{pmatrix}.$$

The metric functions g^{11}, $g^{1\mathcal{A}}$ and $g^{\mathcal{AB}}$ can be conveniently parametrised in terms of real-valued functions U, $X^\mathcal{A}$ and complex-valued functions ω, $\zeta^\mathcal{A}$ on $\mathcal{O}$ such that

$$g^{11} = 2(U - \omega\overline{\omega}), \qquad g^{1\mathcal{A}} = X^\mathcal{A} - (\overline{\omega}\zeta^\mathcal{A} + \omega\overline{\zeta}^\mathcal{A}), \qquad g^{\mathcal{AB}} = -(\zeta^\mathcal{A}\overline{\zeta}^\mathcal{B} + \zeta^\mathcal{B}\overline{\zeta}^\mathcal{A}).$$

Accordingly, setting

$$l^\mu = \delta_1{}^\mu, \qquad n^\mu = \delta_0{}^\mu + U\delta_1{}^\mu + X^\mathcal{A}\delta_\mathcal{A}{}^\mu, \qquad m^\mu = \omega\delta_1{}^\mu + \zeta^\mathcal{A}\delta_\mathcal{A}{}^\mu,$$

one obtains a complex (NP) null tetrad $\{l^a, n^a, m^a, \overline{m}^a\}$ in $\mathcal{O}$. As a result of the vanishing of g_{00} and $g_{0\mathcal{A}}$ on $\mathcal{H}_2$, one has that

$$U = X^\mathcal{A} = \omega = 0, \quad \text{on} \quad \mathcal{H}_2.$$

It follows from the previous discussion that m^a and $\overline{m}^a$ are everywhere tangent to the sections $\mathcal{Z}_u$ of $\mathcal{H}_2$. In general, the complex null vectors m^a and $\overline{m}^a$ are not parallelly propagated along the null generators of $\mathcal{H}_2$.

Associated to the NP null tetrad $\{l^a,\, n^a,\, m^a,\, \overline{m}^a\}$ in $\mathcal{O}$ one has the directional derivatives

$$
\begin{aligned}
D &= \frac{\partial}{\partial r}, \\
\Delta &= \frac{\partial}{\partial u} + U\frac{\partial}{\partial r} + X^{\mathcal{A}}\frac{\partial}{\partial x^{\mathcal{A}}}, \\
\delta &= \omega\frac{\partial}{\partial r} + \zeta^{\mathcal{A}}\frac{\partial}{\partial x^{\mathcal{A}}}.
\end{aligned}
$$

Remark 25. By construction, one has that D is an intrinsic derivative to $\mathcal{H}_1$ pointing along the null generators of this hypersurface. Similarly, Δ is intrinsic to $\mathcal{H}_2$ and points in the direction of its null generators. Finally, $\{\delta, \bar{\delta}\}$ are differential operators which on $\mathcal{H}_2$ are intrinsic to the sections of constant u, $\mathcal{Z}_u$. Observe, however, that while δ restricted to $\mathcal{H}_1$ is still intrinsic to the null hypersurface, it is not intrinsic to the sections of constant r.

The NP null tetrad constructed in the previous paragraph can be specialised further to simplify the associate spin-connection coefficients. By parallelly propagating $\{l^a,\, n^a,\, m^a,\, \overline{m}^a\}$ along the null geodesics with tangent l^a one finds that

$$\kappa = \pi = \epsilon = 0, \tag{4.6a}$$

$$\rho = \overline{\rho}, \qquad \tau = \overline{\alpha} + \beta, \qquad \text{everywhere on} \quad \mathcal{O}. \tag{4.6b}$$

These equations arise from the application of the commutators of the directional derivatives to the chosen coordinate functions. Moreover, from the condition $n^b\nabla_b n^a = 0$ on $\mathcal{H}_2$ it follows that

$$\nu = 0 \qquad \text{on} \quad \mathcal{H}_2. \tag{4.7}$$

Also, using that u is an affine parameter of the generators of $\mathcal{H}_2$ one finds that $\gamma + \overline{\gamma} = 0$ along these generators. One can specialise further by suitably rotating the vectors

$\{m^a, \overline{m}^a\}$ so as to obtain

$$\gamma = 0, \qquad \text{on} \quad \mathcal{H}_2. \tag{4.8}$$

4.2.1.1 Solving the NP constraint equations

The NP Ricci and Bianchi identities split into a subset of intrinsic (constraint) equations to $\mathcal{H}_1 \cup \mathcal{H}_2$ and a subset of transverse (evolution) equations. In [47] the gauge introduced in the previous subsection was used to systematically analyse the constraint equations on $\mathcal{H}_1 \cup \mathcal{H}_2$ with the aim of identifying the freely specifiable data on this pair of intersecting hypersurfaces under the assumption that it is expansion and shear-free. The results from this analysis can be conveniently presented in the form of a table – see Table 4-A, obtained from [47].

$\mathcal{H}_1$	$\mathcal{Z}$	$\mathcal{H}_2$
$\mathrm{D}\zeta^{\mathcal{A}} = 0$	$\zeta^{\mathcal{A}}$ (data)	$\Delta\zeta^{\mathcal{A}} = 0$
$\omega = -r\,\tau$	$\omega = 0$	$\omega = 0$ (geometry)
$X^{\mathcal{A}} = r\,[\tau\,\overline{\zeta}^{\mathcal{A}} + \overline{\tau}\,\zeta^{\mathcal{A}}]$	$X^{\mathcal{A}} = 0$	$X^{\mathcal{A}} = 0$ (geometry)
$U = -r^2\,\left[2\,\tau\,\overline{\tau} + \frac{1}{2}\left(\Psi_2 + \overline{\Psi}_2\right)\right]$	$U = 0$	$U = 0$ (geometry)
$\rho = 0$	$\rho = 0$	$\rho = u\,\left(\overline{\delta}\tau - 2\,\alpha\,\tau - \Psi_2\right)$
$\sigma = 0$	$\sigma = 0$	$\sigma = u\,\left(\delta\tau - 2\,\beta\,\tau\right)$
$\mathrm{D}\tau = 0$	τ (data)	$\Delta\tau = 0$
$\mathrm{D}\alpha = \mathrm{D}\beta = 0$	$\alpha, \beta, \tau = \overline{\alpha} + \beta$	$\Delta\alpha = \Delta\beta = 0$
$\gamma = r\,\left(\tau\,\alpha + \overline{\tau}\,\beta + \Psi_2\right)$	$\gamma = 0$	$\gamma = 0$ (gauge)
$\mu = r\,\Psi_2$	$\mu = 0$	$\mu = 0$
$\lambda = 0$	$\lambda = 0$	$\lambda = 0$
$\nu = \frac{1}{2}\,r^2\,\left(\overline{\delta}\Psi_2 + \overline{\tau}\,\Psi_2\right)$	$\nu = 0$	$\nu = 0$ (gauge)
$\Psi_0 = 0$	$\Psi_0 = 0$	$\Psi_0 = \frac{1}{2}\,u^2\,\widetilde{\Psi}_0$
$\Psi_1 = 0$	$\Psi_1 = 0$	$\Psi_1 = u\,\left(\delta\Psi_2 - 3\,\tau\,\Psi_2\right)$
$\mathrm{D}\Psi_2 = 0$	$\zeta^{A}, \tau \rightarrow\ \alpha, \beta, \Psi_2$	$\Delta\Psi_2 = 0$
$\Psi_3 = r\,\overline{\delta}\Psi_2$	$\Psi_3 = 0$	$\Psi_3 = 0$
$\Psi_4 = \frac{1}{2}\,r^2\,\left(\overline{\delta}^2\,\Psi_2 + 2\,\alpha\,\overline{\delta}\Psi_2\right)$	$\Psi_4 = 0$	$\Psi_4 = 0$

Table 4-A: The full initial data set on $\mathcal{H}_1 \cup \mathcal{H}_2$ (obtained from [47]).

Remark 26. The vacuum field equations and the Bianchi identities written in Newman-Penrose form imply the following relations for the components of the Weyl curvature spinor:

$$\begin{aligned} \Psi_2 &= -\delta\alpha + \overline{\delta}\beta + \alpha\,\overline{\alpha} - 2\,\alpha\,\beta + \beta\,\overline{\beta} \\ \widetilde{\Psi}_0 &= \delta^2\Psi_2 - (7\,\tau + 2\,\beta)\,\delta\Psi_2 + 12\,\tau^2\Psi_2\,. \end{aligned} \tag{4.9}$$

Remark 27. As already mentioned, the following discussion will be mostly interested in the situation where $\mathcal{Z}$ is diffeomorphic to a unit 2-sphere, i.e. $\mathcal{Z} \approx \mathbb{S}^2$. From the definition of the operators $\eth$ and $\overline{\eth}$ as given in (4.2), along with those of the NP spin connection coefficients α and β, it follows that the connection on $\mathcal{Z}$ is encoded in the combination $\overline{\alpha} - \beta$. As discussed in [47], given the freely specifiable data ζ^A and τ one can readily compute the NP coefficients α, β. These, in turn, can be used, together with the NP Ricci equation (4.9), to determine the Weyl spinor component Ψ_2 on $\mathcal{Z}$: from (4.9), it is straightforward to deduce

$$2\,\mathrm{Re}(\Psi_2) = \Psi_2 + \overline{\Psi}_2 = -\delta\,(\,\alpha - \overline{\beta}\,) - \overline{\delta}\,(\,\overline{\alpha} - \beta\,) + 2\,(\,\alpha - \overline{\beta}\,)\,(\,\overline{\alpha} - \beta\,). \tag{4.10}$$

This implies that the real part of Ψ_2 – in accordance with the fact that $-2\,\mathrm{Re}(\Psi_2)$ is the Gaussian curvature K of $\mathcal{Z}$ (see, e.g. Proposition 4.14.21 in [45]) – depends only on the combination $\overline{\alpha} - \beta$, which is completely intrinsic to $\mathcal{Z}$. Analogously, by making use of (4.9), the spin coefficient τ and the imaginary part of Ψ_2 can be shown to be related via

$$2\,\mathrm{i}\,\mathrm{Im}(\Psi_2) = \Psi_2 - \overline{\Psi}_2 = \overline{\delta}\tau - \delta\overline{\tau} - 2\,\left(\beta\,\overline{\tau} - \overline{\beta}\,\tau\right)\,. \tag{4.11}$$

4.3 The Killing spinor data conditions for the characteristic initial problem

In this section, initial data for a set of wave equations will be found on the bifurcated horizons $\mathcal{H}_1 \cup \mathcal{H}_2$, guaranteeing the existence of a Killing spinor in a neighbourhood of the bifurcation surface $\mathcal{Z}$. Once the wave equation system has been established, the required initial data conditions for the zero-quantities will be decomposed in a natural way to obtain conditions on the components of the Killing spinor candidate κ_{AB}.

4.3.1 Killing spinors

As this chapter considers only vacuum spacetimes, the vector $\xi_{AA'}$ constructed from a spinor κ_{AB} satisfying the Killing spinor equation (2.1) via the definition (2.5) is in fact a Killing vector. Furthermore, using the definitions of the zero quantities $H_{A'ABC}$ and $S_{AA'BB'}$ given in (3.4a) and (3.4b) respectively, a calculation (performed for the electrovacuum case in section 3.3, and simplifying when the electromagnetic terms are set to zero)) shows the following:

Proposition 5. *Let κ_{AB} be a solution to equation* (3.3b)*. Then the spinor fields $H_{A'ABC}$ and $S_{AA'BB'}$ satisfy the system of wave equations*

$$\square H_{A'ABC} = 4\big(\Psi_{(AB}{}^{PQ}H_{C)PQA'} + \nabla_{(A}{}^{Q'}S_{BC)Q'A'}\big), \tag{4.12a}$$

$$\square S_{AA'BB'} = -\nabla_{AA'}(\Psi_{B}{}^{PQR}H_{B'PQR}) - \nabla_{BB'}(\Psi_{A}{}^{PQR}H_{A'PQR}) + 2\Psi_{AB}{}^{PQ}S_{PA'QB'} + 2\overline{\Psi}_{A'B'}{}^{P'Q'}S_{AP'BQ'}. \tag{4.12b}$$

Remark 28. As the above equations constitute a system of homogeneous linear wave equations for the fields $H_{A'ABC}$ and $S_{AA'BB'}$, it follows that they readily imply conditions for the existence of a Killing spinor in the development of a given initial value problem for the vacuum Einstein field equations, when sufficient appropriate initial data is provided. In Chapter 3, this initial data (for a more general system of wave equations) was found on a spacelike hypersurface; the calculations performed there can be adapted to the current setting of a characteristic initial data set – see also [28].

4.3.2 Construction of the Killing spinor candidate

As with the case of a spacelike initial hypersurface, it will prove useful to investigate the characteristic initial value problem for the wave equation(3.3b), governing the evolution of the Killing spinor candidate κ_{AB}. An approach to the formulation of the characteristic initial value problem for wave equations on intersecting null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ has been analysed in [48]. This discussion follows the ideas of this analysis closely.

4.3.2.1 Basic set-up

Let $\{o^A, \iota^A\}$ denote a spin dyad normalised according to $o_A\iota^A = 1$. The spinor κ_{AB} can be written as

$$\kappa_{AB} = \kappa_2 o_A o_B - 2\kappa_1 o_{(A}\iota_{B)} + \kappa_0 \iota_A \iota_B.$$

so that

$$\kappa_0 \equiv \kappa_{AB} o^A o^B, \qquad \kappa_1 \equiv \kappa_{AB} o^A \iota^B, \qquad \kappa_2 \equiv \kappa_{AB} \iota^A \iota^B.$$

It can be readily verified that the scalars κ_2, κ_1 and κ_0 have, respectively, spin weights -1, $0, 1$ – i.e. they transform as

$$\kappa_j \mapsto e^{-2(j-1)\mathrm{i}\vartheta}\kappa_j$$

under a rotation $\{o^A, \iota^A\} \mapsto \{e^{\mathrm{i}\vartheta} o^A,\, e^{-\mathrm{i}\vartheta}\iota^A\}$.

A direct decomposition of the wave equation (3.3b) using the NP formalism readily yields the following equations for the independent components κ_0, κ_1 and κ_2 of the spinor

κ_{AB}:

$$
\begin{aligned}
&D\Delta\kappa_2 + \Delta D\kappa_2 - \delta\overline{\delta}\kappa_2 - \overline{\delta}\delta\kappa_2 \\
&\qquad + (\mu + \overline{\mu} + 3\gamma - \overline{\gamma})D\kappa_2 - (\rho + \overline{\rho})\Delta\kappa_2 + (\overline{\tau} - 3\alpha - \overline{\beta})\delta\kappa_2 \\
&\qquad + (\overline{\alpha} - 5\beta + \tau)\overline{\delta}\kappa_2 + (\Psi_2 + 2\alpha\overline{\alpha} - 8\alpha\beta - 2\beta\overline{\beta} - 2\gamma\rho + 2\mu\rho \\
&\qquad - 2\gamma\overline{\rho} + 2\lambda\sigma + 2\alpha\tau + 2\beta\overline{\tau} + 2D\gamma - 2\delta\alpha - 2\overline{\delta}\beta)\kappa_2 + (\Psi_4 - 4\lambda\mu)\kappa_0 = 0, \quad \text{(4.13a)} \\
&D\Delta\kappa_1 + \Delta D\kappa_1 - \delta\overline{\delta}\kappa_1 - \overline{\delta}\delta\kappa_1 \\
&\qquad - 2\tau D\kappa_2 + (\mu + \overline{\mu} - \gamma - \overline{\gamma})D\kappa_1 + 2\nu D\kappa_0 - (\rho + \overline{\rho})\Delta\kappa_1 \\
&\qquad + 2\rho\delta\kappa_2 + (\alpha - \overline{\beta} + \overline{\tau})\delta\kappa_1 - 2\lambda\delta\kappa_2 + 2\sigma\overline{\delta}\kappa_2 + (\overline{\alpha} - \beta + \tau)\overline{\delta}\kappa_1 \\
&\qquad - 2\mu\overline{\delta}\kappa_0 + (-\Psi_1 - \overline{\alpha}\rho + 3\beta\rho + \alpha\sigma + \overline{\beta}\sigma\overline{\rho}\tau - \sigma\overline{\tau} - D\tau + \delta\rho\overline{\delta}\sigma)\kappa_2 \\
&\qquad + (-\Psi_3 + \overline{\alpha}\lambda + \beta\lambda + 3\alpha\mu - \overline{\beta}\mu - \nu\rho - \nu\overline{\rho} + \lambda\tau + \mu\overline{\tau} + D\nu \\
&\qquad - \delta\lambda - \overline{\delta}\mu)\kappa_0 = 0, \qquad \text{(4.13b)} \\
&D\Delta\kappa_0 + \Delta D\kappa_0 - \delta\overline{\delta}\kappa_0 - \overline{\delta}\delta\kappa_0 \\
&\qquad + (\mu + \overline{\mu} - 5\gamma - \overline{\gamma})D\kappa_0 - (\rho + \overline{\rho})\Delta\kappa_0 + (5\alpha - \overline{\beta} + \overline{\tau})\delta\kappa_0 \\
&\qquad + (\overline{\alpha} + 3\beta + \tau)\overline{\delta}\kappa_0 + (\Psi_2 - 2\alpha\overline{\alpha} - 8\alpha\beta + 2\beta\overline{\beta} + 2\gamma\rho + 2\mu\rho + 2\gamma\overline{\rho} \\
&\qquad + 2\lambda\sigma - 2\alpha\tau - 2\beta\overline{\tau} - 2D\gamma + 2\delta\alpha + 2\overline{\delta}\beta)\kappa_0 + (\Psi_0 - 4\rho\sigma)\kappa_2 = 0. \qquad \text{(4.13c)}
\end{aligned}
$$

The above expressions are completely general: no assumption on the spacetime (other than satisfying the vacuum field equations) or the gauge has been made.

Remark 29. At this stage, there is still considerable freedom in the choice of the spin dyad $\{o^A, \iota^A\}$. A natural choice is that of a spin dyad $\{o^A,\ \iota^A\}$ adapted to the NP null tetrad $\{l^a, n^a,\ m^a, \overline{m}^a\}$ – if $\{l^{AA'}, n^{AA'},\ m^{AA'}, \overline{m}^{AA'}\}$ denote the spinorial counterparts of the null tetrad, one has the correspondences

$$
l^{AA'} = o^A\overline{o}^{A'}, \qquad n^{AA'} = \iota^A\overline{\iota}^{A'}, \qquad m^{AA'} = o^A\overline{\iota}^{A'}, \qquad \overline{m}^{AA'} = \iota^A\overline{o}^{A'},
$$

and the gauge conditions (4.6a)-(4.6b), (4.7) and (4.8) hold when computing the corresponding NP spin-connection coefficients by means of derivatives of the spin dyad.

4.3.2.2 The transport equations on $\mathcal{H}_1$

Consider now the restriction of equations (4.13a)-(4.13c) to the null hypersurface $\mathcal{H}_1$ with tangent l^a. It follows then that D is a directional derivative along the null generators of $\mathcal{H}_1$, while Δ is a directional derivative transversal to $\mathcal{H}_1$. Using the NP commutator $[D,\Delta]$ equation to rewrite $\Delta D\kappa_0$, $\Delta D\kappa_1$, $\Delta D\kappa_2$ in terms of $D\Delta\kappa_0$, $D\Delta\kappa_1$ and $D\Delta\kappa_2$, equations (4.13a)-(4.13c) take the form:

$$\begin{aligned}
&2D\Delta\kappa_0 - \delta\overline{\delta}\kappa_0 - \overline{\delta}\delta\kappa_0 + (\overline{\alpha}+3\beta)\overline{\delta}\kappa_0 + (5\alpha-\overline{\beta})\delta\kappa_0 + (\mu+\overline{\mu}-4\gamma)D\kappa_0 \\
&\quad + 4\tau D\kappa_1 + 2\kappa_1 D\tau + (\Psi_2 - 2\alpha\overline{\alpha} - 8\alpha\beta + 2\beta\overline{\beta} - 2\alpha\tau - 2\beta\overline{\tau} \\
&\quad - 2D\gamma + 2\delta\alpha + 2\overline{\delta}\beta)\kappa_0 = 0, \qquad (4.14a)\\
&2D\Delta\kappa_1 - \delta\overline{\delta}\kappa_1 - \overline{\delta}\delta\kappa_1 - 2\nu D\kappa_0 + (\mu+\overline{\mu})D\kappa_1 + 2\tau D\kappa_2 + (\alpha-\overline{\beta})\delta\kappa_1 \\
&\quad + 2\mu\overline{\delta}\kappa_0 + (\overline{\alpha}-\beta)\overline{\delta}\kappa_1 + (\Psi_3 - 3\alpha\mu + \overline{\beta}\mu - \mu\overline{\tau} - D\nu + \overline{\delta}\mu)\kappa_0 \\
&\quad - 2\Psi_2\kappa_1 + \kappa_2 D\tau = 0, \qquad (4.14b)\\
&2D\Delta\kappa_2 - \delta\overline{\delta}\kappa_2 - \overline{\delta}\delta\kappa_2 - 4\nu D\kappa_1 + (4\gamma+\mu+\overline{\mu})D\kappa_2 - (3\alpha+\overline{\beta})\delta\kappa_2 \\
&\quad + 4\mu\overline{\delta}\kappa_1 + (\overline{\alpha}-5\beta)\overline{\delta}\kappa_2 + (\Psi_2 + 2\alpha\overline{\alpha} - 8\alpha\beta - 2\beta\overline{\beta} + 2\alpha\tau + 2\beta\overline{\tau} \\
&\quad + 2D\gamma - 2\delta\alpha - 2\overline{\delta}\beta)\kappa_2 + (2\alpha\mu - 2\Psi_3 + 2\overline{\beta}\mu - 2\mu\overline{\tau} - 2D\nu + 2\overline{\delta}\mu)\kappa_1 \\
&\quad + \Psi_4\kappa_0 = 0. \qquad (4.14c)
\end{aligned}$$

If the value of the components κ_0, κ_1, κ_2 are known on $\mathcal{H}_1$, then the above equations can be read as a system of ordinary differential equations for the transversal derivatives

$$\Delta\kappa_0, \quad \Delta\kappa_1, \quad \Delta\kappa_2,$$

along the null generators of $\mathcal{H}_1$. Initial data for these transport equations is naturally prescribed on $\mathcal{Z}$.

4.3.2.3 The transport equations on $\mathcal{H}_2$

Similarly, one can consider the restriction of equations (4.13a)-(4.13c) to the null hypersurface $\mathcal{H}_2$ with tangent n^a. Thus, Δ is a directional derivative along the null generators of $\mathcal{H}_2$, δ and $\overline{\delta}$ are intrinsic derivatives while D is transversal to $\mathcal{H}_2$. In this case one uses the NP commutator $[D,\Delta]$ to rewrite $D\Delta\kappa_0$, $D\Delta\kappa_1$, $D\Delta\kappa_2$ in terms of $\Delta D\kappa_0$, $\Delta D\kappa_1$, $\Delta D\kappa_2$ and lower order terms so that equations (4.13a)-(4.13c) take the form

$$
\begin{aligned}
&2\Delta D\kappa_0 - \delta\overline{\delta}\kappa_0 - \overline{\delta}\delta\kappa_0 - (\rho+\overline{\rho})\Delta\kappa_0 + 4\tau D\kappa_1 + (5\alpha - \overline{\beta} + 2\overline{\tau})\delta\kappa_0 + \\
&\quad (\overline{\alpha} + 3\beta + 2\tau)\overline{\delta}\kappa_0 + 4\sigma\overline{\delta}\kappa_1 - 4\rho\delta\kappa_1 + (\Psi_2 - 2\alpha\overline{\alpha} - 8\alpha\beta + 2\beta\overline{\beta} \\
&\quad - 2\alpha\tau - 2\beta\overline{\tau} + 2\delta\alpha + 2\overline{\delta}\beta)\kappa_0 + (2\overline{\alpha}\rho + 2\beta\rho + 6\alpha\sigma - 2\overline{\beta}\sigma - 2\overline{\rho}\tau \\
&\quad + 2\sigma\overline{\tau} + 2D\tau - 2\delta\rho - 2\overline{\delta}\sigma - 2\Psi_1)\kappa_1 + (\Psi_0 - 4\rho\sigma)\kappa_2 = 0, && (4.15a) \\
&2\Delta D\kappa_1 - \delta\overline{\delta}\kappa_1 - \overline{\delta}\delta\kappa_1 - (\rho+\overline{\rho})\Delta\kappa_1 + 2\tau D\kappa_2 + (\alpha - \overline{\beta} + 2\overline{\tau})\delta\kappa_1 \\
&\quad + (\overline{\alpha} - \beta + 2\tau)\overline{\delta}\kappa_1 - 2\rho\delta\kappa_2 + 2\sigma\overline{\delta}\kappa_2 - 2\Psi_2\kappa_1 \\
&\quad + (\Psi_1 - \overline{\alpha}\rho - 3\beta\rho - \alpha\sigma - \overline{\beta}\sigma - \overline{\rho}\tau + \sigma\overline{\tau} + D\tau - \delta\rho - \overline{\delta}\sigma)\kappa_2 = 0, && (4.15b) \\
&2\Delta D\kappa_2 - \delta\overline{\delta}\kappa_2 - \overline{\delta}\delta\kappa_2 - (\rho+\overline{\rho})\Delta\kappa_2 + (2\overline{\tau} - 3\alpha - \overline{\beta})\delta\kappa_2 + (\overline{\alpha} - 5\beta + 2\tau)\overline{\delta}\kappa_2 \\
&\quad + (\Psi_2 + 2\alpha\overline{\alpha} - 8\alpha\beta - 2\beta\overline{\beta} + 2\alpha\tau + 2\beta\overline{\tau} - 2\delta\alpha - 2\overline{\delta}\beta)\kappa_2 = 0. && (4.15c)
\end{aligned}
$$

If the values of κ_0, κ_1, κ_2 are known on $\mathcal{H}_2$ then the above equations can be read as a system of ordinary differential equations for the transversal derivatives

$$D\kappa_0, \qquad D\kappa_1, \qquad D\kappa_2,$$

along the null generators of $\mathcal{H}_2$. Initial data for these transport equations is naturally prescribed on $\mathcal{Z}$.

4.3.2.4 Summary: existence of the Killing spinor candidate

The discussion of the previous subsections combined with Theorem 1 from [48] – see also [35] – allows the formulation of the following existence result:

Proposition 6. *Let* $(\mathcal{M}, \boldsymbol{g})$ *denote a spacetime satisfying the assumptions of Theorem 11. Then, given a smooth choice of fields* κ_0, κ_1 *and* κ_2 *on* $\mathcal{H}_1 \cup \mathcal{H}_2$*, there exists a neighbourhood* $\mathcal{O}$ *of* $\mathcal{Z}$ *in* $D(\mathcal{H}_1 \cup \mathcal{H}_2)$ *on which the wave equation* (3.3b) *has a unique solution* κ_{AB}*.*

Proof. Once a basis $\{o^A, \iota^A\}$ has been chosen on $\mathcal{H}_1 \cup \mathcal{H}_2$, the spinor κ_{AB} is determined by the values of $\kappa_0, \kappa_1, \kappa_2$. Furthermore, the equation for κ_{AB} (3.3b) is a quasilinear wave equation of the form needed for Theorem 1 of [48]. By the statement of that theorem, the result follows. □

Remark 30. The assumption of smoothness of the fields κ_0, κ_1 and κ_2 requires, in particular, that the limits of these fields as one approaches to $\mathcal{Z}$ on either $\mathcal{H}_1$ or $\mathcal{H}_2$ coincide.

4.3.3 The NP decomposition of the Killing spinor data conditions

The conditions on the initial data for the Killing spinor candidate κ_{AB} constructed in the previous section which ensure that it is, in fact, a Killing spinor follow from requiring that the propagation system (4.12a)-(4.12b) of Proposition 5 has as a unique solution – the trivial (zero) one.

The purpose of this section is to analyse the characteristic initial value problem for the Killing spinor equation propagation system (4.12a)-(4.12b).

4.3.3.1 Basic observations

We are interested in solutions to the system (4.12a)-(4.12b) ensuring the existence of a Killing spinor on $D(\mathcal{H}_1 \cup \mathcal{H}_2)$. The homogeneity of these equations on the fields $H_{A'ABC}$ and $S_{AA'BB'}$ allows to formulate the following result:

Lemma 19. *Let $(\mathcal{M}, \boldsymbol{g})$ denote a spacetime satisfying the assumptions of Theorem 11. Further, assume that*

$$H_{A'ABC} = 0, \qquad S_{AA'BB'} = 0 \qquad \text{on} \quad \mathcal{H}_1 \cup \mathcal{H}_2.$$

Then there exists a neighbourhood $\mathcal{O}$ of $\mathcal{Z}$ in $D(\mathcal{H}_1 \cup \mathcal{H}_2)$ on which the $H_{A'ABC}$ and $S_{AA'BB'}$ vanish.

Proof. The result follows from using the methods of Section 4.3.2 on the equations (4.12a)-(4.12b), and the uniqueness of the solutions to the characteristic initial value problem. □

From the above lemma and the observations in Section 4.3.1 one directly obtains the following result concerning the existence of Killing spinors on $D(\mathcal{H}_1 \cup \mathcal{H}_2)$:

Proposition 7. *Let $(\mathcal{M}, \boldsymbol{g})$ denote a spacetime satisfying the assumptions of Theorem 11. Assume that initial data κ_0, κ_1, κ_2 on $\mathcal{H}_1 \cup \mathcal{H}_2$ for the wave equation* (3.3b) *can be found such that*

$$H_{A'ABC} = 0, \qquad S_{AA'BB'} = 0 \qquad \text{on} \quad \mathcal{H}_1 \cup \mathcal{H}_2.$$

Then the resulting Killing spinor candidate κ_{AB} is, in fact, a Killing spinor in a neighbourhood $\mathcal{O}$ of $\mathcal{Z}$ on $D(\mathcal{H}_1 \cup \mathcal{H}_2)$.

Remark 31. A straightforward computation shows that the condition

$$H_{A'ABC} = 0$$

is equivalent to the equations

$$D\kappa_0 - 2\epsilon\kappa_0 + 2\kappa\kappa_1 = 0, \tag{4.16a}$$

$$\delta\kappa_0 - 2\beta\kappa_0 + 2\sigma\kappa_1 = 0, \tag{4.16b}$$

$$\bar{\delta}\kappa_0 + 2D\kappa_1 - 2\pi\kappa_0 - 2\alpha\kappa_0 + 2\kappa_1\rho + 2\kappa\kappa_2 = 0, \tag{4.16c}$$

$$\Delta\kappa_0 + 2\delta\kappa_1 + 2\sigma\kappa_2 - 2\mu\kappa_0 + 2\tau\kappa_1 - 2\gamma\kappa_0 = 0, \tag{4.16d}$$

$$D\kappa_2 + 2\bar{\delta}\kappa_1 + 2\rho\kappa_2 - 2\lambda\kappa_0 - 2\pi\kappa_1 + 2\epsilon\kappa_2 = 0, \tag{4.16e}$$

$$\delta\kappa_2 + 2\Delta\kappa_1 + 2\tau\kappa_2 + 2\beta\kappa_2 - 2\mu\kappa_1 - 2\nu\kappa_0 = 0, \tag{4.16f}$$

$$\bar{\delta}\kappa_2 + 2\alpha\kappa_2 - 2\lambda\kappa_1 = 0, \tag{4.16g}$$

$$\Delta\kappa_2 + 2\gamma\kappa_2 - 2\nu\kappa_1 = 0. \tag{4.16h}$$

Remark 32. Defining the basis coefficients of the Killing vector $\xi_{AA'}$ by

$$\xi_{AA'} = \xi_{11'} o_A \bar{o}_{A'} + \xi_{10'} o_A \bar{\iota}_{A'} + \xi_{01'} \iota_A \bar{o}_{A'} + \xi_{00'} \iota_A \bar{\iota}_{A'},$$

equation (2.5) takes the form

$$\xi_{11'} = \Delta\kappa_1 - \delta\kappa_2 - 2\beta\kappa_2 + \tau\kappa_2 + 2\mu\kappa_1 - \nu\kappa_0, \tag{4.17a}$$

$$\xi_{10'} = D\kappa_2 - \bar{\delta}\kappa_1 + 2\epsilon\kappa_2 - \rho\kappa_2 - 2\pi\kappa_1 + \lambda\kappa_0, \tag{4.17b}$$

$$\xi_{01'} = \delta\kappa_1 - \Delta\kappa_0 + 2\gamma\kappa_0 - \mu\kappa_0 - 2\tau\kappa_1 + \sigma\kappa_2, \tag{4.17c}$$

$$\xi_{00'} = \bar{\delta}\kappa_0 - D\kappa_1 - 2\alpha\kappa_0 + \pi\kappa_0 + 2\rho\kappa_1 - \kappa\kappa_2. \tag{4.17d}$$

If $\xi_{AA'}$ is required to be Hermitian so that it corresponds to the spinorial counterpart of a real vector ξ^a then one has the reality conditions

$$\xi_{00'} = \bar{\xi}_{0'0}, \qquad \xi_{11'} = \bar{\xi}_{1'1}, \qquad \xi_{01'} = \bar{\xi}_{1'0}, \qquad \xi_{10'} = \bar{\xi}_{0'1}$$

A further calculation shows that the equation $S_{AA'BB'} = 0$ takes, in NP notation, the form:

$$D\xi_{00'} - \xi_{00'}\epsilon - \xi_{00'}\overline{\epsilon} - \xi_{10'}\kappa - \xi_{01'}\overline{\kappa} = 0, \tag{4.18a}$$

$$\Delta\xi_{11'} + \xi_{11'}\gamma + \xi_{11'}\overline{\gamma} + \xi_{01'}\nu + \xi_{10'}\overline{\nu} = 0, \tag{4.18b}$$

$$\begin{aligned} &D\xi_{11'} + \Delta\xi_{00'} - \xi_{00'}\gamma - \xi_{00'}\overline{\gamma} + \xi_{11'}\epsilon + \xi_{11'}\overline{\epsilon} + \xi_{01'}\pi \\ &\qquad + \xi_{10'}\overline{\pi} - \xi_{10'}\tau - \xi_{01'}\overline{\tau} = 0, \end{aligned} \tag{4.18c}$$

$$\begin{aligned} &\delta\xi_{11'} - \Delta\xi_{01'} + \overline{\alpha}\xi_{11'} + \xi_{11'}\beta + \xi_{01'}\gamma - \xi_{01'}\overline{\gamma} + \xi_{10'}\overline{\lambda} \\ &\qquad + \xi_{01'}\mu - \xi_{00'}\overline{\nu} + \xi_{11'}\tau = 0, \end{aligned} \tag{4.18d}$$

$$\delta\xi_{01'} + \xi_{01'}\overline{\alpha} - \xi_{01'}\beta + \xi_{00'}\overline{\lambda} - \xi_{11'}\sigma = 0, \tag{4.18e}$$

$$\begin{aligned} &\delta\xi_{00'} - D\xi_{01'} - \xi_{00'}\overline{\alpha} - \xi_{00'}\beta + \xi_{01'}\epsilon - \xi_{01'}\overline{\epsilon} + \xi_{11'}\kappa \\ &\qquad - \xi_{00'}\overline{\pi} - \xi_{01'}\overline{\rho} - \xi_{10'}\sigma = 0, \end{aligned} \tag{4.18f}$$

$$\begin{aligned} &\overline{\delta}\xi_{11'} - \Delta\xi_{10'} + \xi_{11'}\alpha + \xi_{11'}\overline{\beta} - \xi_{10'}\gamma + \xi_{10'}\overline{\gamma} + \xi_{01'}\lambda \\ &\qquad + \xi_{10'}\overline{\mu} - \xi_{00'}\nu + \xi_{11'}\overline{\tau} = 0, \end{aligned} \tag{4.18g}$$

$$\overline{\delta}\xi_{10'} + \xi_{10'}\alpha - \xi_{10'}\overline{\beta} + \xi_{00'}\lambda - \xi_{11'}\overline{\sigma} = 0, \tag{4.18h}$$

$$\begin{aligned} &\delta\xi_{10'} + \overline{\delta}\xi_{01'} - \xi_{01'}\alpha - \xi_{10'}\overline{\alpha} + \xi_{10'}\beta + \xi_{01'}\overline{\beta} + \xi_{00'}\mu \\ &\qquad + \xi_{00'}\overline{\mu} - \xi_{11'}\rho - \xi_{11'}\overline{\rho} = 0, \end{aligned} \tag{4.18i}$$

$$\begin{aligned} &\overline{\delta}\xi_{00'} - D\xi_{10'} - \xi_{00'}\alpha - \xi_{00'}\overline{\beta} - \xi_{10'}\epsilon + \xi_{10'}\overline{\epsilon} + \xi_{11'}\overline{\kappa} \\ &\qquad - \xi_{00'}\pi - \xi_{10'}\rho - \xi_{01'}\overline{\sigma} = 0. \end{aligned} \tag{4.18j}$$

The equations (4.16a)-(4.16h) and (4.18a)-(4.18j) are valid at any point in the spacetime. When restricting to the bifurcated horizons, these equations are expected to simplify as a result of the gauge conditions and other specific choices made during the set-up of the characteristic problem - see Table 4-A. Restrictions to the bifurcation surface $\mathcal{Z}$ and ingoing and outgoing null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ will be done successively in the next few sections.

4.3.3.2 The condition $H_{A'ABC} = 0$ on $\mathcal{Z} = \mathcal{H}_1 \cap \mathcal{H}_2$

On $\mathcal{Z} = \mathcal{H}_1 \cap \mathcal{H}_2$ equations (4.16a)-(4.16h) reduce to:

$$D\kappa_0 = 0, \tag{4.19a}$$

$$\Delta\kappa_2 = 0, \tag{4.19b}$$

$$\delta\kappa_0 - 2\beta\kappa_0 = 0, \tag{4.19c}$$

$$\Delta\kappa_0 + 2\delta\kappa_1 + 2\tau\kappa_1 = 0, \tag{4.19d}$$

$$2\Delta\kappa_1 + \delta\kappa_2 + 2\beta\kappa_2 + 2\tau\kappa_2 = 0, \tag{4.19e}$$

$$2D\kappa_1 + \bar{\delta}\kappa_0 - 2\alpha\kappa_0 = 0, \tag{4.19f}$$

$$D\kappa_2 + 2\bar{\delta}\kappa_1 = 0, \tag{4.19g}$$

$$\bar{\delta}\kappa_2 + 2\alpha\kappa_2 = 0. \tag{4.19h}$$

In what follows, regard equations (4.19c) and (4.19h) as intrinsic to $\mathcal{Z}$. Making use of the operators $\eth$ and $\bar{\eth}$ (see (4.2) for their explicit form) these conditions can be concisely rewritten as

$$\eth\kappa_0 = \tau\kappa_0, \tag{4.20a}$$

$$\bar{\eth}\kappa_2 = -\bar{\tau}\kappa_2. \tag{4.20b}$$

Remark 33. Equations (4.19a)-(4.19h) do not constrain the value of the coefficient κ_1 on $\mathcal{Z}$. Instead, given an arbitrary (smooth) choice of κ_1 and coefficients κ_0 and κ_2 satisfying the equations in (4.20a)-(4.20b), equations (4.19b), (4.19d) and (4.19e) are regarded as prescribing the initial values of the derivatives $\Delta\kappa_0$, $\Delta\kappa_1$ and $\Delta\kappa_2$ that need to be provided for the transport equations (4.14a)-(4.14c) along $\mathcal{H}_1$. Similarly, equations (4.19a), (4.19f) and (4.19g) can be used to prescribe the initial values of the derivatives $D\kappa_0$, $D\kappa_1$ and $D\kappa_2$ which are used, in turn, to solve the transport equations (4.15a)-(4.15c) along $\mathcal{H}_2$.

4.3.3.3 The condition $H_{A'ABC} = 0$ on $\mathcal{H}_1$

On $\mathcal{H}_1$ equations (4.16a)-(4.16h) reduce to:

$$D\kappa_0 = 0, \tag{4.21a}$$

$$\Delta\kappa_2 - 2\nu\kappa_1 + 2\gamma\kappa_2 = 0, \tag{4.21b}$$

$$\delta\kappa_0 - 2\beta\kappa_0 = 0, \tag{4.21c}$$

$$\Delta\kappa_0 + 2\delta\kappa_1 - 2(\gamma + \mu)\kappa_0 + 2\tau\kappa_1 = 0, \tag{4.21d}$$

$$2\Delta\kappa_1 + \delta\kappa_2 + 2(\beta + \tau)\kappa_2 - 2\mu\kappa_1 - 2\nu\kappa_0 = 0, \tag{4.21e}$$

$$2D\kappa_1 + \overline{\delta}\kappa_0 - 2\alpha\kappa_0 = 0, \tag{4.21f}$$

$$D\kappa_2 + 2\overline{\delta}\kappa_1 = 0, \tag{4.21g}$$

$$\overline{\delta}\kappa_2 + 2\alpha\kappa_2 = 0. \tag{4.21h}$$

Equations (4.21a), (4.21f) and (4.21g) are interpreted as propagation equations along the null generators of $\mathcal{H}_1$ which are used to propagate the initial values of κ_0, κ_1 and κ_2 at $\mathcal{Z}$. In order to understand the role equations (4.21c) and (4.21h), consider the expressions

$$D(\delta\kappa_0 - 2\beta\kappa_0), \qquad D(\overline{\delta}\kappa_2 + 2\alpha\kappa_2).$$

A direct computation using the NP commutators shows that

$$D(\delta\kappa_0 - 2\beta\kappa_0) = -2\kappa_0 D\beta$$
$$D(\overline{\delta}\kappa_2 + 2\alpha\kappa_2) = 2\kappa_2 D\alpha - 2(\alpha - \overline{\beta})\overline{\delta}\kappa_1 - 2\overline{\eth}^2\kappa_1.$$

Evaluating the Ricci identities on $\mathcal{H}_1$ one finds that $D\alpha = D\beta = 0$ – see also Table 4-A. Thus, it follows that

$$D(\delta\kappa_0 - 2\beta\kappa_0) = 0$$
$$D(\overline{\delta}\kappa_2 + 2\alpha\kappa_2) = -2\overline{\eth}^2\kappa_1.$$

Accordingly, equation (4.21c) holds along $\mathcal{H}_1$ if it is satisfied on $\mathcal{Z}$ – this is equivalent to requiring condition (4.20a) on $\mathcal{Z}$. Observe, however, that in order to obtain the same conclusion for equation (4.21h) one needs $\bar{\eth}^2\kappa_1 = 0$ on $\mathcal{H}_1$.

It remains to consider equations (4.21b), (4.21d) and (4.21e). These prescribe the value of the transversal derivatives $\Delta\kappa_0$, $\Delta\kappa_1$ and $\Delta\kappa_2$. Recall, however, that from the discussion of Section 4.3.2 these derivatives satisfy transport equations along the generators of $\mathcal{H}_1$. Thus, some compatibility conditions will arise. Substituting the value of $\Delta\kappa_0$, given by equation (4.21d) into the transport equation (4.14a), and then using the NP commutators, NP Ricci identities and equations (4.21a), (4.21f) and (4.21g) to simplify one obtains the condition

$$\Psi_2\kappa_0 = 0.$$

Similarly, substituting the value of $\Delta\kappa_1$ given by equation (4.21e) into the transport equation (4.14b) and proceeding in similar manner one finds the further condition

$$\Psi_3\kappa_0 = 0.$$

Finally, the substitution of the value of $\Delta\kappa_2$ as given by equation (4.21b) eventually leads to the condition

$$\Psi_4\kappa_0 + 2\Psi_3\kappa_1 - 3\Psi_2\kappa_2 = 0.$$

One can summarise the discussion of this subsection as follows:

Lemma 20. *Assume that equations* (4.21a), (4.21f) *and* (4.21g) *hold along* $\mathcal{H}_1$ *with initial data for* κ_0 *and* κ_2 *on* $\mathcal{Z}$ *satisfying equations* (4.20a) *and* (4.20b)*, respectively, and that, in addition,*

$$\bar{\eth}^2\kappa_1 = 0, \quad \Psi_2\kappa_0 = 0, \quad \Psi_3\kappa_0 = 0, \quad \Psi_4\kappa_0 + 2\Psi_3\kappa_1 - 3\Psi_2\kappa_2 = 0, \quad \textit{on} \quad \mathcal{H}_1.$$

Then, one has that

$$H_{A'ABC} = 0 \quad on \quad \mathcal{H}_1.$$

4.3.3.4 The condition $H_{A'ABC} = 0$ on $\mathcal{H}_2$

On $\mathcal{H}_2$ equations (4.16a)-(4.16h) reduce to:

$$D\kappa_0 = 0, \tag{4.22a}$$

$$\Delta\kappa_2 = 0, \tag{4.22b}$$

$$\delta\kappa_0 - 2\beta\kappa_0 + 2\sigma\kappa_1 = 0, \tag{4.22c}$$

$$\Delta\kappa_0 + 2\delta\kappa_1 + 2\tau\kappa_1 + 2\sigma\kappa_2 = 0, \tag{4.22d}$$

$$2\Delta\kappa_1 + \delta\kappa_2 + 2(\beta + \tau)\kappa_2 = 0, \tag{4.22e}$$

$$2D\kappa_1 + \bar{\delta}\kappa_0 - 2\alpha\kappa_0 + 2\rho\kappa_1 = 0, \tag{4.22f}$$

$$D\kappa_2 + 2\bar{\delta}\kappa_1 + 2\rho\kappa_2 = 0, \tag{4.22g}$$

$$\bar{\delta}\kappa_2 + 2\alpha\kappa_2 = 0. \tag{4.22h}$$

In analogy with the analysis on $\mathcal{H}_2$, in what follows we regard equations (4.22b), (4.22d) and (4.22e) as propagation equations for the components κ_0, κ_1 and κ_2 along the generators of $\mathcal{H}_2$. Initial data for these equations is naturally prescribed on $\mathcal{Z}$.

Now, regarding equation (4.22h), a direct computation shows that

$$\Delta(\bar{\delta}\kappa_2 + 2\alpha\kappa_2) = 0.$$

Thus, if equation (4.22h) is satisfied on $\mathcal{Z}$ then it holds along the generators of $\mathcal{H}_2$ – this equivalent to requiring (4.20b). A similar computation with equation (4.22c) yields the more complicated relation

$$\Delta(\delta\kappa_0 - 2\beta\kappa_0 + 2\sigma\kappa_1) = -2\eth^2\kappa_1 - 2\kappa_2\delta\sigma - 3\sigma\delta\kappa_2 - 2\bar{\alpha}\sigma\kappa_2.$$

Observe that if $\kappa_2 = 0$ along $\mathcal{H}_2$, then the obstruction to the propagation of equation (4.22c) reduces to the simple condition $\eth^2\kappa_1 = 0$ which is somehow complementary to the condition $\bar{\eth}^2\kappa_1 = 0$ on $\mathcal{H}_1$.

It remains to analyse the compatibility of equations (4.22a), (4.22f) and (4.22g) with the transport equations (4.15a)-(4.15c). Substituting $D\kappa_1$, $\Delta\kappa_0$, $D\kappa_0$ and $\delta\kappa_0$ given by equations (4.22a), (4.22d), (4.22f) and (4.22c) into equation (4.15a) one obtains after some manipulations the condition

$$\Psi_0\kappa_2 + 2\Psi_1\kappa_1 - 3\Psi_2\kappa_0 = 0.$$

Similarly, after substituting $D\kappa_1$, $\Delta\kappa_1$ and $D\kappa_2$ given by equations (4.22f), (4.22e) and (4.22g) into equation (4.15b) one obtains the condition

$$\Psi_1\kappa_2 = 0.$$

Finally, by substituting $D\kappa_2, \Delta\kappa_2$ and $\bar{\delta}\kappa_2$ given by (4.22g), (4.22b) and (4.22h) into equation (4.15c), one obtains the condition

$$\Psi_2\kappa_2 = 0.$$

One can summarise the discussion of this subsection as follows:

Lemma 21. *Assume that equations* (4.22b), (4.22d) *and* (4.22e) *hold along* $\mathcal{H}_2$ *with initial data for* κ_0 *and* κ_2 *on* $\mathcal{Z}$ *satisfying conditions* (4.20a) *and* (4.20b)*, respectively, and that, in addition,*

$$\eth^2\kappa_1 + \kappa_2\delta\sigma + \frac{3}{2}\sigma\delta\kappa_2 + \bar{\alpha}\sigma\kappa_2 = 0, \quad \Psi_2\kappa_2 = 0, \quad \Psi_1\kappa_2 = 0,$$
$$\Psi_0\kappa_2 + 2\Psi_1\kappa_1 - 3\Psi_2\kappa_0 = 0, \quad \textit{on} \quad \mathcal{H}_2.$$

Then, one has that

$$H_{A'ABC} = 0 \quad on \quad \mathcal{H}_2.$$

Remark 34. One can show that the curvature conditions in Lemmas 20 and 21 are in fact components of the equation

$$\Psi_{(ABC}{}^{F}\kappa_{D)F} = 0.$$

The other components of this equation are trivially satisfied. As this is a basis independent expression, the curvature conditions are satisfied in all spin bases, not just the parallely propagated one. One can check this by considering Lorentz transformations and null rotations about l^a and n^a, and showing that these conditions are preserved. The equation above can be shown to be an integrability condition for the Killing spinor equation, so it is unsurprising to find components of it arising naturally from the analysis.

4.3.3.5 The condition $S_{AA'BB'} = 0$ at $\mathcal{Z}$

Using the properties of $\mathcal{Z}$, as given explicitly in Table 4-A, together with the conditions (4.19a)-(4.19h) implied by the equation $H_{A'ABC} = 0$ on $\mathcal{Z}$, equations (4.17a)-(4.17d) reads as

$$\xi_{11'} = -\frac{3}{2}(\eth\kappa_2 + \tau\kappa_2)\,, \tag{4.23a}$$

$$\xi_{10'} = -3\bar{\eth}\kappa_1\,, \tag{4.23b}$$

$$\xi_{01'} = 3\eth\kappa_1\,, \tag{4.23c}$$

$$\xi_{00'} = \frac{3}{2}(\eth\kappa_0 - \bar{\tau}\kappa_0)\,, \tag{4.23d}$$

while on $\mathscr{Z}$ equations (4.18a)-(4.18j) reduce to

$$D\xi_{00'} = 0, \tag{4.24a}$$

$$\Delta\xi_{11'} = 0, \tag{4.24b}$$

$$D\xi_{11'} + \Delta\xi_{00'} - \tau\xi_{10'} - \overline{\tau}\xi_{01'} = 0, \tag{4.24c}$$

$$\Delta\xi_{01'} - \delta\xi_{11'} - 2\tau\xi_{11'} = 0, \tag{4.24d}$$

$$\delta\xi_{01'} + (\overline{\alpha} - \beta)\xi_{01'} = 0, \tag{4.24e}$$

$$D\xi_{01'} - \delta\xi_{00'} + \tau\xi_{00'} = 0, \tag{4.24f}$$

$$\Delta\xi_{10'} - \overline{\delta}\xi_{11'} - 2\overline{\tau}\xi_{11'} = 0, \tag{4.24g}$$

$$\overline{\delta}\xi_{10'} + (\alpha - \overline{\beta})\xi_{10'} = 0, \tag{4.24h}$$

$$\delta\xi_{10'} + \overline{\delta}\xi_{01'} - (\overline{\alpha} - \beta)\xi_{10'} - (\alpha - \overline{\beta})\xi_{01'} = 0, \tag{4.24i}$$

$$D\xi_{10'} - \overline{\delta}\xi_{00'} + \overline{\tau}\xi_{00'} = 0. \tag{4.24j}$$

Equations (4.24e), (4.24h) and (4.24i) can be read as intrinsic equations for $\xi_{01'}$ and $\xi_{10'}$. Expressing these in terms of the $\eth$ and $\overline{\eth}$ operators, observing that the spin-weights of $\xi_{01'}$ and $\xi_{10'}$ are respectively -1 and 1, one has that

$$\eth\xi_{01'} = 0, \tag{4.25a}$$

$$\overline{\eth}\xi_{10'} = 0, \tag{4.25b}$$

$$\eth\xi_{10'} + \overline{\eth}\xi_{01'} = 0. \tag{4.25c}$$

Substituting conditions (4.23b)-(4.23c) into conditions (4.25a)-(4.25b) above yield the simple conditions

$$\eth^2\kappa_1 = 0, \qquad \overline{\eth}^2\kappa_1 = 0.$$

Remark 35. The above expressions indicate that the component κ_1 has a very specific multipolar structure. Note, however, that the $\eth$ and $\overline{\eth}$ above are not the ones corresponding to $\mathbb{S}^2$ but of a 2-manifold diffeomorphic to it.

Remark 36. Equations (4.25a)-(4.25c) are just the components of the 2-dimensional Killing vector equation on the bifurcation surface $\mathcal{Z}$. In Section 5.2 of [39], it is shown that if $\mathcal{Z}$ is diffeomorphic to S^2 then such a Killing vector must correspond geometrically to an axial rotation, and so the surface $\mathcal{Z}$ possesses an axial isometry. This fact will be useful later when trying to find an explicit expression for the Killing spinor component κ_1 on $\mathcal{Z}$.

Crucially, one can also show that equations (4.24a)-(4.24d), (4.24f)-(4.24g) and (4.24j) are implied by equations (4.19a)-(4.19h), the Ricci equations, and the conditions of Lemmas 2 and 3 (which must be satisfied on $\mathcal{Z} = \mathcal{H}_1 \cap \mathcal{H}_2$). Summarising:

Lemma 22. *Assume that equations* (4.19a)-(4.19h) *hold on* $\mathcal{Z}$ *and that, in addition,*

$$\eth^2 \kappa_1 = 0, \qquad \bar{\eth}^2 \kappa_1 = 0, \quad on \quad \mathcal{Z}.$$

Then one has that

$$S_{AA'BB'} = 0 \quad on \quad \mathcal{Z}.$$

4.3.3.6 The Killing vector equation on $\mathcal{H}_1$

On $\mathcal{H}_1$, equations (4.18a)-(4.18j) reduce to:

$$D\xi_{00'} = 0, \tag{4.26a}$$

$$\Delta\xi_{11'} + (\gamma + \overline{\gamma})\xi_{11'} + \nu\xi_{01'} + \overline{\nu}\xi_{10'} = 0, \tag{4.26b}$$

$$D\xi_{11'} + \Delta\xi_{00'} - \tau\xi_{10'} - \overline{\tau}\xi_{01'} - (\gamma + \overline{\gamma})\xi_{00'} = 0, \tag{4.26c}$$

$$\Delta\xi_{01'} - \delta\xi_{11'} - (\gamma - \overline{\gamma} + \mu)\xi_{01'} + \overline{\nu}\xi_{00'} - 2\tau\xi_{11'} = 0, \tag{4.26d}$$

$$\delta\xi_{01'} + (\overline{\alpha} - \beta)\xi_{01'} = 0, \tag{4.26e}$$

$$D\xi_{01'} - \delta\xi_{00'} + \tau\xi_{00'} = 0, \tag{4.26f}$$

$$\Delta\xi_{10'} - \overline{\delta}\xi_{11'} - (\overline{\gamma} - \gamma + \overline{\mu})\xi_{10'} + \nu\xi_{00'} - 2\overline{\tau}\xi_{11'} = 0, \tag{4.26g}$$

$$\overline{\delta}\xi_{10'} + (\alpha - \overline{\beta})\xi_{10'} = 0, \tag{4.26h}$$

$$\delta\xi_{10'} + \overline{\delta}\xi_{01'} + (\mu + \overline{\mu})\xi_{00'} - (\overline{\alpha} - \beta)\xi_{10'} - (\alpha - \overline{\beta})\xi_{01'} = 0, \tag{4.26i}$$

$$D\xi_{10'} - \overline{\delta}\xi_{00'} + \overline{\tau}\xi_{00'} = 0. \tag{4.26j}$$

Substituting the components $\xi_{00'}, \xi_{01'}, \xi_{10'}$ and $\xi_{11'}$, as given by (4.17a)-(4.17d), into these relations (being careful not to discard the Δ derivatives of quantities which vanish on $\mathcal{H}_1$), and using equations (4.21a)-(4.21h) and the Ricci equations, one finds that (4.26a)-(4.26j) reduce to:

$$\eth^2\kappa_1 = \kappa_0(\delta\mu + \mu\tau), \tag{4.27a}$$

$$\overline{\eth}^2\kappa_1 = 0, \tag{4.27b}$$

$$\Psi_2\kappa_0 = 0, \tag{4.27c}$$

$$\Psi_3\kappa_0 = 0, \tag{4.27d}$$

$$\Psi_4\kappa_0 + 2\Psi_3\kappa_1 - 3\Psi_2\kappa_2 = 0. \tag{4.27e}$$

Remark 37. The conditions (4.27b)-(4.27e) are exactly the conditions of Lemma 2. The additional condition (4.27a) must be satisfied on all of $\mathcal{H}_1$. Note, however, that after some manipulations the condition

$$D\left(\eth^2\kappa_1 - \kappa_0(\delta\mu + \mu\tau)\right) = -2\delta(\Psi_2\kappa_0) + 4\beta\Psi_2\kappa_0 = 0$$

can be shown to hold, where in the last step (4.27c) was used. Accordingly, it suffices to guarantee (4.27a) on $\mathcal{Z}$ as then it is satisfied on the whole of $\mathcal{H}_1$ if condition (4.27c) holds on $\mathcal{H}_1$. Furthermore, on $\mathcal{Z}$ the spin coefficient μ vanishes, so (4.27a) reduces to $\eth^2\kappa_1 = 0$ on $\mathcal{Z}$. Note that this is one of the conditions appearing in Lemma 22.

This can be summarised in the following lemma:

Lemma 23. *Assume that equations* (4.21a)-(4.21h) *hold on* $\mathcal{H}_1$*, and the conditions of Lemmas 20 and 22 are satisfied. Then one has that*

$$S_{AA'BB'} = 0 \quad \textit{on} \quad \mathcal{H}_1.$$

4.3.3.7 The Killing vector equation on $\mathcal{H}_2$

On $\mathcal{H}_2$, equations (4.18a)-(4.18j) reduce to:

$$D\xi_{00'} = 0, \tag{4.28a}$$

$$\Delta\xi_{11'} = 0, \tag{4.28b}$$

$$D\xi_{11'} + \Delta\xi_{00'} - \tau\xi_{10'} - \overline{\tau}\xi_{01'} = 0, \tag{4.28c}$$

$$\Delta\xi_{01'} - \delta\xi_{11'} - 2\tau\xi_{11'} = 0, \tag{4.28d}$$

$$\delta\xi_{01'} + (\overline{\alpha} - \beta)\xi_{01'} - \sigma\xi_{11'} = 0, \tag{4.28e}$$

$$D\xi_{01'} - \delta\xi_{00'} + \tau\xi_{00'} + \sigma\xi_{10'} + \rho\xi_{01'} = 0, \tag{4.28f}$$

$$\Delta\xi_{10'} - \overline{\delta}\xi_{11'} - 2\overline{\tau}\xi_{11'} = 0, \tag{4.28g}$$

$$\overline{\delta}\xi_{10'} + (\alpha - \overline{\beta})\xi_{10'} - \overline{\sigma}\xi_{11'} = 0, \tag{4.28h}$$

$$\delta\xi_{10'} + \overline{\delta}\xi_{01'} - (\overline{\alpha} - \beta)\xi_{10'} - (\alpha - \overline{\beta})\xi_{01'} - 2\rho\xi_{11'} = 0, \tag{4.28i}$$

$$D\xi_{10'} - \overline{\delta}\xi_{00'} + \overline{\tau}\xi_{00'} + \overline{\sigma}\xi_{01'} + \rho\xi_{10'} = 0. \tag{4.28j}$$

Substituting the components $\xi_{00'}, \xi_{01'}, \xi_{10'}$ and $\xi_{11'}$, as given by (4.17a)-(4.17d), into these relations (being careful not to discard the D derivatives of quantities which vanish on $\mathcal{H}_2$), and using equations (4.22a)-(4.22h) and the Ricci equations, one finds that (4.28a)-(4.28j) reduce to:

$$\eth^2\kappa_1 + \kappa_2\delta\sigma + \frac{3}{2}\sigma\delta\kappa_2 + \overline{\alpha}\sigma\kappa_2 = 0, \tag{4.29a}$$

$$\overline{\eth}^2\kappa_1 + \kappa_2\overline{\delta}\sigma - \frac{1}{2}\overline{\sigma}\delta\kappa_2 - 3\overline{\alpha}\kappa_2\overline{\sigma} - \overline{\Psi}_1\kappa_2 = 0, \tag{4.29b}$$

$$\Psi_1\kappa_2 = 0, \tag{4.29c}$$

$$\Psi_2\kappa_2 = 0, \tag{4.29d}$$

$$\Psi_0\kappa_2 + 2\Psi_1\kappa_1 - 3\Psi_2\kappa_0 = 0. \tag{4.29e}$$

The conditions (4.29a) and (4.29c)-(4.29e) are exactly the conditions of Lemma 3. The additional condition (4.29b) must be satisfied on all of $\mathcal{H}_2$. This can be summarised in the following lemma:

Lemma 24. *Assume that equations* (4.22a)-(4.22h) *hold on* $\mathcal{H}_2$*, the conditions of Lemma 21 are satisfied, and that in addition,*

$$\overline{\eth}^2\kappa_1 + \kappa_2\overline{\delta}\sigma - \frac{1}{2}\overline{\sigma}\delta\kappa_2 - 3\overline{\alpha}\kappa_2\overline{\sigma} - \overline{\Psi}_1\kappa_2 = 0 \quad \text{on} \quad \mathcal{H}_2.$$

Then one has that

$$S_{AA'BB'} = 0 \quad \text{on} \quad \mathcal{H}_2.$$

4.4 Solving some of the constraints on $\mathscr{Z}$

Now that some necessary conditions for the vanishing of the zero quantities $H_{A'ABC}$ and $S_{AA'BB'}$ on the bifurcated horizon have been found, one can ask whether it is possible to deduce anything about the spinor κ_{AB} from them.

4.4.1 Determining κ_2 on $\mathcal{Z}$

Consider now the restrictions to κ_2 on $\mathcal{Z}$. To satisfy the condition $\Psi_2\kappa_2 = 0$ on $\mathcal{H}_2$, applied in Lemma 3, $\Psi_2\kappa_2$ must necessarily vanish on $\mathcal{Z} \subset \mathcal{H}_2$. Consistent with this condition the following sub-cases can be seen to arise:

i. Assume first that κ_2 is nowhere vanishing on $\mathcal{Z}$. In this case Ψ_2 must vanish throughout $\mathcal{Z}$. Note also that in virtue of Table 4-A all the other Weyl spinor components vanish on $\mathcal{Z}$, and thereby

$$\Psi_{ABCD}|_{\mathcal{Z}} = 0\,.$$

As shown in Table 4-A, Ψ_0 and Ψ_1 vanish on $\mathcal{H}_1$, and Ψ_3 and Ψ_4 vanish on $\mathcal{H}_2$, respectively. Further, observe that the Bianchi identities imply the following relations on $\mathcal{H}_1$:

$$D\Psi_2 = 0,$$
$$D\Psi_3 = \bar{\delta}\Psi_2,$$
$$D\Psi_4 = 2\alpha\Psi_3 + \bar{\delta}\Psi_3.$$

As Ψ_2 vanishes on $\mathcal{Z}$ and D is the directional derivative along the geodesics generating $\mathcal{H}_1$, the first of these equations imply that $\Psi_2 = 0$ on $\mathcal{H}_1$. By the same argument, because the right hand side of the second of the above relations has shown to vanish on $\mathcal{H}_1$, it follows that $\Psi_3 = 0$ on $\mathcal{H}_1$. In turn, this also implies that $\Psi_4 = 0$ on $\mathcal{H}_1$ as a consequence of the last relation. Therefore, along with the vanishing of Ψ_0 and Ψ_1 on $\mathcal{H}_1$ all the Weyl spinor components vanish there – that is one has

$$\Psi_{ABCD}|_{\mathcal{H}_1} = 0\,.$$

Similarly, the Bianchi identities imply the following relations on $\mathcal{H}_2$:

$$\Delta\Psi_0 = \delta\Psi_1 - (4\tau + 2\beta)\Psi_1 + 3\sigma\Psi_2,$$
$$\Delta\Psi_1 = \delta\Psi_2 - 3\tau\Psi_2,$$
$$\Delta\Psi_2 = 0.$$

As Ψ_2 vanishes on $\mathcal{Z}$, and Δ is the directional derivative along the geodesics generating $\mathcal{H}_2$, the third of these equations imply that $\Psi_2 = 0$ on $\mathcal{H}_2$. Thus, the right hand side of the second of the above relations vanishes on $\mathcal{H}_2$, and by the same argument it follows that $\Psi_1 = 0$ on $\mathcal{H}_2$. The first relation then implies that $\Psi_0 = 0$ on $\mathcal{H}_2$. Therefore, along with the vanishing of Ψ_3 and Ψ_4 on $\mathcal{H}_2$ all the Weyl spinor components vanish there. Thus, one has that

$$\Psi_{ABCD}|_{\mathcal{H}_2} = 0\,.$$

Summarising, the non-vanishing of κ_2 on $\mathcal{Z}$ implies that all the Weyl spinor components vanish identically on the union of $\mathcal{Z}$, $\mathcal{H}_1$ and $\mathcal{H}_2$. This, in the vacuum case, implies that all components of the Riemann curvature tensor vanish on $\mathcal{H}_1 \cup \mathcal{H}_2$. It follows then that the neighbourhood $\mathcal{O}$ in $D(\mathcal{H}_1 \cup \mathcal{H}_2)$ spacetime obtained from Theorem 11 is diffeomorphic to a portion of the Minkowski spacetime and the pair intersecting null hypersurfaces has to contains a bifurcate Killing horizon corresponding to a choice of a boost Killing vector field.

ii.* κ_2 *vanishes somewhere on* $\mathcal{Z}$*: It follows from the discussion in the previous subsection that, unless the spacetime is Minkowski, κ_2 must vanish somewhere on $\mathcal{Z}$. It turns out that that if this is the case, then κ_2 must vanish on some open subset of $\mathcal{Z}$. To see this assume, on contrary, that κ_2 vanishes only at isolated points. Choose one of them, say $z \in \mathcal{Z}$ with $\kappa_2(z) = 0$ and a Cauchy sequence $\{z_n\}$ converging to z in the metric topology of $\mathcal{Z} \approx \mathbb{S}^2$. Since κ_2 is assumed to vanish only at isolated points to ensure $\Psi_2\kappa_2 = 0$ on $\mathcal{Z}$, the sequence $\{\Psi_2(z_n)\}$ must be the identically zero sequence in $\mathbb{R}$ which by continuity implies that $\Psi_2(z) = 0$. Applying this argument to any of the isolated points where κ_2 vanishes gives that Ψ_2 must be identically zero on $\mathcal{Z}$. As we saw before, this would imply that the spacetime is Minkowski – in conflict with our assumption that the geometry is not flat. This, in turn, verifies that whenever κ_2 vanishes somewhere on $\mathcal{Z}$ it has to vanish on some (non-empty) open subset of $\mathcal{Z}$.

iii. κ_2 ***vanishes on a (non-empty) open subset of*** $\mathcal{Z}$***.*** It follows from (4.5) that (4.20b), which is valid on $\mathcal{Z}$, can be written:

$$\overline{P}P^{-1}\,\partial_{\overline{z}}\,(P\kappa_2) = -\overline{\tau}\,P^{-1}\,(P\kappa_2) \tag{4.30}$$

implying, in turn, that κ_2 has to be of the form

$$\kappa_2 = \frac{1}{P}\cdot\exp\left[-\int\overline{\tau}\,\overline{P}^{-1}\,d\overline{z} + \varphi(z)\right],$$

where $\varphi(z)$ is an arbitrary holomorphic function. This, however, in virtue of the non-vanishing of P, implies that κ_2 cannot vanish on an open subset of $\mathcal{Z}$ unless it is identically zero on $\mathcal{Z}$, i.e.

$$\kappa_2|_{\mathcal{Z}} = 0$$

as intended. Note also that the condition (4.22b) requires that κ_2 must vanish along the generators of $\mathcal{H}_2$, and so we must also have

$$\kappa_2|_{\mathcal{H}_2} = 0\,.$$

Summarising, the discussion in this section has shown the following:

Lemma 25. *Assume that*

$$\Psi_2\,\kappa_2 = 0 \qquad on \quad \mathcal{Z}.$$

Then, if κ_2 *is nowhere vanishing on* $\mathcal{Z}$*, then the solution to the characteristic initial value problem must be diffeomorphic to the Minkowski spacetime in the domain of dependence of* $D(\mathcal{H}_1\cup\mathcal{H}_2)$*. Otherwise,* $\kappa_2 = 0$ *holds on the whole of* $\mathcal{Z}$*, and then it is also identically* *zero on $\mathcal{H}_2$.*

4.4.2 Determining κ_0 on $\mathcal{Z}$

The analysis of the previous section can be adapted to the component κ_0 by noting that the vanishing of $\Psi_2\kappa_0$ on $\mathcal{H}_1$, one of the conditions in Lemma 20, can be deduced from the vanishing of $\Psi_2\kappa_0$ on $\mathcal{Z}$. Indeed, it can be shown that unless the spacetime is Minkowski, κ_0 must vanish on a non-empty subset of $\mathcal{Z}$. The only difference in the analysis lies on the analogue of equation (4.30). It follows from (4.5) that (4.20a), which is valid on $\mathcal{Z}$, can be written as

$$P\overline{P}^{-1}\,\partial_z\left(\overline{P}\kappa_0\right) = \tau\,\overline{P}^{-1}\left(\overline{P}\kappa_0\right)$$

which implies, in turn, that κ_0 has to be of the form

$$\kappa_0 = \frac{1}{\overline{P}}\cdot\exp\left[\int \tau\, P^{-1}\, dz + \varsigma(\overline{z})\right],$$

where $\varsigma(\overline{z})$ is an arbitrary antiholomorphic function on $\mathcal{Z}$. From here, by an argument analogous to that used for κ_2 one concludes that

$$\kappa_0|_{\mathcal{Z}} = 0$$

and, moreover, as a consequence of equation (4.21a), that

$$\kappa_0|_{\mathcal{H}_1} = 0.$$

Summarising:

Lemma 26. *Assume that*

$$\Psi_2\,\kappa_0 = 0 \qquad on \quad \mathcal{Z}.$$

If κ_0 is nowhere vanishing on $\mathcal{Z}$, then the solution to the characteristic initial value *problem must be diffeomorphic to the Minkowski spacetime in the domain of dependence of $D(\mathcal{H}_1 \cup \mathcal{H}_2)$. Otherwise, $\kappa_0 = 0$ holds on the whole of $\mathcal{Z}$, and it is also identically zero on $\mathcal{H}_1$.*

4.4.3 Eliminating redundant conditions on $\mathcal{H}_1$ and $\mathcal{H}_2$

The first condition in Lemma 20 was

$$\bar{\eth}^2 \kappa_1 = 0 \quad \text{on} \quad \mathcal{H}_1.$$

In theory, one would have to solve this constraint on the whole of $\mathcal{H}_1$. However, one can show that on $\mathcal{H}_1$

$$D(\bar{\eth}^2 \kappa_1) = -\frac{1}{2}\overline{\delta\delta\delta}\kappa_0 + \frac{3}{2}\bar{\tau}\overline{\delta\delta}\kappa_0 + \bar{\delta}\kappa_0 \left(-\alpha^2 - 4\alpha\bar{\beta} - \bar{\beta}^2 + \frac{5}{2}\bar{\delta}\alpha + \frac{1}{2}\overline{\delta\beta}\right)$$
$$+ \kappa_0 \left(2\alpha\bar{\beta}\bar{\tau} - 2\alpha\bar{\delta}\alpha - 3\overline{\beta\delta}\alpha - \alpha\overline{\delta\beta} + \overline{\delta\delta}\alpha\right).$$

Note that as κ_0 vanishes on $\mathcal{H}_1$ (under the assumption that the spacetime is not diffeomorphic to Minkowski), the right hand side of this equation also vanishes on $\mathcal{H}_1$. Therefore, if κ_1 satisfies $\bar{\eth}^2 \kappa_1 = 0$ on $\mathcal{Z}$, then it also satisfies the same condition on the whole of $\mathcal{H}_1$. This was a condition on $\mathcal{Z}$ already present from the requirement that $S_{AA'BB'}|_{\mathcal{Z}} = 0$. Summarising:

Lemma 27. *If $\kappa_0|_{\mathcal{H}_1} = 0$ and $\bar{\eth}^2 \kappa_1|_{\mathcal{Z}} = 0$, then the condition $\bar{\eth}^2 \kappa_1|_{\mathcal{H}_1} = 0$ from Lemma 20 is automatically satisfied.*

A similar procedure can be performed on $\mathcal{H}_2$. The first condition from Lemma 21 was

$$\eth^2 \kappa_1 + \kappa_2 \delta\sigma + \frac{3}{2}\sigma\delta\kappa_2 + \bar{\alpha}\sigma\kappa_2 = 0$$

which must be satisfied on $\mathcal{H}_2$. It has already been shown that necessarily $\kappa_2|_{\mathcal{H}_2} = 0$ if the spacetime is not diffeomorphic to the Minkowski spacetime. Therefore, the aforementioned condition reduces to

$$\eth^2 \kappa_1 = 0 \quad \text{on} \quad \mathcal{H}_2.$$

Now, one can show that on $\mathcal{H}_2$,

$$\begin{aligned}\Delta\left(\eth^2\kappa_1\right) = & -\frac{1}{2}\delta\delta\delta\kappa_2 - \frac{3}{2}\tau\delta\delta\kappa_2 + \delta\kappa_2\left(-\overline{\alpha}^2 - \overline{\alpha}\beta + 2\beta^2 - 2\delta\overline{\alpha} - 4\delta\beta\right) \\ & + \kappa_2\left(-\overline{\alpha}\delta\overline{\alpha} + \beta\delta\overline{\alpha} - 2\overline{\alpha}\delta\beta + 2\beta\delta\beta - \delta\delta\overline{\alpha} - 2\delta\delta\beta\right).\end{aligned}$$

The requirement that κ_2 vanishes on $\mathcal{H}_2$ means that the right hand side of this equation also vanishes on $\mathcal{H}_2$. Therefore, if κ_1 satisfies $\eth^2\kappa_1 = 0$ on $\mathcal{Z}$, then it also satisfies the same condition on the whole of $\mathcal{H}_2$. This was a condition on $\mathcal{Z}$ already present from the requirement that $S_{AA'BB'}|_{\mathcal{Z}} = 0$.

Finally, the condition from Lemma 24 says that

$$\overline{\eth}^2\kappa_1 + \kappa_2\overline{\delta}\sigma - \frac{1}{2}\overline{\sigma}\delta\kappa_2 - 3\overline{\alpha}\kappa_2\overline{\sigma} - \overline{\Psi}_1\kappa_2 = 0 \quad \text{on} \quad \mathcal{H}_2$$

which reduces to $\overline{\eth}^2\kappa_1 = 0$ due to the fact that $\kappa_2|_{\mathcal{H}_2} = 0$ when the spacetime is not diffeomorphic to the Minkowski solution. One can show that on $\mathcal{H}_2$

$$\begin{aligned}\Delta\left(\overline{\eth}^2\kappa_1\right) = \delta\kappa_2\left(\frac{1}{2}\overline{\delta}\overline{\tau} - \overline{\beta}\overline{\tau}\right) + \kappa_2 & \left(-6\alpha^2\beta - 6\alpha\beta\overline{\beta} - 2\alpha\delta\alpha + \overline{\alpha}\overline{\delta}\alpha\right. \\ & \left.+5\beta\overline{\delta}\alpha + 2\alpha\overline{\delta}\overline{\alpha} + \overline{\beta\delta}\,\overline{\alpha} + 7\alpha\overline{\delta}\beta + 2\overline{\beta\delta}\beta + \overline{\delta}\delta\alpha - \overline{\delta\delta}\alpha - 2\overline{\delta\delta}\beta\right)\end{aligned}$$

The requirement that κ_2 vanishes on $\mathcal{H}_2$ means that the right hand side of this equation also vanishes on $\mathcal{H}_2$. So if κ_1 satisfies $\overline{\eth}^2\kappa_1 = 0$ on $\mathcal{Z}$, then it also satisfies the same condition on the whole of $\mathcal{H}_2$. This was a condition already present from the requirement that $S_{AA'BB'}|_{\mathcal{Z}} = 0$. Summarising, we have

Lemma 28. *If* $\kappa_2|_{\mathcal{H}_2} = 0$ *and* $\bar{\eth}^2\kappa_1|_{\mathcal{Z}} = \eth^2\kappa_1|_{\mathcal{Z}} = 0$*, then the conditions*

$$\left(\bar{\eth}^2\kappa_1 + \kappa_2\bar{\delta}\sigma - \frac{1}{2}\bar{\sigma}\delta\kappa_2 - 3\bar{\alpha}\kappa_2\bar{\sigma} - \bar{\Psi}_1\kappa_2\right)|_{\mathcal{H}_2} = 0,$$
$$\left(\eth^2\kappa_1 + \kappa_2\delta\sigma + \frac{3}{2}\sigma\delta\kappa_2 + \bar{\alpha}\sigma\kappa_2\right)|_{\mathcal{H}_2} = 0,$$

applied in Lemmas 21 and 24, are automatically satisfied.

The only remaining condition on $\mathcal{H}_1$ to be considered is from Lemma 20, which reduces to

$$(2\Psi_3\kappa_1 - 3\Psi_2\kappa_2)\,|_{\mathcal{H}_1} = 0 \tag{4.31}$$

due to the requirement that $\kappa_0|_{\mathcal{H}_1} = 0$. One can also use this requirement to show that

$$D^2\,(2\Psi_3\kappa_1 - 3\Psi_2\kappa_2)\,|_{\mathcal{H}_1} = 0.$$

More precisely, the right hand side of this expression can be shown to be homogeneous in κ_0 and derivatives of κ_0 intrinsic to $\mathcal{H}_1$. This can be thought of as a second order ordinary differential equation along the geodesic generators of $\mathcal{H}_1$. Therefore, equation (4.31) is equivalent to the vanishing of $(2\Psi_3\kappa_1 - 3\Psi_2\kappa_2)$ and its first D-derivative on $\mathcal{Z}$. This combination vanishes on $\mathcal{Z}$ if $\kappa_2|_{\mathcal{H}_2} = 0$ as it follows from Table 4-A that $\Psi_3|_{\mathcal{Z}} = 0$. The vanishing of the first derivative on $\mathcal{Z}$ can be shown to be equivalent to

$$\bar{\delta}\left(\kappa_1^3\Psi_2\right)|_{\mathcal{Z}} = 0\,. \tag{4.32}$$

In a similar way, the only remaining condition on $\mathcal{H}_2$ to be analysed is from Lemma 21. This condition reduces to

$$(2\Psi_1\kappa_1 - 3\Psi_2\kappa_0)\,|_{\mathcal{H}_2} = 0 \tag{4.33}$$

due to the requirement that $\kappa_2|_{\mathcal{H}_2} = 0$. One can also use this requirement to show that

$$\Delta^2\,(2\Psi_1\kappa_1 - 3\Psi_2\kappa_0)\,|_{\mathcal{H}_2} = 0.$$

This time, the right hand side of this can be shown to be homogeneous in κ_2 and derivatives of κ_2 intrinsic to $\mathcal{H}_2$. This can be thought of as a second order ordinary differential equation along the geodesic generators of $\mathcal{H}_2$. Therefore, equation (4.33) is equivalent to the vanishing of $(2\Psi_1\kappa_1 - 3\Psi_2\kappa_0)$ and its first Δ derivative on $\mathcal{Z}$. This combination vanishes on $\mathcal{Z}$ if $\kappa_0|_{\mathcal{H}_1} = 0$ as, following Table 4-A, one has that $\Psi_1|_{\mathcal{Z}} = 0$. The vanishing of the first derivative on $\mathcal{Z}$ can be shown to be equivalent to

$$\delta\left(\kappa_1^3\Psi_2\right)|_{\mathcal{Z}} = 0\,. \tag{4.34}$$

Defining the combination in the brackets by

$$\mathfrak{M} \equiv \kappa_1^3\Psi_2 \tag{4.35}$$

it follows from equations (4.32) and (4.34) that $\mathfrak{M} \in \mathbb{C}$ is constant on $\mathcal{Z}$.

We can summarise the discussion of this section in the following:

Lemma 29. *Assume that $\kappa_0|_{\mathcal{H}_1} = \kappa_2|_{\mathcal{H}_2} = 0$. Then $\mathfrak{M} \equiv \kappa_1^3\Psi_2$ is constant on $\mathcal{Z}$ if and only if*

$$\begin{aligned}(2\Psi_3\kappa_1 - 3\Psi_2\kappa_2)\,|_{\mathcal{H}_1} &= 0,\\ (2\Psi_1\kappa_1 - 3\Psi_2\kappa_0)\,|_{\mathcal{H}_2} &= 0.\end{aligned}$$

Remark 38. Note that

$$\begin{aligned}D\mathfrak{M}|_{\mathcal{H}_1} &= \frac{3}{2}\Psi_2\kappa_1^2\left(-\bar{\delta}\kappa_0 + 2\alpha\kappa_0\right)|_{\mathcal{H}_1}\\ &= 0\end{aligned}$$

where equation $D\Psi_2|_{\mathcal{H}_1} = 0$ from Table 4-A, equation (4.21f) and the requirement that

$\kappa_0|_{\mathcal{H}_1} = 0$ have been used. Similarly,

$$\begin{aligned}\Delta\mathfrak{M}|_{\mathcal{H}_2} &= \frac{3}{2}\Psi_2\kappa_1^2\left(-\delta\kappa_2 - 2(\beta+\tau)\kappa_2\right)|_{\mathcal{H}_2} \\ &= 0\end{aligned}$$

where equation $\Delta\Psi_2|_{\mathcal{H}_2} = 0$ from Table 4-A, equation (4.22e) and the requirement that $\kappa_2|_{\mathcal{H}_2} = 0$ have been used. Thus, $\mathfrak{M}$ is constant not merely on $\mathcal{Z}$ but on the whole of $\mathcal{H}_1 \cup \mathcal{H}_2$. Since the Newman-Penrose reduced system coupled to the wave equation for κ_{AB}, equation (3.3b), is a well-posed hyperbolic system we also have that $\mathfrak{M}$ is, in fact, constant throughout the domain of dependence of $\mathcal{H}_1 \cup \mathcal{H}_2$.

4.4.4 Summary

Collecting all the previous lemmas and propositions together one obtains the following:

Theorem 12. *Assume that the spacetime – obtained from the characteristic initial value problem in a neighbourhood $\mathcal{O}$ of $\mathcal{Z}$ in $D(\mathcal{H}_1 \cup \mathcal{H}_2)$ – is not diffeomorphic to the Minkowski spacetime. Then the following two statements are equivalent:*

(*i*) *Given a spin basis $\{o^A, \iota^A\}$ on $\mathcal{Z}$, there exists a constant $\mathfrak{M} \in \mathbb{C}$ such that*

$$\kappa_0 = 0\,, \quad \eth^2\kappa_1 = \bar{\eth}^2\kappa_1 = 0\,, \quad \kappa_2 = 0 \quad \textit{and} \quad \kappa_1^3\Psi_2 = \mathfrak{M} \quad \textit{on} \quad \mathcal{Z}\,.$$

(*ii*) $H_{A'ABC} = 0, \quad S_{AA'BB'} = 0$ *everywhere on* $\mathcal{H}_1 \cup \mathcal{H}_2$.

Recall that the vanishing of the spinors $H_{A'ABC}$ and $S_{AA'BB'}$ on $\mathcal{H}_1 \cup \mathcal{H}_2$ are precisely the conditions of Proposition 7, which along with the assumptions of Theorem 11 imply that the Killing spinor candidate κ_{AB} is in fact a Killing spinor in the causal future (or past) of $\mathcal{Z}$. Summarising these observations gives:

Theorem 13. *Let $(\mathcal{M}, \boldsymbol{g})$ be a vacuum spacetime satisfying the conditions of Theorem 11. Given a spin basis $\{o^A, \iota^A\}$ on $\mathcal{Z}$, assume that there exists a constant $\mathfrak{M} \in \mathbb{C}$ such* *that the following relations*

$$\kappa_0 = 0, \quad \eth^2\kappa_1 = \bar{\eth}^2\kappa_1 = 0\,, \quad \kappa_2 = 0 \quad \textit{and} \quad \kappa_1^3\Psi_2 = \mathfrak{M} \tag{4.36a}$$

hold on $\mathcal{Z}$. Then there exists a neighbourhood $\mathcal{O}$ of $\mathcal{Z}$, in $D(\mathcal{H}_1 \cup \mathcal{H}_2)$, such that the corresponding unique solution κ_{AB} to equation (3.3b) *is a Killing spinor on $\mathcal{O} \cap D(\mathcal{H}_1 \cup \mathcal{H}_2)$.*

Proof. First, note that $H_{A'ABC}$ and $S_{AA'BB'}$ vanish on $\mathcal{H}_1 \cup \mathcal{H}_2$ as a result of Theorem 12. Data for $\kappa_0, \kappa_1, \kappa_2$ on $\mathcal{H}_1$ and $\mathcal{H}_2$ are determined by their values on $\mathcal{Z}$ by (4.21a), (4.21f), (4.21g), (4.22b), (4.22d) and (4.22e), so Proposition 6 says that there exists a unique solution to (3.3b) on $\mathcal{O} \cap D(\mathcal{H}_1 \cup \mathcal{H}_2)$. Proposition 7 then says that this field κ_{AB} satisfies $H_{A'ABC} = 0$ on $\mathcal{O} \cap D(\mathcal{H}_1 \cup \mathcal{H}_2)$, so is in fact a Killing spinor there. □

Remark 39. Condition (4.36a) is a strong restriction on the form of the Weyl spinor component Ψ_2. As already discussed in Remark 27 the Weyl spinor component Ψ_2 is not a basic piece of initial data. In view of (4.11) condition (4.36a), ultimately leads to restrictions on τ and $\zeta^{\mathcal{A}}$.

4.5 Enforcing the Hermiticity of the Killing vector

In Proposition 2, the assumption that the spinor $\xi_{AA'}$ constructed from the Killing spinor κ_{AB} is Hermitian is needed in order to show that the spacetime is isometric to the Kerr solution. Recall that, using equations (4.17a)-(4.17d), the components of $\xi_{AA'}$ can be expressed in terms of derivatives of the Killing spinor components κ_0, κ_1 and κ_2. Accordingly, the Hermiticity condition leads to further restrictions on the components κ_0, κ_1 and κ_2. A consequence of the following proposition is that it suffices to impose restrictions only on the hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

Proposition 8. *Let κ_{AB} be a solution to equation* (3.3b). *Then the spinor field $\xi_{AA'}$*

satisfies the wave equation

$$\square\xi_{AA'} = -\Psi_A{}^{BCD}H_{A'BCD}$$

Proof. Follows by commuting derivatives, and using (3.3b). □

An immediate consequence of this result is that

$$\square\left(\xi_{AA'} - \bar{\xi}_{AA'}\right) = \bar{\Psi}_{A'}{}^{B'C'D'}\bar{H}_{AB'C'D'} - \Psi_A{}^{BCD}H_{A'BCD}.$$

Assuming that the conditions of Lemmas 20 and 21 are satisfied, $H_{A'ABC}$ vanishes in a neighbourhood $\mathcal{O}$ of $\mathcal{Z}$ in $D(\mathcal{H}_1 \cup \mathcal{H}_2)$. Therefore, if the quantity $\xi_{AA'} - \bar{\xi}_{AA'}$ vanishes on $\mathcal{H}_1 \cup \mathcal{H}_2$, there exists a neighbourhood $\mathcal{O}' \subset \mathcal{O}$ of $\mathcal{Z}$ in $D(\mathcal{H}_1 \cup \mathcal{H}_2)$ where $\xi_{AA'} - \bar{\xi}_{AA'}$ vanishes, and thereby the vector $\xi_{AA'}$ is Hermitian there.

4.5.1 Some immediate restrictions

The Hermiticity of the Killing vector $\xi_{AA'}$ is equivalent to the relations

$$\xi_{00'} = \bar{\xi}_{00'}\,, \quad \xi_{01'} = \bar{\xi}_{10'}\,, \quad \xi_{10'} = \bar{\xi}_{01'}\,, \quad \xi_{11'} = \bar{\xi}_{11'}\,. \tag{4.37}$$

These conditions will be imposed on $\mathcal{H}_1$ and $\mathcal{H}_2$ separately.

Conditions on $\mathcal{H}_1$. On $\mathcal{H}_1$, using the explicit expressions (4.17a)-(4.17d), the first condition in (4.37) is trivially satisfied, and the remaining conditions can be shown to be equivalent to

$$\delta(\kappa_1 + \bar{\kappa}_1) = 0, \tag{4.38a}$$

$$\bar{\delta}(\kappa_1 + \bar{\kappa}_1) = 0, \tag{4.38b}$$

$$\Delta\kappa_1 + \tau\kappa_2 \quad \text{real}, \tag{4.38c}$$

on $\mathcal{H}_1$. In fact, it is straightforward to show that on $\mathcal{H}_1$

$$D\delta(\kappa_1 + \overline{\kappa}_1) = D\overline{\delta}(\kappa_1 + \overline{\kappa}_1) = 0\,.$$

Thus, it suffices to impose conditions (4.38a)-(4.38b) only on $\mathcal{Z}$. In other words, the Hermiticity condition on $\mathcal{H}_1$ is equivalent to

$$\begin{aligned} &\mathrm{Re}(\kappa_1) \quad \text{constant on } \mathcal{Z}, \\ &\Delta\kappa_1 + \tau\kappa_2 \quad \text{real} \quad \text{on } \mathcal{H}_1. \end{aligned}$$

Conditions on $\mathcal{H}_2$. Secondly, on $\mathcal{H}_2$, the last condition in (4.37) is trivially satisfied and the remaining conditions are equivalent to

$$\delta(\kappa_1 + \overline{\kappa}_1) = 0, \tag{4.39a}$$

$$\overline{\delta}(\kappa_1 + \overline{\kappa}_1) = 0, \tag{4.39b}$$

$$D\kappa_1 \quad \text{real}, \tag{4.39c}$$

on $\mathcal{H}_2$. Again, it is straightforward to show that on $\mathcal{H}_2$

$$\Delta\delta(\kappa_1 + \overline{\kappa}_1) = \Delta\overline{\delta}(\kappa_1 + \overline{\kappa}_1) = 0.$$

Consequently, it suffices to impose conditions (4.39a)-(4.39b) on $\mathcal{Z}$.

Combining the discussion of the previous two paragraphs one concludes that the spinor field $\xi_{AA'}$ is Hermitian on $\mathcal{H}_1 \cup \mathcal{H}_2$ if and only if the following conditions hold:

$$\kappa_1 + \overline{\kappa}_1 = \text{const on } \mathcal{Z}, \tag{4.40a}$$

$$\Delta\kappa_1 + \tau\,\kappa_2 \quad \text{real on } \mathcal{H}_1, \tag{4.40b}$$

$$D\kappa_1 \quad \text{real on } \mathcal{H}_2. \tag{4.40c}$$

4.5.2 Hermiticity in terms of conditions at $\mathcal{Z}$

In this section it is shown that conditions (4.40b)-(4.40c) can be replaced by restrictions on $\mathcal{Z}$.

Analysis on $\mathcal{H}_2$. Start by considering condition (4.40c). From the transport equation (4.15b) on $\mathcal{H}_2$, and equation (4.19g), it follows that

$$2\Delta D\kappa_1 = \delta\overline{\delta}\kappa_1 + \overline{\delta}\delta\kappa_1 + 4\tau\overline{\delta}\kappa_1 - (3\alpha+\overline{\beta})\delta\kappa_1 - (3\overline{\alpha}+\beta)\overline{\delta}\kappa_1 + 2\Psi_2\kappa_1$$

on $\mathcal{H}_2$. Taking a further Δ-derivative gives

$$2\Delta\Delta D\kappa_1 = \Delta(\delta\overline{\delta}+\overline{\delta}\delta)\kappa_1 + 4\tau\Delta\overline{\delta}\kappa_1 - (3\alpha+\overline{\beta})\Delta\delta\kappa_1 - (3\overline{\alpha}+\beta)\Delta\overline{\delta}\kappa_1 + 2\Psi_2\Delta\kappa_1.$$

We can commute the Δ-derivative with the δ and $\overline{\delta}$ derivatives to obtain

$$2\Delta\Delta D\kappa_1 = (\delta\overline{\delta}+\overline{\delta}\delta)\Delta\kappa_1 + 4\tau\overline{\delta}\Delta\kappa_1 - (3\alpha+\overline{\beta})\delta\Delta\kappa_1 - (3\overline{\alpha}+\beta)\overline{\delta}\Delta\kappa_1 + 2\Psi_2\Delta\kappa_1.$$

Note that all the terms on the right are proportional to intrinsic derivatives of $\Delta\kappa_1$, which by (4.22e) is proportional to κ_2 and its intrinsic derivatives on $\mathcal{H}_2$. As shown in subsection 4.4.1, unless our spacetime is the Minkowski solution, the component κ_2 must vanish on $\mathcal{H}_2$. It follows then that

$$\Delta\Delta D\kappa_1 = 0 \quad \text{on } \mathcal{H}_2\,.$$

This is a second order ordinary differential equation along the generators of $\mathcal{H}_2$. Therefore, the requirement that $D\kappa_1$ is real on $\mathcal{H}_2$ is equivalent to requiring that $D\kappa_1$ and $\Delta D\kappa_1$ are real on $\mathcal{Z}$.

Analysis on $\mathcal{H}_1$. An analogous argument applies in the case of condition (4.40b). Take first a D-derivative along the generators of $\mathcal{H}_1$ and use the transport equation (4.14b) on $\mathcal{H}_1$, along with the assumption that κ_0 vanishes in $\mathcal{H}_1$ to obtain

$$2D(\Delta\kappa_1 + \tau\kappa_2) = \delta\overline{\delta}\kappa_1 + \overline{\delta}\delta\kappa_1 - (\alpha - \overline{\beta})\delta\kappa_1 - (\overline{\alpha} - \beta)\overline{\delta}\kappa_1 + 2\Psi_2\kappa_1.$$

Taking a further D-derivative one gets

$$2DD(\Delta\kappa_1 + \tau\kappa_2) = D(\delta\overline{\delta} + \overline{\delta}\delta)\kappa_1 - (\alpha - \overline{\beta})D\delta\kappa_1 - (\overline{\alpha} - \beta)D\overline{\delta}\kappa_1 + 2\Psi_2 D\kappa_1. \quad (4.41)$$

By commuting the D derivatives with the δ and $\overline{\delta}$ derivatives, it follows that

$$\begin{aligned} 2DD(\Delta\kappa_1 + \tau\kappa_2) =& (\delta\overline{\delta} + \overline{\delta}\delta)D\kappa_1 - (3\alpha + \overline{\beta})\delta D\kappa_1 - (3\overline{\alpha} + \beta)\overline{\delta}D\kappa_1 \\ &+ \left(\delta\overline{\tau} + \overline{\delta}\tau + 4\alpha\overline{\alpha} + 2\alpha\beta + 2\overline{\alpha}\overline{\beta} + 2\Psi_2\right) D\kappa_1. \end{aligned}$$

Note that all terms on the right hand side are proportional to δ and $\overline{\delta}$ derivatives of $D\kappa_1$, which by (4.21f) are proportional to κ_0 and its δ and $\overline{\delta}$ derivatives on $\mathcal{H}_1$. Therefore, again, unless our spacetime is the Minkowski solution, $\kappa_0 = 0$ holds on $\mathcal{H}_1$. Accordingly one has that

$$DD\left(\Delta\kappa_1 + \tau\kappa_2\right) = 0 \quad \text{on } \mathcal{H}_1.$$

Again, the latter is a second order ordinary differential equation along the generators of $\mathcal{H}_1$, and so the requirement that $\Delta\kappa_1 + \tau\kappa_2$ is real on $\mathcal{H}_1$ is equivalent to requiring that $\Delta\kappa_1 + \tau\kappa_2$ and $D\left(\Delta\kappa_1 + \tau\kappa_2\right)$ are real on $\mathcal{Z}$.

Summarising the analyses on both $\mathcal{H}_1$ and $\mathcal{H}_2$:

Lemma 30. *The spinor field $\xi_{AA'}$ is Hermitian on $\mathcal{H}_1 \cup \mathcal{H}_2$, and thereby on the domain* *of dependence of $\mathcal{H}_1 \cup \mathcal{H}_2$, if and only if the conditions*

$$\kappa_1 + \overline{\kappa}_1 = const\,,$$
$$D(\kappa_1 - \overline{\kappa}_1) = 0\,,$$
$$\Delta D(\kappa_1 - \overline{\kappa}_1) = 0\,,$$
$$\Delta(\kappa_1 - \overline{\kappa}_1) + \tau\,\kappa_2 - \overline{\tau}\,\overline{\kappa}_2 = 0,$$
$$D\left(\Delta(\kappa_1 - \overline{\kappa}_1) + \tau\,\kappa_2 - \overline{\tau}\,\overline{\kappa}_2\right) = 0\,,$$

are satisfied on $\mathcal{Z}$.

Note that some of these conditions are redundant. For example, we know that $D\kappa_1$ vanishes on $\mathcal{Z}$ due to equation (4.21f) and the vanishing of κ_0, and so clearly $D(\kappa_1 - \overline{\kappa}_1)$ also vanishes on $\mathcal{Z}$. A similar argument using equation (4.22e) can be used to show that $\Delta(\kappa_1 - \overline{\kappa}_1) + \tau\kappa_2 - \overline{\tau\kappa}_2$ vanishes on $\mathcal{Z}$. We can also use the requirement that $\mathrm{Re}(\kappa_1)$ is constant on $\mathcal{Z}$ to show that the other two conditions are equivalent. Indeed, we have that

$$\begin{aligned}
D\left(\Delta(\kappa_1 - \overline{\kappa}_1) + \tau\kappa_2 - \overline{\tau\kappa}_2\right) &= D\Delta(\kappa_1 - \overline{\kappa}_1) - 2\tau\overline{\delta}\kappa_1 + 2\overline{\tau}\delta\overline{\kappa}_1 \\
&= \Delta D(\kappa_1 - \overline{\kappa}_1) + \tau\overline{\delta}(\kappa_1 - \overline{\kappa}_1) + \overline{\tau}\delta(\kappa_1 - \overline{\kappa}_1) \\
&\qquad - 2\tau\overline{\delta}\kappa_1 + 2\overline{\tau}\delta\overline{\kappa}_1 \\
&= \Delta D(\kappa_1 - \overline{\kappa}_1)
\end{aligned}$$

where (4.19g), the commutator $[\Delta, D]$, and the vanishing of $D\tau$ (see Table 4-A), along with the conditions $\delta\kappa_1 = -\delta\overline{\kappa}_1$ and $\overline{\delta}\kappa_1 = -\overline{\delta}\overline{\kappa}_1$, have been used.

We compute now $\Delta D\kappa_1$. Eliminating $D\kappa_2$ by using (4.21g) the transport equation (4.15b) on $\mathcal{Z}$ can be seen to reduce to

$$\begin{aligned}
2\Delta D\kappa_1 &= (\delta\overline{\delta} + \overline{\delta}\delta)\kappa_1 - (3\alpha + \overline{\beta})\delta\kappa_1 - (3\overline{\alpha} + \beta)\overline{\delta}\kappa_1 - (2\overline{\alpha} + 2\beta)D\kappa_2 + 2\Psi_2\kappa_1 \\
&= (\delta\overline{\delta} + \overline{\delta}\delta)\kappa_1 - (3\alpha + \overline{\beta})\delta\kappa_1 + (\overline{\alpha} + 3\beta)\overline{\delta}\kappa_1 + 2\Psi_2\kappa_1\,.
\end{aligned}$$

Replacing δ and $\overline{\delta}$ derivatives with the $\eth$ and $\overline{\eth}$ operators we obtain

$$2\Delta D\kappa_1 = (\eth\overline{\eth} + \overline{\eth}\eth)\kappa_1 - (2\alpha + 2\overline{\beta})\eth\kappa_1 + (2\overline{\alpha} + 2\beta)\overline{\eth}\kappa_1 + 2\Psi_2\kappa_1\,.$$

The imaginary part of this equation is given by

$$\begin{aligned} 2\Delta D(\kappa_1 - \overline{\kappa}_1) &= \left(\eth\overline{\eth} + \overline{\eth}\eth\right)(\kappa_1 - \overline{\kappa}_1) + 2\Psi_2\kappa_1 - 2\overline{\Psi}_2\overline{\kappa}_1 \\ &= 2\left[\left(\eth\overline{\eth}\kappa_1 + 2\Psi_2\kappa_1\right) - \left(\overline{\eth}\eth\overline{\kappa}_1 + 2\overline{\Psi}_2\overline{\kappa}_1\right)\right], \end{aligned}$$

where in the second step the constancy of $\mathrm{Re}(\kappa_1)$ on $\mathcal{Z}$, along with the commutator (4.3) applied to the spin weight zero quantity κ_1, was used.

Putting these results together gives the following result:

Lemma 31. *The spinorial field $\xi_{AA'}$ is Hermitian on $\mathcal{H}_1 \cup \mathcal{H}_2$ if and only if on $\mathcal{Z}$ the following conditions are satisfied:*

$$\kappa_1 + \overline{\kappa}_1 = const\,, \tag{4.42a}$$

$$\eth\overline{\eth}\kappa_1 + 2\Psi_2\kappa_1 \quad \textit{is real.} \tag{4.42b}$$

Remark 40. The conditions of Lemma 30 involve derivatives off of the bifurcation surface $\mathcal{Z}$, in comparison to the conditions obtained in Lemma 31 which are purely intrinsic to $\mathcal{Z}$.

4.5.3 Summary

We can now integrate the conclusions of Lemma 31 with the conditions provided in Theorems 11 and 13 to give the following characterisation result for the Kerr spacetime:

Theorem 14. *Let $(\mathcal{M}, g)$ be a vacuum spacetime possessing a pair of null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ generated by expansion and shear-free geodesically complete null congruences,* *intersecting on a 2-dimensional spacelike hypersurface $\mathcal{Z} \equiv \mathcal{H}_1 \cap \mathcal{H}_2$. Given a spin basis $\{o^A, \iota^A\}$ on $\mathcal{Z}$, assume that there exist constants c, $\mathfrak{M} \in \mathbb{C}$ such that the following relations hold on $\mathcal{Z}$:*

$$\kappa_0 = \kappa_2 = 0, \tag{4.43a}$$

$$\eth^2\kappa_1 = \bar{\eth}^2\kappa_1 = 0, \tag{4.43b}$$

$$\kappa_1^3\Psi_2 = \mathfrak{M}, \tag{4.43c}$$

$$\kappa_1 + \bar{\kappa}_1 = c, \tag{4.43d}$$

$$\eth\bar{\eth}\kappa_1 + 2\Psi_2\kappa_1 \in \mathbb{R} \tag{4.43e}$$

where $\kappa_0, \kappa_1, \kappa_2$ are the basis components of a spinor κ_{AB} with respect to the spin basis $\{o^A \iota_A\}$. Then, there exist two complex constants $\mathfrak{c}$ and $\mathfrak{l}$ such that

$$\mathcal{H}^2 = -\mathfrak{l}(\mathfrak{c} - \chi)^4$$

in a neighbourhood $\mathcal{O}$ of $\mathcal{Z}$ in $D(\mathcal{H}_1 \cap \mathcal{H}_2)$, where $\mathcal{H}^2 = \mathcal{H}_{ab}\mathcal{H}^{ab}$ is the contraction of the self-dual Killing form with itself (see section 2.2.4 for the full definition) and χ is the Ernst potential (see section 2.2.5.1). Furthermore, if $\mathfrak{c} = 1$ and $\mathfrak{l}$ is real and positive, then $(\mathcal{O}, g)$ is locally isometric to a member of the Kerr family of spacetimes.

Proof. Theorem 11 guarantees the existence of a unique metric in the domain of dependence of the intersecting null hypersurfaces, once initial data is prescribed for the induced metric and connection on $\mathcal{Z}$, and so confirms the well-posedness of the characteristic problem. Due to Theorem 13 and Lemma 31, the conditions (4.43a)-(4.43e) guarantee the existence of a Killing spinor κ_{AB} in a neighbourhood $\mathcal{O}$ of $\mathcal{Z}$ in $D(\mathcal{H}_1 \cap \mathcal{H}_2)$, and that the associated Killing vector (defined by equation (2.5)) is Hermitian. The relation between the self-dual Killing form and the Ernst potential, and the local isometry to a member of the Kerr family once the constants $\mathfrak{c}$, $\mathfrak{l}$ are fixed, follows from Proposition 2. □

Remark 41. In Chapter 3, the asymptotic flatness of the spacetime is used to set the constants $\mathfrak{c}$, $\mathfrak{l}$ to their required values; however, in the characteristic framework this assumption is no longer available. The result above requires that these constants be set manually – this is not physically motivated, but nevertheless must to included to obtain the characterisation. This illustrates the essential nature of the asymptotic flatness assumption for identifying the Kerr spacetime.

4.6 Determining κ_1 on $\mathcal{Z}$

Necessary conditions for the existence of a Killing spinor and the Hermiticity of the associated Killing vector have now been provided. Following on from these, the implications of these conditions can be investigated, allowing one to give a explicit formula for κ_1 and and restriction on the geometry of the bifurcation surface $\mathcal{Z}$. This section proceeds by solving the conditions

$$\eth^2\kappa_1 = 0\,, \qquad \bar{\eth}^2\kappa_1 = 0\,.$$

4.6.1 Solving the conditions $\eth^2\kappa_1 = 0$ and $\bar{\eth}^2\kappa_1 = 0$

Consider first the vanishing of $\eth^2\kappa_1$. As $\eth\kappa_1$ is of spin-weight 1, in virtue of (4.5), we get from

$$\eth^2\kappa_1 = 0$$

that

$$\overline{P}\,\eth\kappa_1 = f(\overline{z})\,, \tag{4.44}$$

where $f(\overline{z})$ is (for the moment) an arbitrary anti-holomorphic function in $\mathcal{Z}$.

Applying once more (4.5) the last relation can also be written as

$$\overline{P}P\,\partial_z\kappa_1 = f(\overline{z})\,. \tag{4.45}$$

As argued in Remark 36, there exists an axial Killing vector field on $\mathcal{Z}$, so it is reasonable to assume without loss of generality that all the geometric quantities including $|P|$ and κ_1 depend only on the modulus $|z|$ of z, and not on the ratio $\overline{z}/z$.

Then, by using the fact that $\partial|z|/\partial z = \overline{z}/(2\,|z|)$ we get from (4.45) that

$$\left[(2\,|z|)^{-1}\,|P|^2\,\partial_{|z|}\kappa_1\right]\overline{z} = f(\overline{z})\,, \tag{4.46}$$

i.e. there should exist a (possibly) complex constant $d^* \in \mathbb{C}$ such that

$$(2\,|z|)^{-1}\,|P|^2\,\partial_{|z|}\kappa_1 = d^* \quad \text{and} \quad f(\overline{z}) = d^*\,\overline{z}\,. \tag{4.47}$$

A completely analogous argument concludes from the vanishing of $\overline{\eth}^2\kappa_1$ that

$$P\,\overline{\eth}\kappa_1 = g(z)\,, \tag{4.48}$$

where $g(z)$ is (for the moment) an arbitrary holomorphic function in $\mathcal{Z}$. This, along with $\partial|z|/\partial\overline{z} = z/(2\,|z|)$, gives as above

$$\left[(2\,|z|)^{-1}\,|P|^2\,\partial_{|z|}\kappa_1\right]z = g(z)\,, \tag{4.49}$$

thereby with the same constant $d^* \in \mathbb{C}$ the relations

$$(2\,|z|)^{-1}\,|P|^2\,\partial_{|z|}\kappa_1 = d^* \quad \text{and} \quad g(z) = d^*\,z\,. \tag{4.50}$$

can be seen to hold.

The first relation in (4.49) or in (4.50) can then be solved for κ_1 as

$$\kappa_1 = 2\,d^* \int \frac{|z|}{|P|^2} d|z| + c\,, \tag{4.51}$$

where c is a constant of integration.

Note that in virtue of (4.42a) κ_1 must have the form

$$\kappa_1 = c + i\,\chi(|z|)\,, \tag{4.52}$$

where c is a real constant and $\chi(|z|)$ is a real function. Thereby, as the integral in (4.51) is real, d^* is purely imaginary, i.e. there exists a real number $d \in \mathbb{R}$ such that $2\,d^* = i\,d$ and, in turn,

$$\kappa_1 = c + id \int \frac{|z|}{|P|^2} d|z|\,. \tag{4.53}$$

Note that given an arbitrary 2-metric on $\mathcal{Z}$, the complex function P can be calculated, and from this the exact form of κ_1 satisfying the equations $\eth^2\kappa_1 = \bar{\eth}^2\kappa_1 = 0$ can be determined. Therefore, the further restrictions placed on κ_1 by enforcing the Hermiticity of the Killing vector must also enforce restrictions on the value of P.

4.6.2 Deriving and solving further conditions on κ_1

Start by the observation that, in virtue of (4.42b), there must exist (a spin-weight zero) real function φ on $\mathcal{Z}$ such that

$$\eth\,\bar{\eth}\,\kappa_1 + 2\,\Psi_2\,\kappa_1 = \varphi \tag{4.54}$$

Taking then the complex conjugate of this relation, using that κ_1 is of spin-weight zero and replacing the complex conjugate of κ_1 by applying (4.52), the following two relations can be seen to hold

$$\begin{aligned} 2\,\varphi &= (\kappa_1 + \overline{\kappa}_1)\,(\Psi_2 + \overline{\Psi}_2) + (\kappa_1 - \overline{\kappa}_1)\,(\Psi_2 - \overline{\Psi}_2) \\ &= 2c\,(\Psi_2 + \overline{\Psi}_2) + 2i\,\chi\,(\Psi_2 - \overline{\Psi}_2)\,, \end{aligned} \tag{4.55}$$

$$2\,\eth\,\bar{\eth}\kappa_1 = 2\,\bar{\eth}\,\eth\kappa_1 = 2\overline{\kappa}_1\,\overline{\Psi}_2 - 2\kappa_1\,\Psi_2 = -2\left[c\,(\Psi_2 - \overline{\Psi}_2) + i\,\chi\,(\Psi_2 + \overline{\Psi}_2)\right]\,. \tag{4.56}$$

Note that (4.56) can also be written as

$$\eth\,\bar{\eth}\kappa_1 = \frac{\overline{\mathfrak{M}}}{\bar{\kappa}_1^2} - \frac{\mathfrak{M}}{\kappa_1^2}\,. \tag{4.57}$$

As $\eth\kappa_1$ is of spin-weight 1, in virtue of (4.5), we get that

$$\bar{\eth}\,\eth\kappa_1 = \overline{P}P\,\partial_{\bar{z}}(P^{-1}\eth\kappa_1) = |P|^2\,\partial_{\bar{z}}\left(P^{-1}\left[P\,\partial_z\kappa_1\right]\right) = |P|^2\,\partial_{\bar{z}}\partial_z\kappa_1\,.$$

As $\partial|z|/\partial z = \bar{z}/(2\,|z|)$ we also have

$$\partial_z\,\kappa_1 = \frac{1}{2}\,\frac{\bar{z}}{|z|}\,\partial_{|z|}\,\kappa_1$$

and

$$\partial_{\bar{z}}\,\partial_z\,\kappa_1 = \frac{1}{4}\left[\partial^2_{|z|}\,\kappa_1 + \frac{1}{|z|}\,\partial_{|z|}\,\kappa_1\right]\,.$$

Using then (4.51) it follows then that

$$\partial_{|z|}\,\kappa_1 = i\,d\,\frac{|z|}{|P|^2}$$

and

$$\partial^2_{|z|}\,\kappa_1 = \frac{i\,d}{|P|^2}\,\left[1 - |z|\,\partial_{|z|}\ln(|P|^2)\right]\,.$$

Therefore, the second order mixed eth derivative of κ_1 can be written as

$$\bar{\eth}\,\eth\kappa_1 = |P|^2\,\partial_{\bar{z}}\partial_z\kappa_1 = \frac{i\,d}{4}\,\left[2 - |z|\,\partial_{|z|}\ln(|P|^2)\right]\,.$$

In virtue of (4.57), this can be reformulated as an "additional" constraint on the conformal factor $|P|^2$:

$$\frac{id}{4}\left(2 - |z|\partial_{|z|}\ln|P|^2\right) = \frac{\overline{\mathfrak{M}}}{\left(c - id\int\frac{|z|}{|P|^2}\mathrm{d}|z|\right)^2} - \frac{\mathfrak{M}}{\left(c + id\int\frac{|z|}{|P|^2}\mathrm{d}|z|\right)^2}.$$

In other words, the requirement that the Killing vector ξ is Hermitian on $\mathcal{H}_1 \cup \mathcal{H}_2$ gives constraints on the allowed geometry of the bifurcation surface $\mathcal{Z}$. This equation can be thought of as an integrability condition on the complex function P for there to exist a solution to the system of equations (4.43a)-(4.43e).

4.7 Identifying the Schwarzschild spacetime

Equation (4.53) provides an explicit expression for κ_1 on $\mathcal{Z}$ in terms of the complex factor P; this factor is determined by the conformal relation between the induced metric σ on $\mathcal{Z}$ and the round metric on $\mathbb{S}^2$. In particular, by making the further assumption that σ is a constant multiple of the round metric (i.e. the round metric on a sphere of radius R), we can hope to be able to single out the Schwarzschild spacetime from the larger class of spacetimes satisfying the conditions of Theorem 13 and Lemma 31. The surface $\mathcal{Z}$ in the exact Schwarzschild spacetime of mass M is metrically $\mathbb{S}^2$ with radius $R = 2M$; this can be seen by writing the Schwarzschild metric in Kruskal-Szekeres coordinates (U, V, θ, ϕ), at which point the radius of the bifurcation sphere (given by the surface $\{U = V = 0\}$) can be simply read off.

By assuming that the metric induced on $\mathcal{Z}$ is

$$\sigma_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = -R^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$

with radius R, equation (4.15.116) of [45] gives the exact form for P:

$$P = \frac{1 + z\overline{z}}{R\sqrt{2}},$$

and equation (4.15.113) gives the relationship between the complex coordinate function z and the standard spherical polar coordinates on the sphere:

$$z = e^{i\phi}\cot\frac{\theta}{2}, \tag{4.58}$$

Using this, the explicit formula for κ_1 in equation (4.53) can be simplified:

$$\begin{aligned}\kappa_1 &= c + idR^2 \int \frac{2|z|}{(1+|z|^2)^2} \mathrm{d}|z| \\ &= c + idR^2 \left(b - \frac{1}{1+|z|^2} \right)\end{aligned}$$

where b is a real constant of integration. Substituting in the form of z given in (4.58) and simplifying,

$$\begin{aligned}\kappa_1 &= c + idR^2 \left(b - \frac{1}{2} \right) + \frac{1}{2} idR^2 \cos\theta \\ &= \tilde{c} + i\tilde{d}\cos\theta\end{aligned}$$

where the constants have been combined into $\tilde{c} \in \mathbb{C}$ and $\tilde{d} \in \mathbb{R}$.

The requirement (4.35) needed for Theorem 13 now gives an explicit form for the Weyl scalar Ψ_2:

$$\Psi_2 = \frac{M}{\left(\tilde{c} + i\tilde{d}\cos\theta\right)^3}.$$

However, as remarked in Section 4.2.1.1, the Gaussian curvature of $\mathcal{Z}$ is $\kappa_G = -2\,\mathrm{Re}(\Psi_2)$; this must be equal to the Gaussian curvature of a metric sphere of radius $R = 2M$, i.e. $\kappa_G = -\frac{1}{4M^2}$. The only way for this to hold identically is for the constant $\tilde{d}$ to vanish, meaning on $\mathcal{Z}$

$$\kappa_1 = \tilde{c}.$$

Now, consider condition (4.54), itself a consequence of Lemma 31. The constancy of κ_1 means that this condition can be written as

$$\mathfrak{M} = \frac{1}{2}\varphi\tilde{c}^2 \tag{4.59}$$

for some real function φ on $\mathcal{Z}$. Recall from equation (2.41) the definition of $\mathfrak{l}$ (the proportionality constant linking the norm of the self-dual Killing form to the Ernst potential) in terms of $\mathfrak{M}$:

$$\begin{aligned}\mathfrak{l} &= \frac{64}{81}\frac{1}{\mathfrak{M}^2} \\ &= \frac{256}{81}\frac{1}{\varphi^2\tilde{c}^4}.\end{aligned}$$

Remark 42. As φ is a real function on $\mathcal{Z}$ (and is, in fact, constant on $\mathcal{Z}$ due to the constancy of both $\mathfrak{M}$ and κ_1), $\mathfrak{l}$ is real and positive if and only if $\tilde{c}$ is either real or pure imaginary. However, from Proposition 2, we know that to identify a member of the Kerr family (in this case, the Schwarzschild metric), this constant must be real and positive. In other words, there exists a large class of solutions to the conditions given in Theorem 14, with metrically spherical bifurcation surfaces, that are *not* isometric to a member of the Kerr family (and in particular the Schwarzschild solution). Therefore, in order to single out the Schwarzschild solution, further conditions must be imposed. This calculation illustrates the essential nature of the asymptotic flatness assumption used in Theorem 3 and the difficulty in finding physically motivated local conditions to achieve the same result.

4.8 Conclusions

The analysis in this chapter identifies a set of conditions (given in (4.43a)-(4.43e)) that must be satisfied on the bifurcation surface $\mathcal{Z}$ of the non-expanding horizon structure $\mathcal{H}_1 \cap \mathcal{H}_2$, in order to guarantee (relying on the results of Rendall [48]) the existence of a Killing spinor in a region $\mathcal{O}$, the intersection of a neighbourhood of $\mathcal{Z}$ with the future development of $\mathcal{H}_1 \cup \mathcal{H}_2$. A result due to Luk [38] has extended the region of existence of a unique solution to the Einstein equations to a neighbourhood of the horizons $\mathcal{H}_1$ and $\mathcal{H}_2$, as long as the constraint equations are satisfied there. One would expect to find that the region of existence of the Killing spinor can also be extended in this way.

Furthermore, by extending the region of existence to the length of the horizon one can investigate the behaviour of the relevant fields in the infinite affine parameter limit; considering the set-up in a conformal setting may be useful for studying this.

However, although the class of spacetimes referred to as 'distorted' black holes is known to *include* the Kerr family, it has been illustrated here that conditions for the existence of a Killing spinor in the spacetime development of the characteristic initial data are insufficient to single out the Kerr family from this larger class. Whereas asymptotic flatness could be used in previous chapters, here there is no obvious physically motivated way to fix the additional local conditions required to identify the Kerr family. A potential solution could be to investigate the behaviour of the relevant fields in the infinite affine parameter limit along the generators of the horizons, as previously mentioned.

Another avenue of investigation would be determining whether the existence result can be extended to a full neighbourhood of the horizon, rather than being restricted to the future development. Theorem 14 is a Killing spinor analogue of the rigidity results in [27], which themselves were extended to a full neighbourhood in [2]. It would be very interesting to see whether the ideas from [2] can be adapted for the results laid out in this chapter, to obtain a statement in the domain of outer communication.

This analysis has considered only for vacuum case, but in principle it could be extended fairly straightforwardly to electrovacuum spacetimes, yielding an existence result for a Killing spinor in a neighbourhood of the horizons and helping to characterise the Kerr-Newman solution via Proposition 3. However, this generalisation would be expected to suffer from the same problem as the vacuum case, namely that the required constants must be set locally rather than using an asymptotic condition.

Chapter 5

Conclusions

In this thesis, the use of Killing spinors as a valuable construction for characterising Kerr and Kerr-Newman spacetimes has been explored. The existence of a Killing spinor corresponds to the presence of a 'hidden symmetry' of the underlying spacetime, so the sufficiency of this condition (along with asymptotic conditions) is evidence that this symmetry is a special feature of the Kerr-Newman family, singling it out from the larger class of spacetimes with these asymptotic properties. In fact, the required asymptotic conditions are not particularly restrictive - a physically motivated, spatially isolated black hole (like those observed in our universe) would be expected to fulfil these requirements.

In Chapter 2, it was shown that the hypotheses of characterisations due to Mars (for the Kerr family) and Wong (for the Kerr-Newman family) can be fulfilled by the existence of a Killing spinor on the spacetime. This takes the form of a local result requiring certain constants to be fixed manually, and global results without this requirement, utilising the assumption of asymptotic flatness. The lack of a simple, physically motivated way of setting these constants without the asymptotic flatness condition suggests that it is an essential feature of Kerr-Newman characterisation results.

In Chapter 3, the justification of the importance of Killing spinors was put to use. In an analogous procedure to the derivation of the KID equations (see [10]), conditions for a symmetric 2-spinor κ_{AB} can be found on an initial data set which guarantees the existence of a Killing spinor on the resulting unique development of the initial data. This is done by constructing a set of wave equations for a set of 'zero quantities' which vanish in the presence of a true Killing spinor; the requirement of trivial initial data for this system gave the desired conditions. A key feature is the fact that these conditions are overdetermined, and so for arbitrary initial data sets do not admit solutions. By extending this system to an elliptic system, which always admits a unique solution for initial data with asymptotic conditions matching those of the Kerr-Newman spacetime, an approximate Killing spinor can be constructed on the initial data set. The fact that this candidate spinor can be found for any initial data set (with suitable asymptotic behaviour) is the key criterion in this analysis, allowing the geometric invariant constructed as the norm of the Killing spinor initial data (under a suitable inner product) to be interpreted as a measure of how much the Killing spinor initial data conditions are violated. For example, to study the behaviour of a perturbed Kerr-Newman black hole, one could calculate this invariant at successive time slices of the evolved spacetime; if the spacetime 'settles' to the exact Kerr-Newman solution, one could conjecture that it will decay to zero. In order for this to be useful in numerical studies, further properties of the constructed geometric invariant need to be established: in particular, its behaviour under time evolution. As mentioned in section 3.8, in order to do this an evolution equation for the approximate Killing spinor must be found which respects the elliptic approximate Killing spinor equation on each leaf of the foliation. If such an equation can be found, then its form would determine the behaviour of the approximate spinor under evolution and provide details of the behaviour of the geometric invariant also.

In Chapter 4, the discussion was moved from spacelike initial data sets to the characteristic problem. The motivation for doing so was provided by the construction of so-called 'distorted black holes', possessing a non-expanding and shear free bifurcate horizon structure. In a similar way to the case of spacelike initial data, conditions on the horizon structure can be found which corresponded to trivial initial data for a system of wave equations for a set of zero-quantities. A key observation in this case is the fact that these conditions can be restricted to the bifurcation surface $\mathcal{Z}$, rather than being required on the extended horizon structure. Furthermore, the Hermiticity of the associated Killing vector can also be guaranteed by conditions only on the bifurcation surface; in particular, all of the assumptions of Proposition 2 can be fulfilled by conditions only on $\mathcal{Z}$. However, it is shown that these conditions are insufficient to fix the constants of Proposition 2, in order to single out the Kerr spacetime specifically. The conclusion reached is that the class of 'distorted' black hole spacetimes includes but is not exhausted by the Kerr family. Further restrictions on the definition of a 'distorted' black hole would be required to do this; at this moment, it is unclear if there exists a physically motivated, or even mathematically satisfying, way to do this.

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