

# Classifying matchbox manifolds

ALEX CLARK  
STEVEN HURDER  
OLGA LUKINA

Matchbox manifolds are foliated spaces with totally disconnected transversals. Two matchbox manifolds which are homeomorphic have return equivalent dynamics, so that invariants of return equivalence can be applied to distinguish non-homeomorphic matchbox manifolds. In this work we study the problem of showing the converse implication: when does return equivalence imply homeomorphism? For the class of weak solenoidal matchbox manifolds, we show that if the base manifolds satisfy a strong form of the Borel Conjecture, then return equivalence for the dynamics of their foliations implies the total spaces are homeomorphic. In particular, we show that two equicontinuous  $\mathbb{T}^n$ -like matchbox manifolds of the same dimension are homeomorphic if and only if their corresponding restricted pseudogroups are return equivalent. At the same time, we show that these results cannot be extended to include the “*adic*-surfaces”, which are a class of weak solenoids fibering over a closed surface of genus 2.

[37B05](#), [54F15](#), [57R30](#), [58H05](#); [20E18](#), [37B45](#)

## 1 Introduction

A matchbox manifold is a compact, connected metrizable space  $\mathfrak{M}$ , equipped with a decomposition into leaves of constant dimension, so that the pair  $(\mathfrak{M}, \mathcal{F})$  is a foliated space as defined in [9, 36], for which the local transversals to the foliation are totally disconnected. In particular, the leaves of  $\mathcal{F}$  are the path connected components of  $\mathfrak{M}$ . A matchbox manifold with 2-dimensional leaves is a lamination by surfaces in the sense of Ghys [25] and Lyubich and Minsky [34]. The “solenoidal spaces” of Sullivan in [44, 45] are examples of matchbox manifolds. The dynamical and topological properties of matchbox manifolds have been studied in a series of works by the authors [12, 13, 14].

Matchbox manifolds arise naturally as exceptional minimal sets for foliations of compact manifolds, for example see [29, 30]; as the tiling spaces associated to repetitive, aperiodic tilings of Euclidean space  $\mathbb{R}^n$  which have finite local complexity, for example see

[3, 41, 42]; and they appear naturally in the study of group representation theory and index theory for leafwise elliptic operators for foliations, as discussed in the books [9, 36]. The classification problem for matchbox manifolds asks for invariants which distinguish their homeomorphism types. For example, in the study of aperiodic tilings and their invariants, the cohomology and K-Theory groups of their associated tiling spaces have been calculated in many instances, as for example in [3, 5, 4, 10, 24].

A matchbox manifold  $(\mathfrak{M}, \mathcal{F})$  is also a type of dynamical system, as discussed in [30] for example. A homeomorphism between matchbox manifolds preserves the leaves, as they are the path connected components of  $\mathfrak{M}$ , and thus many dynamical properties of  $\mathcal{F}$  are invariants of the homeomorphism class of  $\mathfrak{M}$ . For example, the foliation  $\mathcal{F}$  is said to be *minimal* if each leaf  $L \subset \mathfrak{M}$  is dense, and this property is clearly a homeomorphism invariant. For a clopen transversal  $W$  of  $\mathcal{F}$ , the dynamical properties of a minimal foliation  $\mathcal{F}$  are determined by the pseudogroup  $\mathcal{G}_W$  of local holonomy maps acting on the transversal  $W$ . *Return equivalence* of pseudogroup actions on Cantor spaces is the analog of the notion of *Morita equivalence* for groupoids associated to smooth foliations of compact manifolds, as discussed for example by Haefliger in [27, 28]. One then has the following result, whose proof follows along the same method as for the case of smooth foliations:

**Theorem 1.1** *Let  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  be minimal matchbox manifolds. Suppose that there exists a homeomorphism  $h: \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ , then the holonomy pseudogroup actions associated to  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are return equivalent.*

Now consider  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  which are minimal matchbox manifolds whose holonomy pseudogroups are return equivalent. That is, assume there exists clopen transversals  $W_1$  to  $\mathfrak{M}_1$  and  $W_2$  to  $\mathfrak{M}_2$ , and a homeomorphism  $h: W_1 \rightarrow W_2$  which conjugates the restricted holonomy actions. It is natural to ask for assumptions on  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  which are sufficient to guarantee that the transverse map  $h$  extends to a homeomorphism  $H: \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ . In the case of 1-dimensional flows, there is the following result of Aarts and Oversteegen [2, Theorem 17]:

**Theorem 1.2** *Two orientable, minimal, 1-dimensional matchbox manifolds are homeomorphic if and only if they are return equivalent.*

Since any non-orientable, minimal, matchbox manifold admits an orientable double cover, this implies that the local dynamics determines the global topology in dimension one. For a matchbox manifold with leaves of dimension greater than one, the question whether there exists a converse to Theorem 1.1 is much more subtle. Julien and Sadun

studied in [32] the homeomorphism classification for the tiling spaces associated to aperiodic tilings of the Euclidean space  $\mathbb{R}^n$ , and the relation to return equivalence for the associated pseudogroups.

In this work, we consider the converse to Theorem 1.1 when  $\mathfrak{M}$  is homeomorphic to a weak solenoid. A *weak solenoid* is defined as the inverse limit of an infinite sequence of proper finite covering maps of a closed compact manifold, called the base of the solenoid. The properties of weak solenoids are recalled in Section 2. In particular, a weak solenoid is homeomorphic to the suspension of a minimal equicontinuous action of a finitely generated group on a Cantor set, called the global monodromy for the solenoid. In Section 3, the problem of showing that a pair of weak solenoids which are return equivalent are also homeomorphic, is reduced to showing that they have presentations with homeomorphic base manifolds and conjugate global holonomy actions.

There is a special class of solenoidal spaces where the converse to Theorem 1.1 can be proved without further assumptions. We say that  $\mathcal{S}_{\mathcal{P}}$  is a *toroidal solenoid* if it is defined by a presentation  $\mathcal{P}$  as in (1), where each of the manifolds  $M_{\ell}$  is homeomorphic to the  $n$ -torus  $\mathbb{T}^n$ . The toroidal solenoids arise as the minimal sets for smooth foliations, as shown in [11]. For  $n \geq 2$ , we have the following generalization of Theorem 1.1.

**Theorem 1.3** *Suppose that  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are homeomorphic to toroidal solenoids of the same dimension  $n$ . Then  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are homeomorphic if and only if the holonomy pseudogroup actions associated to  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are return equivalent.*

For the toroidal solenoids with base dimension  $n = 1$ , the homeomorphism type of  $\mathcal{S}_{\mathcal{P}}$  is determined by the asymptotic class of a sequence of integers  $\{m_{\ell} \mid \ell > 0\}$ , the covering indices, as shown by Bing [6] and McCord [35, Section 2], and see also Block and Keesling [7, Corollary 2.6]. Moreover, Aarts and Fokkink showed in [1, Section 3] that the asymptotic class of the sequence of covering indices  $\{m_{\ell} \mid \ell > 0\}$  is determined by the return equivalence class of the flow. This result will be discussed further in Section 5 below.

For the toroidal solenoids with base dimension  $n \geq 2$ , the results of Giordano, Putnam and Skau in [26], and Cortez and Medynets in [15], provide complete invariants of the return equivalence class of minimal equicontinuous free  $\mathbb{Z}^n$  actions on Cantor sets. Their invariants combined with the conclusion of Theorem 1.3 yields a classification of toroidal solenoids up to homeomorphism.

In Section 5 below, we introduce the *adic*-surfaces, which are 2-dimensional weak solenoids, and give examples of return equivalent *adic*-surfaces which are non-homeomorphic. For non-toroidal weak solenoids of dimension greater than one,

it is necessary to impose geometric conditions which rule out the examples such as given in Section 5, in order to obtain a converse to Theorem 1.1.

The first condition we impose is that there exists a leaf for the foliation which is simply connected. Secondly, we impose topological restrictions on the base manifolds, in order that their homeomorphism type of their proper coverings are determined by their fundamental groups.

Recall that a finite  $CW$ -complex  $Y$  is *aspherical* if it is connected and its universal covering space is contractible. Let  $\mathcal{A}$  denote the collection of  $CW$ -complexes which are aspherical. Also recall that the *Borel Conjecture* is that if  $Y_1$  and  $Y_2$  are homotopy equivalent, aspherical closed manifolds, then a homotopy equivalence between  $Y_1$  and  $Y_2$  is homotopic to a homeomorphism between  $Y_1$  and  $Y_2$ . The Borel Conjecture has been proven for many classes of aspherical manifolds:

- the torus  $\mathbb{T}^n$  for all  $n \geq 1$ ,
- all infra-nilmanifolds,
- closed Riemannian manifolds  $Y$  with negative sectional curvatures,
- closed Riemannian manifolds  $Y$  of dimension  $n \neq 3, 4$  with non-positive sectional curvatures.

A compact connected manifold  $Y$  is an *infra-nilmanifold* if its universal cover  $\tilde{Y}$  is contractible, and the fundamental group of  $M$  has a nilpotent subgroup with finite index.

The above list is not exhaustive. The history and current status of the Borel Conjecture is discussed in the surveys of Davis [18] and Lück [33]. We introduce the notion of a *strongly Borel* manifold.

**DEFINITION 1.4** A collection  $\mathcal{A}_B$  of closed manifolds is called *Borel* if it satisfies the conditions

- 1) Each  $Y \in \mathcal{A}_B$  is aspherical,
- 2) Any closed manifold  $X$  homotopy equivalent to some  $Y \in \mathcal{A}_B$  is homeomorphic to  $Y$ , and
- 3) If  $Y \in \mathcal{A}_B$ , then any finite covering space of  $Y$  is also in  $\mathcal{A}_B$ .

We say that a closed manifold  $Y$  is *strongly Borel* if the collection  $\mathcal{A}_Y \equiv \langle Y \rangle$  of all finite covers of  $Y$  forms a Borel collection.

Each class of manifolds in the above list is strongly Borel. Here is our second main result:

**Theorem 1.5** *Let  $\mathcal{S}_{\mathcal{P}}$  and  $\mathcal{S}_{\mathcal{Q}}$  be weak solenoids, for which the base manifolds  $M_0$  of the presentation  $\mathcal{P}$  and  $N_0$  of the presentation  $\mathcal{Q}$  are both strongly Borel closed manifolds of the same dimension. Assume that the foliations on  $\mathcal{S}_{\mathcal{P}}$  and  $\mathcal{S}_{\mathcal{Q}}$  each contain a leaf which is simply connected. Then  $\mathcal{S}_{\mathcal{P}}$  and  $\mathcal{S}_{\mathcal{Q}}$  are homeomorphic if and only if the holonomy pseudogroup actions associated to  $\mathcal{S}_{\mathcal{P}}$  and  $\mathcal{S}_{\mathcal{Q}}$  are return equivalent.*

The requirement that there exists a simply connected leaf implies that the global holonomy maps associated to each of these foliations are injective maps. This conclusion yields a connection between return equivalence for the foliations of  $\mathcal{S}_{\mathcal{P}}$  and  $\mathcal{S}_{\mathcal{Q}}$  and the homotopy types of the approximating manifolds in the presentations  $\mathcal{P}$  and  $\mathcal{Q}$ . This requirement need not be imposed for the case of  $Y = \mathbb{T}^n$  in Theorem 1.3, due to the algebraic properties of  $\mathbb{Z}^n$ . We also note that the injectivity of the global holonomy maps implies that the fundamental groups  $\pi_1(M_0, x_0)$  and  $\pi_1(N_0, y_0)$  are residually finite.

A key aspect of the hypotheses in Theorems 1.3 and 1.5, is that the domains of the return equivalence can be taken to have arbitrarily small diameter. Consequently, invariants of return equivalence developed to distinguish actions should have an asymptotic nature, in that they are defined for arbitrarily small transversals.

A homeomorphism between matchbox manifolds induces a quasi-isometry between the leaves of the respective foliations, equipped with the induced metrics. It is a classical result of Plante [37] that the quasi-isometry class of a leaf is determined by its intersection with any transversal, and thus provides a general invariant of asymptotic return equivalence. For example, bounds on the growth rates of the leaves are return equivalence invariants. This observation was used in the work [20] to give growth restrictions on the leaves which imply that the weak solenoid is a homogeneous continuum.

The *asymptotic discriminant* for an equicontinuous minimal Cantor action was defined in [31], and is an invariant of the return equivalence class of the action, essentially by its definition. It thus provides an invariant of the homeomorphism class of the weak solenoid. Using this asymptotic invariant, the constructions of examples of wild solenoids in [31, Section 9] were shown to yield uncountable collections of non-homeomorphic weak solenoids, all with the same compact base manifold whose fundamental group is a higher rank lattice, and in particular is highly non-abelian.

## 2 Standard forms for weak solenoids

Weak solenoids were first introduced by McCord [35], and we recall here the definitions and some of their properties as developed by Schori [43], Rogers and Tollefson [38, 40] and Fokkink and Oversteegen [23]. We then recall the “odometer representation” of a weak solenoid as the suspension of a (non-abelian) group odometer (or subodometer) action.

A *presentation* (for a weak solenoid) is a collection

$$(1) \quad \mathcal{P} = \{p_{\ell+1}: M_{\ell+1} \rightarrow M_\ell \mid \ell \geq 0\} ,$$

where each  $M_\ell$  is a connected compact manifold of dimension  $n$ , and each *bonding* map  $p_{\ell+1}$  is a proper covering map of finite index. The weak solenoid  $\mathcal{S}_\mathcal{P}$  is the inverse limit associated to the presentation  $\mathcal{P}$ :

$$(2) \quad \mathcal{S}_\mathcal{P} \equiv \varprojlim \{p_{\ell+1}: M_{\ell+1} \rightarrow M_\ell\} \subset \prod_{\ell \geq 0} M_\ell .$$

By definition, for a sequence  $\{x_\ell \in M_\ell \mid \ell \geq 0\}$ , we have

$$x = (x_\ell) \equiv (x_0, x_1, \dots) \in \mathcal{S}_\mathcal{P} \iff p_\ell(x_\ell) = x_{\ell-1} \text{ for all } \ell \geq 1 .$$

The set  $\mathcal{S}_\mathcal{P}$  is given the relative (or Tychonoff) topology induced from the product topology. Then  $\mathcal{S}_\mathcal{P}$  is compact and connected. McCord showed in [35] that the space  $\mathcal{S}_\mathcal{P}$  has a local product structure, and moreover we have:

**PROPOSITION 2.1** *Let  $\mathcal{P}$  be a presentation with base space  $M_0$  of dimension  $n \geq 0$ , and let  $\mathcal{S}_\mathcal{P}$  be the associated weak solenoid. Then  $\mathcal{S}_\mathcal{P}$  is a matchbox manifold of dimension  $n$ , and the leaves of the foliation  $\mathcal{F}_\mathcal{S}$  are the path-connected components of  $\mathcal{S}_\mathcal{P}$ .*

Associated to a presentation  $\mathcal{P}$  is a sequence of proper surjective maps

$$(3) \quad q_\ell = p_1 \circ \dots \circ p_{\ell-1} \circ p_\ell: M_\ell \rightarrow M_0 .$$

For each  $\ell > 1$ , projection onto the  $\ell$ -th factor in the product  $\prod_{\ell \geq 0} M_\ell$  in (2) yields a fibration map denoted by  $\Pi_\ell: \mathcal{S}_\mathcal{P} \rightarrow M_\ell$ , for which  $\Pi_0 = q_\ell \circ \Pi_\ell: \mathcal{S}_\mathcal{P} \rightarrow M_0$ .

Fix a choice of a basepoint  $x_0 \in M_0$  and let  $\mathfrak{X}_0 = \Pi_0^{-1}(x_0)$  be the fiber over  $x_0$ . Then  $\mathfrak{X}_0$  is a Cantor set by the assumption that the fibers of each map  $p_\ell$  have cardinality at least 2.

Choose a basepoint  $x \in \mathfrak{X}_0$ , and for  $\ell \geq 1$ , define basepoints  $x_\ell = \Pi_\ell(x) \in M_\ell$ . Then let

$$(4) \quad G_\ell^x = \text{image } \{(q_\ell)_\# : \pi_1(M_\ell, x_\ell) \longrightarrow G_0\}$$

denote the image of the induced map  $(q_\ell)_\#$  on fundamental groups. Associated to the presentation  $\mathcal{P}$  and basepoint  $x \in \mathfrak{X}_0$  we thus obtain a descending chain of subgroups of finite index

$$(5) \quad \mathcal{G}^x \equiv \{G_\ell^x\}_{\ell \geq 0} = \{G_0 = G_0^x \supset G_1^x \supset G_2^x \supset \cdots \supset G_\ell^x \supset \cdots\}.$$

Each quotient  $X_\ell^x = G_0/G_\ell^x$  is a finite set equipped with a left  $G_0$ -action, and the natural surjections  $X_{\ell+1}^x \rightarrow X_\ell^x$  commute with the action of  $G_0$ . Thus, the inverse limit

$$(6) \quad X_\infty^x = \varprojlim \{p_{\ell+1} : X_{\ell+1}^x \rightarrow X_\ell^x\} \subset \prod_{\ell \geq 0} X_\ell^x$$

is a  $G_0$ -space. Give  $X_\infty^x$  the relative topology induced from the product (Tychonoff) topology on the space  $\prod_{\ell \geq 0} X_\ell^x$ , so that  $X_\infty^x$  is a totally disconnected perfect compact set, so is a Cantor space.

Note that the subgroups  $G_\ell^x$  in (4)  $X_\infty^x$  are not assumed be normal in  $G_0$ , and thus  $X_\infty^x$  is not a profinite group in general, without some form of “normality” assumptions on the subgroups in the chain  $\mathcal{G}^x$ . The question of what assumptions are necessary for the limit  $X_\infty^x$  to be a profinite group was first raised in the work [38] by Rogers and Tollefson, and further analyzed by Fokkink and Oversteegen in [23]. The subsequent work by Dyer, Hurder and Lukina in [19] characterized the necessary normality condition in terms of the discriminant invariant of the chain  $G_\ell^x$ .

A sequence  $(g_\ell) \subset G_0$  such that  $g_\ell G_\ell^x = g_{\ell+1} G_{\ell+1}^x$  for all  $\ell \geq 0$  determines a point  $(g_\ell G_\ell^x) \in X_\infty^x$ . Let  $e \in G_0$  denote the identity element, then the sequence  $e_0 = (e G_\ell^x)$  is the *standard basepoint* of  $X_\infty^x$ . The action  $\Phi_x : G_0 \times X_\infty^x \rightarrow X_\infty^x$  is given by coordinate-wise multiplication,  $\Phi_x(g)(g_\ell G_\ell^x) = (gg_\ell G_\ell^x)$ .

We then have the standard observation:

**LEMMA 2.2**  $\Phi_x : G_0 \times X_\infty^x \rightarrow X_\infty^x$  defines an equicontinuous Cantor minimal system  $(X_\infty^x, G_0, \Phi_x)$ .

When  $X_\infty^x$  has the structure of a profinite group, the action  $\Phi_x : G_0 \times X_\infty^x \rightarrow X_\infty^x$  is called an *odometer* by Cortez and Petite in [16], and when  $X_\infty^x$  is simply a Cantor space they call the action a *subodometer*. If the group  $G_0$  is abelian, then  $X_\infty^x$  is a profinite abelian group, and more generally if the chain (5) consists of normal subgroups of  $G_0$ ,

then  $X_\infty^x$  is a profinite group. For simplicity, we will call all of these equicontinuous minimal actions by the nomenclature “odometers”.

Recall that  $\Pi_\ell: \mathcal{S}_\mathcal{P} \rightarrow M_\ell$  is a fibration for each  $\ell \geq 0$ , and so the set  $\mathfrak{X}_\ell^x = \Pi_\ell^{-1}(x_\ell)$  is a clopen subset of  $\mathfrak{X}_0$ . From the relation  $q_{\ell+1} \circ \Pi_{\ell+1} = \Pi_\ell$  we have that  $\mathfrak{X}_{\ell+1}^x \subset \mathfrak{X}_\ell^x$  so we obtain a nested chain of clopen subsets  $\{\mathfrak{X}_{\ell+1}^x \subset \mathfrak{X}_\ell^x \mid \ell \geq 0\}$ . Moreover, by the definition of the topology on the inverse limit  $\mathcal{S}_\mathcal{P}$ , the intersection of these sets is the chosen basepoint  $x \in \mathfrak{X}_0$ .

The global monodromy action  $\Phi_\mathcal{F}: G_0 \times \mathfrak{X}_0 \rightarrow \mathfrak{X}_0$  is then defined as follows. Given a point  $y \in \mathfrak{X}^x$ , let  $L_y \subset \mathcal{S}_\mathcal{P}$  be the leaf containing  $y$ . The restriction  $\Pi_0: L_y \rightarrow M_0$  is a covering map, so given a closed path  $\sigma: [0, 1] \rightarrow M_0$  with basepoint  $x_0$ , there is a unique leafwise path  $\sigma_y$  in  $L_y$  with initial point  $y$  and terminal point  $\sigma_y(1) \in \mathcal{S}_\mathcal{P}$ . The terminal point  $\sigma_y(1)$  depends only on the basepoint-preserving homotopy class of the path  $\sigma$ . Given  $g \in G_0$  and  $y \in \mathfrak{X}_0$  choose a closed path  $\sigma^g$  in  $M_0$  representing  $g$ , choose a lift  $\sigma_y^g$  as above, then set  $\Phi_\mathcal{F}(g)(y) = \sigma_y^g(1)$ . This yields a well-defined group action of  $G_0$  on the Cantor space  $\mathfrak{X}_0$ .

The subgroup  $G_\ell^x \subset G_0 = \pi_1(M_0, x_0)$  is represented by closed paths in  $M_0$  with basepoint  $x_0$  and which admit a lift for the covering  $q_\ell: M_\ell \rightarrow M_0$  to a closed path with endpoint  $x_\ell$ . It follows that for the leaf  $L_x \subset \mathcal{S}_\mathcal{P}$  containing  $x \in \mathfrak{X}_0$ , we can also characterize  $G_\ell^x$  as the subgroup represented by those closed paths which admit a lift to  $L_x$  which start at  $x$ , and terminate at a point in  $L_x \cap \mathfrak{X}_\ell^x$ . Thus we have

$$(7) \quad G_\ell^x = \{g \in G_0 \mid \Phi_\mathcal{F}(g)(\mathfrak{X}_\ell^x) = \mathfrak{X}_\ell^x\}.$$

That is, the action  $\Phi_\mathcal{F}$  of  $g$  fixes the set  $\mathfrak{X}_\ell^x$ , possibly permuting points within this subset.

Let  $g \in G_0$  represent the coset  $[g]_\ell \in G_0/G_\ell$ . It follows from (7) that the image  $\mathfrak{X}_\ell^{x,g} = \Phi_\mathcal{F}(g)(\mathfrak{X}_\ell^x)$  of  $\mathfrak{X}_\ell^x$  under the action of  $g$  either coincides with  $\mathfrak{X}_\ell^x$  or it is disjoint from  $\mathfrak{X}_\ell^x$ . Thus the collection  $\{\mathfrak{X}_\ell^{x,g}\}_{g \in G}$  is a finite collection of disjoint clopen sets which cover  $\mathfrak{X}_0$ . Moreover, for all  $\ell' > \ell > 0$ , the collection of clopen sets  $\{\mathfrak{X}_{\ell'}^{x,g} \mid [g]_{\ell'} = gG_{\ell'}^x \in G_\ell^x/G_{\ell'}^x\}$  is a finite partition of  $X_\ell^x$ .

Given  $y \in \mathfrak{X}_0$  there exists a unique  $(g_\ell G_\ell) \in X_\infty^x$  so that  $y = \bigcap_{\ell \geq 0} \mathfrak{X}_\ell^{x,g_\ell}$ . Define  $\sigma_x: \mathfrak{X}_0 \rightarrow X_\infty^x$  by  $\sigma_x(y) = (g_\ell G_\ell)$ . The map  $\sigma_x$  is surjective, bijective and continuous, hence a homeomorphism. Define  $\tau_x = \sigma_x^{-1}: X_\infty^x \rightarrow \mathfrak{X}_0$  so that  $\tau_x(e_0) = x$ . The map  $\tau_x$  can be viewed as “coordinates” on  $\mathfrak{X}_0$  centered at the chosen basepoint  $x \in \mathfrak{X}_0$ . It follows from the construction of  $\tau_x$  that it commutes with the left  $G_0$ -actions  $\Phi_\mathcal{F}$  on  $\mathfrak{X}_0$  and  $\Phi_x$  on  $X_\infty^x$ .



The group chain (5) and the homeomorphism  $\tau_x$  depend on the choice of a point  $x \in \mathfrak{X}_0$ . For a different basepoint  $y \in \mathfrak{X}_0$  in the fibre over  $x_0$ , for each  $\ell > 0$  there exists  $g_\ell \in G_0$  such that  $y \in \mathfrak{X}_\ell^y \equiv \Phi_{\mathcal{F}(g_\ell)}(\mathfrak{X}_\ell^x)$ , and hence  $y = \bigcap_{\ell \geq 0} \mathfrak{X}_\ell^y$ . Then for each  $\ell > 0$ ,

define  $G_\ell^y = g_\ell G_\ell^x g_\ell^{-1}$  which consists of elements of  $G_0$  that leave the set  $\mathfrak{X}_\ell^y$  invariant. Let  $\mathcal{G}^y = \{G_\ell^y \mid \ell \geq 0\}$  be the resulting group chain, with corresponding inverse limit space  $X_\infty^y$ . Then the map  $\tau_y: X_\infty^y \rightarrow \mathfrak{X}_0$  gives coordinates on  $\mathfrak{X}_0$  centered at the chosen basepoint  $y \in \mathfrak{X}_0$ .

The composition  $\tau_y \circ \tau_x^{-1}: X_\infty^x \rightarrow X_\infty^y$  gives a topological conjugacy between the minimal Cantor actions  $(X_\infty^x, G_0, \Phi_x)$  and  $(X_\infty^y, G_0, \Phi_y)$ , and the composition  $\tau_y \circ \tau_x^{-1}$  can be viewed as a “change of coordinates”. Properties of the minimal Cantor action  $(X_\infty^x, G_0, \Phi_x)$  which are independent of the choice of these coordinates are thus properties of the topological type of  $\mathcal{S}_{\mathcal{P}}$ .

The group chains  $\mathcal{G}^y$  and  $\mathcal{G}^x$  are said to be *conjugate chains*. This notion forms an equivalence relation on group chains which was introduced by Fokkink and Oversteegen [23]. The properties of this equivalence relation were studied in depth in [19, 21].

The map  $\tau_x: X_\infty^x \rightarrow \mathfrak{X}_0$  is used to give the “odometer model” for the solenoid  $\mathcal{S}_{\mathcal{P}}$ . Let  $\tilde{M}_0$  denote the universal covering of the compact manifold  $M_0$ , and let  $(X_\infty^x, G_0, \Phi_x)$  be the minimal Cantor system associated to the presentation  $\mathcal{P}$  and the choice of a basepoint  $x \in \mathfrak{X}_0$ . Associated to the left action  $\Phi_x$  of  $G_0$  on  $X_\infty^x$  is a suspension space

$$(8) \quad \mathfrak{M}_\Phi = \tilde{M}_0 \times X_\infty^x / (z \cdot g^{-1}, y) \sim (z, \Phi_x(g)(y)) \quad \text{for } z \in \tilde{M}_0, g \in G_0, y \in X_\infty^x$$

which is a minimal matchbox manifold. This construction is a generalization of a standard technique for constructing smooth foliations, as discussed in [8, 9] for example.

Moreover, the suspension space  $\mathfrak{M}_\Phi$  of a minimal equicontinuous action  $\varphi$  has an inverse limit presentation, where all of the bonding maps between the coverings  $M_\ell \rightarrow M_0$  are derived from the universal covering map  $\tilde{\pi}: \tilde{M}_0 \rightarrow M_0$ . The following result is given in [12], and its proof is a consequence of the lifting property for maps between coverings:

**Theorem 2.3** *Let  $\mathcal{S}_{\mathcal{P}}$  be a weak solenoid with base space  $M_0$ . Then the suspension of the map  $\tau_x$  yields a foliated homeomorphism  $\tau_x^*: \mathfrak{M}_\Phi \rightarrow \mathcal{S}_{\mathcal{P}}$ .*

**COROLLARY 2.4** *The homeomorphism type of a weak solenoid  $\mathcal{S}_{\mathcal{P}}$  is completely determined by the base manifold  $M_0$  and the associated minimal Cantor system  $(X_\infty^x, G_0, \Phi_x)$ .*

We conclude this discussion of some basic geometry of weak solenoids, by recalling some properties of the holonomy groups of the foliations of weak solenoids. First, recall a basic result of Epstein, Millet and Tischler [22].

**Theorem 2.5** *Let  $(\mathfrak{X}, G, \Phi)$  be a given action, and suppose that  $\mathfrak{X}$  is a Baire space. Then the union of all  $x \in \mathfrak{X}$  such that the germinal holonomy group  $\text{Germ}(\Phi, x)$  at  $x$  is trivial forms a  $G_\delta$  subset of  $\mathfrak{X}$ .*

The main result in [22] is stated in terms of the germinal holonomy groups of leaves of a foliation, but an inspection of the proof shows that it applies directly to a general action  $(\mathfrak{X}, G, \Phi)$ .

We conclude by introducing the following important notion:

**DEFINITION 2.6** *The kernel of the group chain  $\mathcal{G}^x = \{G_\ell^x\}_{\ell \geq 0}$  is the subgroup  $K(\mathcal{G}^x) = \bigcap_{\ell \geq 0} G_\ell^x$ .*

For a weak solenoid  $\mathcal{S}_\mathcal{P}$  with choice of a basepoint  $x_0 \in M_0$  and fiber  $\mathfrak{X}_0 = \Pi_0^{-1}(x_0)$ , the kernel subgroup  $K(\mathcal{G}^x) \subset G_0$  may depend on the choice of the basepoint  $x \in \mathfrak{X}_0$ . The dependence of  $K(\mathcal{G}^x)$  on  $x$  is a natural aspect of the dynamics of the foliation  $\mathcal{F}_\mathcal{S}$  on  $\mathcal{S}_\mathcal{P}$ , when  $K(\mathcal{G}^x)$  is interpreted in terms of the topology of the leaves of  $\mathcal{F}_\mathcal{S}$  as follows.

The map  $\tau_x^*: \mathfrak{M}_\Phi \rightarrow \mathcal{S}_\mathcal{P}$  of Theorem 2.3 sends the quotient space  $\tilde{M}/K(\mathcal{G}^x)$  to the leaf  $L_x \subset \mathcal{S}_\mathcal{P}$  through  $x \in \mathfrak{X}_0$  in  $\mathcal{S}_\mathcal{P}$ , and so  $K(\mathcal{G}^x)$  is naturally identified with the fundamental group  $\pi_1(L_x, x)$ . The global holonomy homomorphism  $\Phi_{\mathcal{F}, x}: \pi_1(L_x, x) \rightarrow \text{Homeo}(\mathfrak{X}_0, x)$  of the leaf  $L_x$  in the suspension foliation  $\mathcal{F}_\mathcal{S}$  of  $\mathcal{S}_\mathcal{P}$  is then conjugate to the left action,  $\Phi_0: K(\mathcal{G}^x) \rightarrow \text{Homeo}(X_\infty^x, e_0)$ .

From the point of view of foliation theory, the leaves of  $\mathcal{F}_\mathcal{S}$  with holonomy are a “small” set by the proof of Theorem 2.5. There always exists leaves without holonomy, while there may exist leaves with holonomy, and so the fundamental groups of the leaves may vary accordingly. This aspect of the foliation dynamics of weak solenoids is discussed further in [21, Section 4.2].

### 3 Return equivalence

The conclusion of Theorem 2.3 is that a weak solenoid is homeomorphic to a suspension space (8) of an equicontinuous action on a Cantor space. In this section, we consider the notion of return equivalence between such suspension spaces.

Let  $\varphi: G \times \mathfrak{X} \rightarrow \mathfrak{X}$  be a minimal action on a Cantor space  $\mathfrak{X}$ . In order to give a precise definition of return equivalence, we introduce the pseudo $\star$ group associated to the action  $\varphi$ . A more general discussion of pseudo $\star$ groups can be found in the works [30] and [31, Section 2.4].

For each  $g \in G$  and open subset  $U \subset \mathfrak{X}$ , let  $\varphi^U(g): U \rightarrow V = \varphi(g)(U)$  denote the restricted homeomorphism. Then the pseudo $\star$ group associated to  $\varphi$  is the collection of maps

$$(9) \quad \Psi^*(\varphi, \mathfrak{X}) \equiv \{ \varphi^U(g) \mid U \subset \mathfrak{X} \text{ open}, g \in G \}.$$

The collection  $\Psi^*(\varphi, \mathfrak{X})$  is not a pseudogroup, as it does not satisfy the “gluing” condition on maps, but  $\Psi^*(\varphi, \mathfrak{X})$  does generate the usual pseudogroup  $\Psi(\varphi, \mathfrak{X})$  associated to the action  $\varphi$  on  $\mathfrak{X}$ .

Given an open subset  $W \subset \mathfrak{X}$ , define the restriction of  $\Psi^*(\varphi, \mathfrak{X})$  to  $W$ ,

$$\Psi^*(\varphi, W) = \{ \varphi^U(g) \mid U \subset W \text{ open}, g \in G, \varphi(g)(U) \subset W \}.$$

**DEFINITION 3.1** Let  $\varphi_i: G_i \times \mathfrak{X}_i \rightarrow \mathfrak{X}_i$  be minimal actions on Cantor spaces  $\mathfrak{X}_i$  for  $i = 1, 2$ . Then  $\varphi_1$  and  $\varphi_2$  are return equivalent if there exists non-empty open sets  $W_1 \subset \mathfrak{X}_1$  and  $W_2 \subset \mathfrak{X}_2$ , and a homeomorphism  $h: W_1 \rightarrow W_2$  which conjugates the restricted pseudo $\star$ group  $\Psi^*(\varphi_1, W_1)$  with the restricted pseudo $\star$ group  $\Psi^*(\varphi_2, W_2)$ .

It is an exercise to show that minimal suspension spaces  $\mathfrak{M}_{\varphi_1}$  and  $\mathfrak{M}_{\varphi_2}$  are return equivalent as foliated spaces, if and only if their associated global monodromy actions satisfy Definition 3.1.

We next introduce a notion which especially pertains to equicontinuous Cantor actions.

**DEFINITION 3.2** Let  $\varphi: G \times \mathfrak{X} \rightarrow \mathfrak{X}$  be an action on a Cantor space  $\mathfrak{X}$ . A non-empty clopen subset  $U \subset \mathfrak{X}$  is adapted to the action  $\varphi$  if for any  $g \in G$ ,  $\varphi(g)(U) \cap U \neq \emptyset$  implies that  $\varphi(g)(U) = U$ . It follows that

$$(10) \quad G_U = \{ g \in G \mid \varphi(g)(U) \cap U \neq \emptyset \}$$

is a subgroup of  $G$ .

**REMARK 3.3** For the action  $\Phi_X: G_0 \times X_\infty^x \rightarrow X_\infty^x$  of Lemma 2.2, for each  $\ell \geq 0$ , the set  $U = \mathfrak{X}_\ell^x$  is adapted with  $G_U = G_\ell^x$  as defined in (7). Note that if  $V \subset U \subset \mathfrak{X}$  are both adapted to an action  $\varphi: G \times \mathfrak{X} \rightarrow \mathfrak{X}$ , with associated groups  $G_V$  and  $G_U$ , then we have  $G_V = \{ g \in G_U \mid \varphi(g)(V) = (V) \}$ . Moreover, if there exists a descending chain of clopen adapted sets  $\{ U_\ell \subset \mathfrak{X} \mid \ell \geq 0 \}$  whose intersection is a point, then it is

an exercise to show that the minimal action  $\varphi$  is equicontinuous. On the other hand, it is also easy to construct examples of actions which are not equicontinuous but admit a proper adapted clopen subset  $U \subset \mathfrak{X}$ . For example, consider any minimal Cantor action  $\varphi_U: G_U \times U \rightarrow U$ , chose a non-trivial finite group  $H$  and set  $G = H \times G_U$ , then extend the action  $\varphi_U$  on  $U$  to  $\varphi: G \times \mathfrak{X} \rightarrow \mathfrak{X}$  acting factor-wise on the product space  $\mathfrak{X} = H \times U$ .

We next establish two technical lemmas which are key for the proofs of Theorems 1.3 and 1.5.

**LEMMA 3.4** *Let  $\varphi_i: G_i \times \mathfrak{X}_i \rightarrow \mathfrak{X}_i$  be minimal actions on Cantor spaces  $\mathfrak{X}_i$ , for  $i = 1, 2$ , and suppose there exists non-empty open sets  $W_i \subset \mathfrak{X}_i$  and a homeomorphism  $h: W_1 \rightarrow W_2$  which conjugates the restricted pseudo $\star$ groups  $\Psi^*(\varphi_1, W_1)$  and  $\Psi^*(\varphi_2, W_2)$ . Then a clopen subset  $U_1 \subset W_1$  is adapted to the action  $\varphi_1$  if and only if  $U_2 = h(U_1) \subset W_2$  is adapted to the action  $\varphi_2$ .*

**Proof** We show that  $U_2$  is adapted to the action of  $\varphi_2$ . The reverse implication follows similarly.

First note that  $U_1$  is an open subset of  $W_1$  and  $h$  is a homeomorphism, hence  $U_2$  is an open subset of  $W_2$  in the relative topology on  $\mathfrak{X}_2$  hence is an open subset of  $\mathfrak{X}_2$ . Also,  $U_2$  is compact as  $U_1$  is compact and all spaces are Hausdorff, thus  $U_2$  is a clopen subset of  $\mathfrak{X}_2$ .

Let  $g_2 \in G_2$  satisfy  $\varphi_2(g_2)(U_2) \cap U_2 \neq \emptyset$ . Let  $h^*: \Psi^*(\varphi_2, W_2) \rightarrow \Psi^*(\varphi_1, W_1)$  be the map induced by  $h: W_1 \rightarrow W_2$  on the restricted pseudo $\star$ groups. By assumption, this map is an isomorphism, and in particular  $h^*(\varphi_2^{U_2}(g_2)) \in \Psi^*(\varphi_1, W_1)$ . Hence, there exists  $g_1 \in G_1$  such that  $\varphi_1^{U_1}(g_1) = h^*(\varphi_2^{U_2}(g_2))$ . Thus,  $\varphi_1(g_1)(U_1) \cap U_1 \neq \emptyset$ . As  $U_1$  is adapted to the action of  $\varphi_1$  this implies that  $\varphi_1(g_1)(U_1) = U_1$ , which implies that  $\varphi_2(g_2)(U_2) \cap U_2 = U_2$  as was to be shown.  $\square$

**LEMMA 3.5** *Let  $\varphi: G \times \mathfrak{X} \rightarrow \mathfrak{X}$  be a minimal action on a Cantor space  $\mathfrak{X}$ , and  $U \subset \mathfrak{X}$  a clopen subset adapted to the action. Then the collection  $\mathcal{S}_U \equiv \{\varphi(g)(U) \mid g \in G\}$  forms a finite disjoint clopen partition of  $\mathfrak{X}$ .*

**Proof** We first show that the images form a disjoint partition. Suppose that for  $g_1, g_2 \in G$  we have  $\varphi(g_1)(U) \cap \varphi(g_2)(U) \neq \emptyset$ . Then  $\varphi(g_2^{-1}g_1)(U) \cap U \neq \emptyset$  hence  $\varphi(g_2^{-1}g_1)(U) = U$ . It follows that  $\varphi(g_1)(U) = \varphi(g_2)(U)$ . Each image  $\varphi(g_1)(U)$  is a clopen subset, and  $\mathfrak{X}$  is compact, so there are only a finite number of disjoint images, which completes the proof.  $\square$

Assume that  $\varphi: G \times \mathfrak{X} \rightarrow \mathfrak{X}$  is a minimal action on a Cantor space  $\mathfrak{X}$ , and  $U \subset \mathfrak{X}$  a clopen subset adapted to the action. Let  $p_U: \mathfrak{X} \rightarrow \mathcal{S}_U$  be the natural map to the elements of the partition of  $\mathfrak{X}$ , which exists by Lemma 3.5. Identify the collection  $\mathcal{S}_U$  with the quotient set  $G/G_U$  via the map  $q_U(\varphi(g)(U)) = gG_U \in G/G_U$ , then the composition  $\pi_U = q_U \circ p_U: \mathfrak{X} \rightarrow G/G_U$  is  $G$ -equivariant.

Given an action  $\varphi: G \times \mathfrak{X} \rightarrow \mathfrak{X}$ , we next construct the suspension foliated space for the action. Let  $M$  be a compact manifold without boundary, with a basepoint  $x_0 \in M$  and let  $G = \pi_1(M, x_0)$  denote its fundamental group based at  $x_0$ . Let  $\tilde{\pi}: \tilde{M} \rightarrow M$  denote the universal covering space of  $M$ , defined by endpoint-fixed homotopy classes of paths in  $M$  with initial point  $x_0$ . Then  $G$  acts on  $\tilde{M}$  on the right by deck transformations. Define the quotient foliated space

$$(11) \quad \mathfrak{M}_\varphi = (\tilde{M} \times \mathfrak{X}) / \{(x \cdot \gamma, w) \sim (z, \varphi(\gamma) \cdot w) \mid z \in \tilde{M}, w \in \mathfrak{X}, \gamma \in G\}.$$

Let  $\pi: \mathfrak{M}_\varphi \rightarrow M$  be the map induced by the projection  $\tilde{\pi}: \tilde{M} \times \mathfrak{X} \rightarrow \tilde{M}$  onto the first factor.

Now assume that the action  $\varphi$  admits a proper adapted clopen subset  $U \subset \mathfrak{X}$ . Then we define

$$(12) \quad M_U = (\tilde{M} \times G/G_U) / \{(x \cdot g, w) \sim (z, g \cdot w) \mid z \in \tilde{M}, w \in G/G_U, g \in G\}.$$

Note that  $M_U$  is naturally identified with the finite covering space  $\tilde{M}/G_U$  of  $M$  associated to the subgroup  $G_U \subset G$ . Let  $x_U \in M_U$  be the basepoint associated with the identity coset of  $G/G_U$ .

The quotient map  $\pi_U: \mathfrak{X} \rightarrow G/G_U$  induces a quotient map  $\Pi_U: \mathfrak{M}_\varphi \rightarrow M_U$  of suspension spaces, with  $U = \Pi_U^{-1}(x_U) \subset \mathfrak{X}$ , and there is a commutative diagram:

$$(13) \quad \begin{array}{ccc} \mathfrak{M}_\varphi & & \\ \pi \downarrow & \searrow \Pi_U & \\ M & \xleftarrow{\pi_{G_U}} & M_U \end{array}$$

Note that the above construction applies to any minimal action with a proper adapted clopen subset. In the case where  $U = \mathfrak{X}_\ell^x$  for an odometer action  $\varphi = \Phi_x: G_0 \times X_\infty^x \rightarrow X_\infty^x$  and  $\ell > 0$ , then  $G_U = G_\ell^x$  as in (7) and the map fibration  $\Pi_U$  is the same as the fibration  $\Pi_\ell$  defined following (3).

We can now give a result which is a key observation for the proofs of Theorems 1.3 and 1.5. For  $i = 1, 2$ , let  $\varphi_i: G_i \times \mathfrak{X}_i \rightarrow \mathfrak{X}_i$  be a minimal action on the Cantor space  $\mathfrak{X}_i$ . Let  $M_i$  be a compact manifold without boundary, with basepoint  $x_i \in M_i$

and  $G_i = \pi_1(M_i, x_i)$  its fundamental group based at  $x_i$ . Assume that the actions  $\varphi_1$  and  $\varphi_2$  return equivalent, so there exists open sets  $W_i \subset \mathfrak{X}_i$  and a homeomorphism  $h: W_1 \rightarrow W_2$  which conjugates the restricted pseudo $\star$ group  $\Psi^*(\varphi_1, W_1)$  with the restricted pseudo $\star$ group  $\Psi^*(\varphi_2, W_2)$ .

Let  $U_1 \subset W_1$  be a clopen subset which is adapted to the action  $\varphi_1$  then by Lemma 3.4 the image  $U_2 = h(U_1)$  is a clopen subset adapted to the action  $\varphi_2$ . For  $i = 1, 2$ , let

$$G_{U_i} = \{g \in G_i \mid \varphi_i(g)(U_i) = U_i\} \subset G_i$$

be the stabilizer group of  $U_i$  for the action  $\varphi_i$ .

The action  $\varphi_i$  induces a homomorphism  $\varphi_{U_i}: G_{U_i} \rightarrow \Lambda_i \subset \text{Homeo}(U_i)$  onto a subgroup  $\Lambda_i$ . Then the inverse of the restriction  $h_{U_1}: U_1 \rightarrow U_2$  induces an isomorphism  $\lambda_h: \Lambda_1 \rightarrow \Lambda_2$ .

Let  $\pi_{G_{U_i}}: M_{U_i} \rightarrow M_i$  be the finite covering associated to  $G_{U_i}$  with basepoint  $x_{U_i} \in M_{U_i}$  over  $x_i$ . A homeomorphism  $f: M_{U_1} \rightarrow M_{U_2}$  is said to *realize*  $\lambda_h$  if the following diagram commutes:

$$(14) \quad \pi_1(M_{U_1}, x_{U_1}) \quad = \quad \begin{array}{ccc} G_{U_1} & \xrightarrow{f\#} & G_{U_2} \\ \varphi_{U_1} \downarrow & & \downarrow \varphi_{U_2} \\ \Lambda_1 & \xrightarrow{\lambda_h} & \Lambda_2 \end{array} \quad = \quad \pi_1(M_{U_2}, x_{U_2})$$

By (13) we can represent  $\mathfrak{M}_i$  as a suspension space over  $M_{U_i}$  with basepoint fiber  $U_i$  and monodromy action  $\varphi_{U_i}: G_{U_i} \rightarrow \text{Homeo}(U_i)$ . Let  $\tilde{f}: \tilde{M}_{U_1} \rightarrow \tilde{M}_{U_2}$  denote the lift of  $f$  to the universal covering spaces. Then the product map

$$(15) \quad \tilde{f} \times h: \tilde{M}_1 \times U_1 \rightarrow \tilde{M}_{U_2} \times U_2$$

is a homeomorphism, and intertwines the diagonal actions of  $G_1$  and  $G_2$ , so descends to a homeomorphism between  $\mathfrak{M}_{\varphi_1}$  and  $\mathfrak{M}_{\varphi_2}$ . We have thus shown:

**PROPOSITION 3.6** *Suppose there exists a homeomorphism  $f: M_{U_1} \rightarrow M_{U_2}$  which realizes the isomorphism  $\lambda_h: \Lambda_1 \rightarrow \Lambda_2$  between the groups of fiber automorphisms induced by return equivalence. Then the suspension spaces  $\mathfrak{M}_{\varphi_1}$  and  $\mathfrak{M}_{\varphi_2}$  are homeomorphic.*

## 4 Proofs of main theorems

In this section, we use Proposition 3.6 to obtain proofs of Theorems 1.3 and 1.5. For  $i = 1, 2$ , let  $\mathfrak{M}_i$  be a matchbox manifold homeomorphic to a weak solenoid  $\mathcal{S}_{\mathcal{P}_i}$  defined

by a presentation

$$(16) \quad \mathcal{P}_i = \{p_{i,\ell+1}: M_{i,\ell+1} \rightarrow M_{i,\ell} \mid \ell \geq 0\},$$

where the base manifolds  $M_{1,0}$  and  $M_{2,0}$  both have dimension  $n \geq 1$ . Let  $\Pi_{\mathcal{P}_i}: \mathcal{S}_{\mathcal{P}_i} \rightarrow M_{i,0}$  denote the projection onto the base manifold.

Let  $x_{i,0} \in M_{i,0}$  be a basepoint, let  $G_{i,0} = \pi_1(M_{i,0}, x_{i,0})$  and set  $\mathfrak{X}_{\mathcal{P}_i} = \Pi_{\mathcal{P}_i}^{-1}(x_{i,0})$ .

The assumption that the holonomy pseudogroups defined by the foliations on  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are return equivalent implies that the foliations of  $\mathcal{S}_{\mathcal{P}_1}$  and  $\mathcal{S}_{\mathcal{P}_2}$  are return equivalent. This in turn implies that the global monodromy actions

$$\Phi_{\mathcal{P}_1}: G_{1,0} \times \mathfrak{X}_1 \rightarrow \mathfrak{X}_1, \quad \Phi_{\mathcal{P}_2}: G_{2,0} \times \mathfrak{X}_2 \rightarrow \mathfrak{X}_2$$

are return equivalent in the sense of Definition 3.1. That is, there exists open sets  $W_1 \subset \mathfrak{X}_1$  and  $W_2 \subset \mathfrak{X}_2$  and a homeomorphism  $h: W_1 \rightarrow W_2$  which conjugates the restricted pseudo $\star$ group  $\Psi^*(\Phi_{\mathcal{P}_1}, W_1)$  with the restricted pseudo $\star$ group  $\Psi^*(\Phi_{\mathcal{P}_2}, W_2)$ .

#### 4.1 Odometer models

Assume we are given weak solenoids  $\mathcal{S}_{\mathcal{P}_1}$  and  $\mathcal{S}_{\mathcal{P}_2}$ . Then as shown in Theorem 2.3, we can assume that the weak solenoids  $\mathcal{S}_{\mathcal{P}_i}$  are homeomorphic to the suspension of odometer actions as in (8). To fix notation, recall the construction of the odometer actions. Choose a basepoint  $x \in W_1 \subset \mathfrak{X}_1$ , and set  $y = h(x) \in W_2 \subset \mathfrak{X}_2$ . Then form the group chains corresponding to the presentations  $\mathcal{P}_1$  at  $x$ , and  $\mathcal{P}_2$  at  $y$ :

$$(17) \quad \mathcal{G}_{\mathcal{P}_1}^x \equiv \{G_{1,\ell}^x\}_{\ell \geq 0} = \{G_{1,0} \supset G_{1,1} \supset G_{1,2} \supset \cdots \supset G_{1,\ell} \supset \cdots\}$$

$$(18) \quad \mathcal{G}_{\mathcal{P}_2}^y \equiv \{G_{2,\ell}^y\}_{\ell \geq 0} = \{G_{2,0} \supset G_{2,1} \supset G_{2,2} \supset \cdots \supset G_{2,\ell} \supset \cdots\}.$$

Let  $\Phi_1: G_{1,0} \times X_{1,\infty} \rightarrow X_{1,\infty}$  be the odometer formed from the chain  $\mathcal{G}_{\mathcal{P}_1}^x$  and let  $\tau_{1,x}: X_{1,\infty} \rightarrow \mathfrak{X}_{\mathcal{P}_1}$  be the  $G_{1,0}$ -equivariant homeomorphism constructed in Section 2. Then we have  $\tau_{1,x}(e_{1,0}) = x$  where  $e_{1,0} = (eG_{1,\ell}^x)$  is the basepoint of  $X_{1,\infty}$ . Moreover, recall from (7) that for  $\ell > 0$ , we have

$$G_{1,\ell}^x = \{g \in G_{1,0} \mid \Phi_{\mathcal{P}_1}(g)(\mathfrak{X}_{i,\ell}) = \mathfrak{X}_{i,\ell}\}.$$

Similarly, let  $\Phi_2: G_{2,0} \times X_{2,\infty} \rightarrow X_{2,\infty}$  be the odometer formed from the chain  $\mathcal{G}_{\mathcal{P}_2}^y$  and let  $\tau_{2,y}: X_{2,\infty} \rightarrow \mathfrak{X}_{\mathcal{P}_2}$  be the corresponding  $G_{2,0}$ -equivariant homeomorphism with  $\tau_{2,y}(e_{2,0}) = y$ .

The preimage  $\tau_{1,x}^{-1}(\mathfrak{X}_{1,\ell}^x)$  is identified with the clopen set

$$(19) \quad U_{1,\ell} = \{(g_k G_{1,k}^x) \mid k \geq 0, g_0 = g_1 = \cdots = g_\ell \in G_{1,\ell}^x\} \subset X_{1,\infty}.$$

The collection  $\{\mathfrak{X}_{1,\ell}^x \mid \ell > 0\}$  is a neighborhood basis around the basepoint  $x \in W_1$ , so there exists  $\ell_1 > 0$  such that  $U_{1,\ell} \subset \tau_{1,x}^{-1}(W_1)$  for  $\ell \geq \ell_1$ . Set  $U_1 = U_{1,\ell_1}$  then the clopen subset  $U_1$  is adapted to the action of  $\Phi_1$  with stabilizer subgroup  $G_{U_1} = G_{\ell_1}^x$  by Remark 3.3. Thus, the action  $\Phi_1$  induces an epimorphism  $\Phi_{U_1}: G_{U_1} \rightarrow \Lambda_1 \subset \text{Homeo}(U_1)$ .

The image  $h \circ \tau_1(U_1) \subset \mathfrak{X}_2$  is a clopen subset adapted to the action of  $\Phi_{\mathcal{P}_2}$  by Lemma 3.4. Set  $U_2 = \tau_2^{-1} \circ h \circ \tau_1(U_1) \subset X_{2,\infty}$ , which is a clopen set adapted to the action  $\Phi_2$ . Let  $G_{U_2} \subset G_{2,0}$  be the stabilizer group of  $U_2$ . Then the action  $\Phi_2$  induces an epimorphism  $\Phi_{U_2}: G_{U_2} \rightarrow \Lambda_2 \subset \text{Homeo}(U_2)$ . Moreover, the homeomorphism  $\tau_2^{-1} \circ h \circ \tau_1: U_1 \rightarrow U_2$  induces an isomorphism  $\lambda_h: \Lambda_1 \rightarrow \Lambda_2$ .

**REMARK 4.1** Before continuing with the proofs of the main theorems, we recall an aspect of the equivalence of weak solenoids from [23] and which is discussed in detail in [19]. The basepoint  $e_{2,0} \in V$  so there exists  $\ell_2 > 0$  such that  $V_{2,\ell} \subset V$  for  $\ell \geq \ell_2$ , where  $V_{2,\ell}$  is defined as in (19). For the action  $\Phi_2$  the group  $G_{2,\ell}$  stabilizes the clopen set  $V_{2,\ell}$  and hence also stabilizes  $V$ . However, it need not be the case that  $G_V$  is equal to one of the subgroups  $G_{2,\ell}$ . It is only possible to conclude that there exists some  $\ell \geq \ell_2$  for which  $G_{2,\ell} \subset G_V$ . This corresponds to the fact that homeomorphic weak solenoids are defined by group chains which are equivalent in the sense of [19, 23], which is to say that their group chains are interlaced up to isomorphism.

By Lemma 3.5, the collection  $\mathcal{S}_2 \equiv \{\Phi_2(g)(U_2) \mid g \in G_{2,0}\}$  is a clopen partition of  $X_{2,\infty}$ . We will apply Proposition 3.6 to show that the suspension spaces  $\mathfrak{M}_{\Phi_1}$  and  $\mathfrak{M}_{\Phi_2}$  are homeomorphic. First, we must construct a map of fundamental groups  $f_*: G_{U_1} \rightarrow G_{U_2}$  so that the diagram (14) is satisfied, and then construct a homeomorphism  $f: M_{U_1} \rightarrow M_{U_2}$  which induces the map  $f_*$ .

## 4.2 Proof of Theorem 1.3

For  $i = 1, 2$ , we are given that  $\mathcal{S}_{\mathcal{P}_i}$  is a toroidal solenoid whose base has dimension  $n$ , so  $M_{i,0} = \mathbb{T}^n$  and hence  $G_{i,0} \cong \mathbb{Z}^n$ . The manifold  $M_{U_i}$  is a covering of  $M_{i,0}$  hence is also a torus, with fundamental group which we identify with  $\mathbb{Z}^n$ . Introduce the subgroups

$$(20) \quad K_i = \ker\{\Phi_{U_i}: G_{U_i} \rightarrow \Lambda_i \subset \text{Homeo}(U_i)\} \subset \mathbb{Z}^n.$$



Each  $K_i$  is a free abelian subgroup with rank  $0 \leq r_i < n$ , and there is a commutative diagram:

$$(21) \quad \begin{array}{ccccc} K_1 & \hookrightarrow & G_{U_1} & \xrightarrow{\Phi_{U_1}} & \Lambda_1 \\ & & \downarrow f_* & & \downarrow \cong \lambda_h \\ K_2 & \hookrightarrow & G_{U_2} & \xrightarrow{\Phi_{U_2}} & \Lambda_2 \end{array}$$

**LEMMA 4.2** *There exists a map  $f_*: G_{U_1} \rightarrow G_{U_2}$  such that the diagram (21) commutes.*

**Proof** This follows because  $G_{U_1} \cong G_{U_2} \cong \mathbb{Z}^n$  are free abelian groups, hence projective  $\mathbb{Z}$ -modules. We give the details of the construction of the map  $f_*$ . Let  $\{a_1, \dots, a_d\} \subset \Lambda_1$  be a minimal set of generators for  $\Lambda_1$ , then  $\{\lambda_h(a_1), \dots, \lambda_h(a_d)\} \subset \Lambda_2$  is a minimal set of generators for  $\Lambda_2$ .

Choose  $\{g_1, \dots, g_d\} \subset G_{U_1}$  so that  $a_i = \Phi_{U_1}(g_i)$  for  $1 \leq i \leq d$ . The kernel  $K_1$  is free abelian, so we can extend this set to a basis  $\{g_1, \dots, g_n\}$  for  $G_{U_1}$  where  $\Phi_{U_1}(g_i)$  is the identity for  $d < i \leq n$ .

Choose elements  $\{g'_1, \dots, g'_d\} \subset G_{U_2}$  so that  $\lambda_h(a_i) = \Phi_{U_2}(g'_i)$  for  $1 \leq i \leq d$ . Note that both  $K_1$  and  $K_2$  are free abelian of rank  $n - d$ , so we can extend this set to a basis  $\{g'_1, \dots, g'_n\}$  for  $G_{U_2}$  where  $\Phi_{U_2}(g'_i)$  is the identity for  $d < i \leq n$ .

Define the group isomorphism  $f_*: G_{U_1} \rightarrow G_{U_2}$  by specifying  $f_*(g_i) = g'_i$  for  $1 \leq i \leq n$ . Then the diagram (21) commutes by our choices of these bases.  $\square$

Finally, to complete the proof of Theorem 1.3, observe that  $f_*$  extends to a linear map  $\widehat{f_*}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ , and so induces a diffeomorphism of the quotient spaces  $f: \mathbb{T}^n \rightarrow \mathbb{T}^n$ . Then the hypotheses of Proposition 3.6 are satisfied.

### 4.3 Proof of Theorem 1.5

The proof of Theorem 1.5 uses the geometric hypotheses on the foliations of the weak solenoids  $\mathcal{S}_{\mathcal{P}_i}$  to show the existence of the map  $f_*$  such that the diagram (21) commutes, in place of the group extension arguments in the proof of Lemma 4.2. In particular, we assume that the foliations on  $\mathcal{S}_{\mathcal{P}_1}$  and  $\mathcal{S}_{\mathcal{P}_2}$  each contain a dense leaf which is simply connected. By the results of Section 4.1, we can assume that  $\mathcal{S}_{\mathcal{P}_1}$  and  $\mathcal{S}_{\mathcal{P}_2}$  are represented as suspensions of odometer actions, and thus it suffices to show that the hypotheses of Proposition 3.6 are satisfied.

We assume that the odometer actions  $\Phi_i: G_{i,0} \times X_{i,\infty} \rightarrow X_\infty$  are return equivalent, for  $i = 1, 2$ , and that open subsets  $W_i \subset X_{i,0}$  are chosen so that the restricted pseudo $\star$ group  $\Psi^*(\Phi_1, W_1)$  is conjugate to the restricted pseudo $\star$ group  $\Psi^*(\Phi_2, W_2)$ . Then let  $U_i \subset W_i$  be chosen as above, with a homeomorphism  $h: U_1 \rightarrow U_2$  conjugating the restricted actions  $\Phi_{U_i}: G_{U_i} \rightarrow \Lambda_i \subset \text{Homeo}(U_i)$ .

Let  $K_i \subset G_{U_i}$  denote the kernel of the map  $\Phi_{U_i}$ , and for  $z \in U_i$  define:

$$(22) \quad K_i(z) = \{g \in G_{U_i} \mid \Phi_{U_i}(g)(z) = z\}.$$

Observe that  $K_i \subset K_i(z)$  for all  $z \in U_i$ .

By the definition (11) of the suspension space  $\mathfrak{M}_{\Phi_{U_i}}$  the leaf  $L_z \subset \mathfrak{M}_{\Phi_{U_i}}$  defined by the point  $z$  is homeomorphic to the covering  $\tilde{M}_i/K_i(z) \rightarrow M_i$ . By assumption, for each  $i = 1, 2$  there exists  $z \in U_i$  so that  $L_z$  is simply connected, which implies that  $K_i(z)$  is the trivial group, which implies that the kernel  $K_i$  is also the trivial group. Thus, the map  $\Phi_{U_i}: G_{U_i} \rightarrow \Lambda_i$  is an isomorphism. Define the map

$$(23) \quad f_* \equiv \Phi_{U_2}^{-1} \circ \lambda_h \circ \Phi_{U_1}: G_{U_1} \rightarrow G_{U_2}$$

which is an isomorphism such that the diagram (21) commutes.

By the hypotheses of Theorem 1.5 the manifolds  $M_1$  and  $M_2$  are both strongly Borel, hence their finite coverings  $M_{U_1}$  and  $M_{U_2}$  satisfy the Borel Conjecture. The map  $f_*$  induces a homotopy equivalence between them, as both have contractible universal covering spaces. Then by the solution of the Borel Conjecture for these spaces, there exists a homeomorphism  $f: M_{U_1} \rightarrow M_{U_2}$  which induces the map  $f_*$  on their fundamental groups. This completes the proof of Theorem 1.5.

**REMARK 4.3** Note that the choice of the clopen set  $U_i$  in the above proofs can be chosen to have arbitrarily small diameter, and hence the degree of the corresponding covering map  $\pi_{U_i}: M_{U_i} \rightarrow M_i$  in (13) can be chosen to be arbitrarily large. As remarked in [18], the homeomorphism  $f$  that is obtained from the solutions of the Borel Conjecture can be assumed to be smooth for a sufficiently large finite covering. It follows that the homeomorphism  $h: \mathcal{S}_{\mathcal{P}_1} \rightarrow \mathcal{S}_{\mathcal{P}_2}$  obtained from Proposition 3.6 can be chosen to be smooth along leaves.

## 5 Examples and counter-examples

In this section, we give several examples to illustrate the necessity of the hypotheses of Theorem 1.5. We first recall a classical result, the classification of Vietoris solenoids

of dimension one. We then consider extensions of this construction to solenoids with dimension  $n \geq 2$  and give examples of solenoids which are return equivalent but not homeomorphic. These examples are essentially the simplest possible constructions. Many other variants on their construction are clearly possible, especially for solenoids of dimensions greater than two, as briefly discussed in Section 5.3.

### 5.1 Vietoris solenoids

A *Vietoris solenoid* [17, 46] is a 1-dimensional solenoid  $\mathcal{S}_{\mathcal{P}}$ , where each  $M_{\ell}$  is a circle, and each  $p_{\ell}: \mathbb{S}^1 \rightarrow \mathbb{S}^1$  in the presentation  $\mathcal{P}$  is an orientation preserving covering map of degree  $m_{\ell} \geq 2$ . Let  $\vec{m} = \{m_1, m_2, \dots\}$  be the list of covering degrees for  $\mathcal{P}$ . Then  $\mathcal{S}_{\mathcal{P}}$  is also called an  $\vec{m}$ -adic solenoid of dimension one, and denoted by  $\mathcal{S}(\vec{m})$ .

Let  $\vec{m} = \{m_{\ell} \mid \ell \geq 1\}$  denote a sequence of positive integers with each  $m_i \geq 2$ . Set  $m_0 = 1$ , then define the profinite group

$$(24) \quad \mathfrak{G}_{\vec{m}} \stackrel{\text{def}}{=} \varprojlim \{ q_{\ell+1}: \mathbb{Z}/m_1 \cdots m_{\ell+1}\mathbb{Z} \rightarrow \mathbb{Z}/m_0 m_1 \cdots m_{\ell}\mathbb{Z} \mid \ell \geq 1 \} \\ = \varprojlim \left\{ \mathbb{Z}/\mathbb{Z} \xleftarrow{m_1} \mathbb{Z}/m_1\mathbb{Z} \xleftarrow{m_2} \mathbb{Z}/m_1 m_2\mathbb{Z} \xleftarrow{m_3} \mathbb{Z}/m_1 m_2 m_3\mathbb{Z} \xleftarrow{m_4} \cdots \right\}$$

where  $q_{\ell+1}$  is the quotient map of degree  $m_{\ell+1}$ . Each of the profinite groups  $\mathfrak{G}_{\vec{m}}$  contains a copy of  $\mathbb{Z}$  embedded as a dense subgroup by  $z \rightarrow ([z]_0, [z]_1, \dots, [z]_k, \dots)$ , where  $[z]_k$  corresponds to the class of  $z$  in the quotient group  $\mathbb{Z}/m_0 \cdots m_k\mathbb{Z}$ . There is a homeomorphism  $a_{\vec{m}}: \mathfrak{G}_{\vec{m}} \rightarrow \mathfrak{G}_{\vec{m}}$  given by “addition of 1” in each finite factor group. The resulting action of  $\mathbb{Z}$  is denoted by  $\Phi_{\vec{m}}: \mathbb{Z} \times \mathfrak{G}_{\vec{m}} \rightarrow \mathfrak{G}_{\vec{m}}$ . The dynamics of  $a_{\vec{m}}$  acting on  $\mathfrak{G}_{\vec{m}}$  is referred to as an *adding machine*, or equivalently as a (classical) *odometer*. We then have the standard result:

**PROPOSITION 5.1** *The Vietoris solenoid  $\mathcal{S}(\vec{m})$  is homeomorphic to the suspension  $\mathfrak{M}_{\Phi_{\vec{m}}}$  of the odometer action  $\Phi_{\vec{m}}$  with base manifold  $M_0 = \mathbb{S}^1$ .*

Two Vietoris solenoids  $\mathcal{S}_{\mathcal{P}}$  and  $\mathcal{S}_{\mathcal{Q}}$  are homeomorphic if and only if their presentations  $\mathcal{P}$  and  $\mathcal{Q}$  yield group chains as in (5) which are equivalent. As all of these are chains of subgroups of the fundamental group  $\mathbb{Z}$  of  $\mathbb{S}^1$ , the equivalence problem for these chains reduces to giving conditions on the sequences of integer covering degrees in  $\mathcal{P}$  and  $\mathcal{Q}$  which imply equivalence of the chains. There are two invariants of sequences which arise in the classification problem. First, consider the function which counts the total number of occurrences of a given prime in the sequence of integers  $\vec{m}$ .

**DEFINITION 5.2** Given a sequence of positive integers  $\vec{m}$  as above, let  $C_{\vec{m}}$  denote the function from the set of prime numbers to the set of extended natural numbers  $\{0, 1, 2, \dots, \infty\}$  given by

$$C_{\vec{m}}(p) = \sum_1^{\infty} m_i(p),$$

where  $m_i(p)$  is the power of the prime  $p$  in the prime factorization of  $m_i$ .

That is,  $C_{\vec{m}}(p) = k$  means that the prime  $p$  occurs a total of  $k$  times in the prime factorization of the integers in the sequence  $\vec{m}$ .

**Theorem 5.3** The Vietoris solenoids  $\mathcal{S}(\vec{m})$  and  $\mathcal{S}(\vec{n})$  are homeomorphic as bundles over the base manifold  $\mathbb{S}^1$  if and only if  $C_{\vec{m}}(p) = C_{\vec{n}}(p)$  for all primes  $p$ .

Next, we recall the notion of “tail equivalence” on sequences. This notion was introduced by Bing in [6], and plays a basic role in the study of return equivalence for Vietoris solenoids in [1].

**DEFINITION 5.4** Two infinite sets of integers,  $\vec{m} = \{m_\ell \mid \ell \geq 1\}$  and  $\vec{n} = \{n_\ell \mid \ell \geq 1\}$ , are said to be tail equivalent, and we write  $\vec{m} \sim \vec{n}$ , if there exists cofinite subsequences  $\vec{m}_* \subset \vec{m}$  and  $\vec{n}_* \subset \vec{n}$  which are in bijective correspondence.

The following observation is a direct consequence of Definitions 5.2 and 5.4.

**LEMMA 5.5** Two sequences of integers  $\vec{m}$  and  $\vec{n}$  as above are tail equivalent if and only if the following two conditions hold:

- (1) For all but finitely many primes  $p$ ,  $C_{\vec{m}}(p) = C_{\vec{n}}(p)$ , and
- (2) for all primes  $p$ ,  $C_{\vec{m}}(p) = \infty$  if and only if  $C_{\vec{n}}(p) = \infty$ .

The classification of Vietoris solenoids up to homeomorphism by Bing [6] and McCord [35], and the study of return equivalence by Aarts and Fokkink in [1] yields:

**Theorem 5.6** [35, 1] The Vietoris solenoids  $\mathcal{S}(\vec{m})$  and  $\mathcal{S}(\vec{n})$  are homeomorphic if and only if they are return equivalent, if and only if  $\vec{m}$  and  $\vec{n}$  are tail equivalent.

## 5.2 $\vec{m}$ -adic solenoids of dimension two

Let  $\Sigma_g$  be a closed surface of genus  $g \geq 1$ , which is obtained by attaching  $g$  torus handles  $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$  to the 2-sphere  $\mathbb{S}^2$ . For example,  $\Sigma_1$  is homeomorphic to the 2-torus  $\mathbb{T}^2$ . Pick a basepoint  $x_0 \in \Sigma_g$  and let  $G_0 = \pi_1(\Sigma_g, x_0)$  be the fundamental group. Choose an epimorphism  $a: G_0 \rightarrow \mathbb{Z}$ , which corresponds to a non-trivial class  $[a] \in H^1(\Sigma_g; \mathbb{Z})$  in integral homology.

Let  $\vec{m} = \{m_\ell \mid \ell \geq 1\}$  denote a sequence of integers with each  $m_i \geq 2$ , and form the profinite  $\vec{m}$ -adic group  $\mathfrak{G}_{\vec{m}}$  as in (24). Let  $\Phi_{\vec{m}}$  denote the odometer action of  $\mathbb{Z}$  described above. Extend this to an action of  $G_0$

$$(25) \quad \Phi_{\vec{m}}^a: G_0 \times \mathfrak{G}_{\vec{m}} \rightarrow \mathfrak{G}_{\vec{m}} \quad : \quad \Phi_{\vec{m}}^a(g)(x) = \Phi_{\vec{m}}(a(g))(x), \quad g \in G_0, \quad x \in \mathfrak{G}_{\vec{m}}$$

**DEFINITION 5.7** *The  $\vec{m}$ -adic surface  $\mathfrak{M}(\Sigma_g, a, \vec{m})$  is the suspension space (11) associated to the action  $\Phi_{\vec{m}}^a$  with base  $\Sigma_g$ .*

We note a consequence of the construction of  $\mathfrak{M}(\Sigma_g, a, \vec{m})$ , which follows immediately from the fact that the action  $\Phi_{\vec{m}}^a$  is induced from the action  $\Phi_{\vec{m}}$  and the results of [1]:

**PROPOSITION 5.8** *Given closed orientable surfaces  $\Sigma_{g_1}$  and  $\Sigma_{g_2}$  of genus  $g_i \geq 1$  for  $i = 1, 2$ , epimorphisms  $a_i: G_{i,0} \rightarrow \mathbb{Z}$  and sequences  $\vec{m}$  and  $\vec{n}$ , then  $\mathfrak{M}(\Sigma_{g_1}, a_1, \vec{m})$  is return equivalent to  $\mathfrak{M}(\Sigma_{g_2}, a_2, \vec{n})$  if and only if  $\vec{m}$  and  $\vec{n}$  are tail equivalent.*

Finally, we consider the problem, given *adic*-surfaces  $\mathfrak{M}(\Sigma_{g_1}, a_1, \vec{m})$  and  $\mathfrak{M}(\Sigma_{g_2}, a_2, \vec{n})$  such that  $\vec{m}$  is tail equivalent to  $\vec{n}$ , when are they homeomorphic as matchbox manifolds? First, consider the case of genus  $g_1 = g_2 = 1$  so that  $\Sigma_{g_1} = \Sigma_{g_2} = \mathbb{T}^2$ . Then Theorem 1.3 and Proposition 5.8 yield:

**Theorem 5.9** *The *adic*-surfaces  $\mathfrak{M}(\mathbb{T}^2, a_1, \vec{m})$  and  $\mathfrak{M}(\mathbb{T}^2, a_2, \vec{n})$  are homeomorphic if and only if  $\vec{m}$  and  $\vec{n}$  are tail equivalent.*

For the general case of *adic*-surfaces where at least one base manifold has higher genus, we next give examples of weak solenoids which are return equivalent but not homeomorphic. Note that in these examples, their base manifolds are compact surfaces hence are strongly Borel, but all their leaves have non-trivial fundamental groups, so the hypotheses of Theorem 1.5 are not satisfied.

**Theorem 5.10** *Let  $\mathfrak{M}_1 = \mathfrak{M}(\Sigma_{g_1}, a_1, \vec{m})$  and  $\mathfrak{M}_2 = \mathfrak{M}(\Sigma_{g_2}, a_2, \vec{n})$  be *adic*-surfaces.*

- (1) *If  $g_1 > 1$  and  $g_2 = 1$ , then  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are never homeomorphic.*

- (2) If  $g_1 = g_2 > 1$  and  $a_1 = a_2$ , then  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are homeomorphic if and only if  $C_{\vec{m}} = C_{\vec{n}}$ .
- (3) If  $g_1 = g_2 > 1$  and  $a_1 = a_2$ , then there exists  $\vec{m} \sim \vec{n}$  but  $\mathfrak{M}_1 \not\approx \mathfrak{M}_2$ .

**Proof** First, recall that the Euler characteristic of the closed surface  $\Sigma_g$  of genus  $g \geq 1$  has Euler characteristic  $\chi(\Sigma_g) = 2 - 2g$ , and the Euler characteristic is multiplicative for coverings. That is, if  $\Sigma'_g$  is a  $k$ -fold covering of  $\Sigma_g$  then  $\chi(\Sigma'_g) = k \cdot \chi(\Sigma_g)$ . In particular, for  $g > 1$ , a proper covering  $\Sigma'_g$  of  $\Sigma_g$  is never homeomorphic to  $\Sigma_g$ .

Next, each of the spaces  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  is homeomorphic to an inverse limit as in (2),

$$(26) \quad \mathfrak{M}_1 = \mathfrak{M}(\Sigma_{g_1}, a_1, \vec{m}) \cong \varprojlim \{f_{\ell+1}: M_{\ell+1} \rightarrow M_\ell\}$$

$$(27) \quad \mathfrak{M}_2 = \mathfrak{M}(\Sigma_{g_2}, a_2, \vec{n}) \cong \varprojlim \{g_{\ell+1}: N_{\ell+1} \rightarrow N_\ell\},$$

where  $M_0 = \Sigma_{g_1}$  and  $N_0 = \Sigma_{g_2}$ . For  $\ell > 0$ , let  $m_\ell$  denote the degree of the covering map  $f_\ell$  and let  $n_\ell$  denote the degree of the covering map  $g_\ell$ .

Now assume there is a homeomorphism  $H: \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ . By the results of Rogers and Tollefson in [39, 40], the map  $H$  is homotopic to a homeomorphism  $\tilde{H}$  which is induced by a map between the inverse limit representations of  $\mathfrak{M}_1$  in (26) and of  $\mathfrak{M}_2$  in (27). Such a map has the following form:

There exists an increasing integer-valued function  $k \rightarrow \ell_k$  for  $k \geq 0$ , and continuous onto maps  $\tilde{H}_k: M_{\ell_k} \rightarrow N_k$  where the collection of maps  $\{\tilde{H}_k \mid k \geq k_0\}$  form a commutative diagram:

$$(28) \quad \begin{array}{ccccccc} M_{\ell_0} & \xleftarrow{f_{\ell_0}^{\ell_1}} & M_{\ell_1} & \xleftarrow{\quad} & \cdots & \xleftarrow{f_{\ell_k}^{\ell_{k+1}}} & M_{\ell_{k+1}} & \xleftarrow{\quad} & \cdots \\ \tilde{H}_0 \downarrow & & \downarrow \tilde{H}_1 & & & \downarrow \tilde{H}_k & & \downarrow \tilde{H}_{k+1} & \\ N_0 & \xleftarrow{g_1} & N_1 & \xleftarrow{\quad} & \cdots & \xleftarrow{g_k} & N_{k+1} & \xleftarrow{\quad} & \cdots \end{array}$$

where the  $f_k$  and  $g_k$  are the bonding maps in the inverse limit representations (26) and (27), and  $f_{\ell_k}^{\ell_{k+1}} = f_{\ell_{k+1}} \circ \cdots \circ f_{\ell_k}$  denotes the composition of bonding maps.

All of the horizontal maps in the diagram (28) are covering maps by construction. Moreover, as the spaces  $M_k$  and  $N_k$  are closed surfaces, we can assume that all of the vertical maps in (28) are also covering maps. Thus, the Euler classes of all surfaces there are related by the covering degrees of the maps. For example,  $\chi(M_{\ell_k}) = d_k \cdot \chi(N_k)$  where  $d_k$  is the covering degree of  $\tilde{H}_k$ .

To show 1) we assume that a homeomorphism  $H$  exists, and so we have diagram (28) as above. Observe that  $g_2 = 1$  implies that  $\chi(\Sigma_2) = \chi(\mathbb{T}^2) = 0$ , hence  $\chi(N_k) = 0$

for all  $k \geq 0$ . Then as  $d_k \geq 1$  for all  $k$ , we obtain  $\chi(M_{\ell_k}) = 0$ . But this contradicts the assumption that  $g_1 > 1$  hence  $\chi(M_{\ell_k}) < 0$  as  $M_{\ell_k}$  is a covering of  $\Sigma_1$  which has  $\chi(\Sigma_1) < 0$ . Thus  $\mathfrak{M}_1 \not\approx \mathfrak{M}_2$ .

To show 2) first assume that  $C_{\vec{m}}(p) = C_{\vec{n}}(p)$  for all primes  $p$ . Then the odometer actions  $\Phi_{\vec{m}}: \mathbb{Z} \times \mathfrak{G}_{\vec{m}} \rightarrow \mathfrak{G}_{\vec{m}}$  and  $\Phi_{\vec{n}}: \mathbb{Z} \times \mathfrak{G}_{\vec{n}} \rightarrow \mathfrak{G}_{\vec{n}}$  are conjugate by an automorphism  $\theta: \mathfrak{G}_{\vec{m}} \rightarrow \mathfrak{G}_{\vec{n}}$ . Then by Proposition 3.6, the suspension spaces  $\mathfrak{M}(\Sigma_{g_1}, a_1, \vec{m})$  and  $\mathfrak{M}_2 = \mathfrak{M}(\Sigma_{g_1}, a_1, \vec{n})$  are homeomorphic.

To show the converse in 2) assume that a homeomorphism  $H$  exists, and suppose that for some prime  $p$  we have  $C_{\vec{m}}(p) \neq C_{\vec{n}}(p)$ . We assume without loss of generality that  $C_{\vec{m}}(p) < C_{\vec{n}}(p)$ . If otherwise, then reverse the roles of  $\vec{m}$  and  $\vec{n}$  and consider the homeomorphism  $H^{-1}$ . Then as  $\chi(\Sigma_1) = \chi(\Sigma_2)$ , for sufficiently large  $k$  the prime factorization of the Euler characteristic  $\chi(M_{\ell_k})$  contains a lower power of  $p$  than the prime factorization of  $\chi(N_k)$ . But this contradicts the fact that  $\chi(M_{\ell_k}) = d_k \cdot \chi(N_k)$  where  $d_k$  is the covering degree of  $\tilde{H}_k$ .

Finally, to show 3) let  $\Sigma = \Sigma_{g_1} = \Sigma_{g_2}$  where  $g = g_1 = g_2 > 1$ . It suffices to chose  $\vec{m}, \vec{n}$  such that  $\vec{m} \sim \vec{n}$ , but  $C_{\vec{m}} \neq C_{\vec{n}}$ . It then follows from 2) that  $\mathfrak{M}_1 \not\approx \mathfrak{M}_2$ . Pick a prime  $p_1 \geq 3$  and let  $\vec{m}$  be any sequence such that  $C_{\vec{m}}(p_1) = 0$ . Then define  $\vec{n}$  by setting  $n_1 = p_1$  and  $n_{k+1} = m_k$  for all  $k \geq 1$ .

Note that  $C_{\vec{m}}(p_1) = 0 \neq 1 = C_{\vec{n}}(p_1)$ , so  $C_{\vec{m}}(p) \neq C_{\vec{n}}(p)$  is satisfied. But clearly  $\vec{m} \sim \vec{n}$ , so the *adic*-surfaces  $\mathfrak{M}(\Sigma_g, a_1, \vec{m})$  and  $\mathfrak{M}(\Sigma_g, a_1, \vec{n})$  are return equivalent by Proposition 5.8, but are not homeomorphic by part 2) above.  $\square$

### 5.3 $\vec{m}$ -adic solenoids of higher dimension

Observe that the requirements on the base manifold  $\Sigma$  used in the proofs of 2) and 3) of Theorem 5.10 are that:

- (1)  $\Sigma$  is a strongly Borel manifold, so that the maps  $\tilde{H}_k$  can be assumed to be coverings;
- (2) the fundamental group  $G_0 = \pi_1(\Sigma, x)$  admits an epimorphism onto  $\mathbb{Z}$ , or equivalently that  $H^1(\Sigma; \mathbb{Z})$  contains a copy of  $\mathbb{Z}$ ;
- (3) the Euler characteristic  $\chi(\Sigma) \neq 0$ .

Thus, the proof of parts 2) and 3) of Theorem 5.10 can be applied almost verbatim to show:

**Theorem 5.11** *Let  $M$  be a closed manifold of dimension  $n \geq 3$ . Assume that  $M$  is strongly Borel, that  $H^1(M; \mathbb{Z})$  has rank at least 1, and the Euler characteristic  $\chi(M) \neq 0$ . Let  $\mathfrak{M}_1 = \mathfrak{M}(M, a, \vec{m})$  and  $\mathfrak{M}_2 = \mathfrak{M}(M, a, \vec{n})$  be the corresponding adic-solenoids, where  $a: \pi_1(M, x) \rightarrow \mathbb{Z}$  is an epimorphism. Then we have:*

- (1)  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are homeomorphic if and only if  $C_{\vec{m}} = C_{\vec{n}}$ .
- (2) there exists  $\vec{m} \sim \vec{n}$  but  $\mathfrak{M}_1 \not\approx \mathfrak{M}_2$ .

Finally, we comment on the requirement in Theorem 1.5 that the base manifolds be strongly Borel. Let  $M$  be a closed  $n$ -manifold where  $n \geq 5$ . Suppose that  $M$  satisfies the conditions of Theorem 5.11.

Let  $N = M \# \mathbb{S}^2 \times \mathbb{S}^{n-2}$  be the closed  $n$ -manifold obtained by attaching the handle  $\mathbb{S}^2 \times \mathbb{S}^{n-2}$ . Then  $\pi_1(M, x) \cong \pi_1(N, x)$  where we choose the basepoint  $x \in M$  disjoint from the disk along which the handle is attached.

Form the adic-solenoids  $\mathfrak{M}_1 = \mathfrak{M}(M, a, \vec{m})$  and  $\mathfrak{M}_2 = \mathfrak{M}(N, a, \vec{m})$  as before, but with bases  $M$  and  $N$ . Then  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are return equivalent, as in fact they have conjugate global monodromy actions. On the other hand, all leaves in  $\mathfrak{M}_1$  have trivial higher homotopy groups, while all leaves in  $\mathfrak{M}_2$  have non-trivial higher homotopy groups. Thus,  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  can not be homeomorphic.

## References

- [1] J M Aarts, R J Fokkink, *The classification of solenoids*, Proc. Amer. Math. Soc. 111 (1991) 1161–1163
- [2] J M Aarts, L G Oversteegen, *Matchbox manifolds*, from “Continua (Cincinnati, OH, 1994)”, Lecture Notes in Pure and Appl. Math. 170, Dekker, New York (1995) 3–14
- [3] J E Anderson, I F Putnam, *Topological invariants for substitution tilings and their associated  $C^*$ -algebras*, Ergodic Theory Dynam. Systems 18 (1998) 509–537
- [4] M Barge, L Sadun, *Quotient cohomology for tiling spaces*, New York J. Math. 17 (2011) 579–599
- [5] M M Barge, R C Swanson, *New techniques for classifying Williams solenoids*, Tokyo J. Math. 30 (2007) 139–157
- [6] R H Bing, *A simple closed curve is the only homogeneous bounded plane continuum that contains an arc*, Canad. J. Math. 12 (1960) 209–230
- [7] L Block, J Keesling, *A characterization of adding machine maps*, Topology Appl. 140 (2004) 151–161



- [8] **C Camacho, A Lins Neto**, *Geometric theory of foliations*, Birkhäuser Boston, Inc., Boston, MA (1985) Translated from the Portuguese by Sue E. Goodman
- [9] **A Candel, L Conlon**, *Foliations. I*, volume 23 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI (2000)
- [10] **A Clark, J Hunton**, *The homology core and invariant measures*, Trans. Amer. Math. Soc. (2018)
- [11] **A Clark, S Hurder**, *Embedding solenoids in foliations*, Topology Appl. 158 (2011) 1249–1270
- [12] **A Clark, S Hurder**, *Homogeneous matchbox manifolds*, Trans. Amer. Math. Soc. 365 (2013) 3151–3191
- [13] **A Clark, S Hurder, O Lukina**, *Shape of matchbox manifolds*, Indag. Math. (N.S.) 25 (2014) 669–712
- [14] **A Clark, S Hurder, O Lukina**, *Manifold-like matchbox manifolds*, Proc. Amer. Math. Soc. (2018)
- [15] **Ma I Cortez, K Medynets**, *Orbit equivalence rigidity of equicontinuous systems*, J. Lond. Math. Soc. (2) 94 (2016) 545–556
- [16] **Ma I Cortez, S Petite**, *G-odometers and their almost one-to-one extensions*, J. Lond. Math. Soc. (2) 78 (2008) 1–20
- [17] **D van Dantzig**, *Über topologisch homogene Kontinua*, Fund. Math. 15 (1930) 102–125
- [18] **J F Davis**, *The work of Tom Farrell and Lowell Jones in topology and geometry*, Pure Appl. Math. Q. 8 (2012) 1–14
- [19] **J Dyer, S Hurder, O Lukina**, *The discriminant invariant of Cantor group actions*, Topology Appl. 208 (2016) 64–92
- [20] **J Dyer, S Hurder, O Lukina**, *Growth and homogeneity of matchbox manifolds*, Indag. Math. (N.S.) 28 (2017) 145–169
- [21] **J Dyer, S Hurder, O Lukina**, *Molino theory for matchbox manifolds*, Pacific J. Math. 289 (2017) 91–151
- [22] **D B A Epstein, K C Millett, D Tischler**, *Leaves without holonomy*, J. London Math. Soc. (2) 16 (1977) 548–552
- [23] **R Fokkink, L Oversteegen**, *Homogeneous weak solenoids*, Trans. Amer. Math. Soc. 354 (2002) 3743–3755
- [24] **A Forrest, J Hunton, J Kellendonk**, *Topological invariants for projection method patterns*, Mem. Amer. Math. Soc. 159 (2002) x+120
- [25] **E Ghys**, *Laminations par surfaces de Riemann*, from “Dynamique et géométrie complexes (Lyon, 1997)”, Panor. Synthèses 8, Soc. Math. France, Paris (1999) ix, xi, 49–95
- [26] **T Giordano, I Putnam, C Skau**,  *$\mathbb{Z}^d$ -odometers and cohomology*, 2017

- [27] **A Haefliger**, *Groupoïdes d'holonomie et classifiants*, from “Transversal structure of foliations (Toulouse, 1982)”, Astérisque (1984) 70–97
- [28] **A Haefliger**, *Foliations and compactly generated pseudogroups*, from “Foliations: geometry and dynamics (Warsaw, 2000)”, World Sci. Publ., River Edge, NJ (2002) 275–295
- [29] **S Hurder**, *Classifying foliations*, from “Foliations, geometry, and topology”, Contemp. Math. 498, Amer. Math. Soc., Providence, RI (2009) 1–65
- [30] **S Hurder**, *Lectures on foliation dynamics*, from “Foliations: dynamics, geometry and topology”, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer, Basel (2014) 87–149
- [31] **S Hurder**, **O Lukina**, *Wild solenoids*, Trans. Amer. Math. Soc. (2018)
- [32] **A Julien**, **L Sadun**, *Tiling deformations, cohomology, and orbit equivalence of tiling spaces* (2015)
- [33] **W Lück**, *Aspherical manifolds*, Bulletin of the Manifold Atlas (2012) 1–17
- [34] **M Lyubich**, **Y Minsky**, *Laminations in holomorphic dynamics*, J. Differential Geom. 47 (1997) 17–94
- [35] **M C McCord**, *Inverse limit sequences with covering maps*, Trans. Amer. Math. Soc. 114 (1965) 197–209
- [36] **C C Moore**, **C L Schochet**, *Global analysis on foliated spaces*, volume 9 of *Mathematical Sciences Research Institute Publications*, second edition, Cambridge University Press, New York (2006)
- [37] **J F Plante**, *Foliations with measure preserving holonomy*, Ann. of Math. (2) 102 (1975) 327–361
- [38] **J T Rogers, Jr**, **J L Tollefson**, *Homogeneous inverse limit spaces with nonregular covering maps as bonding maps*, Proc. Amer. Math. Soc. 29 (1971) 417–420
- [39] **J T Rogers, Jr**, **J L Tollefson**, *Maps between weak solenoidal spaces*, Colloq. Math. 23 (1971) 245–249
- [40] **J T Rogers, Jr**, **J L Tollefson**, *Involutions on solenoidal spaces*, Fund. Math. 73 (1971/72) 11–19
- [41] **L Sadun**, *Tiling spaces are inverse limits*, J. Math. Phys. 44 (2003) 5410–5414
- [42] **L Sadun**, *Topology of tiling spaces*, volume 46 of *University Lecture Series*, American Mathematical Society, Providence, RI (2008)
- [43] **R M Schori**, *Inverse limits and homogeneity*, Trans. Amer. Math. Soc. 124 (1966) 533–539
- [44] **D Sullivan**, *Solenoidal manifolds*, J. Singul. 9 (2014) 203–205
- [45] **A Verjovsky**, *Commentaries on the paper Solenoidal manifolds by Dennis Sullivan [MR3249058]*, J. Singul. 9 (2014) 245–251

- [46] **L Vietoris**, *Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen*, Math. Ann. 97 (1927) 454–472

*Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, United Kingdom*

*Department of Mathematics, University of Illinois at Chicago, 322 SEO (m/c 249), 851 S. Morgan Street, Chicago, IL 60607-7045*

*Department of Mathematics, University of Illinois at Chicago, 322 SEO (m/c 249), 851 S. Morgan Street, Chicago, IL 60607-7045*

[Alex.Clark@leicester.ac.uk](mailto:Alex.Clark@leicester.ac.uk), [hurder@uic.edu](mailto:hurder@uic.edu), [lukina@uic.edu](mailto:lukina@uic.edu)