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Rigid cylindrical frameworks with two coincident points

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Abstract We develop a rigidity theory for graphs whose vertices are constrained to lie on a cylinder and in which two given vertices are coincident. We apply our result to show that the vertex splitting operation preserves the global rigidity of generic frameworks on the cylinder, whenever it satisfies the necessary condition that the deletion of the edge joining the split vertices preserves generic rigidity.

Keywords Infinitesimal rigidity · framework on a surface · count matroid · deletion/contraction characterisation · coincident points

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1 Introduction

A *framework* (G,p) in $\mathbb{R}^d$ is the combination of a finite, simple graph $G=(V,E)$ and a map $p:V\to\mathbb{R}^d$. It is *rigid* if every edge-length preserving continuous motion of the vertices arises as a

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congruence of $\mathbb{R}^d$ (see, for example, [14] for basic definitions and background). The theory of generic rigidity aims to characterise the graphs G for which (G, p) is rigid for all generic choices of p. This was accomplished by Laman [8] for $d = 2$, but is a long-standing open problem for $d \geq 3$.

We are interested in frameworks in $\mathbb{R}^3$ whose vertices are constrained to lie on a fixed surface. Generic rigidity in this context was characterised for graphs on the cylinder and various other surfaces in [10,11]. In this paper we consider frameworks on the cylinder in which two of the vertices are coincident, but are otherwise generic. For such frameworks we give the following combinatorial characterisation of rigidity. Given two vertices u, v of a graph G we use $G - uv$ to denote the graph formed from G by deleting the edge uv if it exists and G/uv to denote the graph which arises from G by contracting the vertices u and v (and deleting any loops and replacing any parallel edges by single edges). We say that G is uv-rigid on a cylinder $\mathcal{Y}$ if there exists a realisation p of G on $\mathcal{Y}$ such that $p(u) = p(v)$, $p|_{V-v}$ is generic on $\mathcal{Y}$, and (G, p) is rigid on $\mathcal{Y}$.

Theorem 1 *Let G be a graph and u, v be distinct vertices of G. Then G is uv-rigid on a cylinder $\mathcal{Y}$ if and only if $G - uv$ and G/uv are both rigid on $\mathcal{Y}$.*

Our proof technique extends that used by Fekete, Jordán and Kaszanitzky [4] to obtain an analogous result for frameworks in $\mathbb{R}^2$.

We apply our result to show that the vertex splitting operation preserves the global rigidity of generic frameworks on the cylinder, whenever it satisfies the necessary condition that the deletion of the edge joining the split vertices preserves generic rigidity. This is a key step in the recent characterisation of generic global rigidity on the cylinder given in [7]. Special position arguments are commonly used to prove that graph operations preserve generic rigidity properties and it is conceivable that our characterisation of generic uv-rigidity on the cylinder may have other such applications.

An outline of the paper is as follows. In Section 2 we provide background for frameworks on a cylinder. In Section 3 we define a count matroid $\mathcal{M}_{uv}(G)$ on a graph G with two distinguished vertices u and v. In Section 4 we derive an inductive construction for graphs whose edge set is independent in $\mathcal{M}_{uv}(G)$. We then use this construction to prove our characterisation of rigidity on a cylinder for frameworks in which u and v are coincident but are otherwise generic. In Section 5 we discuss global rigidity and apply our coincident vertex result to prove that the vertex splitting operation preserves global rigidity for generic frameworks on a cylinder if and only if deletion of the new edge preserves generic rigidity. Finally, in Section 6 we comment on extensions to other surfaces.

2 Frameworks on concentric cylinders

Throughout this paper we will only consider graphs without loops or parallel edges, as loops and parallel edges give rise to trivial distance constraints. Let $G = (V, E)$ be a graph with $V = \{v_1, \dots, v_n\}$. We will consider realisations of G on a family of concentric cylinders $\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \dots \cup \mathcal{Y}_k$ where $\mathcal{Y}_i = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = r_i\}$ and $r = (r_1, \dots, r_k)$ is a vector of positive real numbers.[1] A *framework* (G, p) on $\mathcal{Y}$ is an ordered pair consisting of a graph G and a realisation p such that $p(v_i) \in \mathcal{Y}$ for all $v_i \in V$.

Two frameworks (G, p) and (G, q) on $\mathcal{Y}$ are *equivalent* if $\|p(v_i) - p(v_j)\| = \|q(v_i) - q(v_j)\|$ for all edges $v_i v_j \in E$. Moreover (G, p) and (G, q) on $\mathcal{Y}$ are *congruent* if $\|p(v_i) - p(v_j)\| = \|q(v_i) - q(v_j)\|$

[1] Our proof techniques apply equally well in the cases when there are one or more cylinders.

for all pairs of vertices $v_i, v_j \in V$. The framework (G, p) is *rigid* on $\mathcal{Y}$ if there exists an $\epsilon > 0$ such that every framework (G, q) on $\mathcal{Y}$ which is equivalent to (G, p), and has $\|p(v_i) - q(v_i)\| < \epsilon$ for all $1 \leq i \leq n$, is congruent to (G, p). Moreover (G, p) is *minimally rigid* on $\mathcal{Y}$ if (G, p) is rigid on $\mathcal{Y}$ but $(G-e, p)$ is not for any $e \in E$. The framework (G, p) is *generic* on $\mathcal{Y}$ if $\text{td}[\mathbb{Q}(r, p) : \mathbb{Q}(r)] = 2n$, where $\text{td}[L, K]$ denotes the transcendence degree of the field extension $[L : K]$ i.e. the size of a maximal set of elements of L which are algebraically independent over K.

It was shown in [10] that a generic framework (G, p) on a family of concentric cylinders $\mathcal{Y}$ is rigid if and only if it is infinitesimally rigid in the following sense. An *infinitesimal flex* s of (G, p) on $\mathcal{Y}$ is a map $s : V \to \mathbb{R}^3$ such that $s(v_i)$ is tangential to $\mathcal{Y}$ at $p(v_i)$ for all $v_i \in V$ and $(p(v_j) - p(v_i)) \cdot (s(v_j) - s(v_i)) = 0$ for all $v_j v_i \in E$. The framework (G, p) is *infinitesimally rigid* on $\mathcal{Y}$ if every infinitesimal flex is an infinitesimal isometry of $\mathbb{R}^3$, i.e. an infinitesimal flex corresponding to a combination of translations and rotations of $\mathbb{R}^3$.

The *rigidity matrix* $R^{\mathcal{Y}}(G, p)$ is the $(|E| + |V|) \times 3|V|$ matrix

$$R^{\mathcal{Y}}(G, p) = \begin{pmatrix} R_3(G, p) \\ S(G, p) \end{pmatrix}$$

where: $R_3(G, p)$ has rows indexed by E and 3-tuples of columns indexed by V in which, for $e = v_i v_j \in E$, the submatrices in row e and columns v_i and v_j are $p(v_i) - p(v_j)$ and $p(v_j) - p(v_i)$, respectively, and all other entries are zero; $S(G, p)$ has rows indexed by V and 3-tuples of columns indexed by V in which, for $v_i \in V$, the submatrix in row v_i and column v_i is $\bar{p}(v_i) = (x_i, y_i, 0)$ when $p(v_i) = (x_i, y_i, z_i)$. The *rigidity matroid* $\mathcal{R}^{\mathcal{Y}}(G)$ is the matroid on E in which a set $F \subseteq E$ is independent if and only if the rows of $R^{\mathcal{Y}}(G, p)$ indexed by $F \cup V$ are linearly independent for any generic p. Equivalently $\mathcal{R}^{\mathcal{Y}}(G)$ is the matroid we get from the row matroid of $R^{\mathcal{Y}}(G, p)$ by contracting each element of V. We will use $r^{\mathcal{Y}}$ to denote the rank function of $R^{\mathcal{Y}}(G, p)$.

A graph $G = (V, E)$ is *(k, ℓ)-sparse* if $|E'| \leq k|V'| - \ell$ for all subgraphs (V', E') of G with at least one edge. Moreover G is *(k, ℓ)-tight* if G is (k, ℓ)-sparse and $|E| = k|V| - \ell$.

The following characterisation of generic rigidity on $\mathcal{Y}$ was proved in [10].

Theorem 2 *Let (G, p) be a generic framework on a family of concentric cylinders $\mathcal{Y}$. Then (G, p) is minimally rigid on $\mathcal{Y}$ if and only if G is a complete graph on at most three vertices or G is $(2, 2)$-tight.*

2.1 Coincident realisations on concentric cylinders

Let $G = (V, E)$ be a graph and $u, v \in V$. A framework (G, p) on $\mathcal{Y}$ is *uv-coincident* if $p(u) = p(v)$. A *generic uv-coincident framework* is a uv-coincident framework (G, p) for which $(G - u, p|_{V-u})$ is generic. We denote the uv-coincident cylinder rigidity matroid by $\mathcal{R}^{\mathcal{Y}}_{uv}(G)$ (this is the matroid on E in which a set $F \subseteq E$ is independent if and only if the rows of $R^{\mathcal{Y}}(G, p)$ indexed by $F \cup V$ are linearly independent for any generic uv-coincident realisation p.). Note that the matroid depends on G but not on the choice of generic uv-coincident realisation. That is, for any two generic uv-coincident realisations (G, p) and (G, p') on $\mathcal{Y}$, we get the same matroid. We also use $r^{\mathcal{Y}}_{uv}$ to denote the rank function of $\mathcal{R}^{\mathcal{Y}}_{uv}(G)$. We say that G is *uv-rigid* on $\mathcal{Y}$ if $r^{\mathcal{Y}}_{uv}(G) = 2|V| - 2$ and that G is *minimally uv-rigid* on $\mathcal{Y}$ if G is uv-rigid on $\mathcal{Y}$ and $|E| = 2|V| - 2$.

Note that the terms 'rigid on $\mathcal{Y}$' and 'uv-rigid on $\mathcal{Y}$', and the notations $r^{\mathcal{Y}}$ and $r^{\mathcal{Y}}_{uv}$ appear to depend on $\mathcal{Y}$. Theorems 1 and 2 imply that this is not the case since the characterisations of $\mathcal{R}^{\mathcal{Y}}(G)$

and $\mathcal{R}^{\mathcal{Y}}_{uv}(G)$ given by these results depend only on the graph G and not the family of concentric cylinders $\mathcal{Y}$.

3 A count matroid

In this section we define a count matroid $\mathcal{M}_{uv}(G)$ on the edge set of a graph G with two distinguished vertices u and v. Our approach follows that given in [4]. We will show that $\mathcal{M}_{uv}(G)$ is equal to $\mathcal{R}^{\mathcal{Y}}_{uv}(G)$ in Section 4.

Let $G = (V, E)$ be a graph. For $X \subseteq V$ let $N_G(X)$ be the set of neighbours of X in $V \setminus X$ and put $N_G(x) = N_G(X)$ when $X = \{x\}$. Let $G[X]$ denote the subgraph of G induced by X and let $E_G(X)$ be the set of edges of $G[X]$. Thus $i_G(X) = |E_G(X)|$. For a family $\mathcal{S} = \{S_1, S_2, \dots, S_k\}$, where $S_i \subseteq V$ for all $i = 1, \dots, k$, we define $V(\mathcal{S}) = \bigcup_{i=1}^k S_i$, $E_G(\mathcal{S}) = \bigcup_{i=1}^k E_G(S_i)$ and put $i_G(\mathcal{S}) = |E_G(\mathcal{S})|$. We also define $\mathrm{cov}(\mathcal{S}) = \{xy : x, y \in V, \{x, y\} \subseteq S_i \text{ for some } 1 \le i \le k\}$. We say that $\mathcal{S}$ *covers* a set $F \subseteq E$ if $F \subseteq \mathrm{cov}(\mathcal{S})$. The degree of a vertex w is denoted by $d_G(w)$. We may omit the subscripts referring to G if the graph is clear from the context.

Let $G = (V, E)$ be a graph and $u, v \in V$ be two distinct vertices of G. Let $\mathcal{H} = \{H_1, ..., H_k\}$ be a family with $H_i \subseteq V$, $1 \le i \le k$. We say that $\mathcal{H}$ is *uv-compatible* if $u, v \in H_i$ and $|H_i| \ge 3$ hold for all $1 \le i \le k$. See Figure 1 for an example. We define the *value* of subsets of V and of uv-compatible families as follows. For a nonempty subset $H \subseteq V$, we let

$$\mathrm{val}(H) = 2|H| - t_H,$$

where $t_H = 4$ if $H = \{u, v\}$, $t_H = 3$ if $H \ne \{u, v\}$ and $|H| \in \{2, 3\}$, and $t_H = 2$ otherwise. We will often denote t_{H_i} by t_i for short. For a uv-compatible family $\mathcal{H} = \{H_1, H_2, \dots, H_k\}$ we let

$$\mathrm{val}(\mathcal{H}) = \left(\sum_{i=1}^{k} \mathrm{val}(H_i)\right) - 2(k-1) = \sum_{i=1}^{k} (2|H_i| - t_{H_i} - 2) + 2.$$

Note that if $\mathcal{H} = \{H\}$ is a uv-compatible family containing only one set then the two definitions agree, i.e. $\mathrm{val}(\mathcal{H}) = \mathrm{val}(H)$ holds.

We say that G is *uv-sparse* if for all $H \subseteq V$ with $|H| \ge 2$ we have $i_G(H) \le \mathrm{val}(H)$ and for all uv-compatible families $\mathcal{H}$ we have $i_G(\mathcal{H}) \le \mathrm{val}(\mathcal{H})$. Note that if G is uv-sparse then $uv \notin E$ must hold. A set $H \subseteq V$ of vertices with $|H| \ge 2$ (resp. a uv-compatible family $\mathcal{H} = \{H_1, \dots, H_k\}$) is called *tight* if $i_G(H) = \mathrm{val}(H)$ (resp. $i_G(\mathcal{H}) = \mathrm{val}(\mathcal{H})$) holds. We will show that the edge sets of the uv-sparse subgraphs of G form the independent sets of a matroid $\mathcal{M}_{uv}(G)$.

The following lemmas will enable us to 'uncross' tight sets and tight uv-compatible families in a sparse graph. The first result follows immediately from the definition of the i- and val- functions.

Lemma 1 *Let $X, Y \subseteq V$ be distinct vertex sets in G. Then*
(a) $i(X) + i(Y) \le i(X \cup Y) + i(X \cap Y)$ and
(b) if $X \cap Y \ne \emptyset$, then $\mathrm{val}(X) + \mathrm{val}(Y) + t_X + t_Y = \mathrm{val}(X \cup Y) + \mathrm{val}(X \cap Y) + t_{X \cup Y} + t_{X \cap Y}$.

Lemma 2 *Let $\mathcal{H} = \{H_1, \dots, H_k\}$ be a uv-compatible family in G.*
(a) Suppose $|H_i \cap H_j| \ge 3$ for some pair $1 \le i < j \le k$. Then there is a uv-compatible family $\mathcal{H}'$ with $\mathrm{cov}(\mathcal{H}) \subseteq \mathrm{cov}(\mathcal{H}')$ and $\mathrm{val}(\mathcal{H}') < \mathrm{val}(\mathcal{H})$.
(b) Suppose G is uv-sparse and $\mathcal{H}$ is tight. Then $H_i \cap H_j = \{u, v\}$ for all $1 \le i \le k$.

Proof. (a) We may assume that $i = k-1$, $j = k$. Let $\mathcal{H}' = \{H_1, \ldots, H_{k-2}, H_{k-1} \cup H_k\}$. Using Lemma 1(b) we have $\text{val}(H_{k-1}) + \text{val}(H_k) \geq \text{val}(H_{k-1} \cup H_k) + \text{val}(H_{k-1} \cap H_k)$. Hence

$$\begin{aligned}\text{val}(\mathcal{H}) = \quad & \sum_{l=1}^{k} \text{val}(H_l) - 2(k-1) = \sum_{l=1}^{k-2} \text{val}(H_l) - 2((k-1)-1) + \text{val}(H_{k-1}) + \text{val}(H_k) - 2 \\ \geq \quad & \sum_{l=1}^{k-2} \text{val}(H_l) + \text{val}(H_{k-1} \cup H_k) - 2((k-1)-1) + \text{val}(H_{k-1} \cap H_k) - 2 > \text{val}(\mathcal{H}').\end{aligned}$$

Clearly, we have $\text{cov}(\mathcal{H}) \subseteq \text{cov}(\mathcal{H}')$.
(b) Since $\mathcal{H}$ is tight, if $|H_i \cap H_j| \geq 3$ for some pair $1 \leq i < j \leq k$ then, by (a), we have $\text{val}(\mathcal{H}') < \text{val}(\mathcal{H}) = i(\mathcal{H}) \leq i(\mathcal{H}')$. This contradicts the uv-sparsity of G. Hence $H_i \cap H_j = \{u, v\}$ for all $1 \leq i \leq k$. □

Lemma 3 *Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a uv-compatible family with $H_i \cap H_j = \{u, v\}$ for all $1 \leq i < j \leq k$ and $|H_k| \geq 4$. Then $\mathcal{H}' = \{H_1, \ldots, H_{k-2}, H_{k-1} \cup H_k\}$ is a uv-compatible family with* $\text{cov}(\mathcal{H}) \subset \text{cov}(\mathcal{H}')$ *and for which* $\text{val}(\mathcal{H}') \leq \text{val}(\mathcal{H}) + 1$ *with equality only if $|H_{k-1}| = 3$. Furthermore, if G is uv-sparse, $\mathcal{H}$ is tight and $|H_{k-1}| \geq 4$, then $\mathcal{H}'$ is tight.*

Proof. Using Lemma 1(b) and the facts that $t_k = t_{H_{k-1} \cup H_k} = 2$ and $t_{H_{k-1} \cap H_k} = 4$ we have $\text{val}(H_{k-1}) + \text{val}(H_k) = \text{val}(H_{k-1} \cup H_k) + \text{val}(H_{k-1} \cap H_k) + 4 - t_{k-1} = \text{val}(H_{k-1} \cup H_k) + 4 - t_{k-1}$. Hence

$$\begin{aligned}\text{val}(\mathcal{H}) = \sum_{l=1}^{k} \text{val}(H_l) - 2(k-1) &= \sum_{l=1}^{k-2} \text{val}(H_l) - 2((k-1)-1) + \text{val}(H_{k-1}) + \text{val}(H_k) - 2 \\ &= \sum_{l=1}^{k-2} \text{val}(H_l) + \text{val}(H_{k-1} \cup H_k) - 2((k-1)-1) + 2 - t_{k-1} \\ &= \text{val}(\mathcal{H}') + 2 - t_{k-1}.\end{aligned}$$

Thus $\text{val}(\mathcal{H}') \leq \text{val}(\mathcal{H}) + 1$ with equality only if $|H_{k-1}| = 3$. Clearly, we have $\text{cov}(\mathcal{H}) \subset \text{cov}(\mathcal{H}')$.

Now suppose G is uv-sparse, $\mathcal{H}$ is tight and $|H_{k-1}| \geq 4$. Then $\text{val}(\mathcal{H}') \leq \text{val}(\mathcal{H}) = i(\mathcal{H}) = i(\mathcal{H}')$, so $\mathcal{H}'$ is tight. □

Lemma 4 *Let $G = (V, E)$ be uv-sparse and let $X, Y \subseteq V$ be tight sets in G with $X \cap Y \neq \emptyset$ and $|X|, |Y| \geq 4$. Then $|X \cap Y| \notin \{2, 3\}$ and $X \cup Y$ and $X \cap Y$ are both tight.*

Proof. We have

$$2|X| - 2 + 2|Y| - 2 = i(X) + i(Y) \leq i(X \cup Y) + i(X \cap Y)$$

$$\leq 2|X \cup Y| - t_{X \cup Y} + 2|X \cap Y| - t_{X \cap Y} = 2|X| + 2|Y| - 2 - t_{X \cap Y}.$$

This implies that $t_{X \cap Y} = 2$ and equality holds throughout. Thus $X \cup Y$ and $X \cap Y$ are both tight and either $|X \cap Y| \geq 4$ or $|X \cap Y| = 1$. □

Lemma 5 *Let* $\mathcal{H} = \{H_1, \dots, H_k\}$ *be a uv-compatible family with* $H_j \cap H_l = \{u, v\}$ *for all* $1 \le j < l \le k$*, and let* $Y \subseteq V$ *be a set of vertices with* $|Y| \ge 4$*, and* $|Y \cap \{u, v\}| \le 1$*. Suppose that for some* $1 \le i \le k$ *either* $|Y \cap H_i| \ge 2$*, or* $|Y \cap H_i| = 1$ *and* $|H_i| \ge 4$*. Then there is a uv-compatible family* $\mathcal{H}'$ *with* $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\mathcal{H}')$ *and* $\mathrm{val}(\mathcal{H}') \le \mathrm{val}(\mathcal{H}) + \mathrm{val}(Y)$*. Furthermore, if* G *is uv-sparse and* $\mathcal{H}$ *and* Y *are both tight then* $\mathcal{H}'$ *and* $Y \cap H_i$ *are also tight.*

Proof. Let $S = \{H_i \in \mathcal{H} : |Y \cap H_i| \ge 2$ or $|Y \cap H_i| = 1$ and $|H_i| \ge 4\}$. Renumbering the sets of $\mathcal{H}$, if necessary, we may assume that $S = \{H_i \in \mathcal{H} : j \le i \le k\}$, for some $j \le k$. Let $X = Y \cup (\bigcup_{i=j}^k H_i)$ and $\mathcal{H}' = \{H_1, \dots, H_{j-1}, X\}$. Then $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\mathcal{H}')$ and

$$|X| = \sum_{i=j}^{k} |H_i| + |Y| - 2(k-j) - \sum_{i=j}^{k} |H_i \cap Y| + |Y \cap \{u, v\}|(k-j).$$

This gives

$$\begin{aligned}
\mathrm{val}(\mathcal{H}) + \mathrm{val}(Y) &= \sum_{i=1}^{k} \mathrm{val}(H_i) - 2(k-1) + \mathrm{val}(Y) \\
&= \sum_{i=1}^{j-1} \mathrm{val}(H_i) - 2(j-1) + \sum_{i=j}^{k} (2|H_i| - t_i) - 2(k-j) + (2|Y| - 2) \\
&= \sum_{i=1}^{j-1} \mathrm{val}(H_i) + (2|X| - 2) - 2(j-1) + 4(k-j) - \sum_{i=j}^{k} t_{H_i} \\
&\quad + 2\sum_{i=j}^{k} |Y \cap H_i| - 2(k-j) - 2|Y \cap \{u, v\}|(k-j) \\
&\ge \sum_{i=1}^{j-1} \mathrm{val}(H_i) + \mathrm{val}(X) - 2(j-1) + \sum_{i=j}^{k} (2|Y \cap H_i| - t_{H_i}).
\end{aligned}$$

If $|Y \cap H_i| \ge 2$ then $\mathrm{val}(Y \cap H_i) = 2|Y \cap H_i| - t_{Y \cap H_i} \le 2|Y \cap H_i| - t_{H_i}$. On the other hand, if $|Y \cap H_i| = 1$ and $|H_i| \ge 4$, then $t_{Y \cap H_i} = 2 = t_{H_i}$ and we have $\mathrm{val}(Y \cap H_i) = 2|Y \cap H_i| - t_{H_i}$. Thus, in both cases,

$$\mathrm{val}(\mathcal{H}) + \mathrm{val}(Y) \ge \mathrm{val}(\mathcal{H}') + \sum_{i=j}^{k} \mathrm{val}(Y \cap H_i)$$

and so $\mathrm{val}(\mathcal{H}') \le \mathrm{val}(\mathcal{H}) + \mathrm{val}(Y)$.

Now, suppose that G is uv-sparse and $\mathcal{H}$ and Y are tight. Then we have

$$i(\mathcal{H}') + \sum_{i=j}^{k} i(Y \cap H_i) \ge i(\mathcal{H}) + i(Y) = \mathrm{val}(\mathcal{H}) + \mathrm{val}(Y) \ge$$

$$\ge \mathrm{val}(\mathcal{H}') + \sum_{i=j}^{k} \mathrm{val}(Y \cap H_i) \ge i(\mathcal{H}') + \sum_{i=j}^{k} i(Y \cap H_i),$$

where the first inequality follows from the fact that edges spanned by $\mathcal{H}$ or Y are spanned by $\mathcal{H}'$ and if some edge is spanned by both $\mathcal{H}$ and Y then it is spanned by $Y \cap H_i$ for some i. The equality holds because $\mathcal{H}$ and Y are tight, and the second inequality holds by our calculations above. The last inequality holds because G is uv-sparse. Hence equality must hold everywhere, which implies that $\mathcal{H}'$ is tight and that $Y \cap H_i$ is also tight for all $j \leq i \leq k$. □

Lemma 6 *Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a uv-compatible family with $H_i \cap H_j = \{u, v\}$ for all $1 \leq i < j \leq k$, and let $Y \subseteq V$ be a set of vertices with $|Y| \geq 4$, $Y \cap \{u, v\} = \emptyset$ and $|Y \cap H_i| \leq 1$ for all $1 \leq i \leq k$. Suppose that $|Y \cap H_i| = |Y \cap H_j| = 1$ for some pair $1 \leq i < j \leq k$. Then there is a uv-compatible family $\mathcal{H}'$ with $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\mathcal{H}')$ for which $\mathrm{val}(\mathcal{H}') \leq \mathrm{val}(\mathcal{H}) + \mathrm{val}(Y)$. Furthermore, if G is uv-sparse and $\mathcal{H}$ and Y are both tight, then $\mathcal{H}'$ is tight and $|H_i| = |H_j| = 3$.*

Proof. We may assume that $i = k - 1$ and $j = k$. Let $\mathcal{H}' = \{H_1, \ldots, H_{k-2}, H_{k-1} \cup H_k \cup Y\}$. We have $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\mathcal{H}')$ and

$$\begin{aligned}\mathrm{val}(\mathcal{H}) + \mathrm{val}(Y) &= \sum_{i=1}^{k} \mathrm{val}(H_i) - 2(k-1) + \mathrm{val}(Y)\\ &= \sum_{i=1}^{k-2} \mathrm{val}(H_i) - 2((k-1)-1) - 2 + \mathrm{val}(H_{k-1}) + \mathrm{val}(H_k) + \mathrm{val}(Y).\end{aligned}$$

Using Lemma 1(b) twice and the fact that $|H_{k-1} \cap (H_k \cup Y)| = 3$ we obtain

$$\begin{aligned}\mathrm{val}(H_{k-1}) + \mathrm{val}(H_k) + \mathrm{val}(Y) &= \mathrm{val}(H_{k-1}) + \mathrm{val}(H_k \cup Y) + 2 - t_{H_k}\\ &= \mathrm{val}(H_{k-1} \cup H_k \cup Y) + 8 - t_{H_{k-1}} - t_{H_k}\\ &\geq \mathrm{val}(H_{k-1} \cup H_k \cup Y) + 2,\end{aligned}$$

with equality only if $|H_{k-1}| = |H_k| = 3$. Thus $\mathrm{val}(\mathcal{H}') \leq \mathrm{val}(\mathcal{H}) + \mathrm{val}(Y)$ as claimed.

Now suppose that G is uv-sparse. and $\mathcal{H}$ and Y are both tight. Then we have

$$i(\mathcal{H}) + i(Y) = \mathrm{val}(\mathcal{H}) + \mathrm{val}(Y) \geq \mathrm{val}(\mathcal{H}') \geq i(\mathcal{H}') \geq i(\mathcal{H}) + i(Y)$$

where the last inequality follows since $|Y \cap H_{k-1}| = |Y \cap H_k| = 1$ and $|Y \cap H_i| \leq 1$ for all $1 \leq i \leq k$. Hence equality must hold throughout. Thus $\mathcal{H}'$ is tight and $|H_{k-1}| = |H_k| = 3$. □

Lemma 7 *Let $G = (V, E)$ be uv-sparse and suppose that there is a tight uv-compatible family in G. Then there is a unique tight uv-compatible family $\mathcal{H}_{\max}$ in G for which $\mathrm{cov}(\mathcal{H}) \subseteq \mathrm{cov}(\mathcal{H}_{\max})$ for all tight uv-compatible families $\mathcal{H}$ of G. In addition, if $\mathcal{H}_{\max} = \{H_1, H_2, \ldots, H_k\}$ and $|H_1| \geq |H_2| \geq \ldots \geq |H_k|$, then:*
(a) $H_i \cap H_j = \{u, v\}$ for all $1 \leq i < j \leq k$;
(b) $|H_i| = 3$ for all $2 \leq i \leq k$;
(c) $N(u, v) \subseteq V(\mathcal{H}_{max})$.
Furthermore, if $Y \subseteq V$ is tight, $|Y| \geq 4$, $\mathrm{cov}(Y) \not\subseteq \mathrm{cov}(\mathcal{H}_{\max})$, and $Y \cap H_i \neq \emptyset$ for some $1 \leq i \leq k$, then $|Y \cap H_i| = 1$, $|H_i| = 3$, $Y \cap \{u, v\} = \emptyset$, and $Y \cap H_j = \emptyset$ for all $j \neq i$.

Proof. Let $\mathcal{H}_1 = \{H_1, H_2, \ldots, H_k\}$ be a tight uv-compatible family in G labeled such that $|H_1| \geq |H_2| \geq \ldots \geq |H_k|$ and suppose that $\text{cov}(\mathcal{H}_1)$ is maximal with respect to inclusion. Then Lemmas 2 and 3 imply that $H_i \cap H_j = \{u, v\}$ holds for all $1 \leq i < j \leq k$ and $|H_i| = 3$ for all $2 \leq i \leq k$. Suppose for a contradiction that $\mathcal{H}_2 = \{J_1, J_2, \ldots, J_l\}$ is another tight uv-compatible family whose cover is maximal, labeled so that $|J_1| \geq |J_2| \geq \ldots \geq |J_l|$. We will use the notation $H_i = \{u, v, x_i\}$ for $2 \leq i \leq k$ and $J_j = \{u, v, y_i\}$ for $2 \leq j \leq l$. Without loss of generality we can assume that if $|H_1| = |J_1| = 3$ then $H_1 \neq J_1$.

We define two uv-compatible families as follows: let

$$\mathcal{H}_\cap = \{Z \subseteq V : |Z| \geq 3 \text{ and } H_i \cap J_j = Z \text{ for some } H_i \in \mathcal{H}_1,\ J_j \in \mathcal{H}_2\};$$

let

$$\mathcal{H}_\cup = \{H_1 \cup J_1\} \cup \{H_i : 2 \leq i \leq k \text{ and } x_i \notin H_1 \cup J_1\} \cup \{J_j : 2 \leq j \leq l \text{ and } y_j \notin H_1 \cup J_1\}$$

if $|H_1 \cap J_1| \geq 3$, and

$$\mathcal{H}_\cup = \{H_1\} \cup \{J_1\} \cup \{H_i : 2 \leq i \leq k \text{ and } x_i \notin H_1 \cup J_1\} \cup \{J_j : 2 \leq j \leq l \text{ and } y_j \notin H_1 \cup J_1\}$$

if $|H_1 \cap J_1| = 2$.

It is easy to see that $\mathcal{H}_\cup$ and $\mathcal{H}_\cap$ are both uv-compatible. For convenience we rename the families as $\mathcal{H}_\cup = \{A_1, \ldots, A_p\}$ and $\mathcal{H}_\cap = \{B_1, \ldots, B_q\}$, where $A_1 = H_1 \cup J_1$ and $B_1 = H_1 \cap J_1$ if $|H_1 \cap J_1| \geq 3$, and $A_1 = H_1$ and $A_2 = J_1$ if $|H_1 \cap J_1| = 2$. It follows from their construction that $|A_i| = 3$ for all $3 \leq i \leq p$ and $|B_j| = 3$ for all $2 \leq j \leq q$ and also at least one of $|A_2| = 3$, $|B_1| = 3$ holds. It can be seen easily that $p + q = k + l$. We also have $i(\mathcal{H}_1) + i(\mathcal{H}_2) \leq i(\mathcal{H}_\cup) + i(\mathcal{H}_\cap)$, since the family $\mathcal{H}_\cup$ spans all the edges spanned by $\mathcal{H}_1$ or $\mathcal{H}_2$ and $\mathcal{H}_\cap$ spans all the edges spanned by both $\mathcal{H}_1$ and $\mathcal{H}_2$. Thus

$$\begin{aligned} \text{val}(H_1) + 3(k-1) - 2(k-1) + \text{val}(J_1) + 3(l-1) - 2(l-1) &= \text{val}(\mathcal{H}_1) + \text{val}(\mathcal{H}_2) \\ &= i(\mathcal{H}_1) + i(\mathcal{H}_2) \leq i(\mathcal{H}_\cup) + i(\mathcal{H}_\cap) \leq \text{val}(\mathcal{H}_\cup) + \text{val}(\mathcal{H}_\cap) \\ = \text{val}(A_1) + \max\{\text{val}(A_2), \text{val}(B_1)\} + 3(p-1) &- 2(p-1) + 3(q-1) - 2(q-1). \end{aligned}$$

We will show that equality occurs at both ends of the above inequality. Since $k - 1 + l - 1 = p - 1 + q - 1$, it will suffice to show that $\text{val}(H_1) + \text{val}(J_1) \geq \text{val}(A_1) + \max\{\text{val}(A_2), \text{val}(B_1)\}$. This is immediate if $|H_1 \cap J_1| = 2$ and follows from Lemma 1(b) when $|H_1 \cap J_1| \geq 3$.

Hence equality must hold throughout the displayed inequality. In particular, $\mathcal{H}_\cup$ and $\mathcal{H}_\cap$ are both tight. Since $\text{cov}(\mathcal{H}_1) \cup \text{cov}(\mathcal{H}_2) \subseteq \text{cov}(\mathcal{H}_\cup)$, the maximality of the covers implies that $\text{cov}(\mathcal{H}_1) = \text{cov}(\mathcal{H}_2)$ which in turn gives $\mathcal{H}_1 = \mathcal{H}_2$.

We have now shown that $\mathcal{H}_1 = \mathcal{H}_{\max}$ is unique and that properties (a) and (b) hold. To see that (c) holds choose $x \in N(u, v)$ and suppose that $x \notin V(\mathcal{H}_{max})$. Let $\mathcal{H}' = \mathcal{H}_{\max} + \{u, v, x\}$. Then $i(\mathcal{H}') \geq i(\mathcal{H}_{\max}) + 1$ and $\text{val}(\mathcal{H}') = \text{val}(\mathcal{H}_{\max}) + 1$, so $\mathcal{H}'$ is tight and hence contradicts the maximality of $\mathcal{H}_{\max}$.

To complete the proof we suppose that $Y \subseteq V$ is tight, $|Y| \geq 4$, $\text{cov}(Y) \not\subseteq \text{cov}(\mathcal{H}_{\max})$, and $Y \cap H_i \neq \emptyset$ for some $1 \leq i \leq k$. If $\{u, v\} \subseteq Y$ then $\mathcal{H} = \{Y\}$ would be a uv-compatible family with $\text{cov}(\mathcal{H}) \not\subseteq \text{cov}(\mathcal{H}_{\max})$. This would contradict the maximality of $\mathcal{H}_{\max}$ and hence $\{u, v\} \not\subseteq Y$. If $|Y \cap H_i| \geq 2$ or $|Y \cap H_i| = 1$ and $|H_i| \geq 4$ then Lemma 5 would imply that there exists a uv-compatible family $\mathcal{H}'$ with $\text{cov}(\mathcal{H}_{\max}) \cup \text{cov}(Y) \subseteq \text{cov}(\mathcal{H}')$. Hence $|Y \cap H_i| \leq 1$ and $|H_i| = 3$. This tells us that $|Y \cap H_j| \leq 1$ for all j and hence $\text{cov}(Y) \cap \text{cov}(\mathcal{H}_{\max}) = \emptyset$. If $Y \cap \{u, v\} \neq \emptyset$ then putting

$\mathcal{H}' = \mathcal{H}_{\max} \cup \{Y \cup \{u, v\}\}$ we have $i(\mathcal{H}') \geq i(\mathcal{H}) + 2|Y| - 2$ and $\text{val}(\mathcal{H}') = \text{val}(\mathcal{H}) + 2|Y| - 2$, so $\mathcal{H}'$ would contradict the maximality of $\mathcal{H}_{\max}$. Thus $Y \cap \{u, v\} = \emptyset$. If $Y \cap H_j \neq \emptyset$ for some $j \neq i$ then Lemma 6 now gives us a tight uv-compatible family $\mathcal{H}'$ with $\text{cov}(\mathcal{H}_{\max}) \cup \text{cov}(Y) \subseteq \text{cov}(\mathcal{H}')$. Hence $Y \cap H_j = \emptyset$ for all $j \neq i$. □

Note that Lemma 7 tells us in particular that if G is uv-sparse and $Y \subseteq V$ is tight with $\{u, v\} \cap Y \neq \emptyset$, then $Y \subseteq H_i$ for some $H_i \in \mathcal{H}_{\max}$.

3.1 The matroid and its rank function

It is well known that the edge sets of the $(2, 2)$-sparse subgraphs of a graph $G = (V, E)$ are the independent sets of a matroid on E called the *simple* $(2, 2)$*-sparse matroid* for G. Theorem 2 implies that this matroid is identical to the cylindrical rigidity matroid $\mathcal{R}^{\mathcal{Y}}(G)$. It follows that the rank function of $\mathcal{R}^{\mathcal{Y}}(G)$ can be defined in terms of 'thin covers' where a *cover* of any $F \subseteq E$ is a system $\mathcal{K} = \{H_1, \ldots, H_k\}$ of subsets of V, of cardinality at least 2, such that each edge in F is induced by at least one set in $\mathcal{K}$. This cover is *thin* if $|H_i \cap H_j| \leq 1$ for all pairs $1 \leq i, j \leq k$ with equality only if $|H_i| = 2$ or $|H_j| = 2$. We may use Theorem 2 and a classical result of Edmonds on matroids induced by submodular functions [3] to deduce that the rank of F in $\mathcal{R}^{\mathcal{Y}}(G)$ is given by

$$r^{\mathcal{Y}}(F) = \min_{\mathcal{K}} \left\{ \sum_{H \in \mathcal{K}} (2|H| - 2 - s_H) \right\} \tag{1}$$

where $s_H = 1$ if $|H| = 2$ or 3 and $s_H = 0$ if $|H| > 3$ and the minimum is taken over all thin covers $\mathcal{K}$ of F.

We next define the count matroid $\mathcal{M}_{uv}(G)$. Let $G = (V, E)$ be a graph and $u, v \in V$ be distinct vertices of G. We will prove that the family of sets

$$\mathcal{I}_G = \{F : F \subseteq E \text{ and } (V, F) \text{ is } uv\text{-sparse}\} \tag{2}$$

is the family of independent sets of a matroid $\mathcal{M}_{uv}(G)$ on E and characterise the rank function of this matroid. We need the following definition.

Let $\mathcal{H} = \{X_1, \ldots, X_t\}$ be a uv-compatible family and let $H_1, \ldots, H_k$ be subsets of V of size at least two. The system $\mathcal{K} = \{\mathcal{H}, H_1, \ldots, H_k\}$ is a *uv-cover* of $F \subseteq E$ if $F \subseteq \text{cov}(\mathcal{H}) \cup \text{cov}(\{H_1, \ldots, H_k\})$. It is *thin* if

(i) $\{H_1, \ldots, H_k\}$ is thin,

(ii) $X_i \cap X_j = \{u, v\}$ for all pairs $1 \leq i, j \leq t$, and

(iii) $|H_i \cap X_j| \leq 1$ for all $1 \leq i \leq k$, $1 \leq j \leq t$.

The value of the system $\mathcal{K}$ is given by $\text{val}(\mathcal{K}) = \text{val}(\mathcal{H}) + \sum_{i=1}^{k} \text{val}(H_i)$.

Theorem 3 *Let $G = (V, E)$ be a graph and $u, v \in V$ be distinct vertices of G. Then $\mathcal{M}_{uv}(G) = (E, \mathcal{I}_G)$ is a matroid on E, where $\mathcal{I}_G$ is defined by (2). The rank of a set $F \subseteq E$ in $\mathcal{M}_{uv}(G)$ is given by*

$$r_{uv}(F) = \min\{\text{val}(\mathcal{K}) : \mathcal{K} \text{ is either a thin cover or a thin } uv\text{-cover of } F\}. \tag{3}$$

Proof. Let $\mathcal{I} = \mathcal{I}_G$, let $E' \subseteq E$ and let $F \subseteq E'$ be a maximal subset of E' in $\mathcal{I}$. Since $F \in \mathcal{I}$ we have $|F| \leq \text{val}(\mathcal{K})$ whenever $\mathcal{K}$ is a cover or a uv-cover of E'. We shall prove that there is a thin cover or uv-cover $\mathcal{K}$ of E' with $|F| = \text{val}(\mathcal{K})$, from which the theorem will follow.

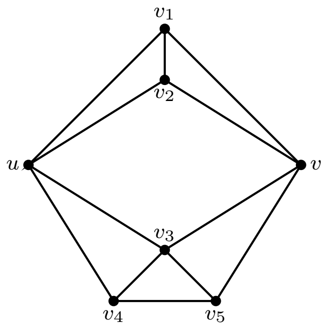

Fig. 1 An example of a $(2,2)$-tight graph $G=(V,E)$ which is not independent in $\mathcal{M}_{uv}(G)$. It is not difficult to see that G is (2,2)-sparse, and hence E is independent in the simple (2,2)-sparse matroid. We will show that E is not independent in $\mathcal{M}_{uv}(G)$. Consider the following sets: $H_1=\{u,v,v_1\}$, $H_2=\{u,v,v_2\}$ and $H_3=\{u,v,v_3,v_4,v_5\}$. Then $\mathcal{H}=\{H_1,H_2,H_3\}$ is a uv-compatible family of G with $\mathrm{val}(\mathcal{H})=\mathrm{val}(H_1)+\mathrm{val}(H_2)+\mathrm{val}(H_3)-2\cdot 2=(2\cdot 3-3)+(2\cdot 3-3)+(2\cdot 5-2)-4=10$ and $\mathrm{cov}(\mathcal{H})=E-v_1v_2$. Hence $i_G(\mathcal{H})=11>\mathrm{val}(\mathcal{H})$ so E is dependent in $\mathcal{M}_{uv}(G)$.

Let $J=(V,F)$ denote the subgraph defined by the edge set F. First suppose that there is no tight uv-compatible family in J and consider the following cover of F:

$$\mathcal{K}_1=\{H_1,H_2,\ldots,H_k\},$$

where $H_1,H_2,\ldots,H_t$ are the maximal tight sets with size at least four in J for some $t\le k$ and $H_{t+1},\ldots,H_k$ are the pairs of end vertices of edges in $J'=(V,F-\bigcup_{i=1}^{t}E(H_i))$. Clearly $\mathcal{K}_1$ is a cover of F. It is thin by Lemma 4. Thus

$$|F|=\sum_{j=1}^{k}|E_J(H_j)|=\sum_{j=1}^{k}(2|H_j|-t_j)=\mathrm{val}(\mathcal{K}_1)$$

follows. We claim that $\mathcal{K}_1$ is a cover of E'. To see this consider an edge $ab=e\in E'-F$. Since F is a maximal subset of E' in $\mathcal{I}$ we have $F+e\notin\mathcal{I}$. By our assumption there is no tight uv-compatible family in J, and hence there must be a tight set X in J with $a,b\in X$. Hence $X\subseteq H_i$ for some $1\le i\le t$ which implies that $\mathcal{K}_1$ covers e. (Recall that our graphs do not contain parallel edges so e is not parallel to any edge in F.)

Next suppose that there is a tight uv-compatible family in J and consider the following uv-cover of F:

$$\mathcal{K}_2=\{\mathcal{H}_{\max},H_1,H_2,\ldots,H_k\},$$

where: $\mathcal{H}_{\max}=\{X_1,X_2,\ldots,X_l\}$ is the tight uv-compatible family of G for which $\mathrm{cov}(\mathcal{H}_{\max})$ is maximal (given by Lemma 7); $H_1,H_2,\ldots,H_t$ are the maximal tight sets with size at least four of $J'=(V,F-E(\mathcal{H}_{\max}))$; and $H_{t+1},\ldots,H_k$ are the pairs of end vertices of edges in $J''=(V,F-E(\mathcal{H}_{\max})-\bigcup_{i=1}^{t}E(H_i))$. Then $\mathcal{K}_2$ is a uv-cover of F. By Lemmas 4 and 7, the uv-cover $\mathcal{K}_2$ is thin, and hence

$$|F|=\sum_{i=1}^{l}|E_J(X_i)|+\sum_{j=1}^{k}|E_J(H_j)|=\sum_{i=1}^{l}(2|X_i|-t_i)-2(l-1)+\sum_{j=1}^{k}(2|H_j|-t_j)=\mathrm{val}(\mathcal{K}_2).$$

We claim that $\mathcal{K}_2$ is a uv-cover of E'. As above, let $ab = e \in E' - F$ be an edge. By the maximality of F we have $F + e \notin \mathcal{I}$. Thus either there is a tight set $X \subseteq V$ in J with $a, b \in \text{cov}(X)$ or there is a tight uv-compatible family $\mathcal{H}' = \{Y_1, \dots, Y_t\}$ in J with $a, b \in Y_i$ for some $1 \leq i \leq t$.

In the latter case Lemma 7 implies that $\text{cov}(\mathcal{H}') \subseteq \text{cov}(\mathcal{H}_{\max})$ and hence e is covered by $\mathcal{K}_2$. In the former case, when $a, b \in X$ for some tight set X in J, we have $|X| \geq 5$ since if $|X| = 2, 3$ or 4 then X induces a complete graph in J and, since G has no parallel edges, $e = ab$ would be an edge of F. Lemma 7 now gives $|X \cap \bigcup_{i=1}^{l} X_i| \leq 1$. Then $E(X) \subseteq E(J')$ and hence $X \subseteq H_i$ for some $1 \leq i \leq k$, since every edge of J' induces a tight set and every tight set is contained in a maximal tight set. Thus e is covered by $\mathcal{K}_2$, as claimed. □

4 Characterisation of the uv-coincident cylinder rigidity matroid

Our aim is to show that the uv-coincident cylinder rigidity matroid $\mathcal{R}_{uv}^{\mathcal{Y}}(G)$ of a graph $G = (V, E)$ is equal to the count matroid $\mathcal{M}_{uv}(G)$. To simplify terminology we will say that G *is independent in* $\mathcal{R}_{uv}^{\mathcal{Y}}$*, respectively* $\mathcal{M}_{uv}$, if E is independent in $\mathcal{R}_{uv}^{\mathcal{Y}}(G)$, respectively $\mathcal{M}_{uv}(G)$.

We first show that independence in $\mathcal{R}_{uv}^{\mathcal{Y}}$ implies independence in $\mathcal{M}_{uv}$. Recall that G/uv denotes the graph obtained from G by contracting the vertex pair u, v into a new vertex which we denote as z_{uv}. Given a uv-coincident realisation (G, p) of G on $\mathcal{Y}$ we obtain a realisation $(G/uv, p_{uv})$ of G/uv on $\mathcal{Y}$ by putting $p_{uv}(z_{uv}) = p(u) = p(v)$ and $p_{uv}(x) = p(x)$ for all $x \in V \setminus \{u, v\}$. Furthermore, each vector in the kernel of $R^{\mathcal{Y}}(G/uv, p_{uv})$ determines a vector in the kernel of $R^{\mathcal{Y}}(G, p)$ in a natural way. It follows that $\dim \text{Ker} R^{\mathcal{Y}}(G, p) \geq \dim \text{Ker} R^{\mathcal{Y}}(G/uv, p_{uv})$ and hence

$$\text{rank}\, R^{\mathcal{Y}}(G, p) \leq \text{rank}\, R^{\mathcal{Y}}(G/uv, p_{uv}) + 3. \tag{4}$$

We can use this fact to prove that independence in $\mathcal{R}_{uv}^{\mathcal{Y}}$ implies independence in $\mathcal{M}_{uv}$.

Lemma 8 *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. If G is independent in $\mathcal{R}_{uv}^{\mathcal{Y}}$ then G is independent in $\mathcal{M}_{uv}$.*

Proof. Let (G, p) be a generic uv-coincident realisation of G on $\mathcal{Y}$. Since G is independent in $\mathcal{R}_{uv}^{\mathcal{Y}}$ the rows of $R^{\mathcal{Y}}(G, p)$ are independent. Since $p(u) = p(v)$, this gives $uv \notin E$. Furthermore if $X \subseteq V$ and $\{u, v\} \not\subseteq X$ then $(G[X], p|_X)$ is a generic realisation of $G[X]$ on $\mathcal{Y}$ and hence $i(X) \leq \text{val}(X)$ by Theorem 2. It remains to show that $i_G(\mathcal{H}) \leq \text{val}(\mathcal{H})$ for all uv-compatible families $\mathcal{H}$ in G. (Note that the case when $X \subseteq V$ and $\{u, v\} \subseteq X$ will be included by taking $\mathcal{H} = \{X\}$.)

Let $\mathcal{H} = \{X_1, \dots, X_k\}$ be a uv-compatible family and consider the subgraph $H = (\bigcup_{i=1}^{k} X_i, \bigcup_{i=1}^{k} E(X_i))$. By contracting the vertex pair u, v in H we obtain the graph H/uv. We have $\mathcal{H}_{uv} = \{X_1/uv, \dots, X_k/uv\}$ is a cover of H where X_i/uv denotes the set that we get from X_i by identifying u and v. Let $U = \bigcup_{i=1}^{k} X_i$ and $F = \bigcup_{i=1}^{k} E(X_i)$. By (1) we have

$$\begin{aligned}\text{rank}\, R^{\mathcal{Y}}(H/uv, p_{uv}) = r^{\mathcal{Y}}(F) + |U| - 1 &\leq \sum_{i=1}^{k} (2|X_i/uv| - 2 - s_{X_i/uv}) + |U| - 1 \\ &= \sum_{i=1}^{k} (2|X_i| - 2 - t_i) + |U| - 1.\end{aligned}$$

Using (4) and the fact that $R^{\mathcal{Y}}(G,p)$ has linearly independent rows, we have

$$|F|+|U| = \operatorname{rank} R^{\mathcal{Y}}(H,p) \le \operatorname{rank} R^{\mathcal{Y}}(H/uv, p_{uv}) + 3 \le \sum_{i=1}^{k}(2|X_i|-2-t_i)+2+|U|$$

$$= \sum_{i=1}^{k} \operatorname{val}(X_i) - (2k-2) + |U| = \operatorname{val}(\mathcal{H}) + |U|.$$

Hence $i_G(\mathcal{H}) = |F| \le \operatorname{val}(\mathcal{H})$. Thus G is independent in $\mathcal{M}_{uv}$, as claimed. □

We next define operations on uv-sparse graphs and use them to show that independence in $\mathcal{M}_{uv}$ implies independence in $\mathcal{R}^{\mathcal{Y}}_{uv}$.

The (two-dimensional versions of) the well-known Henneberg operations are as follows. Let $G = (V,E)$ be a graph. The 0-*extension* operation (on a pair of distinct vertices $a,b \in V$) adds a new vertex z and two edges za, zb to G. The *1-extension* operation (on edge $ab \in E$ and vertex $c \in V \setminus \{a,b\}$) deletes the edge ab, adds a new vertex z and edges za, zb, zc.

We shall need the following specialized versions. Let $u,v \in V$ be two distinct vertices. The *uv-0-extension* operation is a 0-extension on a pair a,b with $\{a,b\} \ne \{u,v\}$. The *uv-1-extension* operation is a 1-extension on some edge ab and vertex c for which $\{u,v\}$ is not a subset of $\{a,b,c\}$. The inverse operations are called *uv-0-reduction* and *uv-1-reduction*, respectively.

We will also need two further moves. The *vertex-to-K_4* move deletes a vertex w and substitutes in a copy of K_4 with $V(K_4) \cap V(G) = \{w\}$ and with an arbitrary replacement of edges xw by edges xy with $y \in V(K_4)$. The inverse operation is known as a *K_4-contraction.* A *vertex-to-4-cycle* move takes a vertex w with neighbours $v_1, v_2, \ldots, v_k$ for any $k \ge 2$, splits w into two new vertices w, w' with $w' \notin V(G)$, adds edges $wv_1, w'v_1, wv_2, w'v_2$ and then arbitrarily replaces edges xw with edges xy where $x \in \{v_3, \ldots, v_k\}$ and $y \in \{w, w'\}$. The inverse move is known as a *4-cycle-contraction.* The only difference in the specialised versions of these moves are that we require $|V(K_4) \cap \{u,v\}| \le 1$ in a uv-K_4-contraction and similarly $|V(C_4) \cap \{u,v\}| \le 1$ in a uv-4-cycle-contraction.

We first consider the 0-extension and 1-extension operations. It was shown in [10] that these operations preserve independence in $\mathcal{R}^{\mathcal{Y}}$. The same arguments can be used to verify analogous results for $\mathcal{R}^{\mathcal{Y}}_{uv}$.

Lemma 9 *Let $G = (V,E)$ be independent in $\mathcal{R}^{\mathcal{Y}}_{uv}$ and suppose that G' is obtained from G by a 0-uv-extension or a uv-1-extension Then G' is independent in $\mathcal{R}^{\mathcal{Y}}_{uv}$.*

In the case of 0-extensions we will also need the following result.

Lemma 10 *Let (G,p) be a generic realisation of a graph $G = (V,E)$ on $\mathcal{Y}$ and $v \in V$. Suppose that $R_{\mathcal{Y}}(G,p)$ has linearly independent rows. Let G' be obtained by performing a 0-extension which adds a new vertex u to G which is not adjacent to v. Put $p'(a) = p(a)$ for all $a \in V$, and put $p'(u) = p(v)$. Then $R^{\mathcal{Y}}(G',p')$ has linearly independent rows.*

Proof. The 0-extension adds 3 rows and 3 columns to $R_{\mathcal{Y}}(G,p)$, the 3 columns being 0 everywhere except the 3 new rows. The genericness of p and the fact that $uv \notin E$ implies the new 3×3 block is invertible. Hence $R_{\mathcal{Y}}(G',p')$ has linearly independent rows so G' is independent in $\mathcal{R}^{\mathcal{Y}}_{uv}$. □

We next consider the vertex-to-4-cycle operation. It was shown in [11] that this operation preserves independence in $\mathcal{R}^{\mathcal{Y}}$. A similar argument would yield the analogous result for $\mathcal{R}^{\mathcal{Y}}_{uv}$ but we will need a stronger result that a vertex-to-4-cycle move which creates two coincident vertices preserves independence in $\mathcal{R}^{\mathcal{Y}}$.

Lemma 11 *Suppose (G,p) is a framework on $\mathcal{Y}$, $R_{\mathcal{Y}}(G,p)$ has linearly independent rows and $w \in V$ with neighbours $v_1, v_2, \dots, v_k$. Suppose further that $p(w)-p(v_1), p(w)-p(v_2)$ and $\bar{p}(w)$ are linearly independent where $\bar{p}(w)$ is the projection of $p(w)$ onto the plane $z=0$. Let G' be obtained by performing a vertex-to-4-cycle operation in G which splits w into two vertices w and w', and is such that v_1 and v_2 are both adjacent to w and w' in G'. Put $p'(a)=p(a)$ for all $a \in V-w$ and put $p'(w)=p'(w')=p(w)$. Then $R^{\mathcal{Y}}(G',p')$ has linearly independent rows.*

Proof. We will construct $R^{\mathcal{Y}}(G',p')$ from $R^{\mathcal{Y}}(G,p)$ by a series of simple matrix operations that preserve the independence of the rows.

We first add three zero columns corresponding to w'. We then add three rows corresponding to the edges $w'v_1, w'v_2$ and the vertex w'. Adding these rows increases the rank by 3 since $p(w)-p(v_1), p(w)-p(v_2)$ and $\bar{p}(w)$ are linearly independent so the 3×3 matrix formed by the entries in the columns corresponding to w' and the rows corresponding to $w'v_1, w'v_2, w'$ is non-singular and the rest of the entries in these columns are zero. The matrix M we obtain by this modification has the following form:

$$
\begin{array}{c|ccc|}
 & \overbrace{\hphantom{p(w)-p(v_1)}}^{w} & \overbrace{\hphantom{p(w)-p(v_1)}}^{w'} & \\
\hline
(wv_1) & p(w)-p(v_1) & \mathbf{0} & \star \\
(wv_2) & p(w)-p(v_2) & \mathbf{0} & \star \\
 & \vdots & \vdots & \vdots \\
(wv_i) & p(w)-p(v_i) & \mathbf{0} & \star \\
 & \vdots & \vdots & \vdots \\
(w'v_1) & \mathbf{0} & p(w)-p(v_1) & \star \\
(w'v_2) & \mathbf{0} & p(w)-p(v_2) & \star \\
 & \vdots & \vdots & \vdots \\
\hline
w & \bar{p}(w) & \mathbf{0} & \mathbf{0} \\
w' & \mathbf{0} & \bar{p}(w) & \mathbf{0} \\
 & \vdots & \vdots & \vdots \\
\hline
\end{array} = M
$$

To obtain $R^{\mathcal{Y}}(G',p')$ from M we need to modify some of the rows in M corresponding to edges (wv_i) into the form of rows corresponding to edges $(w'v_i)$, i.e. we need to move the entries in the columns of w to the columns of w' and replace them with zeros. We will do this one by one.

Since $(p(w)-p(v_1))$, $(p(w)-p(v_2))$ and $\bar{p}(w)$ are linearly independent, for every $3 \le i \le k$ there exist unique values α, β, γ such that $\alpha(p(w)-p(v_1)) + \beta(p(w)-p(v_2)) + \gamma\bar{p}(w) = (p(w)-p(v_i))$. Now subtract the row of (wv_1) multiplied by α, the row of (wv_2) multiplied by β and the row of w multiplied by γ from the row of (wv_i) in M. Then add the row of $(w'v_1)$ multiplied by α, the row of $(w'v_2)$ multiplied by β and the row of w' multiplied by γ to the same row (and change its label from (wv_i) to $(w'v_i)$) for every neighbour v_i of w' in G' to obtain $R^{\mathcal{Y}}(G',p')$. These operations also preserve independence, thus we conclude that the rows of $R^{\mathcal{Y}}(G',p')$ are independent. □

Corollary 1 *Let G be independent in $\mathcal{R}^{\mathcal{Y}}_{uv}$ and suppose that G' is obtained from G by a vertex-to-4-cycle operation. Then G' is independent in $\mathcal{R}^{\mathcal{Y}}_{uv}$.*

Proof. We choose a generic uv-coincident realisation (G,p). Then (G,p) satisfies the hypotheses of Lemma 11. Hence G' has a uv-coincident realisation (G',p') such that $R^{\mathcal{Y}}(G',p')$ has linearly independent rows. It follows that every generic uv-coincident realisation is independent. □

We next consider a generalisation of the vertex-to-K_4 operation which replaces K_4 with an arbitrary minimally rigid subgraph. It was shown in [10] that this operation preserves independence in $\mathcal{R}^{\mathcal{Y}}$. We will need an analogous result for uv-coincident realisations.

Lemma 12 *Let $G = (V, E)$ be a graph with $|E| = 2|V| - 2$ and let $u, v \in V$ be distinct vertices. Suppose $H \subset G$ is chosen so that either:*
(a) $u, v \in V(H)$, H is minimally uv-rigid on $\mathcal{Y}$ and G/H is minimally rigid on $\mathcal{Y}$, or
(b) $|\{u, v\} \cap V(H)| \leq 1$, H is minimally rigid on $\mathcal{Y}$ and G/H is minimally uv-rigid on $\mathcal{Y}$. (Taking u or v to be the vertex of G/H obtained by contracting H when $\{u, v\} \cap V(H) = \{u\}$ or $\{u, v\} \cap V(H) = \{v\}$, respectively.)
Then G is uv-rigid on $\mathcal{Y}$.

Proof. (a) Let $|V| = n$, $|V(H)| = r$ and consider $R_{\mathcal{Y}}(G, p)$ where (G, p) is a generic uv-coincident framework on $\mathcal{Y}$ and $p = (p(v_1), p(v_2), \ldots, p(v_n))$. By reordering rows and columns if necessary we can write $R_{\mathcal{Y}}(G, p)$ in the form

$$\begin{pmatrix} R_{\mathcal{Y}}(H, p|_H) & 0 \\ M_1(p) & M_2(p) \end{pmatrix}$$

where $M_2(p)$ is a square matrix with $3(n - r)$ rows.

Suppose, for a contradiction, that G is not uv-rigid. Then there exists a vector $m \in \ker R_{\mathcal{Y}}(G, p)$ which is not an infinitesimal isometry of $\mathcal{Y}$. Since $(H, p|_H)$ is uv-rigid we may suppose that $m = (0, \ldots, 0, m_{r+1}, \ldots, m_n)$. Consider the realisation (G, p') where $p' = (p(v_r), p(v_r), \ldots, p(v_r), p(v_{r+1}), \ldots, p(v_n))$ and define the realisation $(G/H, p^*)$ by setting $p^* = (p(v_r), p(v_{r+1}), \ldots, p(v_n))$. Since p^* is generic, $(G/H, p^*)$ is infinitesimally rigid on $\mathcal{Y}$ by assumption.

Now, $M_2(p)$ is square with the nonzero vector $(m_{r+1}, \ldots, m_n) \in \ker M_2(p)$. Hence $\operatorname{rank} M_2(p) < 3(n - r)$. Since p is generic, we also have $\operatorname{rank} M_2(p') < 3(n - r)$ and hence there exists a nonzero vector $m' \in \ker M_2(p')$. Therefore we have

$$\left(R_{\mathcal{Y}}(G/H, p^*)\right) \begin{pmatrix} 0 \\ m' \end{pmatrix} = \begin{pmatrix} p(v_r) & 0 \\ \star & M_2(p') \end{pmatrix} \begin{pmatrix} 0 \\ m' \end{pmatrix} = 0,$$

contradicting the infinitesimal rigidity of $(G/H, p^*)$.

(b) A similar proof holds. We choose a generic uv-coincident framework (G, p), a vector $m \in \ker R_{\mathcal{Y}}(G, p)$ which is not an infinitesimal isometry of $\mathbb{R}^3$, and uv-coincident realisations (G, p') and $(G/H, p^*)$ as above. We then use the facts that H is rigid on $\mathcal{Y}$ and G/H is uv-rigid on $\mathcal{Y}$ to obtain a contradiction. $\square$

We next consider the uv-0-reduction, uv-1-reduction, uv-K_4-contraction and uv-4-cycle contraction operations.

Lemma 13 *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Suppose that $|E| = 2|V| - 2$, G is independent in $\mathcal{M}_{uv}$, and $d(w) \geq 3$ for all $w \in V$. Then either there is a vertex $z \in V \setminus \{u, v\}$ with $d(z) = 3$ and $|N(z) \cap \{u, v\}| \leq 1$ or there is a 4-cycle in G which contains both u and v.*

Proof. Since $|E| = 2|V| - 2$ and $d(w) \geq 3$ for all $w \in V$, there are at least 4 vertices of degree 3. Since G is independent in $\mathcal{M}_{uv}$, G has at most two vertices which are adjacent to both u and v. Hence, if there is no vertex $z \in V \setminus \{u, v\}$ with $d(z) = 3$ and $|N(z) \cap \{u, v\}| \leq 1$, then the vertices of degree 3 must induce a C_4 in G which contains both u and v. $\square$

Lemma 14 *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Suppose that G is independent in $\mathcal{M}_{uv}$, and there are vertices a, b such that a, u, b, v is a cycle in G. Then the uv-4-cycle contraction which merges u and v results in a simple graph G' which is $(2,2)$-sparse.*

Proof. The independence of G in $\mathcal{M}_{uv}$ implies that there is no vertex other than a, b that is adjacent with both u and v. Thus G' is simple. Suppose G' is not $(2,2)$-sparse. Then there exists a $(2,2)$-tight set X in G that contains u, v and exactly one of a and b, say a. Let $\{X, \{u, v, b\}\} = \mathcal{H}$. Then $i(\mathcal{H}) = 2|X| - 2 + 2$ and $\mathrm{val}(\mathcal{H}) = 2|X| - 2 + 3 - 2$ which contradicts the independence of G in $\mathcal{M}_{uv}$. □

Lemma 15 *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Suppose that G is independent in $\mathcal{M}_{uv}$ and let $z \in V \setminus \{u, v\}$ with $N(z) = \{v_1, v_2, v_3\}$ and $|N(z) \cap \{u, v\}| \leq 1$. Then either:*
(a) there is a 1-reduction at z which leads to a graph which is independent in $\mathcal{M}_{uv}$, or
(b) z and its neighbours induce a copy of K_4 in G, or
(c) $v_i \in \{u, v\}$ and $v_j v_k \in E$ for some $\{i, j, k\} = \{1, 2, 3\}$, and there is a tight uv-compatible family $\{X_1, X_2, \ldots, X_k\}$ in G such that $X_1 = N(z) \cup \{u, v, z\}$ and $i(X_1) \geq 2|X_1| - 4$.

Proof. Suppose (a) does not occur. Then, for all $1 \leq i < j \leq 3$, either $v_i v_j \in E$, or there exists a tight uv-compatible family $\mathcal{H}_{ij}$ in $G - z$ with $v_i v_j \in \mathrm{cov}(\mathcal{H}_{ij})$ or there exists a tight set X_{ij} in $G - z$ with $\{v_i, v_j\} \subset X_{ij}$ and $\{u, v\} \not\subset X_{ij}$. If the second alternative occurs we may assume that $\mathcal{H}_{ij}$ has been chosen to be the unique tight uv-compatible family in $G - z$ with maximal cover. If $G[v_1, v_2, v_3] \cong K_3$ then (b) occurs. So we may assume that $v_1 v_2 \notin E$.

We first show that

$$v_i v_j \notin E \text{ and that } \mathcal{H}_{ij} \text{ exists for some } 1 \leq i < j \leq 3. \tag{5}$$

Suppose $\mathcal{H}_{12}$ does not exist. Then X_{12} exists. If $v_3 \in X_{12}$ then $X_{12} + z$ contradicts the independence of G in $\mathcal{M}_{uv}$. Hence $v_3 \notin X_{12}$. If $v_1 v_3, v_2 v_3 \in E$ then $X_{12} \cup \{v_3, z\}$ contradicts the independence of G in $\mathcal{M}_{uv}$. Hence suppose that $v_1 v_3 \notin E$. If X_{13} exists, then $X_{12} \cup X_{13} \cup \{z\}$ contradicts the independence of G in $\mathcal{M}_{uv}$. Hence $\mathcal{H}_{13}$ exists. This proves (5).

Relabeling if necessary we assume that $\mathcal{H}_{12} = \{X_1, X_2, \ldots, X_k\}$ exists. Since $v_1 v_2 \in \mathrm{cov}(\mathcal{H}_{12})$ we have $v_1, v_2 \in X_i$ for some $1 \leq i \leq k$. If $v_3 \in X_i$ then $|X_i| \geq 4$, since $|N(z) \cap \{u, v\}| \leq 1$, and the uv-compatible family obtained from $\mathcal{H}_{12}$ by replacing X_i by $X_i + z$ will contradict the independence of G in $\mathcal{M}_{uv}$. Hence $v_3 \notin X_i$.

Suppose that $\{v_1, v_2\} \cap \{u, v\} = \emptyset$. Then $|X_i| \geq 4$. Since $v_3 \notin X_i$, neither $v_1 v_3$ nor $v_2 v_3$ are covered by $\mathcal{H}_{12}$. The maximality of $\mathrm{cov}(\mathcal{H}_{12})$ now implies that $\mathcal{H}_{13}$ and $\mathcal{H}_{23}$ do not exist. If $v_1 v_3, v_2 v_3 \in E$, then the uv-compatible family obtained from $\mathcal{H}_{12}$ by replacing X_i by $X_i + v_3$ would be tight and hence would contradict the maximality of $\mathrm{cov}(\mathcal{H}_{12})$, since the new family would cover $v_1 v_3$ and $v_2 v_3$. Relabeling if necessary, we may suppose that $v_1 v_3 \notin E$, and hence X_{13} exists. Then $X_i \cap X_{13} \neq \emptyset$, $|X_i| \geq 4$, $|X_{13}| \geq 4$ and $v_1 v_3 \in \mathrm{cov}(X_{13}) \setminus \mathrm{cov}(\mathcal{H}_{12})$. This contradicts the final part of Lemma 7. Hence $\{v_1, v_2\} \cap \{u, v\} \neq \emptyset$ and we may assume, without loss of generality, that $u = v_1$.

If $v_3 \notin V(\mathcal{H}_{12})$, then Lemma 7(c) implies that $v_1 v_3 \notin E$ and hence X_{13} exists. This contradicts the final part of Lemma 7 since $u \in X_{13} \cap X_i$. Hence $v_3 \in X_j$ for some $X_j \in \mathcal{H}_{12} - X_i$. The final part of Lemma 7 now implies that X_{23} does not exist and hence $v_2 v_3 \in E$.

Let $X = X_i \cup X_j \cup \{z\}$ and $\mathcal{H} = (\mathcal{H}_{12} \setminus \{X_i, X_j\}) \cup \{X\}$. We have $i_G(\mathcal{H}) \geq i_G(\mathcal{H}_{12}) + 4$ since $zv_1, zv_2, zv_3, v_2 v_3 \in E(X)$ and $\mathrm{val}(\mathcal{H}) = \mathrm{val}(\mathcal{H}_{12}) + t_{x_1} + t_{x_2} - t_X \leq \mathrm{val}(\mathcal{H}_{12}) + 4$ with equality only if $|X_i| = |X_j| = 3$. The facts that G is independent in $\mathcal{M}_{uv}$ and $\mathcal{H}_{12}$ is tight now imply that $|X_i| = 3 = |X_j|$ (so $X = N(z) \cup \{u, v, z\}$), and that $\mathcal{H}$ is a tight uv-compatible family in G with $i(X) \geq 2|X| - 4$. $\square$

Lemma 16 *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Suppose that G is independent in $\mathcal{M}_{uv}$, $\mathcal{H} = \{X_1, X_2, \ldots, X_k\}$ is a tight uv-compatible family in G and that $\mathcal{H} - X_i$ is not tight for all $1 \leq i \leq k$. Then either:*
(a) $k = 1$ and X_1 is tight;
(b) $k = 2$, $|X_1| = |X_2| = 3$ and $i(X_1) = i(X_2) = 2$;
(c) $k = 2$, $|X_1| \geq 4$, $i(X_1) = 2|X_1| - 3$, $|X_2| = 3$ and $i(X_2) = 2$; or
(d) $k = 2$, $|X_i| \geq 4$ and $i(X_i) = 2|X_i| - 3$ for all $i \in \{1, 2\}$.

Proof. We have $i(\mathcal{H} - X_i) = i(H) - i(X_i)$ and $\mathrm{val}(\mathcal{H} - X_i) = \mathrm{val}(\mathcal{H}) - (2|X_i| - 2 - t_i)$. Since $i(\mathcal{H} - X_i) < \mathrm{val}(\mathcal{H} - X_i)$ this gives $i(X_i) \geq 2|X_i| - 2 - t_i$ and hence $i(X_i) \geq 2|X_i| - 3$ if $|X_i| \geq 4$ and $i(X_i) = 2$ if $|X_i| = 3$. In both cases we have $i(X_i) \geq \mathrm{val}(X_i) - 1$. Since G is independent in $\mathcal{M}_{uv}$ we have $i(\mathcal{H}) \leq \mathrm{val}(\mathcal{H}) = \sum_{i=1}^{k}(\mathrm{val}(X_i) - 2) + 2$. This proves that $k = 1$ or $k = 2$. The assertion that X_1 is tight in (a) and the assertions on $i(X_1)$ and $i(X_2)$ in (b), (c) and (d) now follow from the hypothesis that $\mathcal{H}$ is tight. $\square$

Note that if alternative (d) holds then $X_1 \cup X_2$ is tight so we can reduce to alternative (a).

Lemma 17 *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Suppose that G is independent in $\mathcal{M}_{uv}$ and that there exists a subgraph H of G isomorphic to K_4. Then either:*
(a) there is a vertex $x \in V \setminus V(H)$ such that $|N(x) \cap V(H)| = 2$,
(b) $|V(H) \cap \{u, v\}| = 1 = |N(V(H)) \cap \{u, v\}|$,
(c) there is a tight uv-compatible family $\{X_1, X_2, \ldots, X_k\}$ in G such that $X_1 = V(H) \cup \{u, v\}$, $|X_1| = 6$ and $i(X_1) = 8$,
(d) there is a tight uv-compatible family $\{X_1, X_2, \ldots, X_k\}$ in G such that $X_1 = V(H) \cup \{u, v, a\}$ for some $a \in V \setminus (V(H) \cup \{u, v\})$, $|X_1| = 6$ and $i(X_1) = 8$, or
(e) the contraction of H gives a graph G' which is independent in $\mathcal{M}_{uv}$.

Proof. Since G is independent in $\mathcal{M}_{uv}$, $uv \notin E$ and hence $|V(H) \cap \{u, v\}| \leq 1$. Suppose that (a), (b) and (e) fail. Since (a) fails, no vertex of $V \setminus V(H)$ is adjacent to two vertices of H and hence the graph G' obtained by contracting H has no parallel edges. We label the new vertex obtained by contracting H as w (taking $w = u$ if $u \in V(H)$ and $w = v$ if v if $v \in V(H)$). It is easy to check that G' is $(2, 2)$-sparse. Since (b) fails, $uv \notin E(G')$. Since (e) fails, there is a uv-compatible family $\mathcal{H}' = \{X_1', X_2', \ldots, X_k'\}$ for which $\mathrm{val}(\mathcal{H}') < i_{G'}(\mathcal{H}')$ and $w \in V(\mathcal{H}')$. Without loss of generality we may assume $w \in X_1'$. If $|X_1'| \geq 4$ then we get a contradiction as the uv-compatible family $\mathcal{H} = \{(X_1' - w) \cup V(H), X_2', \ldots, X_k'\}$ of G violates independence. If $|X_1'| = 3$ and $V(H) \cap \{u, v\} = \emptyset$ then $\mathcal{H}$ is the uv-compatible family described in (c). Finally if $|X_1'| = 3$ and $|V(H) \cap \{u, v\}| = 1$ then $X_1' = \{u, v, a\}$ for some $a \in V \setminus (V(H) \cup \{u, v\})$ and $\mathcal{H}'' = \{V(H) \cup \{u, v, a\}, X_2, \ldots, X_k\}$ is the uv-compatible family described in (d). $\square$

Lemma 18 *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Suppose that G is independent in $\mathcal{M}_{uv}$, $z \in V \setminus \{u, v\}$ is a vertex of degree 3 with $N(z) = \{v_1, v_2, v_3\}$, $|N(z) \cap \{u, v\}| \leq 1$ and $G[N(z) + z]$ is isomorphic to K_4. Suppose further that there is a vertex $x \in V \setminus \{z, v_1, v_2, v_3\}$ such that $N(x) \cap N(z) = \{v_2, v_3\}$ and $\{v_1, x\} \neq \{u, v\}$. Then the uv-4-cycle contraction operation which contracts x and z into a single vertex x leads to a graph G' which is independent in $\mathcal{M}_{uv}$.*

Proof. Suppose G' is not independent in $\mathcal{M}_{uv}$. Since $G' = G - z + v_1x$ and $xv_1 \notin E$, there exists either a tight uv-compatible family $\mathcal{H}$ in $G - z$ with $xv_1 \in \text{cov}(\mathcal{H})$, or a tight set X in $G - z$ with $\{x, v_1\} \subset X$. Set $Y = \{z, v_1, v_2, v_3, x\}$. Then Y is tight in G.

Suppose X exists. Then $X \cup Y$ and $X \cap Y$ are tight by Lemma 4. Since $\{v_1, x\} \subseteq X \cap Y$ and no proper subset of Y containing v_1 and x is tight, we have $X \cap Y = Y$. This implies that $z \in X$ contradicting the choice of X. Hence $\mathcal{H} = \{X_1, X_2, \ldots, X_k\}$ exists.

Since $xv_1 \in \text{cov}(\mathcal{H})$, we may assume, without loss of generality, that $x, v_1 \in X_1$. Then $x, v_1 \in X_1 \cap Y$. Since $|\{u, v\} \cap Y| \leq 1$ by the hypotheses of the lemma, Lemma 5 implies that $X_1 \cap Y$ is tight. Since no proper subset of Y containing v_1 and x is tight we have $X_1 \cap Y = Y$. This implies that $z \in X_1$ and contradicts the choice of $\mathcal{H}$. □

We can now show that $\mathcal{R}^{\mathcal{Y}}_{uv}(K_n) = \mathcal{M}_{uv}(K_n)$ for all complete graphs K_n with $n \geq 2$. We do this by proving that, for all $G \subseteq K_n$, G is independent in $\mathcal{R}^{\mathcal{Y}}_{uv}$ if and only if G is independent in $\mathcal{M}_{uv}$. Necessity will follow from Lemma 8. We prove sufficiency inductively. We show that a graph G which is independent in $\mathcal{M}_{uv}$ can be reduced to a smaller such graph by the operations of uv-0-extension, uv-1-extension, vertex-to-4-cycle and vertex-to-K_4 and its generalisation. We then apply induction to deduce that the smaller graph is independent in $\mathcal{R}^{\mathcal{Y}}_{uv}$. This will imply that G is independent in $\mathcal{R}^{\mathcal{Y}}_{uv}$ since the inverse operations preserve independence in $\mathcal{R}^{\mathcal{Y}}_{uv}$.

Theorem 4 *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Then G is independent in $\mathcal{R}^{\mathcal{Y}}_{uv}$ if and only if G is independent in $\mathcal{M}_{uv}$.*

Proof. Necessity follows from Lemma 8. Now suppose that G is independent in $\mathcal{M}_{uv}$. We prove that G is independent in $\mathcal{R}^{\mathcal{Y}}_{uv}$ by induction on $|V|$. It is straightforward to check that G is independent in $\mathcal{R}^{\mathcal{Y}}_{uv}$ when $|V| \leq 4$. Hence we may assume that $|V| \geq 5$. By extending $|E|$ to a base of $\mathcal{M}_{uv}(K_{|V|})$ if necessary, we may also assume that $|E| = 2|V| - 2$.

Case 1. *G* contains a vertex of degree 2. First suppose that u has degree 2. Then $G - u$ is $(2, 2)$-sparse. Hence, by Theorem 2, $R_{\mathcal{Y}}(G - u, p)$ has linearly independent rows for any generic p. We can now use Lemma 10 to show that G is independent in $\mathcal{R}^{\mathcal{Y}}_{uv}$.

Now, suppose that there is a vertex $w \in V \setminus \{u, v\}$ with $d(w) = 2$. Let $N(w) = \{a, b\}$. Clearly, $a \neq b$ holds. If $\{a, b\} = \{u, v\}$ then let $\mathcal{H} = \{\{u, v, w\}, \{V - w\}\}$, where $|V - w| \geq 4$. We have

$$2|V| - 2 = |E| = i_E(\mathcal{H}) \leq \text{val}(\mathcal{H}) = 2 \cdot 3 - 3 + 2(|V| - 1) - 2 - 2 = 2|V| - 3,$$

a contradiction. Hence $\{a, b\} \neq \{u, v\}$, which implies that the 0-uv-reduction operation can be applied at w to obtain a graph $G' = (V - w, E')$ that is independent in $\mathcal{M}_{uv}$ and satisfies $|E'| = 2|V - w| - 2$. By induction, G' is independent in $\mathcal{R}^{\mathcal{Y}}_{uv}$. Now Lemma 9 implies that G is independent in $\mathcal{R}^{\mathcal{Y}}_{uv}$.

Case 2. There is a 4-cycle in *G* containing *u* and *v*. By Lemma 14, we may apply a uv-4-cycle-contraction (contracting u and v) to obtain a graph H which is simple and $(2, 2)$-sparse. Theorem 2 implies that any generic realisation (H, p) on $\mathcal{Y}$ is infinitesimally rigid. Now we can use Lemma 11 to show that G is independent in $\mathcal{R}^{\mathcal{Y}}_{uv}$.

Henceforth we assume that Cases 1 and 2 do not occur.

Case 3. There is a proper tight set *X* containing *u* and *v*. Since Case 1 does not occur, we may suppose X is a maximal proper tight set (where proper means $X \neq V$ and maximal means there is no vertex $w \in V \setminus X$ with more than one neighbour in X). Now by the maximality of X, G/X is simple and $|V \setminus X| \geq 3$. Hence G/X is $(2, 2)$-tight. Theorem 2 implies that any generic framework $(G/X_1, p)$ on $\mathcal{Y}$ is infinitesimally rigid. We may now apply Lemma 12(a) to show that G is independent in $\mathcal{R}_{uv}^{\mathcal{Y}}$.

Henceforth we may assume that Case 3 does not occur.

Case 4. There is a degree three vertex z in G which is contained in a subgraph $H \cong K_4$, and a vertex $x \in V \setminus V(H)$ such that $|V(H) \cap N(x)| = 2$. If $\{u, v\} \not\subset V(H) \cup \{x\}$ then we may apply Lemma 18 to find a graph G' which is independent in $\mathcal{M}_{uv}$. We can now use Corollary 1 to show that G is independent in $\mathcal{R}_{uv}^{\mathcal{Y}}$. Thus we may suppose that $\{u, v\} \subset V(H) \cup \{x\}$. Then $H \cup \{x\}$ is tight. This contradicts the assumption that Case 1 (if $H \cup \{x\} = V$) or Case 3 (if $H \cup \{x\} \neq V$) do not occur.

A vertex z of degree 3 in G is *bad* if either

- $z \in \{u, v\}$, or
- z is adjacent to both u and v, or
- z satisfies alternative (c) of Lemma 15 with $X_1 = N(z) \cup \{u, v, z\}$ and $i(X_1) \geq 2|X_1| - 3$, or
- z belongs to a subgraph $H \cong K_4$ satisfying alternative (b) of Lemma 17.

Otherwise we say that z is *good*.

Case 5. All degree three vertices are bad. We may use Lemma 13 and the fact that Case 2 does not occur to deduce there exists a degree three vertex $v_1 \in V \setminus \{u, v\}$ with $|N(v_1) \cap \{u, v\}| \leq 1$. Since v_1 is bad either

(i) v_1 satisfies alternative (c) of Lemma 15 with $X_1 = N(v_1) \cup \{u, v, v_1\}$ and $i(X_1) \geq 2|X_1| - 3$, or
(ii) v_1 belongs to a subgraph $H \cong K_4$ satisfying alternative (b) of Lemma 17.

If (i) occurs then the fact that G is independent in $\mathcal{M}_{uv}$ implies that $i(X_1) \leq 2|X_1| - 2 = 8$ and the fact that Case 2 does not occur tells us equality cannot hold. Hence $i(X_1) = 2|X_1| - 3 = 7$. It follows that we may interchange the labels of u and v and also of v_2 and v_3 such that $L = G[N(v_1) \cup \{u, v, v_1\}]$ is the graph in Figure 2(a) if (i) occurs and the graph in Figure 2(b) if (ii) occurs.

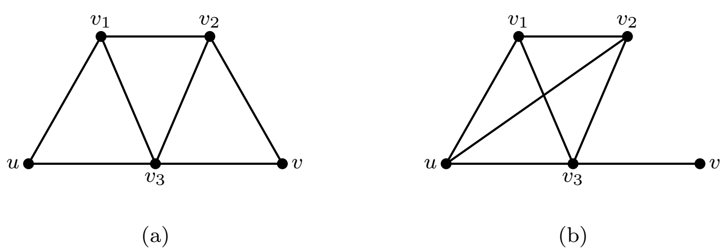

Fig. 2 The two alternatives for L.

The fact that G is $(2, 2)$-sparse implies that, in both cases, there exists a (necessarily bad) degree three vertex $v_4 \in V \setminus V(L)$. Since Case 2 does not occur, v_4 is not adjacent to both u and v. We may now repeat the argument from the previous paragraph to deduce that v_4 also belongs to a subgraph L' which is isomorphic to one of the graphs shown in Figure 2. Let $V(L') = \{v_4, u', v', v_2', v_3'\}$ where $\{u', v'\} = \{u, v\}$. Since Case 2 does not occur, $v_3' = v_3$. If $v_1 \in V(L')$ then we must have $v_1 = v_2'$. Since $v_4 \in N(v_2') = N(v_1) \subseteq V(L)$ this would contradict the fact that $v_4 \in V \setminus V(L)$. Hence $v_1 \notin V(L')$ and $\{u, v, v_3\} \subseteq V(L) \cap V(L') \subseteq \{u, v, v_2, v_3\}$.

We first consider the case when $V(L) \cap V(L') = \{u, v, v_2, v_3\}$. Since Case 2 does not occur v_2 is not adjacent to both u and v and hence $u = u'$ and $v = v'$. Since G is $(2,2)$-sparse $L \cup L'$ is as shown in Figure 3(a) and (b).

We next consider the case when $V(L) \cap V(L') = \{u, v, v_3\}$. Since G is $(2,2)$-sparse $L \cup L'$ is as shown in Figure 3(c), (d) and (e) up to a relabeling of u and v.

Since all five graphs in Figure 3 are tight, we may use the fact that Case 3 does not occur to deduce that $G = L \cup L'$. The fact that Case 1 does not occur now tells us that G is not the graph in Figure 3(a), (b) or (c). The graph in Figure 3(d) cannot be equal to G since $X_1 = N(v_1) \cup \{u, v, v_1\}$ does not belong to a tight uv-compatible family (so v_1 is not bad). Hence G is as shown in Figure 3(e).

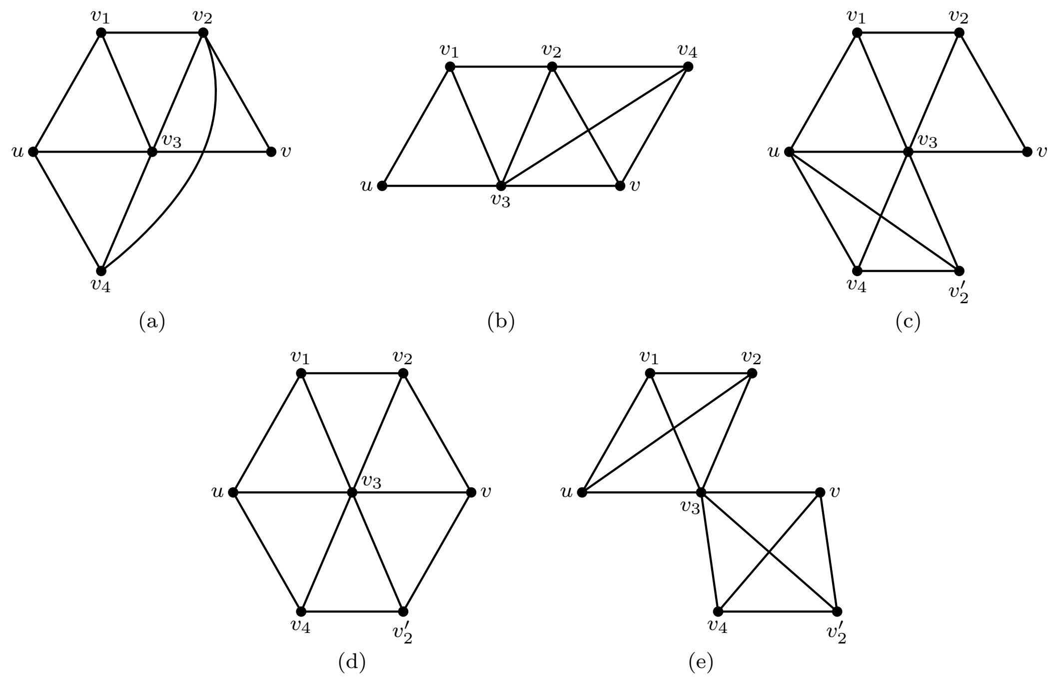

Fig. 3 The five alternatives for G.

We will complete the discussion of this case by showing that G is minimally uv-rigid on $\mathcal{Y}$. Let (G, p) be a generic uv-coincident realisation of G on $\mathcal{Y}$ and m be an infinitesimal motion of (G, p) with $m(u) = 0$. Since K_4 is rigid, $m(w) = 0$ for all $w \in V(L) - v$. In particular $m(v_3) = 0$ and hence $m(w) = 0$ for all $w \in V$.

Case 6. None of the previous cases occur. Let $z_1, z_2, \ldots, z_k$ be the good degree three vertices in G. If the edge set of some 1-reduction of G at z_i is independent in $\mathcal{M}_{uv}$ then we may apply induction to the reduced graph and then apply Lemma 9 to deduce that G is independent in $\mathcal{R}_{uv}^{\mathcal{Y}}$. Hence we may assume that alternative (b) or (c) of Lemma 15 holds for z_i.

Suppose alternative (b) of Lemma 15 holds for z_i. If the contraction of the K_4-subgraph H_i which contains z_i results in a graph which is independent in $\mathcal{M}_{uv}$, then we may apply induction to the reduced graph and then apply Lemma 12 to deduce that G is independent in $\mathcal{R}_{uv}^{\mathcal{Y}}$. (Note that the contracted graph is minimally rigid since Case 4 does not hold and since z_i is good, z_i is adjacent to at most one of $\{u,v\}$ so $|\{u,v\} \cap V(H_i)| \leq 1$. Thus (G, H_i) satisfies the hypotheses of Lemma 12(b).) Hence the contraction of H_i in G is not independent in $\mathcal{M}_{uv}$ and alternative (e) of Lemma 17 does not occur. In addition alternatives (a) and (b) of Lemma 17 do not occur since Case 4 does not hold and z_i is good. Hence there exists a tight uv-compatible family $\mathcal{H}_i$ satisfying alternatives (c) or (d) of Lemma 17.

In summary we have shown that for every good vertex z_i either alternatives (c) or (d) of Lemma 17 or alternative (c) of Lemma 15 hold. We assume that the first alternative holds for all $1 \leq i \leq l$ and that the second alternative holds for $l+1 \leq i \leq k$. Let X_i be the element of $\mathcal{H}_i$ which contains $V(H_i)$ for $1 \leq i \leq l$, where H_i and $\mathcal{H}_i$ are as defined in the previous paragraph. In addition for all $l+1 \leq i \leq k$ alternative (c) of Lemma 15 holds so there exists a tight uv-compatible family $\mathcal{H}_i$ such that $X_i = \{z_i, u, v\} \cup N(z_i)$ belongs to $\mathcal{H}_i$. With these definitions we have $i(X_i) = 2|X_i| - 4$ for all $1 \leq i \leq k$. (This follows from Lemma 17 when $1 \leq i \leq l$ and from Lemma 15 and the fact that z_i is good when $l+1 \leq i \leq k$.)

Let $X = \bigcup_{i=1}^{k} X_i$. We will show by induction that $i(X) \geq 2|X| - 4$. Suppose that we have $i(X') \geq 2|X'| - 4$ for some $X' = \bigcup_{i=1}^{s} X_i$ and some $1 \leq s \leq k$. If $i(X' \cup X_{s+1}) \leq 2|X' \cup X_{s+1}| - 5$, then Lemma 1(a) implies that $i(X' \cap X_{s+1}) \geq 2|X' \cap X_{s+1}| - 3$, this would contradict the fact that G is independent in $\mathcal{M}_{uv}$ since the uv-compatible family $\mathcal{H}'_{s+1}$ which we get from $\mathcal{H}_{s+1}$ by replacing X_{s+1} by $X' \cap X_{s+1}$ would satisfy $i(\mathcal{H}'_{s+1}) - \mathrm{val}(\mathcal{H}'_{s+1}) > i(\mathcal{H}_{s+1}) - \mathrm{val}(\mathcal{H}_{s+1}) = 0$.

We may apply Lemma 16 to a minimal tight uv-compatible subfamily of $\mathcal{H}_i$ for all $1 \leq i \leq k$, and use the facts that Cases 2 and 3 do not occur to deduce that alternatives (a) and (b) of Lemma 16 cannot hold for this family. In addition the remark after Lemma 16 implies that (d) cannot hold either so (c) must hold for this minimal subfamily. Hence there exist sets Y_i and $\{u, v, y_i\}$ in $\mathcal{H}_i$ with $i(Y_i) = 2|Y_i| - 3$ and $i(\{u, v, y_i\}) = 2$. Note that neither set can be equal to X_i since $|X_i| > 3$ and $i(X_i) = 2|X_i| - 4$. Lemma 2(b) implies that $Y_i \cap X_i = \{u, v\} = Y_i \cap \{u, v, y_i\}$ for all $1 \leq i \leq k$. The fact that we are not in Case 2 also implies that $y_i = y_j = y$, say, for all $1 \leq i \leq j \leq k$. Let $Y = \bigcap_{i=1}^{k} Y_i$. Then $Y \cap X = \{u, v\}$ and $y \notin Y$. We can now use Lemma 1(a) and the fact that G contains no proper tight subset containing u and v (since Case 3 does not occur) to prove inductively that $i(Y) = 2|Y| - 3$.

Let $W = V \setminus X$. Since $i(W) \leq 2|W| - 2$ there is an integer t for which $i(W) = 2|W| - 2 - t$. Since $i(Y) = 2|Y| - 3$ and G is $(2, 2)$-sparse, there are at least 3 edges from $Y \setminus \{u, v\}$ to $\{u, v\}$. Since $Y \setminus \{u, v\} \subseteq W$, $y \in W \setminus Y$ and there are two edges from y to $\{u, v\}$, we have at least five edges between $\{u, v\}$ and W. Note that the definition of X tells us that all degree 3 vertices in W are bad.

Suppose that every (bad) degree three vertex in W is adjacent to both u and v. Since Case 2 does not occur we have at most one degree three vertex in W. Since $i(X) \geq 2|X| - 4$, we have $|E| - |E(X)| - |E(W)| \leq 2|V| - 2 - (2|X| - 4) - (2|W| - 2 - t) = 4 + t$. Hence the sum of the degrees of the vertices in W is at most $2(2|W| - 2 - t) + 4 + t = 4|W| - t$. Since there is at most one degree three vertex in W, $t \leq 1$. If $t = 0$, then W is tight and $W + u + v$ violates sparsity since there are at least 5 edges between W and $\{u, v\}$. Hence $t = 1$ and $W + u + v$ is a proper tight set which contradicts the fact that Case 3 does not occur.

Now consider the case when there is a (bad) degree three vertex $z \in W$ which is not adjacent to both u and v. Since z is bad there is either a set $Z \subseteq V$ which satisfies alternative (c) of Lemma 15 and has $i(Z) \geq 2|Z| - 3$, or z belongs to a subgraph $H \cong K_4$ that satisfies alternative (b) of

Lemma 17. We can now deduce, as in Case 5, that $J = G[N(z) \cup \{u, v, z\}]$ is isomorphic to one of the graphs shown in Figure 2, with $v_1 = z$. The vertex labelled v_3 in Figure 2 must be equal to y because Case 2 does not hold. The fact that $y \in V(J) \setminus Y$ implies that $Y \cap V(J) \neq V(J)$. In addition the facts that $i(Y) = 2|Y| - 3$ and no $U \subseteq V(J) - y$, with $\{u, v\} \subset U$, has $i(U) = 2|U| - 3$ implies that $Y \cap V(J) \neq Y$. Hence $Y \cap V(J)$ is a proper subset of both Y and $V(J)$ and hence $i(Y \cap V(J)) \leq 2|Y \cap V(J)| - 4$. Lemma 1(a) now implies that $Y \cup V(J)$ is tight. Since $Y \cup V(J) \neq V$, this contradicts the fact that Case 3 does not occur. □

We can now prove the deletion-contraction characterisation of uv-rigidity stated in the introduction.

Proof of Theorem 1

Necessity follows from the fact that an infinitesimally rigid uv-coincident realisation of G on $\mathcal{Y}$ is an infinitesimally rigid realisation of $G - uv$, and also gives rise to an infinitesimally rigid realisation of G/uv by (4).

To prove sufficiency, suppose, for a contradiction, that $G - uv$ and G/uv are both rigid on $\mathcal{Y}$ but G is not uv-rigid on $\mathcal{Y}$. By Theorems 3 and 4 this implies that there is a thin cover $\mathcal{K}$ of $G - uv$ with $\mathrm{val}(\mathcal{K}) \leq 2|V| - 3$. If $\mathcal{K}$ consists of subsets of V only, then $r^{\mathcal{Y}}(G - uv) \leq 2|V| - 3$ follows, which contradicts the fact that $G - uv$ is rigid on $\mathcal{Y}$.

Hence $\mathcal{K} = \{\mathcal{H}, H_1, \dots, H_k\}$, where $\mathcal{H} = \{X_1, \dots, X_l\}$ is a uv-compatible family. Contract the vertex pair u, v in G into a new vertex z_{uv}. This gives rise to a cover

$$\mathcal{K}' = \{X'_1, \dots, X'_l, H_1, \dots, H_k\}$$

of G/uv, where X'_j is obtained from X_j by replacing u, v by z_{uv}, for $1 \leq j \leq l$. Then we obtain

$$\sum_{i=1}^{k}(2|H_i| - t_{H_i}) + \sum_{j=1}^{l}(2|X'_j| - t(X'_j)) \leq \sum_{i=1}^{k}(2|H_i| - t_{H_i}) +$$

$$+ \sum_{j=1}^{l}(2|X_j| - t(X_j)) - 2l = \mathrm{val}(\mathcal{K}) - 2 \leq 2|V| - 3 - 2 = 2(|V| - 1) - 3,$$

which implies that G/uv is not rigid on $\mathcal{Y}$, a contradiction. This completes the proof. □

A similar proof can be used to verify the following more general result:

Theorem 5 *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Then* $r^{\mathcal{Y}}_{uv}(G) = \min\{r^{\mathcal{Y}}(G - uv), r^{\mathcal{Y}}(G/uv) + 2\}$.

Theorems 2 and 5 show that the polynomial-time algorithms for computing the rank of a count matroid (see e.g. [1,9]) can be used to test whether G is uv-rigid on $\mathcal{Y}$, or more generally, to compute $r^{\mathcal{Y}}_{uv}(G)$.

5 Vertex splitting and global rigidity

Suppose $G = (V, E)$ is a graph with $V = \{v_1, v_2 \ldots, v_n\}$ and (G, p) is a realisation of G on a family of (not necessarily distinct) concentric cylinders $\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \ldots \cup \mathcal{Y}_n$ such that $p(v_i) \in \mathcal{Y}_i$ for $1 \leq i \leq n$. We say that (G, p) is *globally rigid* if every equivalent framework (G, q) on $\mathcal{Y}$, with $q(v_i) \in \mathcal{Y}_i$ for all $1 \leq i \leq n$, is congruent to (G, p).

Let $G = (V, E)$ be a graph and v_1 be a vertex of G with neighbours $v_2, v_3, \ldots, v_t$. A *vertex split* of G at v_1 is a graph $\hat{G}$ which is obtained from G by deleting the edges $v_1v_2, v_1v_3, \ldots, v_1v_k$ and adding a new vertex v_0 and new edges $v_0v_1, v_0v_2, \ldots, v_0v_k$, for some $2 \leq k \leq t$. We will refer to the new edge v_0v_1 as the *bridging edge* of the vertex split. We will show in this section that a vertex splitting operation preserves generic global rigidity on the cylinder if and only if the bridging edge is redundant.

Given a map $p : V \to \mathbb{R}^{3n}$, there is a unique family of concentric cylinders $\mathcal{Y}$ with $p(v_i) \in \mathcal{Y}_i$ for all $1 \leq i \leq n$ as long as $p(v_i)$ does not lie on the z-axis for all $1 \leq i \leq n$. We will refer to $\mathcal{Y}$ as the family of concentric cylinders induced by p and denote it by $\mathcal{Y}^p$.

Connelly and Whiteley [2, Theorem 13] showed that if a framework (G, p) in $\mathbb{R}^d$ is both infinitesimally rigid and globally rigid then all frameworks (G, q) sufficiently close to (G, p) are also infinitesimally rigid and globally rigid. We will adapt their proof technique to obtain an analogous result for the cylinder.

Lemma 19 *If (G, p) is infinitesimally rigid and globally rigid on $\mathcal{Y}$, then there exists an open neighbourhood N_p of p on $\mathcal{Y}$ such that for any $q \in N_p$ the framework (G, q) is infinitesimally rigid and globally rigid on $\mathcal{Y}$.*

Proof. Suppose $|V| \geq 5$ and that for any open neighbourhood N_p, there is a $p^* \in N_p$ such that the framework (G, p^*) is not globally rigid on $\mathcal{Y}$. Then there is a convergent sequence (G, p^k) of non-globally rigid frameworks converging to (G, p). For each framework (G, p^k), let (G, q^k) be an equivalent but non-congruent realisation on $\mathcal{Y}$. We may assume that (G, p^k) and (G, q^k) are in standard position (that is $p^k(v_1) = q^k(v_1) = (0, 1, 0)$ assuming, without loss of generality, that $r_1 = 1$). By the compactness of $\mathbb{R}^{3|V|}$, there is a convergent subsequence (G, q^m) converging to a limiting framework (G, q). As the limits of the respective sequences, (G, q) must be equivalent to (G, p).

If (G, q) is not congruent to (G, p) then we contradict the global rigidity of (G, p). So (G, p) and (G, q) are congruent, i.e. we can transform q to p by a reflection in the plane $x = 0$, a reflection in the plane $z = 0$ or a combination of the two. We apply this same congruence to all the (G, q^m) to obtain a sequence (G, r^m) converging to (G, p) with (G, r^m) being equivalent but not congruent to (G, p^m) for each m.

We next show that $p^m - r^m$ gives an infinitesimal motion of $(G, \frac{p^m+r^m}{2})$ on $\mathcal{Y}^{\frac{p^m+r^m}{2}}$. For each edge v_iv_j we have

$$\left(\frac{p^m(v_i) + r^m(v_i)}{2} - \frac{p^m(v_j) + r^m(v_j)}{2}\right) \cdot ((p^m(v_i) - r^m(v_i)) - (p^m(v_j) - r^m(v_j)))$$

$$= \frac{1}{2}((p^m(v_i) - p^m(v_j)) + (r^m(v_i) - r^m(v_j))) \cdot ((p^m(v_i) - p^m(v_j)) - (r^m(v_i) - r^m(v_j)))$$

$$= \frac{1}{2}\left((p^m(v_i) - p^m(v_j))^2 - (r^m(v_i) - r^m(v_j))^2\right) = 0.$$

Recall that $\bar{p}_m(v_i)$ and $\bar{r}_m(v_i)$ denote the projections of $p_m(v_i)$ and $r_m(v_i)$ onto the plane $z = 0$. Since $p_m(v_i)$ and $r_m(v_i)$ both lie on $\mathcal{Y}_i$, we have $\bar{p}_m(v_i) \cdot \bar{p}_m(v_i) = \bar{r}_m(v_i) \cdot \bar{r}_m(v_i)$. Hence for each vertex v_i,

$$(\bar{p}_m(v_i) + \bar{r}_m(v_i)) \cdot (\bar{p}_m(v_i) - \bar{r}_m(v_i)) = 0.$$

Since p^m and r^m are not congruent, $p^m - r^m$ is a nontrivial infinitesimal motion. This means that the rank of the rigidity matrix for each framework $(G, \frac{p^m+r^m}{2})$ is less than maximal. Since both p^m and r^m converge to p, so does $\frac{p^m+r^m}{2}$. Thus (G, p) is a limit of a sequence of infinitesimally flexible frameworks and hence itself is infinitesimally flexible, a contradiction. (The fact that (G, p) is infinitesimally rigid implies that the rank of $R_{\mathcal{Y}^q}(G, p)$ is maximum for all $q \in \mathbb{R}^{3|V|}$ sufficiently close to p.) □

We can use this lemma and our main result to show that vertex splitting preserves global rigidity on $\mathcal{Y}$ under the additional assumption that the new edge is redundant.

Theorem 6 *Let (G, p) be a generic globally rigid framework on a family of concentric cylinders $\mathcal{Y}$. Let $\hat{G}$ be a vertex split of G at the vertex v_1 with new vertex v_0 and suppose that $\hat{G} - v_0v_1$ is rigid on $\mathcal{Y}$. Let $\hat{p}(v) = p(v)$ for all $v \neq v_0$ and $\hat{p}(v_0) = p(v_1)$. Then for any q on $\mathcal{Y}$ which is sufficiently close to $\hat{p}$, $(\hat{G}, q)$ is globally rigid on $\mathcal{Y}$.*

Proof. Since $(\hat{G}/v_0v_1, p) = (G, p)$ is globally rigid on $\mathcal{Y}$ and p is generic, $\hat{G}/v_0v_1$ is rigid on $\mathcal{Y}$. Since $G - v_0v_1$ is also rigid on $\mathcal{Y}$, Theorem 1 implies that $\hat{G}$ has a v_0v_1-coincident generic rigid realisation $(\hat{G}, \hat{p})$, where $\hat{p}(v) = p(v)$ for all $v \neq v_0$ and $\hat{p}(v_0) = p(v_1)$. Since (G, p) is globally rigid on $\mathcal{Y}$, $(\hat{G}, \hat{p})$ is also globally rigid on $\mathcal{Y}$. We can now use Lemma 19 to deduce that $(\hat{G}, q)$ is globally rigid on $\mathcal{Y}$ for all q sufficiently close to $\hat{p}$. □

Suppose G is a graph which has a generic globally rigid realisation on $\mathcal{Y}$. It was shown in [5] that $G - e$ is rigid on $\mathcal{Y}$ for all $e \in E(G)$. This result and Theorem 6 immediately imply that $\hat{G}$, a vertex split of G with bridging edge e, has a generic globally rigid realisation on $\mathcal{Y}$ if and only if $\hat{G} - e$ is rigid on $\mathcal{Y}$.

6 Concluding remarks

Similarly to our definition of a framework (G, p) on $\mathcal{Y}$ we can define a framework on a family of concentric spheres $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_k$ where $\mathcal{S}_i = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r_i\}$ and $r = (r_1, \ldots, r_k)$ is a vector of positive real numbers. We can project a framework on $\mathcal{S}$ to a framework on the unit sphere by mapping $p(v)$ to $\frac{p(v)}{\|p(v)\|}$ without changing infinitesimal rigidity. We can then map the framework on the unit sphere to a framework on the (affine) plane by central projection. In [12, 13] this process was shown to preserve infinitesimal rigidity for frameworks on the unit sphere. Since the projection also preserves the property that u an v are coincident, the problem of characterising generic rigidity for frameworks with two coincident points on concentric spheres is equivalent to the problem of characterising generic rigidity for frameworks with two coincident points in the plane. We can now use the characterisation of generic uv-rigidity in the plane [4] to give the following result.

Theorem 7 *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Then G is uv-rigid on a family of concentric spheres $\mathcal{S}$ if and only if $G - uv$ and G/uv are both rigid on $\mathcal{S}$.*

Note that a graph $G = (V, E)$ is rigid on $\mathcal{S}$ if and only if it has rank $2|V| - 3$ in the $(2, 3)$-sparse matroid by [10, Theorem 5.1].

We can also replace $\mathcal{Y}$ with other surfaces. In particular if we choose a surface with 1 ambient rigid motion (such as the cone, hyperboloid or torus) then the analogue of Theorem 2 requires the graph to be $(2, 1)$-tight [11]. In the uv-coincident case we would define the value as $\text{val}(H) = 2|H| - t_H$ where $t_H = 3$ if $|H| \in \{2, 3\}$ and $H \neq \{u, v\}$, $t_H = 2$ if $|H| \in \{0, 4\}$ or $H = \{u, v\}$ and $t_H = 1$ if $|H| \geq 5$. We expect that, using similar techniques to Section 3, the appropriate count matroid can be established. However we do not know how to prove an analogue of Theorem 4. To make a start on this problem would require dealing with the case when the only vertices of degree less than 4 are u and v.

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