Quasi-equilibrium models of magnetized compact objects

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Abstract. We report work towards a relativistic formulation for modeling strongly magnetized neutron stars, rotating or in a close circular orbit around another neutron star or black hole, under the approximations of helical symmetry and ideal MHD. The quasi-stationary evolution is governed by the first law of thermodynamics for helically symmetric systems, which is generalized to include magnetic fields. The formulation involves an iterative scheme for solving the Einstein-Maxwell and relativistic MHD-Euler equations numerically. The resulting configurations for binary systems could be used as self-consistent initial data for studying their inspiral and merger.

1. Introduction

A uniformly rotating neutron star is significantly deformed when the ratio of kinetic energy T to gravitational energy W becomes $T/|W| \sim 0.1$. When the magnetic field energy,

$$\mathcal{M} := \frac{1}{8\pi}\int B^2 d^3x, \tag{1}$$

becomes such that $\mathcal{M}/|W| \sim 0.01$, the contribution of the magnetic field to the structure of the neutron srar may not be neglected. This is estimated to occur for

$$B \sim 4.4 \times 10^{16} \left(\frac{M\,[M_\odot]}{1.4\,[M_\odot]}\right)\left(\frac{10\,[\mathrm{km}]}{R\,[\mathrm{km}]}\right)^2 [\mathrm{G}]. \tag{2}$$

Recent observations of anomalous X-ray pulsars, or soft γ-ray repeaters suggest that the neutron stars in these systems may be associated with strong magnetic fields around $10^{14} - 10^{15}$ G at the surface (see, e.g. [1]). We may expect that the interior magnetic field of such a strongly magnetized neutron star, a magnetar, is a few orders of magnitudes stronger.

Such strong magnetic fields have not been found in binary neutron star systems, and may not survive until merger. Hypothetically, however, strongly magnetized neutron stars or black holes may form binary neutron star or black hole - neutron star systems. For instance, the poloidal

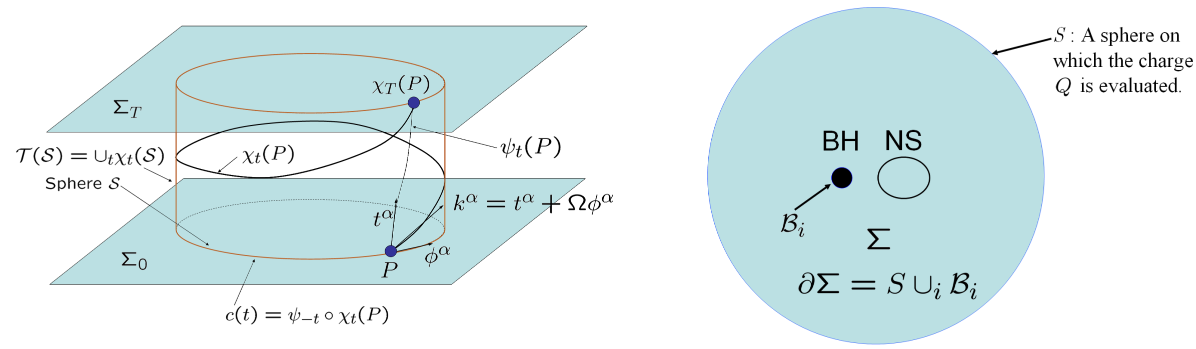

Figure 1. Left panel: the history of a sphere $\mathcal{S}$ generated by the helical vector k^α. Right panel: a domain of integration Σ and its boundaries.

field seen as the surface magnetic field decays but the toroidal field stays strong enough to affect the structure of the compact objects.

In several numerical relativity simulations of magnetized binary neutron stars, it has been shown that, after binary inspiral and merger, the magnetic fields are amplified around 10 times by magnetic winding and the magnetorotational instability in post-merger, pre-collapse objects [2]. Such an object is likely to form a black hole and a magnetized toroid system, which becomes the source of a short γ-ray burst. It is desirable to prepare realistic initial data sets for such merger simulations calculated by solving the Einstein-Maxwell equations and a first integral of the MHD-Euler equation, assuming (quasi-)equilibrium. In this article, we model such magnetized binary compact objects in close circular orbits, assuming that the spacetime and magnetic fields satisfy helical symmetry and that the stars are in equilibrium [3–5].

2. Thermodynamic laws for helically symmetric Einstein-Maxwell spacetimes with charged and magnetized perfect fluids

2.1. Zeroth law

We consider a globally hyperbolic spacetime $(\mathcal{M}, g_{\alpha\beta})$. The field k^α is transverse to each Cauchy surface, not necessarily timelike everywhere, and generates a one-parameter family of diffeomorphisms χ_t. The action of χ_t on a spacelike sphere $\mathcal{S}$ on a Cauchy surface generates a timelike surface, $\mathcal{T}(\mathcal{S}) = \cup_t \chi_t(\mathcal{S})$, called the *history of* $\mathcal{S}$. Then, k^α is a *helical vector* field if there is a smallest $T > 0$ for which P and $\chi_T(P)$ are timelike separated for every point P outside of the history $\mathcal{T}(\mathcal{S})$. A vector k^α written as

$$k^\alpha = t^\alpha + \Omega\phi^\alpha \tag{3}$$

is the helical vector, where t^α is a timelike vector and ϕ^α a spacelike vector that has circular orbits with a parameter length 2π (see, FUS). In Fig. 1, a schematic figure of the history $\mathcal{T}(\mathcal{S})$ generated by the diffeo χ_t is presented.

Each Cauchy surface of a helically symmetric spacetime does not admit flat asymptotics. Therefore, the future (past) horizon $\mathcal{H}^\pm$ is defined as the boundary of the future(past) domain of outer communication $\mathcal{D}^\pm$ of a history $\mathcal{T}(\mathcal{S})$ of each spacelike sphere $\mathcal{S}$. If the history $\mathcal{T}(\mathcal{S})$ of a sphere $\mathcal{S}$ is in $\mathcal{D}^\pm$, the future (past) horizon agrees with the chronological past (future) of the history $\mathcal{T}$, $\mathcal{H}^\pm = \partial I^\mp(\mathcal{T})$.

The existence of a global helical symmetry assures that the horizon is a Killing horizon. With the null energy condition $R_{\alpha\beta} l^\alpha l^\beta \geq 0$ for any null vector l^α, the surface gravity κ, defined on each connected component of the horizon $\mathcal{H}^\pm$ by

$$k^\beta \nabla_\beta k^\alpha = \kappa k^\alpha, \tag{4}$$

is constant. The proof is given in FUS, in which the conditions of theorems by Friedrich, Rácz, and Wald [6] are modified to make them suitable for helically symmetric spacetimes. Also the electric potential $\Phi^{\rm E}$ in the rotating frame, defined by

$$E_\alpha \,=\, F_{\alpha\beta}k^\beta \,=\, -\nabla_\alpha\Phi^{\rm E}, \qquad \Phi^{\rm E} \,=\, A_\alpha k^\alpha + {\rm const}, \tag{5}$$

is constant on $\mathcal{H}^\pm$, assuming the symmetry $\pounds_k A_\alpha \,=\, 0$. Since E_α is null on $\mathcal{H}^\pm$ and $E_\alpha k^\alpha \,=\, 0$, E_α is parallel to the null generator on $\mathcal{H}^\pm$. Hence for any vector η^α tangent to $\mathcal{H}^\pm$, $\eta^\alpha E_\alpha \,=\, -\eta^\alpha\nabla_\alpha\Phi^{\rm E} \,=\, 0$. Therefore, as for the stationary and axisymmetric spacetimes shown by Carter [7,8], the surface gravity κ and the electric potential $\Phi^{\rm E}$ in the rotating frame are constant on the horizon $\mathcal{H}^\pm$.

2.2. First law

Consider a family of spacetimes,

$$\mathcal{Q}(\lambda) := [g_{\alpha\beta}(\lambda), u^\alpha(\lambda), \rho(\lambda), s(\lambda), A_\alpha(\lambda), j^\alpha(\lambda)], \tag{6}$$

whose Lagrangian density is written

$$\mathcal{L} \,=\, \left(\frac{1}{16\pi}R \,-\, \epsilon \,-\, \frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} \,+\, A_\alpha j^\alpha\right)\sqrt{-g}, \tag{7}$$

where u^α, ρ, s, ϵ, and j^α are, respectively, the 4-velocity, baryon rest mass density, entropy per baryon mass, energy density, and electric 4-current. A generalized first law for the spacetime $\mathcal{Q}(\lambda)$ associated with the helical symmetry is derived as a variation formula of the Noether charge associated with the helical Killing vector defined by [9,10]

$$Q(\lambda) = \oint_S Q^{\alpha\beta} dS_{\alpha\beta}, \tag{8}$$

$$\mathfrak{B}^\alpha(\lambda) \,=\, \frac{1}{16\pi}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\beta}g^{\gamma\delta})|_{\lambda=0}\, \overset{\circ}{\nabla}_\beta\, g_{\gamma\delta}(\lambda) \,+\, \frac{1}{4\pi}F^{\beta\alpha}|_{\lambda=0}\Big[A_\beta(\lambda) - \frac{1}{2}A_\beta(0)\Big] \,+\, {\rm O}(\lambda^2), \tag{9}$$

With this choice of $\mathfrak{B}^\alpha$, the charge $Q(\lambda)$ becomes finite and independent of the sphere S used to evaluate the charge $Q(\lambda)$ as long as S encloses all black holes and neutron stars.

To see this, we first check that $Q(0)$ is independent of S. The surface integral of (8) is rewritten in terms of integrals over a spacelike hypersurface Σ transverse to k^α and over the black hole boundary $\mathcal{B}_i$, where $\mathcal{B}_i$ is the i-th connected component of $\Sigma \cap \mathcal{H}^+$. In Fig.1, the region of integration is shown. Since, the boundary of Σ is the union of the sphere S and $\mathcal{B}_i$, $\partial\Sigma = S \cup_i \mathcal{B}_i$, a difference $Q \,-\, \sum_i Q_i$ is calculated at $\lambda = 0$:

$$\begin{aligned} Q \,-\, \sum_i Q_i \,=\, &-\frac{1}{8\pi}\int_\Sigma (G^\alpha{}_\beta k^\beta - 8\pi T_{\rm F}{}^\alpha{}_\beta) dS_\alpha - \frac{1}{16\pi}\int_\Sigma R k^\alpha dS_\alpha. \\ &+ \int_\Sigma \left(\frac{1}{8\pi}\nabla_\gamma F^{\beta\gamma}A_\beta k^\alpha \,-\, \frac{1}{4\pi}k^\gamma A_\gamma \nabla_\beta F^{\alpha\beta}\right) dS_\alpha - \sum_i \frac{1}{4\pi}\int_{\mathcal{B}i} k^\gamma A_\gamma F^{\alpha\beta} dS_{\alpha\beta}. \end{aligned} \tag{10}$$

where $Q_i(\lambda)$ is defined on $\mathcal{B}_i$ by

$$Q_i(\lambda) := \oint_{\mathcal{B}_i} Q^{\alpha\beta} dS_{\alpha\beta}, \tag{11}$$

${T_{\rm F}}^\alpha{}_\beta$, the stress-energy tensor of the electromagnetic field, is defined by,

$$T_{\rm F}^{\alpha\beta} \;=\; \frac{1}{4\pi}\left(F^{\alpha\gamma}F^\beta{}_\gamma - \frac{1}{4}g^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}\right). \tag{12}$$

To derive Eq. (10), we have also used a condition

$$\frac{1}{4\pi}\oint_S k^\gamma A_\gamma F^{\alpha\beta}dS_{\alpha\beta} \;=\; 0 \tag{13}$$

on the boundary sphere S to determine the constant of the electric potential $\Phi^{\rm E}$ in Eq. (5). Eq. (10) implies that $Q(0)$ does not depend on the sphere S as long as it encloses all black holes and neutron stars, because in the region where the sphere S is located, the integrand of the volume integrals in Eq. (10) vanishes when the Einstein and Maxwell equations are satisfied.

Next, the variation $\delta Q := dQ/d\lambda$ in the Noether charge is evaluated in terms of perturbations of the baryon mass, entropy, circulation and electric current of each fluid element, and the surface areas and charges of the black holes. The detailed calculation is described in [11]: the variation δQ becomes

$$\begin{aligned}\delta Q \;=\; & \int_\Sigma \left\{ \frac{T}{u^t}\Delta(s\,\rho u^\alpha\, dS_\alpha) \;+\; \frac{h-Ts}{u^t}\Delta(\rho u^\alpha\, dS_\alpha) \;+\; v^\beta\Delta(hu_\beta\,\rho u^\alpha\, dS_\alpha) \;-\; A_\beta k^\beta\,\Delta(j^\alpha dS_\alpha) \right. \\ & \left. -\; (j^\alpha k^\beta - j^\beta k^\alpha)\Delta A_\beta\, dS_\alpha \right\} \;+\; \sum_i\left(\frac{1}{8\pi}\kappa_i\delta\mathcal{A}_i \;+\; \Phi^{\rm E}_i\,\delta Q^{\rm E}_i\right), \end{aligned} \tag{14}$$

where T is the temperature, h is the relativistic specific enthalpy, and v^α is the spatial velocity defined by the following decomposition of the 4-velocity u^α with respect to the helical vector k^α:

$$u^\alpha = u^t(k^\alpha + v^\alpha) \quad \text{with} \quad v^\alpha\nabla_\alpha t = 0 \quad \text{and} \quad u^t = u^\alpha\nabla_\alpha t. \tag{15}$$

The electric charge of each black hole $Q^{\rm E}_i$ is defined by

$$Q^{\rm E}_i := \frac{1}{4\pi}\oint_{\mathcal{B}i} F^{\alpha\beta}dS_{\alpha\beta}, \tag{16}$$

which is related to the total electric charge of the system $Q^{\rm E}$ by Stokes' theorem:

$$Q^{\rm E} := \frac{1}{4\pi}\oint_S F^{\alpha\beta}dS_{\alpha\beta}. \;=\; \int_\Sigma j^\alpha dS_\alpha \;+\; \sum_i Q^{\rm E}_i. \tag{17}$$

Note that the $\Phi^{\rm E}_i$, defined on each $\mathcal{B}_i$ by

$$\Phi^{\rm E}_i = -A^\alpha k_\alpha = \Phi^{\rm E} + C, \tag{18}$$

is constant. In the case of stationary and axisymmetric spacetimes, the mass variation formula derived by Carter [7, 8] can be derived from Eq. (14).

We can now verify that $Q(\lambda)$ is independent of the location of the 2-surface S, because the charge $Q(\lambda)$ at $\lambda = 0$ is shown to be independent of S, and the variation formula (14) implies that $dQ/d\lambda = \delta Q$ is independent of S as long as it encloses the fluid and black holes. Moreover, in [11] we have shown that the difference,

$$\delta\Big(Q \;-\; \sum_i Q_i \;-\; \frac{1}{4\pi}\int_{\partial\Sigma} k^\gamma A_\gamma F^{\alpha\beta}dS_{\alpha\beta}\Big) \;=\; \delta Q \;-\; \sum_i\Big(\frac{1}{8\pi}\kappa_i\delta\mathcal{A}_i \;+\; \Phi^{\rm E}_i\,\delta Q^{\rm E}_i\Big), \tag{19}$$

is invariant under a gauge transformation that respects the symmetry, and hence we verify that so is δQ.

2.3. Application of the first law to solution sequences in equilibrium

When inspiraling binary systems, or isolated neutron stars are evolving adiabatically – in a timescale much longer than the dynamical timescale – they may be modeled by a sequence of solutions in equilibrium. When the first law for a stationary and axisymmetric perfect fluid spacetime is applied to a sequence of rotating neutron star solutions whose local changes of rest mass and entropy are constant, the first law becomes $\delta M = \Omega\delta J$. This is a condition for applying the turning point theorem [12] to determine the stability of solutions. In this section, we consider an application of the first law (14) to a sequence of helically symmetric solutions that model inspiraling binary black holes and/or neutron stars, assuming that the neutron star matter is a perfect conductor.

For a perfectly conducting medium, the ideal MHD condition,

$$F_{\alpha\beta}u^\beta = 0, \tag{20}$$

is satisfied, and hence its curl is written

$$\pounds_u F_{\alpha\beta} = 0, \tag{21}$$

which is the magnetic flux conservation law, Alfven's law. For each equilibrium solution, rest mass and entropy are conserved:

$$\pounds_u(\rho\sqrt{-g}) = 0, \quad \pounds_u s = 0. \tag{22}$$

When we consider a sequence of solutions along which the rest mass, entropy, and magnetic flux are all conserved, the perturbed conservation laws corresponding to the above Eqs. (21) and (22) have first integrals:

$$\Delta(\rho\sqrt{-g}) = 0, \quad \Delta s = 0, \quad \text{and} \quad \Delta F_{\alpha\beta} = 0. \tag{23}$$

For black holes, we may assume that areas and charges are constant. With these assumptions, the first law (14) is rewritten

$$\delta Q = \int_\Sigma \left\{ v^\beta \Delta(hu_\beta\, \rho u^\alpha\, dS_\alpha) - A_\beta k^\beta\, \Delta(j^\alpha dS_\alpha) \right\}. \tag{24}$$

Here $\Delta F_{\alpha\beta} = 0$ with $F_{\alpha\beta} = (dA)_{\alpha\beta}$, $\Delta(dA)_{\alpha\beta} = (d\Delta A)_{\alpha\beta}$, whence the Poincaré lemma implies that there exists a function Ψ such that $\Delta A_\alpha = \nabla_\alpha\Psi$. From a substitution of this to the last term in the volume integral of Eq. (14), the latter is shown to vanish. The remaining terms in Eq. (24) are related to the circulation of the magnetized flow.

For a perfect fluid without magnetic fields, the conservation of circulation is written as $\pounds_u\hat{\omega}_{\alpha\beta} = 0$ where $\hat{\omega}_{\alpha\beta} := (d(hu))_{\alpha\beta}$ is the vorticity tensor. When circulation is conserved along a sequence of solutions, it implies that $\Delta\hat{\omega}_{\alpha\beta} = 0$, and hence the first term in the integral of Eq. (24) vanishes [13]. In general, there is no such conservation law for the circulation of magnetized flow. However, Bekenstein and Oron [14] (Tarapov and Gorskii [15] for Newtonian MHD) have developed a formulation of ideal MHD, in which a generalized circulation of magnetized flow is conserved. Their theory is based on a Lagrangian in which the interaction term $A_\alpha j^\alpha$ in Eq. (7) is replaced by the term $F_{\alpha\beta}\rho u^\alpha q^\beta$, where the vector q^α is a Lagrange multiplier enforcing the perfect conductivity condition $F_{\alpha\beta}u^\alpha = 0$. Variation with respect to A_α yields the Maxwell equations with a current of the form

$$j^\alpha = \nabla_\beta(\rho u^\alpha q^\beta - \rho u^\beta q^\alpha). \tag{25}$$

When a vector q^α is found to give the current (25), the Lorenz force term in the MHD-Euler equation

$$u^\beta (d(hu))_{\beta\alpha} \,=\, \frac{1}{\rho} F_{\alpha\beta} j^\beta \tag{26}$$

can be absorbed into the left hand side, bringing eq. (26) in the canonical form

$$u^\beta \omega_{\beta\alpha} \,=\, 0, \tag{27}$$

where

$$\omega_{\alpha\beta} := (dw)_{\alpha\beta}, \quad w_\alpha \,:=\, hu_\alpha \,+\, \eta_\alpha, \quad \eta_\alpha := F_{\alpha\beta} q^\beta, \tag{28}$$

because of a relation,

$$\frac{1}{\rho} F_{\alpha\beta} j^\beta \,=\, \frac{1}{\rho} F_{\alpha\beta} \left[\pounds_q (\rho u^\beta) \,+\, \rho u^\beta \nabla_\gamma q^\gamma \right] \,=\, (d\eta)_{\alpha\beta} u^\beta. \tag{29}$$

The one-from w_α and its exterior derivative $\omega_{\alpha\beta}$ can be respectively regarded as a canonical momentum of a magneto-fluid element and a generalized vorticity. The MHD-Euler equation (27) in this case implies conservation of circulation for the magnetized flow, because its curl, with $d\omega = 0$, yields

$$\pounds_u \omega_{\alpha\beta} \,=\, 0, \tag{30}$$

and the Lagrangian perturbation of this conservation law has a first integral

$$\Delta\omega_{\alpha\beta} \,=\, 0. \tag{31}$$

Substituting the current (25) to the first law (14), we have

$$\begin{aligned} \delta Q \;=\; & \int_\Sigma \left\{ \frac{T}{u^t} \Delta dS + \frac{h - Ts}{u^t} \Delta dM_{\rm B} \,+\, v^\alpha \Delta dC_\alpha \,-\, v^\beta q^\gamma \Delta F_{\beta\gamma}\, dM_{\rm B} \,-\, (j^\alpha k^\beta - j^\beta k^\alpha) \Delta A_\beta \, dS_\alpha \right\} \\ & +\; \sum_i \left(\frac{1}{8\pi} \kappa_i \delta \mathcal{A}_i \,+\, \Phi_i^{\rm E}\, \delta Q_i^{\rm E} \right), \end{aligned} \tag{32}$$

where we introduced the notation

$$dM_{\rm B} := \rho u^\alpha dS_\alpha, \quad dS := s\, dM_{\rm B}, \quad dC_\alpha := (hu_\alpha + \eta_\alpha) dM_{\rm B}. \tag{33}$$

Applying the first law (32) to a sequence of solutions along which the quantities are conserved as in Eq. (23) and the circulation of magnetized flow is conserved as in Eq. (31), the first law becomes

$$\delta Q \,=\, 0, \tag{34}$$

or for asymptotically flat systems, such as in the post-Newtonian approximation, $\delta Q = \delta M - \Omega \delta J = 0$.

3. A formulation for computing magnetized neutron star equilibria

To compute solutions of single or binary compact objects in equilibrium, we apply a finite difference scheme, or a pseudo-spectral method, to a system of basic equations and numerically solve the system. Those equations include the Einstein equation, the Maxwell equations, the MHD-Euler equation, and the baryon mass conservation equation. Here, we assume that the flow is homentropic, and hence the neutron-star matter is described by a one-parameter EOS.

For the gravitational field, the Isenberg - Wilson - Mathews formulation, the waveless formulation, or a full set of Einstein's equations assuming helical symmetry may be used. These formulations are based on a 3+1 decomposition of spacetime, and reliable numerical methods to solve these equations have been developed [16–18]. We expect that the Maxwell equations can be solved using analogous formulations and applying one of the above numerical methods.

However, for the MHD-Euler equation, when stationarity or helical symmetry is imposed, it is no longer an evolution equation, and as a result it is difficult to integrate numerically. In self-consistent field methods [16–19], an equilibrium solution is computed using a first integral of the (MHD-) Euler equation which exists when the flow is assumed to be either corotational or irrotational. Therefore, finding the first integral of the MHD-Euler equation is a key, and also a restriction, for computing equilibrium solutions considered in the previous section successfully.

If we assume that the orbit of the stars is closed and quasi-circular, then the system appears stationary in a frame rotating with frequency equal to the orbital frequency Ω, and there exists an approximate helical Killing vector k^α, given by eq. (3), that Lie-derives all variables in (6). The Cartan identity implies

$$k^\beta (dw)_{\beta\alpha} = \pounds_k w_\alpha - \nabla_\alpha (k^\gamma w_\gamma) \tag{35}$$

It should be noticed that $\pounds_k w_\alpha \neq 0$ in general, since q^α is not an observable quantity. One has

$$\pounds_k w_\alpha \;=\; \pounds_k \eta_\alpha \;=\; F_{\alpha\beta} \pounds_k q^\beta. \tag{36}$$

(In fact, if one assumes $\pounds_k w_\alpha = 0$ for a corotational flow, no Lorenz force is exerted on the matter). As shown below (see also [20]), the Cartan identity (35) leads quickly to conserved quantities when certain assumptions for the flow are made.

3.1. Irrotational magneto-flow

If one assumes that the magneto-flow is irrotational, i.e. described by a potential Φ:

$$w_\alpha = h u_\alpha \,+\, \eta_\alpha = \nabla_\alpha \Phi, \tag{37}$$

then the vorticity $\omega = dw$ vanishes and the MHD-Euler equation (27) is automatically satisfied while the left hand side of eq. (35) vanishes as well, so that

$$\pounds_k w_\alpha - \nabla_\alpha (k^\gamma w_\gamma) = 0 \tag{38}$$

The above equation has a first integral iff the term $\pounds_k w_\alpha$ is the gradient of some scalar function. Because of eq. (36), this integrability condition is written as $\pounds_k q^\beta F_{\beta\alpha} = \nabla_\alpha f$, or, using the Cartan identity,

$$\pounds_{[k,q]} F_{\alpha\beta} \;=\; 0, \tag{39}$$

where $[k,q]^\alpha := \pounds_k q^\alpha$. Then, eq. (35) implies that the quantity

$$k^\gamma w_\gamma + f \;=\; \mathcal{E} \tag{40}$$

is constant throughout the magneto-fluid.

In this case, we have four variables for the matter, two thermodynamic variables and two variables for the velocity fields, $\{h, p, \Phi, u^t\}$. The set of equations to solve for these four variables is supplemented by a one-parameter EOS, $p = p(h)$, and the normalization of the 4-velocity $u_\alpha u^\alpha = -1$. The velocity potential Φ is obtained from the rest mass conservation equation which is an elliptic equation for Φ with Neumann boundary conditions on the stellar surface. An additional degree of freedom may be obtained from the first integral (40), if it exists.

It is not trivial to find a vector q^α that satisfies both the integrability condition (39) and the ideal MHD condition (20); q^α is not freely specifiable. Instead of solving for a q^α that satisfies both conditions, let us assume that

$$q^\alpha = q^t k^\alpha \tag{41}$$

with $\pounds_k q^t$ regarded a function of $A_\alpha k^\alpha$. If we select q^α as above, at least instantaneously on an initial hypersurface Σ, then the first integral is written

$$k^\alpha w_\alpha + \int \pounds_k q^t \, d(A_\alpha k^\alpha) = \mathcal{E} \tag{42}$$

It is likely that solutions with neutron stars and strong magnetic fields computed from Eq. (42) may not be strictly in equilibrium. Nonetheless, such solutions with strong magnetic fields that satisfy a set of hydrostationary equations will serve as interesting initial data sets for numerical relativity merger simulations and for further studies of such strongly magnetized compact binaries.

3.2. Corotational magneto-flows

If the flow is corotational, as is the case for a tidally locked binary or a rigidly rotating non-axisymmetric magnetar, then the velocity field can be written it terms of the helical Killing vector (3) as

$$u^\alpha = u^t k^\alpha \tag{43}$$

In this case, the MHD-Euler equation (27) implies that the left-hand side of the Cartan identity (35) vanishes, so that eq. (38) again holds. The integrability condition of the latter equation then is formally identical to eq. (39) and the first integral of eq. (38) again has the form of eq. (40):

$$k^\gamma w_\gamma + f = \mathcal{E} \tag{44}$$

In the corotational case, the perfect MHD condition (20) also admits a first integral:

$$k^\gamma A_\gamma = 0 \tag{45}$$

This follows from the Cartan identity,

$$k^\beta (dA)_{\beta\alpha} = \pounds_k A_\alpha - \nabla_\alpha (k^\gamma A_\gamma), \tag{46}$$

along with the facts that the term $k^\beta (dA)_{\beta\alpha}$ vanishes by virtue of eqs. (43) and (20), and that the term $\pounds_k A_\alpha$ can be made to vanish by a gauge transformation in A_α (c.f. Appendix C in [22]). One can then incorporate the first integrals (44) and (45) in a self-consistent iteration scheme analogous to that outlined for the irrotational case above, coupled to the Einstein-Maxwell equations, to construct equilibrium models of rigidly rotating configurations. A scheme based on eq. (44), with $f = 0$, has been implemented in [23] for constructing non-magnetized triaxial (Jacobi) spheroids in general relativity and can be extended to model magnetars that are triaxially deformed by their magnetic field.

4. Discussion

The thermodynamic laws derived for helically symmetric Einstein-Maxwell spacetimes with magnetized matter can be applied to a non-axisymmetirc single rotating star, or an axisymmetric rotating star beyond the stationary, axisymmetric *and circular* spacetime. Our project involves computing quasi-equilibrium configurations of magnetized compact objects with numerical codes based on self-consistent field methods. Such methods can be used to model not only binary compact objects, but also magnetars, as well as proto neutron stars which are likely to have strong magnetic fields.

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