Relating $\mathrm{U}(N) \times \mathrm{U}(N)$ to $\mathrm{SU}(N) \times \mathrm{SU}(N)$ Chern-Simons Membrane theories

Neil Lambert[1] and Constantinos Papageorgakis[2]

Department of Mathematics
King's College London
The Strand, WC2R 2LS
London, UK

Abstract

By integrating out the $\mathrm{U}(1)_B$ gauge field, we show that the $\mathrm{U}(n) \times \mathrm{U}(n)$ ABJM theory at level k is equivalent to a $\mathbb{Z}_k$ identification of the $(\mathrm{SU}(n) \times \mathrm{SU}(n))/\mathbb{Z}_n$ Chern-Simons theory, but only when n and k are coprime. As a consequence, the $k = 1$ ABJM model for two M2-branes in $\mathbb{R}^8$ can be identified with the $\mathcal{N} = 8$ $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2$ theory. We also conjecture that the $\mathrm{U}(2) \times \mathrm{U}(2)$ ABJM model at $k = 2$ is equivalent to the $\mathcal{N} = 8$ $\mathrm{SU}(2) \times \mathrm{SU}(2)$-theory.

[1]E-mail address: `neil.lambert@kcl.ac.uk`
[2]E-mail address: `costis.papageorgakis@kcl.ac.uk`

Introduction

There has been considerable activity in the past two years leading to a new class of highly supersymmetric three-dimensional conformal Chern-Simons theories which control the dynamics of multiple M2-branes in M-theory. This work started with the papers [1–4], which were the first to construct interacting theories with the correct symmetries; $\mathcal{N}=8$ supersymmetry and SO(8) R-symmetry. These theories have no continuous coupling constant but they do admit a discrete coupling k that arises as the level of the Chern-Simons terms. However, this model is only capable of potentially describing two M2-branes and its spacetime interpretation is unclear. The generalisation to an arbitrary number of n M2-branes in a $\mathbb{R}^8/\mathbb{Z}_k$ orbifold was provided by the celebrated ABJM models [5] which are $\mathrm{U}(n)\times\mathrm{U}(n)$ Chern-Simons-matter theories with $\mathcal{N}=6$ supersymmetry and SU(4) R-symmetry.

The main aim of this note is to elucidate the relation between the $\mathcal{N}=6$ $\mathrm{U}(n)\times\mathrm{U}(n)$ ABJM models and $(\mathrm{SU}(n)\times\mathrm{SU}(n))/\mathbb{Z}_n$ theories. As already noted in [5], the relative $\mathrm{U}(1)_B$ gauge field of the ABJM theories can be naturally integrated out. Since $\mathrm{U}(n)\simeq(\mathrm{U}(1)\times\mathrm{SU}(n))/\mathbb{Z}_n$, naively the effect of this is to reduce the $\mathrm{U}(n)\times\mathrm{U}(n)$ theory to a $\mathbb{Z}_k$ quotient of the $(\mathrm{SU}(n)\times\mathrm{SU}(n))/\mathbb{Z}_n$ theory. However we will see that there is a global obstruction to this $\mathbb{Z}_k$ identification unless n and k are coprime.

We will be particularly interested in the case with $n=2$, where the Lagrangian is precisely the original proposal of [2, 3] and has $\mathcal{N}=8$ supersymmetry and SO(8) R-symmetry. According to the above, the $\mathcal{N}=6$ ABJM $\mathrm{U}(2)\times\mathrm{U}(2)$ theory can be mapped to the $\mathcal{N}=8$, $(\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathbb{Z}_2$ theory along with the $\mathbb{Z}_k$ identification on the fields when k is odd. For $k=1$ the identification is trivial and hence the $(\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathbb{Z}_2$-theory at $k=1$ describes two M2-branes in flat space.

We also seek to clarify statements in [6, 7] which computed the moduli space of the $\mathcal{N}=8$ theory and argued that it corresponded to the IR limit of an SO(5) orbifold in type IIA, obtained by including one unit of discrete torsion for the background 3-form gauge field. In fact the discussion in [6, 7] is insufficient to distinguish between the orbifolds with and without torsion since they both have the same moduli space. Our discussion here shows that at $n=k=2$ the ABJM model cannot be reduced to a $\mathbb{Z}_2$ quotient of the $(\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathbb{Z}_2$ theory. However, the $\mathcal{N}=8$ $\mathrm{SU}(2)\times\mathrm{SU}(2)$ theory at $k=2$ does give the correct moduli space. This, along with the similarity between the two Lagrangians leads us to conjecture that the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ theory obtained from the Lagrangian of [2–4] has an M-theory interpretation at $k=2$ and is equivalent to the $\mathrm{U}(2)\times\mathrm{U}(2)$ Chern-Simons theory of [5], corresponding to the IR fixed point of a 2+1d O(4) orbifold theory. These results should make the connection between the theories of [1–4, 8] and ABJM transparent and explain any aspects of M-theory physics captured by the former.

Note that the Chern-Simons-matter Lagrangians are entirely determined by the 3-algebra data which includes the Lie algebra of the gauge group. In the quantum theory one must also specify the full global gauge group. This choice manifests itself by allowing for different flux quantization conditions which in turn yield distinct quantum theories, with the same symmetry algebra. To account for this we will label the Lagrangian by is Lie algebra but the associated quantum theories will be labeled by the global gauge group.

$\mathcal{N} = 6$ Chern-Simons theories from 3-algebras

Let us start by considering the general form of three-dimensional Lagrangians with scale symmetry and $\mathcal{N} = 6$ supersymmetry [8]:

$$\begin{aligned}\mathcal{L} \;=\;& -\mathrm{Tr}(D_\mu Z^A, D^\mu \bar{Z}_A) - i\mathrm{Tr}(\bar{\psi}^A, \gamma^\mu D_\mu \psi_A) - V + \mathcal{L}_{CS} \\ & -i\mathrm{Tr}(\bar{\psi}^A, [\psi_A, Z^B; \bar{Z}_B]) + 2i\mathrm{Tr}(\bar{\psi}^A, [\psi_B, Z^B; \bar{Z}_A]) \\ & +\frac{i}{2}\varepsilon_{ABCD}\mathrm{Tr}(\bar{\psi}^A, [Z^C, Z^D; \psi^B]) - \frac{i}{2}\varepsilon^{ABCD}\mathrm{Tr}(\bar{Z}_D, [\bar{\psi}_A, \psi_B; \bar{Z}_C]) \ ,\end{aligned} \tag{1}$$

where

$$\begin{aligned} V &= \frac{2}{3}\mathrm{Tr}(\Upsilon^{CD}_B, \bar{\Upsilon}^B_{CD}) \\ \Upsilon^{CD}_B &= [Z^C, Z^D; \bar{Z}_B] - \frac{1}{2}\delta^C_B[Z^E, Z^D; \bar{Z}_E] + \frac{1}{2}\delta^D_B[Z^E, Z^C; \bar{Z}_E],\end{aligned} \tag{2}$$

and $\mathcal{L}_{CS}$ is a Chern-Simons term that we will describe in detail below. The bracket $[\cdot, \cdot; \cdot]$ is antisymmetric in the first two entries and defines the triple product of the 3-algebra where the scalars and fermions take values. Introducing a basis T^a for the 3-algebra, so that $Z^A = Z^A_a T^a$, $\psi_A = \psi_{Aa} T^a$, allows us to use structure constants defined through

$$[T^a, T^b; T_c] = f^{ab}{}_{cd} T^d \ . \tag{3}$$

Here we use notation where complex conjugation raises and lowers both A and a indices (whereas in [8] a raised a index was given a bar).

The supersymmetry transformations are

$$\begin{aligned} \delta Z^A_d &= i\bar{\epsilon}^{AB}\psi_{Bd} \\ \delta\psi_{Bd} &= \gamma^\mu D_\mu Z^A_d \epsilon_{AB} + f^{ab}{}_{cd} Z^C_a Z^A_b \bar{Z}^c_C \epsilon_{AB} + f^{ab}{}_{cd} Z^C_a Z^D_b \bar{Z}^c_B \epsilon_{CD} \\ \delta \tilde{A}_\mu{}^c{}_d &= -i\bar{\epsilon}_{AB}\gamma_\mu Z^A_a \psi^{Bb} f^{ca}{}_{bd} + i\bar{\epsilon}^{AB}\gamma_\mu \bar{Z}_{Ab}\psi^a_B f^{cb}{}_{ad} \ ,\end{aligned} \tag{4}$$

where the covariant derivative is $D_\mu Z^A_d = \partial_\mu Z^A_d - \tilde{A}_\mu{}^c{}_d Z^A_c$ and similarly for the other fields.

One recovers the general form of the ABJM and ABJ Lagrangians [5, 9] by taking the 3-algebra to be $n \times m$ complex matrices with

$$[Z^A, Z^B; \bar{Z}_C] = -\frac{2\pi}{k}(Z^A Z^\dagger_C Z^B - Z^B Z^\dagger_C Z^A) \tag{5}$$

and introducing a metric on the 3-algebra

$$\mathrm{Tr}(T_a, T^b) = \mathrm{tr}(T^\dagger_a T^b) \ , \tag{6}$$

where on the right hand side tr is the ordinary matrix trace.

The gauge symmetry is generated by

$$\delta Z^A = \Lambda^b{}_c[Z^A, T_b; T^c] = M_L Z^A - Z^A M_R \ , \tag{7}$$

where $M_L = \frac{2\pi}{k}\Lambda^b{}_c T_b (T^c)^\dagger$, $M_R = \frac{2\pi}{k}\Lambda^b{}_c (T^c)^\dagger T_b$ and $(\Lambda^b{}_c)^* = -\Lambda^c{}_b$. Thus we see that $M^\dagger_{L/R} = -M_{L/R}$ and hence they can be viewed as generators of $\mathfrak{u}(n) \times \mathfrak{u}(m)$ with Z^A and ψ_A in the bi-fundamental representation.

As a result, the action of the gauge fields $\tilde{A}^a_{\mu b}$ on Z^A_a can be respectively rewritten in terms of left- and right-acting $\mathfrak{u}(n)$ and $\mathfrak{u}(m)$ gauge fields $\tilde{A}^{L/R}_\mu$

$$D_\mu Z^A = \partial_\mu Z^A - i\tilde{A}^L_\mu Z^A + iZ^A \tilde{A}^R_\mu \tag{8}$$

and the term $\mathcal{L}_{CS}$ in (1) is then a level $(k, -k)$ Chern-Simons term for $\mathfrak{u}(n) \times \mathfrak{u}(m)$

$$\mathcal{L}_{CS} = \frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}\left(\mathrm{tr}(\tilde{A}^L_\mu \partial_\nu \tilde{A}^L_\lambda - \frac{2}{3}\tilde{A}^L_\mu \tilde{A}^L_\nu \tilde{A}^L_\lambda) - \mathrm{tr}(\tilde{A}^R_\mu \partial_\nu \tilde{A}^R_\lambda - \frac{2}{3}\tilde{A}^R_\mu \tilde{A}^R_\nu \tilde{A}^R_\lambda)\right) \ . \tag{9}$$

The Chern-Simons level k is integer whenever tr is the trace in the fundamental representation.

However, it is important to note that $\mathrm{tr}(M_L) = \mathrm{tr}(M_R)$. Thus if $M_L = i\theta_L \mathbb{1}_{n\times n}$ and $M_R = i\theta_R \mathbb{1}_{m\times m}$, we have $n\,\theta_L = m\,\theta_R$. Since the action of these Abelian U(1)'s is $Z^A \to e^{i\theta_L} Z^A e^{-i\theta_R} = e^{i(\theta_L - \theta_R)} Z^A$, these cancel for the ABJM case of $m = n$ and hence the gauge algebra is really $\mathfrak{su}(n) \oplus \mathfrak{su}(n) \oplus \mathfrak{u}(1)_Q$, where the overall $\mathrm{U}(1)_Q$ acts trivially on all fields. This is not true in the ABJ case, where $m \neq n$ and the gauge group is an honest $\mathfrak{u}(n) \oplus \mathfrak{u}(m)$. This is in line with the observations of [10, 11].

As an example let us consider the particular choice where Z^A are 2×2 complex matrices. A basis of such matrices is provided by

$$T^a = \left\{-\frac{i}{\sqrt{2}}\sigma_1, -\frac{i}{\sqrt{2}}\sigma_2, -\frac{i}{\sqrt{2}}\sigma_3, \frac{1}{\sqrt{2}}\,\mathbb{1}_{2\times2}\right\} \ , \tag{10}$$

where $a = 1, 2, 3, 4$, σ_i are the Hermitian Pauli matrices: $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma^k$ and the factor of i is chosen to ensure that the structure constants f^{abcd} are real. In particular, using (5) and (6), one sees that

$$f^{abcd} = \frac{\pi}{k}\epsilon^{abcd} \qquad \text{and} \qquad \mathrm{Tr}(T^a T^b) = \delta^{ab} \ . \tag{11}$$

Note that in this case f^{abcd} is real and totally antisymmetric. This means that the Lagrangian $\mathcal{L}_{\mathfrak{su}(2)\times\mathfrak{su}(2)}$ in fact has $\mathcal{N} = 8$ supersymmetry and SO(8) R-symmetry and is precisely the Lagrangian of [2].

From 3-algebras to the ABJM theory

To obtain the $\mathrm{U}(n)\times\mathrm{U}(n)$ ABJM models that describe multiple M2-branes from the above we must gauge the rigid $\mathrm{U}(1)_B$ symmetry $Z^A \to e^{i\theta}Z^A$, $\psi_A \to e^{i\theta}\psi_A$ enjoyed by (1). Given any rigid supersymmetric theory with a global symmetry it is always possible to gauge this symmetry and preserve supersymmetry, provided that the supersymmetries commute with the global symmetries (otherwise the supersymmetries would have to become local and hence one would have to include gravity).

To gauge the $\mathrm{U}(1)_B$ we simply introduce an Abelian gauge field B_μ and redefine the covariant derivative D_μ to be

$$D_\mu Z^A_a = \partial_\mu Z^A_a - \tilde{A}^b_{\mu a} Z^A_b - iB_\mu \delta^b_a Z^A_b \tag{12}$$

and similarly for $D_\mu \psi_{Aa}$ ($\bar{Z}_A$ and ψ^A have the opposite $\mathrm{U}(1)_B$ charge and hence the sign of $\tilde{A}_\mu$ is flipped in $D_\mu \bar{Z}_A$ and $D_\mu \psi^A$). Under the $\mathrm{U}(1)_B$ gauge transformation we have

$$B_\mu \to B_\mu + \partial_\mu \theta \tag{13}$$

and clearly the action is now invariant under $\mathrm{U}(1)_B$ gauge transformations so that the full gauge algebra is $\mathfrak{su}(n) \times \mathfrak{su}(n) \times \mathfrak{u}(1)_Q \times \mathfrak{u}(1)_B$ (although again the $\mathrm{U}(1)_Q$ symmetry is trivial).

Our next step is to make the above action invariant under $\mathcal{N} = 6$ supersymmetry. The transformations of Z^A, ψ_A and $\tilde{A}^a_{\mu b}$ remain the same, except that the covariant derivative now includes the B_μ gauge field. We will need δB_μ which we simply take to be

$$\delta B_\mu = 0 \ . \tag{14}$$

Since locally the theory is the same, the variation of the action is unchanged with the exception of terms in the supervariation of the Fermion kinetic term involving $[D_\mu, D_\nu]$, which now includes a contribution from $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. Indeed we find

$$\begin{aligned}
\delta \mathcal{L}^{gauged}_{\mathfrak{su}(n)\times\mathfrak{su}(n)} &= -\frac{1}{2} G_{\mu\nu} \bar{\epsilon}_{AB} \gamma^{\mu\nu} \psi^{Aa} Z^B_a + \frac{1}{2} G_{\mu\nu} \bar{\epsilon}^{AB} \gamma^{\mu\nu} \psi_{Aa} \bar{Z}^a_B \\
&= -\frac{1}{2} \varepsilon^{\mu\nu\lambda} G_{\mu\nu} \bar{\epsilon}_{AB} \gamma_\lambda \psi^{Aa} Z^B_a + \frac{1}{2} \varepsilon^{\mu\nu\lambda} \bar{\epsilon}^{AB} G_{\mu\nu} \bar{\epsilon} \gamma_\lambda \psi_{Aa} \bar{Z}^a_B \ ,
\end{aligned} \tag{15}$$

where we have used $\gamma^{\mu\nu} = \varepsilon^{\mu\nu\lambda}\gamma_\lambda$. To cancel this we introduce a new field Q_μ and a new term in the Lagrangian

$$\mathcal{L}_{\mathfrak{u}(n)\oplus\mathfrak{u}(n)} = \mathcal{L}^{gauged}_{\mathfrak{su}(n)\oplus\mathfrak{su}(n)} + \frac{k'}{8\pi} \epsilon^{\mu\nu\lambda} G_{\mu\nu} Q_\lambda \ , \tag{16}$$

where in the first term on the right hand side we have included the B_μ gauge field and k' is an as of yet undetermined real constant. We see that this will be supersymmetric if we take

$$\delta Q_\lambda = \frac{4\pi}{k'} \bar{\epsilon}_{AB} \gamma_\lambda \psi^{Aa} Z^B_a - \frac{4\pi}{k'} \bar{\epsilon}^{AB} \gamma_\lambda \psi_{Aa} \bar{Z}^a_B \ . \tag{17}$$

The form for the supersymmetry transformations seems odd since $\delta B_\mu = 0$ and hence $[\delta_1, \delta_2] B_\mu = 0$ so one might worry about closure. However on-shell we have $G_{\mu\nu} = 0$ so that, on-shell,

$$[\delta_1, \delta_2] B_\mu = v^\nu G_{\nu\mu} \qquad v^\nu = \frac{i}{2} (\bar{\epsilon}^{CD}_2 \gamma^\nu \epsilon^1_{CD}) \ , \tag{18}$$

which is a translation and a U(1)$_B$ gauge transformation. We must also check the closure on Q_μ. Here we find that

$$[\delta_1, \delta_2]Q_\mu = \frac{k'}{4\pi} v^\nu \varepsilon_{\mu\nu\lambda}(iZ^A_a D^\lambda \bar{Z}^a_A - iD^\lambda Z^A_a \bar{Z}^a_A - \bar{\psi}^A_a \gamma^\lambda \psi^a_A) + D_\mu \Lambda \ , \tag{19}$$

where $\Lambda = \frac{k'}{4\pi}(\bar{\epsilon}_2^{AC}\epsilon_{1BC} - \bar{\epsilon}_1^{AC}\epsilon_{2BC})\bar{Z}^a_B Z^B_a$. Using the on-shell condition that comes from the Lagrangian

$$H_{\mu\nu} = -\frac{k'}{4\pi}\varepsilon_{\mu\nu\lambda}(iZ^A_a D^\lambda \bar{Z}^a_A - iD^\lambda Z^A_a \bar{Z}^a_A - \bar{\psi}^A_a \gamma^\lambda \psi^a_A) \ , \tag{20}$$

where $H_{\mu\nu} = \partial_\mu Q_\nu - \partial_\nu Q_\mu$, we again find a translation with $\mathfrak{u}(1)_Q \times \mathfrak{u}(1)_B$ gauge transformation

$$[\delta_1, \delta_2]Q_\mu = v^\nu H_{\nu\mu} + D_\mu \Lambda \ . \tag{21}$$

Thus we see that Q_μ, which started off life as a Lagrange multiplier for the constraint $G_{\mu\nu} = 0$, naturally inherits a $\mathfrak{u}(1)$ gauge symmetry of its own. The closure on the other fields remains unchanged from the $\mathfrak{su}(n) \times \mathfrak{su}(n)$ Lagrangian, except that the connection now involves the $\mathfrak{u}(1)_B$ gauge field.

If we write $B_\mu = A^L_\mu - A^R_\mu$ and $Q_\mu = A^L_\mu + A^R_\mu$ then, up to a total derivative, the new term we have added is

$$\mathcal{L}_{\mathfrak{u}(1)\oplus\mathfrak{u}(1)\ CS} = \frac{k'}{4\pi}\epsilon^{\mu\nu\lambda} A^L_\mu \partial_\nu A^L_\lambda - \frac{k'}{4\pi}\epsilon^{\mu\nu\lambda} A^R_\mu \partial_\nu A^R_\lambda \ , \tag{22}$$

which is just the Chern-Simons Lagrangian for a $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$ gauge theory.

We have therefore constructed a family of $\mathcal{N} = 6$ Chern-Simons-matter Lagrangians with gauge fields that take values in a $\mathfrak{u}(1) \oplus \mathfrak{su}(n) \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(n)$ Lie-algebra and are parametrised by k and k'. From the point of view of supersymmetry the levels k and k' are arbitrary and although k must be an integer in the quantum theory, k' need not be (indeed k' can be absorbed into the definition of Q_λ), *e.g.* see [12]. The possibility of choosing different levels was also pointed out in [5].

With the choice[1]

$$k' = nk \ , \tag{23}$$

we see that the addition of the U(1)$\times$U(1) Chern-Simons term simply converts the $\mathfrak{su}(n) \times \mathfrak{su}(n)$ level $(k, -k)$ Chern-Simons term $\mathcal{L}_{CS}$ with connection $\tilde{A}^a{}_b$ in the original Lagrangian (9) into a $\mathfrak{u}(n) \times \mathfrak{u}(n)$ level $(k, -k)$ Chern-Simons term with connection $\tilde{A}^{L/R}_\mu + iA^{L/R}_\mu$. In terms of $A^{R/L}_\mu$, we have

$$\delta A^R_\lambda = \delta A^L_\lambda = \frac{2\pi}{nk}\bar{\epsilon}_{AB}\gamma_\lambda \psi^{Aa} Z^B_a - \frac{2\pi}{nk}\bar{\epsilon}^{AB}\gamma_\lambda \psi_{Aa} \bar{Z}^a_B \ . \tag{24}$$

[1] Here we agree with the literature [11, 13, 14] but normalise the U(n) generators with $T^a \in$ SU(n) for $a = 1, ..., N^2 - 1$ and $T^0 = \mathbb{1}_{N\times N}$, such that the coefficients in the expression for the covariant derivative (12) remain unchanged.

Taking the global gauge group to be $\mathrm{U}(n) \times \mathrm{U}(n)$ we have constructed the $\mathcal{N} = 6$ ABJM theory [5].

Finally we mention a crucial subtlety: the decomposition of $\mathrm{U}(n)$ is not strictly in terms of $\mathrm{SU}(n) \times \mathrm{U}(1)$. In particular given any pair $\omega \in \mathrm{U}(1)$ and $A_0 \in \mathrm{SU}(n)$ we obtain an element $A = \omega A_0 \in \mathrm{U}(n)$. However the inverse map is not unique since, for a given $A \in \mathrm{U}(n)$, we have

$$\omega^n = \det(A) \ , \qquad A_0 = \omega^{-1} A \ , \tag{25}$$

and hence there are n solutions for ω and A_0 related by $\omega \to e^{2\pi i/n}\omega$, $A_0 \to e^{-2\pi i/n} A_0$. Thus the map from $\mathrm{U}(1) \times \mathrm{SU}(n) \to \mathrm{U}(n)$ is an n-fold cover and so the isomorphism is

$$\mathrm{U}(n) \simeq \frac{\mathrm{SU}(n) \times \mathrm{U}(1)}{\mathbb{Z}_n} \ . \tag{26}$$

Although these modifications do not change anything at the level of the Lagrangian or the classical theory, they do change the quantisation conditions for the various fluxes, as we shall see in the next section, which will be important in the next section when we calculate the moduli space of the theory in order to compare with the answer expected from M-theory.

Dual Photon Formulation

Having arrived at the standard form for the ABJM theory we can take a step back and consider the equivalent Lagrangian (16), but once again with $k' = nk$. Integrating by parts and discarding a boundary term leads to

$$\mathcal{L}_{\mathfrak{u}(n)\times\mathfrak{u}(n)} = \mathcal{L}^{gauged}_{\mathfrak{su}(n)\times su(n)} + \frac{nk}{4\pi}\varepsilon^{\mu\nu\lambda} B_\mu \partial_\nu Q_\lambda \ . \tag{27}$$

Next we introduce a Lagrange multiplier term[2]

$$\mathcal{L}_{\mathfrak{u}(n)\times\mathfrak{u}(n)} = \mathcal{L}^{\mathrm{gauged}}_{\mathfrak{su}(n)\times\mathfrak{su}(n)} + \frac{nk}{8\pi}\varepsilon^{\mu\nu\lambda} B_\mu H_{\nu\lambda} + \frac{n}{8\pi}\sigma\varepsilon^{\mu\nu\lambda}\partial_\mu H_{\nu\lambda} \ . \tag{28}$$

Integrating the last term by parts we find

$$\mathcal{L}_{\mathfrak{u}(n)\oplus\mathfrak{u}(n)} = \mathcal{L}^{\mathrm{gauged}}_{\mathfrak{su}(n)\oplus\mathfrak{su}(n)} + \frac{nk}{8\pi}\varepsilon^{\mu\nu\lambda} B_\mu H_{\nu\lambda} - \frac{n}{8\pi}\varepsilon^{\mu\nu\lambda}\partial_\mu\sigma H_{\nu\lambda} \ . \tag{29}$$

We can now integrate out $H_{\mu\nu}$ to see that

$$B_\mu = \frac{1}{k}\partial_\mu\sigma \ . \tag{30}$$

Thus under a $\mathrm{U}(1)_B$ gauge transformation we find

$$\sigma \to \sigma + k\theta \ . \tag{31}$$

[2]Aspects of this procedure have also appeared in [15, 16].

Substituting back we find that the $\mathfrak{u}(n)\oplus\mathfrak{u}(n)$ Lagrangian is equivalent to the $\mathfrak{su}(n)\oplus\mathfrak{su}(n)$ Lagrangian with new variables:

$$\mathcal{L}_{\mathfrak{u}(n)\oplus\mathfrak{u}(n)}(Z^A,\psi_A,\tilde{A}^a_{\mu}b,B_\mu,Q_\mu)\cong\mathcal{L}_{\mathfrak{su}(n)\oplus\mathfrak{su}(n)}(e^{\frac{i}{k}\sigma}Z^A,e^{\frac{i}{k}\sigma}\psi_A,\tilde{A}^a_{\mu}b)\ . \tag{32}$$

In particular the variables $\hat{Z}^A=e^{\frac{i}{k}\sigma}Z^A$ and $\hat{\psi}_A=e^{\frac{i}{k}\sigma}\psi_A$ are $\mathrm{U}(1)\times\mathrm{U}(1)$ gauge invariant.

Finally, we need to determine the periodicity of σ which follows from a quantisation condition on the flux H. Let us review the familiar Dirac quantisation rule. We start by considering the phase induced by the parallel transport over a closed path γ of a field, Ψ, that couples to a U(1) field A_μ through $D_\mu\Psi=\partial_\mu\Psi-iA_\mu\Psi$. We find that the resulting wavefunction is related to the initial wavefunction by a U(1) transformation

$$\Psi_\gamma=e^{i\oint_\gamma A}\Psi_0=e^{i\int_D F}\Psi_0\ , \tag{33}$$

where D is a two-dimensional surface whose boundary is γ. However the choice of D is not unique. Given any two such choices D and D' we require that the phase, viewed as an element of the gauge group U(1), is the same. This implies that

$$e^{i\int_{D-D'}F}=1 \tag{34}$$

and hence $\int_\Sigma F\in 2\pi\mathbb{Z}$, where $\Sigma=D-D'$ is any closed surface. However in our case the gauge group is $(\mathrm{U}(1)\times\mathrm{SU}(n))/\mathbb{Z}_n$ and we need only require that $\int_\Sigma F\in\frac{2\pi}{n}\mathbb{Z}$, *i.e.* the U(1) phases computed by two different paths must be equal modulo $\mathbb{Z}_n$. Thus we see that the quantisation condition is

$$\int dF_{L/R}\in\frac{2\pi}{n}\mathbb{Z}\ . \tag{35}$$

This fractional flux quantization condition arises because the global gauge group is $(\mathrm{SU}(n)\times\mathrm{SU}(n))/\mathbb{Z}_n$ instead of $\mathrm{SU}(n)\times\mathrm{SU}(n)$, with $\mathbb{Z}_n$ the relative centre of the two $\mathrm{SU}(n)$ factors. Thus we refer to the resulting Chern-Simons matter theory as the $(\mathrm{SU}(n)\times\mathrm{SU}(n))/\mathbb{Z}_n$-theory.[3] This should be compared with a theory with the same $\mathcal{L}_{\mathfrak{su}(n)\oplus\mathfrak{su}(n)}$ Lagrangian but global $\mathrm{SU}(n)\times\mathrm{SU}(n)$ gauge symmetry and no fractional flux quantisation which we refer to as the $\mathrm{SU}(n)\times\mathrm{SU}(n)$-theory.

After integrating out H, we are left with the condition $B=\frac{1}{k}d\sigma$. Therefore, locally, $F_L-F_R=dB$ vanishes so that F_L and F_R must have the same flux. Note that we do not require that σ is globally defined so there can be a non-zero Wilson line for the gauge field B. However, since $F_L-F_R=dB=0$ in any open set where σ is single-valued, it follows that $F_L=F_R$ globally. This generalises the flux quantisation argument of [17] to allow for a nonvanishing but trivial gauge field and applies to the full theory, not just the moduli space. Since $H=F_L+F_R$ we have

$$\int dH=\int\frac{1}{2}\epsilon^{\mu\nu\lambda}\partial_\mu H_{\nu\lambda}\in\frac{4\pi}{n}\mathbb{Z} \tag{36}$$

[3] For theories with bifundamental matter the $(\mathrm{SU}(n)\times\mathrm{SU}(n))/\mathbb{Z}_n$ group, where the centre of one $\mathrm{SU}(n)$ factor is identified with the inverse centre of the other, is indistinguishable from $\mathrm{SU}(n))/\mathbb{Z}_n\times\mathrm{SU}(n))/\mathbb{Z}_n\simeq\mathrm{PSU}(n)\times\mathrm{PSU}(n)$.

and σ has period 2π. Note that since $e^{i\theta}$ is a U(n) transformation, θ also has period 2π. Thus we can fix the $\mathrm{U}(1)_B$ symmetry using (31) and set $\sigma = 0 \bmod 2\pi$. However, this periodicity imposes an additional identification on the U(1)-invariant fields

$$\hat{Z}^A \cong e^{\frac{2\pi i}{k}} \hat{Z}^A \qquad \text{and} \qquad \hat{\psi}_A \cong e^{\frac{2\pi i}{k}} \hat{\psi}_A \ . \tag{37}$$

We are therefore told that the $\mathrm{U}(n) \times \mathrm{U}(n)$ ABJM theory is equivalent to a $\mathbb{Z}_k$ identification on the $(\mathrm{SU}(n) \times \mathrm{SU}(n))/\mathbb{Z}_n$-theory. Note that the $\mathbb{Z}_n$ quotient arises here as the relative part of the two $\mathbb{Z}_n$ factors from $\mathrm{U}(n) \simeq (\mathrm{U}(1) \times \mathrm{SU}(n))/\mathbb{Z}_n$.

However we should be careful: Our discussion so far has been largely based on local aspects of the theory and since U(n) is not globally the same as $\mathrm{U}(1) \times \mathrm{SU}(n)$ there could be obstructions at a global level. We will see in the following that the $\mathrm{U}(n) \times \mathrm{U}(n)$ theories can only be viewed as $\mathbb{Z}_k$ identifications when n and k are coprime. In particular, for $k = 1$ the $\mathbb{Z}_k$ identification is clearly trivial and one simply has the $(\mathrm{SU}(n) \times \mathrm{SU}(n))/\mathbb{Z}_n$-theory.

Note that, had we considered instead a $\mathrm{U}(1) \times \mathrm{SU}(n) \times \mathrm{U}(1) \times \mathrm{SU}(n)$ gauge theory, we would not have been able to use the fractional flux quantisation condition and σ would have had period $2\pi/n$. In addition, we would have been free to have any integer value for the U(1) level k' and as a result we would find a $\mathbb{Z}_{k'}$ identification. From this perspective we would arrive at a $\mathrm{SU}(n) \times \mathrm{SU}(n)$-theory by starting with $\mathrm{U}(1) \times \mathrm{SU}(n) \times \mathrm{U}(1) \times \mathrm{SU}(n)$ but take $k' = k$ and the usual Dirac quantisation. However, as we will see in the next section, the moduli space of the resulting theory would then not be the same as the $\mathrm{U}(n) \times \mathrm{U}(n)$ ABJM models due to the different flux quantisation condition on the SU(n) factor. Finally, one might consider other quantisation conditions which lead to different moduli spaces [18].

Moduli Space of $n = 2$ theories

To test the above analysis it is insightful to compute the moduli space of the $(\mathrm{SU}(n) \times \mathrm{SU}(n))/\mathbb{Z}_n$-theory and then compare with the $\mathrm{U}(n) \times \mathrm{U}(n)$ answer. To begin with, we consider the $n = 2$ case in detail.

We observe that the solutions to $V = 0$ are obtained by taking $[Z^A, Z^B; \bar{Z}_C] = 0$ for all A, B, C. This is solved by taking the Z^A, which are 2×2 matrices, to be mutually commuting. Recall that the Z^A are in the bi-fundamental representation so that under a gauge transformation

$$Z^A \cong g_L Z^A g_R^{-1} \ . \tag{38}$$

Thus, modulo gauge transformations, we can take without loss of generality

$$Z^A = \frac{1}{\sqrt{2}} r_1^A - \frac{i}{\sqrt{2}} r_2^A \sigma_3 \ . \tag{39}$$

The gauge symmetries that preserve this form, for generic r_1^A, r_2^A, must satisfy

$$g_L g_R^{-1} = a + ib\sigma_3 \qquad g_L i\sigma_3 g_R^{-1} = c + id\sigma_3 \tag{40}$$

for arbitrary constants a, b, c, d. The first condition can be used to deduce that

$$g_L = e^{i\theta\sigma_3} g_R \ , \tag{41}$$

for an arbitrary θ, whereas the second condition puts a constraint on g_R

$$g_R i\sigma_3 g_R^{-1} = e^{i\theta'\sigma_3}, \tag{42}$$

for an arbitrary θ'. Since the left hand side is traceless we see that this is only possible if $\theta' = \pm\pi/2$ so that

$$g_R i\sigma_3 g_R^{-1} = \pm i\sigma_3. \tag{43}$$

Thus g_R is generating a discrete identification

$$r_2^A \cong -r_2^A \tag{44}$$

and one should think of r_1^A as the centre-of-mass coordinate, while r_2^A as the relative separation between two indistinguishable M2-branes. To this end we write

$$r_1^A = \frac{1}{2}(z_1^A + z_2^A) \qquad \text{and} \qquad r_2^A = \frac{i}{2}(z_1^A - z_2^A) \ , \tag{45}$$

so that the g_R transformation is now $z_1^A \leftrightarrow z_2^A$. In addition we have a continuous U(1) action generated by $g_L = e^{i\theta\sigma_3}$. This acts on z_1^A and z_2^A as

$$z_1^A \to e^{i\theta} z_1^A \ , \qquad z_2^A \to e^{-i\theta} z_2^A \ . \tag{46}$$

The subtle part of the calculation comes from considering the continuous gauge symmetries $g = e^{i\theta}$. Reducing to the moduli space fields with with $A_\mu = \tilde{A}^3_{L\mu}\sigma^3$ and $\tilde{A}_\mu = \tilde{A}^3_{R\mu}\sigma^3$, we find that the Chern-Simons action (9) becomes

$$\mathcal{L} = -\mathcal{D}_\mu z_1^A \mathcal{D}^\mu \bar{z}_{1A} - \mathcal{D}_\mu z_2^A \mathcal{D}^\mu \bar{z}_{2A} + \frac{k}{2\pi}\epsilon^{\mu\nu\lambda}\tilde{A}^3_{L\mu}\partial_\nu \tilde{A}^3_{L\lambda} - \frac{k}{2\pi}\epsilon^{\mu\nu\lambda}\tilde{A}^3_{R\mu}\partial_\nu \tilde{A}^3_{R\lambda} \ , \tag{47}$$

where

$$\mathcal{D}_\mu z_1^A = \partial_\mu z_1^A - i(\tilde{A}^3_{L\mu} - \tilde{A}^3_{R\mu})z_1^A \ , \qquad \mathcal{D}_\mu z_2^A = \partial_\mu z_2^A + i(\tilde{A}^3_{L\mu} - \tilde{A}^3_{R\mu})z_2^A \ . \tag{48}$$

Following the previous discussion we write $\tilde{B}_\mu = \tilde{A}^3_{L\mu} - \tilde{A}^3_{R\mu}$ and $\tilde{Q}_\mu = \tilde{A}^3_{L\mu} + \tilde{A}^3_{R\mu}$ so that the moduli space Lagrangian is

$$\mathcal{L} = -\mathcal{D}_\mu z_1^A \mathcal{D}^\mu \bar{z}_{1A} - \mathcal{D}_\mu z_2^A \mathcal{D}^\mu \bar{z}_{2A} + \frac{2k}{8\pi}\epsilon^{\mu\nu\lambda}\tilde{B}_\mu \tilde{H}_{\nu\lambda} \ , \tag{49}$$

where now $\tilde{H}_{\nu\lambda} = \partial_\nu \tilde{Q}_\lambda - \partial_\lambda \tilde{Q}_\nu$. We can introduce a Lagrange multiplier term

$$\mathcal{L} = -\mathcal{D}_\mu z_1^A \mathcal{D}^\mu \bar{z}_{1A} - \mathcal{D}_\mu z_2^A \mathcal{D}^\mu \bar{z}_{2A} + \frac{2k}{8\pi}\epsilon^{\mu\nu\lambda}\tilde{B}_\mu \tilde{H}_{\nu\lambda} + \frac{2}{8\pi}\chi\epsilon^{\mu\nu\lambda}\partial_\mu \tilde{H}_{\nu\lambda} \ . \tag{50}$$

Integrating out $\tilde{H}_{\mu\nu}$ gives $\tilde{B}_\mu = \frac{1}{k}\partial_\mu\chi$ and the Lagrangian can be written as

$$\mathcal{L} = -\partial_\mu \tilde{z}_1^A \partial^\mu \bar{\tilde{z}}_{1A} - \partial_\mu \tilde{z}_1^A \partial^\mu \bar{\tilde{z}}_{2A} \ , \tag{51}$$

where $\tilde{z}_1^A = e^{i\chi/k} z_1^A$ and $\tilde{z}_2^A = e^{-i\chi/k} z_2^A$ are gauge invariant.

It is once again necessary to determine the periodicity of the dual photon χ. The argument here is identical to what was discussed around Eq. (36). Namely, since $d\tilde{B} = 0$ we have that $\tilde{F}_L^3 = \tilde{F}_R^3$, where $\tilde{F}_{L/R}^3 = d\tilde{A}_{L/R}^3$, and the quantisation condition is

$$\int d\tilde{F}_{L/R}^3 \in \frac{2\pi}{2}\mathbb{Z} \,. \tag{52}$$

The factor of 2 in the denominator arises because the gauge group is $(\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$, in the same manner that n appeared in (35). Thus, since $\tilde{H} = \tilde{F}_L^3 + \tilde{F}_R^3$ we have that

$$\int d\tilde{H} = \int \frac{1}{2}\epsilon^{\mu\nu\lambda}\partial_\mu \tilde{H}_{\nu\lambda} \in \frac{4\pi}{2}\mathbb{Z} \tag{53}$$

and hence χ has period of 2π. We conclude that the vacuum moduli space scalars are subject to the identification

$$z_1^A \cong e^{\frac{2\pi i}{k}} z_1^A \,, \qquad z_2^A \cong e^{-\frac{2\pi i}{k}} z_2^A \,. \tag{54}$$

In summary, we find that the sum of identifications on the vacuum moduli space, including the ones coming from (37), act as

$$\begin{array}{rccl} g_{\mathrm{U}(1)}: & z_1^A & \cong & e^{\frac{2\pi i}{k}} z_1^A \,, \qquad z_2^A \cong e^{\frac{2\pi i}{k}} z_2^A \\ g_{12}: & z_1^A & \cong & z_2^A \\ g_{\mathrm{SU}(2)}: & z_1^A & \cong & e^{\frac{2\pi i}{k}} z_1^A \,, \qquad z_2^A \cong e^{-\frac{2\pi i}{k}} z_2^A \,. \end{array} \tag{55}$$

The first one is a $\mathbb{Z}_k$ coming from integrating out the $\mathrm{U}(1)_B$, and acts on the whole theory, not just the moduli space. The other two are consequences of the $(\mathrm{SU}(n) \times \mathrm{SU}(n))/\mathbb{Z}_n$ gauge symmetry acting on the vacuum moduli space and generate the dihedral group of order $2k$, $\mathbb{D}_k \simeq \mathbb{Z}_2 \ltimes \mathbb{Z}_k$. This is consistent with the calculation in [6, 7] which found $\mathbb{D}_{2k}$, since the difference $k \to 2k$ arises because a fractional quantisation condition was not allowed, corresponding to an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ global gauge group.

We now need to compare these moduli space identifications with the answer for the $\mathrm{U}(2) \times \mathrm{U}(2)$ ABJM theory that describes two indistinguishable M2-branes in $\mathbb{R}^8/\mathbb{Z}_k$, that is

$$\mathcal{M}_k = \frac{(\mathbb{R}^8/\mathbb{Z}_k) \times (\mathbb{R}^8/\mathbb{Z}_k)}{\mathbb{Z}_2} \,. \tag{56}$$

In this case the moduli space quotient group is generated by

$$\begin{array}{rccl} g_1: & z_1^A & \cong & e^{\frac{2\pi i}{k}} z_1^A \,, \qquad z_2^A \cong z_2^A \\ g_{12}: & z_1^A & \cong & z_2^A \\ g_2: & z_1^A & \cong & z_1^A \,, \qquad z_2^A \cong e^{\frac{2\pi i}{k}} z_2^A \,. \end{array} \tag{57}$$

Here we see that $g_{\mathrm{U}(1)} = g_1 g_2$ and $g_{\mathrm{SU}(2)} = g_2^{-1} g_1$. However in order to invert these relations we need to solve $g_1^2 = g_{\mathrm{U}(1)} g_{\mathrm{SU}(2)}$ and $g_2^2 = g_{\mathrm{U}(1)} g_{\mathrm{SU}(2)}^{-1}$, *i.e.* take the square root in the group generated by $g_{\mathrm{U}(1)}$ and $g_{\mathrm{SU}(2)}$. A short calculation shows that this is only possible if k is odd. Thus we conclude the we obtain the correct moduli space only when k is odd.

The value $k = 1$ is special: The orbifold action is trivial and the moduli space of the SU(2) $\times$ SU(2)-theory is the one for 2 M2-branes in flat space. As a by-product we see that for $k = 1$ the $\mathcal{N} = 6$ U(2) $\times$ U(2)-theory in fact has $\mathcal{N} = 8$ supersymmetry. This has also been shown with the help of monopole operators in [5, 15, 16], although here the physics also have a formulation in terms of the manifestly $\mathcal{N} = 8$ supersymmetric, local Lagrangian.

Moduli Space In General

We will now see that the problem we faced for $n = 2$ and k-even extends more generally. For a general n the vacuum moduli space is obtained by setting

$$Z^A = \mathrm{diag}(z_1^A, ..., z_n^A) \ . \tag{58}$$

If we consider gauge transformations of the form $g_L = g_R$ then Z^A behaves as if it were in the adjoint of SU(n) and hence cannot tell the difference between the SU(n) and U(n) theories. The result is that the gauge transformations which preserve the form of Z^A simply interchange the eigenvalues z_i^A leading to the symmetric group acting on the n M2-branes.

Next we can consider transformations in the diagonal subgroup of SU(n) or U(n). These act to rotate the phases of the z_i^A, however in the SU(n)-theory they only do so up to the constraint that the diagonal elements must have unit determinant. In the U(n)-theory this is not the case and there are n independent U(1)'s, one for each z_i^A, and each of these U(1)'s leads to a $\mathbb{Z}_k$ identification on the moduli space. Thus for U(n) we indeed see that we find n commuting copies of $\mathbb{Z}_k$ along with the symmetric group acting on the z_i^A.

For the SU(n)-theory, even including the $\mathbb{Z}_k$ action of U(1)$_B$, this will not always be the case. In particular, note that since the determinant of the gauge transformations coming from SU(n) is always one we have, for an arbitrary element of the moduli space orbifold group,

$$\det(g_{\mathrm{U}(1)}^{l_B} g_0) = \det(g_{\mathrm{U}(1)}^{l_B}) = e^{2\pi i n l_B/k} \ . \tag{59}$$

Here g_0 represents a generic element of the moduli space orbifold group obtained in the (SU(n) $\times$ SU(n))/$\mathbb{Z}_n$-theory. On the other hand, the moduli space orbifold group of the U(n)-theory generated by n independent U(1)'s has

$$\det(g_1^{l_1} ... g_n^{l_n}) = e^{2\pi i(l_1 + ... + l_n)/k} \ . \tag{60}$$

If these two theories are to give the same moduli space then we must be able to have $e^{2\pi i(l_1+...+l_n)/k} = e^{2\pi i n l_B/k}$ for any possible combination of l_i's. Thus we are required to solve

$$l = n\, l_B \bmod k \ , \tag{61}$$

for l_B as a function of l, n, k, where $l = l_1 + ... + l_n$ is arbitrary. Hence, if this equation can be solved for l_B then $g_0 = e^{-2\pi i l_B/k} g_1^{l_1} ... g_n^{l_n}$ is an element of $\mathrm{SU}(n)$ and can arise from the vacuum moduli space quotient group of the$(\mathrm{SU}(n) \times \mathrm{SU}(n))/\mathbb{Z}_n$-theory.

We will now show that (61) has solutions for all l **iff** n and k are coprime. In general the solution is $l_B = (l - pk)/n$ for any $p \in \mathbb{Z}$; however we require that l_B is an integer. It is clear that we may view l, k and p as elements of $\mathbb{Z}/\mathbb{Z}_n$ and we are therefore required to solve the following equation for p

$$l = pk \bmod n \ . \tag{62}$$

This always has solutions if the map $\varphi : p \mapsto pk$ is surjective on $\mathbb{Z}/\mathbb{Z}_n$. Since $\mathbb{Z}/\mathbb{Z}_n$ is a finite set this will be the case **iff** φ is also injective. Thus we wish to show that $pk = p'k \bmod n$ implies $p = p'$. This is equivalent to showing that $qk = 0 \bmod n$ implies $q = 0 \bmod n$. Now suppose that $qk = rn$. If k and n are coprime then all the prime factors of k must be in r and all the prime factors of n must be in q. Thus $q = 0 \bmod n$. On the other hand if k and n have a common factor d then we find a non-zero solution by taking $q = n/d$ and $r = k/d$. Thus $qk = 0 \bmod n$ has no non-trivial solutions for q **iff** n and k are coprime.

This result can been restated as follows: Although locally $\mathrm{U}(n) \simeq \mathrm{U}(1) \times \mathrm{SU}(n)$, this is not true globally. Even though the Lagrangian is defined by local information at the Lie-algebra level, the map we constructed, reducing the $\mathrm{U}(n) \times \mathrm{U}(n)$-theory to a $\mathbb{Z}_k$ quotient of the $(\mathrm{SU}(n) \times \mathrm{SU}(n))/\mathbb{Z}_n$-theory, involves finite gauge transformations and is therefore sensitive to global properties of $\mathrm{U}(n)$. The above discussion shows that the vacuum moduli space quotient group of the $\mathrm{U}(n) \times \mathrm{U}(n)$ theories is not of the form $\mathbb{Z}_k \times G_0$, where $G_0 \subset \mathrm{SU}(n)$, unless n and k are relatively prime.

We have therefore shown that if n and k have a common factor then the vacuum moduli spaces for the two theories do not agree, as there is a global obstruction to mapping the $\mathrm{U}(n) \times \mathrm{U}(n)$-theory to a $\mathbb{Z}_k$ quotient of the $(\mathrm{SU}(n) \times \mathrm{SU}(n))/\mathbb{Z}_n$-theory. On the other hand, if n and k are coprime then the vacuum moduli space calculated in the $(\mathrm{SU}(n) \times \mathrm{SU}(n))/\mathbb{Z}_n$-theory, along with the $\mathbb{Z}_k$ identification coming from $\mathrm{U}(1)_B$, agrees with the vacuum moduli space of the $\mathrm{U}(n) \times \mathrm{U}(n)$-theory. It is therefore natural to conjecture that in these cases the $\mathrm{U}(n) \times \mathrm{U}(n)$ theories are $\mathbb{Z}_k$ quotients of the $(\mathrm{SU}(n) \times \mathrm{SU}(n))/\mathbb{Z}_n$ theories.

The moduli space of the $k = 2$ $\mathrm{SU}(2) \times \mathrm{SU}(2)$-theory

On a related note, the moduli space of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ $\mathcal{N} = 8$ theories was calculated in [6, 7] and found to be $(\mathbb{R}^8 \times \mathbb{R}^8)/\mathbb{D}_{2k}$, where $\mathbb{D}_{2k} \simeq \mathbb{Z}_2 \ltimes \mathbb{Z}_{2k}$ the dihedral group of order $4k$. As we have already mentioned, the extra factor of 2 arises due to the standard Dirac quantisation condition when the global gauge group is $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

For the particular case of $k = 2$ one has [6, 7]

$$\begin{aligned} g_{12} : \quad & z_1^A \ \cong \ z_2^A \\ g_{\mathrm{SU}(2)} : \quad & z_1^A \ \cong \ i z_1^A \ , \qquad z_2^A \cong -i z_2^A \ . \end{aligned} \tag{63}$$

Note that because of the modified flux quantisation condition this agrees with $k = 4$ in (55). Interestingly, by reverting back to the r_1^A, r_2^A variables of (45) we have

$$\begin{aligned} g_{12}: &\quad r_1^A \cong r_1^A \quad , \quad r_2^A \cong -r_2^A \\ g_{\mathrm{SU}(2)}: &\quad r_1^A \cong r_2^A \quad , \quad r_2^A \cong -r_1^A \ . \end{aligned} \tag{64}$$

Although these variables might look contrived from the perspective of the ABJM theory they arise very naturally in the SO(4) formulation [6]. From these we can construct

$$\begin{aligned} g_{12}: &\quad r_1^A \cong r_1^A \quad , \quad r_2^A \cong -r_2^A \\ g_{12}\, g_{\mathrm{SU}(2)}^2: &\quad r_1^A \cong -r_1^A \quad , \quad r_2^A \cong r_2^A \\ g_{\mathrm{SU}(2)}\, g_{12}: &\quad r_1^A \cong r_2^A \quad , \quad r_2^A \cong r_1^A \ . \end{aligned} \tag{65}$$

These identifications are the ones expected for the moduli space $(\mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{R}^8/\mathbb{Z}_2)/\mathbb{Z}_2$ of 2 M2-branes on a $\mathbb{Z}_2$ orbifold singularity of M-theory, as also shown in [6].

For $k = 2$ the SU(2) × SU(2)-theory was interpreted in [6, 7] as the IR limit of an SO(5) gauge theory describing two D2-branes on an $\widetilde{\mathrm{O2}}^+$ orientifold of type IIA string theory, which is an M-theory $\mathbb{Z}_2$ orbifold with discrete torsion. However, there also exists another type IIA orientifold denoted $\mathrm{O2}^-$ and, as was pointed out in [5], corresponding to an O(4)-theory on the D2-brane worldvolume, which in the IR lifts to an M-theory orbifold without torsion. This has an indistinguishable moduli space from the SO(5) case, since the extra fractional brane in the latter is stuck at the fixed point and does not contribute to the moduli space dynamics. The orbifolds with and without torsion are the only expected IR fixed points with $\mathcal{N} = 8$ supersymmetry and $(\mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{R}^8/\mathbb{Z}_2)/\mathbb{Z}_2$ moduli space and correspond to the U(2) × U(2) ABJM and U(2) × U(3) ABJ theories respectively. Given the similarity between the Lagrangians, manifest symmetries (such as Parity) and the agreement between the moduli space calculations, it is also natural to conjecture that the $n = 2, k = 2$ $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2$-theory is equivalent to the $k = 2$ ABJM theory[4] and therefore the IR fixed point of the the maximally supersymmetric O(4) gauge theory in 2+1d.

Summary

In this paper we have discussed the relation of $\mathrm{U}(n) \times \mathrm{U}(n)$ ABJM theories to $(\mathrm{SU}(n) \times \mathrm{SU}(n))/\mathbb{Z}_n$ theories. In particular we showed that locally, at the level of Lagrangians, the $\mathrm{U}(1)_B$ gauge symmetry could be integrated out to give an $(\mathrm{SU}(n) \times \mathrm{SU}(n))/\mathbb{Z}_n$-theory along with a $\mathbb{Z}_k$ identification on the fields. However we also saw that there was a global obstruction to this when n and k are not coprime.

As a result we found that the U(2) × U(2) ABJM theories can be viewed as $\mathbb{Z}_k$ quotients of the $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2$, $\mathcal{N} = 8$ theories when k is odd. In particular for $k = 1$ they are

[4]Note that the ABJM theory at $n = 2$, $k = 2$ is not related to the SU(2) × SU(2)-theory as discussed in the previous section.

identical. However if one considers the SU(2)$\times$SU(2), $\mathcal{N} = 8$-theory of [2, 3] at $k = 2$, then this has the same moduli space, global supersymmetries and manifest Parity as the $k = 2$ U(2) $\times$ U(2) ABJM theory. Thus we conjectured that these two theories are equivalent and the original $\mathcal{N} = 8$, $\mathfrak{su}(2) \times \mathfrak{su}(2)$ Lagrangian of [2, 3] can be used to define quantum theories for two M2-branes on $\mathbb{R}^8$ and $\mathbb{R}^8/\mathbb{Z}_2$ (without discrete torsion) when $k = 1$ or $k = 2$ respectively and with all the symmetries manifest.

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References

[1] J. Bagger and N. Lambert, "Modeling multiple M2's," *Phys. Rev.* **D75** (2007) 045020, arXiv:hep-th/0611108.

[2] J. Bagger and N. Lambert, "Gauge Symmetry and Supersymmetry of Multiple M2-Branes," *Phys. Rev.* **D77** (2008) 065008, arXiv:0711.0955 [hep-th].

[3] A. Gustavsson, "Algebraic structures on parallel M2-branes," *Nucl. Phys.* **B811** (2009) 66–76, arXiv:0709.1260 [hep-th].

[4] J. Bagger and N. Lambert, "Comments On Multiple M2-branes," *JHEP* **02** (2008) 105, arXiv:0712.3738 [hep-th].

[5] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, "N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," *JHEP* **10** (2008) 091, arXiv:0806.1218 [hep-th].

[6] N. Lambert and D. Tong, "Membranes on an Orbifold," *Phys. Rev. Lett.* **101** (2008) 041602, arXiv:0804.1114 [hep-th].

[7] J. Distler, S. Mukhi, C. Papageorgakis, and M. Van Raamsdonk, "M2-branes on M-folds," *JHEP* **05** (2008) 038, arXiv:0804.1256 [hep-th].

[8] J. Bagger and N. Lambert, "Three-Algebras and N=6 Chern-Simons Gauge Theories," *Phys. Rev.* **D79** (2009) 025002, arXiv:0807.0163 [hep-th].

[9] O. Aharony, O. Bergman, and D. L. Jafferis, "Fractional M2-branes," *JHEP* **11** (2008) 043, arXiv:0807.4924 [hep-th].

[10] M. Schnabl and Y. Tachikawa, "Classification of N=6 superconformal theories of ABJM type," arXiv:0807.1102 [hep-th].

[11] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee, and J. Park, "N=5,6 Superconformal Chern-Simons Theories and M2-branes on Orbifolds," *JHEP* **09** (2008) 002, arXiv:0806.4977 [hep-th].

[12] G. V. Dunne, "Aspects of Chern-Simons theory," arXiv:hep-th/9902115.

[13] S. Terashima, "On M5-branes in N=6 Membrane Action," *JHEP* **08** (2008) 080, arXiv:0807.0197 [hep-th].

[14] M. A. Bandres, A. E. Lipstein, and J. H. Schwarz, "Studies of the ABJM Theory in a Formulation with Manifest SU(4) R-Symmetry," arXiv:0807.0880 [hep-th].

[15] A. Gustavsson and S.-J. Rey, "Enhanced N=8 Supersymmetry of ABJM Theory on R(8) and R(8)/Z(2)," arXiv:0906.3568 [hep-th].

[16] O.-K. Kwon, P. Oh, and J. Sohn, "Notes on Supersymmetry Enhancement of ABJM Theory," arXiv:0906.4333 [hep-th].

[17] D. Martelli and J. Sparks, "Moduli spaces of Chern-Simons quiver gauge theories and AdS(4)/CFT(3)," *Phys. Rev.* **D78** (2008) 126005, arXiv:0808.0912 [hep-th].

[18] D. Berenstein and J. Park, "The BPS spectrum of monopole operators in ABJM: towards a field theory description of the giant torus," arXiv:0906.3817 [hep-th].