

On Time-dependent Collapsing Branes and Fuzzy Odd-dimensional Spheres

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Abstract

We study the time-dependent dynamics of a collection of N collapsing/expanding $D0$ -branes in type IIA String Theory. We show that the fuzzy- S^3 and S^5 provide time-dependent solutions to the Matrix Model of $D0$ -branes and its DBI generalisation. Some intriguing cancellations in the calculation of the non-abelian DBI Matrix actions result in the fuzzy- S^3 and S^5 having the same dynamics at large- N . For the Matrix model, we find analytic solutions describing the time-dependent radius, in terms of Jacobi elliptic functions. Investigation of the physical properties of these configurations shows that there are no bounces for the trajectory of the collapse at large- N . We also write down a set of useful identities for fuzzy- S^3 , fuzzy- S^5 and general fuzzy odd-spheres.

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1 Introduction

Fuzzy spheres of even dimensionality [1, 2, 3, 4, 5, 6] have been at the centre of a number of phenomena in String and Matrix Theory [7], as well as the study of the non-abelian DBI action [8, 9, 10, 11, 12, 13, 14]. They enter classical solutions to $D0$ -brane actions giving the microscopic description of time-dependent $D0$ - $D(2k)$ bound states in type IIA, where the $D0$'s expand into a fuzzy- S^{2k} via a time-dependent analog of the Myers effect [15]. They also describe a class of static BIon $D1 \perp D(2k+1)$ brane intersections in type IIB, in which the $D1$'s blow up into a funnel of fuzzy- S^{2k} cross-section [16, 17, 18, 19, 20, 21, 22]. Both cases admit a macroscopic description in terms of the higher dimensional brane worldvolume action, with N units of worldvolume fluxes. The two pictures agree in the large- N limit. This agreement of the classical equations extends to quadratic fluctuations for the $D0$ - $D2$ bound state [23]. A full solution to the $1/N$ corrections for S^2 , coming from the implementation of the symmetrised trace (STr) prescription for the non-abelian D -brane action, has been given in [24]. The time-dependent and static configurations for the fuzzy- S^2 are directly related via an $r \rightarrow 1/r$ duality [25]. It is natural to consider extensions of these ideas to systems involving fuzzy spheres of odd dimensionality. Fuzzy odd-spheres were constructed and studied earlier in [2, 26, 27]. In [28] the fuzzy 3-sphere algebra was expressed as a quantisation of the Nambu bracket. Subsequent work used the fuzzy 3-sphere in the context of $M2 \perp M5$ intersections [29, 30, 31].

This has provided a motivation to revisit fuzzy odd-spheres. In this paper, we study the fuzzy odd-sphere equations in more detail and apply them to the time-dependent process of N $D0$'s blowing-up into a fuzzy- S^3 and S^5 respectively. Compared to the study of fuzzy even-spheres, these phenomena turn out to be significantly more involved. Commutators of fuzzy odd-sphere matrices are not vanishing at large- N , hence calculating the symmetrised trace in that limit requires a non-trivial sum over orderings. After these sums are performed we find the surprising result that the time evolution of the fuzzy- S^3 is identical to that of the fuzzy- S^5 . The rest of the paper is organised as follows. In section 2 we review the fuzzy- S^3 and higher-dimensional fuzzy odd-spheres. A number of useful identities, which apply for odd-spheres of any dimensionality, are presented. Sections 3 and 4 focus on expressions for the particular cases of S^3 and S^5 . Section 5 looks at the dynamics of N coincident $D0$ -branes described by the Matrix DBI action. This is done by using an ansatz involving the fuzzy-sphere Matrices and a time dependent radius. This is inserted into the DBI action, to obtain a reduced action for the radius. It is shown, in Appendix C, that solutions to the reduced action also solve the Matrix equations of motion. In section 6 we proceed to study the physical properties of these configurations, using a definition for the physical radius proposed in [24], and find that there will be no bounces for large- N . The characteristic

length scale of the system is $L = \sqrt{\pi}\ell_s$ and independent of N . In section 7 we show that both the fuzzy- S^3 and S^5 solve the equations of motion in the Matrix Theory limit and yield solutions in terms of Jacobi elliptic functions. In section 7 we discuss a possible dual description of the fuzzy- S^3 , in terms of a non-BPS $D3$ -brane embedded in Euclidean space as a classical three-sphere. Finally, section 8 provides a summary and outlook. Appendix A deals with some of the details on the evaluation of the $SO(4)$ $[X, X]$ term. Appendix B discusses the non-associativity of the projected $SO(2k)$ Matrix algebra, proposed to give a non-associative deformation of the algebra of functions on S^{2k-1} [2]. We find, somewhat surprisingly, that the non-associativity does not vanish in the large- n limit. We describe an alternative product on the projected space of matrices which does become associative at large n .

2 General fuzzy odd-sphere equations with $SO(D)$ symmetry

We start with a quick review of the construction of the fuzzy- S^3 and fuzzy- S^5 [2, 27]. We are working with matrices constructed by taking the symmetric n -fold tensor product of $V = V_+ \oplus V_-$, where V_+ and V_- are the two-dimensional spinor representations of $SO(4)$, of respective positive and negative chirality. There are two projectors P_{\pm} , which project V onto V_{\pm} . In terms of the isomorphism $SO(4) = SU(2) \times SU(2)$ these have respective spin $(2j_L, 2j_R) = (1, 0)$ and $(2j_L, 2j_R) = (0, 1)$. The symmetrised tensor product space $Sym(V^{\otimes n})$, for every odd integer n , contains a subspace \mathcal{R}_n^+ with $(\frac{n+1}{2})$ factors of positive and $(\frac{n-1}{2})$ factors of negative chirality. This is an irreducible representation of $SO(4)$ labelled by $(2j_L, 2j_R) = (\frac{n+1}{2}, \frac{n-1}{2})$. The projector onto this subspace is in $End(Sym(V^{\otimes n}))$ and will be called $\mathcal{P}_{\mathcal{R}_n^+}$. Equivalently, there is a subspace \mathcal{R}_n^- with spins $(2j_L, 2j_R) = (\frac{n-1}{2}, \frac{n+1}{2})$ and projector $\mathcal{P}_{\mathcal{R}_n^-}$. The full space is then defined to be the direct sum $\mathcal{R}_n = \mathcal{R}_n^+ \oplus \mathcal{R}_n^-$. The projector for this space is $\mathcal{P}_{\mathcal{R}_n} = \mathcal{P}_{\mathcal{R}_n^+} \oplus \mathcal{P}_{\mathcal{R}_n^-}$. The matrices X_i are in $End(\mathcal{R}_n)$

$$X_i = \mathcal{P}_{\mathcal{R}_n} \sum_r \rho_r(\Gamma_i) \mathcal{P}_{\mathcal{R}_n} \quad (2.1)$$

where $i = 1, \dots, 4$, mapping \mathcal{R}_{n^+} to \mathcal{R}_{n^-} and vice versa. We can therefore re-express the above as a sum of matrices in $Hom(\mathcal{R}_n^+, \mathcal{R}_n^-)$ and $Hom(\mathcal{R}_n^-, \mathcal{R}_n^+)$

$$X_i = \mathcal{P}_{\mathcal{R}_n^+} X_i \mathcal{P}_{\mathcal{R}_n^-} + \mathcal{P}_{\mathcal{R}_n^-} X_i \mathcal{P}_{\mathcal{R}_n^+} \quad (2.2)$$

The product $X_i^2 = C$ forms the quadratic Casimir of $SO(4)$. There is a set of generators for the Matrix algebra

$$\begin{aligned}
X_i^+ &= \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r(\Gamma_i P_+) \mathcal{P}_{\mathcal{R}_n^+} \\
X_i^- &= \mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r(\Gamma_i P_-) \mathcal{P}_{\mathcal{R}_n^-} \\
X_{ij}^+ &= \mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r(\Gamma_{ij} P_+) \mathcal{P}_{\mathcal{R}_n^+} \\
Y_{ij}^+ &= \mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r(\Gamma_{ij} P_-) \mathcal{P}_{\mathcal{R}_n^+} \\
X_{ij}^- &= \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r(\Gamma_{ij} P_-) \mathcal{P}_{\mathcal{R}_n^-} \\
Y_{ij}^- &= \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r(\Gamma_{ij} P_+) \mathcal{P}_{\mathcal{R}_n^-}
\end{aligned} \tag{2.3}$$

where

$$\Gamma_{ij} = \frac{1}{2}[\Gamma_i, \Gamma_j] \tag{2.4}$$

The coordinates of the sphere can be written as $X_i = X_i^+ + X_i^-$ and one can also define the following combinations

$$\begin{aligned}
X_{ij} &= X_{ij}^+ + X_{ij}^- \\
Y_{ij} &= Y_{ij}^+ + Y_{ij}^- \\
Y_i &= X_i^+ - X_i^- \\
\tilde{X}_{ij} &= X_{ij}^+ - X_{ij}^- \\
\tilde{Y}_{ij} &= Y_{ij}^+ - Y_{ij}^-
\end{aligned} \tag{2.5}$$

The generators above in fact form an over-complete set. It was observed [32] that X_i, Y_i suffice as a set of generators. In the large- n limit, the full Matrix algebra turns out to contain more degrees of freedom than the algebra of functions on the classical three-sphere. However, one can define an appropriate projection operation, which then gives rise to the proper algebra of functions in the large- n limit. This projected Matrix algebra should be commutative and associative¹ at large- n .

For general fuzzy odd-dimensional spheres, S^{2k-1} , the Matrix coordinates are matrices acting in a reducible representation $\mathcal{R}_n^+ \oplus \mathcal{R}_n^-$ of $SO(2k)$. The irreducible representations \mathcal{R}_\pm have respective weights $\vec{r} = (\frac{n}{2}, \dots, \frac{n}{2}, \frac{\pm 1}{2})$, with \vec{r} a k -dimensional vector. The matrices acting on the full space $\mathcal{R} = \mathcal{R}_n^+ \oplus \mathcal{R}_n^-$ can be decomposed into four blocks $End(\mathcal{R}_n^+)$, $End(\mathcal{R}_n^-)$, $Hom(\mathcal{R}_n^+, \mathcal{R}_n^-)$ and $Hom(\mathcal{R}_n^-, \mathcal{R}_n^+)$.

¹A discussion on the definition of this projection and the large- n behaviour of the associator can be found in Appendix B.

We will use the above to construct a number of useful identities for any isometry group $SO(D)$, for $D = 2k$ even. There exist the following basic relationships [2, 26, 27]

$$\begin{aligned}
(\Gamma_i \otimes \Gamma_i)(P_+ \otimes P_+) &= 0 \\
(\Gamma_i \otimes \Gamma_i)(P_- \otimes P_-) &= 0 \\
(\Gamma_i \otimes \Gamma_i)(P_+ \otimes P_-) &= 2(P_- \otimes P_+) \\
(\Gamma_i \otimes \Gamma_i)(P_- \otimes P_+) &= 2(P_+ \otimes P_-)
\end{aligned} \tag{2.6}$$

For completeness, we give the explicit derivation. It is known from fuzzy even-spheres that $\sum_{\mu=1}^{2k+1} (\Gamma_\mu \otimes \Gamma_\mu)$ acting on the irreducible subspace (which requires subtracting traces for $k > 2$) of $Sym(V \otimes V)$ is equal to 1. For any vector v in this subspace, we have

$$(\Gamma_\mu \otimes \Gamma_\mu)v = v \tag{2.7}$$

Separating the sum over μ as $(\Gamma_i \otimes \Gamma_i) + (\Gamma_{2k+1} \otimes \Gamma_{2k+1})$, multiplying by $(P_+ \otimes P_-)$ from the left, and using the Clifford algebra relations proves the fourth equation above. The other equations are obtained similarly, by multiplying with an appropriate tensor product of projectors. From these we derive

$$\begin{aligned}
X_i^2 &= \frac{(n+1)(n+D-1)}{2} \equiv C \\
X_{ij}X_{ij} &= -\frac{D}{4}(n+1)(n+2D-3) \\
Y_{ij}Y_{ij} &= -\frac{D}{4}(n-1)(n+2D-5) \\
X_{ij}Y_{ij} &= \frac{(4-D)}{4}(n^2-1) \\
X_iX_jX_iX_j &= (2-D)C \\
[X_i, X_j][X_j, X_i] &= 2C(C+D-2)
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
[X_i, X_j] &= (n+D-1)X_{ij} - X_{ijk}X_k \\
[X_j, [X_j, X_i]] &= 2(C+D-2)X_i \\
X_jX_iX_j &= (2-D)X_i \\
X_{ki}X_k &= X_kX_{ik} = -\frac{(n+2D-3)}{2}X_i \\
Y_{ki}X_k &= X_kY_{ik} = \frac{(n-1)}{2}X_i \\
X_jX_kX_iX_jX_k &= \frac{1}{4}(n^4 + 2n^3D + (D^2 + 6 - 2D)n^2 \\
&\quad + (6D - 2D^2)n - 3D^2 + 18D - 23)X_i
\end{aligned} \tag{2.9}$$

In the first equation of the second set we have used

$$X_{ijk} = \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r(\Gamma_{ijk} P_+) \mathcal{P}_{\mathcal{R}_n^+} + \mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r(\Gamma_{ijk} P_-) \mathcal{P}_{\mathcal{R}_n^-}$$

where Γ_{ijk} is the normalised anti-symmetric product. It is useful to observe that

$$\mathcal{P}_{\mathcal{R}_n^-} \sum_{r_1 \neq r_2} \rho_{r_1}(\Gamma_k P_-) \rho_{r_2}(\Gamma_j P_+) \mathcal{P}_{\mathcal{R}_n^+} = X_k^- X_j^+ + X_{jk}^+ - \frac{(n+1)}{2} \delta_{jk} \quad (2.10)$$

It also follows that

$$\begin{aligned} & \mathcal{P}_{\mathcal{R}_n^-} \sum_{r_1 \neq r_2} \rho_{r_1}(\Gamma_k P_-) \rho_{r_2}(\Gamma_j P_+) \mathcal{P}_{\mathcal{R}_n^+} + (+ \leftrightarrow -) \\ &= X_k X_j + X_{jk} - \frac{(n+1)}{2} \delta_{jk} \end{aligned} \quad (2.11)$$

where we have added the term obtained by switching the + and - from the first term. These formulae can be used to calculate $X_i X_j X_k X_i$ ²

$$\begin{aligned} X_i X_j X_k X_i &= X_k X_j \left[\frac{(n-1)(n+D+1)}{2} + 2 \right] - 2X_j X_k \\ &+ \frac{(n+1)(n+D-1)}{2} (X_{jk} + Y_{jk} + \delta_{jk}) \\ &= C(-n, -D) X_k X_j + 2[X_k, X_j] + C(n, D) (X_{jk} + Y_{jk} + \delta_{jk}) \end{aligned} \quad (2.12)$$

In the last equality we have recognised that the coefficient of $(X_{jk} + Y_{jk} + \delta_{jk})$ has turned out to be $C = X_i^2$ (2.8). We also made explicit the dependence of C on n, D writing $C = C(n, D)$, and observed that the other numerical coefficient on the RHS is $C(-n, -D)$.

In the large- n limit there are significant simplifications to the above matrix identities

$$\begin{aligned} X_m X_i X_m &= 0 \\ X_m X_i X_j X_m &= C X_j X_i \\ A_{ij} A_{jk} &= C(X_i X_k + X_k X_i) \\ X_i X_{p_1} X_{p_2} \dots X_{p_{2k+1}} X_i &= 0 \\ X_i X_{p_1} X_{p_2} \dots X_{p_{2k}} X_i &= C X_{p_2} \dots X_{p_{2k}} X_{p_1} \\ [X_i X_j, X_k X_l] &= 0 \\ A_{kl} X_m &= -X_m A_{kl} \\ X_i X_j X_k &= X_k X_j X_i \end{aligned} \quad (2.13)$$

where, to avoid clutter, we have denoted

$$A_{ij} = [X_i, X_j] \quad (2.14)$$

²Products of these will appear in the computation of determinants in the following sections.

and $C \sim \frac{n^2}{2}$. From these it follows that

$$\begin{aligned}
X_m A_{ij} X_m &= -C A_{ij} \\
[A_{ij}, A_{kl}] &= 0 \\
A_{ij} A_{ji} &= 2C^2 \\
A_{ij} A_{jk} A_{kl} A_{li} &= 2C^4
\end{aligned} \tag{2.15}$$

As an example of how these simplifications occur, consider the last equality of (2.13). As explained at the beginning of this section, we can decompose a string of operators such as $X_i X_j X_k = X_i^+ X_j^- X_k^+ + X_i^- X_j^+ X_k^-$. Writing out $X_i^+ X_j^- X_k^+$

$$\begin{aligned}
X_i^+ X_j^- X_k^+ &= \mathcal{P}_{\mathcal{R}_n^-} \sum_{r_1, r_2, r_3} \rho_{r_1}(\Gamma_i P_+) \rho_{r_2}(\Gamma_j P_-) \rho_{r_3}(\Gamma_k P_+) \mathcal{P}_{\mathcal{R}_n^+} \\
&= \mathcal{P}_{\mathcal{R}_n^-} \sum_{r_1 \neq r_2 \neq r_3} \rho_{r_1}(\Gamma_i P_+) \rho_{r_2}(\Gamma_j P_-) \rho_{r_3}(\Gamma_k P_+) \mathcal{P}_{\mathcal{R}_n^+} \\
&= \mathcal{P}_{\mathcal{R}_n^-} \sum_{r_1 \neq r_2 \neq r_3} \rho_{r_3}(\Gamma_k P_+) \rho_{r_2}(\Gamma_j P_-) \rho_{r_1}(\Gamma_i P_+) \mathcal{P}_{\mathcal{R}_n^+} \\
&= X_k^+ X_j^- X_i^+
\end{aligned} \tag{2.16}$$

In the second line we used the fact that the terms with coincident r 's, such as $r_1 = r_2$, are sub-leading in the large- n limit. There are $\mathcal{O}(n^3)$ terms of type $r_1 \neq r_2 \neq r_3$ while there are $\mathcal{O}(n^2)$ terms of type $r_1 = r_2 \neq r_3$ and $\mathcal{O}(n)$ terms of type $r_1 = r_2 = r_3$. In the third line, we used the fact that operators acting on non-coincident tensor factors commute. We find

$$\begin{aligned}
X_i X_j X_k &= X_i^+ X_j^- X_k^+ + X_i^- X_j^+ X_k^- \\
&= X_k^+ X_j^- X_i^+ + X_k^- X_j^+ X_i^- \\
&= X_k X_j X_i
\end{aligned} \tag{2.17}$$

Similar manipulations along with the basic relationships (2.6) lead to the rest of the formulae in (2.13).

3 On the equations for fuzzy- S^3

Specialising to the case of the fuzzy- S^3 we can deduct further Matrix identities. Squaring the generators

$$\begin{aligned}
X_i^2 &= \frac{(n+1)(n+3)}{2} \\
X_{ij}^2 &= -(n+1)(n+5) \\
Y_{ij}^2 &= -(n-1)(n+3) \\
X_{ij} Y_{ij} &= 0
\end{aligned}$$

Note that $X_{ij}Y_{ij} = 0$ in the case of $SO(4)$. This product is not zero for general D . We also have

$$\begin{aligned}
[X_i, X_j] &= \frac{(n+3)}{2}X_{ij} - \frac{(n+1)}{2}Y_{ij} \\
X_j X_{ki} X_j &= \frac{(n+1)(n+5)}{2}Y_{ki} \\
X_j Y_{ki} X_j &= \frac{(n-1)(n+3)}{2}X_{ki} \\
X_{ki} X_k &= -\frac{n+5}{2}X_i \\
Y_{ki} X_k &= \frac{n-1}{2}X_i \\
X_k X_j X_k &= -2X_j \\
X_j X_{ki} X_j X_k &= \frac{(n-1)(n+1)(n+5)}{4}X_i \\
X_j Y_{ki} X_j X_k &= -\frac{(n-1)(n+3)(n+5)}{4}X_i \\
X_j X_k X_i X_j X_k &= \frac{1}{4}(n^2 + 4n - 1)^2 X_i
\end{aligned} \tag{3.1}$$

In the second pair of equations of the above set, note that we might have expected $X_j X_{ki} X_j$ to be a linear combination of X_{ki} and Y_{ki} but only Y_{ki} appears. This follows directly from the transformation properties of these operators under $SO(4)$.

We can compute $X_j X_k X_i X_j X_k$ directly and get an answer which works for any D . Alternatively we can make use of the S^3 identities

$$\begin{aligned}
X_j X_k X_i X_j X_k &= X_j ([X_k, X_i] + X_i X_k) X_j X_k \\
&= \frac{(n+3)}{2} X_j X_{ki} X_j X_k - \frac{(n+1)}{2} X_j Y_{ki} X_j X_k + X_j X_i X_k X_j X_k
\end{aligned} \tag{3.2}$$

Using the formulae (3.1) we see that the contributions from the first two terms are equal. The two computations of this object of course agree.

It is worth noting here that the decomposition of the commutator $[X_i, X_j]$ into a sum over X_{ij} and Y_{ij} should be expected. In [2] a complete $SO(4)$ covariant basis of matrices acting on \mathcal{R}_n was given in terms of operators corresponding to self-dual and anti-self dual Young diagrams. According to that analysis, the most general anti-symmetric tensor with two free indices should be a linear combination of the following structure

$$\sum_r \rho_r(\Gamma_{ij}) \tag{3.3}$$

with any allowed combination of P_{\pm} on \mathcal{R}_n^{\pm} , where the $SO(4)$ indices on the Γ 's have been suppressed for simplicity. Note that the coefficients multiplying the above basis elements

include contractions with the appropriate δ and ϵ -tensors. For $SO(4)$ the antisymmetric two-index tensors are (anti)self-dual and $\epsilon_{ijkl}\Gamma_{kl}P_{\pm} = \pm 2\Gamma_{ij}P_{\pm}$, with $\Gamma_5 P_{\pm} = \mp \Gamma_1 \dots \Gamma_4 P_{\pm}$. Contractions with δ are of course ruled out for symmetry reasons. As a consequence, everything can be expressed in terms of X_{ij} , Y_{ij} . The same procedure can be used to show that every composite object with one free $SO(4)$ index i can be reduced to be proportional to X_i^{\pm} . The allowed linearly independent basis elements are

$$\begin{aligned} & \sum_r \rho_r(\Gamma_i) \\ & \sum_{r \neq s} \rho_r(\Gamma_{ij})\rho_s(\Gamma_k)\delta_{jk} \sim \sum_s \rho_s(\Gamma_i) \\ & \sum_{r \neq s} \rho_s(\Gamma_{jk})\rho_s(\Gamma_l)\epsilon_{ijkl} \sim \sum_s \rho_s(\Gamma_i) \end{aligned} \quad (3.4)$$

It is easy to see explicitly that the last two quantities are proportional to X_i , when evaluated on \mathcal{R}_n^{\pm} . Since we should be able to express any object with one free index in terms of this basis, it will necessarily be proportional to X_i .

4 On the equations for fuzzy- S^5

For the fuzzy- S^5 we only present a few specific identities that will appear in the following sections. The commutator decomposes into

$$[X_i, X_j] = (n+5)X_{ij} - X_{ijk}X_k \quad (4.1)$$

Alternatively we can express this as

$$X_i^{\mp}X_j^{\pm} - X_j^{\mp}X_i^{\pm} = (n+1)X_{ij}^{\pm} + \mathcal{P}_{\mathcal{R}_n} \frac{i}{6} \epsilon_{ijklmn} \left[\sum_{r \neq s} \rho_r(\Gamma_{lmn})\rho_s(\Gamma_k) \right] X_7 \mathcal{P}_{\mathcal{R}_n} \quad (4.2)$$

where $\Gamma_7 P_{\pm} = \pm i\Gamma_1 \dots \Gamma_6 P_{\pm}$ and $X_7 \mathcal{P}_{\mathcal{R}_n} = \mathcal{P}_{\mathcal{R}_n^+} - \mathcal{P}_{\mathcal{R}_n^-}$. There is no expression for $[X_i, X_j]$ as a linear combination of only X_{ij} and Y_{ij} , unlike the case of the fuzzy- S^3 . This is not surprising since the $SO(6)$ covariant basis for two-index antisymmetric tensors will now include terms of the form

$$\begin{aligned} & \sum_r \rho_r(\Gamma_{ij}) \\ & \sum_r \rho_r(\Gamma_{klmn})\epsilon_{ijklmn} \sim \sum_s \rho_s(\Gamma_{ij}) \\ & \sum_{r \neq s} \rho_r(\Gamma_{kl})\rho_s(\Gamma_{mn})\epsilon_{ijklmn} \sim \sum_s \rho_s(\Gamma_{ij}) \\ & \sum_{r \neq s} \rho_r(\Gamma_{klm})\rho_s(\Gamma_n)\epsilon_{ijklmn} \end{aligned} \quad (4.3)$$

Note that the last expression is not proportional to $\sum_r \rho_r(\Gamma\Gamma)$. We can once again show that any composite tensor with one free $SO(6)$ index i should be proportional to X_i^\pm , just as in the $SO(4)$ case. We have

$$\begin{aligned}
& \sum_r \rho_r(\Gamma_i) \\
& \sum_{r \neq s} \rho_r(\Gamma_{ij}) \rho_s(\Gamma_k) \delta_{jk} \sim \sum_s \rho_s(\Gamma_i) \\
& \sum_{r \neq s} \rho_r(\Gamma_{jkl}) \rho_s(\Gamma_{lm}) \epsilon_{ijklmn} \sim \sum_s \rho_s(\Gamma_i) \\
& \sum_{r \neq s \neq t} \rho_r(\Gamma_{jk}) \rho_s(\Gamma_{kl}) \rho_t(\Gamma_m) \epsilon_{ijklmn} \sim \sum_t \rho_t(\Gamma_i)
\end{aligned} \tag{4.4}$$

as can be easily verified for any P_\pm combination on \mathcal{R}_n^\pm .

5 The Fuzzy- S^{2k-1} matrices and DBI with symmetrised trace

In this section we will substitute the ansatz

$$\Phi_i = \hat{R}(\sigma, t) X_i \tag{5.1}$$

into the Matrix DBI action of $D1$ -branes to obtain an effective action for \hat{R} . We will show in Appendix C that solutions to the reduced equation of motion also give solutions to the Matrix DBI equations of motion. To begin with, we will give the most general expressions for time-dependent D -strings. Dropping the dependence on the spatial direction σ will reduce the problem to that of time-dependent $D0$ -branes. Assuming a static ansatz will lead to $D1$ -brane fuzzy funnels.

5.1 Fuzzy- S^3

The low energy effective action for N D -strings with no worldvolume gauge field and in a flat background is given by the non-Abelian Dirac-Born-Infeld action [15]

$$S = -T_1 \int d^2\sigma STr \sqrt{-\det \begin{bmatrix} \eta_{ab} & \lambda \partial_a \Phi_j \\ -\lambda \partial_b \Phi_i & Q_{ij} \end{bmatrix}} \equiv -T_1 \int d^2\sigma STr \sqrt{-\det(M)} \tag{5.2}$$

where

$$Q_{ij} = \delta_{ij} + i\lambda [\Phi_i, \Phi_j] \tag{5.3}$$

and $\lambda = 2\pi\ell_s^2$. The determinant can be explicitly calculated keeping in mind the symmetrisation procedure. The result is

$$\begin{aligned}
-\det(M) &= 1 + \frac{\lambda^2}{2}\Phi_{ij}\Phi_{ji} + \lambda^4 \left(\frac{1}{8}(\Phi_{ij}\Phi_{ji})^2 - \frac{1}{4}\Phi_{ij}\Phi_{jk}\Phi_{kl}\Phi_{li} \right) \\
&\quad + \lambda^2\partial^a\Phi_i\partial_a\Phi_i + \lambda^4 \left(\frac{\partial^a\Phi_k\partial_a\Phi_k\Phi_{ij}\Phi_{ji}}{2} - \partial^a\Phi_i\Phi_{ij}\Phi_{jk}\partial_a\Phi_k \right) \quad (5.4)
\end{aligned}$$

Considering the ansatz (5.1) with $i = 1, \dots, 4$ describes the fuzzy- S^3 . The X_i 's are $N \times N$ matrices of $SO(4)$, as defined in section 2. Their size is given by $N = \frac{1}{2}(n+1)(n+3)$. Substituting into (5.4) we get

$$\begin{aligned}
-\det(M) &= 1 + \frac{\lambda^2}{2}\hat{R}^4 A_{ij}A_{ji} + \frac{\lambda^4}{4}\hat{R}^8 \left(\frac{(A_{ij}A_{ji})^2}{2} - A_{ij}A_{jk}A_{kl}A_{li} \right) \\
&\quad + \lambda^2(\partial^a\hat{R})(\partial_a\hat{R})X_iX_i + \lambda^4\hat{R}^4(\partial^a\hat{R})(\partial_a\hat{R}) \left(\frac{X_kX_kA_{ij}A_{ji}}{2} - X_iA_{ij}A_{jk}X_k \right) \quad (5.5)
\end{aligned}$$

At this point we need to implement the symmetrisation of the trace. In order to simplify the problem, this procedure can be carried out in two steps. We first symmetrise the terms that lie under the square root. We then perform a binomial expansion and symmetrise again. The even-sphere cases that were considered in [25] didn't involve this complication, since the commutators $[X_i, X_j]$, $[A_{ij}, A_{kl}]$ and $[X_i, A_{jk}]$ turned out to be sub-leading in n . Thus, for large- N the square root argument was already symmetric and gave a simple result straight away. Here, however, the $[X_i, X_j]$ and $[X_i, A_{jk}]$ yield a leading- n contribution and the symmetrisation needs to be considered explicitly.

From now on, we will focus completely on the time-dependent problem of N type IIA $D0$ -branes and drop the σ -direction. Then the ansatz (5.1) will be describing a dynamical effect of collapsing/expanding branes. Had we chosen to consider the static version of the above action, we would have a collection of coincident D -strings blowing-up into a funnel of higher dimensional matter with an S^{2k-1} cross-section.

5.1.1 Vanishing symmetrised trace contributions

The terms involving only A 's are already symmetric, since the commutator of commutators, $[A_{ij}, A_{kl}]$, is sub-leading in the large- N limit. From (2.13) and (2.15) it follows immediately that the coefficient of \hat{R}^8 in (5.5) vanishes. The latter can be expressed as

$$Sym(A + B) = 0 \quad (5.6)$$

where $A = \frac{A_{ij}A_{ji}A_{kl}A_{lk}}{2} \equiv A_1A_2A_3A_4$, $B = A_{ij}A_{jk}A_{kl}A_{li} \equiv B_1B_2B_3B_4$. In this case, as we have already mentioned, the A_{ij} 's commute and $Sym(B) = B$, $Sym(A) = A$.

When we expand the square root we obtain terms of the form $Sym(C(A+B)^k)$ where C is a product of operators $C_1C_2\dots$, e.g. $C = (X_iX_i)^n$. It is easy to see that these also vanish. The symmetrised expression will contain terms of the form

$$\sum_{l=0}^k \binom{k}{l} \sum_{\sigma} \sigma(C_1\dots C_n(A_1\dots A_4)^l(B_1\dots B_4)^{k-l}) \quad (5.7)$$

The sum over σ will contain terms where the l copies of $A_1\dots A_4$ and the $k-l$ copies of $B_1\dots B_4$ are permuted amongst each other and also amongst the C factors. Due to the relations in (2.13) and (2.15), we can permute the $4l$ elements from A^l through the C 's and B 's to collect them back in the form of A^l . Likewise for B^{k-l} . Since A_{ij} elements commute with other A_{kl} and anti-commute with X_k , we will pick up, in this re-arrangement, a factor of (-1) raised to the number of times an A_{ij} type factor crosses an X_k factor. This is a factor that depends on the permutation σ and on k but not on l , since the number of A_{ij} 's coming from A and B do not depend on l . We call this factor $N(\sigma, k)$. The above sum takes the form

$$\sum_{\sigma} N(\sigma, k)(C_1\dots C_n) \sum_{l=0}^k \binom{k}{l} (A_1\dots A_4)^l(B_1\dots B_4)^{k-l} \quad (5.8)$$

This contains the expansion of $(A+B)^k$ with no permutations that could mix the A, B factors. Therefore it is zero.

Similarly, we can show that the coefficient of $\hat{R}^2\dot{\hat{R}}^2$ in the determinant (5.5) is zero. This requires a small calculation of summing over 24 permutations³. The relevant formulae are

$$\begin{aligned} X_iX_iA_{jk}A_{kj} &= 2C^3 \\ X_iA_{jk}X_iA_{kj} &= -2C^3 \\ X_iX_jA_{ik}A_{kj} &= C^3 \\ X_iA_{ik}X_jA_{kj} &= -C^3 \end{aligned} \quad (5.9)$$

which follow from (2.13). The outcome is again $Sym(A+B) = 0$, where $A = \frac{X_kX_kA_{ij}A_{ji}}{2} \equiv A_1A_2A_3A_4$ and $B = X_kX_iA_{ij}A_{jk} \equiv B_1B_2B_3B_4$. In this case, we do not have $Sym(A) = A$, $Sym(B) = B$, since the factors within A, B do not commute. We can repeat the above arguments to check $Sym(A+B)^k$. Start with a sum of the form

$$\sum_{l=0}^k \binom{k}{l} \sum_{\sigma} \sigma(A^lB^{k-l}) \quad (5.10)$$

³Or 6 if we fix one element using cyclicity.

Because of the permutation in the sum, there will be terms where $X_i X_i A_{jk} A_{kj}$ has extra X 's interspersed in between. Such a term can be re-collected into the original form at the cost of introducing a sign factor, a factor $E(n_i) = (1 + (-1)^{n_i})/2$, where n_i is the number of X 's separating the two X_i , and also introducing a permutation of the remaining X 's. We will describe this process in more detail, but the important fact is that these factors will be the same when we are re-collecting $X_i X_k A_{ij} A_{jk}$, i.e. the index structure doesn't affect the combinatorics of the re-shuffling. The first step is to move the $A_{jk} A_{kj}$ all the way to the right, thus picking up a sign for the number of X 's one moves through during the process. We then have

$$\begin{aligned}
& X_i (X_{p_1} \dots X_{p_m}) X_i (X_{q_1} \dots X_{q_n}) A_{jk} A_{kj} \\
&= E(m) (X_{p_2} \dots X_{p_m} X_{p_1}) X_i X_i (X_{q_1} \dots X_{q_n}) A_{jk} A_{kj} \\
&= E(m) (X_{p_2} \dots X_{p_m} X_{p_1}) (X_{q_1} \dots X_{q_n}) X_i X_i A_{jk} A_{kj}
\end{aligned} \tag{5.11}$$

In the last line $E(m) = (1 + (-1)^m)/2$ is 1 if m is even and zero otherwise. If instead we are considering $X_i X_k A_{ij} A_{jk}$ we have

$$\begin{aligned}
& X_i (X_{p_1} \dots X_{p_m}) X_k (X_{q_1} \dots X_{q_n}) A_{ij} A_{jk} \\
&= X_i (X_{p_1} \dots X_{p_m}) X_k (A_{ij} A_{jk}) (X_{q_1} \dots X_{q_n}) \\
&= X_i (X_{p_1} \dots X_{p_m}) X_k (C(X_i X_k + X_k X_i)) (X_{q_1} \dots X_{q_n}) \\
&= X_i (X_{p_1} \dots X_{p_m}) C^2 X_i (X_{q_1} \dots X_{q_n}) \\
&= E(m) C^3 (X_{p_2} \dots X_{p_m} X_{p_1}) (X_{q_1} \dots X_{q_n}) \\
&= E(m) (X_{p_2} \dots X_{p_m} X_{p_1}) (X_{q_1} \dots X_{q_n}) X_i X_k A_{ij} A_{jk}
\end{aligned} \tag{5.12}$$

The last lines of (5.11) and (5.12) show that the rules for re-collecting A and B from more complicated expressions, where their components have been separated by a permutation, are the same. Hence,

$$\sum_{l=0}^k \binom{k}{l} \sum_{\sigma} \sigma(A^l B^{k-l}) \tag{5.13}$$

can be re-written, taking advantage of the fact that terms with different values of l only differ in substituting A with B . This does not affect the combinatorics of re-collecting. Finally, we get

$$\sum_{\sigma} F(\sigma, k) \sum_{l=0}^k \binom{k}{l} A^l B^{k-l} = 0 \tag{5.14}$$

The $F(\sigma, k)$ is obtained from collecting all the sign and E -factors that appeared in the discussion of the above re-shuffling.

When other operators, such as some generic C , are involved

$$\sum_{l=0}^k \binom{k}{l} \sum_{\sigma} \sigma(CA^l B^{k-l}) \quad (5.15)$$

the same argument shows that

$$\sum_{\sigma} \tilde{\sigma}(C) \sum_{l=0}^k \binom{k}{l} A^l B^{k-l} = 0 \quad (5.16)$$

Note that σ has been replaced by $\tilde{\sigma}$ because the process of re-collecting the powers of A, B , as in (5.12) and (5.11), involve a re-shuffling of the remaining operators. The proof of $Sym(A+B) = 0$ can also be presented along the lines of the above argument. Then

$$\begin{aligned} X_i A_{jk} X_i A_{kj} &= -X_i X_i A_{jk} A_{kj} \\ X_i A_{ij} X_k A_{jk} &= -X_i X_k A_{ij} A_{jk} \end{aligned} \quad (5.17)$$

which, combined with $X_i X_i A_{jk} A_{jk} = 2X_i X_k A_{ij} A_{jk}$, gives the vanishing result.

5.1.2 Non-vanishing symmetrised trace contributions

After the discussion in the last section, we are only left to consider

$$STr \sqrt{-\det(M)} = \sum_{m=0}^{\infty} STr \left(\frac{\lambda^2}{2} \hat{R}^4 A_{ij} A_{ji} + \lambda^2 (\partial^a \hat{R})(\partial_a \hat{R}) X_i X_i \right)^m \binom{1/2}{m} \quad (5.18)$$

We derive the following formulae for symmetrised traces

$$\begin{aligned} \frac{STr(XX)^m}{N} &= C^m \frac{(m!)^2 2^m}{(2m)!} \\ \frac{STr((XX)^{m_1} (AA)^{m_2})}{N} &= 2^{m_1+m_2} C^{m_1+2m_2} \frac{(m_1+m_2)!}{m_2!} \frac{m_1!(2m_2)!}{(2m_1+2m_2)!} \\ \frac{STr(AA)^m}{N} &= 2^m C^{2m} \end{aligned} \quad (5.19)$$

To calculate the first line, note that we have to sum over all possible permutations of $X_{i_1} X_{i_1} X_{i_2} X_{i_2} \dots X_{i_m} X_{i_m}$. For all terms where the two X_{i_1} 's are separated by an even number of other X 's we can replace the pair by C . Whenever the two are separated by an odd number of X 's they give a sub-leading contribution and therefore can be set to zero in the large- N expansion. In doing the averaging we treat the two i_1 's as distinct objects and sum over $(2m)!$ permutations. For any ordering let us label the positions from 1 to $2m$. To get a non-zero answer, we need one set $i_1 \dots i_m$ to be distributed amongst m even places in $m!$

ways and another set of the same objects to be distributed amongst the odd positions in $m!$ ways. There is a factor 2^m for permutations of the two copies for each index. As a result we obtain the $C^m \frac{(m!)^{2^m}}{(2m)!}$.

Now consider the second line. By cyclicity we can always fix the first element to be an X . There are then $(2m_1 + 2m_2 - 1)!$ permutations of the $(2m_1 - 1)$ X -factors. As we have seen, in the large- N limit $AX = -XA$. Reading towards the right, starting from the first X , suppose we have p_1 A 's followed by an X , then p_2 A 's followed by an X , etc. This is weighted by $(-1)^{p_1+p_3+\dots+p_{2m_1-1}}$. Therefore, we sum over all partitions of $2m_2$, including a multiplicity for different orderings of the integers in the partition, and weighted by the above factor. This can be done by a mathematical package such as Maple in a variety of cases and gives

$$\frac{(m_1 + m_2 - 1)! (2m_2)!(2m_1 - 1)!}{m_2!(m_1 - 1)! (2m_1 + 2m_2 - 1)!} \quad (5.20)$$

The denominator $(2m_1 + 2m_2 - 1)!$ comes from the number of permutations which keep one X fixed. The above can be re-written in a way symmetric under the exchange of m_1 with m_2

$$\frac{(m_1 + m_2)! (2m_1)!(2m_2)!}{m_1!m_2! (2m_1 + 2m_2)!} \quad (5.21)$$

The factor $\frac{(m_1!)^{2^{2m_1}}}{(2m_1)!}$ comes from the sum over permutations of the X 's.

We describe another way to derive this result. This time we will not use cyclicity to fix the first element in the permutations of $(XX)^{m_1}(AA)^{m_2}$ to be X . Let there be p_1 A 's on the left, then one X , and p_2 A 's followed by another X and so on, until the last X is followed by p_{2m_1+1} X 's. We will evaluate this string by moving all the A 's to the left, picking up a sign factor $(-1)^{p_2+p_4+\dots+p_{2m_1}}$ in the process. This leads to a sum over $p_1 \dots p_{2m_1+1}$ which can be re-arranged by defining $P = p_2 + p_4 + \dots + p_{2m_1}$. P ranges from 0 to $2m_2$ and is the total number of A 's in the even slots. For each fixed P there is a combinatoric factor of $\tilde{C}(P, m_1) = \frac{(P+m_1-1)!}{P!m_1!}$ from arranging the P objects into m_1 slots. There is also a factor $\tilde{C}(2m_2 - P, m_1 + 1)$ from arranging the remaining $(2m_2 - P)$ A 's into the $m_1 + 1$ positions. These considerations lead to

$$\begin{aligned} & \sum_{p_{2m_1}=0}^{2m_2} \sum_{p_{2m_1-1}=0}^{2m_2-p_{2m_1}} \dots \sum_{p_1=0}^{2m_2-p_2-\dots-p_{2m_1}} (-1)^{p_2+p_4+\dots+p_{2m_1}} \\ &= \sum_{P=0}^{2m_2} (-1)^P \tilde{C}(P, m_1) \tilde{C}(2m_2 - P, m_1 + 1) \\ &= \frac{(m_1 + m_2)!}{m_1!m_2!} \end{aligned}$$

The factor obtained above is multiplied by $(2m_2)!$ since all permutations among the A 's give

the same answer. Summing over permutations of X 's give the extra factor $2^{m_1}(m_1!)^2$. Finally, there is a normalising denominator of $(2m_1 + 2m_2)!$. Collecting these and the appropriate power of C gives

$$2^{m_2} C^{m_1+2m_2} \frac{(m_1 + m_2)! 2^{m_1}(m_1!)^2 (2m_2)!}{m_1! m_2! (2m_1 + 2m_2)!} \quad (5.22)$$

which agrees with the second line of (5.19).

5.2 Fuzzy- S^5

We will now turn to the case of the fuzzy- S^5 . The starting action will be the same as in (5.2). However, the ansatz incorporates six non-trivial transverse scalars $\Phi_i = \hat{R}(\sigma, t) X_i$, where $i = 1, \dots, 6$. The X_i 's are $N \times N$ matrices of $SO(6)$, as defined in section 2, with their size given by $N = \frac{1}{192}(n+1)(n+3)^2(n+5)$. By truncating the problem to the purely time-dependent configuration, this system represents a dynamical process of N $D0$ -branes expanding into a fuzzy- S^5 and then collapsing towards a point. The static truncation provides an analogue of the static fuzzy- S^3 funnel, with a collection of N D -strings blowing-up into a funnel with a fuzzy- S^5 cross-section.

The calculation of the determinant yields the following result

$$\begin{aligned} -\det(M) = & 1 + \frac{\lambda^2}{2} \Phi_{ij} \Phi_{ji} + \lambda^4 \left(\frac{1}{8} (\Phi_{ij} \Phi_{ji})^2 - \frac{1}{4} \Phi_{ij} \Phi_{jk} \Phi_{kl} \Phi_{li} \right) \\ & + \lambda^6 \left(\frac{(\Phi_{ij} \Phi_{ji})^3}{48} - \frac{\Phi_{mn} \Phi_{nm} \Phi_{ij} \Phi_{jk} \Phi_{kl} \Phi_{li}}{8} + \frac{\Phi_{ij} \Phi_{jk} \Phi_{kl} \Phi_{lm} \Phi_{mn} \Phi_{ni}}{6} \right) \\ & + \lambda^2 \partial^a \Phi_i \partial_a \Phi_i + \lambda^4 \left(\frac{\partial^a \Phi_k \partial_a \Phi_k \Phi_{ij} \Phi_{ji}}{2} - \partial^a \Phi_i \Phi_{ij} \Phi_{jk} \partial_a \Phi_k \right) \\ & - \lambda^6 \left(\frac{\partial^a \Phi_m \partial_a \Phi_m \Phi_{ij} \Phi_{jk} \Phi_{kl} \Phi_{li}}{4} - \frac{\partial^a \Phi_i \partial_a \Phi_i (\Phi_{jk} \Phi_{kj})^2}{8} \right. \\ & \left. + \frac{\partial^a \Phi_i \Phi_{ij} \Phi_{jk} \partial_a \Phi_k \Phi_{ml} \Phi_{lm}}{2} - \partial^a \Phi_i \Phi_{ij} \Phi_{jk} \Phi_{kl} \Phi_{lm} \partial_a \Phi_m \right) \end{aligned} \quad (5.23)$$

Once again we will need to implement the symmetrisation procedure, just as we did for the fuzzy- S^3 . The structure of the terms within the square root for $D = 6$ is almost the same as for $D = 4$. The only difference is that we will now need to include expressions of the type $Sym(A + B + C)$ and $Sym(A + B + C + D)$, coming from the two new $\mathcal{O}(\lambda^6)$ terms. Consider the first of these for example. After expanding the square root, we will need the multinomial series expansion

$$\sum_{n_1, n_2, n_3 \geq 0} \frac{n!}{n_1! n_2! n_3!} A^{n_1} B^{n_2} C^{n_3} \quad \text{with} \quad n_1 + n_2 + n_3 = n \quad (5.24)$$

and analogously for the expression with four terms. The initial multinomial coefficient will separate out and we will then have to deal with the permutations, just as we did for the binomial terms. Note that the symmetrisation discussion in the former section did not make any use of the fact that $D = 4$. All the simplifications that occurred by taking the large- N limit and the combinatoric factors, which came from the re-shuffling of the operators, were derived for representations of $SO(2k)$ with k not specified. It is straightforward to see that the former arguments will carry through to this case. The effect of the permutations for any possible combination of terms will simply introduce a common pre-factor, multiplied by the multinomial coefficient of the original term. One can easily verify that, with the help of the large- N matrix identities from section 2, all the terms multiplying powers of λ higher than 2 in (5.23) will give a zero contribution. Therefore, the substitution of the ansatz will give

$$STr \sqrt{-\det(M)} = \sum_{m=0}^{\infty} STr \left(\frac{\lambda^2}{2} \hat{R}^4 A_{ij} A_{ji} + \lambda^2 (\partial^a \hat{R}) (\partial_a \hat{R}) X_i X_i \right)^m \binom{1/2}{m}$$

This is, somewhat surprisingly, exactly what appeared in (5.18). It is intriguing that there is such a universal description for both the $D = 4$ and $D = 6$ problems.

6 The large- N dynamics of fuzzy odd-spheres.

The discussion of the previous section allows us to write the Lagrangian governing the collapse/expansion of the fuzzy 3-sphere as well as the fuzzy 5-sphere.

$$\begin{aligned} \mathcal{L} &= - \sum_{m=0}^{\infty} \sum_{k=0}^m \binom{1/2}{m} \binom{m}{k} (-1)^{m-k} \left(\frac{\lambda^2 \hat{R}^4}{2} \right)^k (\lambda^2 \dot{\hat{R}}^2)^{m-k} STr [(A_{ij} A_{ji})^k (X_i X_i)^{m-k}] \\ &= - \sum_{m=0}^{\infty} \sum_{k=0}^m \binom{1/2}{m} \binom{m}{k} (-1)^{m-k} s^{2(m-k)} r^{4k} \frac{m!}{(2m)!} \frac{(2k)!}{k!} 2^{m-k} (m-k)! \end{aligned} \quad (6.1)$$

where we have defined

$$\begin{aligned} r^4 &= \lambda^2 \hat{R}^4 C^2 \\ s^2 &= \lambda^2 \dot{\hat{R}}^2 C \end{aligned} \quad (6.2)$$

Alternatively we can express this as two infinite sums with $n + k = m$

$$\mathcal{L} = - \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{1/2}{n+k} \binom{n+k}{k} (-1)^n s^{2n} r^{4k} \frac{(n+k)!}{(2(n+k))!} \frac{(2k)!}{k!} 2^n n! \quad (6.3)$$

From this we can calculate the energy of the configuration to get

$$E = - \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (2n-1) \binom{1/2}{n+k} \binom{n+k}{k} (-1)^n s^{2n} r^{4k} \frac{(n+k)!}{(2(n+k))!} \frac{(2k)!}{k!} 2^n n! \quad (6.4)$$

The first sum can be done explicitly and gives

$$E = \sum_{n=0}^{\infty} \left(\frac{s^2}{2}\right)^n {}_2F_1\left(\frac{1}{2}, n - \frac{1}{2}, n + \frac{1}{2}; -r^4\right) \quad (6.5)$$

There is an identity for the ${}_2F_1$ function

$${}_2F_1(a, b, c; z) = (1-z)^{-a} {}_2F_1\left(a, c-b, c; \frac{z}{z-1}\right), \quad \text{for } z \notin (1, \infty)$$

We can use this to re-express the energy sum as

$$E = \sum_{n=0}^{\infty} \left(\frac{s^2}{2}\right)^n \frac{1}{\sqrt{1+r^4}} {}_2F_1\left(\frac{1}{2}, 1, n + \frac{1}{2}; \frac{r^4}{1+r^4}\right) \quad (6.6)$$

We need one more step to complete the evaluation. The integral representation for the hypergeometric function is

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \rho^{b-1} (1-\rho)^{-b+c-1} (1-\rho z)^{-a} d\rho$$

for $Re(c) > Re(b) > 0$ and $|Arg(1-z)| < \pi$. These conditions are satisfied for $n \neq 0$ so we will split the sum into two parts. The $n = 0$ part simplifies to just $\sqrt{1+r^4}$, while the rest is

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\frac{s^2}{2}\right)^n \frac{1}{\sqrt{1+r^4}} {}_2F_1\left(\frac{1}{2}, 1, n + \frac{1}{2}; \frac{r^4}{1+r^4}\right) \\ &= \sum_{n=1}^{\infty} \left(\frac{s^2}{2}\right)^n \frac{1}{\sqrt{1+r^4}} \left(n - \frac{1}{2}\right) \int_0^1 \frac{(1-\rho)^{n-\frac{3}{2}}}{\sqrt{(1-\rho)\frac{r^4}{1+r^4}}} d\rho \end{aligned} \quad (6.7)$$

By first summing over n and then performing the integration over ρ we get a result, which when added to the $n = 0$ piece gives the full answer for the energy

$$\begin{aligned} E &= \frac{\sqrt{1+r^4} \sqrt{s^2/2+r^4} (r^4(s^2/2-1) - s^2/2) + r^4(s^2/2-1)(s^2/2) \tanh^{-1}\left(\sqrt{\frac{r^4+s^2/2}{r^4+1}}\right)}{(r^4+s^2/2)^{3/2}(s^2/2-1)} \\ &= \sqrt{1+r^4} \left(1 - \frac{s^4}{(s^2-2)(s^2+2r^4)}\right) + \frac{r^4 s^2}{2(r^4+s^2/2)^{3/2}} \tanh^{-1}\left(\sqrt{\frac{r^4+s^2/2}{r^4+1}}\right) \end{aligned} \quad (6.8)$$

We can use the same method to obtain the explicit form of the Lagrangian (6.3), restoring all the appropriate dimensional parameters

$$S = -\frac{N}{g_s \ell_s} \int dt \left(\sqrt{1+r^4} - \frac{s^2}{2\sqrt{r^4+s^2/2}} \tanh^{-1} \frac{\sqrt{r^4+s^2/2}}{\sqrt{1+r^4}} \right) \quad (6.9)$$

We can perform expansions of the above expression for small values of the r and s variables

$$\begin{aligned}
E(r \sim 0, s) &= \frac{1}{(1 - s^2/2)} + r^4 \frac{1}{s^2 - 2} + \frac{\sqrt{2}r^4}{s} \tanh^{-1} \frac{s}{\sqrt{2}} + \dots \\
E(r, s \sim 0) &= \sqrt{1 + r^4} + \frac{s^2}{2r^2} \tanh^{-1} \left(\sqrt{\frac{r^4}{r^4 + 1}} \right) + \dots
\end{aligned} \tag{6.10}$$

Also for large r and small s

$$\begin{aligned}
E(r, s) &= \left(r^2 + \frac{1}{2r^2} - \frac{1}{8r^6} \right) + \left(\frac{1}{8r^6} + \frac{\ln(2r^2)}{2r^2} \right) s^2 \\
&\quad + \left(\frac{3}{16r^6} - \frac{3\ln(2r^2)}{8r^6} + \frac{3}{8r^2} \right) s^4 + \left(\frac{5}{32r^2} - \frac{5}{32r^6} \right) s^6 + \dots
\end{aligned} \tag{6.11}$$

One of the features of the fuzzy even-spheres, namely the fact that they admit $r = t$ type solutions is also true here. This statement translates into having an $s^2/2 = 1$ solution to the $\partial_t E = 0$ equations of motion. It is easy to check that this holds. It is unfortunate that the energy formula includes an inverse hyperbolic tangent of (r, s) . This leads to a transcendental relationship between the two variables and prevents us from conducting an analysis similar to [25], which would give the time of collapse, a possible solution to the radial profile and the configuration's periodicities.

6.1 Constancy of the speed of light

In order to explore the physical properties of our configurations, we will use a new definition of the physical radius for any S^{2k-1} fuzzy sphere

$$\begin{aligned}
R_{phys}^2 &= \lambda^2 \lim_{m \rightarrow \infty} \frac{STr(\Phi_i \Phi_i)^{m+1}}{STr(\Phi_i \Phi_i)^m} \\
&= \lambda^2 \hat{R}^2 \lim_{m \rightarrow \infty} \frac{STr(X_i X_i)^{m+1}}{STr(X_i X_i)^m} \\
&= \frac{\lambda^2 C \hat{R}^2}{2}
\end{aligned} \tag{6.12}$$

where we have evaluated the matrix products in the large- N limit. This will guarantee that the series defining the Lagrangian will converge for $\dot{R}_{phys} = 1$. The definition can be interpreted as an application of the principle of constancy of the speed of light. It was introduced in [24] where it also gave the correct results for the fuzzy even-sphere problems at both large and finite- N . We would like to highlight that the above is not the same thing as requiring local Lorentz invariance. This is because the form of the summed series is not the same as in special relativity. For example the action takes the form $\int \frac{dt}{1 - \dot{R}_{phys}^2}$ at

$R_{phys} = 0$, rather than the standard $\int dt \sqrt{1 - \dot{R}_{phys}^2}$ which appears in the large- N limit of the $D0$ - $D(2k)$ systems. Modifications of the standard relativistic form arise in the study of fuzzy even-spheres at finite- N [24].

In terms of the (r, s) variables, which appear in the expressions for the energy in the previous section, we have

$$\begin{aligned} r &= \frac{\sqrt{2}R_{phys}}{\sqrt{\lambda}} = \frac{R_{phys}}{L_{\text{odd}}} \\ s &= \sqrt{2}\dot{R}_{phys} \end{aligned} \tag{6.13}$$

The physical singularity at $\dot{R}_{phys} = 1$ corresponds to $s^2 = 2$. For later convenience, we will further define the normalised dimensionless variables $\sqrt{2}(\hat{r}, \hat{s}) = (r, s)$. This implies that $\hat{s} = 1$ is the speed of light. The characteristic length scale of the system for fuzzy odd spheres, appearing in (6.13), is $L_{\text{odd}} = \sqrt{\frac{\lambda}{2}}$. This should be contrasted with $L_{\text{even}} = \sqrt{\frac{N\lambda}{2}}$. In the fuzzy- S^2 problem we were able to keep L finite when $\ell_s \rightarrow 0$ while $\sqrt{N} \rightarrow \infty$. In the present case we cannot do so. If we take $\ell_s \rightarrow 0$ we lose the physics of the fuzzy odd-spheres. This is compatible with the idea that they should be related to some tachyonic configuration on an unstable higher dimensional dual brane. These tachyonic modes become infinitely massive as $\ell_s \rightarrow 0$.

6.2 Derivatives, no-bounce results, accelerations

We can use the above results to get a picture of the nature of the collapse/expansion for the $D0$'s blown-up into a fuzzy- S^{2k-1} . We can explore whether there will be a bounce in the trajectory, as was done in [14, 24] for the finite- N dynamics of fuzzy even-spheres, by looking for zeros of constant energy contour plots in (r, s) . This can be simply seen by a zero of the first derivative of the energy with respect to s , for constant r . For our case this is

$$\begin{aligned} \frac{\partial E}{\partial \hat{s}} \Big|_{\hat{r}} &= \frac{2\sqrt{(1+4\hat{r}^4)}\hat{s}^3(\hat{s}^2 + \hat{r}^4(10 - 6\hat{s}^2))}{(1 - \hat{s}^2)^2(4\hat{r}^4 + \hat{s}^2)^2} \\ &\quad + \frac{4\hat{r}^4 \hat{s} (8\hat{r}^4 - \hat{s}^2) \tanh^{-1}\left(\frac{\sqrt{4\hat{r}^4 + \hat{s}^2}}{\sqrt{1+4\hat{r}^4}}\right)}{(4\hat{r}^4 + \hat{s}^2)^{5/2}} \end{aligned} \tag{6.14}$$

We have checked numerically that the above expression is non-zero for $0 < \hat{s} < 1$ and $\hat{r} > 0$. Hence there will be no bounce and the configuration will classically collapse all the way to zero radius. In our treatment so far, we have assumed that higher derivative α' corrections can be neglected. This statement translates into requiring that higher commutators should be small. This condition gives $[\ell_s \Phi_i, [\ell_s \Phi_i,]] \ll 1$ and, with the use of $[X_i, [X_i,]] = n^2$ for

large- n , it implies that $\hat{r} \ll \sqrt{\frac{\pi}{2}} \sim 1$. This corresponds to

$$R_{phys} < \ell_s \quad (6.15)$$

The other relevant quantity in investigating the validity of the action is the proper acceleration, which should be small. This is defined as

$$\alpha = \gamma^3 \frac{d^2 \hat{r}}{d\tau^2} = \gamma^3 \hat{s} \frac{d\hat{s}}{d\hat{r}} = -\gamma^3 \frac{\partial_{\hat{r}} E}{\partial_{\hat{s}} E} \quad (6.16)$$

with $\gamma = (1 - \hat{s}^2)^{-1/2}$. The derivative of the energy with respect to \hat{r} is

$$\begin{aligned} \frac{\partial E}{\partial \hat{r}} \Big|_{\hat{s}} &= \frac{8\hat{r}^3(\hat{s}^4 + 16\hat{r}^8(\hat{s}^2 - 1) + 4\hat{r}^4\hat{s}^2(4\hat{s}^2 - 3))}{\sqrt{1 + 4\hat{r}^4}(\hat{s}^2 - 1)(4\hat{r}^4 + \hat{s}^2)^2} \\ &\quad + \frac{16\hat{r}^3\hat{s}^2(\hat{s}^2 - 2\hat{r}^4) \tanh^{-1}\left(\sqrt{\frac{4\hat{r}^4 + \hat{s}^2}{1 + 4\hat{r}^4}}\right)}{(4\hat{r}^4 + \hat{s}^2)^{5/2}} \end{aligned} \quad (6.17)$$

From the above we get a complicated expression for the proper acceleration in \hat{r} and \hat{s} . Since the energy relation, combined with the boundary condition that the velocity at \hat{r}_0 is zero, gives a transcendental equation for \hat{r} and \hat{s} , we can't predict the behaviour of the velocity for the duration of the collapse. However, for small \hat{r} the proper acceleration simplifies to

$$\alpha = \frac{4\hat{r}^3 (\hat{s} + 2(\hat{s}^2 - 1) \tanh^{-1}(\hat{s}))}{\hat{s}\sqrt{1 - \hat{s}^2}} + \mathcal{O}(\hat{r}^5) \quad (6.18)$$

In the small \hat{r} limit, we can see that the velocity \hat{s} will also be small throughout the trajectory⁴ and we will be within the Matrix Theory limit, which we describe in the next section. For these values the proper acceleration is small enough and the action is valid throughout the collapse. If we were to give the configuration some large initial velocity, we have numerical evidence that there will be a region where the proper acceleration is small enough for the action to be valid but could change sign. We leave the further investigation of the relativistic regime for the future.

7 The Matrix Theory (Yang-Mills) limit

It is interesting to consider the Matrix Theory limit of the action for both the S^3 and the S^5 cases. Consider equation (5.18)

$$S = -\frac{1}{g_s \ell_s} \int dt \text{STr} \sqrt{1 + \frac{\lambda^2}{2} \hat{R}^4 A_{ij} A_{ji} - \lambda^2 \dot{\hat{R}}^2 X_i X_i}$$

⁴If we take $r^4 \ll 1$ in (6.8) we find that $s \sim 0$.

For small \hat{R} and small $\dot{\hat{R}}$, the above yields

$$S = -\frac{N}{g_s \ell_s} \int dt \left(1 + \frac{\lambda^2}{4} \hat{R}^4 A_{ij} A_{ji} - \frac{\lambda^2}{2} \dot{\hat{R}}^2 X_i X_i \right) \quad (7.1)$$

The equations of motion for both S^3 and S^5 will be in dimensionless variables

$$\ddot{r} = -2r^3 \quad (7.2)$$

and will be solved exactly by radial profiles of the form

$$r(\tau) = r_0 C n \left(\sqrt{2} r_0 \tau, \frac{1}{\sqrt{2}} \right) \quad (7.3)$$

where r_0 is a parameter indicating the value of the initial radius at the beginning of the collapse. However, we need to prove that having a solution to the above reduced action is equivalent to solving the Matrix equations of motion. Starting from equation (5.2) and doing a small λ expansion, we arrive at

$$S = -\frac{1}{g_s \ell_s} \int dt Tr \left(1 + \frac{\lambda^2}{4} [\Phi_i, \Phi_j] [\Phi_j, \Phi_i] - \frac{\lambda^2}{2} \partial_t \Phi_i \partial_t \Phi_i \right) \quad (7.4)$$

The equations of motion are

$$\frac{\partial^2 \Phi_i}{\partial t^2} = -[\Phi_j, [\Phi_j, \Phi_i]] \quad (7.5)$$

and upon substituting the ansatz, $\Phi_i = \hat{R}(t) X_i$, and using the matrix identities for any D we get

$$\ddot{\hat{R}} = -\hat{R}^3 (C + D - 2) \quad (7.6)$$

It is easy to verify that, by directly substituting the ansatz into (7.4) and then calculating the equations of motion, we will get the same result. Therefore, any solution of the reduced action will also be a solution of the full Matrix action for any N .

The time of collapse can be calculated by using conservation of energy, in the same way as for S^2 [14, 25]. Expressed in terms of the characteristic length-scale of the theory, the answer will be $T \sim L^2/R_0$. However, as we have already noted, the parameter L in the odd-sphere case does not contain a factor of \sqrt{N} . As a consequence, the collapse of fuzzy- S^3 and S^5 spheres at large- N and for $R_0 \ll L$, will occur much faster than for the fuzzy- S^2 . It will also not be experiencing any finite- N modifications of the kind observed in [24].

8 Towards a dual 3-brane picture for fuzzy- S^3

In the spirit of the dualities that have been established between higher and lower dimensional branes for the case of even spheres [6, 14, 18, 19, 20], it is reasonable to expect

that there should be a dual description for the system of $D0$ -branes blowing-up into a fuzzy- S^3 . The $D0$'s couple to the RR three-form potential, via a term proportional to $\int dt \text{STr}(C_{tij}^{(3)}[\Phi_j, \Phi_i] + C_{kij}^{(3)} \dot{\Phi}_k[\Phi_j, \Phi_i])$ in the Chern-Simons part of the action. They also couple to the five-form potential via $\int dt \text{STr}(C_{tijk}^{(5)}[\Phi_i, \Phi_j][\Phi_k, \Phi_l])$, although by expanding the commutator terms and checking that the leading order term⁵ in $\epsilon_{ijkl} X_i X_j X_k X_l$ evaluates to zero, one sees that this is sub-leading in n . The overall trace will ensure that the net higher brane charge will be zero. Therefore, at large- n we will only see an effective multi-pole coupling to the $D2$ -brane charge.

The simplest candidate for a dual construction is a collection of coincident $D3$'s embedded in Euclidean space as a classical 3-sphere, which in Type IIA are non-BPS. As a consequence, we expect to have an effective field theory on the compact unstable $D3$ ($UD3$) worldvolume, which will incorporate a real tachyon field and some $SU(M)$ non-trivial worldvolume flux, where M is the number of $UD3$'s. The most general description will be in terms of the non-abelian, tachyonic Dp -brane action proposed in [33]. In the case at hand, there are no non-trivial transverse scalars and we are working in a flat background, in a spherical embedding. The DBI part of the $UD3$ -brane action will then reduce to

$$\begin{aligned} S &= -\frac{\mu_3}{g_s} \int d^4\sigma \text{STr} \left(V(T) \sqrt{-\det(P[G]_{ab} + \lambda F_{ab} + \lambda D_a T D_b T)} \right) \\ &= -\frac{\mu_3}{g_s} \int d^4\sigma \text{STr} \left(V(T) \sqrt{-\det(M)} \right) \end{aligned} \quad (8.1)$$

The symmetrisation procedure should be implemented amongst all non-abelian expressions of the form F_{ab} and $D_a T$ and on T of the potential, which is well approximated by $V(T) \sim e^{-T^2}$. The gauge field strength and covariant derivative of tachyons are

$$\begin{aligned} F_{ab} &= \partial_a A_b - \partial_b A_a - i[A_a, A_b] \\ D_a T &= \partial_a T - i[A_a, T] \end{aligned} \quad (8.2)$$

It is straightforward to calculate the determinant. The result, in spherical coordinates $(\alpha_1, \alpha_2, \alpha_3)$, is

$$\begin{aligned} \sqrt{-\det(M)} &= \sqrt{g} \left[(1 - \dot{R}^2 - \lambda D_0 T D_0 T) \left(R^6 - \frac{\lambda^2 R^2}{2} F_{ij} F^{ji} \right) - R^4 \lambda^2 F_{0i} F_0^i \right. \\ &\quad \left. + (1 - \dot{R}^2) \left(\lambda R^4 D_i T D^i T - \lambda^3 \left(\frac{D_i T D^i T F_{jk} F^{kj}}{2} - D^i T F_{ij} F^{jk} D_k T \right) \right) \right. \\ &\quad \left. - 2\lambda^3 R^2 D_0 T D_j T F_{0i} F^{ij} + \frac{\lambda^4}{4} \left(\frac{F_{0i} F_0^i F_{jk} F^{kj}}{2} - F_{0i} F^{ij} F_{jk} F_0^k \right) \right]^{1/2} \end{aligned} \quad (8.3)$$

⁵That is the one with no coincidences.

where $g = \sin^4 \alpha_1 \sin^2 \alpha_2$ is the determinant of the unit three-sphere metric.

The Chern-Simons part of the non-BPS brane action which is of interest to us, has been discussed in [34, 35] and is given in terms of the curvature of the super-connection

$$\begin{aligned}
S_{CS} &\sim \mu_3 \int C \wedge STr e^{\frac{F}{2\pi}} \\
&\sim \mu_3 \int C \wedge STr e^{\frac{1}{2\pi}(F-T^2+DT)} \\
&\sim \frac{\mu_3}{8\pi^2} \int C^{(1)} \wedge STr ((F \wedge DT) W(T))
\end{aligned} \tag{8.4}$$

where only odd-forms are kept in the exponential expansion, the $DT \wedge DT \wedge DT$ term vanishes because of the overall symmetrisation and $W(T) = e^{-\frac{T^2}{2\pi}}$. There will also be a term proportional to $\int C^{(3)} \wedge STr(DT)$. If we want to match the two descriptions, we should require that the overall $D0$ -charge is conserved. We therefore impose that

$$\frac{1}{8\pi^2} STr ((F \wedge DT)W(T)) = \frac{N}{\text{Vol}_{S^3}} \Omega_3 \tag{8.5}$$

where Ω_3 is the $SO(4)$ invariant volume form on the 3-sphere with angles $(\alpha_1, \alpha_2, \alpha_3)$ and N is an integer. By restricting to the worldvolume of a single brane, i.e. having a $U(1)$ gauge symmetry for the gauge and tachyon fields, this condition translates in components into the following expression

$$F_{[ij}\partial_{k]}T W(T) = 4N\epsilon_{ijk} \tag{8.6}$$

where $\epsilon_{\alpha_1\alpha_2\alpha_3} = \sqrt{g}$. Contraction with ϵ^{ijk} results in

$$\epsilon^{ijk}\partial_i T F_{kj} = -\frac{24N}{W(T)} \tag{8.7}$$

while contraction with $\partial^i T F^{kj}$ and then use of the above equation gives

$$\frac{\partial_i T \partial^i T F_{jk} F^{kj}}{2} - \partial^i T F_{ij} F^{jk} \partial_k T = -\frac{144N^2}{W(T)^2} \tag{8.8}$$

Using this last expression we can simplify the $UD3$ action just by $SO(4)$ symmetry of the charge conservation condition. We obtain

$$\begin{aligned}
S &= -\frac{\mu_3}{g_s} \int d\sigma^4 \sqrt{g} V(T) \left[(1 - \dot{R}^2 - \lambda \dot{T}^2) \left(R^6 - \frac{\lambda^2 R^2}{2} F_{ij} F^{ji} \right) \right. \\
&\quad - 2\lambda^3 R^2 \dot{T} \partial_j T F_{0i} F^{ij} - R^4 \lambda^2 F_{0i} F_0^i + (1 - \dot{R}^2) \left(\lambda R^4 \partial_i T \partial^i T + \frac{144\lambda^3 N^2}{W(T)^2} \right) \\
&\quad \left. + \frac{\lambda^4}{4} \left(\frac{F_{0i} F_0^i F_{jk} F^{kj}}{2} - F_{0i} F^{ij} F_{jk} F_0^k \right) \right]^{1/2}
\end{aligned} \tag{8.9}$$

The equations of motion for this configuration are involved. Nevertheless, note that all the individual terms in the above action should be scalars of $SO(4)$. This means that we should have both $\partial_i T \partial^i T$ and T^2 independent of the angles and a constant T cannot satisfy the spherical symmetry condition (8.5). Hence, there is no worldvolume theory for a compact unstable brane system in three dimensions with $SO(4)$ symmetry and we should be looking at least at higher numbers of coincident non-BPS branes in order to find a dual description.

We may expect the matching between the $D0$ and the higher brane pictures to be more tricky than in the even-sphere case. In the even-sphere, the upper bound on the validity of the lower brane description increases with N , and allows an overlap with the higher brane description [18]. As can be seen from (6.15), the upper bound does not increase with N for the fuzzy odd-sphere case. The separation of the degrees of freedom of the fuzzy- S^3 Matrix algebra [27] into geometrical and internal also suggests that a simple relation to a non-abelian theory is not possible. However, this does not preclude a relation via a non-trivial renormalisation group flow, analogous to that proposed in [29] in the context of the application of the fuzzy three-sphere to the $M2 \perp M5$ intersection.

9 Summary and Outlook

In this paper we provided a set of formulae for general fuzzy odd-spheres and studied them as solutions of Matrix DBI $D0$ -brane actions and their Matrix Theory (Yang-Mills) limit. After implementing the symmetrised trace, which in the fuzzy-odd sphere case requires a non-trivial sum over orderings even at large- N , we found the same equations of motion for the fuzzy- S^3 and S^5 . We proved that solutions to the reduced DBI action also solve the full Matrix equations of motion. For the Matrix Theory limit, we gave exact expressions for these solutions in terms of Jacobi elliptic functions. The study of the physical properties of these systems showed that the classical collapse will proceed all the way to the origin.

Given that we have now established the fuzzy- S^3 (and S^5) as solutions to stringy Matrix Models, we can study the action for fluctuations. Using the remarks in [27] on the geometrical structure of the Matrix algebras, we expect that it should be possible to write the action in terms of fields on a higher dimensional geometry: $S^2 \times S^2$ for the S^3 case and $SO(6)/(U(2) \times U(1))$ in the case of S^5 . It will be intriguing to clarify the geometry and symmetries of this action.

A very interesting open problem is to identify a macroscopic large- N dual description of the fuzzy- S^3 and S^5 systems that we have described. The main difficulty lies in constructing $SO(2k)$ -invariant finite energy time-dependent solutions describing tachyons coupled to gauge fields on odd-dimensional spheres. The spherical symmetry restrictions that we dis-

cussed here should facilitate the task of investigating a non-abelian solution. A dual description, and agreement with the microscopic picture at the level of the action and/or equations of motion, would not only provide us with a new check of the current effective worldvolume actions for non-BPS branes, but also give the possibility of constructing cosmological toy models of bouncing universes with three spatial dimensions. The gravitational back-reaction of such time-dependent spherical brane bound states would also be of interest as a possible avenue towards physically interesting time-dependent versions of the AdS/CFT duality. A non-trivial extension to the problem would be the addition of angular momentum. This could provide a stabilisation mechanism for fuzzy odd-spheres in the absence of the right RR fluxes, which provide a simple stabilisation mechanism for fuzzy even-spheres. The study of finite- N effects and the embedding of these systems in more general backgrounds [36, 37] provide other possible avenues for future research. Finally it would be interesting to get a better understanding of the relation between the fuzzy-odd sphere constructions in this paper and those of [38, 39] involving fibrations over projective spaces.

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A Derivation of $[X_i, X_j]$ for fuzzy- S^3

Here we include part of the calculation that led to the commutator decomposition for the fuzzy- S^3 , using two possible ways of evaluating $[X_i, X_j]^2$. One is a direct method and starts by writing out the commutators

$$\begin{aligned} [X_i, X_j][X_j, X_i] &= 2X_i X_j X_i X_j - 2X_i X_j X_j X_i \\ &= 2X_i^+ X_j^- X_i^+ X_j^- + 2X_i^- X_j^+ X_i^- X_j^+ - 2C^2 \end{aligned} \quad (\text{A.1})$$

A straightforward calculation for any D gives

$$X_i^\mp X_j^\pm X_i^\mp X_j^\pm = -\frac{(D-2)(n+1)(n+D-1)}{2} \quad (\text{A.2})$$

The two combine into

$$[X_i, X_j][X_j, X_i] = -\frac{(n+1)(N+D-1)}{2} [2D-4 + (n+1)(n+D-1)] \quad (\text{A.3})$$

The other is based on the decomposition of any antisymmetric tensor of $SO(4)$ with two free indices in terms of the appropriate basis elements

$$[X_i, X_j] = \alpha X_{ij} + \beta Y_{ij} \quad (\text{A.4})$$

where α and β are some yet undetermined coefficients. Then

$$[X_i, X_j][X_j, X_i] = \alpha^2 X_{ij}X_{ji} + \beta^2 Y_{ij}Y_{ji} \quad (\text{A.5})$$

since for $D = 4$ we have seen that $X_{ij}Y_{ij} = 0$. Deriving the identities for $X_{ij}X_{ji}$ and $Y_{ij}Y_{ji}$ is a simple task. Using these results, it is easy to evaluate the above expression and compare against what we get from the straightforward calculation. The outcome is

$$\alpha = \frac{(n+3)}{2} \quad \text{and} \quad \beta = -\frac{(n+1)}{2} \quad (\text{A.6})$$

This result can also be checked against calculations of $X_{ij}[X_i, X_j]$ and $Y_{ij}[X_i, X_j]$.

B The large- n limit of the fuzzy- S^3 projected algebra

In [27] it was proposed that there is a simple prescription for obtaining the space of functions on S^{2k-1} by performing a projection of the Matrix algebra onto symmetric and traceless representations of $SO(2k)$. The remaining representations should also be invariant under a \mathbb{Z}_2 action, which interchanges the positive and negative chiralities. This process projects out the Y_i 's, X_{ij}^\pm 's and Y_{ij}^\pm 's, while leaving the X_i 's and their symmetric products. The projected Matrix algebra is non-associative but commutative at finite- n , as is the case for even dimensional fuzzy-spheres. Here we will show that in the large- n limit non-associativity persists, unlike the case of S^{2k} for which it vanishes.

We will begin by calculating the simplest associator, $X_i * X_j * X_k$, where $*$ stands for the standard non-associative product. This is

$$(X_i * X_j) * X_k - X_i * (X_j * X_k) \quad (\text{B.1})$$

with the implementation of the projection performed every time a product is calculated. The first matrix product gives

$$X_i \cdot X_j = \mathcal{P}_{\mathcal{R}_n^+} \left[\sum_{r=s} \rho_r(\Gamma_i \Gamma_j P_+) + \sum_{r \neq s} \rho_r(\Gamma_i P_-) \rho_s(\Gamma_j P_+) \right] \mathcal{P}_{\mathcal{R}_n^+} \quad + \quad (+ \leftrightarrow -) \quad (\text{B.2})$$

and after the projection

$$\begin{aligned} X_i * X_j &= \mathcal{P}_{\mathcal{R}_n^+} \sum_{r=s} \rho_r(\delta_{ij} P_+) \mathcal{P}_{\mathcal{R}_n^+} + \frac{1}{2} \mathcal{P}_{\mathcal{R}_n^+} \sum_{r \neq s} \rho_r(\Gamma_i P_-) \rho_s(\Gamma_j P_+) \mathcal{P}_{\mathcal{R}_n^+} \\ &\quad + \frac{1}{2} \mathcal{P}_{\mathcal{R}_n^+} \sum_{r \neq s} \rho_r(\Gamma_j P_-) \rho_s(\Gamma_i P_+) \mathcal{P}_{\mathcal{R}_n^+} \quad + \quad (+ \leftrightarrow -) \end{aligned} \quad (\text{B.3})$$

We then proceed to take the ordinary product with X_k .

Consider the 1-coincidence terms, where the Γ_k acts on the same tensor factor as Γ_i or Γ_j . The symmetric part of the product of Γ 's is clearly kept in the projected product defined group theoretically above. The antisymmetric part has a traceless part which transforms according to the Young diagram of row lengths (2, 1). The trace part transforms in (1, 0) and has to be kept. The decomposition into traceless and trace for a 3-index tensor antisymmetric in two indices is

$$\begin{aligned} A_{i[jk]} &= \left(A_{i[jk]} - \frac{1}{3}\delta_{ij}A_{l[lk]} - \frac{1}{3}\delta_{ik}A_{l[jl]} \right) + \left(\frac{1}{3}\delta_{ij}A_{l[lk]} + \frac{1}{3}\delta_{ik}A_{l[jl]} \right) \\ &= A'_{i[jk]} + \left(\frac{1}{3}\delta_{ij}A_{l[lk]} + \frac{1}{3}\delta_{ik}A_{l[jl]} \right) \end{aligned} \quad (\text{B.4})$$

with the normalisation fixed by taking extra contractions of the above. Using this, we obtain from the 1-coincidence terms⁶

$$\frac{(2n+1)}{3}\delta_{ij}X_k + \frac{(n-1)}{6}(\delta_{jk}X_i + \delta_{ik}X_j) \quad (\text{B.5})$$

The terms with no coincidences, where the $\Gamma_i, \Gamma_j, \Gamma_k$ all act in different tensor factors, can be decomposed as

$$A_{(ij);k} = \frac{1}{3}(A_{(ij);k} + A_{(ik);j} + A_{(jk);i}) + \frac{1}{3}(2A_{(ij);k} - A_{(ik);j} - A_{(jk);i}) \quad (\text{B.6})$$

It can be verified, by applying the Young Symmetriser, that the first term corresponds to a symmetric Young diagram, while the second to a mixed symmetry one. The traceless part of the tensor $A_{(ij);k}$ in four dimensions can be evaluated to be

$$A_{(ij);k} - \frac{2\delta_{ij}}{9}A_{(ll);k} - \frac{\delta_{ik}}{9}A_{(lj);l} - \frac{\delta_{jk}}{9}A_{(il);l} \quad (\text{B.7})$$

Keeping the mixed symmetry trace part from the non-coincident terms, obtained when (B.3) multiplies X_k from the left, gives additional contributions⁷. Adding these to (B.5) we get

$$(X_i * X_j) * X_k = \frac{(n^2 + 10n + 7)}{18}\delta_{ij}X_k - \frac{(n^2 - 8n + 7)}{36}(\delta_{jk}X_i + \delta_{ik}X_j) + S_{ijk} \quad (\text{B.8})$$

where S_{ijk} is the explicitly symmetrised product with no coincidences

$$S_{ijk} = \mathcal{P}_{\mathcal{R}_n^-} \sum_{r \neq s \neq t} \rho_r(\Gamma_i P_+) \rho_s(\Gamma_j P_-) \rho_t(\Gamma_k P_+) \mathcal{P}_{\mathcal{R}_n^+} \quad + \quad (+ \leftrightarrow -) \quad (\text{B.9})$$

⁶We thank Neil Copland (see also [40]) for a discussion which helped fix an error in the corresponding formula appearing in the first version of this paper.

⁷These contributions were missed in the first version of this paper.

Similarly we find that

$$X_i * (X_j * X_k) = \frac{(n^2 + 10n + 7)}{18} \delta_{jk} X_i - \frac{(n^2 - 8n + 7)}{36} (\delta_{ij} X_k + \delta_{ik} X_j) + S_{ijk} \quad (\text{B.10})$$

The difference is

$$(X_i * X_j) * X_k - X_i * (X_j * X_k) = \frac{n^2 + 4n + 7}{12} (\delta_{ij} X_k - \delta_{jk} X_i) \quad (\text{B.11})$$

The X 's should be renormalised in order to correspond to the classical sphere co-ordinates in the large- n limit. Since we have $X_i^2 \sim n^2/2$ for large- n and any D from (2.8), we define the normalised matrices as

$$Z_i = \frac{\sqrt{2}}{n} X_i \quad (\text{B.12})$$

which gives $Z_i^2 = 1$. In this normalisation the associator becomes

$$(Z_i * Z_j) * Z_k - Z_i * (Z_j * Z_k) = \frac{1}{6} \left(1 + \frac{4}{n} + \frac{7}{n^2} \right) (\delta_{ij} Z_k - \delta_{jk} Z_i) \quad (\text{B.13})$$

and is obviously non-vanishing in the large- n limit.

More generally one can consider multiplying $(S_{i_1 \dots i_p} * S_{j_1 \dots j_q}) * S_{k_1 \dots k_r}$ and $S_{i_1 \dots i_p} * (S_{j_1 \dots j_q} * S_{k_1 \dots k_r})$. It is clear from the above discussion that the only terms of order one (after the normalisation) that can appear in the associator are the ones coming from terms with no coincidences. These products will, in the large- n limit, include terms which match the classical product on the space of functions on the sphere. But, as illustrated here, they will also include additional terms responsible for non-associativity even in the large- n limit. The $*$ -product (discussed above) on the projected space of Matrices transforming as symmetric representations is the most obvious one available: the matrix product followed by projection. We have shown that it does not become associative in the large- n limit. There is, however, another way to modify the matrix product which does become associative in the large- n limit. This involves keeping only the completely symmetric (in i, j, k etc.) part from the completely non-coincident terms and is a mild modification of the $*$ -product discussed above. One can imagine yet other modifications. An alternative method for defining a non-associative product for the fuzzy odd-sphere, which approaches the associative one in the large- n limit, would be to start with the even sphere case for general even dimensions D (where the prescription of matrix product followed by multiplication does give vanishing non-associativity at large n) and then continue in D . Whether the latter product is related to the alternative product contemplated above is another question we will leave unanswered. The eventual usefulness of such products would be proven if we could use them in an appropriate way to illuminate dualities between the zero-brane and higher brane constructions. In the simplest the $D0$ - $D2$ system, the non-commutative matrix product plays a role in the comparison of $D0$ and $D2$ actions (see for example [23]).

C Solutions to reduced action and solutions to DBI

In the Matrix Theory limit, in section 7, we saw that solving the Matrix equations of motion with the fuzzy odd-sphere ansatz is equivalent to solving the equations for the reduced action. We show here that the same is true for the full DBI equations of motion. We will discuss the fuzzy- S^3 for concreteness, but the same proof applies to the case of the fuzzy- S^5 . Consider the action arising from the expansion of (5.4). The full Lagrangian will comprise of an infinite sum of ‘words’ W , consisting of products of $[\Phi, \Phi]$ ’s and $\partial_t \Phi$ ’s, $S = -T_1 \int dt \text{STr}(\sum W)$. Therefore, if

$$\frac{\partial W(\Phi_i = \hat{R}X_i)}{\partial \hat{R}} = X_l \frac{\partial W}{\partial \Phi_l} \Big|_{\Phi_l = \hat{R}X_l} \quad (\text{C.1})$$

and

$$\partial_t \left(\frac{\partial W(\Phi_i = \hat{R}X_i)}{\partial \dot{\hat{R}}} \right) = X_l \partial_t \left(\frac{\partial W}{\partial (\partial_t \Phi_l)} \right) \Big|_{\Phi_l = \hat{R}X_l} \quad (\text{C.2})$$

then

$$\left[\frac{\partial}{\partial \hat{R}} - \partial_t \left(\frac{\partial}{\partial \dot{\hat{R}}} \right) \right] \sum W(\Phi_i = \hat{R}X_i) = X_l \left[\left(\frac{\partial}{\partial \Phi_l} \right) \Big|_{\Phi_l = \hat{R}X_l} - \partial_t \left(\frac{\partial}{\partial (\partial_t \Phi_l)} \right) \Big|_{\Phi_l = \hat{R}X_l} \right] \sum W \quad (\text{C.3})$$

and a solution to the equations for the reduced action would also be a solution to the equations coming from the full Matrix action.

We will proceed by proving (C.1) and (C.2). Take a word consisting only of $[\Phi, \Phi]$ ’s and expand all the commutators. The result will be $W = (\Phi_{A_1} \dots \Phi_{A_m})$, where m is even and set equal to the contracted pairs of indices. Then define

$$\frac{\partial W}{\partial \Phi_l} \Big|_{\Phi = \hat{R}X} = \sum_{k=1}^m \hat{D}_{(1)}^l \left(\hat{C}_k W \right) \Big|_{\Phi = \hat{R}X} \quad (\text{C.4})$$

where the operator \hat{C}_k uses the cyclic property of the trace to rotate the k -th element, that is to be differentiated, to the first slot. $\hat{D}_{(1)}^l$ takes the derivative of the first term in the word with respect to Φ_l , then sets the index of its contracted partner equal to l . We have shown that any composite operator of $SO(4)$ and $SO(6)$ with one free index i will be proportional to X_i . Therefore we will have that

$$\frac{\partial W}{\partial \Phi_l} \Big|_{\Phi = \hat{R}X} = \hat{R}^{m-1} \sum_{k=1}^m \alpha_k(W) X_l \quad (\text{C.5})$$

where $\alpha_k(W)$ is some constant factor, which in general depends on the word W and k . One can see that $\alpha_k(W) = \alpha(W)$, is actually independent of k . If we multiply the contribution of the k -th term by X_l from the left

$$X_l \left[\hat{D}_{(1)}^l \left(\hat{C}_k W \right) \right] \Big|_{\Phi = \hat{R}X} = \hat{R}^{m-1} \alpha_k(W) X_l X_l \quad (\text{C.6})$$

On the LHS we will now have again a Casimir of $SO(D)$

$$\hat{R}^{m-1} X_l \left[\hat{D}'_{(1)} \left(\hat{C}_k W(\Phi \rightarrow X) \right) \right] = C \hat{R}^{m-1} \alpha_k(W) \quad (\text{C.7})$$

As such it will obey the cyclicity property can be rotated back to form the original word with $\Phi \rightarrow X$

$$\begin{aligned} \alpha_k(W) &= \frac{1}{C} X_l \left[D'_{(1)} \left(\hat{C}_k W(\Phi \rightarrow X) \right) \right] \\ &= \frac{1}{C} W(\Phi \rightarrow X) \end{aligned} \quad (\text{C.8})$$

$$= \frac{1}{C} (\alpha(W) C) \quad (\text{C.9})$$

As a consequence, every contribution in the sum (C.4) is going to be the same and

$$X_l \frac{\partial W}{\partial \Phi_l} \Big|_{\Phi=\hat{R}X} = m \hat{R}^{m-1} \alpha(W) C \quad (\text{C.10})$$

It is much easier to evaluate the LHS of (C.1) to get

$$\frac{\partial W(\Phi_i = \hat{R}X_i)}{\partial \hat{R}} = m C \alpha(W) \hat{R}^{m-1} \quad (\text{C.11})$$

Exactly the same procedure can be applied to words containing time derivatives, where m is now the number of Φ 's coming just from commutators and therefore (C.1) holds. Similar steps can be carried out for the words with m $\partial_t \Phi$ terms and n Φ terms coming from the expansion of commutators. We will have

$$\partial_t \left(\frac{\partial W}{\partial (\partial_t \Phi_l)} \right) \Big|_{\Phi_l = \hat{R}X_l} = \partial_t \left(\sum_{k=1}^m \hat{S}'_{(1)} \left(\hat{C}_k W \right) \Big|_{\Phi=\hat{R}X} \right) \quad (\text{C.12})$$

where $\hat{S}'_{(1)}$ takes the derivative of the first term in the word with respect to $\partial_t \Phi_l$, then sets the index of its contracted partner equal to l . This will become

$$\partial_t \left(\sum_{k=1}^m \hat{S}'_{(1)} \left(\hat{C}_k W \right) \Big|_{\Phi=\hat{R}X} \right) = \left(m \hat{R}^n \dot{\hat{R}}^{m-1} \alpha(W) X_l \right) \cdot \quad (\text{C.13})$$

and, when multiplied by X_l , will result into what one would get from evaluation of the LHS of (C.2), namely

$$X_l \partial_t \left(\frac{\partial W}{\partial (\partial_t \Phi_l)} \right) \Big|_{\Phi_l = \hat{R}X_l} = C \alpha(W) m \left(\dot{\hat{R}}^{m-1} \hat{R}^n \right) \cdot \quad (\text{C.14})$$

This completes the proof that any solutions to the reduced DBI equations of motion for S^3 and S^5 will also be solutions to the full Matrix equations of motion, for any N .

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