# Vector Effective Field Theories from Soft Limits

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We present a bottom-up construction of vector effective field theories using the infrared structure of scattering amplitudes. Our results employ two distinct probes of soft kinematics: multiple soft limits and single soft limits after dimensional reduction applicable in four and general dimensions, respectively. Both approaches uniquely specify the Born-Infeld (BI) model as the only theory of vectors completely fixed by certain infrared conditions which generalize the Adler zero for pions. These soft properties imply new recursion relations for on-shell scattering amplitudes in BI theory and suggest the existence of a wider class of vector effective field theories.

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*Introduction.*—On-shell scattering amplitudes are fundamental physical observables in quantum field theory. In recent years, these objects have spurred many exciting developments: unexpected simplifications, hidden symmetries, and new mathematical structures invisible in the standard approach of Feynman diagrams. While most progress has centered on theories of maximal supersymmetry (SUSY) at high loop orders, surprises have arisen even in the case of tree-level effective field theories (EFTs).

As is well known, on-shell tree amplitudes in gauge theory and gravity are completely fixed by gauge invariance and proper factorization on poles,

$$\lim_{P^2\to 0} A = \sum \frac{A_L A_R}{P^2}, \tag{1}$$

where the sum runs over all internal states. Alas, this approach does not uniquely specify EFTs, which exhibit higher-dimensional contact terms in the Lagrangian that are invisible on factorization kinematics. This obstacle was overcome in Ref. [1], which showed how tree amplitudes in a broad class of scalar EFTs are completely fixed once factorization is supplemented by the additional physical criterion that the amplitude vanishes as

$$\lim_{p\to 0} A = \mathcal{O}(p^\sigma) \tag{2}$$

in the soft limit [2]. By building an ansatz that factorizes properly and *by assumption* conforms to Eq. (2), one discovers a remarkable class of *exceptional* theories: the nonlinear sigma model (NLSM), Dirac-Born-Infeld (DBI) theory, and the special Galileon. These theories exhibit maximally strong soft behavior, exposing them as the EFT analogs of gauge theory and gravity [4,5].

These scalar EFTs appear in a variety of disparate contexts, e.g., the Cachazo-He-Yuan formalism [6–8], certain world sheet models [9], the Bern-Carrasco-Johansson double-copy construction [10,11], and the web of unifying relations for massless theories [12,13].

Notably, within this same orbit of topics appears ubiquitously a certain *vector* EFT: the Born-Infeld (BI) model. This theory is a nonlinear extension of Maxwell theory, which in $D$ dimensions has the Lagrangian

$$\mathcal{L}_{\rm BI} = 1 - \sqrt{(-1)^{D-1}\det(\eta_{\mu\nu} + F_{\mu\nu})} \tag{3}$$

working in natural units with mostly minus metric convention and without using explicit normalization. The purpose of this Letter is to show that BI theory is also uniquely specified by certain infrared properties and that this methodology generalizes to a broader class of vector EFTs. Our results are built around two distinct soft limits: a multiple chiral soft limit applicable to $D = 4$ dimensions and dimensional reduction to scalars applicable in any $D$.

To begin, consider a massless vector degree of freedom, which is described by a general Lagrangian which is a function of the gauge invariant Abelian field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, so

$$\mathcal{L} = -\frac{1}{4}\langle FF\rangle + g_4^{(1)}\langle FFFF\rangle + g_4^{(2)}\langle FF\rangle^2 + \cdots, \tag{4}$$

where $\langle FF\rangle = F_{\mu\nu}F^{\mu\nu}$, $\langle FFFF\rangle = F_{\mu\nu}F^{\rho\nu}F_{\rho\sigma}F^{\mu\sigma}$, etc., and all odd traces are identically zero. Here, gauge invariance is not an additional assumption but simply encodes the existence of massless vectors. While this narrows the form of the Lagrangian ansatz, we are still left with an infinite number of free coefficients, $g_n^{(m)}$. From the above Lagrangian, we then compute a tree-level $n$-point amplitude $A_n$ and fix the numerical coefficients $g_n^{(m)}$ by demanding certain soft properties of $A_n$.

*Uniqueness from multichiral soft limits.*—First, let us focus on the case of $D = 4$ where all possible interactions can be expressed in terms of two basic building blocks,

$$f = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad \text{and} \quad g = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{5}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. This follows from Cayley-Hamilton relation for four-by-four matrices. Assuming parity, we construct a general effective Lagrangian for a massless vector,

$$\mathcal{L} = f + a_1 f^2 + a_2 g^2 + b_1 f^3 + b_2 f g^2 + \cdots, \tag{6}$$

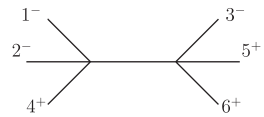
where $g$ enters only in even powers. This Lagrangian covers a huge range of EFTs, including, e.g., the well-known Euler-Heisenberg theory describing quantum electrodynamics at low energy, as well as our target BI theory, whose action is $\mathcal{L}_{\rm BI} = 1 - \sqrt{1 - 2f - g^2}$ in this basis.

Next, let us consider the amplitudes corresponding to this general Lagrangian. Starting at 4 points, there are three possible helicity configurations modulo helicity conjugation: $----$, $---+$, and $--++$. In our conventions, all particles are outgoing, so $+$ $(-)$ denotes positive- (negative-) helicity particles, respectively. The 4-point amplitudes for the all-but-one same helicity configuration $(---+)$ are zero. However, we are still left with two independent on-shell amplitudes, which in spinor helicity variables are

$$\begin{aligned} A_{----} &= \frac{1}{2}(a_1 - a_2)(\langle 12\rangle^2\langle 34\rangle^2 + \text{perm}), \\ A_{--++} &= \frac{1}{2}(a_1 + a_2)\langle 12\rangle^2[34]^2. \end{aligned} \tag{7}$$

For the moment, we assume that the only nonvanishing amplitudes are *helicity conserving*; i.e., they have equal numbers of positive- and negative-helicity particles, so $a_1 = a_2$. This criterion does not fix the theory completely, but it will simplify our analysis. As we will see later on, helicity conservation can actually be dropped as an assumption in favor of a special infrared property of amplitudes.

Next, we compute the 6-point amplitude $A_{---+++}$, recycling the 4-point on-shell amplitude as a 4-point Feynman vertex while adding contributions from a general 6-point contact term. However, the latter does not exist: because of the considerations of little group weight and mass dimension, the only allowed contact term is $\langle 12\rangle\langle 23\rangle\langle 31\rangle[45][56][64]$, which vanishes identically upon symmetrization on (123) and (456). Hence, the 6-point amplitude is given uniquely by factorization diagrams involving the 4-point vertex,

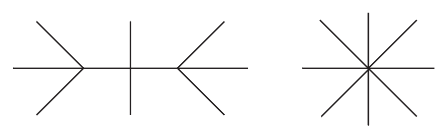



$$A_{---+++} = \frac{\langle 12\rangle^2[56]^2\langle 3|1+2|4]^2}{s_{124}} + \text{perm.}, \tag{8}$$

dropping the overall normalization $a_1^2$ with permutations in the diagram tacitly assumed. This amplitude scales as $\mathcal{O}(1)$ in the single soft limit, so it is not interesting in this respect. However, we discover highly nontrivial infrared behavior if we take a *multichiral soft limit* defined by sending $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3 \to \epsilon$ or $\lambda_4, \lambda_5, \lambda_6 \to \epsilon$,

$$\lim_{\tilde{\lambda}_- \to \epsilon \ \text{ or } \ \lambda_+ \to \epsilon} A_{---+++} = \mathcal{O}(\epsilon), \tag{9}$$

where the $+$ or $-$ subscripts on the spinors are shorthand for all legs of a given helicity. Alternatively, we could instead send $\lambda_1, \lambda_2, \lambda_3 \to \epsilon$ or $\tilde{\lambda}_4, \tilde{\lambda}_5, \tilde{\lambda}_6 \to \epsilon$, which gives analogous behavior $\mathcal{O}(\epsilon^7)$ with the extra $\epsilon^6$ suppression trivially entering through $\lambda$'s or $\tilde{\lambda}$'s in the polarization vectors. Interestingly, similar behavior can be achieved when only two of three spinors of given type are sent to zero. In this case, individual terms scale as $\mathcal{O}(1)$, so a cancellation must occur between diagrams. The crucial test of this approach is the 8-point amplitude given by the set of Feynman graphs:



While the contact 6-point helicity conserving amplitude does not exist, there is an 8-point contact term with an unfixed coefficient,

$$\begin{aligned} A_{----++++} = &\frac{\langle 3|1+2|5]^2\langle 4|7+8|6]^2\langle 12\rangle^2[78]^2}{s_{125}s_{478}} \\ &+ \frac{1}{2}\frac{[5|(1+2)(3+4)|6]^2\langle 12\rangle^2\langle 34\rangle^2[78]^2}{s_{125}s_{346}} \\ &+ (- \leftrightarrow +) + k\langle 12\rangle^2\langle 34\rangle^2[56]^2[78]^2 + \text{perm.} \end{aligned} \tag{10}$$

As it turns out, this expression does not have any special behavior for the single- or double-chiral soft limit, but if we send $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4 \to \epsilon$ or $\lambda_5, \lambda_6, \lambda_7, \lambda_8 \to \epsilon$, we again obtain vanishing behavior [14] only if the coefficient of the contact term is set to $k = -1$. Analogously, for 10-point amplitude there are no contact terms allowed, so it is automatically $\mathcal{O}(\epsilon)$ in the chiral soft limit when four or five appropriate spinors are set to zero.

For 12-point amplitude, there is a single contact term $(\langle 12\rangle^2\langle 34\rangle^2\langle 56\rangle^2[78]^2[910]^2[1112]^2 + \text{perm})$, whose coefficient is fixed by demanding the $\mathcal{O}(\epsilon)$ behavior in the multichiral soft limit.

The amplitudes constructed in the way described above up to the 12-point one coincide with the amplitudes obtained by direct calculation of Feynman graphs within the BI theory. This leads us to the following two conjectures: (i) all tree-level amplitudes beyond 4 points in BI theory have the enhanced $\mathcal{O}(\epsilon)$ behavior under the prescribed multichiral soft limit, and (ii) demanding this multichiral soft limit and the standard unitarity, the amplitudes are fixed uniquely modulo over all normalization. Combining the two conjectures implies that BI theory is a unique theory with such a multichiral soft behavior. Let us prove the two conjectures in turn.

Interestingly, the initial assumption of helicity conservation for $n > 4$ can be dropped if we apply a generalization of the above multichiral soft behavior. For an amplitude with $l-$helicity and $m+$helicity with $l \le m$, it is sufficient to require that

$$A[1^-2^-\ldots l^-(l+1)^+\ldots(l+m)^+] = \mathcal{O}(\epsilon) \qquad (11)$$

for the antiholomorphic soft limit, $\tilde{\lambda}_i \to \epsilon$ for $i = 1\ldots l$. The case with $l = 0$ must be trivially zero, and the combinations of helicities with $l > m$ are obtained simply by the helicity conjugation.

*Proof of the multichiral soft limits of BI.*—The validity of the multichiral soft limit for the BI theory can be derived using SUSY. As is well known, BI theory corresponds to the pure bosonic sector of the EFT describing spontaneous symmetry breaking of $\mathcal{N} = 2$ to $\mathcal{N} = 1$ SUSY [15–17]. The full set of physical degrees of freedom of this SUSY extension are the BI photon $A_\mu$ and Goldstino $\psi$; however, the tree-level amplitudes involving only photons coincide in both BI and SUSY BI. The unbroken $\mathcal{N} = 1$ SUSY implies a Ward identity relating the pure photon amplitudes to the ones with two Goldstinos,

$$\tilde{\lambda}_1^{\dot{\alpha}} A[1^-2^-\ldots n/2^-(n/2+1)^+\ldots n^+] = -\sum_{i=n/2+1}^{n} \tilde{\lambda}_i^{\dot{\alpha}} A[\psi_1^- 2^-\ldots n/2^-(n/2+1)^+\ldots\psi_i^+\ldots n^+]. \qquad (12)$$

The broken SUSY generators are realized through nonlinear transformations which act on the Goldstino as $\psi_A \to \psi_A + \eta_A + \cdots$, so amplitudes exhibit a vanishing $\mathcal{O}(p)$ soft limit for the Goldstino when $p \to 0$. In the soft limit defined by $\lambda_i \to 0$ for all $i > n/2$, the right-hand side of Eq. (12) is zero because the amplitude exhibits the Goldstino soft zero. Naively, there is the subtlety that the multichiral soft limit could induce a soft pole to cancel this Adler zero. However, such a pole does not appear because the factorization channel either vanishes by helicity conservation or is nonsingular due to the specific form of the 4-point vertices. Thus, we conclude that the left-hand side of Eq. (12) vanishes, which is our conjectured soft theorem. Contracting both sides of Eq. (12) with $\tilde{\lambda}_j^{\dot{\alpha}}$ for any $j$ of a positive-helicity photon, we find that the BI amplitude also vanishes in the multichiral soft limit $\lambda_i \to 0$ for the $(n/2-1)$ positive-helicity photons, as we have also discussed in previous sections. Finally, we remark that it would be of interest to understand the multichiral soft behavior directly from the BI theory without resorting to the hidden supersymmetry.

*Proof of uniqueness and recursion relations.*—Let us now show that the multichiral soft limit and unitarity fix the theory uniquely modulo the normalization of 4-point amplitude. We prove this by constructing on-shell recursion relations which determine the amplitudes uniquely. First, we deform the spinors in $n$-point kinematics [8,18] into

$$\tilde{\lambda}_i \to \tilde{\lambda}_i(1-z) \quad \text{and} \quad \lambda_k \to \lambda_k + z\eta_k, \qquad (13)$$

for $i = 1, \ldots, n/2$ and $k = n-1, n$. The shift of $\tilde{\lambda}_i$ probes the multichiral soft limit while shifting $\lambda_k$ to ensure momentum conservation, provided

$$\eta_{n-1} = -\frac{1}{[n-1n]}\sum_{i=1}^{n/2}[in]\lambda_i, \qquad \eta_n = \frac{1}{[n-1n]}\sum_{i=1}^{n/2}[in-1]\lambda_i.$$

Because of the multichiral soft limit behavior, the deformed amplitude scales as $A(z) = \mathcal{O}(1-z)$ for $z = 1$ and as $A(z) = \mathcal{O}(1)$ for $z = \infty$, which can also be checked by the inspection of individual Feynman diagrams. Next, the Cauchy formula implies the contour integral

$$\int \frac{dzA(z)}{z(1-z)} = 0,$$

where the pole at $z = 1$ is canceled by the coincident zero in $A(z)$. Summing over all other poles of $A(z)$, which are factorization channels, yields the recursion formula,

$$A_n = \sum_I \frac{A_L(z_{I_-})A_R(z_{I_-})}{P_I^2(1-z_{I_-}/z_{I_+})(1-z_{I_-})} + (z_{I_-} \leftrightarrow z_{I_+}), \qquad (14)$$

where the sum is over factorization channels $I$, and $z_{I_\pm}$ are roots of equation $\hat{P}_I^2(z) = 0$. The recursion, therefore, fixes the theory up to the seed, i.e., the 4-point amplitude, which is fixed already in Eq. (7) modulo normalization. Combining with the previous discussion, we find such a unique solution is BI theory.

*Uniqueness from dimensional reduction.*—BI theory can also be fixed uniquely by a combination of soft limits and dimensional reduction. Specifically, we constrain a general amplitude for a massless vector demanding that its dimensionally reduced amplitudes describe DBI scalars, whose dynamics are, in turn, completely specified by enhanced soft behavior. To begin, we take an $n$-point amplitude in general $D$, partitioning all $n$ legs into $p$ sets, $\{I_1|\ldots|I_p\}$. Here, each set is interpreted as an extra dimension in which a subset of vectors is polarized and are, thus, scalars under dimensional reduction. Since these extra-dimensional polarizations are orthogonal to the momenta, we set $(e_i \cdot p_j) = 0$ and

$$(e_i \cdot e_j) = 1, i, j \in I_a, \quad (e_i \cdot e_j) = 0, \quad \text{otherwise}. \tag{15}$$

The dimensionally reduced amplitude describes $p$ flavors of scalar particles whose momenta are restricted to $(D - p)$ dimensions. Because the interactions are written as field strengths, the resulting scalars are derivatively coupled and trivially exhibit $\mathcal{O}(p)$ soft behavior. However, by demanding an enhanced $\mathcal{O}(p^2)$ soft limit, these amplitudes are constrained to be scalar DBI amplitudes, and we deduce that the original theory is BI theory.

We label every possible dimensional reduction by the notation $\{a_1|a_2|\ldots\}$, where $a_i$ denotes the number of photons reduced corresponding to the same set $I_i$. For example, we can reduce an $n$-point photon amplitude in $D$ dimensions by $\{n\}$, yielding a single scalar theory in $D - 1$ dimensions, or by $\{a|b\}$ yielding $a$ scalars of one flavor and $b$ scalars of another flavor in $D - 2$ dimensions, etc.

For concreteness, consider the example of 4 points, where the general Lagrangian is

$$\mathcal{L}_4 = c_1 \langle FFFF \rangle + c_2 \langle FF \rangle^2 \tag{16}$$

stipulating that dimension-reduced amplitudes $\{4\}$ and $\{2|2\}$ have enhanced soft behavior, $A_4 = \mathcal{O}(p^2)$. This fixes the relative coefficients, $c_2 = c_1/4$. For the 6-point case, the relevant Lagrangian is

$$\mathcal{L}_6 = d_1 \langle FFFFFF \rangle + d_2 \langle FFFF \rangle \langle FF \rangle + d_3 \langle FF \rangle^3. \tag{17}$$

If we demand that the amplitude $A_6 = \mathcal{O}(p^2)$ for all possible dimensional reductions by one $\{6\}$, two $\{4|2\}$, or three extra dimensions $\{2|2|2\}$, the solution is unique. For 8 points, it is sufficient to fix the amplitude uniquely by demanding the $\mathcal{O}(p^2)$ soft limits for dimensional reductions by one, two, three, and four extra dimensions. We expect this holds for general point: enhanced soft limits in all possible dimensional reductions fix BI theory uniquely.

A more restricted operation also uniquely fixes the BI Lagrangian in Eq. (4): reduce only a single pair of photons $e_i$, $e_j$ to scalars rather than all photons. Here, we set $(e_{i,j} \cdot p) = (e_{i,j} \cdot e_k) = 0$ and $(e_i \cdot e_j) = 1$, where $k$ denotes all other labels, yielding an amplitude of two scalars and $n - 2$ photons. This is the limit $\{2\}$. Demanding the soft limit behavior $\mathcal{O}(p^2)$ for either of the scalars also fixes the BI action, which we have checked explicitly up to 8 points. This directly implies that the original vector amplitude can be expressed purely in terms of amplitudes involving two scalars, so

$$A_n = \sum_{i<j} (e_i \cdot e_j) A(i,j)|_{(e\cdot e)^m \to [(e\cdot e)^m/m]}, \tag{18}$$

where each term of the form $(e \cdot e)^m$ is rescaled by a symmetry factor $1/m$ to eliminate overcounting, and $A(i, j)$ is the amplitude with photons $i$ and $j$ dimensionally reduced to scalars [19]. Since $A(i, j)$ has two DBI scalars, it is uniquely fixed by its enhanced soft behavior [1], so Eq. (18) defines all tree amplitudes in BI theory.

That BI theory is uniquely fixed from its dimensionally reduced DBI amplitudes is obvious in hindsight. Any tree amplitude of vectors is a polynomial in $(e_i \cdot e_j)$, with coefficients that depend on $(e_i \cdot p_j)$ and $(p_i \cdot p_j)$. The reduced amplitudes are obtained from the original expression by applying derivatives with respect to $\partial/\partial(e_i \cdot e_j)$ [12]. Moreover, derivatives of the amplitude fix the original amplitude up to a "constant," which depends only on $(e_i \cdot p_j)$ and $(p_i \cdot p_j)$. However, such a term cannot be gauge invariant by itself, so it is related to terms involving $(e_i \cdot e_j)$, which have already been fixed.

*Vector Galileon-like theories.*—It is straightforward to generalize the construction of previous sections to a vector theory with even more derivatives. While BI theory has one derivative per field, the next interesting case corresponds to a Lagrangian of the schematic form,

$$\mathcal{L} = F^2 + \partial^2 F^4 + \partial^4 F^6 + \partial^6 F^8 + \cdots \tag{19}$$

In detail, there are three terms of the form $\partial^2 F^4$ and 64 terms of the form $\partial^4 F^6$ in general $D$. The obvious extension of our previous results is to constrain Eq. (19) with a stronger $\mathcal{O}(\epsilon^3)$, in analogy with the soft behavior of the special Galileon.

Notably, there is a no-go theorem forbidding vector particles with a Galileon symmetry [21] (see, also, Ref. [22]). However, this obstruction is evaded [23] if one considers multiple flavors of scalar Galileon or $p$-form Galileons for even $p$. More important, in our case here, we do not seek a theory with a bona fide Galileon symmetry but rather a theory of "Galileon"-like interacting vectors with the same power counting as the scalar Galileon and similar exceptional infrared properties. Here, we offer partial evidence of the existence of a Galileon-like vector theory.

In $D = 4$, we can construct an analog of Eq. (6) and impose more severe vanishing under chiral multisoft limits. We again demand that only helicity conserving amplitudes are nonzero, and for 4 points we get a single term

$$A_{--++} = \langle 12 \rangle^2 [34]^2 s_{12}, \tag{20}$$

while for 6 points, we obtain five independent contact terms, in contrast to zero for BI power counting. Constructing the 6-point amplitude from factorization terms and contact term, we find five free coefficients. By imposing the chiral multisoft limit $\lambda_4, \lambda_5, \lambda_6 \to 0$, we find $\mathcal{O}(\epsilon^4)$ for the factorization term and one of the contact terms, with all other contact terms behaving worse. However, there is no choice of contact term coefficients that can accommodate an even stronger multichiral soft limit, so we cannot uniquely fix the amplitude from this procedure unless more constraints are imposed.

Finally, let us discuss how to constrain Eq. (19) from single soft limits via dimensional reduction. Consider the case where all the vectors are dimensionally reduced to scalars which have the unique $\mathcal{O}(\epsilon^3)$ soft behavior of the special Galileon. Unlike DBI, the special Galileon does not have a multifield analog corresponding to multiple extra dimensions, so we are forced to dimensionally reduce all vectors to a single extra dimension, $\{n\}$. This corresponds to the setting $(e_i \cdot p_j) = 0$ and $(e_i \cdot e_j) = 1$ for all indices $i$, $j$. Perhaps unsurprisingly, this procedure yields *multiple* vector theories satisfying these constraints. For example, this is achieved by the Lagrangian,

$$\mathcal{L} = \sum_n c_n F^2 \epsilon^{\alpha_1 \dots \alpha_D} \epsilon^{\beta_1 \dots \beta_D} \partial_{\alpha_1} F_{\beta_1 \alpha_D} \partial_{\beta_2} F_{\alpha_2 \beta_D} \times \prod_{i=2}^{n} \partial_{\alpha_{2i-1}} F_{\beta_{2i-1}\mu_i} \partial_{\alpha_{2i}} F^{\mu_i}_{\beta_{2i}} \prod_{j=2n+1}^{D-1} \eta_{\alpha_j \beta_j}. \tag{21}$$

Under dimensional reduction, this trivially reduces to a special Galileon in $d = D - 1$ dimensions,

$$\mathcal{L} = \sum_n c_n (\partial\phi)^2 \epsilon^{\mu_1 \dots \mu_d} \epsilon^{\nu_1 \dots \nu_d} \prod_{k=1}^{2n} (\partial_{\mu_k} \partial_{\nu_k} \phi) \prod_{j=2n+1}^{d} \eta_{\mu_j \nu_j},$$

where $c_n$ are certain combinatorial factors given in Ref. [24]. Nevertheless, applying the simple replacement $\partial_{\alpha_k} F_{\beta_k \mu} \to \partial_{\beta_k} F_{\alpha_k \mu}$ to Eq. (21) yields a different physically distinct vector Lagrangian whose dimensionally reduced scalar amplitudes are the same. Hence, the constraint of the soft limit and dimensional reduction into a single direction do not uniquely fix the amplitude.

That said, imposing constraints from $\mathcal{O}(\epsilon^2)$ soft zeros for combinations of dimensional reduction actually fixes the 4-point amplitude uniquely from $\{2\}$. However, the 6-point amplitude still has free parameters after applying constraints from $\{4|2\}$, $\{2|2|2\}$, $\{4\}$, $\{2|2\}$, and $\{2\}$. So, while this gives extra conditions, there are still not enough to fix the action completely. The question of whether there is a unique theory of this type given additional constraints is left for future work.

*Conclusions.*—In summary, we have applied modern amplitude methods to EFTs of massless vector particles. We have unambiguously identified BI theory as a theory uniquely fixed by certain infrared conditions. These conditions include the multiple chiral soft limit in four dimensions or dimensional reductions of vector amplitudes to scalar amplitudes in lower dimensions. We plan to apply the same method to tree-level amplitudes of multiple vectors or particles of spin-2.

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