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A random walk approach to linear statistics in random tournament ensembles

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Abstract

We investigate the linear statistics of random matrices with purely imaginary Bernoulli entries of the form $H_{pq} = \overline{H}_{qp} = \pm i$, that are either independently distributed or exhibit global correlations imposed by the condition $\sum_q H_{pq} = 0$. These are related to ensembles of so-called random tournaments and random regular tournaments respectively. Specifically, we construct a random walk within the space of matrices and show that the induced motion of the first k traces in a Chebyshev basis converges to a suitable Ornstein-Uhlenbeck process. Coupling this with Stein's method allows us to compute the rate of convergence to a Gaussian distribution in the limit of large matrix dimension.

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1 Introduction

The idea of using a stochastic dynamical evolution to unearth the spectral properties of random matrices was first proposed by Dyson [15]. His insight was that, by initiating a suitable Brownian motion within the space of certain invariant matrix ensembles, one could induce a corresponding motion in the eigenvalues, which is independent of the eigenvectors. Thus, solving the associated Fokker-Planck equation for the stationary solution would recover the joint probability density function for the eigenvalues. Dyson Brownian motion (DBM), as it is now known, has since become a powerful tool in random matrix theory (RMT) (see for instance [1, 18, 16]). In [26] the present authors advocated an approach in which the idea of using stochastic dynamics to obtain spectral statistics was extended to Bernoulli matrix ensembles. In particular, we argued heuristically, that by initiating a suitable discrete random walk in the space of matrices, the induced motion of the eigenvalues would tend, in some fashion, to DBM in the limit of large matrix size. Then, as a consequence, the spectral properties of Bernoulli matrices would converge to those of the Gaussian orthogonal ensemble (GOE). In the present article we apply this approach to the linear-statistics of matrices associated to random tournaments

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and random regular tournaments. Tournament graphs are widely studied objects in combinatorics, with results and open questions regarding, their enumeration, score sequences, cycle properties and Perron-Frobenius eigenvalues for instance [47, 35, 36, 19, 20, 31, 30]. However, beyond [45], there appears to be little analysis from an RMT perspective.

For a random (self-adjoint) matrix M of size $N \times N$, the linear-statistic, for some function h, refers to the following random variable,

$$\Phi_h(M) := \mathrm{Tr}\left(h(M)\right) = \sum_{\mu=1}^{N} h\left(\lambda_\mu(M)\right), \tag{1.1}$$

where $\lambda_\mu(M)$ are the eigenvalues of M. For random matrices M with appropriately scaled and suitably chosen iid elements, Wigner showed that as $N \to \infty$ the expectation for polynomial functions h converges to the semicircle distribution [50, 51], i.e.

$$\frac{1}{N}\mathbb{E}[\mathrm{Tr}\left(h(M)\right)] \to \frac{2}{\pi}\int_{-1}^{1} h(\lambda)\sqrt{1-\lambda^2}d\lambda \qquad N \to \infty. \tag{1.2}$$

In addition, Wigner showed the variance satisfies $\frac{1}{N^2}\mathrm{Var}[h(M)] = \mathcal{O}(N^{-2})$. This result is therefore, in some respects, analogous to the law of large numbers in standard probability theory.

One is therefore led to the question regarding fluctuations about this mean, i.e. what is the distribution of $\Phi_h(M) - \mathbb{E}[\Phi_h(M)]$ for some particular random matrix ensemble? This was first addressed by Jonsson [25] in the case of Wishart matrices, showing this random variable is Gaussian in the large N limit. Later, this was also shown to be the case for Wigner matrices [29, 43] and also for β-ensembles with appropriate potentials [23] for various forms of the test function h. Notice there is no obvious analogy with the classic CLT, since the eigenvalues in (1.1) are highly correlated, meaning the usual $1/\sqrt{N}$ normalisation is not required. Proving this behaviour has become a key part of the universality hypothesis within RMT, since it addresses the global spectral behaviour, and has thus garnered much attention since the first results were established. For instance, many authors have attempted to classify for which test functions h the Gaussian behaviour is retained [3, 11, 33, 42, 46]. Others have investigated large deviation aspects [22], rates of convergence [9] or different kinds of random matrix ensembles such as band matrices [2] or those with non-trivial correlations [9, 41].

To show the convergence of (1.1) for all polynomial test functions of degree k one may, instead, show the joint convergence for a polynomial basis. A particularly convenient choice are the Chebyshev polynomials of the first kind

$$T_n(x) := \cos(n\arccos(x)) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} d_r^{(n)} x^{n-2r}, \qquad d_r^{(n)} = (-1)^r \frac{n}{2}\frac{(n-r-1)!}{r!(n-2r)!}2^{n-2r}. \tag{1.3}$$

If one takes the traces

$$\mathrm{Tr}(T_n(M)) := \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} d_r^{(n)}\,\mathrm{Tr}(M^{n-2r}), \tag{1.4}$$

then it was first observed by Johansson [23] that if M is chosen from one of the standard Gaussian ensembles, then in the limit of large matrix size the random variables $(\mathrm{Tr}(T_1(M)), \ldots, \mathrm{Tr}(T_k(M)))$ converge to independent Gaussian random variables.

A Brownian motion approach has already been used to show convergence to independent Gaussian random variables of $\mathrm{Tr}(T_n(M))$ in the Gaussian unitary ensemble [8] and more general β-ensembles [32], as well as traces of unitary matrices $\mathrm{Tr}(U^n)$ in the classical compact groups [13] and the circular β-ensembles [49]. In particular, the works [32, 13, 49] utilised a multivariate form of Stein's method, developed by Chatterjee & Meckes and Reinert & Röllin [10, 39, 37], to obtain rates of convergence: Something which, beyond [9], is often neglected in the analysis of linear statistics. However the scenarios [8, 13, 49] have involved invariant matrix ensembles, which have permitted the use of exact expressions for the eigenvalue motion, which are not available in this context. We therefore turn to an alternative combinatorial approach, similar to that applied in [14, 24] for random regular graphs and [27] for the unimodular ensemble. In particular, we express the variables $\mathrm{Tr}(T_n(M))$ in terms of sums over non-backtracking cycles and analyse how these behave under the random walk. The difficulties arise in providing accurate bounds for the remainder terms, which involve the expectations of certain products of matrix elements with respect to the appropriate ensembles.

The article is outlined as follows: In Section 2.1 we discuss the ensembles of random tournaments and random regular tournaments, which lead to Definition 2.1 and Definition 2.2 for the matrix ensembles we call the imaginary tournament ensemble (ITE) and regular imaginary tournament ensemble (RITE) respectively. We then present our main results, given in Theorem 2.3 and Theorem 2.4, which provide rates of convergence to independent Gaussian random variables of the first k traces of Chebyshev polynomials for matrices in the ITE and RITE respectively. In Section 2.2 we attempt to give an intuitive explanation of the random walk approach, including Theorem 2.6 (due to [10, 39, 37]) regarding the multidimensional exchangeable pairs approach to Stein's method, and briefly outline the the methods used to evaluate the appropriate remainder terms.

In Section 3 we introduce some graph theoretical tools required for subsequent analysis. Sections 4 and 5 are dedicated to showing how to construct suitable random walks for the ITE and RITE respectively. Specifically, we prove Propositions 4.1 and 5.2 (respectively) which show the remainders contained in Theorem 2.6 are small enough to allow for the results of Theorem 2.3 and Theorem 2.4. In particular, although interesting in its own right, the ITE will serve as an illustrative example that the approach works in simple settings and will help introduce ideas needed for the more complicated RITE.

Finally, in Section 6 we offer some concluding thoughts and remarks about possible extensions and in the appendix we collect some necessary theorems, proofs and identities. In particular, Appendix B contains a proof for the growth rate of expectation of products of matrix elements in the RITE. This is adapted from the work of McKay [35] on the number of regular tournaments and is critical in estimating the remainders in Proposition 5.2.

2 Main results

2.1 Definitions and results

A tournament graph on N vertices is a complete graph in which every edge has a specific orientation (see e.g. Figure 1). Player p is said to win against player q (equivalently player q loses against player p) if there is a directed edge from p to q. This is represented by an a adjacency matrix A admitting the property that $A_{pq} = 1 - A_{qp} = 1$ (resp. 0) if player p wins (resp. loses). Since a player can't play themselves the diagonal $A_{pp} = 0$. We denote the set of tournaments on N vertices as $\mathfrak{T}_N$, with cardinality $|\mathfrak{T}_N| = 2^{N(N-1)/2}$ - the number of possible choices of direction for each edge.

If all players win the same number of games, or equivalently the number of incoming edges into a vertex is equal to the number of outgoing edges for every vertex, then the tournament graph is said to be *regular* (see e.g. Figure 2). This is characterised by the condition $\sum_q A_{pq} = (N-1)/2$ for all $p = 1, \ldots, N$, which enforces N to be odd. We denote the set of *regular tournaments* on N vertices by $\mathfrak{R}_N$. An exact formula for $|\mathfrak{R}_N|$ is not available however McKay showed [35] (improving on an earlier estimate of Spencer [47]) that for large N

$$|\mathfrak{R}_N| = \frac{2^{(N^2-1)/2}e^{-1/2}}{\pi^{(N-1)/2}N^{N/2-1}}\left(1+\mathcal{O}(N^{-1/2+\epsilon})\right). \tag{2.1}$$

In particular, one observes that $|\mathfrak{R}_N|/|\mathfrak{T}_N| \to 0$ as $N \to \infty$ and therefore one cannot immediately infer properties of regular tournaments from tournaments by ergodicity arguments. Hence, the restriction of the row-sums must be dealt with another manner.

Due to the non-symmetric nature of the adjacency matrices the eigenvalues are, in general, complex. However applying the simple transformation $H = i(2A - (\mathbf{E}_N - \mathbf{I}_N))$ (where $i = \sqrt{-1}$ and $\mathbf{E}_N$ is the $N \times N$ matrix in which every element is 1) brings the matrices into a self-adjoint form. Thus $A_{pq} = 0$ (resp. 1) corresponds to $H_{pq} = +i$ (resp. $-i$) for all off-diagonal elements $p \neq q$ and $H_{pp} = 0$ for all $p = 1, \dots, N$. Importantly this means taking complex conjugation yields $\overline{H} = -H$, which in turn implies that if λ is an eigenvalue of H then so is $-\lambda$, with the eigenvectors being complex conjugates of each other. This spectral symmetry implies

$$\mathrm{Tr}(H^n) \equiv 0 \ \ \forall\, n \text{ odd}. \tag{2.2}$$

In order to make a distinction we say that H is an *imaginary tournament matrix* (resp. *regular imaginary tournament matrix*) if $A = \frac{1}{2}(\mathbf{E}_N - \mathbf{I}_N - iH)$ is a tournament (resp. regular tournament). Therefore, with a slight abuse of notation, we will write either $H \in \mathfrak{T}_N$ or $H \in \mathfrak{R}_N$.

Definition 2.1 (Imaginary tournament ensemble). *Let $\mathfrak{T}_N$ be the set of imaginary tournament matrices of size N. Then the imaginary Bernoulli ensemble (ITE) is given by the set of $H \in \mathfrak{T}_N$ with the uniform probability measure $P(H) = |\mathfrak{T}_N|^{-1}$.*

Definition 2.2 (Regular imaginary tournament ensemble). *Let $\mathfrak{R}_N$ be the set of regular imaginary tournament matrices of size N (with N being odd). Then the random imaginary tournament ensemble (RITE) is given by the set of $H \in \mathfrak{R}_N$ with the uniform probability measure $P(H) = |\mathfrak{R}_N|^{-1}$.*

Note that Definition 2.1 is equivalent to choosing the entires H_{pq} equal to $\pm i$ independently and with equal probability, whereas Definition 2.2 is equivalent to choosing H_{pq} equal to $\pm i$ with equal probability subject to the constraint that $\sum_q H_{pq} = 0$ for all $p = 1, \dots, N$.

Due to the independence of the elements in the ITE, many of the techniques developed to treat Wigner matrices are directly applicable, for example the universality of local statistics has been established in this case [45]. Moreover, since H is related to A by a (complex) rank one perturbation, the spectral properties of the ITE can be related to the complex eigenvalues of random tournaments [45]. However, to the best of our knowledge, there are no such results for the RITE, although linear statistics [14, 24], local semicircle estimates [7, 6] and local universality results [5] have been obtained for random regular graphs using switching methods.

Theorem 2.3 (Convergence for ITE). *Let $Z = (Z_2, Z_3, \dots, Z_k)$ be a collection of independent random Gaussian variables with mean 0 and variance $\sigma_n^2 = \mathbb{E}[Z_n^2] = n$. Let H be chosen according to the ITE and define the random variables*

$$Y_n(H) := \mathrm{Tr}\left(T_{2n}\left(\frac{H}{\sqrt{4N}}\right)\right) - \mathbb{E}\left[\mathrm{Tr}\left(T_{2n}\left(\frac{H}{\sqrt{4N}}\right)\right)\right]. \tag{2.3}$$

Then, for $Y(H) = (Y_2(H), Y_3(H)\dots, Y_k(H))$, $\phi \in C^2(\mathbb{R}^{k-1})$ with k fixed and N sufficiently large

$$|\mathbb{E}[\phi(Y(H))] - \mathbb{E}[\phi(Z)]| \leq \mathcal{O}(N^{-1})\|\phi\| + \mathcal{O}(N^{-1})\|\nabla\phi\| + \mathcal{O}(N^{-1})\|\nabla^2\phi\|.$$

where

$$\|\nabla^j\phi\| := \sup_{Q\in\mathbb{R}^j}\max_{n_1,\dots,n_j}\left|\frac{\partial^j\phi(Q)}{\partial Q_{n_1}\dots\partial Q_{n_j}}\right|. \tag{2.4}$$

Theorem 2.4 (Convergence for RITE)**.** *Let Z and $Y_n(H)$ be as in Theorem 2.3 and let H be chosen according to the RITE. Then, for $Y(H) = (Y_2(H), Y_3(H)\dots, Y_k(H))$, $\phi \in C^2(\mathbb{R}^{k-1})$ with k fixed and N sufficiently large. Then*

$$|\mathbb{E}[\phi(Y(H))] - \mathbb{E}[\phi(Z)]| \le \mathcal{O}(N^{-1/2})\|\phi\| + \mathcal{O}(N^{-1})\|\nabla\phi\| + \mathcal{O}(N^{-1})\|\nabla^2\phi\|. \tag{2.5}$$

Proof. The proof of Theorem 2.3 requires incorporating the results of Proposition 4.1 into Theorem 2.6. For Theorem 2.4 we incorporate the results of Proposition 5.2 into Theorem 2.6. □

Note that we exclude all the odd Chebyshev polynomials since they are comprised entirely of odd traces (see Equation (1.3)) and so by (2.2) they are identically zero. In addition we have $\mathrm{Tr}(H^2) = \sum_{p,q} H_{pq}H_{qp} = N(N-1)$ for all H, which means $\mathrm{Tr}(T_2(H/\sqrt{4N}))$ is constant.[1]

2.2 Outline of ideas and methods

In order to prove Theorems 2.3 and 2.4 we introduce random walks within $\mathfrak{T}_N$ and $\mathfrak{R}_N$ with two properties. Firstly, the stationary distributions correspond to $P(H) = |\mathfrak{T}_N|^{-1}$ and $P(H) = |\mathfrak{R}_N|^{-1}$, as per Definitions 2.1 and 2.2 respectively. Secondly, the induced motion of the random variable $Y(H)$ will be closely described by a process, whose stationary distribution is given by $Z = (Z_2, Z_3, \dots, Z_k)$, as in Theorems 2.3 and 2.4.

More precisely, suppose that at some discrete-time $t \in \mathbb{N}$ our random walker is situated at the matrix H, then we have a transition probability $\rho(H \to H')$ for the walker to be at the matrix H' at time $t+1$ later. From this one may track how the corresponding variable $Y_n(H)$ changes to $Y_n(H')$, i.e.

$$\mathbb{E}[\delta Y_n|H] := \sum_{H'}\rho(H\to H')[Y_n(H') - Y_n(H)]. \tag{2.6}$$

Similarly, fluctuations are obtained by calculating the second moment

$$\mathbb{E}[\delta Y_n\delta Y_m|H] := \sum_{H'}\rho(H\to H')[Y_n(H') - Y_n(H)][Y_m(H') - Y_m(H)]. \tag{2.7}$$

Now suppose that, if we design our random walk correctly, we observe that the moments take the form

$$\mathbb{E}[\delta Y_n|H] = \alpha_N[-nY_n(H) + R_n(H)] \tag{2.8}$$

$$\mathbb{E}[\delta Y_n\delta Y_m|H] = \alpha_N[2n^2\delta_{nm} + R_{nm}(H)], \tag{2.9}$$

where α_N is a certain constant depending only on N and $R_n(H), R_{nm}(H)$ are small remainders (the nature of small will be clarified later). Then, for arbitrary test functions $f \in C^3(\mathbb{R}^{k-1})$, expanding $\delta f := f(Y(H) + \delta Y(H,H')) - f(Y(H))$ in a Taylor series gives

$$\begin{aligned}\frac{\mathbb{E}[\delta f|H]}{\alpha_N} &= \sum_{n=2}^{k}\frac{\mathbb{E}[\delta Y_n|H]}{\alpha_N}\frac{\partial f}{\partial Y_n} + \frac{1}{2}\sum_{n,m=2}^{k}\frac{\mathbb{E}[\delta Y_n\delta Y_m|H]}{\alpha_N}\frac{\partial^2 f}{\partial Y_n\partial Y_m} + \frac{\mathbb{E}[S_f(H,H')|H]}{\alpha_N}\\ &= \mathcal{A}f(Y(H)) + \sum_{n=2}^{k}R_n(H)\frac{\partial f}{\partial Y_n} + \frac{1}{2}\sum_{n,m=2}^{k}R_{nm}(H)\frac{\partial^2 f}{\partial Y_n\partial Y_m} + \frac{\mathbb{E}[S_f(H,H')|H]}{\alpha_N},\end{aligned} \tag{2.10}$$

[1] One may consider this at odds with the Gaussian case but it was shown in [34] the first two moments in the Gaussian β-ensembles can be scaled in such a way that they may be considered independently of all other moments.

with some remainder $S_f(H,H')$ and the operator $\mathcal{A}$ is given by

$$\mathcal{A} := \sum_{n=2}^{k} n\left[n\frac{\partial^2}{\partial X_n^2} - X_n\frac{\partial}{\partial X_n}\right]. \tag{2.11}$$

If the Markov process is started from a stationary state, then the distributions of the random variables H and H' will be the same, in which case they are referred to as an *exchangeable pair*. The expected change in f will therefore satisfy $\mathbb{E}[\delta f] = \mathbb{E}[f(Y(H'))] - \mathbb{E}[f(Y(H))] = 0$, which, in turn, means $0 = \alpha_N^{-1}\mathbb{E}[\delta f] = \mathbb{E}[\mathcal{A}f(Y(H))] + \mathbb{E}[\mathcal{R}(H)]$, where $\mathcal{R}(H)$ denotes the total remainder in (2.10). The connection with the Gaussian distribution Z now emerges, since if it were the case the remainder $\mathbb{E}[\mathcal{R}(H)]$ is equal to 0 for all test functions f then we would have the following result, known as Stein's Lemma.

Lemma 2.5 (Stein's Lemma)**.** *Let $\mathcal{A}$ be the operator given in (2.11). Then $\mathbb{E}[\mathcal{A}f(Z)] = 0$ for all $f \in C^2(\mathbb{R}^k)$ if and only if $Z = (Z_2, Z_3, \ldots, Z_k)$, where $Z_n \sim N(0,n)$.*

Proof. One should consult e.g. Lemma 1 in [37] for details. Although briefly - integration by parts yields $\mathbb{E}[\mathcal{A}f(Z)] := \int dZ\; P(Z)\mathcal{A}f(Z) = \int dZ\; f(Z)\mathcal{A}^*P(Z) = 0$ for any $f \in C^2(\mathbb{R}^k)$ and thus establishes the first implication. For the converse one requires the exact form of the solution to equation (2.12) presented in Proposition A.1 in Appendix A. □

Of course the remainder will not, in general, be zero but one might expect that if it is close (in some appropriate manner) then the corresponding variable $Y(H)$ will be close to Z. Stein's realisation was that $\mathcal{A}$ and f could be connected via an auxiliary test function ϕ in what is now known as *Stein's equation*

$$\mathcal{A}f(x) = \phi(x) - \mathbb{E}[\phi(Z)], \tag{2.12}$$

with Z as in Lemma 2.5. Taking the expectation with respect to $Y(H)$ gives $|\mathbb{E}[\phi(Y)] - \mathbb{E}[\phi(Z)]| = |\mathbb{E}[\mathcal{A}f(Y)]|$. The aim is therefore to find a bound for $|\mathbb{E}[\mathcal{A}f(Y)]|$ in terms of ϕ, as this will allow for an estimate on the distributional distance between Y and Z. This idea was initially developed by Charles Stein as an alternative method for proving the classical CLT [48]. Stein's method now refers to the overall technique of recovering the distributional distance from bounding the quantity $\mathbb{E}[\mathcal{A}f(Z)]$. For readers unfamiliar with the basics of Stein's method, the review by Ross [40] provides an excellent introduction and overview of the different ways this may be achieved.

The work of Götze [21] and Barbour [4] in the early 90s allowed for an extension of Stein's method to multivariate Gaussian distributions and established an explicit connection between Stein's method and Markov processes. Using these ideas a number of authors adapted the use of the exchangeable pairs mechanism to multivariate Gaussian distributions [10, 39, 37] (the thesis of Döbler offers an excellent overview of this [12]), from which the following theorem is obtained.

Theorem 2.6. *Let (M,M') be an exchangeable pair of $N \times N$ random self-adjoint matrices with α_N a constant depending only on N and Z the multi-dimensional Gaussian random variable in Theorem 2.3. If the random variable $Y(M) = (Y_2(M), \ldots, Y_n(M))$ satisfies*

$$\frac{1}{\alpha_N}\mathbb{E}[\delta Y_n|M] = -nY_n(M) + R_n(M) \tag{2.13}$$

$$\frac{1}{\alpha_N}\mathbb{E}[\delta Y_n \delta Y_m|M] = 2n^2\delta_{nm} + R_{nm}(M) \tag{2.14}$$

$$\frac{1}{\alpha_N}\mathbb{E}[|\delta Y_n \delta Y_m \delta Y_l||M] = R_{nml}(M). \tag{2.15}$$

Then for all $\phi \in C^2(\mathbb{R}^k)$ *we have*

$$|\mathbb{E}[\phi(Y(M))] - \mathbb{E}[\phi(Z)]| \leq c_1 \mathcal{R}^{(1)} \|\phi\| + c_2 \mathcal{R}^{(2)} \|\nabla \phi\| + c_3 \mathcal{R}^{(3)} \|\nabla^2 \phi\|, \tag{2.16}$$

where $\|\nabla^j \phi\|$ *is given in (2.4),* c_j *are fixed positive constants and*

$$\mathcal{R}^{(j)} := \sum_{n_1,\ldots,n_j} \mathbb{E}|R_{n_1\ldots n_j}(M)|.$$

Proof. Theorem 2.6 is a specific form of Theorem 3 in [37], except that we have decided to use the alternative quantities $\|\nabla^k \phi\|$. We have therefore decided to include the proof of Theorem 2.6 in Appendix A for completeness and to aid the understanding of the interested reader, even though, beyond minimal adjustments, there is nothing new. □

Remark 2.7. As was first noted by Götze [21] and Barbour [4], the operator $\mathcal{A}$ is the generator for a specific multi-dimensional Ornstein-Uhlenbeck (OU) process. Thus, in essence, Theorem 2.6 is stating that if the random walk is close (i.e. the remainders R_n, R_{nm} etc and the constant α_N go to 0 in the limit of large N) to that of the associated OU-process then the corresponding stationary distributions will also be close - in the distributional sense of (2.16).

Remark 2.8. In principle one could remove the factor of n present in (2.11) and achieve the same stationary distribution but it will transpire the evolution of our observables $Y_n(B)$, given in (2.3), can only be analysed if it is included. This is because this factor corresponds to rescaling the time $t \to nt$, which is independent of the random variable in question. Thus, in general, the linear statistic $\Phi_h(H)$ will not evolve according to a single one-dimensional OU process, but rather a linear combination of independent one-dimensional OU processes evolving at different rates.

The novel aspect of our work concerns the evaluation of the remainders $R_n(H)$, $R_{nm}(H)$ and $R_{nml}(H)$. For comparison, the CLT results in [12, 49, 32], whilst slight stronger, heavily utilise Dyson Brownian motion, which affords a closed form expression for the evolution of spectrum. In other words, the remainders are functions of the eigenvalues, i.e. $R_n(H) = R_n(\lambda_1(H), \ldots, \lambda_N(H))$ etc. However, since our ensembles are not invariant under, say unitary or orthogonal transformations, we do not have this luxury. We therefore use alternative combinatorial methods to obtain estimates in terms of the matrix dimension N. These are similar to those previously utilised for random regular graphs [14, 24], however, in the current setting, we must compute matrix averages in addition to counting cycles.

The starting point of these methods comes from a generalised form of the Bartholdi identity, developed in [38] to obtain a trace formula for the eigenvalues of (magnetic) regular graphs. This allows us to relate the centred Chebyshev Polynomials $Y_n(H)$ to sums of products of matrix elements, like $H_{p_1p_2}H_{p_2p_3}\ldots$, associated to non-backtracking cycles (see Section 3). The change of such products under the appropriate random walks leads to remainder terms comprised of, again, certain classes of matrix products. Estimating the remainders consists of bounding the expectations of this quantities with respect to either the ITE and RITE.

For the ITE the estimates are relatively straightforward because the matrix elements are independent. It means the contributions from many types of cycles are precisely zero. Those cycles that remain only give contributions tending to 0 in the large N limit. For the RITE, however, a more complicated random walk leads, inevitably, to more complicated expressions for the remainder terms. Moreover, the lack of independence means the expectations of matrix products that were identically zero for the ITE are no longer so for the RITE. A key part of our analysis is therefore showing the correlations are small enough so the expectations go to zero sufficiently fast in N, as proved in Lemma 5.3.

3 Graph theoretical tools

Before proceeding to our random walks we first introduce some necessary terminology and simple results. A graph G consists of a set of vertices $V(G)$ and edges $E(G)$ connecting these vertices. G is said to be *simple* if every pair of vertices is connected by at most one edge and there are no vertices connected to themselves. G is also said to be *complete* if every pair of vertices has precisely one edge connecting them.

A *walk* ω of length n on a graph G is an ordered sequence of vertices $\omega = (p_0, p_1, \ldots, p_{n-1}, p_n)$ such that $p_{i+1} \neq p_i$ and all pairs $(p_i, p_{i+1}) \in E(G), i = 0, \ldots, n-1$ are edges on the graph. If $p_{i+2} = p_i$ for some $i = 0, \ldots, n-2$ then the walk is said to be *backtracking*. Otherwise ω is *non-backtracking*. A walk is also a *cycle* (of length n) if the first and last vertices are the same, i.e. $p_0 = p_n$. Note that, in the present article, cycles will be distinguished by the starting vertex, so for example, $\omega = (1,2,3,4,1) \neq (2,3,4,1,2) = \omega'$. Again, the cycle is *backtracking* if there exists some i such that $p_i = p_{i+2(n)}$ and *non-backtracking* otherwise.

We use the notations V_ω and E_ω to denote the set of distinct vertices and (undirected) edges[2] in a walk ω and $\nu_\omega(e)$ for the number of times the edge $e = (p, q) = (q, p)$ appears in ω. Therefore, in terms of the tournament matrix H, a walk ω corresponds to the product over all matrix elements associated to (directed) edges in ω, i.e

$$H_\omega := H_{p_0 p_1} H_{p_1 p_2} \ldots H_{p_{n-1} p_n}. \tag{3.1}$$

In addition, for a collection of walks $\omega_1, \omega_2, \ldots, \omega_n$ on G we define

$$V_{\omega_1,\ldots,\omega_n} := V_{\omega_1} \cup V_{\omega_2} \cup \ldots \cup V_{\omega_n}, \qquad E_{\omega_1,\ldots,\omega_n} := E_{\omega_1} \cup E_{\omega_2} \cup \ldots \cup E_{\omega_n} \tag{3.2}$$

and $\nu_{\omega_1,\ldots,\omega_n}(e)$ for the number of times the edge e is traversed by the walks $\omega_1, \ldots, \omega_n$. Similarly

$$H_{\omega_1,\ldots,\omega_n} := H_{\omega_1} H_{\omega_2} \ldots H_{\omega_n}. \tag{3.3}$$

If an edge (p, q) appears an even number of times in $\omega_1, \ldots, \omega_n$ then the corresponding matrix element will disappear from (3.3) since we have identically $H_{pq}^2 = -H_{pq}H_{qp} = 1$ for every $H \in \mathfrak{T}_N$. It is therefore convenient to define the set of 'free' edges as

$$F_{\omega_1,\ldots,\omega_n} := \{e \in E_{\omega_1,\ldots,\omega_n} : \nu_{\omega_1,\ldots,\omega_n}(e) \equiv 1 \mod (2)\},$$

i.e. the set of edges that are traversed an odd number of times by $\omega_1, \ldots, \omega_n$. This will be especially useful when evaluating remainders for the RITE in Section 5.

We say that two walks ω and ω' are equivalent if ω' can be obtained from ω by simply relabelling the vertices and we will use the notation $\omega \sim \omega'$ to denote that is the case. For example $\omega = (1,2,3,1,4) \sim (2,3,9,2,6) = \omega'$. We will write $[\omega] := \{\omega' : \omega \sim \omega'\}$ to denote the associated equivalence class and if Ω is a set of walks then $[\Omega] := \{[\omega] : \omega \in \Omega\}$ is the set of equivalence classes. Moreover, we shall use the notation $\omega \cong \omega'$ if $\omega \sim \omega'$ and $F_\omega = F_{\omega'}$. For example $\omega = (1,2,3,4,5,6,7,5,4,3,1) \cong (1,3,2,8,6,5,7,6,8,2,1) = \omega'$. The above notions immediately generalise to collections of walks $(\omega_1, \ldots, \omega_n)$.

Lemma 3.1. *Let G be a simple, connected graph with vertex set $V(G)$ and edge set $E(G)$. Let $G' \subseteq G$ be a subgraph with $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. Then*

$$|E(G')| - |V(G')| \leq |E(G)| - |V(G)|, \tag{3.4}$$

provided $|V(G')| \geq 1$.

[2] i.e. (p, q) and (q, p) will be associated with the same edge.

Proof. Let C denote the number of connected components of G'. We can create a new graph $\tilde{G} \subseteq G$ by adding a minimal number of edges to G' such that $\tilde{G}$ is connected, then

$$|E(G')| - |V(G')| = |E(\tilde{G})| - |V(\tilde{G})| - C + 1 = \beta(\tilde{G}) - C.$$

Here $\beta(\tilde{G}) = |E(\tilde{G})| - |V(\tilde{G})| + 1$ is the first Betti number of $\tilde{G}$, which counts the number of fundamental cycles. However, since $\tilde{G}$ is a subgraph of G it cannot have more fundamental cycles than G and so

$$|E(G')| - |V(G')| \leq |E(G)| - |V(G)| + (1 - C).$$

The condition $|V(G')| \geq 1$ ensures that $C \geq 1$, which completes the result. □

Corollary 3.2. *Let $\bar{\omega} = (\omega_1, \dots, \omega_n)$ be a collection of walks and define the subgraph $G = (V_{\bar{\omega}}, F_{\bar{\omega}})$. If G is disconnected with C components then we write $G_i = (V_{\bar{\omega}}^{(i)}, F_{\bar{\omega}}^{(i)}), i = 1, \dots, C$ to denote the subgraphs of these components and $\beta_i = |F_{\bar{\omega}}^{(i)}| - |V_{\bar{\omega}}^{(i)}| + 1$ the associated first Betti numbers. Suppose $\bar{\omega} \sim \bar{\omega}'$ then*

$$|V_{\bar{\omega},\bar{\omega}'}| - \frac{|F_{\bar{\omega},\bar{\omega}'}|}{2} \leq |V_{\bar{\omega}}| + \sum_{i=1}^{C} \delta_{\beta_i,0}.$$

Proof. By construction we have

$$|V_{\bar{\omega},\bar{\omega}'}| = |V_{\bar{\omega}}| + |V_{\bar{\omega}'}| - |V_{\bar{\omega}} \cap V_{\bar{\omega}'}| = 2|V_{\bar{\omega}}| - |V_{\bar{\omega}} \cap V_{\bar{\omega}'}| \tag{3.5}$$

$$|F_{\bar{\omega},\bar{\omega}'}| = |F_{\bar{\omega}}| + |F_{\bar{\omega}'}| - 2|F_{\bar{\omega}} \cap F_{\bar{\omega}'}| = 2|F_{\bar{\omega}}| - 2|F_{\bar{\omega}} \cap F_{\bar{\omega}'}| \tag{3.6}$$

and therefore

$$|V_{\bar{\omega},\bar{\omega}'}| - \frac{|F_{\bar{\omega},\bar{\omega}'}|}{2} = 2|V_{\bar{\omega}}| - |F_{\bar{\omega}}| + |F_{\bar{\omega}} \cap F_{\bar{\omega}'}| - |V_{\bar{\omega}} \cap V_{\bar{\omega}'}|. \tag{3.7}$$

Now, let us define $G_i' = (V_{\bar{\omega}}^{(i)} \cap V_{\bar{\omega}'}, F_{\bar{\omega}}^{(i)} \cap F_{\bar{\omega}'}) \subseteq G_i$. If $|V_{\bar{\omega}}^{(i)} \cap V_{\bar{\omega}'}| \geq 1$ then by Lemma 3.1 we have

$$|F_{\bar{\omega}}^{(i)} \cap F_{\bar{\omega}'}| - |V_{\bar{\omega}}^{(i)} \cap V_{\bar{\omega}'}| \leq |F_{\bar{\omega}}^{(i)}| - |V_{\bar{\omega}}^{(i)}| = \beta_i - 1, \tag{3.8}$$

whereas, if $|V_{\bar{\omega}}^{(i)} \cap V_{\bar{\omega}'}| = 0$ then $|F_{\bar{\omega}}^{(i)} \cap F_{\bar{\omega}'}| = 0$ also. Therefore, since all the G_i are disconnected we have $|V_{\bar{\omega}}| - |F_{\bar{\omega}}| = \sum_i (1 - \beta_i)$, and so (3.7) becomes

$$|V_{\bar{\omega},\bar{\omega}'}| - \frac{|F_{\bar{\omega},\bar{\omega}'}|}{2} = |V_\omega| + \sum_{i=1}^{C} \left\{ 1 - \beta_i + |F_{\bar{\omega}}^{(i)} \cap F_{\bar{\omega}'}| - |V_{\bar{\omega}}^{(i)} \cap V_{\bar{\omega}'}| \right\} \leq |V_\omega| + \sum_{i=1}^{C} \delta_{\beta_i,0}.$$

□

For our imaginary tournament matrices there is an intimate connection between the traces of Chebyshev polynomials (1.4) and the sets of non-backtracking cycles. This is given by the following lemma.

Lemma 3.3. *Let M be an $N \times N$ self-adjoint matrix with elements of the form*

$$M_{pq} = e^{i\phi_{pq}} = \overline{M}_{qp}, \quad \phi_{pq} \in [0, 2\pi), \forall q \neq p \tag{3.9}$$

and $M_{pp} = 0$ for all p. Then

$$\mathrm{Tr}\left(T_n\left(\frac{M}{2\sqrt{N-2}}\right)\right) = \frac{1}{2}\frac{1}{(N-2)^{\frac{n}{2}}}\left[\sum_{\omega \in \Omega_n} M_\omega - \frac{1}{2}N(N-3)(1+(-1)^n)\right], \tag{3.10}$$

where Ω_n denotes the set of non-backtracking cycles of length $2n$ and M_ω is given in (3.1).

Proof. We are aware of two related methods for proving the validity of this statement that we shall not recount here. The first approach is to make a generalisation of the so-called Bartholdi identity (see e.g. [38, 27]) that relates the spectrum of M to another matrix associated to non-backtracking walks in the edge space. This connection is applicable since M can be considered as a magnetic adjacency matrix of a complete graph on N vertices. The second approach is based upon showing that polynomials associated to non-backtracking walks obey the same recursion relations as the Chebyshev polynomials (see e.g. [44] and references therein). □

4 Imaginary tournament ensemble

We now construct the random walk process in $\mathfrak{T}_N$. Many of the intricate details of this walk are discussed in [26] and so we attempt to keep to the essential points. Suppose that at time $t \in \mathbb{N}$ we select a matrix $H \in \mathfrak{T}_N$, then at time $t+1$ we randomly choose another matrix $H' \in \mathfrak{T}_N$ by selecting with equal probability one of the upper triangular elements of H (say H_{pq} with $p < q$) and, together with its symmetric partner H_{qp}, we change its sign $H_{pq} \to -H_{pq}$. We will write

$$\delta H^{pq} := H' - H = -2H_{pq}[\mathbf{e}_p \mathbf{e}_q^T - \mathbf{e}_q \mathbf{e}_p^T], \tag{4.1}$$

to denote the $N \times N$ rank 2 difference matrix obtained as a result of performing this change of sign. Here $\mathbf{e}_p$ is the column vector with a 1 in entry p and 0 everywhere else. This switch corresponds to changing the direction of an edge (see Figure 1) in the associated tournament graph, as described in Section 2.1.

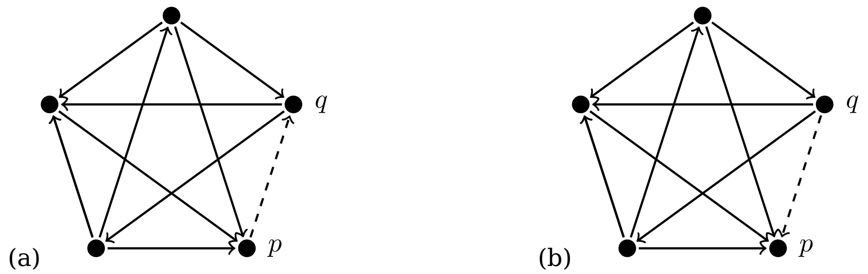

Figure 1: The Markov process consists of choosing an edge (p,q) uniformly at random in the tournament graph (a) and then switching the orientation to obtain the tournament graph in (b). In this example the (p,q)-th element of the associated adjacency matrix $A_{pq} = 1 - A_{qp} = 1$ in (a) is updated to $A_{pq} = 0$ in (b). Hence $H_{pq} = i(2A - (\mathbf{E}_N - \mathbf{I}_N))_{pq} = -H_{qp} = i \mapsto H_{pq} = -i$ when making the switch from (a) to (b).

Interpreting this in terms of a random walk we say that if the walker is at H at time t then in each unit time step we let the walker move to any matrix $H' \in \mathfrak{T}_N$ which is exactly a Hamming distance[3] one away with equal probability - giving us the transition probability

$$\rho(H \to H') = \begin{cases} \frac{1}{d_N} & |H - H'| = 1 \\ 0 & |H - H'| \neq 1, \end{cases} \tag{4.2}$$

where $d_N = N(N-1)/2$ is the number of independent elements of H. Therefore, if $P_t(H)$ is the probability for the random walker to be at matrix H at time t then the probability

[3]The Hamming distance between two matrices $H, H' \in \mathfrak{T}_N$ is given by $|H - H'| = \frac{1}{2}\sum_{p<q} |H'_{pq} - H_{pq}|$, which counts the number of differences in signs of the free matrix elements.

to be at some other matrix $H' \in \mathfrak{T}_N$ is given by

$$P_{t+1}(H') = \sum_{H \in \mathfrak{T}_N} \rho(H \to H') P_t(H) = \sum_{H:|H'-H|=1} \frac{P_t(H)}{d_N} = \frac{1}{d_N} \sum_{p<q} P_t(H' - \delta H^{(pq)}).$$

One may easily verify that $P_t(H') = |\mathfrak{T}_N|^{-1}$ (the measure of the ITE in Definition 2.1) is the stationary distribution of this process. In this instance the random matrices H and H' have the same distribution and are thus an exchangeable pair.

The expected change of some observable $f(H)$ with respect to this random walk is hence given by

$$\mathbb{E}[\delta f|H] := \sum_{H' \in \mathfrak{T}_N} \rho(H \to H')[f(H') - f(H)] = \frac{1}{d_N} \sum_{p<q} [f(H + \delta H^{pq}) - f(H)]. \tag{4.3}$$

Similarly, higher moments are obtained by taking the expectation of products of changes, i.e. for $f_1(H), f_2(H), \dots, f_k(H)$

$$\mathbb{E}[\delta f_1 \dots \delta f_k|H] := \frac{1}{d_N} \sum_{p<q} [f_1(H + \delta H^{pq}) - f_1(H)] \dots [f_k(H + \delta H^{pq}) - f_k(H)]. \tag{4.4}$$

We are now in position to state how the observables $Y_n(H)$, given in (2.3), behave under this random walk.

Proposition 4.1 (ITE Random walk)**.** *Let (H', H) be an exchangeable pair from the ITE and connected via (4.2). Let $Y_n(H)$ be as defined in (2.3). Then*

- *(a)* $\frac{d_N}{4}\mathbb{E}[\delta Y_n|H] = -nY_n(H) + R_n(H)$ *(Drift term)*
- *(b)* $\frac{d_N}{4}\mathbb{E}[\delta Y_n \delta Y_m|H] = 2n^2\delta_{nm} + R_{nm}(H)$ *(Diffusion term)*
- *(c)* $\frac{d_N}{4}\mathbb{E}[|\delta Y_n \delta Y_m \delta Y_l||H] = R_{nml}(H)$ *(Remainder term)*

with $\mathbb{E}|R_n(H)| = \mathcal{O}(N^{-1})$, $\mathbb{E}|R_{nm}(H)| = \mathcal{O}(N^{-1})$ *and* $\mathbb{E}|R_{nml}(H)| = \mathcal{O}(N^{-1})$ *for all* $n, m, l = 2, \dots, k$.

Proof. The proofs for Parts (a), (b) and (c) will be presented in Sections 4.1, 4.2 and 4.3 respectively. □

To show Proposition 4.1 we utilise Lemma 3.3, which expresses the trace of Chebyshev polynomials of H in terms of non-backtracking cycles to write $Y_n(H)$ (see Equation (2.3)) in the following form

$$Y_n(H) = \frac{1}{2}\frac{1}{(N-2)^n} \sum_{\omega \in \Omega_{2n}} H_\omega - \mathbb{E}[H_\omega] = \frac{1}{2}\frac{1}{(N-2)^n} \sum_{\omega \in \Lambda_{2n}} H_\omega. \tag{4.5}$$

Here $\Lambda_{2n} := \{\omega \in \Omega_{2n} : \exists e \in E_\omega \text{ s.t. } \nu_\omega(e) = 1 \mod (2)\}$ is the set of non-backtracking cycles $\omega \in \Omega_{2n}$ for which there is at least one edge that is traversed an odd number of times. Hence, since the matrix elements are independent, $\mathbb{E}[H_\omega] = 0$ for all $\omega \in \Lambda_{2n}$, which is why the expectation term disappears in (4.5).

From (4.1), since only the (p,q)-th element of H changes sign, a general $e = (p', q')$ element of $H' = H + \delta H^{pq}$ is given by

$$H'_e = H_e(1 - 2\chi_{e,pq}) = H_e(-1)^{\chi_{e,pq}}, \tag{4.6}$$

where $\chi_{e,pq} = (\delta_{pp'}\delta_{qq'} + \delta_{qp'}\delta_{pq'})$ is equal to 1 if $e = (p,q)$ or (q,p) and 0 otherwise. Therefore, if $\omega = (p_0, \dots, p_{2n-1}, p_0)$ then

$$\delta H^{pq}_\omega := H'_\omega - H_\omega = \prod_{i=0}^{2n-1} H'_{p_i p_{i+1(2n)}} - H_\omega = H_\omega \prod_{i=0}^{2n-1} (-1)^{\chi_{p_i p_{i+1(2n)}, pq}} - H_\omega = -2H_\omega \phi_{\omega,pq}$$

where

$$\phi_{\omega,pq} = \frac{1}{2}((-1)^{\chi_{\omega,pq}} - 1), \qquad \chi_{\omega,pq} = \sum_{i=0}^{2n-1} \chi_{p_i p_{i+1(2n)},pq}.$$

Hence $\phi_{\omega,pq} = 1$ if $\nu_\omega((p,q)) = 1 \mod (2)$ and 0 otherwise. In other words $\phi_{\omega,pq}$ is only non-zero when the cycle ω traverses the undirected edge (p,q) an odd number of times.

Therefore $\delta Y_n^{pq} := Y_n(H + \delta H^{pq}) - Y_n(H)$ is given by

$$\delta Y_n^{pq} = \frac{1}{2}\frac{1}{(N-2)^n} \sum_{\omega\in\Lambda_{2n}} \delta H_\omega^{pq} = -\frac{1}{(N-2)^n} \sum_{\omega\in\Lambda_{2n}} H_\omega \phi_{\omega,pq}. \tag{4.7}$$

4.1 Proof of Proposition 4.1 Part (a) - Drift term

Inserting the form (4.7) for δY_n^{pq} into the expression (4.3) for the expected change of an observable undergoing this random walk leads to

$$\mathbb{E}[\delta Y_n|H] = \frac{1}{d_N}\sum_{p<q} \delta Y_n^{pq} = -\frac{1}{d_N}\frac{1}{(N-2)^n}\sum_{p<q}\sum_{\omega\in\Lambda_{2n}} H_\omega \phi_{\omega,pq} = \frac{4}{d_N}[-nY_n(H) + R_n(H)]. \tag{4.8}$$

Using the expression (4.5) for $Y_n(H)$ therefore gives the remainder

$$R_n(H) = \frac{n}{2}\frac{1}{(N-2)^n}\sum_{\omega\in\Lambda_{2n}} H_\omega \left(1 - \frac{1}{2n}\sum_{p<q}\phi_{\omega,pq}\right). \tag{4.9}$$

Our aim is to show that $|\mathbb{E}[R_n(H)]| = \mathcal{O}(N^{-1})$. We now write $\Lambda_{2n}^\star = \{\omega \in \Lambda_{2n} : |F_\omega| = 2n\}$, i.e. the set of non-backtracking cycles in Λ_{2n} in which all edges are traversed exactly once. (Note this does not exclude the possibility of ω visiting a particular vertex more than once). We also write $\Lambda_{2n}^\circ = \Lambda_{2n} \setminus \Lambda_{2n}^\star$ for the set of non-backtracking cycles in which at least one edge is traversed more than once. For convenience, let us write $\Phi_\omega := \sum_{p<q}\phi_{\omega,pq}$, which counts the number of edges in ω that are traversed an odd number of times. Importantly, for all $\omega \in \Lambda_{2n}^\star$ we have

$$\Phi_\omega := \sum_{p<q}\phi_{\omega,pq} = 2n, \tag{4.10}$$

and in general $\Phi_\omega \le 2n$. Therefore the sum over ω in Λ_{2n} in (4.9) can be reduced to the lesser sum over Λ_{2n}°. As outlined in Section 3, let us write $[\omega]$ for the equivalence class of ω, which simply corresponds to relabelling the vertices, and $[\Lambda_{2n}^\circ]$ for the set of such equivalence classes in Λ_{2n}°. Since all we are doing is relabelling, the quantity Φ_ω is the same for all $\omega \in [\omega]$, meaning we can write $\Phi_{[\omega]}$ instead. Hence

$$\begin{aligned}\mathbb{E}|R_n(H)| &\le \mathcal{O}(N^{-n})\mathbb{E}\left|\sum_{[\omega]\in[\Lambda_{2n}^\circ]}\left(1 - \frac{\Phi_{[\omega]}}{2n}\right)\sum_{\omega\in[\omega]} H_\omega\right| \\ &\le \mathcal{O}(N^{-n})\sum_{[\omega]\in[\Lambda_{2n}^\circ]}\left|1 - \frac{\Phi_{[\omega]}}{2n}\right|\mathbb{E}\left|\sum_{\omega\in[\omega]} H_\omega\right|. \end{aligned} \tag{4.11}$$

Thus, using that $0 \le (1 - \Phi_{[\omega]}/2n) \le 1$ and the inequality $\mathbb{E}|A| \le \sqrt{\mathbb{E}[A^2]}$, leads to

$$\mathbb{E}|R_n(H)| \le \mathcal{O}(N^{-n})\sum_{[\omega]\in[\Lambda_{2n}^\circ]}\sqrt{\sum_{\omega,\omega'\in[\omega]}\mathbb{E}[H_{\omega,\omega'}]}, \tag{4.12}$$

where $H_{\omega,\omega'} := H_\omega H_{\omega'}$, as in (3.3). The quantity $\mathbb{E}[H_{\omega,\omega'}] \neq 0$ only when $\omega \cong \omega'$. Let us suppose that every edge in $E_{\omega,\omega'}$ is traversed exactly twice, then $|E_{\omega,\omega'}| = 2n$ and because

$\omega \in \Lambda_{2n}^{\circ}$ the subgraph $G = (V_{\omega,\omega'}, E_{\omega,\omega'})$ is connected with at least three fundamental cycles, i.e. $\beta(G) \geq 3$. Thus $|V_{\omega,\omega'}| = |E_{\omega,\omega'}| - \beta(G) + 1 \leq 2n - 2$. Alternatively, if there exists a least one edge that is traversed four times or more then $|E_{\omega,\omega'}| \leq 2n - 1$ and $\beta(G) \geq 2$, meaning $|V_{\omega,\omega'}| = |E_{\omega,\omega'}| - \beta(G) + 1 \leq 2n - 2$ also.

The contribution from the term inside the square-root in (4.12) is obtained by labelling the independent vertices in $V_{\omega,\omega'}$. Up to a constant, we have $N(N-1)\dots(N-|V_{\omega,\omega'}|-1) = \mathcal{O}(N^{|V_{\omega,\omega'}|})$ pairs $\omega \cong \omega' \in [\omega]$, so taking the square root we have $\mathbb{E}|R_n(H)| \leq \mathcal{O}(N^{-n})|[\Lambda_{2n}^{\circ}]|\sqrt{\mathcal{O}(N^{2n-2})} = |[\Lambda_{2n}^{\circ}]|\mathcal{O}(N^{-1})$. We are thus left to evaluate $|[\Lambda_{2n}^{\circ}]|$, the number of *unlabelled* non-backtracking cycles $\omega \in \Lambda_{2n}^{\circ}$. However, since the labelling has been removed this quantity is now independent of N, and so $|[\Lambda_{2n}^{\circ}]| = \mathcal{O}(1)$, meaning $\mathbb{E}|R_n(H)| = \mathcal{O}(N^{-1})$, as desired.

4.2 Proof of Proposition 4.1 Part (b) - Diffusion term

Inserting the form (4.7) for δY_n^{pq} into the expression (4.4) leads to the following diffusion term

$$\mathbb{E}[\delta Y_n \delta Y_m | H] = \frac{1}{d_N} \sum_{p<q} \delta Y_n^{pq} \delta Y_m^{pq} = \frac{1}{d_N} \frac{1}{(N-2)^{n+m}} \sum_{p<q} \sum_{\omega_1 \in \Lambda_{2n}} \sum_{\omega_2 \in \Lambda_{2m}} H_{\omega_1,\omega_2} \phi_{\omega_1,pq} \phi_{\omega_2,pq}.$$

Therefore writing $\mathbb{E}[\delta Y_n \delta Y_m | H] = \frac{4}{d_N}[2n^2 \delta_{nm} + R_{nm}(H)]$ and rearranging gives

$$R_{nm}(H) = \frac{1}{4} \frac{1}{(N-2)^{n+m}} \sum_{p<q} \sum_{\omega_1 \in \Lambda_{2n}} \sum_{\omega_2 \in \Lambda_{2m}} H_{\omega_1,\omega_2} \phi_{\omega_1,pq} \phi_{\omega_2,pq} - 2n^2 \delta_{nm}. \tag{4.13}$$

We estimate the cases $n = m$ and $n \neq m$ separately. For the former case let us take $\Lambda_{2n}^{\star}$ as in Section 4.1 and define $\Gamma_{2n}^{\star} = \{(\omega_1, \omega_2) \in \Lambda_{2n}^{\star} \times \Lambda_{2n}^{\star} : \omega_1 \cong \omega_2\}$, with the complement $\Gamma_{2n}^{\circ} = (\Lambda_{2n} \times \Lambda_{2n}) \setminus \Gamma_{2n}^{\star}$. For walks $\omega_1 \cong \omega_2$ we have $\phi_{\omega_1,pq} = \phi_{\omega_2,pq}$ and $H_{\omega_1} = H_{\omega_2}$ (so $H_{\omega_1,\omega_2} = 1$). Moreover, if $\omega_1 \in \Lambda_{2n}^{\star}$ then from (4.10) $\sum_{p<q} \phi_{\omega_1,pq}^2 = \sum_{p<q} \phi_{\omega_1,pq} = 2n$. In addition, if $|V_{\omega_1}| = |V_{\omega_2}| = 2n$ (ω is a single loop) then for a fixed ω_1 there are $4n$ possible ω_2 such that $\omega_1 \cong \omega_2$ - obtained by choosing the $2n$ possible starting vertices of the cycle and the 2 possible orientations. Labelling the independent vertices of ω_1 leads to an overall contribution to $|\Gamma_{2n}^{\star}|$ of $4nN(N-1)\dots(N-(2n-1)) = 4nN^{2n} + \mathcal{O}(N^{2n-1})$. If, in contrast, $|V_{\omega_1}| = |V_{\omega_2}| < 2n$ then the contribution to $|\Gamma_{2n}^{\star}|$ will be of order $\mathcal{O}(N^{2n-1})$. Therefore

$$\begin{aligned} R_{nn}(H) &= \frac{1}{4} \frac{1}{(N-2)^{2n}} \sum_{(\omega_1,\omega_2) \in \Gamma_{2n}^{\circ}} H_{\omega_1,\omega_2} \Phi_{\omega_1,\omega_2} + \frac{n}{2(N-2)^{2n}} |\Gamma_{2n}^{\star}| - 2n^2 \\ &= \frac{1}{4} \frac{1}{(N-2)^{2n}} \sum_{(\omega_1,\omega_2) \in \Gamma_{2n}^{\circ}} H_{\omega_1,\omega_2} \Phi_{\omega_1,\omega_2} + \mathcal{O}(N^{-1}), \end{aligned} \tag{4.14}$$

where $\Phi_{\omega_1,\omega_2} := \sum_{p<q} \phi_{\omega_1,pq} \phi_{\omega_2,pq}$. Similarly $\Phi_{[\omega_1,\omega_2]}$ takes the value Φ_{ω_1,ω_2} for any $(\omega_1, \omega_2) \in [\omega_1, \omega_2]$. Therefore, splitting the sum into equivalence classes gives

$$\begin{aligned} \mathbb{E}|R_{nn}(H)| &\leq \mathcal{O}(N^{-2n}) \sum_{[\omega_1,\omega_2] \in [\Gamma_{2n}^{\circ}]} \Phi_{[\omega_1,\omega_2]} \mathbb{E}\left| \sum_{(\omega_1,\omega_2) \in [\omega_1,\omega_2]} H_{\omega_1,\omega_2} \right| + \mathcal{O}(N^{-1}) \\ &\leq \mathcal{O}(N^{-2n}) \sum_{[\omega_1,\omega_2] \in [\Gamma_{2n}^{\circ}]} \Phi_{[\omega_1,\omega_2]} \sqrt{\sum_{\substack{(\omega_1,\omega_2),(\omega_1',\omega_2') \\ \in [\omega_1,\omega_2]}} \mathbb{E}[H_{\omega_1,\omega_2,\omega_1',\omega_2'}]} + \mathcal{O}(N^{-1}). \end{aligned} \tag{4.15}$$

Since we want to maximise the number of independent vertices we can assume that the edges in E_{ω_1,ω_2} are traversed at most twice by (ω_1, ω_2), which implies that $2|E_{\omega_1,\omega_2}| -$

$|F_{\omega_1,\omega_2}| = 4n$. In addition, ω_1 and ω_2 must share at least one edge (otherwise $\Phi_{\omega_1,\omega_2} = 0$) and we cannot have $V_{\omega_1} = V_{\omega_2}$, otherwise $\omega_1 \cong \omega_2$ meaning $(\omega_1, \omega_2) \notin \Gamma^\circ_{2n}$. This means the subgraph $\hat{G} = (V_{\omega_1,\omega_2}, E_{\omega_1,\omega_2})$ is connected and has $\beta(\hat{G}) \geq 2$.

Now, since $(\omega_1, \omega_2) \cong (\omega_1', \omega_2')$ in (4.15) (otherwise we would have $\mathbb{E}[H_{\omega_1,\omega_2,\omega_1',\omega_2'}] = 0$) we get $|F_{\omega_1,\omega_2,\omega_1',\omega_2'}| = 0$ and so Corollary 3.2 gives $|V_{\omega_1,\omega_2,\omega_1',\omega_2'}| \leq |V_{\omega_1,\omega_2}| + |V_I|$, where V_I are the isolated vertices of the graph $G = (V_{\omega_1,\omega_2}, F_{\omega_1,\omega_2})$. But by construction $|V_I| \leq (|E_{\omega_1,\omega_2}| - |F_{\omega_1,\omega_2}|) - 1$, since $(|E_{\omega_1,\omega_2}| - |F_{\omega_1,\omega_2}|)$ is the number of edges traversed precisely twice. Therefore

$$\begin{aligned}
|V_{\omega_1,\omega_2,\omega_1',\omega_2'}| &\leq |V_{\omega_1,\omega_2}| + |V_I| \\
&\leq |V_{\omega_1,\omega_2}| + |E_{\omega_1,\omega_2}| - |F_{\omega_1,\omega_2}| - 1 \\
&= 2|E_{\omega_1,\omega_2}| - |F_{\omega_1,\omega_2}| - \beta(\hat{G}) \leq 4n - 2.
\end{aligned}$$

Again, we have $|[\Gamma^\circ_{2n}]| = \mathcal{O}(1)$, since it is independent of N and also $\Phi_{[\omega_1,\omega_2]} = \mathcal{O}(1)$, since it is equal to, at most, the number of shared edges of ω_1 and ω_2. Hence, $\mathbb{E}|R_{nn}(H)| \leq \mathcal{O}(N^{-2n})\sqrt{\mathcal{O}(N^{4n-2})} + \mathcal{O}(N^{-1}) = \mathcal{O}(N^{-1})$.

It thus remains to evaluate $\mathbb{E}|R_{nm}(H)|$ for $n \neq m$. In this instance we have, from (4.13)

$$\begin{aligned}
\mathbb{E}|R_{nm}(H)| &\leq \mathcal{O}(N^{-(n+m)})\mathbb{E}\Bigg| \sum_{\substack{(\omega_1,\omega_2) \\ \in \Lambda_{2n}\times\Lambda_{2m}}} H_{\omega_1,\omega_2}\Phi_{\omega_1,\omega_2} \Bigg| \\
&\leq \mathcal{O}(N^{-(n+m)}) \sum_{\substack{[\omega_1,\omega_2] \\ \in[\Lambda_{2n}\times\Lambda_{2m}]}} \Phi_{[\omega_1,\omega_2]} \sqrt{\sum_{\substack{(\omega_1,\omega_2),(\omega_1',\omega_2') \\ \in[\omega_1,\omega_2]}} \mathbb{E}[H_{\omega_1,\omega_2,\omega_1',\omega_2'}]}. \qquad (4.16)
\end{aligned}$$

Again, the main contribution will come from cycles ω_1 and ω_2 in which all vertices are distinct, i.e. $|V_{\omega_1}| = 2n$ and $|V_{\omega_2}| = 2m$. However, since $n \neq m$, ω_1 and ω_2 cannot share all the same edges. The condition $\alpha_{\omega_1,\omega_2} > 0$ only if ω_1 and ω_2 share at least one edge, and therefore, for the same reasons as above, those contributing collections of cycles $(\omega_1, \omega_2, \omega_1', \omega_2')$ for which $\mathbb{E}[H_{\omega_1,\omega_2,\omega_1',\omega_2'}]$ is non-zero satisfy $|V_{\omega_1,\omega_2,\omega_1',\omega_2'}| \leq 2n + 2m - 2$. Hence, $\mathbb{E}|R_{nm}(H)| \leq \mathcal{O}(N^{-(n+m)})|[\Lambda_{2n} \times \Lambda_{2m}]|\sqrt{\mathcal{O}(N^{2n+2m-2})} = \mathcal{O}(N^{-1})$.

4.3 Proof of Proposition 4.1 Part (c) - Remainder term

For the remainder term we again insert the expression (4.7) into (4.4), which gives us

$$\begin{aligned}
\mathbb{E}[|\delta Y_n \delta Y_m \delta Y_l||H] &= \frac{1}{d_N}\sum_{p<q}|\delta Y_n^{pq}\delta Y_m^{pq}\delta Y_l^{pq}| \\
&\leq \frac{1}{d_N}\frac{1}{(N-2)^{n+m+l}}\sum_{p<q}\Bigg| \sum_{\substack{\omega_1,\omega_2,\omega_3 \\ \in\Lambda_{2n}\times\Lambda_{2m}\times\Lambda_{2l}}} H_{\omega_1,\omega_2,\omega_3}\phi_{\omega_1,pq}\phi_{\omega_2,pq}\phi_{\omega_3,pq} \Bigg|. \qquad (4.17)
\end{aligned}$$

Let us define $\Gamma^{pq}_{2n,2m,2l} := \{(\omega_1, \omega_2, \omega_3) \in \Lambda_{2n} \times \Lambda_{2m} \times \Lambda_{2l} : \phi_{\omega_1,pq}\phi_{\omega_2,pq}\phi_{\omega_3,pq} = 1\}$ as the set of non-backtracking cycles that all traverse the edge (p, q) an odd number of times. Since $\mathbb{E}|R_{nml}(H)| = \frac{d_N}{4}\mathbb{E}[\mathbb{E}[|\delta Y_n \delta Y_m \delta Y_l||H]]$, taking the expectation over the ITE subsequently leads to

$$\begin{aligned}
\mathbb{E}|R_{nml}(H)| &= \mathcal{O}(N^{-(n+m+l)})\sum_{p<q}\mathbb{E}\Bigg| \sum_{(\omega_1,\omega_2,\omega_3)\in\Gamma^{pq}_{2n,2m,2l}} H_{\omega_1,\omega_2,\omega_3} \Bigg| \\
&\leq \mathcal{O}(N^{-(n+m+l)})\sum_{p<q}\sum_{[\omega_1,\omega_2,\omega_3]\in[\Gamma^{pq}_{2n,2m,2l}]}\sqrt{\sum_{\substack{(\omega_1,\omega_2,\omega_3),(\omega_1',\omega_2',\omega_3') \\ \in[\omega_1,\omega_2,\omega_3]}} \mathbb{E}[H_{\omega_1,\omega_2,\omega_3,\omega_1',\omega_2',\omega_3'}]}. \qquad (4.18)
\end{aligned}$$

The main contribution to the above will again come from non-backtracking cycles in which all vertices are distinct ($|V_{\omega_1}| = |V_{\omega'_1}| = 2n$ etc.), as this maximises the number of vertices. In this case all the cycles $\omega_i, \omega'_i, i = 1, 2, 3$ must traverse the edge p, q precisely once. The expectation $\mathbb{E}[H_{\omega_1,\omega_2,\omega_3,\omega'_1,\omega'_2,\omega'_3}]$ is only non-zero when every edge in $E_{\omega_1,\omega_2,\omega_3,\omega'_1,\omega'_2,\omega'_3}$ is traversed an even number of times by $(\omega_1, \omega_2, \omega_3, \omega'_1, \omega'_2, \omega'_3)$. Therefore the number of vertices will be maximised when every edge (other than (p, q)) is traversed precisely twice, in which case $|V_{\omega_1,\omega_2,\omega_3,\omega'_1,\omega'_2,\omega'_3}| = 2n + 2m + 2l - 4$. However the two vertices p and q are fixed, so when obtaining the contribution inside the square root above by labelling the vertices we get $\mathbb{E}|R_{nml}(H)| = \sum_{p<q} \mathcal{O}(N^{-(n+m+l)})\sqrt{\mathcal{O}(N^{2n+2m+2l-4-2})} = \sum_{p<q} \mathcal{O}(N^{-3}) = \mathcal{O}(N^{-1})$.

5 Regular imaginary tournament ensemble

In a similar manner to the previous section we shall introduce a random walk within $\mathfrak{R}_N$, which in turn induces a random walk in the variables $Y_n(H)$. Obviously this must be different to that of ITE in the previous section, for if we simply change the sign of one element of H then we no longer have $\sum_q H_{pq} = 0$ for all p and therefore the new matrix $H' \notin \mathfrak{R}_N$. To remedy this situation we use a random walk that has already been investigated previously in the literature [28]. In order to describe this Markov process we first note that every regular tournament on N vertices contains directed cycles $\boldsymbol{q} = (q_0, q_1, q_2)$ of length 3, i.e. $H_{q_0q_1} = H_{q_1q_2} = H_{q_2q_0}$ (see e.g. Figure 2 (a)). We shall refer to such directed cycles as *triangles*, for which there are precisely

$$d_N = \frac{N(N-1)(N+1)}{4} \tag{5.1}$$

in every regular tournament. Note that we distinguish labelled triangles, so $(1, 2, 3, 1) \neq (2, 3, 1, 2)$ for example.

Proof of (5.1). Let us introduce the following indicator function

$$\Theta_{\boldsymbol{q}}(H) = \frac{1}{8}(1 - H_{q_0q_1}H_{q_1q_2})(1 - H_{q_1q_2}H_{q_2q_0})(1 - H_{q_2q_0}H_{q_0q_1})(1 - \delta_{q_0q_1}\delta_{q_1q_2}\delta_{q_2q_0}), \tag{5.2}$$

which satisfies

$$\Theta_{\boldsymbol{q}}(H) = \begin{cases} 1 & H_{q_0q_1} = H_{q_1q_2} = H_{q_2q_0} \text{ and } q_0 \neq q_1 \neq q_2 \neq q_0 \\ 0 & \text{otherwise} . \end{cases} \tag{5.3}$$

Summing over $\boldsymbol{q}$ and using that $H_{pq}H_{pq} = -1$ and $\sum_{r:r\neq p,q} H_{qr} = -H_{qp}$ gives

$$\begin{aligned} \sum_{\boldsymbol{q}} \Theta_{\boldsymbol{q}}(H) &= \frac{1}{8} \sum_{q_0\neq q_1\neq q_2\neq q_0} (1 - H_{q_0q_1}H_{q_1q_2})(1 - H_{q_1q_2}H_{q_2q_0})(1 - H_{q_2q_0}H_{q_0q_1}) \quad (5.4) \\ &= \frac{1}{8} \sum_{q_0\neq q_1\neq q_2\neq q_0} (2 - 2H_{q_0q_1}H_{q_1q_2} - 2H_{q_1q_2}H_{q_2q_0} - 2H_{q_2q_0}H_{q_0q_1}) \\ &= \frac{1}{4}\Bigg[\sum_{q_0\neq q_1\neq q_2\neq q_0} 1 + 3\sum_{p\neq q} H_{qp}H_{pq}\Bigg] = \frac{1}{4}\Big[N(N-1)(N-2) + 3N(N-1)\Big], \end{aligned}$$

which is equal to d_N. □

For functions invariant under cyclic permutations of these indices the following simplification occurs

Lemma 5.1. *Let $\Theta_{\boldsymbol{q}}(H)$ be as in (5.2) and $h(H,\boldsymbol{q})$ a function such that $h(H,q_0,q_1,q_2) = h(H,q_1,q_2,q_0) = h(H,q_2,q_0,q_1)$. Then*

$$\sum_{\boldsymbol{q}} \Theta_{\boldsymbol{q}}(H) h(H,\boldsymbol{q}) = \sum_{\boldsymbol{q}}{}' \frac{(1-3H_{q_0q_1}H_{q_1q_2})}{4} h(H,\boldsymbol{q}) \ , \tag{5.5}$$

where the prime in the sum denotes that $q_0 \neq q_1 \neq q_2 \neq q_0$.

Proof. Starting with the expression (5.2) for $\Theta_{\boldsymbol{q}}(H)$ we can remove the factor $(1 - \delta_{q_0q_1}\delta_{q_1q_2}\delta_{q_2q_0})$ provided we assume that $q_0 \neq q_1 \neq q_2 \neq q_0$. Therefore, expanding in the same way as (5.4)

$$\begin{aligned}
\sum_{\boldsymbol{q}} \Theta_{\boldsymbol{q}}(H) h(H,\boldsymbol{q}) &= \frac{1}{8} \sum_{\boldsymbol{q}}{}' (1 - H_{q_0q_1}H_{q_1q_2})(1 - H_{q_1q_2}H_{q_2q_0})(1 - H_{q_2q_0}H_{q_0q_1}) h(H,\boldsymbol{q}) \\
&= \frac{1}{4} \sum_{\boldsymbol{q}}{}' (1 - H_{q_0q_1}H_{q_1q_2} - H_{q_1q_2}H_{q_2q_0} - H_{q_2q_0}H_{q_0q_1}) h(H,\boldsymbol{q}) \\
&= \frac{1}{4} \sum_{\boldsymbol{q}}{}' (1 - 3H_{q_0q_1}H_{q_1q_2}) h(H,\boldsymbol{q}),
\end{aligned} \tag{5.6}$$

where in the last line we have cyclicly permuted the indices in q_0, q_1 and q_2. □

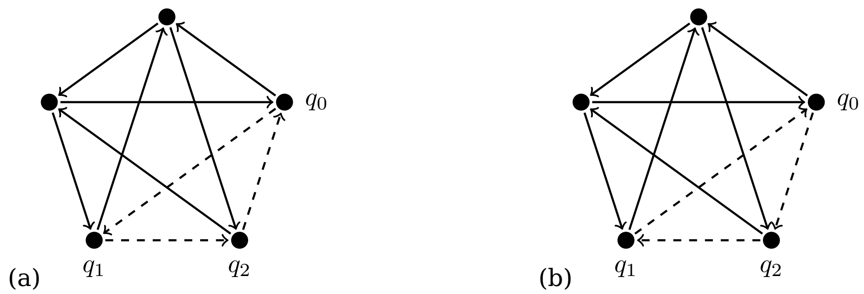

Figure 2: The Markov process consists of choosing uniformly at random one of the d_N triangles in the regular tournament graph (a) and then reversing the orientation in order to obtain (b). This preserves the number of incoming and outgoing edges to all vertices, or, in terms of the corresponding adjacency matrix, this preserves the condition $\sum_q A_{pq} = (N-1)/2$ for all p.

The random walk is performed by choosing one of these d_N triangles $\boldsymbol{q}$ uniformly at random and then reversing the orientation, i.e. $H_{q_0q_1}, H_{q_1q_2}, H_{q_2q_0} \to -H_{q_0q_1}, -H_{q_1q_2}, -H_{q_2q_0}$ (see Figure 2). This guarantees the new matrix $H' = H + \delta H^{\boldsymbol{q}}$ is contained in $\mathfrak{R}_N$ as it satisfies $\sum_q H'_{pq} = 0$ for all p. The difference matrix is given explicitly by

$$\delta H^{\boldsymbol{q}} := H' - H = \sum_{i=0}^{2} (-2H_{q_iq_{i+1(3)}})(\mathbf{e}_{q_i}\mathbf{e}^T_{q_{i+1(3)}} - \mathbf{e}_{q_{i+1(3)}}\mathbf{e}^T_{q_i}). \tag{5.7}$$

We may summarise this random walk in the following transition probability for $H, H' \in \mathfrak{R}_N$

$$\rho(H \to H') = \begin{cases} \frac{1}{d_N} & |H - H'|_{\mathfrak{R}_N} = 1 \\ 0 & |H - H'|_{\mathfrak{R}_N} \neq 1, \end{cases} \tag{5.8}$$

where $|H - H'|_{\mathfrak{R}_N} = \frac{1}{6}\sum_{p,q} |H_{pq} - H'_{pq}|$ is equal to 1 if and only if $H, H' \in \mathfrak{R}_N$ differ by the reversal of exactly one triangle. Starting at any tournament $H \in \mathbb{R}_N$, one may reach any other tournament $H' \in \mathbb{R}_N$ by performing successive reversals. Moreover, this Markov process is known to be mixing [28].

If $P_t(H)$ is the probability of the random walker to be at H at time t then

$$P_{t+1}(H') = \sum_{H \in \mathfrak{R}_N} \rho(H \to H') P_t(H) = \sum_{H \in \mathfrak{R}_N : |H - H'|_{\mathfrak{R}_N} = 1} \frac{1}{d_N} P_t(H).$$

Thus $P_t(H) = |\mathfrak{R}_N|^{-1}$ implies that $P_{t+1}(H') = |\mathfrak{R}_N|^{-1}$ also, i.e. H and H' are an exchangeable pair.

Using the indicator function (5.3) and Lemma 5.1 the expected change of some observable $f(H)$ under this random walk is therefore

$$\begin{aligned} \mathbb{E}[\delta f | H] &:= \sum_{H' \in \mathfrak{R}_N} \rho(H \to H')[f(H') - f(H)] \\ &= \sum_{\boldsymbol{q}} \frac{\Theta_{\boldsymbol{q}}(H)}{d_N} \delta f^{\boldsymbol{q}} = \sum_{\boldsymbol{q}}{}' \frac{(1 - 3H_{q_0 q_1} H_{q_1 q_2})}{4 d_N} \delta f^{\boldsymbol{q}}, \end{aligned} \tag{5.9}$$

where $\delta f^{\boldsymbol{q}} := f(H + \delta H^{\boldsymbol{q}}) - f(H)$. Similarly, higher moments are obtained by taking the expectation of products of changes, i.e. for $f_1(H), f_2(H), \ldots, f_k(H)$

$$\mathbb{E}[\delta f_1 \ldots \delta f_k | H] := \sum_{\boldsymbol{q}} \frac{(1 - 3H_{q_0 q_1} H_{q_1 q_2})}{4 d_N} \delta f_1^{\boldsymbol{q}} \ldots \delta f_k^{\boldsymbol{q}}. \tag{5.10}$$

Here we are again interested in the particular observables $Y_n(H)$ given in (2.3). Using Lemma 3.3 this can be expressed in terms of non-bactracking cycles as

$$Y_n(H) = \frac{1}{2} \frac{1}{(N-2)^n} \sum_{\omega \in \Omega_{2n}} H_\omega - \mathbb{E}[H_\omega] = \frac{1}{2} \frac{1}{(N-2)^n} \sum_{\omega \in \Lambda_{2n}} H_\omega - \mathbb{E}[H_\omega] \ ,$$

with Ω_{2n} and Λ_{2n} the same as in Section 4. Note, however, that in contrast to the analogous expression (4.5) for the ITE the expectation term is not identically zero. This is precisely due to the global correlations enforced by demanding the row sums of H are zero and will require the use of Lemma 5.3 below to evaluate.

The following proposition describes how the $Y_n(H)$ behave under the aforementioned random walk.

Proposition 5.2. *Let (H', H) be an exchangeable pair from the RITE connected via (5.8). Let $Y_n(H)$ be as defined in (2.3) with N sufficiently large. Then*

- *(a)* $\frac{d_N}{6N} \mathbb{E}[\delta Y_n | H] = -n Y_n(H) + R_n(H)$ *(Drift term)*
- *(b)* $\frac{d_N}{6N} \mathbb{E}[\delta Y_n \delta Y_m | H] = 2n^2 \delta_{nm} + R_{nm}(H)$ *(Diffusion term)*
- *(c)* $\frac{d_N}{6N} \mathbb{E}[|\delta Y_n \delta Y_m \delta Y_l| | H] = R_{nml}(H)$ *(Remainder term)*

with $\mathbb{E}|R_n(H)| = \mathcal{O}(N^{-\frac{1}{2}})$, $\mathbb{E}|R_{nm}(H)| = \mathcal{O}(N^{-1})$ and $\mathbb{E}|R_{nml}(H)| = \mathcal{O}(N^{-1})$ for all $n, m, l = 2, \ldots, k$.

Proof. Parts (a), (b) and (c) of Proposition 5.2 will be proved in Sections 5.1, 5.2 and 5.3 respectively. □

Before progressing we first outline some necessary requirements.

Lemma 5.3. *Let $\boldsymbol{p} = (p_0, \ldots, p_{v-1})$ be v distinct vertices and $E = \{(p_i, p_j)\}$ be a collection of k edges on these vertices. Let us write $H_E := \prod_{(p,q) \in E} H_{pq}$ for the product of matrix elements over these edges and $\mathbb{E}$ the expectation over the RITE. Then*

$$\mathbb{E}[H_E] = \mathcal{O}(N^{-\frac{k}{2}}). \tag{5.11}$$

Proof. See Appendix B. □

We stress the above lemma provides a key part in our subsequent analysis of the remainder terms in Proposition 5.2. Lemma 5.3 shows that whilst we do not have $\mathbb{E}[H_E] = \prod_{(p,q)\in E} \mathbb{E}[H_{pq}] = 0$, as in the Wigner case, the correlations for a fixed number of elements are sufficiently weak as to allow for convergence to universal behaviour in the large N limit.

From (5.7), the $e = (p', q')$-th element of the matrix $H' = H + \delta H^{\boldsymbol{q}}$ is given by

$$H'_e = H_e(-1)^{\chi_{e,\boldsymbol{q}}},$$

where $\chi_{e,\boldsymbol{q}} = \sum_{i=0}^{2}(\delta_{p'q_i}\delta_{q'q_{i+1}} + \delta_{p'q_i}\delta_{q'q_{i+1}})$ is an indicator function equal to 1 if the edge $e = (p', q')$ corresponds to one of the (undirected) edges $\{(q_0, q_1), (q_1, q_0), (q_1, q_2)\}$ and 0 otherwise. The change in H_ω for the non-backtracking cycle $\omega = (p_0, p_1, \dots p_{2n-1}, p_0)$ is therefore

$$\delta H^{\boldsymbol{q}}_\omega := H'_\omega - H_\omega = H_\omega \left[\prod_{i=0}^{2n-1}(-1)^{\chi_{p_i p_{i+1(2n)},\boldsymbol{q}}} - 1\right] = -2H_\omega \phi_{\omega,\boldsymbol{q}}, \tag{5.12}$$

where we can also write

$$\phi_{\omega,\boldsymbol{q}} = \frac{1}{2}(1 - (-1)^{\chi_{\omega,\boldsymbol{q}}}), \qquad \chi_{\omega,\boldsymbol{q}} = \sum_{i=0}^{2n-1} \chi_{p_i p_{i+1(2n)},\boldsymbol{q}}.$$

Thus $\phi_{\omega,\boldsymbol{q}}$ is an indicator function equal to 1 if the edges in the triangle $\boldsymbol{q}$ are traversed an odd number of times by ω and 0 otherwise. The corresponding change in $Y_n(H)$ is

$$\delta Y^{\boldsymbol{q}}_n := Y_n(H+\delta H^{\boldsymbol{q}}) - Y_n(H) = \frac{1}{2}\frac{1}{(N-2)^n}\sum_{\omega\in\Lambda_{2n}} \delta H^{\boldsymbol{q}}_\omega = -\frac{1}{(N-2)^n}\sum_{\omega\in\Lambda_{2n}} H_\omega \phi_{\omega,\boldsymbol{q}}. \tag{5.13}$$

5.1 Proof of Proposition 5.2 Part (a) - Drift term

Inserting the expression (5.13) for $\delta Y^{\boldsymbol{q}}_n$ into (5.9) gives

$$\mathbb{E}[\delta Y_n|H] = -\frac{1}{2}\frac{1}{(N-2)^n}\frac{1}{2d_N}\sum_{\omega\in\Lambda_{2n}}\sideset{}{'}\sum_{\boldsymbol{q}}(1 - 3H_{q_0q_1}H_{q_1q_2})H_\omega\phi_{\omega,\boldsymbol{q}}. \tag{5.14}$$

Therefore we may write

$$\mathbb{E}[\delta Y_n|H] = \frac{6N}{d_N}[-nY_n(H) + R_n(H)],$$

with the remainder given by

$$R_n(H) = \frac{1}{2}\frac{n}{(N-2)^n}\sum_{\omega\in\Lambda_{2n}}\left[H_\omega\left(1 - \frac{1}{12nN}\sideset{}{'}\sum_{\boldsymbol{q}}\phi_{\omega,\boldsymbol{q}}\right)\right.$$
$$\left. + \left(\frac{1}{4nN}\sideset{}{'}\sum_{\boldsymbol{q}}\phi_{\omega,\boldsymbol{q}}H_{q_0q_1}H_{q_1q_2}H_\omega - \mathbb{E}[H_\omega]\right)\right]. \tag{5.15}$$

Now, crucially, by splitting the sum over Λ_{2n} into $\Lambda^\star_{2n} = \{\omega \in \Lambda_{2n} : |F_\omega| = 2n\}$ (recalling that F_ω are the 'free' edges defined in Section 3) and $\Lambda^\circ_{2n} = \Lambda_{2n} \setminus \Lambda^\star_{2n}$ (see Section 4) the constant expectation term in the above can be expressed in the following alternative manner, which each step subsequently explained,

$$\sum_{\omega\in\Lambda_{2n}}\mathbb{E}[H_\omega]=\sum_{\omega\in\Lambda_{2n}^\star}\mathbb{E}[H_\omega]+\sum_{\omega\in\Lambda_{2n}^\circ}\mathbb{E}[H_\omega] \tag{5.16}$$

$$=\frac{1}{12nN}\sum_{\omega\in\Lambda_{2n}^\star}\sum_{\boldsymbol{q}}{}^{'}\phi_{\omega,\boldsymbol{q}}\mathbb{E}[H_\omega]+\mathcal{O}(N^{n-1}) \tag{5.17}$$

$$=\frac{1}{12nN}\sum_{\omega\in\Lambda_{2n}}\sum_{\boldsymbol{q}}{}^{'}\phi_{\omega,\boldsymbol{q}}\mathbb{E}[H_\omega]+\mathcal{O}(N^{n-1}) \tag{5.18}$$

$$=\frac{1}{4nN}\sum_{\omega\in\Lambda_{2n}}\sum_{\boldsymbol{q}}{}^{'}\phi_{\omega,\boldsymbol{q}}\mathbb{E}[H_{q_0q_1}H_{q_1q_2}H_\omega]+\mathcal{O}(N^{n-1}). \tag{5.19}$$

- (5.16) to (5.17) - First note that Lemma 5.3 implies that $\sum_{\omega\in\Lambda_{2n}^\circ}\mathbb{E}[H_\omega]=\mathcal{O}(N^\Psi)$, where $\Psi:=\max_{\omega\in\Lambda_{2n}^\circ}\{|V_\omega|-|F_\omega|/2\}$, with the contribution of $\mathcal{O}(N^{|V_\omega|})$ coming from the number of possibilities of labelling the vertices in ω. Let us consider those ω in which every edge is traversed at most twice (all other cycles will give a negligible contribution in comparison) and form the graph $\hat{G}=(V_\omega,E_\omega)$. Since ω is a cycle the graph $\hat{G}$ is connected and satisfies $2|E_\omega|-|F_\omega|=2n$, with the first Betti number $\beta(\hat{G})=|E_\omega|-|V_\omega|+1$. Thus $|V_\omega|-|F_\omega|/2=n+1-\beta(\hat{G})$. Now $\beta(\hat{G})>0$, otherwise the ω would be backtracking. In addition, suppose $\beta(\hat{G})=1$, then $\hat{G}$ must be a loop (there can be no dangling edges since ω is non-backtracking), however this is only possible for walks ω in which $|F_\omega|=2n$ or $|F_\omega|=0$, which means $\omega\notin\Lambda_{2n}^\circ$. Hence $\beta(\hat{G})\geq 2$ and therefore $|V_\omega|-|F_\omega|/2\leq n-1$, meaning the second term in (5.16) is of order $\mathcal{O}(N^{n-1})$.

  Now, in addition, for all $\omega\in\Lambda_{2n}^\star$ we have

  $$\sum_{\boldsymbol{q}}{}^{'}\phi_{\omega,\boldsymbol{q}}=12nN+\mathcal{O}(1)\ , \tag{5.20}$$

  which comes from counting all triangles $\boldsymbol{q}$ that share a single edge with ω: If we fix, for instance, $(q_0,q_1)=(p_0,p_1)$ (the first edge in ω) then there are $N+\mathcal{O}(1)$ possible values for q_2 for which $\phi_{\omega,\boldsymbol{q}}=1$. Noting there are 6 possible orientations of $\boldsymbol{q}$ for each edge of ω and $2n$ edges gives (5.20).

  Then, for all $\omega\in\Lambda_{2n}^\star$ we have $|F_\omega|=2n$ and $|V_\omega|\leq 2n$, so $|V_\omega|-|F_\omega|/2\leq n$ and thus $\sum_{\omega\in\Lambda_{2n}^\star}\mathbb{E}[H_\omega]=\mathcal{O}(N^n)$. Combining this with (5.20) leads to (5.17).
- (5.17) to (5.18) - By following the same reasoning as in (5.20), we have that $\frac{1}{12nN}\sum_{\boldsymbol{q}}'\phi_{\omega,\boldsymbol{q}}=\mathcal{O}(1)$ for all $\omega\in\Lambda_{2n}^\circ$. Therefore, since $\sum_{\omega\in\Lambda_{2n}^\circ}\mathbb{E}[H_\omega]=\mathcal{O}(N^{n-1})$ we can extend the sum from $\Lambda_{2n}^\star$ to Λ_{2n}.
- (5.18) to (5.19) - Since $\mathbb{E}[\delta Y_n]=\mathbb{E}[\mathbb{E}[\delta Y_n|H]]=0$, taking the expectation in (5.14) gives (5.19)

Therefore, inserting (5.19) into the expression (5.15) leads to

$$R_n(H)=\frac{1}{2}\frac{n}{(N-2)^n}[S_n^{(1)}(H)+S_n^{(2)}(H)]+\mathcal{O}(N^{-1}), \tag{5.21}$$

where

$$S_n^{(1)}(H)=\sum_{\omega\in\Lambda_{2n}}H_\omega\left(1-\frac{1}{12nN}\sum_{\boldsymbol{q}}{}^{'}\phi_{\omega,\boldsymbol{q}}\right) \tag{5.22}$$

and

$$S_n^{(2)}(H) = \frac{1}{4nN} \sum_{\omega\in\Lambda_{2n}} \sum_{\boldsymbol{q}}' \phi_{\omega,\boldsymbol{q}} \Big(H_{q_0q_1}H_{q_1q_2}H_\omega - \mathbb{E}[H_{q_0q_1}H_{q_1q_2}H_\omega] \Big). \tag{5.23}$$

Part (a) of Proposition 5.2 thus follows immediately from the triangle equality and the following lemma

Lemma 5.4. *Let $S_n^{(1)}(H)$ and $S_n^{(2)}(H)$ be as defined in (5.22) and (5.23) respectively and $\mathbb{E}$ denote the expectation over the RITE. Then*

- *(a)* $\mathbb{E}|S^{(1)}(H)| = \mathcal{O}(N^{n-1})$
- *(b)* $\mathbb{E}|S^{(2)}(H)| = \mathcal{O}(N^{n-1/2})$.

Proof of Lemma 5.4 Part (a). Firstly, for notational convenience, let us write $\kappa_\omega := (1 - \frac{1}{12nN}\sum_{\boldsymbol{q}}' \phi_{\omega,\boldsymbol{q}}) \geq 0$ and $\kappa_{[\omega]}$ to denote the value of all $\omega \in [\omega]$. Splitting the sum over Λ_{2n} in (5.22) into $\Lambda_{2n}^\star$ and Λ_{2n}° leads to

$$\begin{aligned}
\mathbb{E}|S_n^{(1)}(H)| &\leq \sum_{[\omega]\in[\Lambda_{2n}^\star]} \kappa_{[\omega]} \mathbb{E}\Big|\sum_{\omega\in[\omega]} H_\omega\Big| + \sum_{[\omega]\in[\Lambda_{2n}^\circ]} \kappa_{[\omega]} \mathbb{E}\Big|\sum_{\omega\in[\omega]} H_\omega\Big| \\
&\leq \sum_{[\omega]\in[\Lambda_{2n}^\star]} \kappa_{[\omega]} \sqrt{\sum_{\omega,\omega'\in[\omega]} \mathbb{E}[H_{\omega,\omega'}]} + \sum_{[\omega]\in[\Lambda_{2n}^\circ]} \kappa_{[\omega]} \sqrt{\sum_{\omega,\omega'\in[\omega]} \mathbb{E}[H_{\omega,\omega'}]},
\end{aligned} \tag{5.24}$$

For a particular equivalence class $[\omega]$, if $\Psi_{[\omega]} = \max_{\omega,\omega'\in[\omega]}\{|V_{\omega,\omega'}| - |F_{\omega,\omega'}|/2\}$ then, using Lemma 5.3, the quantity $\sum_{\omega,\omega'\in[\omega]} \mathbb{E}[H_{\omega,\omega'}]$ is of order $\mathcal{O}(N^{\Psi_{[\omega]}})$.

For $\omega \in \Lambda_{2n}^\star$ we have $|F_\omega| = 2n$, meaning the graph $G = (V_\omega, F_\omega)$ will have one connected component and $\beta(G) \geq 1$. Therefore, by Corollary 3.2, we find that for $\omega \sim \omega' \in [\Lambda_{2n}^\star]$, $|V_{\omega,\omega'}| - |F_{\omega,\omega'}|/2 \leq |V_\omega| \leq 2n$. In addition (5.20) implies that $\kappa_{[\omega]} = \mathcal{O}(N^{-1})$. Hence the first term in (5.24) is of order $\mathcal{O}(N^{-1})\sqrt{\mathcal{O}(N^{2n})} = \mathcal{O}(N^{n-1})$.

For $\omega \in \Lambda_{2n}^\circ$ we have $0 < |F_\omega| < 2n$, which implies the graph $G = (V_\omega, F_\omega)$ will have multiple connected components, which we can label $i = 1, \ldots, C$. However, since ω is a cycle, those components satisfying $\beta_i = 0$ must be isolated vertices. Let us suppose that all edges in ω are traversed a maximum of twice (more than twice will give lower order contributions), then the graph $\hat{G} = (V_\omega, E_\omega) \supseteq G$ must be connected and satisfy $2|E_\omega| - |F_\omega| = 2n$. Now, if $|V_I| = \sum_i \delta_{\beta_i,0}$ counts the number of isolated vertices then we must have $|V_I| \leq |E_\omega| - |F_\omega| - 1$, since for $\omega \in \Lambda_{2n}^\circ$ the number of edges traversed twice (given by $|E_\omega| - |F_\omega|$) must be at least one more than the number of isolated vertices. Hence, $|V_I| \leq 2n - |E_\omega| - 1 = 2n - |V_\omega| - \beta(\hat{G})$. Thus using Corollary 3.2, we have $|V_{\omega,\omega'}| - |F_{\omega,\omega'}|/2 \leq |V_\omega| + |V_I| \leq 2n - \beta(\hat{G}) \leq 2n - 2$ because $\beta(\hat{G}) \geq 2$ for all $\omega \in \Lambda_{2n}^\circ$. In addition $\kappa_\omega = O(1)$ for $\omega \in \Lambda_{2n}^\circ$ so the second term in (5.24) is of order $\mathcal{O}(1)\sqrt{\mathcal{O}(N^{2n-2})} = \mathcal{O}(N^{n-1})$. □

Proof of Lemma 5.4 Part (b). Let us define the following sets of walks

$$A_r = \{(p_0, \ldots, p_{r-1}, p_0, q, p_{r-1}) : p_i \text{ distinct}, q \neq p_0, p_1, p_{r-2}, p_{r-1}\} \tag{5.25}$$
$$B_r = \{(p_0, \ldots, p_{r-1}, p_1) : p_i \text{ distinct}\} \tag{5.26}$$
$$C_r = \{(p_0, \ldots, p_{r-1}) : p_i \text{ distinct}\} \tag{5.27}$$
$$D_r = \{(p_0, \ldots, p_{r-1}, q) : p_i \text{ distinct}, q = p_0, \ldots, p_{r-3}\}. \tag{5.28}$$

These lead to the following proposition, which shall be proved later.

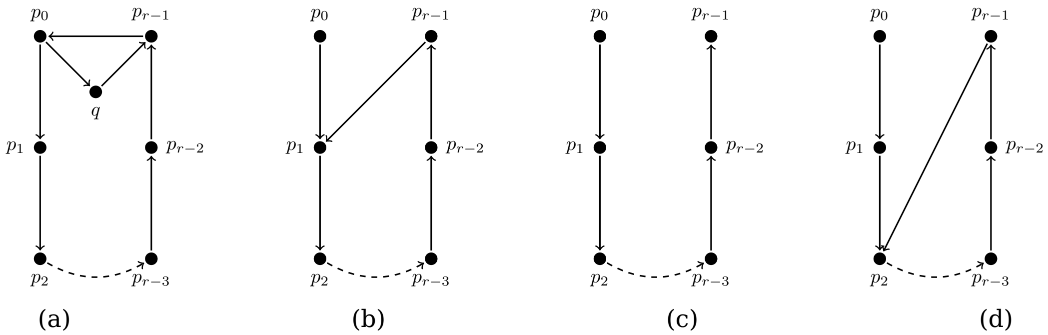

Figure 3: Examples of walks in (a) A_r, where q may also be equal to $p_2, p_3, \dots, p_{r-3}$, (b) B_r, (c) C_r and (d) D_r, where the last vertex in the walk may be any of $p_0, \dots, p_{r-3}$.

Proposition 5.5. *Define $\Lambda_{2n}^{\dagger} := \{\omega \in \Lambda_{2n} : |V_\omega| = 2n\}$ and $\Lambda_{2n}^{\times} = \Lambda_{2n} \setminus \Lambda_{2n}^{\dagger}$. Then, splitting the sum over Λ_{2n} in (5.23) leads to*

$$S_n^{(2)}(H) = \frac{1}{4nN}\Bigg[2n \sum_{\omega \in A_{2n}} H_\omega - 4n \sum_{\omega \in \Lambda_{2n}^{\dagger}} H_\omega - 4n \sum_{\omega \in B_{2n}} H_\omega + 4n \sum_{j=1}^{n-2} \mathcal{O}(N^j) \sum_{\omega \in D_{2n-2j}} H_\omega + \sum_{\omega \in \Lambda_{2n}^{\times}} \sideset{}{'}\sum_{\boldsymbol{q}} \phi_{\omega,\boldsymbol{q}} H_{q_0 q_1} H_{q_1 q_2} H_\omega \Bigg] + \mathcal{O}(N^{n-1}). \quad (5.29)$$

Given Proposition 5.5 we have by the triangle inequality

$$\mathbb{E}|S_n^{(2)}(H)| \le \mathcal{O}(N^{-1}) \Bigg[\mathbb{E}\Big|\sum_{\omega \in A_{2n}} H_\omega\Big| + \mathbb{E}\Big|\sum_{\omega \in \Lambda_{2n}^{\dagger}} H_\omega\Big| + \mathbb{E}\Big|\sum_{\omega \in B_{2n}} H_\omega\Big| + \sum_{j=1}^{n-2} \mathcal{O}(N^j)\mathbb{E}\Big|\sum_{\omega \in D_{2n-2j}} H_\omega\Big| + \mathbb{E}\Big|\sum_{\omega \in \Lambda_{2n}^{\times}} \sideset{}{'}\sum_{\boldsymbol{q}} \phi_{\omega,\boldsymbol{q}} H_{q_0 q_1} H_{q_1 q_2} H_\omega\Big|\Bigg] + \mathcal{O}(N^{n-1}). \quad (5.30)$$

The result of Lemma 5.4 Part (a) is then obtained by showing all the terms within the square brackets are at most $\mathcal{O}(N^{n+\frac{1}{2}})$.

We start with walks $\omega \in A_{2n}$. As before, we note that

$$\mathbb{E}\Big|\sum_{\omega \in A_{2n}} H_\omega\Big| \le \sum_{[\omega] \in [A_{2n}]} \sqrt{\sum_{\omega, \omega' \in [\omega]} \mathbb{E}[H_{\omega,\omega'}]}. \quad (5.31)$$

From Corollary 3.2 we have, for all $\omega \sim \omega' \in A_{2n}$, that $|V_{\omega,\omega'}| - |F_{\omega,\omega'}|/2 \le |V_\omega| \le 2n+1$, giving a contribution of order $\sqrt{\mathcal{O}(N^{2n+1})} = \mathcal{O}(N^{n+1/2})$. Similarly, taking the same inequality for walks $\omega \in D_{2n-2j}$ and noting that $|V_{\omega,\omega'}| - |F_{\omega,\omega'}|/2 \le |V_\omega| = 2n-2j$ for all $\omega \sim \omega' \in D_{2n-2j}$ leads to a contribution $\mathbb{E}|\sum_{\omega \in D_{2n-2j}} H_\omega| = \mathcal{O}(N^{n-j})$. Finally the inclusions $B_{2n}, \Lambda_{2n}^{\dagger} \subset D_{2n}$, immediately imply the respective terms in (5.30) are $O(N^n)$.

It thus remains to estimate the term involving walks in $\Lambda_{2n}^{\times}$. Let us define $\tilde{\Lambda}_{2n}^{\times} := \{(\boldsymbol{q}, \omega) : q_0 \ne q_1 \ne q_2 \ne q_0, \omega \in \Lambda_{2n}^{\times}, \phi_{\omega,\boldsymbol{q}} = 1\}$ as those combinations of ω and $\boldsymbol{q}$ with an odd number of intersecting edges and $H_{\boldsymbol{q},\omega} := H_{q_0q_1}H_{q_1q_2}H_\omega$. Examples of $(\boldsymbol{q},\omega)$ in $\tilde{\Lambda}_{2n}^{\times}$ are given in Figure 4. Using this notation the final term in (5.30) satisfies

$$\mathbb{E}\Big|\sum_{\omega \in \Lambda_{2n}^{\times}} \sideset{}{'}\sum_{\boldsymbol{q}} \phi_{\omega,\boldsymbol{q}} H_{q_0 q_1} H_{q_1 q_2} H_\omega\Big| \le \sum_{[\boldsymbol{q},\omega] \in [\tilde{\Lambda}_{2n}^{\times}]} \sqrt{\sum_{\substack{(\boldsymbol{q},\omega),(\boldsymbol{q}',\omega') \\ \in [\boldsymbol{q},\omega]}} \mathbb{E}[H_{\boldsymbol{q},\omega} H_{\boldsymbol{q}',\omega'}]}. \quad (5.32)$$

If we write $V_{\boldsymbol{q}} = \{q_0, q_1, q_2\}$ and $E_{\boldsymbol{q}} = \{(q_0, q_1), (q_1, q_2)\}$ then $V_{\boldsymbol{q},\omega}$, $E_{\boldsymbol{q},\omega}$ and $F_{\boldsymbol{q},\omega}$ are defined in the usual manner. Let us define the graph $\hat{G} = (V_{\boldsymbol{q},\omega}, E_{\boldsymbol{q},\omega})$ and assume that edges in $E_{\boldsymbol{q},\omega}$ are traversed a maximum of twice by $(\boldsymbol{q}, \omega)$, meaning $2|E_{\boldsymbol{q},\omega}| - |F_{\boldsymbol{q},\omega}| = 2n+2$. Note that $\hat{G}$ must be connected. Let V_I be the set of isolated vertices in $G = (V_{\boldsymbol{q},\omega}, F_{\boldsymbol{q},\omega})$. For example, in Figure 4 (a) $V_I = \{p_{2n-1}\}$ since edges $(p_{2n-1}, p_0), (p_1, p_{2n-1}) \notin F_{\boldsymbol{q},\omega}$, whereas there are no isolated vertices in Figure 4 (b) and (c).

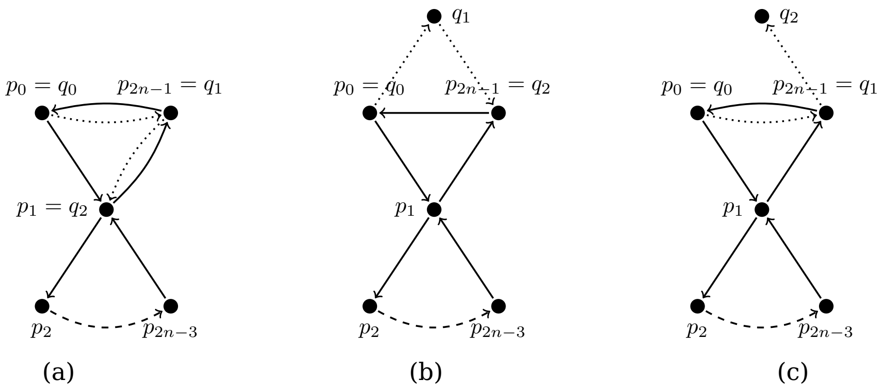

Figure 4: Examples of pairs of walks and triangles $(\boldsymbol{q}, \omega) \in \tilde{\Lambda}_{2n}^{\times}$. Here $\boldsymbol{q}$ is given by the triangle $q_0 \to q_1 \to q_2 \to q_0$ (the edges $E_{\boldsymbol{q}}$ are highlighted by dotted lines) and ω is given by solid lines. In (a) there are 3 edges of $\boldsymbol{q}$ overlapping ω, whereas in (b) and (c) only 1 edge of $\boldsymbol{q}$ overlaps with ω.

The condition $\phi_{\omega,\boldsymbol{q}} = 1$ implies that ω and $\boldsymbol{q}$ must share an odd number of edges, i.e. $|E_\omega \cap \{(q_0, q_1), (q_1, q_2), (q_2, q_0)\}| = 1 \mod (2)$. This leads to two scenarios: Either $E_\omega \cap E_{\boldsymbol{q}} \neq \emptyset$ (see e.g Figure 4 (a) and (c)) or $E_\omega \cap E_{\boldsymbol{q}} = \emptyset$ (see e.g. Figure 4 (b)).

In the first scenario, since at least one edge in $E_{\boldsymbol{q},\omega}$ must be traversed twice, the number of isolated vertices satisfies $|V_I| \leq |E_{\boldsymbol{q},\omega}| - |F_{\boldsymbol{q},\omega}| - 1$. Therefore, $|V_I| \leq 2n + 2 - |E_{\boldsymbol{q},\omega}| - 1 = 2n + 2 - |V_{\boldsymbol{q},\omega}| - \beta(\hat{G})$. By Corollary 3.2 we therefore have $|V_{\boldsymbol{q},\omega,\boldsymbol{q}',\omega'}| - |F_{\boldsymbol{q},\omega,\boldsymbol{q}',\omega'}|/2 \leq |V_{\boldsymbol{q},\omega}| + |V_I| \leq 2n + 2 - \beta(\hat{G}) \leq 2n$, since $\omega \in \Lambda_{2n}^{\times}$ implies $\beta(\hat{G}) \geq 2$.

In the second scenario one may construct instances (Figure 4 (b) for example) in which all edges are free, i.e. $F_{\boldsymbol{q},\omega} = E_{\boldsymbol{q},\omega}$, which would in turn imply that there are no isolated vertices. Hence, in all situations we find the number of isolated vertices satisfies $|V_I| \leq |E_{\boldsymbol{q},\omega}| - |F_{\boldsymbol{q},\omega}| = 2n + 2 - |V_{\boldsymbol{q},\omega}| - \beta(\hat{G})$, which, via Corollary 3.2, leads to the inequality $|V_{\boldsymbol{q},\omega,\boldsymbol{q}',\omega'}| - |F_{\boldsymbol{q},\omega,\boldsymbol{q}',\omega'}|/2 \leq 2n + 3 - \beta(\hat{G})$. However, in contrast to the first scenario, taking the union $E_{\boldsymbol{q}} \cap E_\omega$ leads to an additional fundamental cycle, i.e. $\beta(\hat{G}) \geq 3$, implying that $|V_{\boldsymbol{q},\omega,\boldsymbol{q}',\omega'}| - |F_{\boldsymbol{q},\omega,\boldsymbol{q}',\omega'}|/2 \leq 2n$ once again. Thus (5.32) and, in turn, the final term within the square brackets in (5.30) is of order $\mathcal{O}(N^n)$. □

Proof of Proposition 5.5. The idea will be to split the sum over Λ_{2n} into those sets $\Lambda_{2n}^{\dagger}$ and $\Lambda_{2n}^{\times}$, then show the sum over $\Lambda_{2n}^{\dagger}$ can be re-expressed in the form of the first four terms in (5.29), up to a correction of $\mathcal{O}(N^{-1})$. Let us start with the first term in (5.23), since $|V_\omega| = 2n$ for all $\omega \in \Lambda_{2n}^{\dagger}$, the condition $\phi_{\omega,\boldsymbol{q}} = 1$ means that $\boldsymbol{q}$ and ω must share a single edge (i.e. they cannot share three). Therefore writing out explicitly all those terms in which the edge $(p_i, p_{i+1(2n)}) = (q_j, q_{j+1(3)})$ or $(q_{j+1(3)}, q_j)$ for $i = 0, \ldots, 2n-1$,

$j = 0, 1, 2$ gives

$$\sum_{\omega \in \Lambda_{2n}^{\dagger}} \sum_{\boldsymbol{q}}{}' \phi_{\omega,\boldsymbol{q}} H_{q_0 q_1} H_{q_1 q_2} H_\omega = \sum_{p_0,\dots,p_{2n-1}}{}' H_\omega \sum_{i=0}^{2n-1} \sum_{q \notin \mathcal{P}_i} (H_{p_i q} H_{q p_{i+1(2n)}} + H_{p_{i+1(2n)} q} H_{q p_i})$$
$$+ \sum_{p_0,\dots,p_{2n-1}}{}' H_\omega \sum_{i=0}^{2n-1} \sum_{q \notin \mathcal{P}_i} (H_{p_i p_{i+1(2n)}} H_{p_{i+1(2n)} q} + H_{p_{i+1(2n)} p_i} H_{p_i q}$$
$$+ H_{q p_i} H_{p_i p_{i+1(2n)}} + H_{q p_{i+1(2n)}} H_{p_{i+1(2n)} p_i})$$
$$= 4n \sum_{\omega \in A_{2n}} H_\omega + 8n \sum_{p_0,\dots,p_{2n-1}}{}' \sum_{q \notin \mathcal{P}_{2n-1}} H_{p_0 p_1} \dots H_{p_{2n-2} p_{2n-1}} H_{p_{2n-1} q}, \quad (5.33)$$

where $\mathcal{P}_i = \{p_{i-1(2n)}, p_i, p_{i+1(2n)}, p_{i+2(2n)}\}$, A_{2n} is defined in (5.25) and the primed summation denotes that all p_i are distinct. The second term in (5.33) may be further modified by using the regularity of H.

$$\sum_{p_0,\dots,p_{2n-1}}{}' \sum_{q \notin \mathcal{P}_{2n-1}} H_{p_0 p_1} \dots H_{p_{2n-2} p_{2n-1}} H_{p_{2n-1} q}$$
$$= - \sum_{p_0,\dots,p_{2n-1}}{}' H_{p_0 p_1} \dots H_{p_{2n-2} p_{2n-1}} (H_{p_{2n-1} p_0} + H_{p_{2n-1} p_1} + H_{p_{2n-1} p_{2n-2}})$$
$$= - \sum_{\omega \in \Lambda_{2n}^{\dagger}} H_\omega - \sum_{\omega \in B_{2n}} H_\omega - (N - (2n-1)) \sum_{\omega \in C_{2n-1}} H_\omega, \quad (5.34)$$

with B_{2n} and C_{2n-1} given in (5.25) and (5.26) respectively. The summation of walks in C_{2n-1} is obtained by noting that we have $H_{p_{2n-2} p_{2n-1}} H_{p_{2n-1} p_{2n-2}} = 1$ and summing over the free variable $p_{2n-1} \neq p_0, \dots, p_{2n-2}$. If $n = 2$ the summation in the final term is precisely

$$\sum_{\omega \in C_3} H_\omega = \sum_{p_0,p_1}{}' \sum_{p_2 \neq p_0,p_1} H_{p_0 p_1} H_{p_1 p_2} = - \sum_{p_0,p_1}{}' H_{p_0 p_1} H_{p_1 p_0} = -N(N-1). \quad (5.35)$$

If $n > 2$ then we apply the regularity condition to the index p_{2n-2} in order to obtain

$$\sum_{\omega \in C_{2n-1}} H_\omega := \sum_{p_0,\dots,p_{2n-3}}{}' \sum_{p_{2n-2} \neq p_0,\dots,p_{2n-3}} H_{p_0 p_1} \dots H_{p_{2n-4} p_{2n-3}} H_{p_{2n-3} p_{2n-2}}$$
$$= - \sum_{p_0,\dots,p_{2n-3}}{}' \sum_{q = p_0,\dots,p_{2n-5}} H_{p_0 p_1} \dots H_{p_{2n-3} q} - \sum_{p_0,\dots,p_{2n-3}}{}' H_{p_0 p_1} \dots H_{p_{2n-4} p_{2n-3}} H_{p_{2n-3} p_{2n-4}}$$
$$= - \sum_{\omega \in D_{2n-2}} H_\omega - (N - (2n-3)) \sum_{\omega \in C_{2n-3}} H_\omega, \quad (5.36)$$

with D_{2n-2} given in (5.28). Therefore, by recursion we establish that for all $n \geq 2$

$$(N - (2n-1)) \sum_{\omega \in C_{2n-1}} H_\omega = \sum_{j=1}^{n-2} (-1)^j \left(\prod_{r=0}^{j-1} (N - (2n-1) + 2r) \right) \sum_{\omega \in D_{2n-2j}} H_\omega + K_{n,N} \quad (5.37)$$

with the first term equal to 0 if $n = 2$ and

$$K_{n,N} = (-1)^n \left(\prod_{r=0}^{n-2} (N - (2n-1) + 2r) \right) \sum_{\omega \in C_3} H_\omega$$
$$= (-1)^{n+1} N(N-1) \left(\prod_{r=0}^{n-2} (N - (2n-1) + 2r) \right) \quad (5.38)$$

is a constant independent of H. Therefore, inserting (5.37) into (5.34) and then (5.34) into (5.33) leads to the following expression

$$\sum_{\omega \in \Lambda_{2n}^{\dagger}} \sum_{\boldsymbol{q}}{}' \phi_{\omega,\boldsymbol{q}} H_{q_0 q_1} H_{q_1 q_2} H_\omega$$
$$= 4n \sum_{\omega \in A_{2n}} H_\omega - 8n \sum_{\omega \in \Lambda_{2n}^{\dagger}} H_\omega - 8n \sum_{\omega \in B_{2n}} H_\omega + \sum_{j=1}^{n-2} \mathcal{O}(N^j) \sum_{\omega \in D_{2n-2j}} H_\omega - \tilde{K}_{n,N}. \quad (5.39)$$

The constant $\tilde{K}_{n,N} = 8nK_{n,N}$ is of order $\mathcal{O}(N^{n+1})$, which would lead to a larger result in Lemma 5.4 Part (b), however by subtracting the expectation of the same quantity, as in (5.23), this leading order is removed. Hence, splitting the sum in (5.23) into walks over $\Lambda_{2n}^{\dagger}$ and $\Lambda_{2n}^{\times}$ and inserting (5.39) gives

$$S_2(H) = \frac{1}{N} \Bigg[\sum_{\omega \in A_{2n}} H_\omega - 2 \sum_{\omega \in \Lambda_{2n}^{\dagger}} H_\omega - 2 \sum_{\omega \in B_{2n}} H_\omega + \sum_{j=1}^{n-2} \mathcal{O}(N^j) \sum_{\omega \in D_{2n-2j}} H_\omega$$
$$- \sum_{\omega \in A_{2n}} \mathbb{E}[H_\omega] + 2 \sum_{\omega \in \Lambda_{2n}^{\dagger}} \mathbb{E}[H_\omega] + 2 \sum_{\omega \in B_{2n}} \mathbb{E}[H_\omega] - \sum_{j=1}^{n-2} \mathcal{O}(N^j) \sum_{\omega \in D_{2n-2j}} \mathbb{E}[H_\omega]$$
$$+ \sum_{\omega \in \Lambda_{2n}^{\times}} \sum_{\boldsymbol{q}}{}' \phi_{\omega,\boldsymbol{q}} H_{q_0 q_1} H_{q_1 q_2} H_\omega - \sum_{\omega \in \Lambda_{2n}^{\times}} \sum_{\boldsymbol{q}}{}' \phi_{\omega,\boldsymbol{q}} \mathbb{E}[H_{q_0 q_1} H_{q_1 q_2} H_\omega] \Bigg]. \quad (5.40)$$

The result is therefore obtained once we show all terms involving expectations are at most $\mathcal{O}(N^n)$. We start with D_{2n-2j}. In this case each walk ω has $|V_\omega| = |F_\omega| = 2n - 2j$ and therefore $|V_\omega| - |F_\omega|/2 = n - j$. Hence

$$\sum_{\omega \in D_{2n-2j}} \mathbb{E}[H_\omega] = \mathcal{O}(N^{n-j}).$$

The same holds for B_{2n} and $\Lambda_{2n}^{\dagger}$ since they are both contained in D_{2n}. For walks in A_{2n} we have $|V_\omega| \leq 2n + 1$ and $|F_\omega| = 2n + 2$, giving $|V_\omega| - |F_\omega|/2 \leq n$ and so the same result follows.

In the final term the walk ω must share at least one edge with $\boldsymbol{q}$ otherwise $\phi_{\omega,\boldsymbol{q}} = 0$. As in the proof of Lemma 5.4 Part (b) let us take those $(\boldsymbol{q}, \omega) \in \tilde{\Lambda}_{2n}^{\times}$ such that every edge is traversed at most twice (so $2|E_{\boldsymbol{q},\omega}| - |F_{\boldsymbol{q},\omega}| = 2n+2$) and define the graph $\hat{G} = (V_{\boldsymbol{q},\omega}, E_{\boldsymbol{q},\omega})$, which must be connected. Then $|V_{\boldsymbol{q},\omega}| - |F_{\boldsymbol{q},\omega}|/2 = |V_{\boldsymbol{q},\omega}| - |E_{\boldsymbol{q},\omega}| + n + 1 = n + 2 - \beta(\hat{G}) \leq n$, since $\beta(\hat{G}) \geq 2$ for all $(\boldsymbol{q}, \omega) \in \tilde{\Lambda}_{2n}^{\times}$. Therefore

$$\sum_{\omega \in \Lambda_{2n}^{\times}} \sum_{\boldsymbol{q}}{}' \phi_{\omega,\boldsymbol{q}} \mathbb{E}[H_{q_0 q_1} H_{q_1 q_2} H_\omega] = \sum_{(\boldsymbol{q},\omega) \in \tilde{\Lambda}_{2n}^{\times}} \mathbb{E}[H_{\boldsymbol{q},\omega}] = \mathcal{O}(N^n),$$

which completes the result. □

5.2 Proof of Proposition 5.2 Part (b) - Diffusion term

Inserting the expression (5.13) for $\delta Y_n^{\boldsymbol{q}}$ into (5.10) gives

$$\begin{aligned}\mathbb{E}[\delta Y_n \delta Y_m | H] &= \frac{1}{d_N}\sum_{\boldsymbol{q}} \Theta_{\boldsymbol{q}}(H)\delta Y_n^{\boldsymbol{q}} \delta Y_m^{\boldsymbol{q}} \\ &= \frac{1}{(N-2)^{n+m}}\frac{1}{4d_N}\sum_{\omega_1\in\Lambda_{2n}}\sum_{\omega_2\in\Lambda_{2m}} H_{\omega_1,\omega_2}\sum_{\boldsymbol{q}}{}' (1-3H_{q_0q_1}H_{q_1q_2})\phi_{\omega_1,\boldsymbol{q}}\phi_{\omega_2,\boldsymbol{q}} \\ &= \frac{6N}{d_N}[2n^2\delta_{nm} + R_{nm}(H)]. \qquad (5.41)\end{aligned}$$

Let us treat the cases $n=m$ and $n\neq m$ separately. Starting with the former, we define $\Gamma_{2n}^\star := \{(\omega_1,\omega_2)\in\Lambda_{2n}^\star\times\Lambda_{2n}^\star : \omega_1\cong\omega_2\}$ and $\Gamma_{2n}^\circ = (\Lambda_{2n}\times\Lambda_{2n})\setminus\Gamma_{2n}^\star$, exactly as in Section 4.2 for the ITE. For $(\omega_1,\omega_2)\in\Gamma_{2n}^\star$ we have $H_{\omega_1,\omega_2}=1$ and $\phi_{\omega_1,\boldsymbol{q}} = \phi_{\omega_2,\boldsymbol{q}}$ for all $\boldsymbol{q}$. Therefore,

$$\begin{aligned}R_{nn}(H) = \frac{1}{(N-2)^{2n}}\frac{1}{24N}&\sum_{(\omega_1,\omega_2)\in\Gamma_{2n}^\star}\sum_{\boldsymbol{q}}{}'(1-3H_{q_0q_1}H_{q_1q_2})\phi_{\omega_1,\boldsymbol{q}}^2 \\ &+\frac{1}{(N-2)^{2n}}\frac{1}{24N}\sum_{(\omega_1,\omega_2)\in\Gamma_{2n}^\circ} H_{\omega_1,\omega_2}\sum_{\boldsymbol{q}}{}'(1-3H_{q_0q_1}H_{q_1q_2})\phi_{\omega_1,\boldsymbol{q}}\phi_{\omega_2,\boldsymbol{q}} - 2n^2. \qquad (5.42)\end{aligned}$$

The first term can be simplified by noting that $\phi_{\omega,\boldsymbol{q}}^2 = \phi_{\omega,\boldsymbol{q}}$ and $\sum_{\boldsymbol{q}}' \phi_{\omega,\boldsymbol{q}} = 12nN + \mathcal{O}(1)$ if $\omega\in\Lambda_{2n}^\star$ (see Equation (5.20)). Moreover, if $\omega_1\in\Lambda_{2n}^\star$ is fixed then there are $4n$ possible walks ω_2 such that $\omega_1\cong\omega_2$, which comes from the $2n$ possible starting points and the 2 possible orientations. Therefore, since $|V_{\omega_1}|\leq 2n$, counting the number of ways of labelling the vertices in ω_1 leads to $|\Gamma_{2n}^\star| = 4n(N^{2n}+\mathcal{O}(N^{2n-1}))$. These observations mean

$$\frac{1}{(N-2)^{2n}}\frac{1}{24N}\sum_{(\omega_1,\omega_2)\in\Gamma_{2n}^\star}\sum_{\boldsymbol{q}}{}'\phi_{\omega_1,\boldsymbol{q}}^2 = \frac{n}{2(N-2)^{2n}}|\Gamma_{2n}^\star|(1+\mathcal{O}(N^{-1})) = 2n^2+\mathcal{O}(N^{-1}).$$

In addition, if the triangle $\boldsymbol{q}$ is fixed and $\omega\in\Lambda_{2n}^\star$ traverses one of its edges (i.e. $\phi_{\omega,\boldsymbol{q}} = 1$) then we have $|V_\omega|-2$ remaining vertices of ω that are free to be relabelled, so $\sum_{\omega\in\Lambda_{2n}^\star}\phi_{\omega,\boldsymbol{q}} = \mathcal{O}(N^{2n-2})$. Hence, using again that for a fixed ω_1 there are $4n$ possible ω_2 such that $(\omega_1,\omega_2)\in\Gamma_{2n}^\star$ gives

$$\begin{aligned}\frac{1}{24N}\sum_{(\omega_1,\omega_2)\in\Gamma_{2n}^\star}\sum_{\boldsymbol{q}}{}' 3H_{q_0q_1}H_{q_1q_2}\phi_{\omega_1,\boldsymbol{q}} &= \frac{12n}{24N}\sum_{\omega\in\Lambda_{2n}^\star}\sum_{\boldsymbol{q}}{}' H_{q_0q_1}H_{q_1q_2}\phi_{\omega,\boldsymbol{q}} \\ = \mathcal{O}(N^{2n-3})\sum_{q_0,q_1}{}'\sum_{q_2\neq q_0,q_1} H_{q_0q_1}H_{q_1q_2} &= \mathcal{O}(N^{2n-3})\sum_{q_0,q_1}{}' H_{q_0q_1}H_{q_1q_0} = \mathcal{O}(N^{2n-1}), \qquad (5.43)\end{aligned}$$

where the last equality requires the regularity of H. The remainder (5.42) therefore reduces to

$$R_{nn}(H) = \frac{1}{(N-2)^{2n}}\frac{1}{24N}\sum_{\boldsymbol{q}}{}'(1-3H_{q_0q_1}H_{q_1q_2})\sum_{(\omega_1,\omega_2)\in\Gamma_{2n}^\circ} H_{\omega_1,\omega_2}\phi_{\omega_1,\boldsymbol{q}}\phi_{\omega_2,\boldsymbol{q}} + \mathcal{O}(N^{-1}). \qquad (5.44)$$

Let us define $\tilde{\Gamma}_{2n}^\circ := \{(\boldsymbol{q},\omega_1,\omega_2) : q_0\neq q_1\neq q_2\neq q_0, (\omega_1,\omega_2)\in\Gamma_{2n}^\circ, \phi_{\omega_1,\boldsymbol{q}}\phi_{\omega_2,\boldsymbol{q}}=1\}$ and $H_{\boldsymbol{q},\omega_1,\omega_2} = H_{q_0q_1}H_{q_1q_2}H_{\omega_1}H_{\omega_2}$ as usual. Similarly let $[\boldsymbol{q},\omega_1,\omega_2]$ be the equivalence class of labellings of the vertices of $(\boldsymbol{q},\omega_1,\omega_2)$ and $[\tilde{\Gamma}^\circ_{2n}]$ the set of such equivalence classes. Then showing that $\mathbb{E}|R_{nn}(H)| = \mathcal{O}(N^{-1})$ reduces to showing the following is of order $\mathcal{O}(N^{2n})$.

$$\begin{aligned}\mathbb{E}\Bigg|\sum_{(\omega_1,\omega_2)\in\Gamma^\circ_{2n}} H_{\omega_1,\omega_2}\sum_{\boldsymbol{q}}{}^{'}(1-3H_{q_0q_1}H_{q_1q_2})\phi_{\omega_1,\boldsymbol{q}}\phi_{\omega_2,\boldsymbol{q}}\Bigg| \\ \leq \sum_{[\omega_1,\omega_2]\in[\Gamma^\circ_{2n}]}\Phi_{[\omega_1,\omega_2]}\sqrt{\sum_{\substack{(\omega_1,\omega_2),(\omega_1',\omega_2')\\ \in[\omega_1,\omega_2]}}\mathbb{E}[H_{\omega_1,\omega_2,\omega_1',\omega_2'}]} \\ +\sum_{[\boldsymbol{q},\omega_1,\omega_2]\in[\tilde{\Gamma}^\circ_{2n}]}3\sqrt{\sum_{\substack{(\boldsymbol{q},\omega_1,\omega_2),(\boldsymbol{q}',\omega_1',\omega_2')\\ \in[\boldsymbol{q},\omega_1,\omega_2]}}\mathbb{E}[H_{\boldsymbol{q},\omega_1,\omega_2,\boldsymbol{q}',\omega_1',\omega_2'}]},\end{aligned}\tag{5.45}$$

where we have written $\Phi_{\omega_1,\omega_2} := \sum_{\boldsymbol{q}}' \phi_{\omega_1,\boldsymbol{q}}\phi_{\omega_2,\boldsymbol{q}} \geq 0$ and $\Phi_{[\omega_1,\omega_2]}$ for any $(\omega_1,\omega_2)\in[\omega_1,\omega_2]$ for convenience.

We start by evaluating the first term in (5.45). For a fixed (ω_1,ω_2) the quantity $\Phi_{[\omega_1,\omega_2]} > 0$ only if ω_1 and ω_2 are connected, i.e. $|V_{\omega_1}\cap V_{\omega_2}| > 0$, giving rise to three scenarios we must consider

(a) $|E_{\omega_1}\cap E_{\omega_2}| = 0$ (ω_1 and ω_2 do not share any edges).

(b) $|E_{\omega_1}\cap E_{\omega_2}| > 0$ and $\omega_1 \ncong \omega_2$ (ω_1 and ω_2 share at least one edge but not all their free edges).

(c) $|E_{\omega_1}\cap E_{\omega_2}| > 0$ and $\omega_1 \cong \omega_2$ (ω_1 and ω_2 share all of their free edges).

As before, let us define $G = (V_{\omega_1,\omega_2}, F_{\omega_1,\omega_2})$ and $\hat{G} = (V_{\omega_1,\omega_2}, E_{\omega_1,\omega_2})$. The condition $\Phi_{\omega_1,\omega_2} > 0$ implies that $\hat{G}$ is connected and we can assume that edges are traversed a maximum of twice in $(\omega_1,\omega_2)\in\Gamma^\circ_{2n}$, meaning that $2|E_{\omega_1,\omega_2}| - |F_{\omega_1,\omega_2}| = 4n$. Also, since ω_1 and ω_2 are non-backtracking cycles, if any of the various connected components of the subgraph $G = (V_{\omega_1,\omega_2}, F_{\omega_1,\omega_2})$ have $\beta_i = 0$ they must be isolated vertices, as in Section 5.1.

Scenario (a) is only possible if ω_1 and ω_2 contain different edges of the triangle $\boldsymbol{q}$, which means all vertices $V_{\boldsymbol{q}}$ are contained in V_{ω_1,ω_2}. Therefore, for a fixed (ω_1,ω_2) we have $\Phi_{\omega_1,\omega_2} = \mathcal{O}(1)$. Moreover, we have $|F_{\omega_1,\omega_2}| \leq 4n$ free edges and so the number of isolated vertices satisfies $|V_I| \leq |E_{\omega_1,\omega_2}| - |F_{\omega_1,\omega_2}|$. Hence $|V_I| \leq 4n - |E_{\omega_1,\omega_2}| = 4n - |V_{\omega_1,\omega_2}| - \beta(\hat{G}) + 1$ and so, by Corollary 3.2 we have $|V_{\omega_1,\omega_2,\omega_1',\omega_2'}| - |F_{\omega_1,\omega_2,\omega_1',\omega_2'}|/2 \leq |V_{\omega_1,\omega_2}| + |V_I| \leq 4n-1$, as $\beta(\hat{G}) \geq 2$ for all $(\omega_1,\omega_2)\in\Gamma^\circ_{2n}$. The contribution to (5.45) is therefore $\mathcal{O}(1)\sqrt{\mathcal{O}(N^{4n-1})} = \mathcal{O}(N^{2n-1/2})$.

In Scenario (b) ω_1 and ω_2 can share the same edge of $\boldsymbol{q}$, say (q_0,q_1), leaving a free vertex, say q_2, which can be summed over to obtain $\Phi_{\omega_1,\omega_2} = \mathcal{O}(N)$. At least one edge must be traversed twice in (ω_1,ω_2) so the number of free edges satisfies $|F_{\omega_1,\omega_2}| \leq 4n-2$. Thus, the number of isolated vertices $|V_I| \leq |E_{\omega_1,\omega_2}| - |F_{\omega_1,\omega_2}| - 1 \leq 4n - |V_{\omega_1,\omega_2}| - \beta(\hat{G})$. Moreover we cannot have $\beta(\hat{G}) = 1$, as this is only possible if ω_1 and ω_2 are single loops and satisfy $\omega_1 \cong \omega_2$ (but this would mean $(\omega_1,\omega_2)\notin\Gamma^\circ_{2n}$). Thus $|V_{\omega_1,\omega_2,\omega_1',\omega_2'}| - |F_{\omega_1,\omega_2,\omega_1',\omega_2'}|/2 \leq |V_{\omega_1,\omega_2}| + |V_I| \leq 4n-2$. The contribution to (5.45) is therefore $\mathcal{O}(N)\sqrt{\mathcal{O}(N^{4n-2})} = \mathcal{O}(N^{2n})$.

In Scenario (c) we again have $\Phi_{\omega_1,\omega_2} = \mathcal{O}(N)$ for the same reason. However this time the condition $\omega_1 \cong \omega_2$ implies that $F_{\omega_1,\omega_2} = \emptyset$ and thus all vertices in G are isolated, i.e. $V_I = V_{\omega_1} = V_{\omega_2}$. However $\omega_1 \cong \omega_2$ implies that $\omega_1 \notin \Lambda^\star_{2n}$ by the definition of $\Gamma^\star_{2n}$ and therefore $|V_{\omega_1}| \leq 2n-2$, so $|V_{\omega_1,\omega_2,\omega_1',\omega_2'}| - |F_{\omega_1,\omega_2,\omega_1',\omega_2'}|/2 = |V_{\omega_1,\omega_2,\omega_1',\omega_2'}| \leq 2|V_I| \leq 4n-4$, which gives a contribution of $\mathcal{O}(N)\sqrt{\mathcal{O}(N^{4n-4})} = \mathcal{O}(N^{2n-1})$ to (5.45).

We now turn our attention to the second term in (5.45). For those $[\boldsymbol{q},\omega_1,\omega_2]\in[\tilde{\Gamma}^{\circ}_{2n}]$ we define the graph $\hat{G}=(V_{\boldsymbol{q},\omega_1,\omega_2},E_{\boldsymbol{q},\omega_1,\omega_2})$, with $E_{\boldsymbol{q}}=\{(q_0,q_1),(q_1,q_2)\}$. If each of the edges are traversed at most twice then we have $2|E_{\boldsymbol{q},\omega_1,\omega_2}|-|F_{\boldsymbol{q},\omega_1,\omega_2}|=4n+2$. Moreover, since there must be a least one edge that is traversed twice we find the number of isolated vertices satisfies $|V_I|\leq|E_{\boldsymbol{q},\omega_1,\omega_2}|-|F_{\boldsymbol{q},\omega_1,\omega_2}|-1=4n+2-|V_{\boldsymbol{q},\omega_1,\omega_2}|-\beta(\hat{G})$. Thus, using that $\beta(\hat{G})\geq 2$ and Corollary 3.2 we find $|V_{\boldsymbol{q},\omega_1,\omega_2,\boldsymbol{q}',\omega_1',\omega_2'}|-|F_{\boldsymbol{q},\omega_1,\omega_2,\boldsymbol{q}',\omega_1',\omega_2'}|/2\leq 4n$ for all $(\boldsymbol{q},\omega_1,\omega_2)\sim(\boldsymbol{q}',\omega_1',\omega_2')\in\tilde{\Gamma}^{\circ}_{2n}$. The contribution of the second term is therefore $\mathcal{O}(N^{2n})$, as desired.

It thus remains to evaluate $\mathbb{E}|R_{nm}(H)|$ for $n\neq m$. If we define $\tilde{\Gamma}_{2n,2m}=\{(\boldsymbol{q},\omega_1,\omega_2):\omega_1\in\Lambda_{2n},\omega_2\in\Lambda_{2m},\phi_{\omega_1,\boldsymbol{q}}\phi_{\omega_2,\boldsymbol{q}}=1\}$ then from (5.41) we have

$$\mathbb{E}|R_{nm}(H)|\leq\frac{1}{(N-2)^{n+m}}\frac{1}{24N}\Bigg\{\sum_{[\omega_1,\omega_2]\in[\Lambda_{2n}\times\Lambda_{2m}]}\Phi_{[\omega_1,\omega_2]}\sqrt{\sum_{\substack{(\omega_1,\omega_2),(\omega_1',\omega_2')\\ \in[\omega_1,\omega_2]}}\mathbb{E}[H_{\omega_1,\omega_2,\omega_1',\omega_2'}]}$$
$$+\sum_{[\boldsymbol{q},\omega_1,\omega_2]\in[\tilde{\Gamma}_{2n,2m}]}\sqrt{\sum_{\substack{(\boldsymbol{q},\omega_1,\omega_2),(\boldsymbol{q}',\omega_1',\omega_2')\\ \in[\boldsymbol{q},\omega_1,\omega_2]}}\mathbb{E}[H_{\boldsymbol{q},\omega_1,\omega_2,\boldsymbol{q}',\omega_1',\omega_2'}]}\Bigg\}. \quad (5.46)$$

For the first term we can use the same arguments as for $n=m$. In particular, if $\Phi_{\omega_1,\omega_2}>0$ then ω_1 and ω_2 are connected. If they do not share an edge (i.e. $|E_{\omega_1}\cap E_{\omega_2}|=0$) then it is possible the number of non-free edges $|E_{\omega_1,\omega_2}|-|F_{\omega_1,\omega_2}|=0$. Hence $|V_I|\leq|E_{\omega_1,\omega_2}|-|F_{\omega_1,\omega_2}|$. If we assume the edges are traversed no more than twice by (ω_1,ω_2)) then $2|E_{\omega_1,\omega_2}|-|F_{\omega_1,\omega_2}|=2n+2m$ and therefore Corollary 3.2 gives $|V_{\omega_1,\omega_2,\omega_1',\omega_2'}|-|F_{\omega_1,\omega_2,\omega_1',\omega_2'}|/2\leq|V_{\omega_1,\omega_2}|+|V_I|\leq 2n+2m+1-\beta(\hat{G})\leq 2n+2m-1$, as $\beta(\hat{G})\geq 2$. Moreover, $\Phi_{\omega_1,\omega_2}=\mathcal{O}(1)$ if $|E_{\omega_1}\cap E_{\omega_2}|=0$ so the contribution is of order $\mathcal{O}(1)\sqrt{\mathcal{O}(N^{2n+2m-1})}=\mathcal{O}(N^{n+m-1/2})$. Alternatively, if ω_1 and ω_2 share at least one edge (i.e. $|E_{\omega_1}\cap E_{\omega_2}|>0$) then we must have $|V_I|\leq|E_{\omega_1,\omega_2}|-|F_{\omega_1,\omega_2}|-1$, which in turn implies that $|V_{\omega_1,\omega_2,\omega_1',\omega_2'}|-|F_{\omega_1,\omega_2,\omega_1',\omega_2'}|/2\leq|V_{\omega_1,\omega_2}|+|V_I|\leq 2n+2m-\beta(\hat{G})\leq 2n+2m-2$, as $\beta(\hat{G})\geq 2$. Therefore, since $\Phi_{\omega_1,\omega_2}=\mathcal{O}(N)$ in this case, we attain a contribution of $\mathcal{O}(N)\sqrt{\mathcal{O}(N^{2n+2m-2})}=\mathcal{O}(N^{n+m})$.

Similarly for the second term we write $\hat{G}=(V_{\boldsymbol{q},\omega_1,\omega_2},E_{\boldsymbol{q},\omega_1,\omega_2})$. If we assume the edges are traversed at most twice then $2|E_{\boldsymbol{q},\omega_1,\omega_2}|-|F_{\boldsymbol{q},\omega_1,\omega_2}|=2n+2m+2$. Then, since $|V_I|\leq|E_{\boldsymbol{q},\omega_1,\omega_2}|-|F_{\boldsymbol{q},\omega_1,\omega_2}|-1$ we get $|V_{\boldsymbol{q},\omega_1,\omega_2}|+|V_I|\leq 2n+2m+2-\beta(\hat{G})$. Thus Corollary 3.2 implies that $|V_{\boldsymbol{q},\omega_1,\omega_2,\boldsymbol{q}',\omega_1',\omega_2'}|-|F_{\boldsymbol{q},\omega_1,\omega_2,\boldsymbol{q}',\omega_1',\omega_2'}|/2\leq 2n+2m$, as $\beta(\hat{G})\geq 2$. Therefore the contribution of the second term is of order $\mathcal{O}(N^{n+m})$ also.

Finally noting that there is a factor of order $O(N^{-n-m-1})$ means that $\mathbb{E}|R_{nm}(H)|=\mathcal{O}(N^{-1})$.

5.3 Proof of Proposition 5.2 Part (c) - Remainder term

Using the expression (5.13) for $\delta Y_n^{\boldsymbol{q}}$ and working backwards in the proof of Lemma 5.1 we obtain

$$\mathbb{E}[|\delta Y_n\delta Y_m\delta Y_l||H]=\frac{1}{4d_N}\sum_{\boldsymbol{q}}{}'\frac{(1-3H_{q_0q_1}H_{q_1q_2})}{4d_N}|\delta Y_n^{\boldsymbol{q}}\delta Y_m^{\boldsymbol{q}}\delta Y_l^{\boldsymbol{q}}|$$
$$=\sum_{\boldsymbol{q}}{}'\frac{\left(1-\sum_{i=0}^{2}H_{q_iq_{i+1(2)}}H_{q_{i+1(2)}q_{i+2(2)}}\right)}{4d_N(N-2)^{n+m+l}}\Bigg|\sum_{\substack{(\omega_1,\omega_2,\omega_3)\\ \in\Lambda_{2n}\times\Lambda_{2m}\times\Lambda_{2l}}}H_{\omega_1,\omega_2,\omega_3}\phi_{\omega_1,\boldsymbol{q}}\phi_{\omega_2,\boldsymbol{q}}\phi_{\omega_3,\boldsymbol{q}}\Bigg|.$$

Therefore, since $0 \le (1 - H_{q_0q_1}H_{q_1q_2} - H_{q_1q_2}H_{q_2q_0} - H_{q_2q_0}H_{q_0q_1}) \le 2$ we find, using $\mathbb{E}|R_{nml}(H)| = \frac{d_N}{6N}\mathbb{E}[\mathbb{E}[|\delta Y_n \delta Y_m \delta Y_l||H]]$

$$\mathbb{E}|R_{nml}(H)| \le \frac{1}{(N-2)^{n+m+l}} \frac{1}{12N} \sum_{\boldsymbol{q}}{}' \, \mathbb{E}\left| \sum_{\substack{(\omega_1,\omega_2,\omega_3) \\ \in \Gamma^{\boldsymbol{q}}_{2n,2m,2l}}} H_{\omega_1,\omega_2,\omega_3} \right|, \tag{5.47}$$

where $\Gamma^{\boldsymbol{q}}_{2n,2m,2l} = \{(\omega_1,\omega_2,\omega_3) \in \Lambda_{2n} \times \Lambda_{2m} \times \Lambda_{2l} : \phi_{\omega_1,\boldsymbol{q}}\phi_{\omega_2,\boldsymbol{q}}\phi_{\omega_3,\boldsymbol{q}} = 1\}$. Using the standard inequality for the expectation means we must compute the quantity

$$\mathbb{E}\left| \sum_{\substack{(\omega_1,\omega_2,\omega_3) \\ \in \Gamma^{\boldsymbol{q}}_{2n,2m,2l}}} H_{\omega_1,\omega_2,\omega_3} \right| \le \sum_{\substack{[\omega_1,\omega_2,\omega_3] \\ \in \left[\Gamma^{\boldsymbol{q}}_{2n,2m,2l}\right]}} \sqrt{\sum_{\substack{(\omega_1,\omega_2,\omega_3),(\omega_1',\omega_2',\omega_3') \\ \in [\omega_1,\omega_2,\omega_3]}} \mathbb{E}[H_{\omega_1,\omega_2,\omega_3,\omega_1',\omega_2',\omega_3'}]}. \tag{5.48}$$

The condition $\phi_{\omega_1,\boldsymbol{q}}\phi_{\omega_2,\boldsymbol{q}}\phi_{\omega_3,\boldsymbol{q}} = 1$ imposes the restriction that ω_1, ω_2 and ω_3 must share an odd number of edges with $\boldsymbol{q}$. Let us restrict ourselves to those $(\omega_1,\omega_2,\omega_3) \in \Lambda^\dagger_{2n} \times \Lambda^\dagger_{2m} \times \Lambda^\dagger_{2l}$ (i.e. $|V_{\omega_1}| = |V_{\omega_2}| = |V_{\omega_3}| = 2n$) as this maximises the number of vertices and therefore gives the main contribution to (5.48). Note that for all $\omega \in \Lambda^\dagger_{2n}$, $\phi_{\boldsymbol{q},\omega} = 1$ if and only if $|E_\omega \cap \{(q_0,q_1),(q_1,q_2),(q_2,q_0)\}| = 1$. There are (up to the relabelling of vertices) three scenarios

(a) $|E_{\omega_1} \cap \{(q_0,q_1)\}| = |E_{\omega_2} \cap \{(q_1,q_2)\}| = |E_{\omega_3} \cap \{(q_2,q_0)\}| = 1$
(b) $|E_{\omega_1} \cap \{(q_0,q_1)\}| = |E_{\omega_2} \cap \{(q_0,q_1)\}| = |E_{\omega_3} \cap \{(q_1,q_2)\}| = 1$
(c) $|E_{\omega_1} \cap \{(q_0,q_1)\}| = |E_{\omega_2} \cap \{(q_0,q_1)\}| = |E_{\omega_3} \cap \{(q_0,q_1)\}| = 1$.

In Scenario (a) we assume that each edge is traversed at most twice by $(\omega_1,\omega_2,\omega_3)$, so $2|E_{\omega_1,\omega_2,\omega_3}| - |F_{\omega_1,\omega_2,\omega_3}| = 2n + 2m + 2l$. Then, following the same arguments as in Sections 5.1 and 5.2, the number of isolated vertices $|V_I| \le |E_{\omega_1,\omega_2,\omega_3}| - |F_{\omega_1,\omega_2,\omega_3}| \le 2n + 2m + 2l - |V_{\omega_1,\omega_2,\omega_3}| + 1 - \beta(\hat{G})$, where $\hat{G} = (V_{\omega_1,\omega_2,\omega_3}, E_{\omega_1,\omega_2,\omega_3})$. By construction we must have $\beta(\hat{G}) \ge 4$ and thus $|V_{\omega_1,\omega_2,\omega_3}| + |V_I| \le 2m + 2n + 2l - 3$.

In Scenario (b) we also assume that each edge is traversed at most twice by $(\omega_1,\omega_2,\omega_3)$. However at least one edge (given by (q_0,q_1)) must be traversed twice, which means the number of isolated vertices satisfies $|V_I| \le |E_{\omega_1,\omega_2,\omega_3}| - |F_{\omega_1,\omega_2,\omega_3}| - 1$ (it is always at least one less than the number of non-free edges). Therefore we find $|V_{\omega_1,\omega_2,\omega_3}| + |V_I| \le 2n + 2m + 2l - \beta(\hat{G}) = 2n + 2m + 2l - 3$, since we must have $\beta(\hat{G}) \ge 3$.

Using Corollary 3.2 we thus determine that in Scenarios (a) and (b)

$$|V_{\omega_1,\omega_2,\omega_3,\omega_1',\omega_2',\omega_3'}| - \frac{|F_{\omega_1,\omega_2,\omega_3,\omega_1',\omega_2',\omega_3'}|}{2} \le 2n + 2m + 2l - 3.$$

However all three vertices q_0, q_1 and q_2 are fixed for a particular $\Gamma^{\boldsymbol{q}}_{2n,2m,2l}$ so the contribution to (5.48) is of order $\sqrt{\mathcal{O}(N^{2n+2m+2l-3-3})} = \mathcal{O}(N^{n+m+l-3})$.

In Scenario (c) the edge (q_0,q_1) is traversed 3 times. We assume that all others are traversed at most twice, which implies that $2|E_{\omega_1,\omega_2,\omega_3}| - |F_{\omega_1,\omega_2,\omega_3}| = 2n + 2m + 2l - 2$. Now, either $\omega_i \cong \omega_j$ for some $i \ne j = 1,2,3$ or not. Let us suppose the former case arises (we can say $\omega_1 \cong \omega_2$ for instance) then the number of isolated vertices satisfies $|V_I| \le |E_{\omega_1,\omega_2,\omega_3}| - |F_{\omega_1,\omega_2,\omega_3}| - 1$, which means $|V_{\omega_1,\omega_2,\omega_3}| + |V_I| = 2n + 2m + 2l - \beta(\hat{G}) - 2 \le 2n + 2m + 2l - 4$, since $\beta(\hat{G}) \ge 2$ in this case.

Alternatively, if $\omega_i \ncong \omega_j$ for all $i \ne j$ then $|V_I| \le |E_{\omega_1,\omega_2,\omega_3}| - |F_{\omega_1,\omega_2,\omega_3}|$, but we must have $\beta(\hat{G}) \ge 3$, which means again, $|V_{\omega_1,\omega_2,\omega_3}| + |V_I| \le 2n + 2m + 2l - \beta(\hat{G}) - 1 \le 2n + 2m + 2l - 4$. Corollary 3.2 therefore implies that for Scenario (c) we have

$$|V_{\omega_1,\omega_2,\omega_3,\omega_1',\omega_2',\omega_3'}| - \frac{|F_{\omega_1,\omega_2,\omega_3,\omega_1',\omega_2',\omega_3'}|}{2} \le 2n + 2m + 2l - 4.$$

However this time there are only two vertices of $\boldsymbol{q}$ contained in $V_{\omega_1,\omega_2,\omega_3}$, which means the contribution to (5.48) is of order $\sqrt{\mathcal{O}(N^{2n+2m+2l-4-2})} = \mathcal{O}(N^{n+m+l-3})$ once again.

Returning to (5.47) and noting that the sum over $\boldsymbol{q}$ gives a contribution of $\mathcal{O}(N^3)$ means that $\mathbb{E}|R_{nml}(H)| = \mathcal{O}(N^{-1})$, as desired.

6 Conclusions

We have used a combination of appropriate random walks and Stein's method to provide rates of convergence for the traces of random Bernoulli ensembles derived from both tournaments and regular tournaments. Specifically we have shown that under this random walk the traces, in a basis of Chebyshev polynomials, behave like independent Ornstein-Uhlenbeck processes in the limit of large matrix size. Subsequently, this allows to use the results of Chatterjee & Meckes [10], Reinert & Röllin [39] and Meckes [37], regarding the multivariate version of the exchangeable pairs mechanism for Stein's method, in order to obtain rates of convergence to an appropriate Gaussian distribution. In particular, we are able to obtain these results using combinatorial methods, closely related to previous calculations for showing distributional convergence, but without explicit rates (see e.g. [41, 44]). Moreover, this approach only requires estimates involving third order moments to show distributional convergence.

We would like to finish with a couple of comments. Firstly, we note that in the bound for the distributional distance (2.5) of the RITE in Theorem 2.4, the first term is of order $\mathcal{O}(N^{-1/2})$. This comes from a single set of walks, arising due to the regularity of the matrix H (see the proof of Lemma 5.4 Part (b)). It is not clear whether this can be improved to $\mathcal{O}(N^{-1})$ in order to match the corresponding result in Theorem 2.3 for the ITE. Secondly, we believe the results could be easily applied to other types of matrix ensembles such as Wigner matrices, or tournaments with different score sequences. For Wigner matrices the random walk would be very similar - one may choose a matrix element at random and then resample from the appropriate distribution. However Lemma 3.3 is not immediately applicable and would therefore have to be amended. Although results in this direction have already been achieved [17]. For tournaments with different score sequences similar random walks to the RITE have already been analysed [28] and the number of such tournaments have been asymptotically estimated [36], expanding on the technques developed by McKay for regular tournaments [35], which suggests a result akin to Lemma 5.3 would also be possible.

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Appendix

A Stein solution

Since we detail a slightly different (and more specific) version of Theorem 2.6 to that of Theorem 2.1 in [39] and Theorem 3 in [37] we have decided to include a short proof for the aid of the reader. In particular, [39, 37] allow for a multivariate Gaussian distribution with general covariance matrix Σ, which we have decided to specify to our situation for clarity. Moreover, the bounds in [39] are also in terms of the derivatives $\|\nabla^j \phi\|$ but of an order one more than presented here. This realisation that the order can be reduced by one through integration by parts (see Lemma A.2) is presented in [37] but this is done with a more complicated type of function bound and so we keep with derivatives of the form $\|\nabla^j \phi\|$ for simplicity. The rate of converge in terms of N is unaffected by this.

Proposition A.1 (Stein solution)**.** *Let $\mathcal{A}$ be the operator given in (2.11) and define Stein's equation (see (2.12)) as*

$$\mathcal{A}f(x) = \phi(x) - \mathbb{E}[\phi(Z)], \tag{A.1}$$

with $Z = (Z_2, \dots, Z_k)$, $Z_n \sim N(0, n)$. Then, for all $\phi : \mathbb{R}^{k-1} \to \mathbb{R}$ which are (finitely) twice differentiable on $\mathbb{R}$, the function

$$f(x) := -\int_0^\infty dt \ \left[\int d\mu(Z)[\phi(Z') - \phi(Z)]\right], \tag{A.2}$$

is in $C^2(\mathbb{R}^{k-1})$ and solves Equation (A.1). Here $Z' = (Z'_2, Z'_3, \dots, Z'_k)$ is the random variable given by

$$Z'_n = xe^{-nt} + \sqrt{1 - e^{-2t}} Z_n,$$

and μ is the measure of Z.

Proof. For a proof utilising that $\mathcal{A}$ is the generator of a suitable OU process one may consult the works of [4, 21]. Alternatively Meckes delivers a direct proof using integration by parts in [37]. An excellent exposition on the latter is provided in Section 3 of [12]. ☐

Lemma A.2. *Let f be connected to $\phi \in C^3(\mathbb{R}^{k-2})$ as in Proposition A.1. Then*

$$\left\|\nabla^j f\right\| \le \frac{1}{\sqrt{\pi}} \frac{2^{j-3}\Gamma(\frac{k}{2})^2}{(k-1)!} \left\|\nabla^{j-1}\phi\right\|, \tag{A.3}$$

with $\|\nabla^j f\|$ and $\|\nabla^j \phi\|$ defined in (2.4).

Proof. We have, writing $d\mu(Z) = dZP(Z)$ and changing variables of the derivatives

$$\begin{aligned}\frac{\partial^j f(X)}{\partial X_{n_1} \dots \partial X_{n_j}} &= \int_0^\infty dt \int dZ P(Z) \frac{\partial^j \phi(Z')}{\partial X_{n_1} \dots \partial X_{n_j}} \\ &= -\int_0^\infty dt \ \frac{e^{-(n_1 + \dots + n_j)t}}{\sqrt{1 - e^{-2n_j t}}} \int dZ P(Z) \frac{\partial^j \phi(Z')}{\partial Z'_{n_1} \dots \partial Z'_{n_{j-1}} \partial Z_{n_j}} \ , \end{aligned}\tag{A.4}$$

where $n_i = 2, \dots, k$. Integration by parts may therefore be performed on the Z_{n_j} variable

$$\begin{aligned}&\frac{\partial^j f(X)}{\partial X_{n_1} \dots \partial X_{n_j}} \\ &= \int_0^\infty dt \frac{e^{-(n_1 + \dots + n_j)t}}{\sqrt{1 - e^{-2n_j t}}} \left\{ \int dZ \frac{\partial P(Z)}{\partial Z_{n_j}} \frac{\partial^{j-1} \phi(Z')}{\partial Z'_{n_1} \dots \partial Z'_{n_j}} - \left[P(Z) \frac{\partial^{j-1} \phi(Z')}{\partial Z'_{n_1} \dots \partial Z'_{n_{j-1}}} \right]_{-\infty}^{\infty} \right\} \\ &= -\int_0^\infty dt \frac{e^{-(n_1 + \dots + n_j)t}}{\sqrt{1 - e^{-2n_j t}}} \int dZ P(Z) \frac{Z_{n_j}}{n_j} \frac{\partial^{j-1} \phi(Z')}{\partial Z'_{n_1} \dots \partial Z'_{n_{j-1}}}. \end{aligned}\tag{A.5}$$

Thus, using $\mathbb{E}[|Z_{n_j}|] = \sqrt{\frac{2n_j}{\pi}}$ for $Z_{n_j} \sim N(0, n_j)$, gives

$$\left| \frac{\partial^j f(X)}{\partial X_{n_1} \dots \partial X_{n_j}} \right| \le \sup_{Z'} \left| \frac{\partial^{j-1} \phi(Z')}{\partial Z'_{n_1} \dots \partial Z'_{n_{j-1}}} \right| \sqrt{\frac{2}{\pi n_j}} \int_0^\infty dt \frac{e^{-(n_1 + \dots + n_j)t}}{\sqrt{1 - e^{-2n_j t}}}.$$

Finally, since $n_i \ge 2$ we have $e^{-(n_1 + \dots + n_j)t} \le e^{-2jt}$, $(1 - e^{-2n_j t})^{-\frac{1}{2}} \le (1 - e^{-4t})^{-\frac{1}{2}}$ and

$$\int_0^\infty dt \frac{e^{-2jt}}{\sqrt{1 - e^{-4t}}} = \frac{1}{2} \frac{2^{j-2}\Gamma(j/2)^2}{(j-1)!},$$

which leads directly to (A.3). ☐

Proof of Theorem 2.6. Let f be connected to ϕ via the Stein equation (2.12). Since (M, M') are an exchangeable pair so are the random variables $Y' := Y(M')$ and $Y := Y(M)$, hence $\mathbb{E}[\delta f] = \mathbb{E}[f(Y')] - \mathbb{E}[f(Y)] = 0$. Therefore, expanding $f(Y')$ in a Taylor series about Y and substituting for the expressions (2.13) and (2.14) we get

$$\begin{aligned} 0 &= \frac{1}{\alpha_N}(\mathbb{E}[f(Y')] - \mathbb{E}[f(Y)]) \\ &= \frac{1}{\alpha_N}\mathbb{E}\left[\sum_{n=2}^{k}\mathbb{E}[\delta Y_n|M]\frac{\partial f}{\partial Y_n} + \frac{1}{2}\sum_{n,m=2}^{k}\mathbb{E}[\delta Y_n \delta Y_m|M]\frac{\partial^2 f}{\partial Y_n \partial Y_m} + \mathbb{E}[S_f(M,M')|M]\right] \\ &= \mathbb{E}[\mathcal{A}f(Y(M))] + \mathbb{E}\left[\sum_{n=2}^{k}R_n(M)\frac{\partial f}{\partial Y_n} + \frac{1}{2}\sum_{n,m=2}^{k}R_{nm}(M)\frac{\partial^2 f}{\partial Y_n \partial Y_m}\right. \\ &\qquad\qquad\left. + \frac{1}{\alpha_N}\mathbb{E}[S_f(M,M')|M]\right], \end{aligned}$$

where $S_f(M, M')$ is the integral form of the remainder obtained in Taylor's theorem

$$S_f(M,M') = \frac{1}{3!}\sum_{n,m,l=2}^{k}\delta Y_n \delta Y_m \delta Y_l \int_0^1 dv(1-v)^2 \frac{\partial^3 f((1-v)Y + vY')}{\partial Y_n \partial Y_m \partial Y_l}.$$

Using $\int_0^1 dv(1-v)^2 = \frac{1}{3}$ means $|S_f(M,M')| \leq \frac{1}{3!}\frac{1}{3}\|\nabla^3 f\| \sum_{n,m,l=2}^{k} |\delta Y_n \delta Y_m \delta Y_l|$ and so a direct substitution of Stein's equation (2.12) yields

$$\begin{aligned} &|\mathbb{E}[\phi(Y)] - \mathbb{E}[\phi(Z)]| \\ &\leq \|\nabla f\| \sum_{n=2}^{k}\mathbb{E}|R_n(M)| + \frac{1}{2}\|\nabla^2 f\| \sum_{n,m=2}^{k}\mathbb{E}|R_{nm}(M)| + \frac{1}{18}\|\nabla^3 f\| \sum_{n,m,l=2}^{k}\mathbb{E}|R_{nml}(M)| \\ &= \mathcal{R}^{(1)}\|\nabla f\| + \frac{1}{2}\mathcal{R}^{(1)}\|\nabla^2 f\| + \frac{1}{18}\mathcal{R}^{(1)}\|\nabla^3 f\|. \end{aligned}$$

Finally, using Lemma A.2 we have $\|\nabla^j f\| \leq r_j \|\nabla^{j-1}\phi\|$ with explicit values for the r_j. □

B Expectations in the RITE

Proof of Lemma 5.3. In order to prove the lemma we use the ideas of McKay [35], who was originally interested in establishing the asymptotic number of regular tournaments. This was achieved via what he describes as a saddle-point argument, which we adapt here for our current purposes. The main idea is to rewrite the expectation $\mathbb{E}[H_E]$ in terms of a trigonometric integral (see Equation (B.4)), with N angles θ_p corresponding to each of the N rows in the matrix H. Crucially the integrand depends only on the differences $\theta_p - \theta_q$ of these angles and is maximised when all angles are equal. Therefore we show the main contribution comes from the region where $\theta_p \approx \theta_q$ for all p, q and the remaining regions are negligible in the limit of large N.

To construct the appropriate integral expression we begin with the following characteristic function

$$\chi_{\mathfrak{R}_N}(H) = \begin{cases} 0 & H \notin \mathfrak{R}_N \\ 1 & H \in \mathfrak{R}_N. \end{cases}$$

An analytical expression for $\chi_{\mathfrak{R}_N}(H)$ may be achieved via the Kronecker delta function. If we let $S_p = -\sum_q iH_{pq}$ be the row sums then our matrix H belongs to $\mathfrak{R}_N$ only if $S_p = 0$

for all p. Therefore

$$\chi_{\mathfrak{R}_N}(H) = \prod_p \delta_{S_p,0} = \prod_p \frac{1}{2\pi}\int_0^{2\pi} d\theta_p \exp\left(iS_p\theta_p\right) = \frac{1}{(2\pi)^N}\int_0^{2\pi} d^N\theta \prod_{p<q} \exp\left(H_{pq}(\theta_p-\theta_q)\right), \tag{B.1}$$

where we have used that $H_{pq} = -H_{qp}$. We notice in the expressions above that, since S_p is always even, the integrand is invariant under the shift $\theta_p \mapsto \theta_p + \pi$ for any p, and so

$$\chi_{\mathfrak{R}_N}(H) = \frac{1}{\pi^N}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d^N\theta \prod_{p<q} \exp\left(H_{pq}(\theta_p-\theta_q)\right). \tag{B.2}$$

Summing over all possible matrices $H \in \mathfrak{T}_N$ and weighting by this characteristic function leads to the following integral expression for the number of regular tournaments and evaluated by McKay [35]

$$\begin{aligned}|\mathfrak{R}_N| = \sum_{H\in\mathfrak{T}_N} \chi_{\mathfrak{R}_N}(H) &= \frac{2^{N(N-1)/2}}{\pi^N}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d^N\theta \prod_{p<q} \cos(\theta_p-\theta_q) \\ &= \frac{2^{N(N-1)/2}}{\pi^{N-1}}\left(\frac{N}{e}\right)^{\frac{1}{2}}\left(\frac{2\pi}{N}\right)^{\frac{N-1}{2}}(1+\mathcal{O}(N^{-\frac{1}{2}+\epsilon})).\end{aligned} \tag{B.3}$$

Using the same approach we can evaluate the expectation in Lemma 5.3. Using the characteristic function (B.1) the expectation (5.11) is therefore

$$\mathbb{E}_{\mathfrak{R}_N}[H_E] = \frac{1}{|\mathfrak{R}_N|}\sum_{H\in\mathfrak{T}_N} H_E\chi_{\mathfrak{R}_N}(H) = \frac{i^k\, 2^{N(N-1)/2}}{\pi^N|\mathfrak{R}_N|} I, \tag{B.4}$$

where

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d^N\theta \prod_{(p,q)\in E} \sin(\theta_p-\theta_q) \prod_{(p,q)\in E^c} \cos(\theta_p-\theta_q)$$

and $E^c = \{(p,q) : 1 \le p < q \le N\} \setminus E$. Note that k will be taken to be even so (B.4) is real.

To evaluate the integral I we split the integration range into those parts which are dominant and subdominant. To this end let us define the following quantities

- $A_s = [(s-4)\pi/8, (s-3)\pi/8]$, so in particular $[-\frac{\pi}{2}, \frac{\pi}{2}] = \bigcup_{s=0}^{7} A_s$.
- We therefore write $\boldsymbol{s} = (s_1, \dots, s_N)$, for $s_p = 0, 1, \dots, 7$ to signify that the N-tuple of angles is in the specific region $(\theta_1, \theta_2, \dots, \theta_N) \in V(\boldsymbol{s}) = A_{s_1} \times A_{s_2} \times \dots \times A_{s_N}$.
- $n_j = n_j(\boldsymbol{s}) = \#\{p : s_p = j\}$. This counts the number of angles θ_p in the segment A_j.
- $D^{(1)} = \{\boldsymbol{s} : n_j + n_{j+1(8)} + n_{j+2(8)} + n_{j+3(8)} = N \text{ for some } j = 0, \dots, 7\}$ (where $i+1(8)$ refers to $i+1$ modulo 8 etc.). Thus if $\boldsymbol{s} \in D^{(1)}$ this means all angles θ_p are contained in the region $A_j \cup A_{j+1(8)} \cup A_{j+2(8)} \cup A_{j+3(8)}$, for some j.
- $D^{(2)} = \{\boldsymbol{s} \in \{0, \dots, 7\}^N\} \setminus D^{(1)}$ denotes all other possible placements of the angles $(\theta_1, \theta_2, \dots, \theta_N)$.

Thus, $I = J^{(1)} + J^{(2)}$, where

$$J^{(t)} = \sum_{\boldsymbol{s}\in\mathcal{D}^{(t)}} \int_{V(\boldsymbol{s})} d^N\theta \prod_{(p,q)\in E} \sin(\theta_p-\theta_q) \prod_{(p,q)\in E^c} \cos(\theta_p-\theta_q). \tag{B.5}$$

We will show subsequently that

$$|J^{(1)}| \le \sqrt{N}\left(\frac{2\pi}{N}\right)^{\frac{N-1}{2}} \mathcal{O}\left(N^{-\frac{k}{2}}\right), \tag{B.6}$$

where $k = |E|$ is the number of edges in E, and $|J^{(2)}|$ is negligible in comparison in the large N limit. Hence, inserting the expressions (B.6) and (B.3) into (B.4) gives $\mathbb{E}_{\mathfrak{R}_N}[H_E] = \mathcal{O}(N^{-k/2})$, as desired.

We begin by showing the result (B.6) for $J^{(1)}$ which provides the leading contribution. From the form of $D^{(1)}$ we see that, after suitable translations that do not change the value of the integral[4], all angles are contained in the range $[-\frac{\pi}{4}, \frac{\pi}{4}]$. The sets $D_i^{(1)} := \{\boldsymbol{s} : n_i + n_{i+1(8)} + n_{i+2(8)} + n_{i+3(8)} = N\}$ are not necessarily disjoint for $i \neq j$ so $D^{(1)} \subset \bigcup_{i=0}^{7} D_i^{(1)}$. But the sum in (B.5) is the same when restricted to any of the $D_i^{(1)}$. Thus

$$\begin{aligned}|J^{(1)}| &\leq \sum_{i=0}^{7} \sum_{\boldsymbol{s}\in\mathcal{D}_i^{(1)}} \int_{V(\boldsymbol{s})} d^N\theta \prod_{(p,q)\in E} |\sin(\theta_p - \theta_q)| \prod_{(p,q)\in E^c} |\cos(\theta_p - \theta_q)| \\ &= 8\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d^N\theta \prod_{(p,q)\in E} |\sin(\theta_p - \theta_q)| \prod_{(p,q)\in E^c} |\cos(\theta_p - \theta_q)|. \qquad \text{(B.7)}\end{aligned}$$

We are now in a position to use the following bounds

$$|\sin(x)| \leq |x| \exp\left(-\frac{1}{2}x^2\right) \quad \text{and} \quad |\cos(x)| \leq \exp\left(-\frac{1}{2}x^2\right), \qquad \text{(B.8)}$$

which are valid for $|x| \leq \frac{\pi}{2}$. Inserting these, employing the transition $\theta_p \mapsto \theta_p + \theta_N$ for all $p = 1, \ldots, N-1$, integrating over the redundant θ_N and extending the integration range to the whole real line leads to

$$\begin{aligned}|J^{(1)}| &\leq 8\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d^N\theta \prod_{(p,q)\in E} |\theta_p - \theta_q| \exp\left(-\frac{1}{2}\sum_{1\leq p<q\leq N} (\theta_p - \theta_q)^2\right) \\ &\leq 8\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta_N \int_{-\frac{\pi}{4}+\theta_N}^{\frac{\pi}{4}+\theta_N} d^{N-1}\theta \prod_{(p,q)\in E} |\theta_p - \theta_q| \exp\left(-\sum_{p<q}^{N-1} \frac{(\theta_p - \theta_q)^2}{2} - \sum_{p=1}^{N-1} \frac{\theta_p^2}{2}\right) \\ &\leq \frac{8\pi}{2}\int_{-\infty}^{\infty} d^{N-1}\theta \prod_{(p,q)\in E} |\theta_p - \theta_q| e^{-\frac{1}{2}\theta^T\Sigma^{-1}\theta}, \qquad \text{(B.9)}\end{aligned}$$

where $\theta = (\theta_1, \ldots, \theta_{N-1})^T$ and the covariance matrix is $\Sigma^{-1} = N\mathbf{I}_{N-1} - \mathbf{E}_{N-1}$ ($\mathbf{I}_r$ denotes the $r \times r$ identity matrix and $\mathbf{E}_r$ the $r \times r$ matrix in which every element is 1). The inverse can be easily verified to be

$$\Sigma = \frac{1}{N}(\mathbf{I}_{N-1} + \mathbf{E}_{N-1}) \qquad \text{(B.10)}$$

and thus, via Sylvester's determinant identity (with $\mathbf{1}_{N-1}$ the column vector of ones of length $N-1$)

$$\det(\Sigma) = \frac{1}{N^{N-1}} \det(I_{N-1} + \mathbf{1}_{N-1}\mathbf{1}_{N-1}^T) = \frac{1}{N^{N-1}}(1 + \mathbf{1}_{N-1}^T\mathbf{1}_{N-1}) = N^{1-(N-1)}.$$

[4]For example, if we have $D_6^{(1)} = A_6 \cup A_7 \cup A_0 \cup A_1$ then we can first make a translation of $\theta_p \mapsto \theta_p - \pi$ to all angles in $A_6 \cup A_7$ which does not change the value of the integral, so $D_6^{(1)} \mapsto [-3\pi/4, -\pi/4]$ and then make a simultaneous shift of $\pi/2$ for all angles.

Hence, using Hölder's inequality,

$$|J^{(1)}| \leq \sqrt{N} 4\pi \left(\frac{2\pi}{N}\right)^{\frac{N-1}{2}} \mathbb{E}_\theta \Big[\prod_{(p,q)\in E_{\boldsymbol{p}}} |\theta_p - \theta_q| \Big]$$

$$\leq \sqrt{N} 4\pi \left(\frac{2\pi}{N}\right)^{\frac{N-1}{2}} \prod_{(p,q)\in E_{\boldsymbol{p}}} \sqrt{\mathbb{E}_\theta[(\theta_p-\theta_q)^2]}, \quad \text{(B.11)}$$

where

$$\mathbb{E}_\theta[f(\theta)] := \frac{1}{\sqrt{(2\pi)^{N-1}\det(\Sigma)}} \int_{-\infty}^{\infty} d^{N-1}\theta f(\theta) e^{-\frac{1}{2}\theta^T\Sigma^{-1}\theta}.$$

Using the form of the covariance matrix (B.10) the Gaussian expectation of two random variables is $\mathbb{E}_\theta[\theta_p\theta_q] = \Sigma_{pq} = \frac{1}{N}(\delta_{pq}+1)$. Therefore $\mathbb{E}_\theta[(\theta_p-\theta_q)^2] = \mathbb{E}_\theta[\theta_p^2] - 2\mathbb{E}[\theta_p\theta_q] + \mathbb{E}[\theta_q^2] = 2/N$ and so

$$|J^{(1)}| \leq \sqrt{N} 4\pi \left(\frac{2\pi}{N}\right)^{\frac{N-1}{2}} \left(\frac{2}{N}\right)^{k/2}.$$

We now turn to the evaluation of $J^{(2)}$. Due to the condition $\sum_j n_j = N$, we have that at least one of n_7+n_0, n_1+n_2, n_3+n_4 and n_5+n_6 is greater than or equal to $N/4$. Suppose this is the case for n_3+n_4. Let us denote $A = A_3 \cup A_4 = [-\frac{\pi}{8}, \frac{\pi}{8}]$, $B = A_2 \cup A_5 = [-\frac{\pi}{4}, -\frac{\pi}{8}] \cup [\frac{\pi}{8}, \frac{\pi}{4}]$ and $C = A_0 \cup A_1 \cup A_6 \cup A_7 = [-\frac{\pi}{2}, -\frac{\pi}{4}] \cup [\frac{\pi}{4}, \frac{\pi}{2}]$, with $n_A = n_3+n_4, n_B = n_2+n_5$ and $n_C = n_0+n_1+n_6+n_7$ accordingly. If we write $\mathcal{F} := \{\boldsymbol{s} \in \mathcal{D}^{(2)} : n_A \geq N/4\}$ and account for the four possibilities of having at least $N/4$ angles in the particular segment then

$$|J^{(2)}| \leq 4 \sum_{\boldsymbol{s}\in\mathcal{F}} \int_{V(\boldsymbol{s})} d^N\theta \prod_{(p,q)\in E} |\sin(\theta_p-\theta_q)| \prod_{(p,q)\in E^c} |\cos(\theta_p-\theta_q)|. \quad \text{(B.12)}$$

In addition, we split $\mathcal{F} = \mathcal{F}_> \cup \mathcal{F}_\leq$, where for some $\epsilon > 0$ we have $\mathcal{F}_> = \{\boldsymbol{s} \in \mathcal{F} : n_C > N^\epsilon\}$ and $\mathcal{F}_\leq = \{\boldsymbol{s} \in \mathcal{F} : n_C \leq N^\epsilon\}$ and evaluate each part separately.

We start with $\mathcal{F}_>$. If $\theta_p \in A$ and $\theta_q \in C$ (or vice versa) then $\pi/8 \leq |\theta_p - \theta_q| \leq 5\pi/8$ and so $|\cos(\theta_p-\theta_q)| \leq \cos(\pi/8) = e^{-c}$ for $c = -\log(\cos(\pi/8)) > 0$. In addition, for $\theta_p, \theta_q \in A\cup B$ and $\theta_p,\theta_q \in C$ we can employ the bounds (B.8), and for all others write $|\sin(\theta_p-\theta_q)| \leq 1$ and $|\cos(\theta_p-\theta_q)| \leq 1$, so

$$\sum_{\boldsymbol{s}\in\mathcal{F}_>} \int_{V(\boldsymbol{s})} d^N\theta \prod_{(p,q)\in E} |\sin(\theta_p-\theta_q)| \prod_{(p,q)\in E^c} |\cos(\theta_p-\theta_q)| \leq$$

$$e^{-c(n_An_C-k_{AC})} \sum_{\boldsymbol{s}\in\mathcal{F}_>} \left(\int_{-\pi/4}^{\pi/4} d^{n_A+n_B}\theta \prod_{(p,q)\in E_{A\cup B}} |\theta_p-\theta_q| \exp\left(-\frac{1}{2}\sum_{p<q}(\theta_p-\theta_q)^2\right)\right)$$

$$\times \left(\int_{-\pi/4}^{\pi/4} d^{n_C}\theta \prod_{(p,q)\in E_C} |\theta_p-\theta_q| \exp\left(-\frac{1}{2}\sum_{p<q}(\theta_p-\theta_q)^2\right)\right). \quad \text{(B.13)}$$

Here $E_{A\cup B} = \{(p,q)\in E : \theta_p,\theta_q \in A\cup B\}$, $E_C = \{(p,q)\in E : \theta_p,\theta_q \in C\}$ and $k_{AC} = \#\{(p,q) \in E : \theta_p \in A, \theta_q \in C\}$. Therefore, using the previous arguments above for

Gaussian integrals, the above expression is less than or equal to

$$\frac{\pi^2}{4}\sum_{\boldsymbol{s}\in\mathcal{F}_>} e^{-c(n_A n_C - k_{AC})}\sqrt{(n_A+n_B)n_C}\left(\frac{2\pi}{n_A+n_B}\right)^{\frac{n_A+n_B-1}{2}} \leq \\ \times\left(\frac{2}{n_A+n_B}\right)^{\frac{k_{A\cup B}}{2}}\left(\frac{2\pi}{n_C}\right)^{\frac{n_C-1}{2}}\left(\frac{2}{n_C}\right)^{\frac{k_C}{2}}, \quad \text{(B.14)}$$

where $k_{A\cup B} = |E_{A\cup B}|$ and $k_C = |E_C|$. Now, given that $k_{AC} \leq k$, we have $e^{ck_{AC}} \leq e^{ck}$. Also, since $k_{A\cup B} + k_C \leq k$ and $n_A + n_B$ and n_C cannot be equal to zero for $\boldsymbol{s} \in \mathcal{F}$ (otherwise $\boldsymbol{s} \in \mathcal{D}^{(1)}$)

$$\left(\frac{2}{n_A+n_B}\right)^{\frac{k_{A\cup B}}{2}}\left(\frac{2}{n_C}\right)^{\frac{k_C}{2}} \leq 2^{k/2}.$$

In addition, $n_A n_C \geq \frac{1}{4}N^{1+\epsilon}$ for $\boldsymbol{s} \in \mathcal{F}_>$, so the expression in (B.14) is less than or equal to

$$\frac{\pi^2}{4}F_k e^{-\frac{c}{4}N^{1+\epsilon}}\sum_{\boldsymbol{s}\in\mathcal{F}_>}\sqrt{(n_A+n_B)n_C}\left(\frac{2\pi}{n_A+n_B}\right)^{\frac{n_A+n_B-1}{2}}\left(\frac{2\pi}{n_C}\right)^{\frac{n_C-1}{2}} \\ \leq \frac{\pi^2}{4^2}F_k e^{-\frac{c}{4}N^{1+\epsilon}}\sum_{r=N^\epsilon}^{N}\binom{N}{r}\sqrt{(N-r)r}\left(\frac{2\pi}{N-r}\right)^{\frac{N-r-1}{2}}\left(\frac{2\pi}{r}\right)^{\frac{r-1}{2}}, \quad \text{(B.15)}$$

where $F_k = 2^{k/2}e^{ck}$ and we have used $r = n_C$ and $N - r = n_A + n_B$. The factor $\binom{N}{r}$ accounts for the number ways of placing r angles in C and $N-r$ angles in $A\cup B$. The summand is maximised when $r = N/2$ and so, using the bound $\sqrt{2\pi n}n^n e^{-n} \leq n! \leq 2\sqrt{2\pi n}n^n e^{-n}$ when $n \geq 1$ for the factorial, we get the contribution from $\mathcal{F}_>$ is less than or equal to

$$(N-N^\epsilon)\frac{\pi^2}{4}F_k e^{-\frac{c}{4}N^{1+\epsilon}}\frac{2\sqrt{2\pi N}N^N}{(N\pi)(N/2)^N}\frac{N}{2}\left(\frac{2\pi}{N}\right)^{\frac{N-1}{2}}\left(\frac{2\pi}{N}\right)^{-\frac{1}{2}}2^{N/2-1} \\ = \mathcal{O}(N^{3/2})\sqrt{N}\left(\frac{2\pi}{N}\right)^{\frac{N-1}{2}}\exp\left(-\frac{c}{4}N^{1+\epsilon}+\frac{3\ln(2)}{2}N\right), \quad \text{(B.16)}$$

which is negligible in comparison to the contribution from $J^{(1)}$ given in (B.6).

This leaves the evaluation of $\mathcal{F}_\leq$. If we restrict the expression (B.12) to $\mathcal{F}_\leq$ and follow exactly the same steps as for the contribution from $\mathcal{F}_>$ above we get that, since $n_A \geq N/4$ and $n_C \geq 1$,

$$\sum_{\boldsymbol{s}\in\mathcal{F}_\leq}\int_{V(\boldsymbol{s})}d^N\theta \prod_{(p,q)\in E}|\sin(\theta_p-\theta_q)|\prod_{(p,q)\in E^c}|\cos(\theta_p-\theta_q)| \\ \leq \frac{\pi^2}{4}F_k e^{-\frac{c}{4}N}\sum_{r=1}^{N^\epsilon}\binom{N}{r}\sqrt{(N-r)r}\left(\frac{2\pi}{N-r}\right)^{\frac{N-r-1}{2}}\left(\frac{2\pi}{r}\right)^{\frac{r-1}{2}}, \quad \text{(B.17)}$$

which matches (B.15), except for the exponential factor and the summation range. Now, using the bound for the factorial and removing a factor of $\sqrt{N}(2\pi/N)^{(N-1)/2}$ gives that (B.17) is less than or equal to

$$\leq \frac{\pi^2}{4}F_k e^{-\frac{c}{4}N}\frac{\sqrt{N}}{2\pi}\left(\frac{2\pi}{N}\right)^{\frac{N-1}{2}}\sum_{r=1}^{N^\epsilon}((1-r/N)^{-N})^{3/2}\left(\frac{N-r}{r}\right)^{\frac{3r}{2}}\sqrt{r}\left(\frac{N-r}{N}\right)^{1/2}.$$

Finally, since $r \le N^\epsilon$ we have $(1 - r/N)^{-N} \le (1 - N^{\epsilon-1})^{-N} = \exp(-N \ln(1 - N^{\epsilon-1})) = \exp(N^\epsilon + \mathcal{O}(N^{2\epsilon-1}))$ and $((N - r)/r)^{3r/2} \le \exp(3N^\epsilon \ln(N)/2)$. Therefore the contribution from (B.17) is of the form

$$\mathcal{O}(N^{3\epsilon/2})\sqrt{N}\left(\frac{2\pi}{N}\right)^{\frac{N-1}{2}} \exp\left(-\frac{c}{4}N + \frac{3}{2}N^\epsilon + \frac{3}{2}N^\epsilon \ln(N) + \mathcal{O}(N^{2\epsilon-1})\right),$$

which, again, is negligible in comparison to (B.6). □

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