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Proof-terms for classical and intuitionistic resolution (extended abstract)

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Proof-terms for classical and intuitionistic resolution (extended abstract)*

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Abstract. We exploit a system of realizers for classical logic, and a translation from resolution into the sequent calculus, to assess the intuitionistic force of classical resolution for a fragment of intuitionistic logic. This approach is in contrast to formulating locally intuitionistically sound resolution rules. The techniques use the $\lambda\mu\epsilon$-calculus, a development of Parigot's $\lambda\mu$-calculus.

1 Introduction

1.1 Local methods for intuitionistic logic

It is standard practice to draw a sharp distinction between *local* methods of automated deduction for classical logic, inspired by techniques such as Robinson's resolution [17] and Maslov's inverse method [9], and *global* methods, those inspired by Gentzen's sequent calculus [8] and Smullyan's tableaux systems [18].

For a non-classical logic, such as intuitionistic propositional logic, global methods are more easily developed (see *e.g.*, [6, 14]) and, as Mints points out in his [11], many attempts to formulate local methods fail to preserve the essential properties of local methods for classical systems. He goes on to formulate a list of criteria by which a system can qualify as "resolution," and to present a local method which satisfies them.

It is the propositional structure of the resolution method that gives it its combinatorial strength. The viewpoint outlined above suggests that in obtaining a local method for a non-classical logic we try to preserve the propositional structure of the standard method as far as is possible, modifying only the condition under which a particular clash or connection is sound. The complexity of the local soundness check should be small compared with the complexity of the propositional search space.

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For intuitionistic propositional logic this approach is of particular significance. Gentzen [8] formulates the intuitionistic sequent calculus LJ as a *restriction* of the sequent calculus for classical logic LK. The restriction concerns the use of *weakening* on the right. Since LJ is a restriction of LK, the latter is complete for intuitionistic logic, but not sound. By studying the structure of LK derivations under permutation of inference rules, it is possible to assess their intuitionistic force and hence use classical search to determine intuitionistic provability. In [16], we used terms of the $\lambda\mu\epsilon$-calculus, a variant of Parigot's $\lambda\mu$-calculus, as a system of *realizers* for sequent derivations to present such an analysis, and hence give a characterization of the search space for intuitionistic logic in terms of that for classical logic. Operations on the realizers were instrumental in enlarging the set of classical derivations that could be considered to have non-trivial intuitionistic force. This in turn simplified the search space.

In this paper, we show how to extend this analysis to resolution. We do this by reconsidering Mints' Maslov-inspired translations between resolution systems and the sequent calculus. Unlike Mints, our goal is not to *modify* resolution to make it locally intuitionistically sound, but to express the intuitionistic force of standard (classical) resolution, and thereby give a characterization of the search space for intuitionistic logic in terms of that for classical resolution.

1.2 Overview of technical results

In § 2, we summarize the results of [16]. A translation of sequent derivations into $\lambda\mu\epsilon$ terms is given for the disjunction-free fragment of classical logic. The terms are seen as realizers for the derivations. A realizer is said to be *intuitionistic* if it satisfies a certain structural condition related to weakening and rule permutation. A sequent is intuitionistically provable if there is a classical sequent proof of it whose realizer is intuitionistic (Theorem 1). By defining a (finite) restricted operation of permutation on realizers we obtain a completeness result (Theorem 2): if a sequent is intuitionistically provable, then there is a classical derivation for which some permutation of its realizer is intuitionistic. True to the spirit of the outline above, the intuitionistic search space (for the fragment considered) is rendered as a restriction of the classical search space together with a computable test for intuitionistic soundness.

In § 3, we show that (inessential variants of) Mints' translations establish tight connections between *uniform proofs* and resolution derivations (Lemma 8). We also show that for classical logic permutations in the resolution search space correspond to permutations in the sequent search space (Proposition 11). The results of [16] then give realizers for (classical) resolution derivations.

In § 4, we use the results summarized in § 2 to assess the intuitionistic force of (classical) resolution derivations. An intuitionistic soundness result for resolution is proved as Theorem 13. By a careful analysis, and modification of Mints' translation, the class of resolution derivations that can be deemed to have non-trivial intuitionistic force can be extended. With respect to this extended class of derivations a completeness result is established as Theorem 16.

2 The intuitionistic force of classical search

In [16], we presented a characterization of when search by means of the classical sequent calculus yields sufficient evidence for provability in a fragment of intuitionistic logic, namely propositional intuitionistic logic with *implication*, *negation* and *conjunction*. The characterization takes account of the rule permutability properties of both logics. We defined a translation, $[\![-]\!]$, from classical sequent derivations into terms of the $\lambda\mu\epsilon$-calculus [16, 13] and gave a combinatorial criterion for the corresponding $\lambda\mu\epsilon$-term to determine intuitionistic validity of the endsequent.[1] Based on this characterization, we defined a proof procedure for intuitionistic logic considered which extends the notion of uniform proof as defined by Miller. The procedure was shown to be sound and complete for the fragment.

The $\lambda\mu\epsilon$-calculus thus provides realizers for multiple-conclusioned sequents $\Gamma \longrightarrow A, \Delta$, where A is a distinguished, or active, formula. The formulae in Δ are indexed by names $\alpha, \beta, \ldots$, which are different from variables and which may also occur in terms. The $\lambda\mu\epsilon$-calculus differs from Parigot's $\lambda\mu$-calculus in that it makes use of *explicit substitutions* to represent the $\supset L$-rule.[2] The rules of $\lambda\mu\epsilon$ are presented in Appendix A in Figures 1 and 2.

A $\lambda\mu\epsilon$-term is identified as *intuitionistic* if the free names of the term model weakening. For example, consider the sequent $B \longrightarrow A \supset B, D \supset E$. An intuitionistic search for a proof of this sequent based on LJ will be successful only if we reduce the formula $A \supset B$ first; *i.e.*, closer to the root of the derivation. This is not so classically. If a search according to the multiple-conclusioned classical rules of LK results in a (classical) proof in which the formula $D \supset E$ is reduced at the root of the proof, we would need to detect from the corresponding $\lambda\mu\epsilon$-term whether this reduction can be considered superfluous. Therefore, if we can judge this property, we can use the classical proof to determine that the sequent is intuitionistically provable. In fact, an intuitionistic proof of the sequent can be constructed from the data to hand. In the $\lambda\mu\epsilon$-term $\lambda x\colon A.\mu\beta.[\gamma]\lambda y\colon D.\mu\epsilon.[\beta]b$, corresponding to the obvious classical proof outlined above, this amounts to determining that certain subterms, here the abstraction $\lambda y\colon D$, model weakening. Full details are provided in [16] in which we prove the following theorem for the fragment considered:

Theorem 1. *Let ϕ be a classical sequent derivation of $\Gamma \longrightarrow A, \Delta$. If $[\![\phi]\!]$ is an intuitionistic $\lambda\mu\epsilon$-term, then $\Gamma \longrightarrow A, \Delta$ is intuitionistically provable.*

We continued by giving a proof procedure for intuitionistic logic by extending a definition of Miller *et al.* [10]: we defined a *uniform* proof to be a sequent derivation where right rules are closer to the root than left rules. We call a proof *fully uniform* if right rules are preferred even over putative axioms, thereby

[1] The restriction to a fragment of intuitionistic logic arises from the simple fact that $\lambda\mu\epsilon$ has not been extended to disjunction; we see few difficulties in providing an extension suitable for our purposes as we are not intending to ensure properties of cut-elimination that $\lambda\mu$ was introduced to regulate.

[2] This formulation is closely related to the notion of *construction* introduced in [15].

ensuring that the succedents of all axioms consist of atomic formulae only. We then established the following completeness result for the fragment:

Theorem 2. *If the sequent $\Gamma \longrightarrow A, \Delta$ is intuitionistically provable, then there exists a fully uniform proof ψ of this sequent such that $[\![\psi]\!]$ is intuitionistic.*

Hence to check provability of a sequent it is enough to construct a uniform proof and then to check, for all possible axiom instances and for all possible exchanges of $\supset L$ and $\neg L$-rules, whether any of the corresponding $\lambda\mu\epsilon$-terms are intuitionistic.

3 Resolution in classical logic

In this section, we show that, under inessential modifications, Mints' translations between resolution systems and the sequent calculus establish tight connections between uniform proofs and resolution derivations in both classical and intuitionistic logic (Lemma 8). We also show that, for classical logic, permutations in the resolution search space correspond to permutations in the search space generated by the sequent calculus (Proposition 11). The results of [16] then give realizers for (classical) resolution derivations.

We begin by recalling from [11] the construction of a set of clauses of bounded complexity from an arbitrary propositional formula.

Definition 3. *A formula A is a* classical clause *if it is a disjunction $A_1 \vee \cdots \vee A_n$, with $n > 0$ and each A_i, $1 \leq i \leq n$, a literal. Clauses which differ only in the numbering or order of literals are identified.*

Lemma 4. *For any propositional formula A, a set X_A of clauses of length ≤ 3 can be constructed in linear space and time (of the length of A) such that A is valid if and only if X_A is inconsistent.*

Proof. We construct, by induction over the structure of A, a set of clauses for the formulae $\neg X \vee A$ and $\neg A \vee X$, where X is a propositional variable; for details see [11].

Resolution is defined as a calculus for deriving a judgement $\Gamma \vdash C$, where Γ is a set of clauses and C is a clause. The precise definition follows below.

Definition 5. *Let Γ be a set of clauses, let C be a clause and let A and B be atoms. A* resolution derivation *of a judgement $\Gamma \vdash C$ is given by:*

$$\frac{}{\Gamma, C, \Gamma' \vdash C}\ Ax \qquad \frac{}{\Gamma \vdash A \vee \neg A}\ EM \qquad \frac{\Gamma \vdash \neg A_1 \vee C_1 \cdots \Gamma \vdash \neg A_n \vee C_n}{\Gamma, A_1 \vee \cdots \vee A_n \vdash C_1 \vee \cdots \vee C_n}\ Res.$$

In the last case, we call the formula $A_1 \vee \cdots \vee A_n$ the input formula *of the resolution rule.*

Note that weakening is admissible in this system: whenever $\Gamma \vdash C$ and also $\Gamma \subseteq \Gamma'$, then also $\Gamma' \vdash C$.

Mints [11] proves the following theorem:

Theorem 6. *A formula A is classically provable if and only if there is a resolution derivation $X_A \vdash \emptyset$.*

This is proved by transforming a resolution derivation into a sequent derivation where formulae consist only of disjunction and negation and vice versa.

Because our analysis of resolution is based on translating resolution derivations into sequent derivations, which in turn are translated into $\lambda\mu\epsilon$-terms, and because the $\lambda\mu\epsilon$-calculus has no construction for disjunction, we replace disjunction in favour of conjunction, implication and negation. We will show that Theorem 6 also holds for the appropriate modification of these translations.

We start with a translation of a resolution proof into a derivation in the classical sequent calculus LK without cut.

Definition 7. *Define the concatenation of n sequents $\Gamma_1 \longrightarrow \Delta_1, \ldots, \Gamma_n \longrightarrow \Delta_n$ to be the sequent $\Gamma_1, \ldots, \Gamma_n \longrightarrow \Delta_1, \ldots, \Delta_n$.*

(i) *By induction over the structure of clauses we define a sequent derivation of $\Gamma \longrightarrow \Delta$, for each clause C with a polarity $\{+,-\}$. If C is the clause $C_1^+ \vee C_2^+$, then we define $[\![C_1^+ \vee C_2^+]\!]$ to be the concatenation of the two sequents $[\![C_1^+]\!] = \Gamma_1 \longrightarrow \Delta_1$ and $[\![C_2^+]\!] = \Gamma_2 \longrightarrow \Delta_2$. For the remainder of the clauses the definition is as follows:*

$$\begin{aligned}
[\![(\neg A \vee \neg B \vee \neg C)^-]\!] &= \neg(A \wedge B \wedge C) \longrightarrow \\
[\![(\neg A \vee \neg B \vee C)^-]\!] &= (A \wedge B) \supset C \longrightarrow \\
[\![(\neg A \vee B \vee C)^-]\!] &= (A \wedge \neg B) \supset C \longrightarrow \\
[\![(A \vee B \vee C)^-]\!] &= (\neg A \wedge \neg B) \supset C \longrightarrow \\
[\![(\neg A \vee \neg B)^-]\!] &= \neg(A \wedge B) \longrightarrow \\
[\![(\neg A \vee B)^-]\!] &= A \supset B \longrightarrow \\
[\![(A \vee B)^-]\!] &= \neg A \supset B \longrightarrow
\end{aligned}
\qquad
\begin{aligned}
[\![(\neg A)^-]\!] &= \longrightarrow A \\
[\![(A)^-]\!] &= A \longrightarrow \\
[\![(\neg A)^+]\!] &= A \longrightarrow \\
[\![(A)^+]\!] &= \longrightarrow A
\end{aligned}$$

(ii) *If X is a set of clauses $C_1, \ldots, C_n$ and C is a clause, we denote the sequence resulting from concatenation of $[\![C_1^-]\!], \ldots, [\![C_n^-]\!]$ and $[\![C^+]\!]$ by $[\![X^-]\!] \longrightarrow [\![C^+]\!]$. By induction over the derivation of $X \vdash C$, we define a classical sequent derivation of $[\![X^-]\!] \longrightarrow [\![C^+]\!]$ as follows:*

- *With each axiom $\Gamma, C, \Gamma' \vdash C$, associate the appropriate derivation from the axioms;*
- *With each axiom $X \vdash p \vee \neg p$, associate the derivation consisting of the axiom $[\![X^-]\!], p \longrightarrow p$;*
- *If the input formula is $\neg A \vee \neg B$ and if we have resolution derivations of $X \vdash A \vee C_1$ and $X \vdash B \vee C_2$, then we construct the following derivation:*

$$\dfrac{\dfrac{[\![X^-]\!] \longrightarrow A, [\![C_1^+]\!] \qquad [\![X^-]\!] \longrightarrow B, [\![C_2^+]\!]}{[\![X^-]\!] \longrightarrow A \wedge B, [\![C_1^+]\!], [\![C_2^+]\!]}\wedge R}{[\![X^-]\!], \neg(A \wedge B) \longrightarrow [\![C_1^+]\!], [\![C_2^+]\!]}\neg L;$$

- *If the input formula is* $\neg A_1 \vee \neg A_2 \vee \neg A_3$*, then the construction is similar;*
- *If the input formula is* $\neg A \vee \neg B \vee C$ *and if we have resolution derivations of* $X \vdash A \vee C_1$*,* $X \vdash B \vee C_2$ *and* $X \vdash \neg C \vee C_3$*, then we construct the following derivation:*

$$\dfrac{\dfrac{[\![X^-]\!] \longrightarrow A, [\![C_1^+]\!] \qquad [\![X^-]\!] \longrightarrow B, [\![C_2^+]\!]}{[\![X^-]\!] \longrightarrow A \wedge B, [\![C_1^+]\!], [\![C_2^+]\!]} \wedge R \qquad [\![X^-]\!], C \longrightarrow [\![C_3^+]\!]}{[\![X^-]\!], (A \wedge B) \supset C \longrightarrow [\![C_1^+]\!], [\![C_2^+]\!], [\![C_3^+]\!]} \supset L;$$

- *If the input formula is* $\neg A \vee B \vee C$ *and if we have resolution derivations of* $X \vdash A \vee C_1$*,* $X, B \vdash C_2$ *and* $X \vdash \neg C \vee C_3$*, then we construct the following derivation:*

$$\dfrac{\dfrac{[\![X^-]\!] \longrightarrow A, [\![C_1^+]\!] \qquad \dfrac{[\![X^-]\!], B \longrightarrow [\![C_2^+]\!]}{[\![X^-]\!] \longrightarrow \neg B, [\![C_2^+]\!]} \neg R}{[\![X^-]\!] \longrightarrow A \wedge \neg B, [\![C_1^+]\!], [\![C_2^+]\!]} \wedge R \qquad [\![X^-]\!], C \longrightarrow [\![C_3^+]\!]}{[\![X^-]\!], (A \wedge \neg B) \supset C \longrightarrow [\![C_1^+]\!], [\![C_2^+]\!], [\![C_3^+]\!]} \supset L;$$

- *If the input formula is* $\neg A \vee B$ *and we have resolution derivations* $X \vdash A \vee C_1$ *and* $X \vdash \neg B \vee C_2$*, then we construct the following derivation:*

$$\dfrac{[\![X^-]\!] \longrightarrow A, [\![C_1^+]\!] \qquad [\![X^-]\!], B \longrightarrow [\![C_2^+]\!]}{[\![X^-]\!], A \supset B \longrightarrow [\![C_1^+]\!], [\![C_2^+]\!]} \supset L;$$

- *If the input formula is* $A \vee B$ *and if we have resolution derivations* $X \vdash \neg A \vee C_1$ *and* $X \vdash \neg B \vee C_2$*, we obtain the following derivation:*

$$\dfrac{\dfrac{[\![X^-]\!], A \longrightarrow [\![C_1^+]\!]}{[\![X^-]\!] \longrightarrow \neg A, [\![C_1^+]\!]} \neg R \qquad [\![X^-]\!], B \longrightarrow [\![C_2^+]\!]}{[\![X^-]\!], \neg A \supset B \longrightarrow [\![C_1^+]\!], [\![C_2^+]\!]} \supset L;$$

- *If the input formula is* $A_1 \vee A_2 \vee A_3$*, then the construction is similar;*
- *If the input formula is* A *and if we have a resolution derivation* $X \vdash \neg A \vee C$*, then we have a sequent derivation of* $[\![X^-]\!], A \longrightarrow [\![C^+]\!]$*, by assumption, which we simply take;*
- *If the input formula is* $\neg A$ *and if we have a resolution derivation* $X \vdash A \vee C$*, then we have a sequent derivation of* $[\![X^-]\!] \longrightarrow A, [\![C^+]\!]$*, by assumption, which we simply take.*

By applying the translation of sequent derivations into $\lambda\mu\epsilon$-terms, as given in [16], we obtain a $\lambda\mu\epsilon$-term for each resolution derivation. Moreover, this sequent derivation is *uniform* in the sense that right rules are closer to the root than left rules. In [16] we also define a corresponding notion for $\lambda\mu\epsilon$-terms which captures the uniformity constraint by suitable constraints on the occurrence of the term constructors in the $\lambda\mu\epsilon$-calculus.

Lemma 8. (i) *The sequent derivation associated with a resolution derivation is uniform.*
(ii) *The $\lambda\mu\epsilon$-term associated with each resolution derivation is uniform.*

Proof. (i) Note that the right-hand side of all root sequents of such a sequent derivation is atomic. Furthermore any intermediate non-atomic formula on the right is reduced as soon as it occurs. Hence the sequent derivation is uniform.
(ii) See [16].

As an example, we will give the resolution derivation and the corresponding $\lambda\mu\epsilon$-term for the formula $A \supset A$. According to Lemma 4 the set $X_{A \supset A}$ is the set $\{\neg X \vee \neg A \vee A, A \vee X, \neg A \vee X, \neg X\}$. A resolution derivation of the empty clause from a subset of these clauses can be obtained as follows:

$$\dfrac{\dfrac{\neg X \vdash \neg A \vee A \qquad \neg X \vdash \neg X}{\neg X, A \vee X \vdash A} \qquad \neg X \vdash \neg X}{\neg X, A \vee X, \neg A \vee X \vdash \emptyset}.$$

The corresponding sequent derivation is

$$\dfrac{\dfrac{\dfrac{A \longrightarrow A, X}{\longrightarrow \neg A, A, X}\neg R \qquad X \longrightarrow X}{\neg A \supset X \longrightarrow A, X}\supset L \qquad X \vdash X}{\neg A \supset X, A \supset X \longrightarrow X}\supset L.$$

The corresponding $\lambda\mu\epsilon$-term, which is obtained by extending the translation of sequent calculus into natural deduction to the case of multiple formulae on the right-hand side, is $\mu\beta.[\beta]w(\mu\alpha.[\beta]u(\lambda a\colon A.\mu\delta.[\alpha]a))$.

Observe that the sequent derivations obtained by translating resolution derivations do not use weakening. Moreover, these derivations can be rewritten in such a way that the axioms have the form $A \longrightarrow A$, but at the expense of introducing weakening at the root of the derivation. These properties are a consequence of the absence of a weakening rule in the resolution calculus. A translation of classical sequent derivations into resolution derivations can be given only for sequents without weakening in the middle of the derivation. Mints [11] gives such a translation. Because every sequent derivation where all formulae are either clauses or elements of $[\![X-]\!]$ can be transformed into one in which weakening occurs at the root of the derivation only, for each derivable sequent $\Gamma \longrightarrow \Delta$ with this property there is a subsequent $\Gamma' \longrightarrow \Delta'$ which has a resolution proof. This translation is part of the following theorem.

Theorem 9. *Consider a uniform sequent derivation of $[\![X^-]\!], \Delta_1 \longrightarrow \Delta_2$ such that (a) Δ_1 and Δ_2 consist of atoms; (b) all weakenings occur only at the root of the derivation; and (c) all axioms have the form $A \longrightarrow A$. Then there is a resolution derivation of $X \vdash \neg\Delta_1' \vee \Delta_2'$, in which Δ_1' and Δ_2' are subsets of Δ_1 and Δ_2 respectively. Furthermore, all of the formulae in Δ_1 and Δ_2 that are not obtained by weakening are in Δ_1' and Δ_2' respectively.*

Proof. We give only a sketch. Let Ψ be the subderivation above the last weakening rule; proceed by induction over Ψ. The hypothesis about weakening ensures that we can always construct a resolution derivation with $[\![A^-]\!]$ as the resolution formula whenever the last rule was an A-L-rule.

Theorem 6 can now be obtained as a corollary.

Corollary 10. *A formula A is classically provable if and only if there is a resolution derivation $X_A \vdash \emptyset$.*

Proof. Suppose A is classically provable. By Lemma 4, the set X_A is inconsistent, hence there is a derivation of $[\![X_A^-]\!] \longrightarrow$. Theorem 9 implies the existence of a resolution derivation $X_A \vdash \emptyset$.

Conversely, given a resolution derivation of $X_A \vdash \emptyset$. The second part of Definition 7 yields a derivation $[\![X_A^-]\!] \longrightarrow$; hence X_A is inconsistent. So A is provable.

A central idea of [16] is to investigate when permutations transform a uniform sequent derivation which is non-intuitionistic into an intuitionistic derivation. Here we show how permutations in the sequent calculus are related to the choice of input formulae in the resolution calculus. Later on we will transfer this connection to intuitionistic logic. Because the formulae occurring in sequent derivations arising from resolution derivations have a rather simple structure, it suffices to consider exchanges of $\supset L$-rules and $\neg L$-rules. These are the only two rules whose exchange leads from a uniform derivation to another uniform derivation. The details are contained in the following proposition:

Proposition 11. (i) *Let*

$$\dfrac{\dfrac{X \vdash \neg A_1 \vee C_1 \qquad X \vdash \neg A_n \vee C_n}{X, A_1 \vee \cdots \vee A_n \vdash C_1 \vee \cdots C_n} \qquad X \vdash \neg B_1 \vee D_1 \cdots X \vdash \neg B_m \vee D_m}{X, A_1 \vee \cdots \vee A_n, \neg C_1 \vee B_1 \cdots \vee B_m \vdash C_2 \vee \cdots C_n \vee D_1 \vee \cdots \vee D_m}$$

be a resolution derivation and let

$$\dfrac{\dfrac{X \vdash \neg A_1 \vee C_1 \qquad X \vdash \neg B_1 \vee D_1 \cdots X \vdash \neg B_m \vee D_m}{X, \neg C_1 \vee B_1 \cdots \vee B_m \vdash \neg A_1 \vee D_1 \vee \cdots \vee D_m} \qquad X \vdash \neg A_2 \vee C_2 \cdots X \vdash \neg A_n \vee C_n}{X, A_1 \vee \cdots \vee A_n, \neg C_1 \vee B_1 \cdots \vee B_m \vdash C_2 \vee \cdots C_n \vee D_1 \vee \cdots \vee D_m}$$

be the derivation in which the application of the two instances of the resolution rule are exchanged. The translation of the second resolution derivation into a sequent derivation is obtained by exchanging the two left-rules to which the two applications of the resolution rule are translated.
(ii) *Conversely, given a uniform sequent derivation of a sequent $\Gamma \longrightarrow \Delta$, where Γ consists only of clauses and Δ only of atoms, the exchange of $\neg L$ and $\supset L$-rules corresponds to the exchange of two resolution rules.*

Proof. For first part, check each resolution formula in turn. For the second part, calculate the resolution derivations for all possible exchanges.

Intuitively, this proposition indicates that the search for a uniform derivation of a sequent with formulae in clausal form is as complicated as the search for a resolution derivation of the corresponding clauses. In other words, this proposition shows that the essential aspect of the resolution method is the transformation of formulae into clausal form; the complexity of finding the right input formula is the same as finding the right permutation in the sequent derivation.

This analysis carries over to the intuitionistic case (see next section), including the case of a resolution formula $(A \supset B) \supset C$. This is important because, in contrast to the classical case, in intuitionistic logic permutations of inferences do matter. Classically, but not intuitionistically, any permutation of a sequent derivation transforms a proof only into a proof and a non-proof only into a non-proof.

4 Resolution in intuitionistic logic

In this section, we develop a resolution calculus for a fragment of intuitionistic logic without disjunction based on the ideas above. The idea is to retain the resolution calculus for classical logic, because this calculus has no constraints on the order in which input formulae are taken. The translation of such resolution derivations into $\lambda\mu\epsilon$-terms is used to decide when the derivation provides sufficient evidence that the formula is intuitionistically provable.

4.1 Mints' intuitionistic resolution

Mints [11] also defines a resolution calculus for intuitionistic logic. It is easily seen that his calculus corresponds to constructing uniform proofs in LJ, with weakening pushed as close to the root as much as possible. It is important to note that Mints' calculus is not a restriction of classical resolution, but has special rules for each connective of the logic. Moreover, clauses are no longer formulae, but sequents of the form $A \supset B \rightarrow C$, $A \rightarrow B \vee C$ and $A_1, \ldots, A_n \rightarrow B$ with $n \leq 3$, where all formulae are propositional variables. Mints constructs, for every formula A, a set of clauses X_A, the translation of these clauses into one formula Y_A and an atom F such that A is intuitionistically provable if and only if $Y_A \longrightarrow F$ is provable in LJ.

Mints then gives translations between resolution derivations and LJ derivations with weakening pushed down to the root as much as possible, and obtains as a corollary that a formula A is intuitionistically provable if and only if $X_A \vdash F$ is derivable in the resolution calculus.

The rules for implication and negation cannot be obtained as special cases of the rules for classical resolution, hence it is not immediately possible to transfer the implementations of classical resolution to the intuitionistic case. The reason is that derivations may contain weakening at places other than at their roots. As an example, consider the LK-derivation

$$\cfrac{\cfrac{\cfrac{\Gamma \longrightarrow B, \Delta}{\Gamma, A \longrightarrow B, \Delta}\, W\, L}{\Gamma \longrightarrow A \supset B, \Delta}\supset R \qquad C \longrightarrow C}{\Gamma, (A \supset B) \supset C \longrightarrow C, \Delta}\supset L,$$

where the weakening of the formula A cannot be pushed to the root of the derivation. Because the construction of Theorem 9 works only for derivations where weakening is applied only as the last rule of the derivation, there can be no resolution derivation corresponding to this sequent derivation. Indeed, the method of the previous section, which uses the (classical) equivalence $(A \supset B) \supset C \equiv (A \vee C) \wedge (\neg B \vee C)$, yields only the following resolution derivation:

$$\frac{\Gamma \vdash B \vee \Delta \qquad \emptyset \vdash \neg C \vee C}{\Gamma, \neg B \vee C \vdash \Delta},$$

where Δ is interpreted as the disjunction of its members, and the input formula $A \vee C$ is added by weakening at the end and not obtained by a resolution step.

4.2 The intuitionistic force of classical resolution

In this section, we exploit the results given above and in [16] to assess the intuitionistic force of classical resolution. We take the association of $\lambda\mu\epsilon$ terms with resolution derivations, as developed in the previous section, and identify when they provide evidence for intuitionistic provability.

We restrict our treatment to intuitionistic formulae containing negation, conjunction and implication, since as formulated, the $\lambda\mu\epsilon$-calculus has no coproducts and disjunction is a primitive connective in intuitionistic logic. A treatment of disjunction in $\lambda\mu\epsilon$, sufficient for our purposes, should present no problems as we are not constrained to achieve strong normalization with respect to cut-elimination: the original motivation for the calculus.

The translation of formulae into clauses, referred to in Lemma 4, produces clauses given by the BNF

$$C ::= A_1 \vee A_2 \mid \neg A_1 \vee A_2 \mid \neg A_1 \vee \neg A_2 \vee A_3 \mid \neg A_1 \vee \neg A_2 \mid \neg A_1 \supset A_2,$$

where A_1, A_2, A_3 are all atomic. In the sequel we restrict attention to such clauses. Note that the transformations leading from formulae to clauses are no longer equivalences: the formula $(A \supset B) \supset C$ intuitionistically implies $(A \vee C) \wedge (\neg B \vee C)$, but not vice versa. In all other cases, the transformations that lead from formulae to clauses are intuitionistic equivalences.

The correspondence between the $\lambda\mu\epsilon$-calculus and intuitionistic logic is based on a sequent calculus with multiple conclusions for intuitionistic logic, as presented in [5, 21]. This calculus is the same as the calculus LK [8] for classical logic except for the $\supset R$ and $\neg R$-rules:

$$\frac{\Gamma, A, \longrightarrow B}{\Gamma \longrightarrow A \supset B, \Delta} \supset R \quad \frac{\Gamma, A \longrightarrow}{\Gamma \longrightarrow \neg A, \Delta} \neg R$$

The translation of resolution derivations into $\lambda\mu\epsilon$-terms leads directly to a criterion when a resolution derivation gives rise to an intuitionistic proof.

Definition 12. *A resolution derivation is said to be* intuitionistic *if it translates into an intuitionistic $\lambda\mu\epsilon$-term.*

The soundness theorem, *i.e.*, that an intuitionistic resolution derivation indeed gives rise to an intuitionistic sequent derivation is as follows:

Theorem 13. *A formula A is intuitionistically provable if there is an intuitionistic resolution derivation $X_A \vdash \emptyset$.*

Proof. The translation of the resolution derivation produces a derivation of $[\![X_A^-]\!] \longrightarrow$. By assumption the $\lambda\mu\epsilon$-term corresponding to this derivation is intuitionistic, hence the sequent is intuitionistically provable by Theorem 1.

To establish completeness, the translation of resolution derivations into sequent derivations must be modified to enable a larger class of the former to be recognized as having non-trivial intuitionistic force. For example, consider the resolution derivation for the formula $A \supset A$, shown after Lemma 8. The corresponding sequent derivation is a classical sequent proof, but the resulting $\lambda\mu\epsilon$-term is not intuitionistic. However, we can modify the translation of resolution derivations such that this resolution derivation is translated into a sequent proof whose $\lambda\mu\epsilon$-term is intuitionistic. The modification is to map resolutions of this form,

$$\frac{\dfrac{X \vdash \neg A \vee B \vee C_1 \qquad X \vdash \neg C \vee C}{X, A \vee C \vdash B \vee C_1 \vee C} \qquad X \vdash \neg C \vee C.}{X, A \vee C, \neg B \vee C \vdash C_1 \vee C}$$

where both formulae $A \vee C$ and $\neg B \vee C$ occur as input formulae, to the sequent derivation

$$\frac{\dfrac{[X^-], A \longrightarrow B, [C_1^+]}{[X^-] \longrightarrow A \supset B, [C_1^+]} \supset R \qquad C \longrightarrow C}{[X^-], (A \supset B) \supset C \longrightarrow [C_1^+], C} \supset L.$$

The modified transformation maps resolution derivations $X \vdash C$ into sequent derivations of $\Gamma \longrightarrow [\![C]\!]^+$, in which Γ is the result of replacing some choice of pairs of clauses $A \vee C$ and $\neg B \vee C$ by $(A \supset B) \supset C$ in $[\![X^-]\!]$.

The soundness theorem for the modified translation is as follows:

Theorem 14. *A formula A is intuitionistically provable if there is a resolution derivation of $X_A \vdash \emptyset$ such that the $\lambda\mu\epsilon$-term corresponding to the modified translation into the sequent calculus is intuitionistic.*

Proof. The translation of the resolution derivation produces either a derivation $[\![X_A^-]\!] \longrightarrow$ or a derivation $\Gamma \longrightarrow$ where Γ results from replacing pairs of clauses $\neg A \supset C$ and $B \supset C$ by $(A \supset B) \supset C$. By assumption the $\lambda\mu\epsilon$-term corresponding to this derivation is intuitionistic, hence there is an intuitionistic derivation of this sequent [16]. The following lemma, a modification of Lemma 4, now yields the claim.

Lemma 15. *A formula A is intuitionistically provable if and only if there is an intuitionistic sequent derivation of $\Gamma \longrightarrow$, where Γ is the result of replacing some choice of pairs of clauses $A \vee C$ and $\neg B \vee C$ by $(A \supset B) \supset C$ in $[\![X_A^-]\!]$.*

Looking at the example of the resolution derivation for the formula $A \supset A$ again, we see that the modified translation yields a derivation

$$\dfrac{\dfrac{A \longrightarrow A}{\longrightarrow A \supset A}\supset R \qquad X \longrightarrow X}{(A \supset A) \supset X \longrightarrow X}\supset L,$$

with the $\lambda\mu\epsilon$-term $w(\lambda a\colon A.a)$, which is in fact a λ-term and hence an intuitionistic $\lambda\mu\epsilon$-term.

We need one extra step for the completeness proof. In our previous paper [16] we show that a sequent $\Gamma \longrightarrow \Delta$ is intuitionistically provable if there is a uniform classical sequent derivation such that the corresponding $\lambda\mu\epsilon$-term is intuitionistic. We now have:

Theorem 16. *Suppose we have a uniform classical sequent derivation of a sequent $[\![X^-]\!], \Delta_1 \longrightarrow \Delta_2$ such that the corresponding $\lambda\mu\epsilon$-term is intuitionistic, all formulae in X are clauses or $(A \supset B) \supset C$, all formulae in Δ_1 and Δ_2 are atoms, weakening is pushed as far as possible to the root of the derivation, and all axioms have the form $A \longrightarrow A$. Then there is an intuitionistic resolution derivation $X \vdash \neg\Delta_1' \vee \Delta_2'$, where Δ_1' and Δ_2' are subsets of Δ_1 and Δ_2 respectively. Furthermore, all of the formulae in Δ_1 and Δ_2 that are not obtained by weakening are in Δ_1' and Δ_2' respectively.*

Proof. We use the proof of Theorem 9 to construct a resolution derivation except for the case of the principal formulae $\neg A \supset C$ and $B \supset C$, if they arise from the translation of $(A \supset B) \supset C$. If neither A nor B is obtained by weakening, we construct a resolution derivation with the last two resolution formulae being $A \vee C$ and $\neg B \vee C$. If A is obtained by weakening, we construct a resolution derivation with $\neg B \vee C$ as the last resolution formulae, and if B is obtained by weakening, we construct a resolution derivation with $A \vee C$ as the last resolution derivation. The modified translation ensures that the translation of the constructed resolution derivation is also an intuitionistic sequent derivation.

Soundness and completeness now follow in exactly the same way as shown for classical logic.

Corollary 17. *A formula A is intuitionistically provable if and only if there is an intuitionistic resolution derivation of* $X_A \vdash \emptyset$.

Proof. One direction has already been shown; see Theorem 14. For the other, the argument as in Corollary 10 works for the modified translation.

Now we turn to the connection between the choice of input formulae in the resolution calculus and permutations in the sequent calculus. Consider the translation of a resolution derivation and examine all the permutations of $\supset L$-rules and $\neg L$-rules. If one permutation yields an intuitionistic $\lambda\mu\epsilon$-term, then permutation of the order of introducing the input formulae yields the image of an intuitionistic resolution derivation under the translation. Hence, the soundness and completeness properties (Corollary 17) imply that the search for an intuitionistic resolution derivation amounts essentially to the search for a permutation of the $\supset L$ and $\neg L$-rules which yields an intuitionistic $\lambda\mu\epsilon$-term. As an example of this phenomenon, consider the formula $(A \supset B \wedge (A \supset B) \supset B) \supset B$. This example is the same one we gave in our previous paper [16] to demonstrate how a permutation can transform a classical sequent derivation with no intuitionistic force into one with such force. The crucial point is that in order to obtain a uniform intuitionistic proof, the $\supset L$-rule with principal formula $(A \supset B) \supset B$ has to be the rule closest to the root of the derivation. This is also true for the resolution derivation of the formula $(A \supset B \wedge (A \supset B) \supset B) \supset B$ in that the resolution step that uses the input formula corresponding to $(A \supset B) \supset B$ must occur as late as possible; this gives rise to a $\lambda\mu\epsilon$-term which is intuitionistic.

5 Application to a logical framework

In an earlier paper [15], we argued that a λ-calculus with explicit substitutions could be use to provide a characterization, via normal forms, of the search space of SLD-resolution for minimal implicational horn-clause logic. The analysis of proof-search presented here constitutes a unified framework for both classical and intuitionistic resolution, applicable to larger propositional fragments, in which the search space of resolution is again characterized by properties of λ-terms.

Our general programme is concerned with representing logics together with (the search spaces of) associated proof-procedures, in order to build a framework for defining computational logics. Such a framework might allow the specification of a system of logic and the derivation of a logic programming language based upon it. The framework we use for defining logics, LF, is based on a dependently typed λ-calculus, $\lambda\Pi$, and the representation of logics via *judgements-as-types*.

One application of the present paper would be to consider an extension of the $\lambda\Pi$-calculus to handle multiple conclusions, perhaps via the type-theoretic connective $+$. By introducing such structure, it should be possible to exploit many of the results presented here in the constructive type-theoretic setting.

Moreover, it should be possible to lift search procedures for classical object-logics, such as resolution, to the framework. In order to deal with the quantifiers of dependent type theories such as $\lambda\Pi$, it will be necessary to exploit the techniques introduced in [14], based on Herbrand's theorem.

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A Review of the $\lambda\mu\epsilon$-calculus

We describe our variation on Parigot's $\lambda\mu$-calculus [13], which we call the $\lambda\mu\epsilon$-calculus [16]. In addition to implicational types, we also include conjunctive (or product) types and explicit substitutions $\{t/x\}$. The latter are crucial for our analysis of search in that we shall have representatives within proof-objects for possibly incomplete sequent derivations, thereby forcing a lazy approach to substitution. In addition to the constructs of $\lambda\mu\epsilon$ presented in [16], we include here rules for $\perp$.

The raw terms of the $\lambda\mu\epsilon$-calculus are given by the following grammar:

$$t ::= x \mid \lambda x{:}A\,.\,t \mid tt \mid [\alpha]t \mid \mu\alpha.t \mid t\{xt/y\} \mid \langle t,t\rangle \mid \pi(t) \mid \pi'(t)$$

The rules for well-formed terms are given in Figure 1. The reduction rules are given in Figure 2.

The $\lambda\mu\epsilon$-calculus provides an account of classical natural deduction: *i.e.*, realizers for a calculus in which multiple-conclusioned sequents can be derived without impure constraints [5]. Consequently, the form of the typing judgement in the $\lambda\mu\epsilon$-calculus is $\Gamma \vdash t : A\,,\ \Delta$, where Γ is a context familiar from the typed λ-calculus and Δ is a context containing types indexed by names, $\alpha, \beta, \ldots$, distinct from variables. The idea is that each $\lambda\mu\epsilon$-sequent has exactly one principal formula, A, on the right-hand side, *i.e.*, the leftmost one, which is the formula upon which all introduction and elimination rules operate. This formula is the type of the term t.

The term $[\alpha]t$ realizes the introduction of a name. The term $\mu\alpha.[\beta]t$ realizes the exchange operation: if A^{α} was part of Δ before the exchange, then A is the principal formula of the succedent after the exchange. Taken together, these terms also provide a notation for the realizers of contractions and weakenings on the right of a multiple-conclusioned calculus. It is also easy to detect whether a formula B^{β} in the right-hand side is, in fact, superfluous, *i.e.*, there is a derivation of $\Gamma \vdash t\colon A, \Delta'$ where Δ' does not contain B; it is superfluous if β is not a free name in t.

$$\frac{}{\Gamma, x\colon A \vdash x\colon A, \Delta}\ Ax$$

$$\frac{\Gamma, x\colon A \vdash t\colon B, \Delta}{\Gamma \vdash \lambda x\colon A.t\colon A \to B, \Delta} \to I \qquad \frac{\Gamma \vdash t\colon A \to B, \Delta \qquad \Gamma \vdash s\colon A, \Delta}{\Gamma \vdash ts\colon B, \Delta} \to E$$

$$\frac{\Gamma \vdash t\colon A, \Delta}{\Gamma \vdash [\alpha]t\colon A^{\alpha}, \Delta}\ [\text{-}] \qquad \frac{\Gamma \vdash t\colon A^{\alpha}, \Delta}{\Gamma \vdash \mu\alpha.t\colon A, \Delta}\ \mu$$

$$\frac{\Gamma \vdash t\colon A, A^{\alpha}, \Delta}{\Gamma \vdash [\alpha]t\colon A^{\alpha}, \Delta}\ [\text{-}] \qquad \frac{\Gamma \vdash t\colon \Delta}{\Gamma \vdash \mu\alpha.t\colon A, \Delta}\ \mu$$

$$\frac{\Gamma \vdash t\colon \bot, \Delta}{\Gamma \vdash [\gamma]t\colon \Delta}\ \bot \qquad \frac{\Gamma \vdash t\colon \Delta}{\Gamma \vdash \mu\delta.t\colon \bot, \Delta}\ \bot$$

$$\frac{\Gamma, w\colon B \vdash t\colon C, \Delta \qquad \Gamma \vdash s\colon A, \Delta}{\Gamma, x\colon A \to B \vdash t\,\{xs/w\}\colon C, \Delta}\ \epsilon L$$

$$\frac{\Gamma \vdash t\colon A, \Delta \qquad \Gamma \vdash s\colon B, \Delta}{\Gamma \vdash \langle t, s\rangle\colon A \wedge B, \Delta} \wedge I \qquad \frac{\Gamma \vdash t\colon A \wedge B, \Delta}{\Gamma \vdash \pi(t)\colon A, \Delta} \wedge E \qquad \frac{\Gamma \vdash t\colon A \wedge B, \Delta}{\Gamma \vdash \pi'(t)\colon B, \Delta} \wedge E$$

The second instances of the rules [-] and μ model contraction and weakening respectively.

Fig. 1. Rules for well-typed $\lambda\mu\epsilon$-terms

β	$(\lambda x\colon A.t)s \leadsto t[s/x]$	$(\lambda x\colon A.t)\{ys/z\} \leadsto \lambda x\colon A.t\{ys/z\}$
$\mu-\nu$	$(\mu\alpha^{A\to B}.t)s \leadsto \mu\beta^{B}.t[[\beta]us/[\alpha]u]$	$(ts)\{yu/z\} \leadsto t\{yu/z\}s\{yu/z\}$
$\mu-\eta$	$\mu\alpha.[\alpha]s \leadsto s$ if α not free in s	$([\alpha]t)\{ys/z\} \leadsto [\alpha]t\{ys/z\}$
$\mu-\beta$	$[\gamma](\mu\alpha.s) \leadsto s[\gamma/\alpha]$	$(\mu\alpha.t)\{ys/z\} \leadsto \mu\alpha.t\{ys/z\}$
$\mu-prod$	$\pi(\mu\alpha^{A\times B}.s) \leadsto \mu\beta^{A}.t[[\beta]\pi(u)/[\alpha]u]$	Also obvious cases for conjunctive
	$\pi'(\mu\alpha^{A\times B}.s) \leadsto \mu\gamma^{B}.t[[\gamma]\pi'(u)/[\alpha]u]$	terms. Standard variable-capture
$proj$	$\pi(\langle t, s\rangle) \leadsto t$	conditions assumed.
	$\pi'(\langle t, s\rangle) \leadsto s$	

The term $t[s/[\alpha]u]$ is t with all occurrences of a subterm of the form $[\alpha]u$ replaced by s.

Fig. 2. Reduction rules of the $\lambda\mu\epsilon$-calculus

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