Killing spinor data on distorted black hole horizons and the uniqueness of stationary vacuum black holes

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Abstract

We make use of the black hole holograph construction of [I. Rácz, Stationary black holes as holographs, Class. Quantum Grav. 31, 035006 (2014)] to analyse the existence of Killing spinors in the domain of dependence of the horizons of distorted black holes. In particular, we provide conditions on the bifurcation sphere ensuring the existence of a Killing spinor. These conditions can be understood as restrictions on the curvature of the bifurcation sphere and ensure the existence of an axial Killing vector on the 2-surface. We obtain the most general 2-dimensional metric on the bifurcation sphere for which these curvature conditions are satisfied. Remarkably, these conditions are found to be so restrictive that, in the considered particular case, the free data on the bifurcation surface (determining a distorted black hole spacetime) is completely determined by them. In addition, we formulate further conditions on the bifurcation sphere ensuring that the Killing vector associated to the Killing spinor is Hermitian. Once the existence of a Hermitian Killing vector is guaranteed, one can use a characterisation of the Kerr spacetime due to Mars to identify the particular subfamily of 2-metrics giving rise to a member of the Kerr family in the black hole holograph construction. Our analysis sheds light on the role of asymptotic flatness and curvature conditions on the bifurcation sphere in the context of the problem of uniqueness of stationary black holes. The Petrov type of the considered distorted black hole spacetimes is also determined.

Contents

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1 Introduction

In [34] it is shown that the 4-dimensional geometry of a spacetime admitting a pair of expansion- and shear-free null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ intersecting on a two-surface $\mathcal{Z} \equiv \mathcal{H}_1 \cap \mathcal{H}_2$ can uniquely be determined in the domain of dependence of $\mathcal{H}_1 \cup \mathcal{H}_2$, once suitable data —consisting of three complex functions— has been prescribed on $\mathcal{Z} = \mathcal{H}_1 \cap \mathcal{H}_2$. This set-up provides a basis for the use of the *characteristic initial value problem* in the investigation of a variety of black hole configurations by inspecting the freedom in the specification of the data on the bifurcation surface $\mathcal{Z}$ of the horizons only. In the following, we will often refer to this set-up as *Rácz's black hole holograph construction*. In fact, the set-up introduced in [33, 34] is suitable to host all of the *stationary distorted electrovacuum black hole spacetimes* —within the class of solutions to the Einstein-Maxwell equations with non-zero cosmological constant.

As it was proposed already in [33, 34] the black hole holograph construction should open a new avenue in the black hole uniqueness problem. To this end note that the Kerr-Newman family of solutions (describing a charged, rotating black hole) is an example of a family of exact solutions to the Einstein-Maxwell equations satisfying these conditions, and so it belongs this class of distorted black hole solutions. Thus, one can naturally ask what further conditions are necessary to impose on the horizons in order to single out the Kerr-Newman family from the more general class, and how restrictive these conditions are.

In the present article, we make use of a characterisation of the Kerr solution by Killing spinors to to identify the appropriate set of conditions on the data at the bifurcation sphere $\mathcal{Z}$. Killing spinors are known to represent *hidden symmetries* of a spacetime, and the existence of such a field on the Kerr spacetime is directly related to the existence of the Carter constant, which allows the geodesic equations to be completely integrated [4] —see also [6, 39]. In recent work, it has been shown that the existence of a Killing spinor on a spacetime, along with the assumption of asymptotic flatness, can be used to identify the spacetime as a member of the Kerr or Kerr-Newman families [7]. These ideas have been used in previous work to determine whether initial data corresponds to exact Kerr data. The assumption on the existence of a Killing spinor can be recast as an initial value problem, producing a set of *Killing spinor initial data equations* that must be satisfied on a spacelike initial hypersurface. These constraint equations can be used, for example, to determine whether the initial data set on the hypersurface corresponds to initial data for the exact Kerr spacetime. In a similar way, in the present article, we show it is possible to guarantee the existence of a Killing spinor on the domain of dependence $D(\mathcal{H}_1 \cup \mathcal{H}_2)$ of the intersecting expansion and shear-free horizons $\mathcal{H}_1 \cup \mathcal{H}_2$ by prescribing data for the Killing spinor and, in accordance with the black hole holograph construction, this data need only be given on the intersection surface $\mathcal{Z}$. The only restriction on the background spacetime is the prescription of the only gauge invariant Weyl spinor component Ψ_2 in terms of this initial data.

In this article, we consider the vacuum case and set the goal of identifying the Kerr family of solutions to the Einstein equations from the general class of *stationary* distorted vacuum black hole spacetimes. We give a set of conditions which must be satisfied on the bifurcation sphere $\mathcal{Z}$ to ensure the existence of a Killing spinor on the domain of dependence $D(\mathcal{H}_1 \cup \mathcal{H}_2)$ of $\mathcal{H}_1 \cup \mathcal{H}_2$ —which is nothing but the interior of the black hole region in the smooth setting, whereas it also contains the domain of outer communication if analyticity is allowed— and describe further conditions which must be given there to single out the Kerr solution. Our main results can be described as follows:

(i) *We identify the conditions that need to be imposed on the initial data surface —comprised by two expansion- and shear-free* horizons intersecting on a two-surface $\mathcal{Z}$— to ensure the existence of a Killing spinor in the domain of dependence of the horizons, $\mathcal{H}_1 \cup \mathcal{H}_2$. These conditions are stated in Lemmas 1, 2, 3, 4 and 5. These conditions set restrictions on both the free specifiable data for the Killing spinor on $\mathcal{H}_1 \cup \mathcal{H}_2$ and on the components of the Weyl spinor and some of the spin connection coefficients.

(ii) We show that the conditions obtained in (i) can be imposed by satisfying a set of *Killing spinor constraints* at the bifurcation sphere $\mathcal{Z}$. In particular, it turns out that the whole Killing spinor data can be propagated along $\mathcal{H}_1 \cup \mathcal{H}_2$ from some *basic Killing spinor data* on $\mathcal{Z}$. The Killing spinor constraints on $\mathcal{Z}$ are given in Proposition 4.

(iii) Using Rácz's black hole holograph construction, it follows that *if the Killing spinor constraints are satisfied, then one can ensure the existence of a Killing spinor everywhere in the domain of dependence of the horizons of distorted black hole.* This result is stated more precisely in Theorem 3.

(iv) All of the above results are local —i.e. independent of the topology of $\mathcal{Z}$. Note, however, if one restrict considerations to black holes, in virtue of Hawking's black hole topology theorem

[15, 16] (see also Corollary 4.2 of [33] relevant for generic distorted black holes) $\mathcal{Z}$ has to have the topology of a 2-sphere. *Using this assumption, it is shown that the Killing spinor constraints imply, in particular, that the Killing vector field —that always comes together with the existence of a generic Killing spinor— gives rise to be an axial Killing vector field on the bifurcation surface.* This result is given in Proposition 7.

(v) *We show how to encode, in terms of further constraints on $\mathcal{Z}$, that the Killing vector field determined by the Killing spinor is Hermitian (i.e. real)* in the domain of dependence of $\mathcal{H}_1 \cup \mathcal{H}_2$. The relevant conditions are spelled out in full details in Lemma 12.

(vi) *We determine the most general (regular) two-metric and associated curvature scalar on the bifurcation sphere $\mathcal{Z}$ ensuring that the Killing spinor constraints are satisfied.* This result is presented in Proposition 6.

(vii) *Notably, the aforementioned Killing spinor constraints, can be seen to be geometrically equivalent to the freedom one has in choosing initial data in Rácz's black hole holograph construction.* The corresponding results are presented in Section 6.1.

(viii) *It is also shown that the existence of the Killing spinor, in the generic case, implies that the domain of dependence of $\mathcal{H}_1 \cup \mathcal{H}_2$ must be of Petrov type D.* This result is presented in Subsection 6.3 (see also Corollary 1).

(ix) *Finally, we also give a clear identification of that subclass of basic initial data on $\mathcal{Z}$ which gives rise to a member of the Kerr family of spacetimes in the domain of dependence of $\mathcal{H}_1 \cup \mathcal{H}_2$.* The basic idea behind this calculation is to make use of Mars's invariant characterisation of the Kerr spacetime that is summarised in Theorem 1. The conditions on the freely specifiable data leading to a development isomorphic to Kerr are spelled out in Proposition 9.

Remark 1. The integrability conditions for the Killing spinor equation readily imply that a vacuum spacetime endowed with a Killing spinor must have a Weyl tensor which is of Petrov type D or has more special algebraic type —see e.g. [29]. In [21] all vacuum Petrov type D spacetimes have been described and expressed in terms of some local coordinates. Kinnersley's solution's posess, at least, two Killing vectors. Unfortunately, *a priori*, it is very hard to transform these expressions into the gauge used in Rácz's holograph construction as, in general, there need not exist more than a horizon Killing vector field in that setup. Therefore, the interrelation of the aforementioned two representations of Petrov D type solutions can only be done after all the implications of the existence of a non-trivial Killing spinor are explored —see Sections 6–10 below for more details.

Overview

This paper is structured as follows: in Section 2, we recall the results of [7], illustrating how the existence of a Killing spinor can be used to characterise the Kerr spacetime. This is done in the form of a local result requiring the evaluation of two constants.

In Section 3, we summarise the construction of the characteristic problem in [34], used to define the class of distorted black holes to be considered in this article.

In Section 4, we decompose the wave equation for the Killing spinor into equations intrinsic to the horizons, providing a system of transport equations for the components of the Killing spinor. Furthermore, by finding a system of homogeneous wave equations for a collection of zero-quantity fields and imposing appropriate initial data for the system, we find further conditions (differential and algebraic constraints) for the components of the Killing spinor and their first derivatives on the bifurcate horizon $\mathcal{H}_1 \cup \mathcal{H}_2$ —the *Killing spinor data conditions* on $\mathcal{Z}$.

In Section 5, we investigate these constraints. We show that the conditions intrinsic to the bifurcation sphere $\mathcal{Z}$ imply a specific form for the components of the Killing spinor. We also show that the constraints intrinsic to $\mathcal{H}_1$ or $\mathcal{H}_2$ satisfy ordinary differential equations along the generators of the relevant horizons, and so can be replaced with conditions on the bifurcation sphere, or become redundant. In this way, conditions on the extended horizon construction are reduced to conditions only on the bifurcation surface $\mathcal{Z}$.

In Section 7 it is shown that the Killing spinor data conditions on $\mathcal{Z}$ imply that the bifurcation sphere is an axially symmetri 2-surface.

In Section 9 further conditions on $\mathcal{Z}$ are obtained which ensure that the Killing vector associated to the Killing spinor is a Hermitian (i.e. real) vector.

Section 8 discusses the more general solution to the constraints on $\mathcal{Z}$. This solution fixes the Gaussian curvature of the bifurcation sphere $\mathcal{Z}$ and, in turn, also the functional form of the metric of the 2-surface and the spin coefficient τ. Using this metric one can use Rácz's black hole holograph construction to (locally in a neighbourhood of $\mathcal{Z}$) obtain the most general family of vacuum (with vanishing cosmological constant) distorted black holes with Killing spinors.

Section 10 is devoted to the task of identifying the Kerr out of the class of spacetimes constructed in the previous section. The main tools for this is a characterisation of Kerr spacetime in terms of Killing spinors based on a more general result by Mars [24] —see Theorem 1 in Section 2.

We provide some concluding remarks in Section 11.

Notation and conventions

In what follows $(\mathcal{M}, \boldsymbol{g})$ will denote a vacuum spacetime. The metric $\boldsymbol{g}$ is assumed to have signature $(+,-,-,-)$. The Latin letters a, $b, \ldots$ are used as abstract tensorial spacetime indices. The script letters $\mathcal{A}$, $\mathcal{B}, \ldots$ are used to denote *angular coordinates*. The Latin capital letters A, $B, \ldots$ are used as abstract spinorial indices.

We make systematic use of the standard Newman-Penrose (NP) formalism as discussed in, say, [28, 37]. Standard NP notation and conventions will be used —see e.g. [37]. In particular, if η is a smooth scalar on a 2-surface $\mathcal{Z}$ with spin-weight s, the action of the $\eth$ and $\overline{\eth}$ operators on is defined by

$$\eth\eta = \delta\eta + s\,(\overline{\alpha} - \beta)\,\eta\,, \qquad \overline{\eth}\eta = \overline{\delta}\,\eta - s\,(\alpha - \overline{\beta})\,\eta\,. \tag{1}$$

One also has that

$$(\overline{\eth}\eth - \eth\overline{\eth})\,\eta = s\,K_{\mathcal{G}}\,\eta\,, \tag{2}$$

where $K_{\mathcal{G}}$ denotes the Gaussian curvature of $\mathcal{Z}$.

We shall also make use of the representation of $\eth$ and $\overline{\eth}$ operators following the construction in Section 4.14 of [28]. In particular, by choosing an arbitrary holomorphic function z the 2-metric $\boldsymbol{\sigma}$ on $\mathcal{Z}$ can be given as

$$\boldsymbol{\sigma} = -\frac{1}{P\overline{P}}\big(\mathbf{d}z \otimes \mathbf{d}\overline{z} + \mathbf{d}\overline{z} \otimes \mathbf{d}z\big), \tag{3}$$

where P is a complex function on $\mathcal{Z}$. If $\mathcal{Z}$ was the unit sphere $\mathbb{S}^2$, then the coefficient P would have the form $P = \frac{1}{2}(1 + z\overline{z})$.

The operators $\eth$ and $\overline{\eth}$—acting on a scalar η of spin-weight s—are defined as (see (4.14.3)-(4.14.4) in [28])

$$\eth\eta \equiv P\overline{P}^{-s}\frac{\partial}{\partial z}\big(\overline{P}^{s}\eta\big)\,, \qquad \overline{\eth}\eta \equiv \overline{P}P^{s}\frac{\partial}{\partial \overline{z}}\big(P^{-s}\eta\big)\,. \tag{4}$$

As the complex coordinates z and $\overline{z}$ have no spin-weight direct calculations readily verify that

$$\eth z = P, \qquad \eth\overline{z} = 0\,,$$

and that

$$\eth\overline{P} = 0\,, \qquad \overline{\eth}P = 0\,.$$

Note, finally, that in the generic setup for the characteristic initial value problem the initial data is given on a pair of intersecting null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$. The solution to the Einstein's equations is known to exist in certain domains. We shall denote the domain of dependence

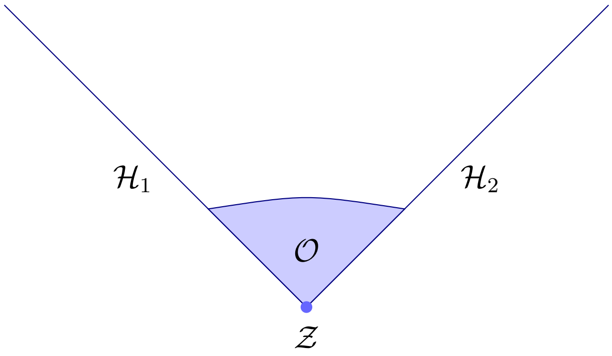

Figure 1: The possible extents of the domain of dependence of the initial data surface, comprised by a pair of null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ intersecting on a two-surface $\mathcal{Z} = \mathcal{H}_1 \cap \mathcal{H}_2$ in the characteristic initial value problem, is indicated.

of $\mathcal{H}_1 \cup \mathcal{H}_2$ by $D(\mathcal{H}_1 \cup \mathcal{H}_2)$. The extent of $D(\mathcal{H}_1 \cup \mathcal{H}_2)$ is known to depend on the techniques used in verifying the existence of solutions. According to the claims in [35] it is covering only a neighbourhood $\mathcal{O}$ of the spacelike 2-surface $\mathcal{Z}$ (indicated by the blue coloured area on Fig.1). Nevertheless, when techniques of energy estimates are used, as e.g. in [23], the domain of dependence can be seen to be larger covering (at least certain) neighbourhood of the two intersecting null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Hereafter, we shall refer to the domain of dependence without explicit mentioning of its extent. This is done to simplify the presentation. The size of this domain does not play a significant role in our discussions.

2 An invariant characterisation of the Kerr spacetime

In this section we provide a brief overview of a characterisation of the Kerr spacetime by means of Killing spinors.

2.1 Killing spinors

A *Killing spinor* is a symmetric rank 2 spinor $\kappa_{AB} = \kappa_{(AB)}$ satisfying the *Killing spinor equation*

$$\nabla_{A'(A}\kappa_{BC)} = 0. \tag{5}$$

Given a spinor κ_{AB}, the spinor

$$\xi_{AA'} \equiv \nabla^P{}_{A'}\kappa_{AP} \tag{6}$$

is the spinorial counterpart of a (possibly complex) Killing vector. Thus, it satisfies the equation

$$\nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'} = 0.$$

Conditions on a spacelike hypersurface $\mathcal{S}$ ensuring the existence of a Killing spinor on the future domain of dependence of $\mathcal{S}$, $D^+(\mathcal{S})$, have been analysed in [2, 13]. In view of the subsequent discussion it will be convenient to define the following *zero quantities:*

$$H_{A'ABC} \equiv 3\nabla_{A'(A}\kappa_{BC)}\,,$$
$$S_{AA'BB'} \equiv \nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'}\,.$$

A straightforward consequence of the Killing spinor equation is the wave equation

$$\Box\kappa_{AB} + \Psi_{ABCD}\kappa^{CD} = 0, \tag{7}$$

where Ψ_{ABCD} denotes the Weyl spinor.

A calculation then yields the following:

Proposition 1. *Let κ_{AB} be a solution to equation (7). Then the spinor fields $H_{A'ABC}$ and $S_{AA'BB'}$ satisfy the system of wave equations*

$$\square H_{AA'BC} = 4\big(\Psi_{(AB}{}^{PQ}H_{C)PQA'} + \nabla_{(A}{}^{Q'}S_{BC)Q'A'}\big), \tag{8a}$$

$$\begin{aligned}\square S_{AA'BB'} = &-\nabla_{AA'}(\Psi_{B}{}^{PQR}H_{B'PQR}) - \nabla_{BB'}(\Psi_{A}{}^{PQR}H_{A'PQR}) \\ &+2\Psi_{AB}{}^{PQ}S_{PA'QB'} + 2\overline{\Psi}_{A'B'}{}^{P'Q'}S_{AP'BQ'}.\end{aligned} \tag{8b}$$

Remark 2. As the above equations constitute a system of homogeneous linear wave equations for the fields $H_{A'ABC}$ and $S_{AA'BB'}$, it follows that they readily imply conditions for the existence of a Killing spinor in the development of a given initial value problem for the Einstein field equations.

2.2 The Killing form and the Ernst potential

In this section let $\xi_{AA'}$ denote the spinorial counterpart of a real Killing vector ξ^a. Accordingly, $\xi_{AA'}$ is assumed to be Hermitian. The spinorial counterpart of the Killing form of ξ^a, namely,

$$H_{ab} \equiv \nabla_{[a}\xi_{b]} = \nabla_a\xi_b$$

is given by

$$H_{AA'BB'} \equiv \nabla_{AA'}\xi_{BB'}.$$

As a consequence of the antisymmetry in the pairs ${}_{AA'}$ and ${}_{BB'}$, the latter can be decomposed into irreducible parts as

$$H_{AA'BB'} \equiv \eta_{AB}\epsilon_{A'B'} + \overline{\eta}_{A'B'}\epsilon_{AB},$$

where η_{AB} is a symmetric spinor. In the following we will make use of the self-dual part of $H_{AA'BB'}$, denoted by $\mathcal{H}_{AA'BB'}$, and defined by

$$\mathcal{H}_{AA'BB'} \equiv H_{AA'BB'} + \mathrm{i}H^*_{AA'BB'}.$$

It follows readily that

$$\mathcal{H}_{AA'BB'} = 2\eta_{AB}\epsilon_{A'B'}.$$

The spinor η_{AB} can be expressed in terms of $\xi_{AA'}$ as

$$\eta_{AB} = \frac{1}{2}\nabla_{AA'}\xi_B{}^{A'}.$$

If, moreover, $\xi_{AA'}$ is obtained from a Killing spinor κ_{AB} using formula (6), then one has that

$$\eta_{AB} = -\frac{3}{4}\Psi_{ABCD}\kappa^{CD}.$$

For later use we also define

$$\mathcal{H}^2 \equiv \mathcal{H}_{ab}\mathcal{H}^{ab} = 8\eta_{AB}\eta^{AB}.$$

The *Ernst form* of the Killing vector ξ^a is defined as

$$\chi_a = 2\xi^b\mathcal{H}_{ba}.$$

It is well-known that in vacuum, the Ernst form closed and thus, locally exact. Therefore, there exists a complex function, the *Ernst potential* χ, that satisfies

$$\chi_a = \nabla_a\chi.$$

A calculation then readily yields that

$$\chi_{AA'} = 4\eta_{AB}\xi^B{}_{A'} = 3\kappa^{CF}\Psi_{ABCF}\nabla_{DA'}\kappa^{DB}.$$

2.3 Mars's characterisation of the Kerr spacetime

In [24] is has been shown that the Kerr spacetime can be characterised in terms of an alignment condition of the Weyl tensor and the Killing form of the stationary Killing vector of the spacetime. This invariant characterisation admits both local and semi-global versions. In [7] it has been shown that the alignment condition follows if the spacetime is assumed to have a Killing spinor. More precisely, one has the following:

Theorem 1. *Let $(\mathcal{M}, \boldsymbol{g})$ denote a smooth vacuum spacetime endowed with a Killing spinor κ_{AB} satisfying $\kappa_{AB}\kappa^{AB} \neq 0$, such that the spinor $\xi_{AA'} \equiv \nabla^{B}{}_{A'}\kappa_{AB}$ is Hermitian. Then there exist two complex constants $\mathfrak{l}$ and $\mathfrak{c}$ such that*

$$\mathcal{H}^2 = -\mathfrak{l}(\mathfrak{c} - \chi)^4. \tag{9}$$

If, in addition, $\mathfrak{c} = 1$ and $\mathfrak{l}$ is real positive, then $(\mathcal{M}, \boldsymbol{g})$ is locally isometric to the Kerr spacetime.

3 The characteristic initial value problem on expansion and shear-free hypersurfaces

In [34], by adopting and slightly generalising results of [11], a systematic analysis of the null characteristic initial value problem for the Einstein-Maxwell equations in terms of the Newman-Penrose formalism was carried out. In particular, it was shown how to obtain a system of reduced evolution equations forming a first order symmetric hyperbolic system of equations. Moreover, it was shown that the solutions to these evolution equations imply, in turn, a solution to the full Einstein-Maxwell system provided that the inner (constraint) equations on the initial null hypersurfaces hold. For this type of setting, the theory for the characteristic initial value problem developed in [35] applies and ensures the local existence and uniqueness of a solution of the reduced evolution equations.

The general results described in the previous paragraph were then used to investigate electrovacuum spacetimes $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ possessing a pair of null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ generated by expansion and shear-free geodesically complete null congruences, with intersection on a two dimensional spacelike hypersurface $\mathcal{Z} \equiv \mathcal{H}_1 \cap \mathcal{H}_2$. The configuration formed by $\mathcal{H}_1$ and $\mathcal{H}_2$ constitute a *bifurcate horizon*. In general, the freely specifiable data on $\mathcal{Z}$ do not possess any symmetry in addition to the horizon Killing vector (implied by the non-expanding character of these horizons). Thus, these spacetimes constitute the generic class of *stationary* distorted electrovacuum spacetimes. The key observation resulting from the analysis in [34] is, for the *vacuum case*, summarised in the following:

Theorem 2.

(i) Suppose that $(\mathcal{M}, \boldsymbol{g})$ is a vacuum spacetime with a vanishing Cosmological constant possessing a pair of null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ generated by expansion and shear-free geodesically complete null congruences, intersecting on a 2-dimensional spacelike surface $\mathcal{Z} \equiv \mathcal{H}_1 \cup \mathcal{H}_2$. Then, the metric $\boldsymbol{g}$ is uniquely determined (up to diffeomorphisms) on a neighbourhood $\mathcal{O}$ of $\mathcal{Z}$ contained in the domain of dependence $D(\mathcal{H}_1 \cap \mathcal{H}_2)$ of $\mathcal{H}_1$ and $\mathcal{H}_2$, once a complex vector field $\zeta^{\mathcal{A}}$ determining the induced metric $\boldsymbol{\sigma}$ on $\mathcal{Z}$ and the spin connection coefficient τ are specified on $\mathcal{Z}$.

(ii) Conversely, given a Riemannian metric $\boldsymbol{\sigma}$ and a complex scalar field τ defined on a 2-dimensional surface $\mathcal{Z}$ generated as the intersection of two null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$, there is a neighbourhood $\mathcal{O}$ of $\mathcal{Z}$ contained in the domain of dependence $D(\mathcal{H}_1 \cup \mathcal{H}_2)$ where there exists a 4-dimensional metric $\boldsymbol{g}$ solution of the vacuum Einstein field equations with vanishing Cosmological constant such that the hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are generated by expansion and shear-free congruences.

(iii) Two pairs of basic data $(\boldsymbol{\sigma}, \tau)$ and $(\boldsymbol{\sigma}', \tau')$ on $\mathcal{Z}$ give rise to metrics $\boldsymbol{g}$ and $\boldsymbol{g}'$ which are isometric on $\mathcal{O}$ if and only if $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}$ are isometric and there exists a real function ϑ over $\mathcal{Z}$ such that $\tau' = \tau + \eth \log \vartheta$.

3.1 Summary of the construction

In the remainder of this article we will require further information concerning the construction in [34]. Throughout, let $(\mathcal{M}, \boldsymbol{g})$ denote a vacuum spacetime and let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote two null hypersurfaces in $(\mathcal{M}, \boldsymbol{g})$ intersecting on a spacelike 2-surface $\mathcal{Z}$.

Remark 3. In the remaining of this section, the topology of $\mathcal{Z}$ will not be relevant in the discussion. The situation will, however, change when we attempt to single out the Kerr spacetime.

Let n^a denote a smooth future-directed null vector on $\mathcal{Z}$ tangent to $\mathcal{H}_2$, which is extended to $\mathcal{H}_2$ by requiring it to satisfy $n^b\nabla_b n_a = 0$ on $\mathcal{H}_2$. Moreover, let u be an affine parameter along the null generators of $\mathcal{H}_2$, so that $u = 0$ on $\mathcal{Z}$ and $\mathcal{Z}_u$ are the associated 1-parameter family of smooth cross sections of $\mathcal{H}_2$. We choose a further null vector l^a as the unique future-directed null vector field on $\mathcal{H}_2$ which is orthogonal to the 2-dimensional cross sections $\mathcal{Z}_u$ and satisfies the normalisation condition $n_a l^a = 1$. Consider now the null geodesics starting on $\mathcal{H}_2$ with tangent l^a. Since $\mathcal{H}_2$ is assumed to be smooth and the vector fields n^a and l^a are smooth on $\mathcal{H}_2$ by construction, these geodesics do not intersect in a sufficiently small open neighbourhood $\mathcal{O} \subset \mathcal{M}$ of $\mathcal{H}_2$. Let now r denote the affine parameter along the null geodesics starting on $\mathcal{H}_2$ with tangent l^a, chosen such that $r = 0$ of $\mathcal{H}_2$. By construction one has that $l^a = (\partial/\partial r)^a$. The affine parameter defines a smooth function $r : \mathcal{O} \to \mathbb{R}$. The function $\mathcal{H}_2 \to \mathbb{R}$ defined by the affine parameter of the integral curves of n^a can be extended to a smooth function $u : \mathcal{O} \to \mathbb{R}$ by requiring it to be constant along the null geodesics with tangent l^a.

The construction of the previous paragraph is complemented by choosing suitable coordinates $(x^{\mathcal{A}})$ on patches of $\mathcal{Z}$ and extending them to $\mathcal{O}$ by requiring them to be constant along the integral curves of the vectors l^a and n^a. In this manner one obtains a system of *Gaussian null coordinates* $(u, r, x^{\mathcal{A}})$ on patches of $\mathcal{O}$. In each of these patches the spacetime metric $\boldsymbol{g}$ takes the form

$$\boldsymbol{g} = g_{00}\mathbf{d}u \otimes \mathbf{d}u + (\mathbf{d}u \otimes \mathbf{d}r + \mathbf{d}r \otimes \mathbf{d}u) + g_{0\mathcal{A}}(\mathbf{d}u \otimes \mathbf{d}x^{\mathcal{A}} + \mathbf{d}x^{\mathcal{A}} \otimes \mathbf{d}u) + g_{\mathcal{A}\mathcal{B}}\mathbf{d}x^{\mathcal{A}} \otimes \mathbf{d}x^{\mathcal{B}}, \quad (10)$$

where g_{00}, $g_{0\mathcal{A}}$, $g_{\mathcal{A}\mathcal{B}}$ are smooth functions of the coordinates $(u, r, x^{\mathcal{A}})$ such that

$$g_{00} = g_{0\mathcal{A}} = 0, \quad \text{on} \quad \mathcal{H}_2, \quad (11)$$

and $g_{\mathcal{A}\mathcal{B}}$ is a negative definite 2×2 matrix. Observe that by construction in $\mathcal{O}$ the $u = 0$ and $r = 0$ hypersurfaces coincide with $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

In the following it will be convenient to consider the components of the contravariant form of the metric associated to the line element (10). A calculation shows that components of the contravariant metric g^{ab} in the Gaussian null coordinates $(u, r, x^{\mathcal{A}})$ can be given as

$$g^{ab} \rightleftharpoons \begin{pmatrix} 0 & 1 & 0 \\ 1 & g^{11} & g^{1\mathcal{B}} \\ 0 & g^{\mathcal{A}1} & g^{\mathcal{A}\mathcal{B}} \end{pmatrix}.$$

The metric functions g^{11}, $g^{1\mathcal{A}}$ and $g^{\mathcal{A}\mathcal{B}}$ can be conveniently parametrised in terms of real-valued functions U, $X^{\mathcal{A}}$ and complex-valued functions ω, $\zeta^{\mathcal{A}}$ on $\mathcal{O}$ such that

$$g^{11} = 2(U - \omega\overline{\omega}), \qquad g^{1\mathcal{A}} = X^{\mathcal{A}} - (\overline{\omega}\zeta^{\mathcal{A}} + \omega\overline{\zeta}^{\mathcal{A}}), \qquad g^{\mathcal{A}\mathcal{B}} = -(\zeta^{\mathcal{A}}\overline{\zeta}^{\mathcal{B}} + \zeta^{\mathcal{B}}\overline{\zeta}^{\mathcal{A}}).$$

Accordingly, setting

$$l^a = (\partial_r)^a, \qquad n^a = (\partial_u)^a + U\,(\partial_r)^a + X^{\mathcal{A}}\,(\partial_{x^{\mathcal{A}}})^a, \qquad m^a = \omega(\partial_r)^a + \zeta^{\mathcal{A}}\,(\partial_{x^{\mathcal{A}}})^a,$$

one obtains a complex (NP) null tetrad $\{l^a, n^a, m^a, \overline{m}^a\}$ in $\mathcal{O}$. As a result of the conditions in (11) one has that

$$U = X^{\mathcal{A}} = \omega = 0, \quad \text{on} \quad \mathcal{H}_2.$$

It follows from the previous discussion that m^a and $\overline{m}^a$ are everywhere tangent to the sections $\mathcal{Z}_u$ of $\mathcal{H}_2$. In general, the complex null vectors m^a and $\overline{m}^a$ are not parallelly propagated along the null generators of $\mathcal{H}_2$.

Associated to the NP null tetrad $\{l^a,\, n^a,\, m^a,\, \overline{m}^a\}$ in $\mathcal{O}$ one has the directional derivatives

$$
\begin{aligned}
D &= \frac{\partial}{\partial r}, \\
\Delta &= \frac{\partial}{\partial u} + U\frac{\partial}{\partial u} + X^{\mathcal{A}}\frac{\partial}{\partial x^{\mathcal{A}}}, \\
\delta &= \omega\frac{\partial}{\partial r} + \zeta^{\mathcal{A}}\frac{\partial}{\partial x^{\mathcal{A}}}.
\end{aligned}
$$

Remark 4. By construction, one has that D is an intrinsic derivative to $\mathcal{H}_1$ pointing along the null generators of this hypersurface. Similarly, Δ is intrinsic to $\mathcal{H}_2$ and points in the direction of its null generators. Finally, $\{m^a,\, \overline{m}^a\}$ are differential operators which on $\mathcal{H}_2$ are intrinsic to the sections of constant u, $\mathcal{Z}_u$. Observe, however, that while δ restricted to $\mathcal{H}_1$ is still intrinsic to the null hypersurface, it is not intrinsic to the sections of constant r.

The NP null tetrad constructed in the previous paragraph can be specialised further to simplify the associate spin-connection coefficients. By parallelly propagating $\{l^a,\, n^a,\, m^a,\, \overline{m}^a\}$ along the null geodesics with tangent l^a one finds that

$$\kappa = \pi = \epsilon = 0, \tag{12a}$$

$$\rho = \overline{\rho}, \qquad \tau = \overline{\alpha} + \beta, \qquad \text{everywhere on} \quad \mathcal{O}. \tag{12b}$$

Moreover, from the condition $n^b\nabla_b n^a = 0$ on $\mathcal{H}_2$ it follows that

$$\nu = 0 \qquad \text{on} \quad \mathcal{H}_2. \tag{13}$$

Also, using that u is an affine parameter of the generators of $\mathcal{H}_2$ one finds that $\gamma + \overline{\gamma} = 0$ along these generators. One can specialise further by suitably rotating the vectors $\{m^a,\, \overline{m}^a\}$ so as to obtain

$$\gamma = 0, \qquad \text{on} \quad \mathcal{H}_2. \tag{14}$$

3.1.1 Solving the NP constraint equations

The NP Ricci and Bianchi identities split into a subset of intrinsic (constraint) equations to $\mathcal{H}_1 \cup \mathcal{H}_2$ and a subset of transverse (evolution) equations. In [34] the gauge introduced in the previous subsection was used to systematically analyse the constraint equations on $\mathcal{H}_1 \cup \mathcal{H}_2$ with the aim of identifying the freely specifiable data on this pair of intersecting hypersurfaces under the assumption that it is expansion and shear-free. The results from this analysis can be conveniently presented in the form of a table —see Table 1.

Remark 5. As already mentioned, in what follows we will mostly be interested in the situation where $\mathcal{Z}$ is diffeomorphic to a unit 2-sphere, i.e. $\mathcal{Z} \approx \mathbb{S}^2$. From the definition of the operators $\eth$ and $\overline{\eth}$ as given in (1), along with those of the NP spin connection coefficients α and β, it follows that the connection on $\mathcal{Z}$ is encoded in the combination $\overline{\alpha} - \beta$. As discussed in [34], given the freely specifiable data ζ^A and τ one can readily compute the NP coefficients α, β. These, in turn, can be used, together with the NP Ricci equation

$$\Psi_2 = -\delta\alpha + \overline{\delta}\beta + \alpha\,\overline{\alpha} - 2\,\alpha\,\beta + \beta\,\overline{\beta} \tag{15}$$

to determine the Weyl spinor component Ψ_2 on $\mathcal{Z}$. From the latter it is straightforward to deduce

$$2\,\mathrm{Re}(\Psi_2) = \Psi_2 + \overline{\Psi}_2 = -\delta\,(\,\alpha - \overline{\beta}\,) - \overline{\delta}\,(\,\overline{\alpha} - \beta\,) + 2\,(\,\alpha - \overline{\beta}\,)\,(\,\overline{\alpha} - \beta\,) \tag{16}$$

which implies that the real part of Ψ_2 —in accordance with the fact that $-2\,\mathrm{Re}(\Psi_2)$ is the Gaussian curvature $K_{\mathcal{G}}$ of $\mathcal{Z}$ (see, i.e. Proposition 4.14.21 in [28])— depends only on the combination $\overline{\alpha} - \beta$ which is completely intrinsic to $\mathcal{Z}$. Analogously, by making use of (15) τ and the imaginary part of Ψ_2 can be seen to be closely related to each other via

$$2\,\mathrm{i}\,\mathrm{Im}(\Psi_2) = \Psi_2 - \overline{\Psi}_2 = \overline{\delta}\tau - \delta\overline{\tau} - 2\,\left(\,\beta\,\overline{\tau} - \overline{\beta}\,\tau\,\right)\,. \tag{17}$$

$\mathcal{H}_1$	$\mathcal{Z}$	$\mathcal{H}_2$
$\mathrm{D}\zeta^{\mathcal{A}} = 0$	$\zeta^{\mathcal{A}}$ (data)	$\Delta\zeta^{\mathcal{A}} = 0$
$\omega = -r\,\tau$	$\omega = 0$	$\omega = 0$ (geometry)
$X^{\mathcal{A}} = r\,[\tau\,\overline{\zeta}^{\mathcal{A}} + \overline{\tau}\,\zeta^{\mathcal{A}}]$	$X^{\mathcal{A}} = 0$	$X^{\mathcal{A}} = 0$ (geometry)
$U = -r^2\,\left[2\,\tau\,\overline{\tau} + \frac{1}{2}\left(\Psi_2 + \overline{\Psi}_2\right)\right]$	$U = 0$	$U = 0$ (geometry)
$\rho = 0$	$\rho = 0$	$\rho = u\,\left(\overline{\delta}\tau - 2\,\alpha\,\tau - \Psi_2\right)$
$\sigma = 0$	$\sigma = 0$	$\sigma = u\,(\,\delta\tau - 2\,\beta\,\tau\,)$
$\mathrm{D}\tau = 0$	τ (data)	$\Delta\tau = 0$
$\mathrm{D}\alpha = \mathrm{D}\beta = 0$	$\alpha, \beta, \tau = \overline{\alpha} + \beta$	$\Delta\alpha = \Delta\beta = 0$
$\gamma = r\,(\,\tau\,\alpha + \overline{\tau}\,\beta + \Psi_2\,)$	$\gamma = 0$	$\gamma = 0$ (gauge)
$\mu = r\,\Psi_2$	$\mu = 0$	$\mu = 0$
$\lambda = 0$	$\lambda = 0$	$\lambda = 0$
$\nu = \frac{1}{2}\,r^2\,\left(\overline{\delta}\Psi_2 + \overline{\tau}\,\Psi_2\,\right)$	$\nu = 0$	$\nu = 0$ (gauge)
$\Psi_0 = 0$	$\Psi_0 = 0$	$\Psi_0 = \frac{1}{2}\,u^2\,(\delta^2\Psi_2 - (7\,\tau + 2\,\beta)\,\delta\Psi_2 + 12\,\tau^2\Psi_2)$
$\Psi_1 = 0$	$\Psi_1 = 0$	$\Psi_1 = u\,(\,\delta\Psi_2 - 3\,\tau\,\Psi_2\,)$
$\mathrm{D}\Psi_2 = 0$	$\zeta^{\mathcal{A}}, \tau \rightarrow\ \alpha, \beta, \Psi_2$	$\Delta\Psi_2 = 0$
$\Psi_3 = r\,\overline{\delta}\Psi_2$	$\Psi_3 = 0$	$\Psi_3 = 0$
$\Psi_4 = \frac{1}{2}\,r^2\,\left(\overline{\delta}^2\Psi_2 + 2\,\alpha\,\overline{\delta}\Psi_2\right)$	$\Psi_4 = 0$	$\Psi_4 = 0$

Table 1: The full initial data set on the intersecting null hypersurfaces $\mathcal{H}_1 \cup \mathcal{H}_2$.

4 The Killing spinor data conditions for a characteristic initial problem

In this section we adapt the analysis of Killing spinor initial data in [2] to the setting of a characteristic initial data set —see also [14].

4.1 Construction of the Killing spinor candidate

In this section we investigate the characteristic initial value problem for the wave equation, equation (7), governing the evolution of the Killing spinor candidate κ_{AB}. An approach to the formulation of the characteristic initial value problem for wave equations on intersecting null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ has been analysed in [35]. Our discussion follows the ideas of this analysis closely.

4.1.1 Basic set-up

Let $\{o^A, \iota^A\}$ denote a spin dyad normalised according to $o_A\iota^A = 1$. The spinor κ_{AB} can be written as

$$\kappa_{AB} = \kappa_2 o_A o_B - 2\kappa_1 o_{(A}\iota_{B)} + \kappa_0 \iota_A \iota_B.$$

so that

$$\kappa_0 \equiv \kappa_{AB} o^A o^B, \qquad \kappa_1 \equiv \kappa_{AB} o^A \iota^B, \qquad \kappa_2 \equiv \kappa_{AB} \iota^A \iota^B.$$

It can be readily verified that the scalars κ_2, κ_1 and κ_0 have, respectively, spin weights $-1, 0, 1$ — i.e. they transform as

$$\kappa_j \mapsto e^{-2(j-1)\mathrm{i}\vartheta}\kappa_j$$

under a rotation $\{o^A, \iota^A\} \mapsto \{e^{\mathrm{i}\vartheta} o^A,\, e^{-\mathrm{i}\vartheta}\iota^A\}$.

A direct decomposition of the wave equation (7) using the NP formalism readily yields the following equations for the independent components κ_0, κ_1 and κ_2 of the spinor κ_{AB}:

$$\begin{aligned}
&D\Delta\kappa_2 + \Delta D\kappa_2 - \delta\overline{\delta}\kappa_2 - \overline{\delta}\delta\kappa_2 \\
&\qquad +(\mu+\overline{\mu}+3\gamma-\overline{\gamma})D\kappa_2 - (\rho+\overline{\rho})\Delta\kappa_2 + (\overline{\tau}-3\alpha-\overline{\beta})\delta\kappa_2 + (\overline{\alpha}-5\beta+\tau)\overline{\delta}\kappa_2 \\
&\qquad +(\Psi_2 + 2\alpha\overline{\alpha} - 8\alpha\beta - 2\beta\overline{\beta} - 2\gamma\rho + 2\mu\rho - 2\gamma\overline{\rho} + 2\lambda\sigma + 2\alpha\tau + 2\beta\overline{\tau} + 2D\gamma - 2\delta\alpha - 2\overline{\delta}\beta)\kappa_2 \\
&\qquad +(\Psi_4 - 4\lambda\mu)\kappa_0 = 0, &&\text{(18a)}\\
&D\Delta\kappa_1 + \Delta D\kappa_1 - \delta\overline{\delta}\kappa_1 - \overline{\delta}\delta\kappa_1 \\
&\qquad -2\tau D\kappa_2 + (\mu+\overline{\mu}-\gamma-\overline{\gamma})D\kappa_1 + 2\nu D\kappa_0 - (\rho+\overline{\rho})\Delta\kappa_1 + 2\rho\delta\kappa_2 + (\alpha-\overline{\beta}+\overline{\tau})\delta\kappa_1 \\
&\qquad -2\lambda\delta\kappa_2 + 2\sigma\overline{\delta}\kappa_2 + (\overline{\alpha}-\beta+\tau)\overline{\delta}\kappa_1 - 2\mu\overline{\delta}\kappa_0 \\
&\qquad +(-\Psi_1 - \overline{\alpha}\rho + 3\beta\rho + \alpha\sigma + \overline{\beta}\sigma\overline{\rho}\tau - \sigma\overline{\tau} - D\tau + \delta\rho\overline{\delta}\sigma)\kappa_2 \\
&\qquad +(-\Psi_3 + \overline{\alpha}\lambda + \beta\lambda + 3\alpha\mu - \overline{\beta}\mu - \nu\rho - \nu\overline{\rho} + \lambda\tau + \mu\overline{\tau} + D\nu - \delta\lambda - \overline{\delta}\mu)\kappa_0 = 0, &&\text{(18b)}\\
&D\Delta\kappa_0 + \Delta D\kappa_0 - \delta\overline{\delta}\kappa_0 - \overline{\delta}\delta\kappa_0 \\
&\qquad +(\mu+\overline{\mu}-5\gamma-\overline{\gamma})D\kappa_0 - (\rho+\overline{\rho})\Delta\kappa_0 + (5\alpha-\overline{\beta}+\overline{\tau})\delta\kappa_0 + (\overline{\alpha}+3\beta+\tau)\overline{\delta}\kappa_0 \\
&\qquad +(\Psi_2 - 2\alpha\overline{\alpha} - 8\alpha\beta + 2\beta\overline{\beta} + 2\gamma\rho + 2\mu\rho + 2\gamma\overline{\rho} + 2\lambda\sigma - 2\alpha\tau - 2\beta\overline{\tau} - 2D\gamma + 2\delta\alpha + 2\overline{\delta}\beta)\kappa_0 \\
&\qquad +(\Psi_0 - 4\rho\sigma)\kappa_2 = 0. &&\text{(18c)}
\end{aligned}$$

The above expressions are completely general: no assumption on the spacetime (other than satisfying the vacuum field equations) or the gauge has been made.

Remark 6. In the sequel we investigate the consequences of these equations on the hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$. For this we consider a spin dyad $\{o^A, \iota^A\}$ adapted to the NP null tetrad $\{l^a, n^a, m^a, \overline{m}^a\}$ —if $\{l^{AA'}, n^{AA'}, m^{AA'}, \overline{m}^{AA'}\}$ denote the spinorial counterparts of the null tetrad, one has the correspondences

$$l^{AA'} = o^A\overline{o}^{A'}, \qquad n^{AA'} = \iota^A\overline{\iota}^{A'}, \qquad m^{AA'} = o^A\overline{\iota}^{A'}, \qquad \overline{m}^{AA'} = \iota^A\overline{o}^{A'},$$

and the gauge conditions (12a)-(12b), (13) and (14) hold when computing the corresponding NP spin-connection coefficients by means of derivatives of the spin dyad.

4.1.2 The transport equations on $\mathcal{H}_1$

Consider now the restriction of equations (18a)-(18c) to the null hypersurface $\mathcal{H}_1$ with tangent l^a. It follows then that D is a directional derivative along the null generators of $\mathcal{H}_1$, while Δ is a directional derivative transversal to $\mathcal{H}_1$. Using the NP commutator $[D,\Delta]$ equation to rewrite $\Delta D\kappa_0$, $\Delta D\kappa_1$, $\Delta D\kappa_2$ in terms of $D\Delta\kappa_0$, $D\Delta\kappa_1$ and $D\Delta\kappa_2$, equations (18a)-(18c) take the form:

$$\begin{aligned}
&2D\Delta\kappa_0 - \delta\overline{\delta}\kappa_0 - \overline{\delta}\delta\kappa_0 + (\overline{\alpha}+3\beta)\overline{\delta}\kappa_0 + (5\alpha-\overline{\beta})\delta\kappa_0 + (\mu+\overline{\mu}-4\gamma)D\kappa_0 + 4\tau D\kappa_1 \\
&\qquad +2\kappa_1 D\tau + (\Psi_2 - 2\alpha\overline{\alpha} - 8\alpha\beta + 2\beta\overline{\beta} - 2\alpha\tau - 2\beta\overline{\tau} - 2D\gamma + 2\delta\alpha + 2\overline{\delta}\beta)\kappa_0 = 0, &&\text{(19a)}\\
&2D\Delta\kappa_1 - \delta\overline{\delta}\kappa_1 - \overline{\delta}\delta\kappa_1 - 2\nu D\kappa_0 + (\mu+\overline{\mu})D\kappa_1 + 2\tau D\kappa_2 + (\alpha-\overline{\beta})\delta\kappa_1 + 2\mu\overline{\delta}\kappa_0 + (\overline{\alpha}-\beta)\overline{\delta}\kappa_1 \\
&\qquad +(\Psi_3 - 3\alpha\mu + \overline{\beta}\mu - \mu\overline{\tau} - D\nu + \overline{\delta}\mu)\kappa_0 - 2\Psi_2\kappa_1 + \kappa_2 D\tau = 0, &&\text{(19b)}\\
&2D\Delta\kappa_2 - \delta\overline{\delta}\kappa_2 - \overline{\delta}\delta\kappa_2 - 4\nu D\kappa_1 + (4\gamma+\mu+\overline{\mu})D\kappa_2 - (3\alpha+\overline{\beta})\delta\kappa_2 + 4\mu\overline{\delta}\kappa_1 + (\overline{\alpha}-5\beta)\overline{\delta}\kappa_2 \\
&\qquad +(\Psi_2 + 2\alpha\overline{\alpha} - 8\alpha\beta - 2\beta\overline{\beta} + 2\alpha\tau + 2\beta\overline{\tau} + 2D\gamma - 2\delta\alpha - 2\overline{\delta}\beta)\kappa_2 \\
&\qquad +(2\alpha\mu - 2\Psi_3 + 2\overline{\beta}\mu - 2\mu\overline{\tau} - 2D\nu + 2\overline{\delta}\mu)\kappa_1 + \Psi_4\kappa_0 = 0. &&\text{(19c)}
\end{aligned}$$

If the value of the components κ_0, κ_1, κ_2 are known on $\mathcal{H}_1$, then the above equations can be read as a system of ordinary differential equations for the transversal derivatives

$$\Delta\kappa_0, \quad \Delta\kappa_1, \quad \Delta\kappa_2,$$

along the null generators of $\mathcal{H}_1$. Initial data for these transport equations is naturally prescribed on $\mathcal{Z}$.

4.1.3 The transport equations on $\mathcal{H}_2$

Similarly, one can consider the restriction of equations (18a)-(18c) to the null hypersurface $\mathcal{H}_2$ with tangent n^a. Thus, Δ is a directional derivative along the null generators of $\mathcal{H}_2$, δ and $\overline{\delta}$ are intrinsic derivatives while D is transversal to $\mathcal{H}_2$. In this case one uses the NP commutator $[D,\Delta]$ to rewrite $D\Delta\kappa_0$, $D\Delta\kappa_1$, $D\Delta\kappa_2$ in terms of $\Delta D\kappa_0$, $\Delta D\kappa_1$, $\Delta D\kappa_2$ and lower order terms so that equations (18a)-(18c) take the form

$$\begin{aligned}
&2\Delta D\kappa_0 - \delta\overline{\delta}\kappa_0 - \overline{\delta}\delta\kappa_0 - (\rho+\overline{\rho})\Delta\kappa_0 + 4\tau D\kappa_1 + (5\alpha - \overline{\beta} + 2\overline{\tau})\delta\kappa_0 + (\overline{\alpha} + 3\beta + 2\tau)\overline{\delta}\kappa_0 \\
&\quad + 4\sigma\overline{\delta}\kappa_1 - 4\rho\delta\kappa_1 + (\Psi_2 - 2\alpha\overline{\alpha} - 8\alpha\beta + 2\beta\overline{\beta} - 2\alpha\tau - 2\beta\overline{\tau} + 2\delta\alpha + 2\overline{\delta}\beta)\kappa_0 \\
&\quad + (2\overline{\alpha}\rho + 2\beta\rho + 6\alpha\sigma - 2\overline{\beta}\sigma - 2\overline{\rho}\tau + 2\sigma\overline{\tau} + 2D\tau - 2\delta\rho - 2\overline{\delta}\sigma - 2\Psi_1)\kappa_1 \\
&\quad + (\Psi_0 - 4\rho\sigma)\kappa_2 = 0, \qquad (20a) \\
&2\Delta D\kappa_1 - \delta\overline{\delta}\kappa_1 - \overline{\delta}\delta\kappa_1 - (\rho+\overline{\rho})\Delta\kappa_1 + 2\tau D\kappa_2 + (\alpha - \overline{\beta} + 2\overline{\tau})\delta\kappa_1 + (\overline{\alpha} - \beta + 2\tau)\overline{\delta}\kappa_1 \\
&\quad - 2\rho\delta\kappa_2 + 2\sigma\overline{\delta}\kappa_2 - 2\Psi_2\kappa_1 \\
&\quad + (\Psi_1 - \overline{\alpha}\rho - 3\beta\rho - \alpha\sigma - \overline{\beta}\sigma - \overline{\rho}\tau + \sigma\overline{\tau} + D\tau - \delta\rho - \overline{\delta}\sigma)\kappa_2 = 0, \qquad (20b) \\
&2\Delta D\kappa_2 - \delta\overline{\delta}\kappa_2 - \overline{\delta}\delta\kappa_2 - (\rho+\overline{\rho})\Delta\kappa_2 + (2\overline{\tau} - 3\alpha - \overline{\beta})\delta\kappa_2 + (\overline{\alpha} - 5\beta + 2\tau)\overline{\delta}\kappa_2 \\
&\quad + (\Psi_2 + 2\alpha\overline{\alpha} - 8\alpha\beta - 2\beta\overline{\beta} + 2\alpha\tau + 2\beta\overline{\tau} - 2\delta\alpha - 2\overline{\delta}\beta)\kappa_2 = 0. \qquad (20c)
\end{aligned}$$

If the values of κ_0, κ_1, κ_2 are known on $\mathcal{H}_2$ then the above equations can be read as a system of ordinary differential equations for the transversal derivatives

$$D\kappa_0, \qquad D\kappa_1, \qquad D\kappa_2,$$

along the null generators of $\mathcal{H}_2$. Initial data for these transport equations is naturally prescribed on $\mathcal{Z}$.

4.1.4 Summary: existence of the Killing spinor candidate

The discussion of the previous subsections combined with the methods of [35] —see also [20]— allows to formulate the following existence result:

Proposition 2. *Let $(\mathcal{M},\boldsymbol{g})$ denote a spacetime satisfying the assumptions of Theorem 2. Then, given a smooth choice of fields κ_0, κ_1 and κ_2 on $\mathcal{H}_1\cup\mathcal{H}_2$, there exists a neighbourhood $\mathcal{O}$ of $\mathcal{Z}$ in $D(\mathcal{H}_1\cup\mathcal{H}_2)$ on which the wave equation (7) has a unique solution κ_{AB}.*

Remark 7. The assumption of smoothness of the fields κ_0, κ_1 and κ_2 require, in particular, that the limits of these fields as one approaches to $\mathcal{Z}$ on either $\mathcal{H}_1$ or $\mathcal{H}_2$ coincide.

4.2 The NP decomposition of the Killing spinor data conditions

The conditions on the initial data for the Killing spinor candidate κ_{AB} constructed in the previous section which ensure that it is, in fact, a Killing spinor follow from requiring that the propagation system (8a)-(8b) of Proposition 1 has as a unique solution —the trivial (zero) one.

The purpose of this section is to analyse the characteristic initial value problem for the Killing spinor equation propagation system (8a)-(8b).

4.2.1 Basic observations

We are interested in solutions to the system (8a)-(8b) ensuring the existence of a Killing spinor on $D(\mathcal{H}_1\cup\mathcal{H}_2)$. The homogeneity of these equations on the fields $H_{AA'BC}$ and $S_{AA'BB'}$ allows to formulate the following result:

Proposition 3. *Let $(\mathcal{M},\boldsymbol{g})$ denote a spacetime satisfying the assumptions of Theorem 2. Assume that initial data κ_0, κ_1, κ_2 on $\mathcal{H}_1\cup\mathcal{H}_2$ for the wave equation (7) can be found such that*

$$H_{AA'BC}=0, \qquad S_{AA'BB'}=0 \qquad \text{on} \quad \mathcal{H}_1\cup\mathcal{H}_2.$$

Then there exists a neighbourhood $\mathcal{O}$ of $\mathcal{Z}$ on the domain of dependence of $\mathcal{H}_1\cup\mathcal{H}_2$ where $H_{AA'BC}$ and $S_{AA'BB'}$ vanish and the resulting Killing spinor candidate κ_{AB} is, in fact, a Killing spinor.

Proof. The result follows from using the methods of Section 4.1 on the equations (8a)-(8b), and the uniqueness of the solutions to the characteristic initial value problem. □

Remark 8. A standard computation shows that the condition

$$H_{AA'BC} = 0$$

is equivalent to the equations

$$\begin{aligned}
&D\kappa_0 - 2\epsilon\kappa_0 + 2\kappa\kappa_1 = 0, && (21a)\\
&\delta\kappa_0 - 2\beta\kappa_0 + 2\sigma\kappa_1 = 0, && (21b)\\
&\overline{\delta}\kappa_0 + 2D\kappa_1 - 2\pi\kappa_0 - 2\alpha\kappa_0 + 2\kappa_1\rho + 2\kappa\kappa_2 = 0, && (21c)\\
&\Delta\kappa_0 + 2\delta\kappa_1 + 2\sigma\kappa_2 - 2\mu\kappa_0 + 2\tau\kappa_1 - 2\gamma\kappa_0 = 0, && (21d)\\
&D\kappa_2 + 2\overline{\delta}\kappa_1 + 2\rho\kappa_2 - 2\lambda\kappa_0 - 2\pi\kappa_1 + 2\epsilon\kappa_2 = 0, && (21e)\\
&\delta\kappa_2 + 2\Delta\kappa_1 + 2\tau\kappa_2 + 2\beta\kappa_2 - 2\mu\kappa_1 - 2\nu\kappa_0 = 0, && (21f)\\
&\overline{\delta}\kappa_2 + 2\alpha\kappa_2 - 2\lambda\kappa_1 = 0, && (21g)\\
&\Delta\kappa_2 + 2\gamma\kappa_2 - 2\nu\kappa_1 = 0. && (21h)
\end{aligned}$$

Remark 9. Using the notation

$$\xi_{AA'} = \xi_{11'}o_A\overline{o}_{A'} + \xi_{10'}o_A\overline{\iota}_{A'} + \xi_{01'}\iota_A\overline{o}_{A'} + \xi_{00'}\iota_A\overline{\iota}_{A'},$$

equation (6) takes the form

$$\begin{aligned}
\xi_{11'} &= \Delta\kappa_1 - \delta\kappa_2 - 2\beta\kappa_2 + \tau\kappa_2 + 2\mu\kappa_1 - \nu\kappa_0, && (22a)\\
\xi_{10'} &= D\kappa_2 - \overline{\delta}\kappa_1 + 2\epsilon\kappa_2 - \rho\kappa_2 - 2\pi\kappa_1 + \lambda\kappa_0, && (22b)\\
\xi_{01'} &= \delta\kappa_1 - \Delta\kappa_0 + 2\gamma\kappa_0 - \mu\kappa_0 - 2\tau\kappa_1 + \sigma\kappa_2, && (22c)\\
\xi_{00'} &= \overline{\delta}\kappa_0 - D\kappa_1 - 2\alpha\kappa_0 + \pi\kappa_0 + 2\rho\kappa_1 - \kappa\kappa_2. && (22d)
\end{aligned}$$

If $\xi_{AA'}$ is required to be Hermitian so that it corresponds to the spinorial counterpart of a real vector ξ^a then one has the reality conditions

$$\xi_{00'} = \overline{\xi}_{0'0}, \qquad \xi_{11'} = \overline{\xi}_{1'1}, \qquad \xi_{01'} = \overline{\xi}_{1'0}, \qquad \xi_{10'} = \overline{\xi}_{0'1}$$

A further calculation shows that the equation $S_{AA'BB'} = 0$ takes, in NP notation the form

$$\begin{aligned}
&D\xi_{00'} - \xi_{00'}\epsilon - \xi_{00'}\overline{\epsilon} - \xi_{10'}\kappa - \xi_{01'}\overline{\kappa} = 0, && (23a)\\
&\Delta\xi_{11'} + \xi_{11'}\gamma + \xi_{11'}\overline{\gamma} + \xi_{01'}\nu + \xi_{10'}\overline{\nu} = 0, && (23b)\\
&D\xi_{11'} + \Delta\xi_{00'} - \xi_{00'}\gamma - \xi_{00'}\overline{\gamma} + \xi_{11'}\epsilon + \xi_{11'}\overline{\epsilon} + \xi_{01'}\pi + \xi_{10'}\overline{\pi} - \xi_{10'}\tau - \xi_{01'}\overline{\tau} = 0, && (23c)\\
&\delta\xi_{11'} - \Delta\xi_{01'} + \overline{\alpha}\xi_{11'} + \xi_{11'}\beta + \xi_{01'}\gamma - \xi_{01'}\overline{\gamma} + \xi_{10'}\overline{\lambda} + \xi_{01'}\mu - \xi_{00'}\overline{\nu} + \xi_{11'}\tau = 0, && (23d)\\
&\delta\xi_{01'} + \xi_{01'}\overline{\alpha} - \xi_{01'}\beta + \xi_{00'}\overline{\lambda} - \xi_{11'}\sigma = 0, && (23e)\\
&\delta\xi_{00'} - D\xi_{01'} - \xi_{00'}\overline{\alpha} - \xi_{00'}\beta + \xi_{01'}\epsilon - \xi_{01'}\overline{\epsilon} + \xi_{11'}\kappa - \xi_{00'}\overline{\pi} - \xi_{01'}\overline{\rho} - \xi_{10'}\sigma = 0, && (23f)\\
&\overline{\delta}\xi_{11'} - \Delta\xi_{10'} + \xi_{11'}\alpha + \xi_{11'}\overline{\beta} - \xi_{10'}\gamma + \xi_{10'}\overline{\gamma} + \xi_{01'}\lambda + \xi_{10'}\overline{\mu} - \xi_{00'}\nu + \xi_{11'}\overline{\tau} = 0, && (23g)\\
&\overline{\delta}\xi_{10'} + \xi_{10'}\alpha - \xi_{10'}\overline{\beta} + \xi_{00'}\lambda - \xi_{11'}\overline{\sigma} = 0, && (23h)\\
&\delta\xi_{10'} + \overline{\delta}\xi_{01'} - \xi_{01'}\alpha - \xi_{10'}\overline{\alpha} + \xi_{10'}\beta + \xi_{01'}\overline{\beta} + \xi_{00'}\mu + \xi_{00'}\overline{\mu} - \xi_{11'}\rho - \xi_{11'}\overline{\rho} = 0, && (23i)\\
&\overline{\delta}\xi_{00'} - D\xi_{10'} - \xi_{00'}\alpha - \xi_{00'}\overline{\beta} - \xi_{10'}\epsilon + \xi_{10'}\overline{\epsilon} + \xi_{11'}\overline{\kappa} - \xi_{00'}\pi - \xi_{10'}\rho - \xi_{01'}\overline{\sigma} = 0. && (23j)
\end{aligned}$$

In the remainder of this section we investigate these conditions on $\mathcal{H}_1 \cup \mathcal{H}_2$.

4.2.2 The condition $H_{AA'BC} = 0$ on $\mathscr{Z}$

On $\mathscr{Z} = \mathcal{H}_1 \cap \mathcal{H}_2$ equations (21a)-(21h) reduce to:

$$D\kappa_0 = 0, \tag{24a}$$
$$\Delta\kappa_2 = 0, \tag{24b}$$
$$\delta\kappa_0 - 2\beta\kappa_0 = 0, \tag{24c}$$
$$\Delta\kappa_0 + 2\delta\kappa_1 + 2\tau\kappa_1 = 0, \tag{24d}$$
$$2\Delta\kappa_1 + \delta\kappa_2 + 2\beta\kappa_2 + 2\tau\kappa_2 = 0, \tag{24e}$$
$$2D\kappa_1 + \overline{\delta}\kappa_0 - 2\alpha\kappa_0 = 0, \tag{24f}$$
$$D\kappa_2 + 2\overline{\delta}\kappa_1 = 0, \tag{24g}$$
$$\overline{\delta}\kappa_2 + 2\alpha\kappa_2 = 0. \tag{24h}$$

In what follows, we regard equations (24c) and (24h) as intrinsic to $\mathscr{Z}$. Making use of the operators $\eth$ and $\overline{\eth}$ (see (1) for their explicit form) these conditions can be concisely rewritten as

$$\eth\kappa_0 = \tau\kappa_0, \tag{25a}$$
$$\overline{\eth}\kappa_2 = -\overline{\tau}\kappa_2. \tag{25b}$$

Remark 10. Equations (24a)-(24h) do not constrain the value of the coefficient κ_1 on $\mathscr{Z}$. Instead, given an arbitrary (smooth) choice of κ_1 and coefficients κ_0 and κ_2 satisfying the equations in (25a)-(25b), we regard equations (24b), (24d) and (24e) as prescribing the initial values of the derivatives $\Delta\kappa_0$, $\Delta\kappa_1$ and $\Delta\kappa_2$ that need to be provided for the transport equations (19a)-(19c) along $\mathcal{H}_1$. Similarly, we use equations (24a), (24f) and (24g) to prescribe the initial values of the derivatives $D\kappa_0$, $D\kappa_1$ and $D\kappa_2$ which are used, in turn, to solve the transport equations (20a)-(20c) along $\mathcal{H}_2$.

4.2.3 The condition $H_{AA'BC} = 0$ on $\mathcal{H}_1$ and on $\mathcal{H}_2$

On $\mathcal{H}_1$ equations (21a)-(21h) reduce to:

$$D\kappa_0 = 0, \tag{26a}$$
$$\Delta\kappa_2 - 2\nu\kappa_1 + 2\gamma\kappa_2 = 0, \tag{26b}$$
$$\delta\kappa_0 - 2\beta\kappa_0 = 0, \tag{26c}$$
$$\Delta\kappa_0 + 2\delta\kappa_1 - 2(\gamma + \mu)\kappa_0 + 2\tau\kappa_1 = 0, \tag{26d}$$
$$2\Delta\kappa_1 + \delta\kappa_2 + 2(\beta + \tau)\kappa_2 - 2\mu\kappa_1 - 2\nu\kappa_0 = 0, \tag{26e}$$
$$2D\kappa_1 + \overline{\delta}\kappa_0 - 2\alpha\kappa_0 = 0, \tag{26f}$$
$$D\kappa_2 + 2\overline{\delta}\kappa_1 = 0, \tag{26g}$$
$$\overline{\delta}\kappa_2 + 2\alpha\kappa_2 = 0. \tag{26h}$$

Equations (26a), (26f) and (26g) are interpreted as propagation equations along the null generators of $\mathcal{H}_1$ which are used to propagate the initial values of κ_0, κ_1 and κ_2 at $\mathscr{Z}$. In order to understand the role equations (26c) and (26h) we consider the expressions

$$D(\delta\kappa_0 - 2\beta\kappa_0), \qquad D(\overline{\delta}\kappa_2 + 2\alpha\kappa_2).$$

A direct computation using the NP commutators shows that

$$D(\delta\kappa_0 - 2\beta\kappa_0) = -2\kappa_0 D\beta \qquad D(\overline{\delta}\kappa_2 + 2\alpha\kappa_2) = 2\kappa_2 D\alpha - 2(\alpha - \overline{\beta})\overline{\delta}\kappa_1 - 2\overline{\delta}^2\kappa_1.$$

Evaluating the Ricci identities on $\mathcal{H}_1$ one finds that $D\alpha = D\beta = 0$ —see also Table 1. Thus, it follows that

$$D(\delta\kappa_0 - 2\beta\kappa_0) = 0, \qquad D(\overline{\delta}\kappa_2 + 2\alpha\kappa_2) = -2\overline{\eth}^2\kappa_1.$$

Accordingly, equation (26c) holds along $\mathcal{H}_1$ if it is satisfied on $\mathcal{Z}$ —this is equivalent to requiring condition (25a) on $\mathcal{Z}$. Observe, however, that in order to obtain the same conclusion for equation (26h) one needs $\bar{\eth}^2\kappa_1 = 0$ on $\mathcal{H}_1$.

It remains to consider equations (26b), (26d) and (26e). These prescribe the value of the transversal derivatives $\Delta\kappa_0$, $\Delta\kappa_1$ and $\Delta\kappa_2$. Recall, however, that from the discussion of Section 4.1 these derivatives satisfy transport equations along the generators of $\mathcal{H}_1$. Thus, some compatibility conditions will arise. Substituting the value of $\Delta\kappa_0$, given by equation (26d) into the transport equation (19a), and then using the NP commutators, NP Ricci identities and equations (26a), (26f) and (26g) to simplify one obtains the condition

$$\Psi_2\kappa_0 = 0.$$

Similarly, substituting the value of $\Delta\kappa_1$ given by equation (26e) into the transport equation (19b) and proceeding in similar manner one finds the further condition

$$\Psi_3\kappa_0 = 0.$$

Finally, the substitution of the value of $\Delta\kappa_2$ as given by equation (26b) eventually leads to the condition

$$\Psi_4\kappa_0 + 2\Psi_3\kappa_1 - 3\Psi_2\kappa_2 = 0.$$

One can summarise the discussion of this subsection as follows:

Lemma 1. *Assume that equations* (26a)*,* (26f) *and* (26g) *hold along* $\mathcal{H}_1$ *with initial data for* κ_0 *and* κ_2 *on* $\mathcal{Z}$ *satisfying equations* (25a) *and* (25b)*, respectively, and that, in addition,*

$$\bar{\eth}^2\kappa_1 = 0, \quad \Psi_2\kappa_0 = 0, \quad \Psi_3\kappa_0 = 0, \quad \Psi_4\kappa_0 + 2\Psi_3\kappa_1 - 3\Psi_2\kappa_2 = 0, \quad on \quad \mathcal{H}_1.$$

Then, one has that

$$H_{A'ABC} = 0 \quad on \quad \mathcal{H}_1.$$

On $\mathcal{H}_2$ equations (21a)-(21h) reduce to:

$$D\kappa_0 = 0, \tag{27a}$$
$$\Delta\kappa_2 = 0, \tag{27b}$$
$$\delta\kappa_0 - 2\beta\kappa_0 + 2\sigma\kappa_1 = 0, \tag{27c}$$
$$\Delta\kappa_0 + 2\delta\kappa_1 + 2\tau\kappa_1 + 2\sigma\kappa_2 = 0, \tag{27d}$$
$$2\Delta\kappa_1 + \delta\kappa_2 + 2(\beta+\tau)\kappa_2 = 0, \tag{27e}$$
$$2D\kappa_1 + \bar{\delta}\kappa_0 - 2\alpha\kappa_0 + 2\rho\kappa_1 = 0, \tag{27f}$$
$$D\kappa_2 + 2\bar{\delta}\kappa_1 + 2\rho\kappa_2 = 0, \tag{27g}$$
$$\bar{\delta}\kappa_2 + 2\alpha\kappa_2 = 0. \tag{27h}$$

In analogy with the analysis on $\mathcal{H}_1$, in what follows we regard equations (27b), (27d) and (27e) as propagation equations for the components κ_0, κ_1 and κ_2 along the generators of $\mathcal{H}_2$. Initial data for these equations is naturally prescribed on $\mathcal{Z}$. Proceeding in an analogous manner to $\mathcal{H}_1$ one obtains the following:

Lemma 2. *Assume that equations* (27b)*,* (27d) *and* (27e) *hold along* $\mathcal{H}_2$ *with initial data for* κ_0 *and* κ_2 *on* $\mathcal{Z}$ *satisfying conditions* (25a) *and* (25b)*, respectively, and that, in addition,*

$$\eth^2\kappa_1 + \kappa_2\delta\sigma + \frac{3}{2}\sigma\delta\kappa_2 + \bar{\alpha}\sigma\kappa_2 = 0, \quad \Psi_2\kappa_2 = 0, \quad \Psi_1\kappa_2 = 0, \quad \Psi_0\kappa_2 + 2\Psi_1\kappa_1 - 3\Psi_2\kappa_0 = 0, \quad on \quad \mathcal{H}_2.$$

Then, one has that

$$H_{A'ABC} = 0 \quad on \quad \mathcal{H}_2.$$

Remark 11. One can show that the curvature conditions in Lemmas 1 and 2 are in fact components of the equation

$$\Psi_{(ABC}{}^{F}\kappa_{D)F} = 0.$$

The other components of this equation are trivially satisfied. As this is a basis independent expression, the curvature conditions are satisfied in all spin bases, not just the parallelly propagated one. One can check this by considering Lorentz transformations and null rotations about l^a and n^a, and show that these conditions are preserved. The equation above can be shown to be an integrability condition for the Killing spinor equation, so it is unsurprising to find components of it arising naturally from the analysis.

4.2.4 The condition $S_{AA'BB'} = 0$ at $\mathscr{Z}$

Using the properties of $\mathscr{Z}$, as given explicitly in Table 1, together with the conditions (24a)-(24h) implied by the equation $H_{AA'BC} = 0$ on $\mathscr{Z}$, equations (22a)-(22d) reads as

$$\xi_{11'} = -\frac{3}{2}(\eth\kappa_2 + \tau\kappa_2)\,, \tag{28a}$$
$$\xi_{10'} = -3\bar{\eth}\kappa_1\,, \tag{28b}$$
$$\xi_{01'} = 3\eth\kappa_1\,, \tag{28c}$$
$$\xi_{00'} = \frac{3}{2}(\eth\kappa_0 - \bar{\tau}\kappa_0)\,, \tag{28d}$$

while on $\mathscr{Z}$ equations (23a)-(23j) reduce to

$$D\xi_{00'} = 0, \tag{29a}$$
$$\Delta\xi_{11'} = 0, \tag{29b}$$
$$D\xi_{11'} + \Delta\xi_{00'} - \tau\xi_{10'} - \bar{\tau}\xi_{01'} = 0, \tag{29c}$$
$$\Delta\xi_{01'} - \delta\xi_{11'} - 2\tau\xi_{11'} = 0, \tag{29d}$$
$$\delta\xi_{01'} + (\bar{\alpha} - \beta)\xi_{01'} = 0, \tag{29e}$$
$$D\xi_{01'} - \delta\xi_{00'} + \tau\xi_{00'} = 0, \tag{29f}$$
$$\Delta\xi_{10'} - \bar{\delta}\xi_{11'} - 2\bar{\tau}\xi_{11'} = 0, \tag{29g}$$
$$\bar{\delta}\xi_{10'} + (\alpha - \bar{\beta})\xi_{10'} = 0, \tag{29h}$$
$$\delta\xi_{10'} + \bar{\delta}\xi_{01'} - (\bar{\alpha} - \beta)\xi_{10'} - (\alpha - \bar{\beta})\xi_{01'} = 0, \tag{29i}$$
$$D\xi_{10'} - \bar{\delta}\xi_{00'} + \bar{\tau}\xi_{00'} = 0. \tag{29j}$$

Equations (29e), (29h) and (29i) can be read as intrinsic equations for $\xi_{01'}$ and $\xi_{10'}$. Expressing these in terms of the $\eth$ and $\bar{\eth}$ operators, observing that the spin-weight of $\xi_{01'}$ and $\xi_{10'}$ are respectively -1 and 1, one has that

$$\eth\xi_{01'} = 0, \tag{30a}$$
$$\bar{\eth}\xi_{10'} = 0, \tag{30b}$$
$$\eth\xi_{10'} + \bar{\eth}\xi_{01'} = 0. \tag{30c}$$

Substituting conditions (28b)-(28c) into conditions (30a)-(30b) above yield the simple conditions

$$\eth^2\kappa_1 = 0, \qquad \bar{\eth}^2\kappa_1 = 0\,, \tag{31}$$

whereas (30c)—as κ_1 is of zero spin-weight quantity—reduces to the commutation relation

$$\eth\bar{\eth}\kappa_1 - \bar{\eth}\eth\kappa_1 = 0\,.$$

Remark 12. The above expressions indicate that the component κ_1 has a very specific multipolar structure. Note, however, that the $\eth$ and $\bar{\eth}$ above are not the ones corresponding to $\mathbb{S}^2$ but of a 2-manifold diffeomorphic to it. Thus, in order to further the discussion one needs to consider the conformal properties of the operators.

Crucially, one can also show that equations (29a)-(29d), (29f)-(29g) and (29j) are implied by equations (24a)-(24h), the Ricci equations, and the conditions of Lemmas 2 and 3 (which must be satisfied on $\mathcal{Z} = \mathcal{H}_1 \cap \mathcal{H}_2$. Summarising:

Lemma 3. *Assume that equations* (24a)-(24h) *hold on* $\mathcal{Z}$ *and that, in addition,*

$$\eth^2 \kappa_1 = 0, \qquad \bar{\eth}^2 \kappa_1 = 0, \quad \text{on} \quad \mathcal{Z}.$$

Then one has that

$$S_{AA'BB'} = 0 \quad \text{on} \quad \mathcal{Z}.$$

4.2.5 The Killing vector equation on $\mathcal{H}_1$ and on $\mathcal{H}_2$

On $\mathcal{H}_1$, equations (23a)-(23j) reduce to:

$$D\xi_{00'} = 0, \tag{32a}$$
$$\Delta\xi_{11'} + (\gamma + \bar{\gamma})\xi_{11'} + \nu\xi_{01'} + \bar{\nu}\xi_{10'} = 0, \tag{32b}$$
$$D\xi_{11'} + \Delta\xi_{00'} - \tau\xi_{10'} - \bar{\tau}\xi_{01'} - (\gamma + \bar{\gamma})\xi_{00'} = 0, \tag{32c}$$
$$\Delta\xi_{01'} - \delta\xi_{11'} - (\gamma - \bar{\gamma} + \mu)\xi_{01'} + \bar{\nu}\xi_{00'} - 2\tau\xi_{11'} = 0, \tag{32d}$$
$$\delta\xi_{01'} + (\bar{\alpha} - \beta)\xi_{01'} = 0, \tag{32e}$$
$$D\xi_{01'} - \delta\xi_{00'} + \tau\xi_{00'} = 0, \tag{32f}$$
$$\Delta\xi_{10'} - \bar{\delta}\xi_{11'} - (\bar{\gamma} - \gamma + \bar{\mu})\xi_{10'} + \nu\xi_{00'} - 2\bar{\tau}\xi_{11'} = 0, \tag{32g}$$
$$\bar{\delta}\xi_{10'} + (\alpha - \bar{\beta})\xi_{10'} = 0, \tag{32h}$$
$$\delta\xi_{10'} + \bar{\delta}\xi_{01'} + (\mu + \bar{\mu})\xi_{00'} - (\bar{\alpha} - \beta)\xi_{10'} - (\alpha - \bar{\beta})\xi_{01'} = 0, \tag{32i}$$
$$D\xi_{10'} - \bar{\delta}\xi_{00'} + \bar{\tau}\xi_{00'} = 0. \tag{32j}$$

Substituting the components $\xi_{00'}, \xi_{01'}, \xi_{10'}$ and $\xi_{11'}$, as given by (22a)-(22d), into these relations (being careful not to discard the Δ derivatives of quantities which vanish on $\mathcal{H}_1$), and using equations (26a)-(26h) and the Ricci equations, one finds that (32a)-(32j) reduce to:

$$\eth^2 \kappa_1 = \kappa_0(\delta\mu + \mu\tau), \tag{33a}$$
$$\bar{\eth}^2 \kappa_1 = 0, \tag{33b}$$
$$\Psi_2\kappa_0 = 0, \tag{33c}$$
$$\Psi_3\kappa_0 = 0, \tag{33d}$$
$$\Psi_4\kappa_0 + 2\Psi_3\kappa_1 - 3\Psi_2\kappa_2 = 0. \tag{33e}$$

Remark 13. The conditions (33b)-(33e) are exactly the conditions of Lemma 2. The additional condition (33a) must be satisfied on all of $\mathcal{H}_1$. Note, however, that after some manipulations the condition

$$D\left(\eth^2\kappa_1 - \kappa_0(\delta\mu + \mu\tau)\right) = -2\delta(\Psi_2\kappa_0) + 4\beta\Psi_2\kappa_0 = 0$$

can be shown to hold, where in the last step (33c) was used. Accordingly, it suffices to guarantee (33a) on $\mathcal{Z}$ as then it is satisfied on the whole of $\mathcal{H}_1$ if condition (33c) holds on $\mathcal{H}_1$. Furthermore, on $\mathcal{Z}$ the spin coefficient μ vanishes, so (33a) reduces to $\eth^2\kappa_1 = 0$ on $\mathcal{Z}$. Note that this is one of the conditions appearing in Lemma 3.

Summarising, we have the following lemma:

Lemma 4. *Assume that equations* (26a)-(26h) *hold on* $\mathcal{H}_1$*, and the conditions of Lemmas 1 and 3 are satisfied. Then one has that*

$$S_{AA'BB'} = 0 \quad \text{on} \quad \mathcal{H}_1.$$

On $\mathcal{H}_2$, equations (23a)-(23j) reduce to:

$$D\xi_{00'} = 0, \tag{34a}$$
$$\Delta\xi_{11'} = 0, \tag{34b}$$
$$D\xi_{11'} + \Delta\xi_{00'} - \tau\xi_{10'} - \overline{\tau}\xi_{01'} = 0, \tag{34c}$$
$$\Delta\xi_{01'} - \delta\xi_{11'} - 2\tau\xi_{11'} = 0, \tag{34d}$$
$$\delta\xi_{01'} + (\overline{\alpha} - \beta)\xi_{01'} - \sigma\xi_{11'} = 0, \tag{34e}$$
$$D\xi_{01'} - \delta\xi_{00'} + \tau\xi_{00'} + \sigma\xi_{10'} + \rho\xi_{01'} = 0, \tag{34f}$$
$$\Delta\xi_{10'} - \overline{\delta}\xi_{11'} - 2\overline{\tau}\xi_{11'} = 0, \tag{34g}$$
$$\overline{\delta}\xi_{10'} + (\alpha - \overline{\beta})\xi_{10'} - \overline{\sigma}\xi_{11'} = 0, \tag{34h}$$
$$\delta\xi_{10'} + \overline{\delta}\xi_{01'} - (\overline{\alpha} - \beta)\xi_{10'} - (\alpha - \overline{\beta})\xi_{01'} - 2\rho\xi_{11'} = 0, \tag{34i}$$
$$D\xi_{10'} - \overline{\delta}\xi_{00'} + \overline{\tau}\xi_{00'} + \overline{\sigma}\xi_{01'} + \rho\xi_{10'} = 0. \tag{34j}$$

Substituting the components $\xi_{00'}, \xi_{01'}, \xi_{10'}$ and $\xi_{11'}$, as given by (22a)-(22d), into these relations (being careful not to discard the D derivatives of quantities which vanish on $\mathcal{H}_2$), and using equations (27a)-(27h) and the Ricci equations, one finds that (34a)-(34j) reduce to:

$$\eth^2\kappa_1 + \kappa_2\delta\sigma + \frac{3}{2}\sigma\delta\kappa_2 + \overline{\alpha}\sigma\kappa_2 = 0, \tag{35a}$$
$$\overline{\eth}^2\kappa_1 + \kappa_2\overline{\delta}\sigma - \frac{1}{2}\overline{\sigma}\delta\kappa_2 - 3\overline{\alpha}\kappa_2\overline{\sigma} - \overline{\Psi}_1\kappa_2 = 0, \tag{35b}$$
$$\Psi_1\kappa_2 = 0, \tag{35c}$$
$$\Psi_2\kappa_2 = 0, \tag{35d}$$
$$\Psi_0\kappa_2 + 2\Psi_1\kappa_1 - 3\Psi_2\kappa_0 = 0. \tag{35e}$$

The conditions (35a) and (35c)-(35e) are exactly the conditions of Lemma 3. The additional condition (35b) must be satisfied on all of $\mathcal{H}_2$. Summarising, we have the following lemma:

Lemma 5. *Assume that equations* (27a)-(27h) *hold on $\mathcal{H}_2$, the conditions of Lemma 2 are satisfied, and that in addition,*

$$\overline{\eth}^2\kappa_1 + \kappa_2\overline{\delta}\sigma - \frac{1}{2}\overline{\sigma}\delta\kappa_2 - 3\overline{\alpha}\kappa_2\overline{\sigma} - \overline{\Psi}_1\kappa_2 = 0 \quad \text{on} \quad \mathcal{H}_2.$$

Then one has that

$$S_{AA'BB'} = 0 \quad \text{on} \quad \mathcal{H}_2.$$

5 Analysis the constraints on $\mathcal{Z}$

In this section we analyse the constraints on $\mathcal{Z}$ obtained in the previous section.

5.1 Determining κ_2 on $\mathcal{Z}$

Consider now the restrictions we have concerning κ_2 on $\mathcal{Z}$. To satisfy the condition $\Psi_2\kappa_2 = 0$ on $\mathcal{H}_2$, applied in Lemma 3, we have that $\Psi_2\kappa_2 = 0$ has to vanish on $\mathcal{Z} \subset \mathcal{H}_2$, as well. Consistent with this condition the following subcases can be seen to arise:

i. Assume first that κ_2 is nowhere vanishing on $\mathcal{Z}$. In this case Ψ_2 must vanish throughout $\mathcal{Z}$. Note also that in virtue of Table 1 all the other Weyl spinor components vanish on $\mathcal{Z}$, and thereby

$$\Psi_{ABCD}|_{\mathcal{Z}} = 0\,.$$

As shown in Table 1, Ψ_0 and Ψ_1 vanish on $\mathcal{H}_1$, and Ψ_3 and Ψ_4 vanish on $\mathcal{H}_2$, respectively. Further, observe that the Bianchi identities imply the following relations on $\mathcal{H}_1$:

$$D\Psi_2 = 0,$$
$$D\Psi_3 = \overline{\delta}\Psi_2,$$
$$D\Psi_4 = 2\alpha\Psi_3 + \overline{\delta}\Psi_3.$$

As Ψ_2 vanishes on $\mathcal{Z}$ and D is the directional derivative along the geodesics generating $\mathcal{H}_1$, the first of these equations imply that $\Psi_2 = 0$ on $\mathcal{H}_1$. By the same argument, because the right hand side of the second of the above relations has shown to vanish on $\mathcal{H}_1$, we have that $\Psi_3 = 0$ on $\mathcal{H}_1$. In turn, this also implies that $\Psi_4 = 0$ on $\mathcal{H}_1$ as a consequence of the last relation. Therefore, along with the vanishing of Ψ_0 and Ψ_1 on $\mathcal{H}_1$ all the Weyl spinor components vanish there —that is one has

$$\Psi_{ABCD}|_{\mathcal{H}_1} = 0\,.$$

Similarly, the Bianchi identities imply the following relations on $\mathcal{H}_2$:

$$\begin{aligned}
&\Delta\Psi_0 = \delta\Psi_1 - (4\tau + 2\beta)\Psi_1 + 3\sigma\Psi_2,\\
&\Delta\Psi_1 = \delta\Psi_2 - 3\tau\Psi_2,\\
&\Delta\Psi_2 = 0.
\end{aligned}$$

As Ψ_2 vanishes on $\mathcal{Z}$, and Δ is the directional derivative along the geodesics generating $\mathcal{H}_2$, the third of these equations imply that $\Psi_2 = 0$ on $\mathcal{H}_2$. Thus, the right hand side of the second of the above relations vanishes on $\mathcal{H}_2$, and by the same argument we must have $\Psi_1 = 0$ on $\mathcal{H}_2$. The first relation then implies that $\Psi_0 = 0$ on $\mathcal{H}_2$. Therefore, along with the vanishing of Ψ_3 and Ψ_4 on $\mathcal{H}_2$ all the Weyl spinor components vanish there. Thus, one has that

$$\Psi_{ABCD}|_{\mathcal{H}_2} = 0\,.$$

Summarising, the non-vanishing of κ_2 on $\mathcal{Z}$ implies that all the Weyl spinor components vanish identically on the union of $\mathcal{Z}$, $\mathcal{H}_1$ and $\mathcal{H}_2$. This, in the vacuum case, implies that all components of the Riemann curvature tensor vanish on $\mathcal{H}_1 \cup \mathcal{H}_2$. It follows then that the spacetime obtained in Theorem 2 is diffeomorphic to a portion of the Minkowski spacetime and the pair intersecting null hypersurfaces has to contains a bifurcate Killing horizon corresponding to a choice of a boost Killing vector field.

ii. ***κ_2 vanishes on a (non-empty) open subset of $\mathcal{Z}$.*** It follows from the discussion in the previous subsection that, unless the spacetime is Minkowski, κ_2 must vanish somewhere on $\mathcal{Z}$. It turns out that that if this is the case, then κ_2 must vanish on some open subset of $\mathcal{Z}$ —this fact can readily verified by a contradiction argument together with the condition $\Psi_2\kappa_2 = 0$. Keeping the latter observation in mind, it follows from (25b), and from equation (4), that

$$\bar{\eth}\kappa_2 = -\bar{\tau}\,\kappa_2\,,$$

can be written as

$$\overline{P}P\,\partial_{\bar{z}}\,(P^{-1}\kappa_2) = -\bar{\tau}\,P\,(P^{-1}\,\kappa_2) \tag{36}$$

implying, in turn, that κ_2 has to be of the form

$$\kappa_2 = P\cdot\exp\left(-\int\bar{\tau}\,\overline{P}^{-1}\,d\bar{z} + \varphi(z)\right)\,, \tag{37}$$

where $\varphi(z)$ is an arbitrary holomorphic function. This, however, in virtue of the non-vanishing of P, implies that κ_2 cannot vanish on an open subset of $\mathcal{Z}$ unless it is identically zero on $\mathcal{Z}$, i.e.

$$\kappa_2|_{\mathcal{Z}} = 0$$

as we intended to show. Note also that the condition (27b) requires then the vanishing of κ_2 along the generators of $\mathcal{H}_2$, thereby we have

$$\kappa_2|_{\mathcal{H}_2} = 0\,.$$

Summarising, in this subsection we have shown the following:

Lemma 6. *Assume that*

$$\Psi_2\,\kappa_2 = 0 \qquad \textit{on} \quad \mathcal{Z}.$$

Then, if κ_2 is nowhere vanishing on $\mathcal{Z}$, then the solution to the characteristic initial value problem must be diffeomorphic to the Minkowski spacetime in the domain of dependence of $D(\mathcal{H}_1 \cup \mathcal{H}_2)$. Otherwise, $\kappa_2 = 0$ holds on $\mathcal{Z}$, and then it is also identically zero on $\mathcal{H}_2$.

5.2 Determining κ_0 on $\mathcal{Z}$

The analysis of the previous section can be adapted, *mutatis mutandi*, to the component κ_0 by noting that the vanishing of $\Psi_2\kappa_0$ on $\mathcal{H}_1$, one of the conditions in Lemma 1, can be traced back to the vanishing of $\Psi_2\kappa_0$ on $\mathcal{Z}$. Indeed, it can be shown that unless the spacetime is Minkowski κ_0 must vanish on a non-empty subset of $\mathcal{Z}$. The only difference in the analysis lies on the analogue of equation (36). From equations (25a) and (4) along with the fact that κ_0 is of spin weight -1, it follows that

$$\eth\kappa_0 = \tau\,\kappa_0\,,$$

can be written as

$$P\overline{P}\,\partial_z\,(\overline{P}^{-1}\kappa_0) = \tau\,\overline{P}\,(\overline{P}^{-1}\kappa_0) \tag{38}$$

which implies, in turn, that κ_0 has to be of the form

$$\kappa_0 = \overline{P}\cdot\exp\left(\int \tau\,P^{-1}\,dz + \varsigma(\overline{z})\right)\,, \tag{39}$$

where $\varsigma(\overline{z})$ is an arbitrary antiholomorphic function on $\mathcal{Z}$. From here, by an argument analogous to that used for κ_2 one concludes that

$$\kappa_0|_{\mathcal{Z}} = 0$$

and, moreover, as a consequence of equation (26a), also that

$$\kappa_0|_{\mathcal{H}_1} = 0.$$

Summarising:

Lemma 7. *Assume that*

$$\Psi_2\,\kappa_0 = 0 \qquad \textit{on} \quad \mathcal{Z}.$$

Then, if κ_0 is nowhere vanishing on $\mathcal{Z}$, then the solution to the characteristic initial value problem must be diffeomorphic to the Minkowski spacetime in the domain of dependence of $D(\mathcal{H}_1\cup\mathcal{H}_2)$. Otherwise, $\kappa_0 = 0$ holds on $\mathcal{Z}$, and then it is also identically zero on $\mathcal{H}_1$.

5.3 Eliminating redundant conditions on $\mathcal{H}_1$ and $\mathcal{H}_2$

The first condition in Lemma 1 was

$$\bar{\eth}^2\kappa_1 = 0 \quad \text{on} \quad \mathcal{H}_1.$$

In theory, one would have to solve this constraint on the whole of $\mathcal{H}_1$. However, one can show that on $\mathcal{H}_1$

$$\begin{aligned} D(\bar{\eth}^2\kappa_1) = &-\frac{1}{2}\overline{\delta\delta\delta}\kappa_0 + \frac{3}{2}\overline{\tau}\,\overline{\delta\delta}\kappa_0 + \overline{\delta}\kappa_0\left(-\alpha^2 - 4\alpha\overline{\beta} - \overline{\beta}^2 + \frac{5}{2}\overline{\delta}\alpha + \frac{1}{2}\overline{\delta\beta}\right) \\ &+ \kappa_0\left(2\alpha\overline{\beta}\,\overline{\tau} - 2\alpha\overline{\delta}\alpha - 3\overline{\beta\delta}\alpha - \alpha\overline{\delta\beta} + \overline{\delta\delta}\alpha\right). \end{aligned}$$

Note that as κ_0 vanishes on $\mathcal{H}_1$ (under the assumption that the spacetime is not diffeomorphic to Minkowski), the right hand side of this equation also vanishes on $\mathcal{H}_1$. Therefore, if κ_1 satisfies $\bar{\eth}^2\kappa_1 = 0$ on $\mathcal{Z}$, then it also satisfies the same condition on the whole of $\mathcal{H}_1$. This was a condition on $\mathcal{Z}$ already present from the requirement that $S_{AA'BB'}|_{\mathcal{Z}} = 0$. Summarising:

Lemma 8. *If $\kappa_0|_{\mathcal{H}_1} = 0$ and $\bar{\eth}^2\kappa_1|_{\mathcal{Z}} = 0$, then the condition $\bar{\eth}^2\kappa_1|_{\mathcal{H}_1} = 0$ from Lemma 1 is automatically satisfied.*

A similar procedure can be performed on $\mathcal{H}_2$. The first condition from Lemma 2 was

$$\eth^2\kappa_1 + \kappa_2\delta\sigma + \frac{3}{2}\sigma\delta\kappa_2 + \overline{\alpha}\sigma\kappa_2 = 0$$

which must be satisfied on $\mathcal{H}_2$. As we already we have shown that necessarily $\kappa_2|_{\mathcal{H}_2} = 0$ unless the spacetime is diffeomorphic to the Minkowski spacetime. Therefore, the aforementioned condition reduces to

$$\eth^2\kappa_1 = 0 \quad \text{on} \quad \mathcal{H}_2.$$

Now, one can show that on $\mathcal{H}_2$,

$$\begin{aligned}\Delta\left(\eth^2\kappa_1\right) = &-\frac{1}{2}\delta\delta\delta\kappa_2 - \frac{3}{2}\tau\delta\delta\kappa_2 + \delta\kappa_2\left(-\overline{\alpha}^2 - \overline{\alpha}\beta + 2\beta^2 - 2\delta\overline{\alpha} - 4\delta\beta\right)\\ &+ \kappa_2\left(-\overline{\alpha}\delta\overline{\alpha} + \beta\delta\overline{\alpha} - 2\overline{\alpha}\delta\beta + 2\beta\delta\beta - \delta\delta\overline{\alpha} - 2\delta\delta\beta\right).\end{aligned}$$

The requirement that κ_2 vanishes on $\mathcal{H}_2$ means that the right hand side of this equation also vanishes on $\mathcal{H}_2$. Therefore, if κ_1 satisfies $\eth^2\kappa_1 = 0$ on $\mathcal{Z}$, then it also satisfies the same condition on the whole of $\mathcal{H}_2$. This was a condition on $\mathcal{Z}$ already present from the requirement that $S_{AA'BB'}|_{\mathcal{Z}} = 0$.

Finally, the condition from Lemma 5 says that

$$\overline{\eth}^2\kappa_1 + \kappa_2\overline{\delta}\sigma - \frac{1}{2}\overline{\sigma}\delta\kappa_2 - 3\overline{\alpha}\kappa_2\overline{\sigma} - \overline{\Psi}_1\kappa_2 = 0 \quad \text{on} \quad \mathcal{H}_2$$

which reduces to $\overline{\eth}^2\kappa_1 = 0$ due to the fact that $\kappa_2|_{\mathcal{H}_2} = 0$ when the spacetime is not diffeomorphic to the Minkowski solution. One can show that on $\mathcal{H}_2$

$$\begin{aligned}\Delta\left(\overline{\eth}^2\kappa_1\right) = \delta\kappa_2\left(\frac{1}{2}\overline{\delta}\overline{\tau} - \overline{\beta}\overline{\tau}\right) &+ \kappa_2\left(-6\alpha^2\beta - 6\alpha\beta\overline{\beta} - 2\alpha\delta\alpha + \overline{\alpha}\overline{\delta}\alpha\right.\\ &\left.+5\beta\overline{\delta}\alpha + 2\alpha\overline{\delta}\overline{\alpha} + \overline{\beta\delta\overline{\alpha}} + 7\alpha\overline{\delta}\beta + 2\overline{\beta\delta}\beta + \overline{\delta}\delta\alpha - \overline{\delta\delta}\alpha - 2\overline{\delta\delta}\beta\right).\end{aligned}$$

The requirement that κ_2 vanishes on $\mathcal{H}_2$ means that the right hand side of this equation also vanishes on $\mathcal{H}_2$. So if κ_1 satisfies $\overline{\eth}^2\kappa_1 = 0$ on $\mathcal{Z}$, then it also satisfies the same condition on the whole of $\mathcal{H}_2$. This was a condition already present from the requirement that $S_{AA'BB'}|_{\mathcal{Z}} = 0$. Summarising, we have

Lemma 9. *If $\kappa_2|_{\mathcal{H}_2} = 0$ and $\overline{\eth}^2\kappa_1|_{\mathcal{Z}} = \eth^2\kappa_1|_{\mathcal{Z}} = 0$, then the conditions*

$$\left(\overline{\eth}^2\kappa_1 + \kappa_2\overline{\delta}\sigma - \frac{1}{2}\overline{\sigma}\delta\kappa_2 - 3\overline{\alpha}\kappa_2\overline{\sigma} - \overline{\Psi}_1\kappa_2\right)|_{\mathcal{H}_2} = 0,$$

$$\left(\eth^2\kappa_1 + \kappa_2\delta\sigma + \frac{3}{2}\sigma\delta\kappa_2 + \overline{\alpha}\sigma\kappa_2\right)|_{\mathcal{H}_2} = 0,$$

applied in Lemmas 2 and 5, are automatically satisfied.

The only remaining condition on $\mathcal{H}_1$ to be considered is from Lemma 1, which reduces to

$$\left(2\Psi_3\kappa_1 - 3\Psi_2\kappa_2\right)|_{\mathcal{H}_1} = 0 \tag{40}$$

due to the requirement that $\kappa_0|_{\mathcal{H}_1} = 0$. One can also use this requirement to show that

$$D^2\left(2\Psi_3\kappa_1 - 3\Psi_2\kappa_2\right)|_{\mathcal{H}_1} = 0.$$

In fact, the right hand side of this expression can be shown to be homogeneous in κ_0 and derivatives of κ_0 intrinsic to $\mathcal{H}_1$. This can be thought of as a second order ordinary differential equation along the geodesic generators of $\mathcal{H}_1$. Therefore, equation (40) is equivalent to the vanishing of $(2\Psi_3\kappa_1 - 3\Psi_2\kappa_2)$ and its first D-derivative on $\mathcal{Z}$. This combination vanishes on $\mathcal{Z}$ if $\kappa_2|_{\mathcal{H}_2} = 0$ as it follows from Table 1 that $\Psi_3|_{\mathcal{Z}} = 0$. The vanishing of the first derivative on $\mathcal{Z}$ can be shown to be equivalent to

$$\overline{\delta}\left(\kappa_1^3\Psi_2\right)|_{\mathcal{Z}} = 0\,. \tag{41}$$

In a similar way, the only remaining condition on $\mathcal{H}_2$ to be analysed is from Lemma 2. This condition reduces to

$$\left(2\Psi_1\kappa_1 - 3\Psi_2\kappa_0\right)|_{\mathcal{H}_2} = 0 \tag{42}$$

due to the requirement that $\kappa_2|_{\mathcal{H}_2} = 0$. One can also use this requirement to show that

$$\Delta^2 \left(2\Psi_1\kappa_1 - 3\Psi_2\kappa_0\right)|_{\mathcal{H}_2} = 0.$$

In fact, the right hand side of this can be shown to be homogeneous in κ_2 and derivatives of κ_2 intrinsic to $\mathcal{H}_2$. This can be thought of as a second order ordinary differential equation along the geodesic generators of $\mathcal{H}_2$. Therefore, equation (42) is equivalent to the vanishing of $(2\Psi_1\kappa_1 - 3\Psi_2\kappa_0)$ and its first Δ derivative on $\mathcal{Z}$. This combination vanishes on $\mathcal{Z}$ if $\kappa_0|_{\mathcal{H}_1} = 0$ as, following 1, one has that $\Psi_1|_{\mathcal{Z}} = 0$. The vanishing of the first derivative on $\mathcal{Z}$ can be shown to be equivalent to

$$\delta\left(\kappa_1^3\Psi_2\right)|_{\mathcal{Z}} = 0\,. \tag{43}$$

It follows then from equations (43) and (41),

$$\mathfrak{K} \equiv \kappa_1^3\Psi_2 \tag{44}$$

is constant $\mathfrak{K} \in \mathbb{C}$ on $\mathcal{Z}$.

Summarising the discussion of this section one has that:

Lemma 10. *Assume that $\kappa_0|_{\mathcal{H}_1} = \kappa_2|_{\mathcal{H}_2} = 0$. Then $\mathfrak{K} \equiv \kappa_1^3\Psi_2$ is constant on $\mathcal{Z}$ if and only if*

$$\begin{aligned}(2\Psi_3\kappa_1 - 3\Psi_2\kappa_2)\,|_{\mathcal{H}_1} = 0,\\ (2\Psi_1\kappa_1 - 3\Psi_2\kappa_0)\,|_{\mathcal{H}_2} = 0.\end{aligned}$$

Remark 14. Note that

$$\begin{aligned}D\mathfrak{K}|_{\mathcal{H}_1} &= \frac{3}{2}\Psi_2\kappa_1^2\left(-\bar{\delta}\kappa_0 + 2\alpha\kappa_0\right)|_{\mathcal{H}_1}\\ &= 0\end{aligned}$$

where we have used equation $D\Psi_2|_{\mathcal{H}_1} = 0$ from Table 1, equation (26f) and the requirement that $\kappa_0|_{\mathcal{H}_1} = 0$. Similarly,

$$\begin{aligned}\Delta\mathfrak{K}|_{\mathcal{H}_2} &= \frac{3}{2}\Psi_2\kappa_1^2\left(-\delta\kappa_2 - 2(\beta+\tau)\kappa_2\right)|_{\mathcal{H}_2}\\ &= 0\end{aligned}$$

where we have used equation $\Delta\Psi_2|_{\mathcal{H}_2} = 0$ from Table 1, equation (27e) and the requirement that $\kappa_2|_{\mathcal{H}_2} = 0$. Thus, $\mathfrak{K}$ is constant not merely on $\mathcal{Z}$ but on the whole of $\mathcal{H}_1 \cup \mathcal{H}_2$. Since the Newman-Penrose reduced system coupled to the wave equation for κ_{AB}, equation (7), is a well-posed hyperbolic system we also have that $\mathfrak{K}$ is, in fact, constant throughout the domain of dependence of $\mathcal{H}_1 \cup \mathcal{H}_2$.

5.4 Summary

Collecting all the previous results together one obtains the following:

Proposition 4. *Assume that the spacetime obtained from the characteristic initial value problem in $D(\mathcal{H}_1\cup\mathcal{H}_2)$ is not diffeomorphic to the Minkowski spacetime. Then the following two statements are equivalent:*

(i) *Given a spin basis $\{o^A, \iota^A\}$ on $\mathcal{Z}$, there exist a constant $\mathfrak{K} \in \mathbb{C}$ such that*

$$\kappa_0 = 0\,,\quad \eth^2\kappa_1 = \bar{\eth}^2\kappa_1 = 0\,,\quad \kappa_2 = 0 \quad \textit{and} \quad \kappa_1^3\Psi_2 = \mathfrak{K} \quad \textit{on} \quad \mathcal{Z}\,.$$

(ii) $H_{A'ABC} = 0, \quad S_{AA'BB'} = 0$ *everywhere on* $\mathcal{H}_1 \cup \mathcal{H}_2$.

Recall that the vanishing of the spinors $H_{A'ABC}$ and $S_{AA'BB'}$ on $\mathcal{H}_1 \cup \mathcal{H}_2$ are precisely the conditions of Proposition 3, which along with the assumptions of Theorem 2 imply that the Killing spinor candidate κ_{AB} is in fact a Killing spinor in the causal future (or past) of $\mathcal{Z}$. By summing up these observations we get:

Theorem 3. *Let $(\mathcal{M}, \boldsymbol{g})$ be a vacuum spacetime satisfying the conditions of Theorem 2. Given a spin basis $\{o^A, \iota^A\}$ on $\mathcal{Z}$, assume that there exist a constant $\mathfrak{K} \in \mathbb{C}$ such that the conditions*

$$\kappa_0 = 0, \quad \eth^2\kappa_1 = \bar{\eth}^2\kappa_1 = 0\,, \quad \kappa_2 = 0 \quad \textit{and} \quad \kappa_1^3\Psi_2 = \mathfrak{K} \tag{45}$$

hold on $\mathcal{Z}$. Then the corresponding unique solution κ_{AB} to equation (7) *is a Killing spinor everywhere on the domain of dependence of $\mathcal{H}_1 \cup \mathcal{H}_2$.*

Remark 15. Condition (45) is a strong restriction on the form of the Weyl spinor component Ψ_2 —and thus, also of the curvature of the 2-surface $\mathcal{Z}$. As it will be seen in Section 8, it fixes its functional form up to some constants. As already discussed in Remark 5 the Weyl spinor component Ψ_2 is not a basic piece of initial data. In view of (17) condition (45), ultimately leads to restrictions on τ and $\zeta^{\mathcal{A}}$.

6 On the role of κ_1

The purpose of this section is to discuss some of the consequences of the existence of a Killing spinor field. The extent of these implications is remarkable.

6.1 Restrictions on the initial data of distorted black holes

Note first that once κ_1 is fixed on $\mathcal{Z}$, the component Ψ_2 gets also to be determined by the relation (44) as

$$\Psi_2 = \mathfrak{K}\,\kappa_1^{-3} \tag{46}$$

where $\mathfrak{K}$ is some complex number. In turn, we also get restrictions on the free data —comprised by the complex vector field ζ^A and the spin coefficient τ on $\mathcal{Z}$— as given in Rácz's black hole holograph construction in [33, 34].

Now, observe that once Ψ_2 is known, the metric $\boldsymbol{\sigma}$ is restricted in a great extent. To see this recall first the definition of $\eth$ and $\bar{\eth}$ given by (4) in terms of the function P on $\mathcal{Z}$. By applying the commutation relation relevant for P, that is of spin-weight one, we get

$$(\eth\,\bar{\eth} - \bar{\eth}\,\eth)\,P = (\Psi_2 + \overline{\Psi}_2)\,P\,,$$

which by using $\bar{\eth}\,P = 0$, and also by using explicit z- and $\bar{z}$-derivatives (16) can be seen to take the form

$$P\overline{P}\,\partial_z\partial_{\bar{z}}\big(\log(P\overline{P})\big) = -2\,\mathrm{Re}(\Psi_2)\,. \tag{47}$$

Similarly, (17) can be seen to impose very strong restrictions on the spin-coefficient τ. Indeed, using the above notation (17) takes the form

$$P\,(\partial_z\bar{\tau}) - \overline{P}\,(\partial_{\bar{z}}\tau) - \big(\bar{\tau}\,P\,(\partial_z \log \overline{P}) - \tau\,\overline{P}\,(\partial_{\bar{z}} \log P)\big) = -2\,\mathrm{i}\,\mathrm{Im}(\Psi_2)\,. \tag{48}$$

By applying the substitutions $\tau \mapsto \tau_1 + \mathrm{i}\,\tau_2$, $P \mapsto P_1 + \mathrm{i}\,P_2$ and $z \mapsto z_1 + \mathrm{i}\,z_2$ we get, by a direct calculation, that the real part of (48) reduces to a homogeneous linear equation for τ_1 and τ_2, whereas the vanishing of the imaginary part can be seen to be a first order linear partial differential equation for the variables τ_1 and τ_2 on $\mathbb{R}^2$, with coordinates (z_1, z_2). Once, say, τ_1 is eliminated by the linear algebraic relation, the corresponding linear partial differential equation can always be solved for τ_2. This completes then the verification of the claim that whenever a Killing spinor exists on a distorted vacuum black hole spacetime, the specification of κ_1 on the bifurcation surface is equivalent to the freely specifiable data comprised by ζ^A and τ there.

Remark 16. It is worth mentioning that under a boost transformation

$$l^a \mapsto \vartheta\, l^a, \qquad n^a \mapsto \vartheta^{-1} n^a,$$

where ϑ is a smooth positive real function on $\mathcal{Z}$, the spin connection coefficient τ transforms as

$$\tau \mapsto \tau + \delta \log \vartheta.$$

$\mathcal{H}_1$	$\mathcal{Z}$	$\mathcal{H}_2$
$\kappa_0 = 0$	$\kappa_0 = 0$	$\kappa_0 = -2\,u\,(\eth\kappa_1 + \tau\,\kappa_1)$
$\kappa_1 = \kappa_1\vert_{\mathcal{Z}}$	$\kappa_1 :\ \eth^2\kappa_1 = \overline{\eth}^2\kappa_1 = 0$	$\kappa_1 = \kappa_1\vert_{\mathcal{Z}}$
$\kappa_2 = -2\,r\,\overline{\eth}\kappa_1$	$\kappa_2 = 0$	$\kappa_2 = 0$

Table 2: The components of the Killing spinor field κ_{AB} on the null hypersurface $\mathcal{H}_1 \cup \mathcal{H}_2$.

$\mathcal{H}_1$	$\mathcal{Z}$	$\mathcal{H}_2$
$\xi_{11'} = -3\,r\,(\tau\,\overline{\eth}\kappa_1 - \overline{\tau}\,\eth\kappa_1)$	$\xi_{11'} = 0$	$\xi_{11'} = 0$
$\xi_{10'} = -3\,\overline{\eth}\kappa_1$	$\xi_{10'} = -3\,\overline{\eth}\kappa_1$	$\xi_{10'} = -3\,\overline{\eth}\kappa_1$
$\xi_{10'} = 3\,\eth\kappa_1$	$\xi_{10'} = 3\,\eth\kappa_1$	$\xi_{10'} = 3\,\eth\kappa_1$
$\xi_{00'} = 0$	$\xi_{00'} = 0$	$\xi_{00'} = -3\,u\,(\tau\,\overline{\eth}\kappa_1 - \overline{\tau}\,\eth\kappa_1)$

Table 3: The components of the Killing vector field $\xi_{AA'}$ on the null hypersurface $\mathcal{H}_1 \cup \mathcal{H}_2$.

This gauge freedom has been left open in the black hole holograph construction [33, 34]. Thereby, by solving the first-order quasilinear partial differential equation

$$\tau_1 + \mathrm{Re}(\delta \log \vartheta) = 0$$

for ϑ, the real part of τ could be transformed out to the expense of having the imaginary part changing as

$$\tau_2 \mapsto \tau_2 + \mathrm{Im}(\delta \log \vartheta).$$

For a simple application of such a gauge transformation see the argument below (64) in Section 10.

6.2 The explicit form of κ_{AB} and $\xi_{AA'} = \nabla^P{}_{A'}\kappa_{AP}$ on the horizon

It is also instructive to compute the explicit form of the Killing spinor κ_{AB} and the associated Killing vector field $\xi_{AA'} = \nabla^P{}_{A'}\kappa_{AP}$ on the horizon $\mathcal{H}_1 \cup \mathcal{H}_2$.

As for the explicit form of the Killing spinor note first that, in virtue of Theorem 3, on $\mathcal{Z}$ we have

$$\kappa_0 = 0, \quad \eth^2\kappa_1 = \overline{\eth}^2\kappa_1 = 0\,, \quad \kappa_2 = 0\,. \tag{49}$$

Using then (26a), (26f) and (26g), and by commuting D and δ derivatives, we get that on $\mathcal{H}_1$

$$\kappa_0 = 0, \quad \kappa_1 = \kappa_1\vert_{\mathcal{Z}}\,, \quad \kappa_2 = -2\,r\,\overline{\eth}\kappa_1\,. \tag{50}$$

Analogously, by (27b), (27d) and (27e), and by commuting Δ and δ derivatives, we get on $\mathcal{H}_2$

$$\kappa_0 = -2\,u\,(\eth\kappa_1 + \tau\,\kappa_1), \quad \kappa_1 = \kappa_1\vert_{\mathcal{Z}}\,, \quad \kappa_2 = 0\,. \tag{51}$$

These observations are summarised in Table 2.

In exactly the same way, the components of the Killing vector field $\xi_{AA'} = \nabla^P{}_{A'}\kappa_{AP}$ can be determined by equations (28a)-(28d) on $\mathcal{Z}$, by (32a), (32c), (32f) and (32j) on $\mathcal{H}_1$, as well as, by (34b), (34c), (34d) and (34g) on $\mathcal{H}_2$, along with commuting derivatives on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The corresponding explicit formulas are collected in Table 3.

Remark 17. In order to proceed with the interpretation of the above expressions recall, first, that any of the the distorted black hole configurations was shown [33, 34] to admit a horizon Killing vector field of the form

$$K^a = \begin{cases} -r\,(\partial/\partial r)^a\,, & \text{on } \mathcal{H}_1\,, \\ \ u\,(\partial/\partial u)^a\,, & \text{on } \mathcal{H}_2\,. \end{cases}$$

Note also, in Gaussian null coordinates $(u, r, x^{\mathcal{A}})$ the coordinate functions u and r are affine parameters along he generators of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Accordingly, the components $\xi_{11'}$ and $\xi_{00'}$ of the Killing vector field are not constant in these coordinates. They would be constant if they were to be expressed in terms of the associated Killing parameters instead of the affine ones. Notably, this behaviour of $\xi_{AA'}$ on the horizon $\mathcal{H}_1 \cup \mathcal{H}_2$ resembles that of the asymptotically time translational Killing vector field $(\partial/\partial t)^a$ of the Kerr solution. Recall that the orbits of $(\partial/\partial t)^a$ repeatedly and periodically intersect the generators of the horizons $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and also that $(\partial/\partial t)^a$ reduces to an axial Killing vector field on the bifurcation surface.

6.3 The Petrov type of the domain of dependence

It has been known for long [39] (see also [28]) that the existence of the Killing spinor κ_{AB} satisfying (5) imposes strong restrictions on the self-dual Weyl spinor Ψ_{ABCD} via the integrability condition

$$\Psi_{(ABC}{}^{F}\kappa_{D)F} = 0\,.$$

Namely, if neither Ψ_{ABCD} nor κ_{AB} vanishes there exist some scalar ψ such that

$$\Psi_{ABCD} = \psi\,\kappa_{(AB}\kappa_{CD)}\,,$$

implying, in particular, that Ψ_{ABCD} *must be of Petrov type D or N.* It is also known that if the Killing spinor field κ_{AB} is generic, i.e. $\kappa_{AB} = \alpha_{(A}\beta_{B)}$, for some $\alpha_A \neq \beta_A$ spinors, Ψ_{ABCD} *is of Petrov type D.*

As this integrability condition had been used (see Remark 11) in the previous sections in identifying the conditions on the initial data for κ_{AB} on $\mathcal{H}_1 \cup \mathcal{H}_2$, and both κ_{AB} and Ψ_{ABCD} are known to satisfy wave equations that are linear and homogeneous in these variables (see e.g. [2, 27]) the integrability condition immediately holds everywhere in the domain of dependence of $\mathcal{H}_1 \cup \mathcal{H}_2$.

Note also that, as for the Killing spinor field $\kappa_{AB} = \kappa_2 o_A o_B - 2\kappa_1 o_{(A}\iota_{B)} + \kappa_0 \iota_A \iota_B$ holds, whenever κ_1 is non identically zero (which could only happen in the flat case) κ_{AB} is guaranteed to be generic. All these observations verify the following :

Corollary 1. *Let $(\mathcal{M}, \boldsymbol{g})$ be a vacuum spacetime satisfying the conditions of Theorem 2. If κ_1 is not identically zero on the bifurcation surface $\mathcal{Z}$ then the corresponding distorted black hole spacetime is of Petrov type D everywhere in the domain of dependence of $\mathcal{H}_1 \cup \mathcal{H}_2$.*

Remark 18. As mentioned in the introduction, in [21] all vacuum Petrov type D spacetimes have been described and expressed in terms of some local coordinates. The difficulty of bringing these explicit solutions into the gauge used in Rácz holograph construction motivate much of the rest of this article.

7 Axial symmetry of the bifurcation surface $\mathcal{Z} \approx \mathbb{S}^2$

As argued below, whenever the bifurcation surface $\mathcal{Z}$ possesses the topology of a two-sphere, $\mathbb{S}^2$, the conditions in (45) immediately imply the existence of an axial Killing vector field on $\mathcal{Z}$.

7.1 Existence of a Killing vector on $\mathcal{Z}$

We begin by observing that as a consequence of equations (28a) and (28d) then if $\kappa_0 = \kappa_2 = 0$ then necessarily $\xi_{00'} = \xi_{11'} = 0$. Thus, under the assumptions of Theorem 3, the Killing vector $\xi_{AA'}$ is tangent to $\mathcal{Z}$ —i.e. its only non-vanishing components are $\xi_{01'}$ and $\xi_{10'}$. As we have seen in Subsection 4.2.4 the existence of a (possibly complex) Killing vector field on $\mathcal{Z}$ is equivalent to (30a)-(30c) on $\mathcal{Z}$ which had also been seen to be equivalent to the vanishing of $S_{AA'BB'} = \nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'}$ on $\mathcal{Z}$. Thereby, the conditions of Theorem 3 are equivalent to the existence of a (possibly complex) Killing vector field on $\mathcal{Z}$.

Remark 19. As all the geometric quantities including the spin coefficients, as well as, the Weyl spinor components Ψ_{ABCD} are constructed from the metric $\boldsymbol{g}$, given by (10), and $\xi_{AA'} = \nabla^P{}_{A'}\kappa_{AP}$ is known to be a Killing vector field everywhere in $D(\mathcal{H}_1 \cup \mathcal{H}_2)$ we immediately have that

$$\xi^{AA'}\nabla_{AA'}\tau = 0\,, \quad \xi^{AA'}\nabla_{AA'}\Psi_2 = 0$$

and, in virtue of (44) and the argument in Remark 14 above, that

$$\xi^{AA'}\nabla_{AA'}\kappa_1 = 0$$

everywhere in $D(\mathcal{H}_1 \cup \mathcal{H}_2)$. Thus we shall use from now on, without loss of generality, that τ, Ψ_2 and κ_1, when they are restricted to $\mathscr{Z}$, they all respect the axial symmetry of the metric on $\mathscr{Z}$.

7.2 The axial Killing vector

By our assumption on the underlying smoothness of the setting the Killing vector field $\xi_{AA'}$ is smooth on $\mathscr{Z}$. If $\xi_{AA'}$ was also Hermitian—i.e.

$$\xi_{01'} = \overline{\xi}_{10'}\,,$$

would hold, then, by appealing to the *hairy ball theorem* this Killing vector field vanished at some point, say at $p \in \mathscr{Z}$. As —apart from the trivial case when κ_1 is constant on $\mathscr{Z}$— $\xi_{AA'}$ was not identically zero on $\mathscr{Z}$, and by applying the argument of Wald—see pages 119-120 in [38]—$\xi_{AA'}$ had to be an axial Killing vector field on $\mathscr{Z}$ with closed orbits, with some fixed periodicity, around the *fixed point* $p \in \mathscr{Z}$.

In returning now to the generic case note that the argument just outlined applies to the real and imaginary parts of $\xi_{AA'}$, separately. Thereby, whenever $\xi_{AA'}$ is non-Hermitian the metric on $\mathscr{Z}$ has to admit both $\xi_{AA'} + \overline{\xi}_{AA'}$ and $\mathrm{i}\,(\xi_{AA'} - \overline{\xi}_{AA'})$ as real Killing vector fields. If both of the are non-trivial they either vanish at the same location on $\mathscr{Z}$ or not. If both vanish at $p \in \mathscr{Z}$ they must be proportional and the factor of proportionality is determined by the ration of their individual periodicities. If their vanishing occurs at two different points of $\mathscr{Z}$ then $\xi_{AA'} + \overline{\xi}_{AA'}$ and $\mathrm{i}\,(\xi_{AA'} - \overline{\xi}_{AA'})$ must be linearly independent real axial Killing vector fields on $\mathscr{Z}$ implying, in virtue of (45), that $\kappa_1 = \mathit{const}$ and, in turn, that $\Psi_2 = \mathit{const}$ and $\tau = 0$ which implies then that the metric $\boldsymbol{\sigma}$ on $\mathscr{Z}$ is spherically symmetric.

We can summarise the discussion of this section in the following:

Proposition 5. *Assume that the spacetime obtained from the characteristic initial value problem in $D(\mathcal{H}_1 \cup \mathcal{H}_2)$ admits a Killing spinor κ_{AB} such that (45) hold on $\mathscr{Z}$. Then $\xi_{AA'} = \nabla^P{}_{A'}\kappa_{AP}$ gives rise to a (possibly complex) axial Killing vector field on $\mathscr{Z}$.*

8 Determining κ_1 on $\mathscr{Z}$

As we have seen in Section 7, once $\mathscr{Z}$ is assumed to have the topology of a 2-sphere, the spinor $\xi_{AA'}$ is guaranteed to be an axial Killing vector field on $\mathscr{Z}$. By restricting our considerations to this case, the purpose of this section is to explicitly determine κ_1 satisfying the equations

$$\eth^2\kappa_1 = 0\,, \qquad \overline{\eth}^2\kappa_1 = 0\,. \tag{52}$$

This can be done in the most effective way by using *coordinates adapted to the axial symmetry* of $\boldsymbol{\sigma}$ as done in [9]. Therefore, for the sake of completeness, we shall outline the argument applied Section IV of [9] in the following subsection. We note the use of these coordinates can be traced back to [36] —see also [10] for a further application of this method.

8.1 Coordinates adapted to axial symmetry

Recall, first, that by assumption the 2-dimensional manifold $\mathcal{Z}$ has the topology of the 2-sphere $\mathbb{S}^2$. Thus, by the *Riemann mapping theorem* the metric $\boldsymbol{\sigma}$ is conformal to the standard round metric of $\mathbb{S}^2$. Taking into account the assumption of axial symmetry one writes then

$$\boldsymbol{\sigma} = \varpi^2 \big(\mathbf{d}\theta \otimes \mathbf{d}\theta + \sin^2\theta \mathbf{d}\varphi \otimes \mathbf{d}\varphi\big) \tag{53}$$

where (θ, φ) are standard *spherical coordinates* on $\mathbb{S}^2$ and $\varpi = \varpi(\theta)$ is a suitable conformal factor depending only on the *colatitude* θ. If $\boldsymbol{\sigma}$ is a smooth metric, then the conformal factor is also a *strictly positive* scalar field over $\mathcal{Z}$. The key idea behind the explicit integration of the equations in (52) is to introduce a new coordinate ψ given by the condition

$$\mathbf{d}\psi = \frac{\varpi^2}{R^2}\sin\theta \mathbf{d}\theta,$$

where R is the *area radius* defined by

$$\begin{aligned} 4\pi R^2 &= \int_{\mathcal{Z}} \varpi^2 \sin\theta \mathbf{d}\theta \wedge \mathbf{d}\varphi \\ &= R^2 \int_{\mathcal{Z}} \mathbf{d}\psi \wedge \mathbf{d}\varphi = 2\pi R^2(\psi_1 - \psi_0). \end{aligned}$$

Setting, without loss of generality, $\psi_0 = -1$ one has that $\psi_1 = 1$ —these coordinate values correspond, respectively, to the North and South poles of $\mathcal{Z}$ defined by the conditions $\theta = 0$ and $\theta = \pi$. Thus, the coordinate ψ is defined on the range $[-1, 1]$. Defining, for convenience, the function $Q = Q(\psi)$ by

$$Q^2 \equiv \frac{\varpi^2 \sin^2\theta}{R^2}, \tag{54}$$

the metric (53) takes, in terms of the coordinates (ψ, ϕ) the form

$$\boldsymbol{\sigma} = R^2\left(\frac{1}{Q(\psi)^2}\mathbf{d}\psi \otimes \mathbf{d}\psi + Q(\psi)^2 \mathbf{d}\varphi \otimes \mathbf{d}\varphi\right). \tag{55}$$

In particular, from (54) we have that

$$Q(-1) = Q(1) = 0. \tag{56}$$

A direct computation then shows that the Levi-Civita connection of $\boldsymbol{\sigma}$ —encoded in the combination $\alpha - \overline{\beta}$ [see e.g. (1)]— is given in terms of the function Q by

$$\alpha - \overline{\beta} = -\frac{1}{\sqrt{2}R}\partial_\psi Q \equiv -\frac{1}{\sqrt{2}R}Q'. \tag{57}$$

8.2 Integration of the equations for κ_1

We now make use of the coordinates introduced in the previous subsection to integrate the equations in (52).

Consistent with the discussion in Section 7 we look for solutions which are axially symmetric. To this end we observe that, in terms of the coordinates (ψ, φ), the directional derivatives δ and $\overline{\delta}$ acting on scalars are given by

$$\delta = \frac{1}{\sqrt{2}R}\left(Q\partial_\psi + \frac{\mathrm{i}}{Q}\partial_\varphi\right), \qquad \overline{\delta} = \frac{1}{\sqrt{2}R}\left(Q\partial_\psi - \frac{\mathrm{i}}{Q}\partial_\varphi\right).$$

As it follows from the argument applied in Remark 19 κ_1 is also axially symmetric, i.e. $\partial_\varphi \kappa_1 = 0$. Therefore the two conditions $\eth^2\kappa_1 = 0$ and $\overline{\eth}^2\kappa_1 = 0$ are no longer independent (in fact they are equivalent!). Then, in virtue of (1),

$$\begin{aligned} \eth^2\kappa_1 &= \left(\frac{1}{\sqrt{2}R}Q\partial_\psi - \frac{1}{\sqrt{2}R}\partial_\psi Q\right)\left(\frac{1}{\sqrt{2}R}Q\partial_\psi\kappa_1\right) \\ &= \frac{Q^2}{2R^2}\partial^2_\psi \kappa_1 = 0\,, \end{aligned}$$

from which one readily obtains the solution

$$\kappa_1 = \mathfrak{c}\psi + \mathfrak{b}, \qquad \mathfrak{c},\, \mathfrak{b} \in \mathbb{C}. \tag{58}$$

From this solution, recalling the relation (44) one readily obtains the following expression for Ψ_2:

$$\Psi_2 = \frac{\mathfrak{K}}{(\mathfrak{c}\psi + \mathfrak{b})^3}. \tag{59}$$

Remark 20. Observe that in view of the linearity of the Killing spinor equation, there is a freedom in the choice of the normalisation of the solution (58). In contexts where there is an asymptotic end present, there exist a natural choice for the normalisation —namely, choosing it so that the associated Killing vector is, to leading order, ∂_t. In the present context we do not have this natural normalisation. However, one can always use this freedom to make the constant $\mathfrak{b}$ equal to any arbitrary (non-zero) real constant. Accordingly, the solution (58) actually only contains one essential complex constant —i.e. two real parameters.

8.2.1 The Gauss-Bonnet condition

The Weyl scalar is related to the Gaussian curvature of 2-surfaces —see [28], Section 4.14. In particular, for the 2-surface $\mathcal{Z}$ one has that it is given by $K_{\mathcal{G}} = -2\mathrm{Re}\,\Psi_2$. It follows then that the Gauss-Bonnet formula applied to $\mathcal{Z} \approx \mathbb{S}^2$ implies

$$\int_{\mathcal{Z}} \Psi_2 \mathrm{d}S = -2\pi \tag{60}$$

—see equation (4.14.44) in [28]. Taking into account the line element (55) one finds that

$$\begin{aligned}\int_{\mathcal{Z}} \Psi_2 \mathrm{d}S &= R^2 \mathfrak{K} \int_{-1}^{1}\int_0^{2\pi} \frac{\mathrm{d}\varphi \mathrm{d}\psi}{(\mathfrak{b} + \mathfrak{c}\psi)^3} \\ &= \frac{4\pi R^2 \mathfrak{K}\mathfrak{b}}{(\mathfrak{b}^2 - \mathfrak{c}^2)^2}.\end{aligned}$$

Thus, from (60) one obtains the condition

$$\frac{2R^2\mathfrak{K}\mathfrak{b}}{(\mathfrak{b}^2 - \mathfrak{c}^2)^2} = -1. \tag{61}$$

Remark 21. Condition (61), being a consequence of the Gauss-Bonnet formula, is a necessary condition for $-2\mathrm{Re}\,\Psi_2$ to be the Gaussian curvature of a smooth 2-surface. It, can be used to fix the value of the radius R. Observe, also that it implies that the combination

$$\frac{\mathfrak{K}\mathfrak{b}}{(\mathfrak{b}^2 - \mathfrak{c}^2)^2}$$

must be real. It will be seen in Subsection 10.2 that for the Kerr spacetime one necessarily has that $\mathfrak{K}$ must be real and $\mathfrak{c}$ pure imaginary. If this is the case, then $\mathfrak{b}$ must also be real.

8.3 Integrating the function Q

Equation (16) can be used to compute the explicit form of the function Q appearing in the line element (55). Taking into account (57) and (59), equation (16) implies then

$$(QQ')' = \frac{\mathfrak{K}R^2}{(\mathfrak{b} + \mathfrak{c}\psi)^3} + \frac{\overline{\mathfrak{K}}R^2}{(\overline{\mathfrak{b}} + \overline{\mathfrak{c}}\psi)^3}.$$

This expression can be readily integrated to get

$$Q^2 = C_2 + C_1\psi + \frac{\mathfrak{K}R^2}{\mathfrak{c}^2(\mathfrak{b} + \mathfrak{c}\psi)} + \frac{\overline{\mathfrak{K}}R^2}{\overline{\mathfrak{c}}^2(\overline{\mathfrak{b}} + \overline{\mathfrak{c}}\psi)} \tag{62}$$

with C_1 and C_2 some real integration constants which are fixed using the conditions in (56). A direct computation shows then that

$$C_1 = \frac{\mathfrak{K}R^2}{\mathfrak{c}(\mathfrak{b}^2 - \mathfrak{c}^2)} + \frac{\overline{\mathfrak{K}}R^2}{\overline{\mathfrak{c}}(\overline{\mathfrak{b}}^2 - \overline{\mathfrak{c}}^2)}, \tag{63a}$$

$$C_2 = -\frac{\mathfrak{K}R^2\mathfrak{b}}{\mathfrak{c}^2(\mathfrak{b}^2 - \mathfrak{c}^2)} - \frac{\overline{\mathfrak{K}}R^2\overline{\mathfrak{b}}}{\overline{\mathfrak{c}}^2(\overline{\mathfrak{b}}^2 - \overline{\mathfrak{c}}^2)}. \tag{63b}$$

Remark 22. The constants R, $\mathfrak{K}$, $\mathfrak{b}$ and $\mathfrak{c}$ in (62) and (63a)-(63b) are subject to the constraint (61) arising from the Gauss-Bonnet identity.

Remark 23. In order to ensure the regularity of the function Q and of the associated curvature of $\mathcal{Z}$, it is necessary that the ratio $-\mathfrak{b}/\mathfrak{c} \in \mathbb{C} \setminus [-1, 1]$ —that is, $-\mathfrak{b}/\mathfrak{c}$ can lie in any point of the complex plane except the interval $[-1, 1]$ on the real axis. Moreover, in order for the expression (62) to be well defined, the constants R, $\mathfrak{K}$, $\mathfrak{b}$ and $\mathfrak{c}$ have to be such that the left hand side of the expression is non-negative for $\psi \in [-1, 1]$.

Remark 24. Note that by writing out equation (17) explicitly, and by making use of the present setup —along with the explicit ψ dependence of Q and Ψ_2—the imaginary part of τ gets to be uniquely determined as

$$\mathrm{Im}(\tau) = -\frac{\sqrt{2}\,R}{Q(\psi)} \int_{-1}^{\psi} \mathrm{Im}(\Psi_2(\psi'))\mathrm{d}\psi' . \tag{64}$$

Note, finally, that by making use of the gauge freedom we have in the black hole holograph construction —for a related discussion and an application see the last paragraph of Subsection 6.1— the real part of τ can be set to zero by performing an axially symmetric boost transformation with parameter $\vartheta = \vartheta(\psi)$ given by

$$\vartheta = \exp\left(-\frac{\sqrt{2}\,R}{Q(\psi)} \int_{-1}^{\psi} \mathrm{Re}(\tau(\psi'))\,\mathrm{d}\psi' \right) . \tag{65}$$

Remarkably, the axial symmetry of the setup guarantees that $\mathrm{Im}(\delta \log \vartheta) = 0$ and, in turn, that the imaginary part of τ remains unchanged. This, in particular, implies that (64) holds independently of the choice made for the axially symmetric boost transformation or, equivalently, for the real part of τ.

8.4 Summary: distorted black holes with Killing spinors

Summarising the discussion of the previous section we get the following:

Proposition 6. *There exists a four (real) parameter family of smooth axial symmetric 2-metrics* $\boldsymbol{\sigma}$ *on* $\mathcal{Z} \approx \mathbb{S}^2$ *such that* κ_1 *is a solution to the constraints*

$$\eth^2\kappa_1 = 0 \quad \text{and} \quad \overline{\eth}^2\kappa_1 = 0 \, ,$$

and such that the curvature condition

$$\kappa_1^3\Psi_2 = \mathfrak{K}$$

also holds.

Remark 25. By appealing now to the general black hole holograph construction [34], as summarised in Theorem 2, it follows then that from the four parameter family of initial data—comprised by the metrics referred to in Proposition 6, along with pertinent form of τ determined by (64)—on $\mathcal{Z}$, there exists a *four parameter family of distorted black hole configurations* the members of which are uniquely determined everywhere in the domain of dependence of the initial data surface, $\mathcal{H}_1 \cup \mathcal{H}_2$. This figure is consistent with the count of independent constants found in Kinnersley's analysis of vacuum Petrov type D solutions—see [21].

Remark 26. An extensive study of spacetimes admitting Petrov type D isolated null surfaces have also been carried out recently [8, 9, 22]. Notably, using our notation, the basic equations therein, relevant for a pair of such intersecting Petrov type D surfaces in the pure vacuum case, were found to take the form

$$\eth^2 \Psi_2^{-1/3} = 0 \quad \text{and} \quad \bar{\eth}^2 \Psi_2^{-1/3} = 0 \,.$$

The authors in [8, 9, 22] also claimed that if the intersection of the two null hypersurfaces is a topological 2-sphere, the 4-metric (determined only on these null hypersurfaces) can uniquely parametrized by the horizon area and angular momentum. It is not clear then where the other two-parameters —from among those four ones characterizing generic Petrov type D solutions [21]— are lost in arriving to this conclusion.

9 Enforcing the Hermiticity of the Killing vector

In Theorem 1, the assumption that the spinor $\xi_{AA'}$ constructed from the Killing spinor κ_{AB} is Hermitian is needed in order to show that the spacetime is isometric to the Kerr solution. Recall that using equations (22a)-(22d) the components of $\xi_{AA'}$ can be expressed in terms of derivatives of the Killing spinor components κ_0, κ_1 and κ_2. Accordingly, the Hermiticity condition leads to further restrictions on the components κ_0, κ_1 and κ_2. A consequence of the following proposition is that it suffices to impose restrictions only on the hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

Proposition 7. *Let κ_{AB} be a solution to equation* (7)*. Then the spinor field $\xi_{AA'}$ satisfies the wave equation*

$$\square \xi_{AA'} = -\Psi_A{}^{BCD} H_{A'BCD} \tag{66}$$

Proof. Follows by commuting derivatives, and using (7). □

An immediate consequence of this result is that

$$\square \left(\xi_{AA'} - \bar{\xi}_{AA'} \right) = \bar{\Psi}_{A'}{}^{B'C'D'} \bar{H}_{AB'C'D'} - \Psi_A{}^{BCD} H_{A'BCD} \tag{67}$$

Assuming that the conditions of Lemmas 1 and 2 are satisfied and $H_{A'ABC}$ vanishes $\mathcal{H}_1 \cup \mathcal{H}_2$. Then, in virtue of (67), $\xi_{AA'} - \bar{\xi}_{AA'}$ must also vanish everywhere on the domain of dependence of $\mathcal{H}_1 \cup \mathcal{H}_2$, guaranteeing thereby that the vector $\xi_{AA'}$ is Hermitian there.

This verifies then the following:

Proposition 8. *Assume that the spacetime obtained from the characteristic initial value problem in $D(\mathcal{H}_1 \cup \mathcal{H}_2)$ admits a Killing spinor κ_{AB} such that conditions* (45) *and*

$$\eth(\kappa_1 + \bar{\kappa}_1) = 0$$

hold on $\mathcal{Z}$. Then $\xi_{AA'} = \nabla^P{}_{A'} \kappa_{AP}$ is a real axial Killing vector field on $\mathcal{Z}$.

9.1 Some immediate restrictions

The Hermiticity of the Killing vector $\xi_{AA'}$ is equivalent to the relations

$$\xi_{00'} = \bar{\xi}_{00'} \,, \quad \xi_{01'} = \bar{\xi}_{10'} \,, \quad \xi_{10'} = \bar{\xi}_{01'} \,, \quad \xi_{11'} = \bar{\xi}_{11'} \,. \tag{68}$$

These conditions will be imposed on $\mathcal{H}_1$ and $\mathcal{H}_2$ separately.

Conditions on $\mathcal{H}_1$. On $\mathcal{H}_1$, using the explicit expressions (22a)-(22d), the first condition in (68) is trivially satisfied, and the remaining conditions can be shown to be equivalent to

$$\delta(\kappa_1 + \bar{\kappa}_1) = 0, \tag{69a}$$
$$\bar{\delta}(\kappa_1 + \bar{\kappa}_1) = 0, \tag{69b}$$
$$\Delta\kappa_1 + \tau\kappa_2 \quad \text{real}, \tag{69c}$$

on $\mathcal{H}_1$. In fact, it is straightforward to show that on $\mathcal{H}_1$

$$D\delta(\kappa_1+\overline{\kappa}_1) = D\overline{\delta}(\kappa_1+\overline{\kappa}_1) = 0\,.$$

Thus, it suffices to impose conditions (69a)-(69b) only on $\mathcal{Z}$. In other words, the Hermiticity condition on $\mathcal{H}_1$ is equivalent to

$$\begin{aligned}&\mathrm{Re}(\kappa_1) \quad \text{constant on } \mathcal{Z},\\ &\Delta\kappa_1+\tau\kappa_2 \quad \text{real} \quad \text{ on } \mathcal{H}_1.\end{aligned}$$

Conditions on $\mathcal{H}_2$. Secondly, on $\mathcal{H}_2$, the last condition in (68) is trivially satisfied and the remaining conditions are equivalent to

$$\delta(\kappa_1+\overline{\kappa}_1) = 0, \tag{70a}$$

$$\overline{\delta}(\kappa_1+\overline{\kappa}_1) = 0, \tag{70b}$$

$$D\kappa_1 \quad \text{real}, \tag{70c}$$

on $\mathcal{H}_2$. Again, it is straightforward to show that on $\mathcal{H}_2$

$$\Delta\delta(\kappa_1+\overline{\kappa}_1) = \Delta\overline{\delta}(\kappa_1+\overline{\kappa}_1) = 0.$$

Consequently, it suffices to impose conditions (70a)-(70b) on $\mathcal{Z}$.

Combining the discussion of the previous two paragraphs one concludes that the spinor field $\xi_{AA'}$ is Hermitian on $\mathcal{H}_1 \cup \mathcal{H}_2$ if and only if we have

$$\kappa_1+\overline{\kappa}_1 = \ \text{const on } \mathcal{Z}, \tag{71a}$$

$$\Delta\kappa_1+\tau\,\kappa_2 \quad \text{real on } \mathcal{H}_1, \tag{71b}$$

$$D\kappa_1 \quad \text{real on } \mathcal{H}_2. \tag{71c}$$

9.2 Hermiticity in terms of conditions at $\mathcal{Z}$

In this section it is shown that conditions (71b)-(71c) can be replaced by restrictions on $\mathcal{Z}$.

Analysis on $\mathcal{H}_2$. Start by considering condition (71c). From the transport equation (20b) on $\mathcal{H}_2$, and equation (24g), we have that

$$2\Delta D\kappa_1 = \delta\overline{\delta}\kappa_1+\overline{\delta}\delta\kappa_1+4\tau\overline{\delta}\kappa_1-(3\alpha+\overline{\beta})\delta\kappa_1-(3\overline{\alpha}+\beta)\overline{\delta}\kappa_1+2\Psi_2\kappa_1$$

on $\mathcal{H}_2$. Taking a further Δ-derivative we obtain

$$2\Delta\Delta D\kappa_1 = \Delta(\delta\overline{\delta}+\overline{\delta}\delta)\kappa_1+4\tau\Delta\overline{\delta}\kappa_1-(3\alpha+\overline{\beta})\Delta\delta\kappa_1-(3\overline{\alpha}+\beta)\Delta\overline{\delta}\kappa_1+2\Psi_2\Delta\kappa_1.$$

We can commute the Δ-derivative with the δ and $\overline{\delta}$ derivatives to obtain

$$2\Delta\Delta D\kappa_1 = (\delta\overline{\delta}+\overline{\delta}\delta)\Delta\kappa_1+4\tau\overline{\delta}\Delta\kappa_1-(3\alpha+\overline{\beta})\delta\Delta\kappa_1-(3\overline{\alpha}+\beta)\overline{\delta}\Delta\kappa_1+2\Psi_2\Delta\kappa_1.$$

Note that all the terms on the right are proportional to intrinsic derivatives of $\Delta\kappa_1$, which by (27e) is proportional to κ_2 and its intrinsic derivatives on $\mathcal{H}_2$. As shown in subsection 5.1, unless our spacetime is the Minkowski solution, the component κ_2 must vanishes on $\mathcal{H}_2$. It follows then that

$$\Delta\Delta D\kappa_1 = 0 \quad \text{ on } \mathcal{H}_2\,.$$

This is a second order ordinary differential equation along the generators of $\mathcal{H}_2$. Therefore, the requirement that $D\kappa_1$ is real on $\mathcal{H}_2$ is equivalent to requiring that $D\kappa_1$ and $\Delta D\kappa_1$ are real on $\mathcal{Z}$.

Analysis on $\mathcal{H}_1$. An analogous argument apply in case of condition (71b). Take first a D-derivative along the generators of $\mathcal{H}_1$ and use the transport equation (19b) on $\mathcal{H}_1$, along with the assumption that κ_0 vanishes in $\mathcal{H}_1$ to obtain

$$2D(\Delta\kappa_1+\tau\kappa_2) = \delta\overline{\delta}\kappa_1+\overline{\delta}\delta\kappa_1-(\alpha-\overline{\beta})\delta\kappa_1-(\overline{\alpha}-\beta)\overline{\delta}\kappa_1+2\Psi_2\kappa_1.$$

Taking a further D-derivative one gets

$$2DD(\Delta\kappa_1 + \tau\kappa_2) = D(\delta\overline{\delta} + \overline{\delta}\delta)\kappa_1 - (\alpha - \overline{\beta})D\delta\kappa_1 - (\overline{\alpha} - \beta)D\overline{\delta}\kappa_1 + 2\Psi_2 D\kappa_1. \tag{72}$$

By commuting the D derivatives with the δ and $\overline{\delta}$ derivatives, we obtain

$$\begin{aligned} 2DD(\Delta\kappa_1 + \tau\kappa_2) =& (\delta\overline{\delta} + \overline{\delta}\delta)D\kappa_1 - (3\alpha + \overline{\beta})\delta D\kappa_1 - (3\overline{\alpha} + \beta)\overline{\delta}D\kappa_1 \\ &+ \left(\delta\overline{\tau} + \overline{\delta}\tau + 4\alpha\overline{\alpha} + 2\alpha\beta + 2\overline{\alpha}\overline{\beta} + 2\Psi_2\right) D\kappa_1. \end{aligned}$$

Note that all terms on the right hand side are proportional to δ and $\overline{\delta}$ derivatives of $D\kappa_1$, which by (26f) are proportional to κ_0 and its δ and $\overline{\delta}$ derivatives on $\mathcal{H}_1$. Therefore, again, unless our spacetime is the Minkowski solution, $\kappa_0 = 0$ holds on $\mathcal{H}_1$. Accordingly one has that

$$DD\left(\Delta\kappa_1 + \tau\kappa_2\right) = 0 \quad \text{on } \mathcal{H}_1.$$

Again, the latter is a second order ordinary differential equation along the generators of $\mathcal{H}_1$, and so the requirement that $\Delta\kappa_1 + \tau\kappa_2$ is real on $\mathcal{H}_1$ is equivalent to requiring that $\Delta\kappa_1 + \tau\kappa_2$ and $D\left(\Delta\kappa_1 + \tau\kappa_2\right)$ are real on $\mathcal{Z}$.

Summarising the results of this section we have:

Lemma 11. *The spinor field $\xi_{AA'}$ is Hermitian on $\mathcal{H}_1 \cup \mathcal{H}_2$, and thereby on the domain of dependence of $\mathcal{H}_1 \cup \mathcal{H}_2$, if and only if the conditions*

$$\begin{aligned} &\kappa_1 + \overline{\kappa}_1 = const\,, \\ &D(\kappa_1 - \overline{\kappa}_1) = 0\,, \\ &\Delta D(\kappa_1 - \overline{\kappa}_1) = 0\,, \\ &\Delta(\kappa_1 - \overline{\kappa}_1) + \tau\,\kappa_2 - \overline{\tau}\,\overline{\kappa}_2 = 0, \\ &D\left(\Delta(\kappa_1 - \overline{\kappa}_1) + \tau\,\kappa_2 - \overline{\tau}\,\overline{\kappa}_2\right) = 0\,, \end{aligned}$$

are satisfied on $\mathcal{Z}$.

Note that some of these conditions are redundant. For example, we know that $D\kappa_1$ vanishes on $\mathcal{Z}$ due to equation (26f) and the vanishing of κ_0, and so clearly $D(\kappa_1 - \overline{\kappa}_1)$ also vanishes on $\mathcal{Z}$. A similar argument using equation (27e) can be used to show that $\Delta(\kappa_1 - \overline{\kappa}_1) + \tau\kappa_2 - \overline{\tau}\overline{\kappa}_2$ vanishes on $\mathcal{Z}$. We can also use the requirement that $\mathrm{Re}(\kappa_1)$ is constant on $\mathcal{Z}$ to show that the other two conditions are equivalent. Indeed, we have that

$$\begin{aligned} D\left(\Delta(\kappa_1 - \overline{\kappa}_1) + \tau\kappa_2 - \overline{\tau}\overline{\kappa}_2\right) &= D\Delta(\kappa_1 - \overline{\kappa}_1) - 2\tau\overline{\delta}\kappa_1 + 2\overline{\tau}\delta\overline{\kappa}_1 \\ &= \Delta D(\kappa_1 - \overline{\kappa}_1) + \tau\overline{\delta}(\kappa_1 - \overline{\kappa}_1) + \overline{\tau}\delta(\kappa_1 - \overline{\kappa}_1) - 2\tau\overline{\delta}\kappa_1 + 2\overline{\tau}\delta\overline{\kappa}_1 \\ &= \Delta D(\kappa_1 - \overline{\kappa}_1) \end{aligned}$$

where (24g), the commutator $[\Delta, D]$, and the vanishing of $D\tau$ (see Table 1), along with the conditions $\delta\kappa_1 = -\delta\overline{\kappa}_1$ and $\overline{\delta}\kappa_1 = -\overline{\delta}\overline{\kappa}_1$, have been used. We compute now $\Delta D\kappa_1$. Eliminating $D\kappa_2$ by using (26g) the transport equation (20b) on $\mathcal{Z}$ can be seen to reduce to

$$\begin{aligned} 2\Delta D\kappa_1 &= (\delta\overline{\delta} + \overline{\delta}\delta)\kappa_1 - (3\alpha + \overline{\beta})\delta\kappa_1 - (3\overline{\alpha} + \beta)\overline{\delta}\kappa_1 - (2\overline{\alpha} + 2\beta)D\kappa_2 + 2\Psi_2\kappa_1 \\ &= (\delta\overline{\delta} + \overline{\delta}\delta)\kappa_1 - (3\alpha + \overline{\beta})\delta\kappa_1 + (\overline{\alpha} + 3\beta)\overline{\delta}\kappa_1 + 2\Psi_2\kappa_1\,. \end{aligned}$$

Replacing δ and $\overline{\delta}$ derivatives with the $\eth$ and $\overline{\eth}$operators we obtain

$$2\Delta D\kappa_1 = (\eth\overline{\eth} + \overline{\eth}\eth)\kappa_1 - (2\alpha + 2\overline{\beta})\eth\kappa_1 + (2\overline{\alpha} + 2\beta)\overline{\eth}\kappa_1 + 2\Psi_2\kappa_1\,.$$

The imaginary part of this equation is given by

$$\begin{aligned} 2\Delta D(\kappa_1 - \overline{\kappa}_1) &= \left(\eth\overline{\eth} + \overline{\eth}\eth\right)(\kappa_1 - \overline{\kappa}_1) + 2\Psi_2\kappa_1 - 2\overline{\Psi}_2\overline{\kappa}_1 \\ &= 2\left[\left(\eth\overline{\eth}\kappa_1 + 2\Psi_2\kappa_1\right) - \left(\overline{\eth}\eth\overline{\kappa}_1 + 2\overline{\Psi}_2\overline{\kappa}_1\right)\right], \end{aligned}$$

where in the second step the constancy of $\mathrm{Re}(\kappa_1)$ on $\mathcal{Z}$, along with the commutator (2) applied to the spin weight zero quantity κ_1, was used.

Summarising, we have that:

Lemma 12. *The spinorial field $\xi_{AA'}$ is Hermitian on $\mathcal{H}_1 \cup \mathcal{H}_2$ if and only if on $\mathcal{Z}$ we have*

$$\kappa_1 + \bar{\kappa}_1 = const\,, \tag{73a}$$

$$\eth\bar{\eth}\kappa_1 + 2\Psi_2\kappa_1 \quad \text{is real.} \tag{73b}$$

10 Identifying the Kerr spacetime

In this section we make use of Theorem 1 to identify the values of the parameters $\mathfrak{K}$, $\mathfrak{b}$ and $\mathfrak{c}$ defining the coefficient κ_1 on $\mathcal{Z}$ which correspond to the Kerr solution. To this end, we first identify conditions ensuring that the Killing vector associated to the Killing spinor is Hermitian.

10.1 Imposing the Hermiticity of $\xi_{AA'}$

Recall that the conditions ensuring the Hermiticity of the spinor $\xi_{AA'}$ have been given in Lemma 12. Accordingly, we now proceed to evaluate conditions (73a)-(73b) in the explicit solution (58).

Condition (73a)**.** From (58) it readily follows that

$$\kappa_1 + \bar{\kappa}_1 = (\mathfrak{c} + \bar{\mathfrak{c}})\psi + (\mathfrak{b} + \bar{\mathfrak{b}}).$$

Thus, condition (73a) requires that $\mathfrak{c} + \bar{\mathfrak{c}} = 0$ so that one can write

$$\mathfrak{c} = \mathrm{i}c, \qquad c \in \mathbb{R}.$$

Accordingly, expression (58) simplifies to

$$\kappa_1 = \mathrm{i}c\psi + \mathfrak{b}$$

so that

$$\Psi_2 = \frac{\mathfrak{K}}{(\mathfrak{b} + \mathrm{i}c\psi)^3}. \tag{74}$$

Condition (73b)**.** A direct computation shows that

$$\begin{aligned}
\eth\bar{\eth}\kappa_1 &= \left(\frac{1}{\sqrt{2}R}Q\partial_\psi + \frac{1}{\sqrt{2}R}\partial_\psi Q\right)\left(\frac{1}{\sqrt{2}R}Q\partial_\psi\kappa_1\right) \\
&= \frac{Q^2}{2R^2}\partial_\psi^2\kappa_1 + \frac{Q}{R^2}\partial_\psi Q\partial_\psi\kappa_1. \\
&= \frac{Q}{R^2}\partial_\psi Q\partial_\psi\kappa_1,
\end{aligned}$$

where in the last line it has used that the expression for κ_1 given by (58) satisfies $\partial_\psi^2\kappa_1 = 0$. As $\partial_\psi\kappa_1 = \mathrm{i}c$, it readily follows that

$$\eth\bar{\eth}\kappa_1 + 2\Psi_2\kappa_1 = \frac{\mathrm{i}c}{R^2}QQ' + \frac{2\mathfrak{K}}{(\mathfrak{b} + \mathrm{i}c\psi)^2}.$$

Thus, condition (73b) implies that

$$(Q^2)' = \frac{2\mathrm{i}R^2\mathfrak{K}}{c(\mathfrak{b} + \mathrm{i}c\psi)^2} - \frac{2\mathrm{i}R^2\bar{\mathfrak{K}}}{c(\bar{\mathfrak{b}} - \mathrm{i}c\psi)^2}. \tag{75}$$

As a consequence of the solution (62), the above expression is automatically satisfied so that condition (73b) does not add any further restrictions.

Lemma 13. *For the family of 2-metrics on $\mathcal{Z}$ given by Proposition 6, the spinor $\xi_{AA'}$ associated to the Killing spinor κ_{AB} is Hermitian if and only if the coefficient κ_1 on $\mathcal{Z}$ is of the form*

$$\kappa_1 = \mathfrak{b} + \mathrm{i}c\psi\,.$$

10.2 Applying Mars's characterisation

If the spinor $\xi_{AA'}$ is Hermitian, then the associated Killing form is well defined and on $\mathcal{Z}$ the norm of the self-dual Killing form, $\mathcal{H}^2$, associated to the Killing spinor κ_{AB} is given by

$$\mathcal{H}^2 = -36\kappa_1^2\Psi_2^2.$$

Moreover, the Ernst potential χ takes on $\mathcal{Z}$, up to a (possibly complex) constant $\mathfrak{x} \in \mathbb{C}$, the form

$$\chi = \mathfrak{x} - 18\kappa_1^2\Psi_2.$$

Making use of the relation (44) to eliminate Ψ_2 one obtains

$$\mathcal{H}^2 = -\frac{36\mathfrak{K}^2}{\kappa_1^4}, \qquad \chi = \mathfrak{x} - \frac{18\mathfrak{K}}{\kappa_1}.$$

Now, in order to identify the Kerr spacetime via Theorem 1, we set $\mathfrak{x} = 1$ so that

$$1 - \chi = \frac{18\,\mathfrak{K}}{\kappa_1}$$

from which, in turn, one readily obtains that

$$\begin{aligned}(1-\chi)^4 &= \frac{18^4\,\mathfrak{K}^4}{\kappa_1^4}\\ &= -\left(\frac{18^4\mathfrak{K}^2}{36}\right)\mathcal{H}^2.\end{aligned}$$

The previous expression allows to identify the constant $\mathfrak{l}$ in Theorem 1 given, in terms of the parameters used above, as

$$\mathfrak{l} \equiv \frac{36}{18^4\,\mathfrak{K}^2}\,.$$

Thus, in order to have the Kerr spacetime $\mathfrak{l}$ must be real and positive which can only be satisfied if $\mathfrak{K}$ is non-zero and real, i.e. $\mathfrak{K} = K \in \mathbb{R}\setminus\{0\}$. Finally, a direct computation using the constraint (61) shows that $\mathfrak{b} = b \in \mathbb{R}$ —cfr. Remark 61.

We summarise the discussion in the following:

Proposition 9. *The members of the family of 2-metrics given in Proposition 6 giving rise to solutions to the vacuum Einstein field equations on $D(\mathcal{H}_1\cup\mathcal{H}_2)$, which are isometric to a member of the 2-parameter Kerr family of metrics are characterised by the conditions*

$$\mathfrak{b},\ \mathfrak{K} \in \mathbb{R}, \qquad \mathfrak{c} \in \mathbb{C}\setminus\mathbb{R}.$$

These conditions fix the value of the component of the Weyl tensor Ψ_2 on $\mathcal{Z}$.

10.2.1 Relation to the standard parameters of the Kerr family

From the previous discussion it follows that one can write

$$\Psi_2 = \frac{K}{(b + \mathrm{i}\,c\,\psi)^3}\,, \qquad b,\ c,\ K \in \mathbb{R}\,. \tag{76}$$

Clearly, Ψ_2 as given above is regular everywhere on $\mathcal{Z}$ —and accordingly, also the Gaussian curvature of $\mathcal{Z}$. Now, observing that b is an arbitrary normalisation constant of the Killing spinor we conclude that the representation of the Kerr family of spacetimes has two independent constants —as it should be expected!

In order to relate the real parameters b, c, K with the standard mass (m) and angular momentum (a) parameters of the Kerr family, we recall that in a dyad $\{o^A,\ \iota^A\}$ consisting of principal spinors of Ψ_{ABCD}, the only non-zero component of the Weyl spinor is given, in terms of standard *Boyer-Lindquist coordinates*, by

$$\Psi_2 = -\frac{m}{(r - \mathrm{i}a\cos\theta)^3}$$

—see e.g. [1]. In this dyad the Killing spinor takes the form

$$\kappa_{AB} = \frac{2}{3}(r - \mathrm{i}a\cos\theta)o_{(A}\iota_{B)}.$$

The normalisation in the above expression of the Killing spinor is chosen so that the associated Killing vector has the form

$$\xi^a = (\partial_t)^a.$$

In Boyer-Lindquist coordinates the bifurcation sphere is determined by the condition

$$r = r_+, \qquad r_+ \equiv m + \sqrt{m^2 - a^2}.$$

Thus, at the bifurcation sphere the component Ψ_2 of the Weyl tensor takes the form

$$\Psi_2 = -\frac{m}{(r_+ - \mathrm{i}a\cos\theta)^3}. \tag{77}$$

Now, in order to make contact with the framework of Rácz's holograph construction we observe that although the spin dyad associated to the null tetrad $\{l^a, n^a, m^a, n^a\}$ introduced in Section 3 is, in general, not aligned with the principal directions of the Weyl tensor, it happens to be aligned at the bifurcation sphere $\mathcal{Z}$. As the component Ψ_2 is invariant under spin-boosts one can readily identify the expressions (76) and (77) —that is, one has

$$\frac{K}{(b + \mathrm{i}\,c\,\psi)^3} = -\frac{m}{(r_+ - \mathrm{i}a\cos\theta)^3},$$

so that, essentially, the constants K, c and b correspond, respectively, to the values of the mass parameter, angular momentum parameter and the value of the radial Boyer-Lindquist coordinate at the event horizon.

11 Final remarks

As mentioned earlier, all the distorted electrovaccum black hole spacetimes can be represented within Rácz's black hole holograph construction [33, 34]. In this paper a systematic investigation of a specific subset of these spacetimes was carried out. This subset was chosen by requiring the existence of a Killing spinor field in the pure vacuum case. The primary aim was to identifying the freedom we have in choosing initial data for the Killing spinor on the horizon of the underlying distorted vacuum black hole. In accordance with Rácz's black hole holograph construction by fixing merely one of the Killing spinor components on the bifurcation surface the Killing spinor gets to be uniquely determined everywhere in the domain of dependence of the horizons.

The motivation for the use of a Killing spinor field can be traced back to the following conceptual issue raised already in [33, 34]: Recall first that the Kerr family of vacuum black holes represents only a critical point in the space of the distorted vacuum black hole spacetimes. It is natural to ask then, what sort of geometric selection rule, imposed only on the space bifurcation surface, singles out the only asymptotically flat stationary vacuum black hole spacetimes distinguished by the black hole uniqueness theorems?

To get a clearer perspective of the results of the present paper it is worth recalling some of the details of the black hole uniqueness proofs. Note, first, that asymptotic flatness as an assumption is a completely natural requirement if one is interested in the properties of black holes which are completely isolated in space. It is not surprising then that the black hole uniqueness theorems (see e.g. Refs. [5, 6, 3, 16]) all assume asymptotic flatness of the domain of outer communications of the selected vacuum spacetimes. Indeed, the black holes uniqueness proofs —using the black hole rigidity theorem of Hawking [15, 16] (claiming that an asymptotically flat stationary electrovac black hole spacetime is either static or stationary axisymmetric)— can be traced back to proving the uniqueness of solutions to an elliptic boundary value problem [18, 19, 5, 6, 26, 3]. The relevant elliptic equations are derived from the Einstein's equations on "$t = const$" hypersurfaces (or—based on Hawking's black hole rigidity theorem—on a suitable factor space of them), whereas the boundary conditions are specified at the bifurcation surface and at spacelike infinity [3].

In view of the great detail of information on the geometry of the bifurcation surface provided by the presented investigations, one may ask which part of them were actually used in the black hole uniqueness proofs. The short answer is that almost none. More precisely, it was only assumed that the geometry at the bifurcation surface is regular and that the "$t = const$" hypersurfaces smoothly extend to this surface. The validity of this latter assumption had been verified in a series of papers either for generic metric theories of gravity [30, 31] or in general relativity with the inclusion of various matter fields [12, 32]. Nevertheless, the assumptions concerning the geometry were never as detailed as given in the present paper. One might be puzzled by this, but from the perspective of the black hole holograph construction [33, 34] it becomes clear immediately that in identifying the Kerr family of black hole solutions in the black hole holography construction one cannot refer to the asymptotic properties. Accordingly, all the information we may use must be restricted to the bifurcation surfaces which plays the role of "holograms", as these compact two-dimensional carriers store all the information concerning the geometry of the associated four-dimensional stationary black hole spacetimes.

After having the selection rules identified in case of vacuum configurations it is of obvious interest to get them also in the electrovaccum case. In this way a completely new, quasi-local, type of black hole uniqueness proofs can be established in the four dimensional case. Note, however, that—as it was also proposed in [33, 34]— in virtue of the large variety of stationary black hole, black ring and other type of "black" objects in higher dimensions it would be even more important to generalize the techniques and concepts applied here to higher dimensions. The corresponding investigations and constructions would definitely deserve further attention.

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