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Parametricity and Mulry's Strong Dinaturality

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Abstract

We express Mulry's notion of 'strong dinaturality' in terms of binary relational parametricity. This leads to an interesting category of binary relations based on cospans in place of spans. Such relations admit a natural, componentwise formula for smash products. They also admit a very intuitive formula for function spaces. Our account of binary relational parametricity is based on a graph category framework which allows for a notion of 'diparametric' transformation between mixed variance operations. We show that Mulry's 'strong dinaturality with restricted variation', which characterizes canonical fixed point operators in an axiomatic setting, is equivalent to a diparametricity condition. We find that the 'restricted variation' can be dealt with smoothly using the bijective-on-objects/full-and-faithful factorization system in Cat. Along the way, we consider a pleasant category of push-me-pull-you's which abstract from both pull-back's and push-out's.

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1 Introduction

In categories of monotone maps between complete partial orders, least fixed points are obtained using least upper bounds and are characterized by Plotkin's Axiom. In Mulry's axiomatic setting [4], a Fixed Point Object induces a fixed point operation characterized by it's 'strong dinaturality with variation restricted to algebra morphisms'. Despite the terminology, this characterization is best seen as a categorical reformulation of Plotkin's Axiom. Ordinary dinaturality and 'strong dinaturality with restricted variation', although both implied by 'strong dinaturality', are generally incomparable. The 'restricted variation' corresponds to the restriction on g in Plotkin's Axiom.

Axiom 1 (Plotkin) *If g is strict and continuous and $g \circ f_0 = f_1 \circ g$, then $g \circ \mathrm{lfp} f_0 = \mathrm{lfp} f_1$.*

$$\begin{array}{ccc} d_0 & \xrightarrow{f_0} & d_0 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ d_1 & \xrightarrow{f_1} & d_1 \end{array} \quad\Rightarrow\quad \begin{array}{ccc} 1 & \xrightarrow{\mathrm{lfp} f_0} & d_0 \\ | & & \downarrow{\scriptstyle g} \\ 1 & \xrightarrow{\mathrm{lfp} f_1} & d_1 \end{array}$$

Let us put this in terms of binary relational parametricity. We say a pair of functions $f_i : a_i \to a'_i$ is *parametric* with respect to the relations $r : a_0 \leftrightarrow a_1$ and $r' : a'_0 \leftrightarrow a'_1$ if it carries pairs in r to pairs in r' and we write $[r, r'] : [a_0, a'_0] \leftrightarrow [a_1, a'_1]$ for the binary relation containing such pairs of functions. Plotkin's Axiom says that the least fixed point operator is parametric with respect to the graphs of strict, continuous functions.

$$\begin{array}{ccc} [d_0, d_0] & \xrightarrow{\mathrm{lfp}_{d_0}} & d_0 \\ \updownarrow{\scriptstyle [\langle g \rangle, \langle g \rangle]} & & \updownarrow{\scriptstyle \langle g \rangle} \\ [d_1, d_1] & \xrightarrow{\mathrm{lfp}_{d_1}} & d_1 \end{array}$$

This means that for every strict, continuous function $g : d_0 \to d_1$ the pair of functions lfp_{d_0} and lfp_{d_1} is parametric with respect to the relations $[\langle g \rangle, \langle g \rangle]$ and $\langle g \rangle$ or, expanding further, the pair of functions lfp_{d_0} and lfp_{d_1} carries each pair of endo-functions f_0 and f_1 that is parametric with respect to $\langle g \rangle$ and $\langle g \rangle$ to a pair in $\langle g \rangle$. By the properties of graphs of functions and the properties of function spaces, this is equivalent to Axiom 1.

$$\begin{array}{ccc} d_0 & \xrightarrow{f_0} & d_0 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ d_1 & \xrightarrow{f_1} & d_1 \end{array} \quad\Leftrightarrow\quad \begin{array}{ccc} d_0 & \xrightarrow{f_0} & d_0 \\ \updownarrow{\scriptstyle \langle g \rangle} & & \updownarrow{\scriptstyle \langle g \rangle} \\ d_1 & \xrightarrow{f_1} & d_1 \end{array} \quad\Leftrightarrow\quad \begin{array}{ccc} 1 & \xrightarrow{\hat{f}_0} & [d_0, d_0] \\ | & & \updownarrow{\scriptstyle [\langle g \rangle, \langle g \rangle]} \\ 1 & \xrightarrow{\hat{f}_1} & [d_1, d_1] \end{array}$$

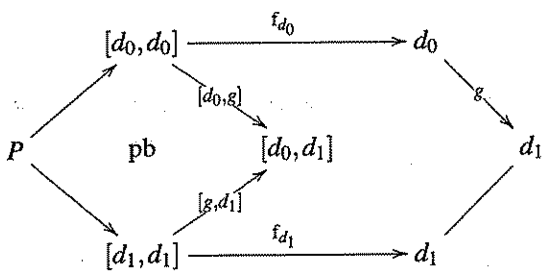
$$\Rightarrow\quad \begin{array}{ccc} 1 & \xrightarrow{\mathrm{lfp}_{d_0} \circ \hat{f}_0} & d_0 \\ | & & \updownarrow{\scriptstyle \langle g \rangle} \\ 1 & \xrightarrow{\mathrm{lfp}_{d_1} \circ \hat{f}_1} & d_1 \end{array} \quad\Leftrightarrow\quad \begin{array}{ccc} 1 & \xrightarrow{\mathrm{lfp} f_0} & d_0 \\ | & & \downarrow{\scriptstyle g} \\ 1 & \xrightarrow{\mathrm{lfp} f_1} & d_1 \end{array}$$

Just as Axiom 1 fails without conditions on g, the least fixed point operator is not parametric with respect to all relations. A relation $r : d_0 \leftrightarrow d_1$ is *strict* if it contains the pair $(\bot_{d_0}, \bot_{d_1})$ and is *complete* if it is complete as a subset of the partial order $d_0 \times d_1$.

Axiom 2 *The least fixed point operator is parametric with respect to strict, complete relations.*

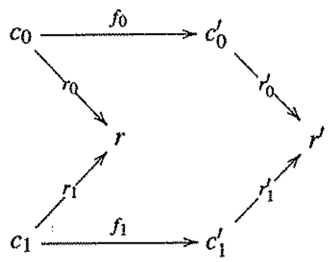
The graph of a strict, continuous function is a strict, complete relation, so Axiom 2 implies Axiom 1. Axiom 2 therefore also characterizes the least fixed point operator and so, in the concrete setting of complete partial orders, Axiom 2 is equivalent to Axiom 1. We would like an abstract, categorical treatment of these Axioms.

For a family of operators $\mathrm{f}_d : [d,d] \to d$, Mulry's definition of 'strong dinaturality with variation restricted to algebra morphisms' says that for every algebra morphism g the following diagram commutes.



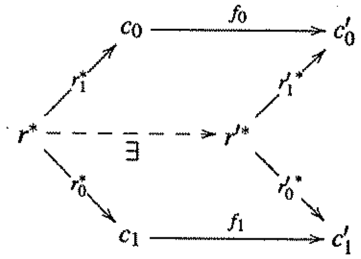
Applied to categories of partial orders, algebra morphisms are strict, continuous functions, the pull-back P contains just those pairs (f_0, f_1) such that $g \circ f_0 = f_1 \circ g$ and we have a direct internalisation of Plotkin's Axiom.

At this point we could turn to standard categorical treatments of relations and compare the resulting notions of parametricity with the above notion of strong dinaturality. However, the above diagram suggests a specific category of binary relations with cospans for objects.

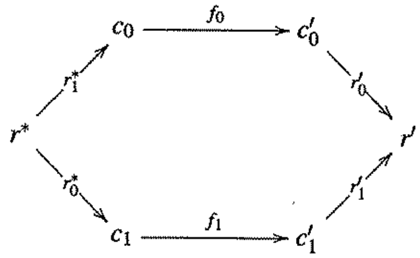


We think of a cospan as relating those pairs of points it sends to the same point in the vertex and we call such relations 'quotient relations'. Conceptually, quotient relations can be viewed as those induced by abstractions: the elements of two concrete domains, c_0 and c_1, are related just when they represent, via r_0 and r_1, the same element of some abstract domain, r. A jointly monic span representing the same relation is obtained by taking a pull-back of the cospan. This is how we decide if the pair of maps f_0 and f_1

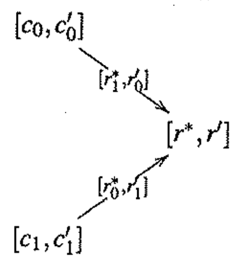
is parametric. We take pull-backs of the two cospans and look for a commuting vertex component.



However, the existence of such a vertex component is equivalent to the commutativity of the first pull-back followed by the pair of maps followed by the second cospan.



Not only does this avoid one pull-back operation and an existential quantification, but it matches up nicely with the way quotient relations express relatedness: pairs of maps in $[c_0, c'_0] \times [c_1, c'_1]$ are parametric *iff* the cospan $([r^*_1, r'_0], [r^*_0, r'_1])$ sends them to the same map in $[r^*, r']$.



This formula works for any closed structure $[-,+]$, including strict and ordinary function spaces over categories of domains. The various other operations on domains also lift without undue complication. In particular, the componentwise lifting of smash product gives the correct smash product on such relations. Compare this with the formula in [1] for the smash product of general binary relations.

There remains the question of the strength of the associated notion of parametricity. Are quotient relations adequate for the logical relations arguments we would like to apply to our models? We have yet to consider precise equivalence results, but there are grounds for optimism: quotient relations include the graphs of functions, they admit good liftings of the usual operations on domains, and, as expected, quotient parametricity characterizes canonical fixed point operators such as least fixed points or those induced by a Fixed Point Object.

Contents. In Section 7 we use the bijective-on-objects/full-and-faithful factorization system in Cat to describe abstractly a setting for restricted variation. The setting specializes to Mulry's monadic setting, and hence, to partial order settings, and provides an exact correspondence between quotient parametricity with restricted variation and Mulry's strong dinaturality with restricted variation. This derives from the general correspondence, described in Section 6, between quotient parametricity and strong dinaturality. Our notion of parametricity requires that we know how to lift the relevant operations, function spaces and projections in the case of fixed point operators, from the base category to the category of relations. In Section 5 we explain how function spaces lift using an interesting category of push-me-pull-you's. The case of function spaces is special in that they lift regardless of their effect on the pull-backs we use to construct our categories of quotient relations. When an operation preserves these pull-backs, we have the usual extension from categories to functors of any construction based on those pull-backs including quotient relations as shown in Section 4. This allows us to lift the usual covariant operations on domains, including the smash product. In Section 3 we give a categorical description of quotient relations using pull-backs in the base category. In Section 2 we go over a 'graph' framework for binary relational parametricity. This follows the 'reflexive graph' framework introduced by O'Hearn and Tennent [2]. We include an explicit definition of diparametric transformation following the implicit definition in [1].

2 Binary Relational Parametricity

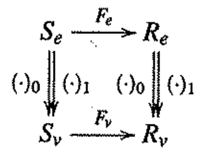
A *graph category* consists of a category of edges R_e together with a *source* functor $(\cdot)_0$ and a *target* functor $(\cdot)_1$ to a category of vertices R_v.

$$\begin{array}{c} R_e \\ {\scriptstyle (\cdot)_0}\Big\downarrow\Big\downarrow{\scriptstyle (\cdot)_1} \\ R_v \end{array}$$

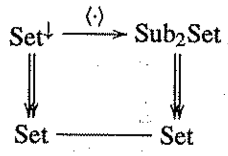
For example, the arrow category $C^{\downarrow}$ can be viewed as the edge category of a graph category with source and target given by domain and codomain, respectively. Another example is $\mathrm{Sub_2Set}$, the pull-back of the subset fibration SubSet along binary products in Set. This category has binary relations for objects and parametric pairs of functions for arrows.

$$\begin{array}{ccccc} \mathrm{Sub_2Set} & & \mathrm{Sub_2Set} & \longrightarrow & \mathrm{SubSet} \\ \Big\downarrow\Big\downarrow & \longleftarrow & \Big\downarrow & \mathrm{pb} & \Big\downarrow \\ \mathrm{Set} & & \mathrm{Set}\times\mathrm{Set} & \xrightarrow{\times} & \mathrm{Set} \end{array}$$

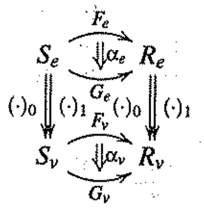
A *graph functor* consists of an edge functor F_e and a vertex functor F_v such that $(F_e s)_0 = F_v(s_0)$ and $(F_e s)_1 = F_v(s_1)$.



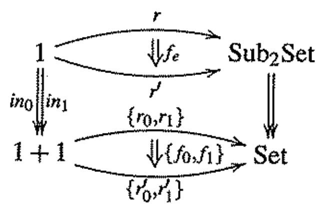
For example, there is the graph graph functor $\langle\cdot\rangle$ from $\text{Set}^{\downarrow}$ to Sub_2Set. The vertex functor is the identity on Set and edge functor takes a function f to its graph $\langle f\rangle$, the set of pairs (x, fx).



A *graph transformation* consists of an edge natural transformation α_e and a vertex natural transformation α_v such that $(\alpha_{es})_0 = \alpha_{v(s_0)}$ and $(\alpha_{es})_1 = \alpha_{v(s_1)}$.



For example, given binary relations r and r' over Set, each pair of functions that is parametric with respect to r and r' corresponds to a graph transformation f thus:



A graph category R can be viewed either as an internal directed graph in the large category of categories, as presented above, or as an internal category in the large category of directed graphs, in which case it has an arrow graph R^1 and an object graph R^0 together with graph morphisms for domain, codomain, composition and identities. Similarly, graph functors and graph transformations can be viewed as internal, directed graph morphisms and directed graph transformations (replace 1-cells with 2-cells in the definition of directed graph morphism) or as internal functors and internal natural transformations. Either way, graph categories, graph functors and graph transformations form a large 2-category GCat.

We must be careful to distinguish $C^{\downarrow}$ from the graph category $C^{\uparrow}$ with source and target given by codomain and domain, respectively. The graph category $C^{\downarrow}$ of *down arrows* is not generally isomorphic to the graph category $C^{\uparrow}$ of *up arrows*. The two are equivalent as graph categories over the identity on C *iff* C is a groupoid. Note that there *is* a graph isomorphism between the graph of down arrows on the opposite of C and opposite of the graph of up arrows on C.

$$\begin{array}{ccc}
(C^{\mathrm{op}})^{\downarrow} & \overset{\Downarrow\Uparrow}{\underset{\Uparrow\Downarrow}{\rightleftarrows}} & (C^{\uparrow})^{\mathrm{op}} \\
\Downarrow\!\!\Downarrow & & \Downarrow\!\!\Downarrow \\
C^{\mathrm{op}} & \text{———} & C^{\mathrm{op}}
\end{array}$$

The graphs $C^{\downarrow}$ and $C^{\uparrow}$ both embed fully and faithfully into $C^{>}$, the graph of cospans over C. This has the domain of the first cospan component for source and the domain of the second for target.

$$\begin{array}{ccccc}
C^{\uparrow} & \longrightarrow & C^{>} & \longleftarrow & C^{\downarrow} \\
\Downarrow\!\!\Downarrow & & \Downarrow\!\!\Downarrow & & \Downarrow\!\!\Downarrow \\
C & \text{———} & C & \text{———} & C
\end{array}$$

A *graph operator* from S to R consists of an edge function F_e from the objects of S_e to the objects of R_e and a vertex function F_v from the objects of S_v to the objects of R_v such that $(F_e s)_0 = F_v(s_0)$ and $(F_e s)_1 = F_v(s_1)$. In other words, a graph operator is a graph morphism from the object graph of S to the object graph of R. For example, the identity on R^0, the object graph of R, gives a graph operator from R^{op} to R and the diagonal graph morphism $\Delta : R^0 \to R^0 \times R^0$ gives a graph operator from R to $R^{\mathrm{op}} \times R$. Note that, while every graph functor restricts to a graph operator, operators such as those above do not generally extend to graph functors.

Given graph operators F and G on S, a *parametric transformation* from F to G is a family of maps $\alpha_d : Fd \to Gd$ indexed by the objects of S_v that lifts to a family of maps $\alpha_s : Fs \to Gs$ indexed by the objects of S_e, meaning $(\alpha_s)_0 = \alpha_{s_0}$ and $(\alpha_s)_1 = \alpha_{s_1}$ for all s.

$$\begin{array}{ccccc}
Fs_0 & \text{——} & \alpha_{s_0} & \longrightarrow & Gs_0 \\
\uparrow & & \uparrow & & \uparrow \\
Fs & \text{——} & \alpha_s & \longrightarrow & Gs \\
\downarrow & & \downarrow & & \downarrow \\
Fs_1 & \text{——} & \alpha_{s_1} & \longrightarrow & Gs_1
\end{array}$$

For graph categories with at most one such α_s we just draw a *parametricity square*:

$$\begin{array}{ccc}
Fs_0 & \xrightarrow{\alpha_{s_0}} & Gs_0 \\
{\scriptstyle Fs}\updownarrow & & \updownarrow{\scriptstyle Gs} \\
Fs_1 & \xrightarrow{\alpha_{s_1}} & Gs_1
\end{array}$$

For example, the familiar naturality square is a parametricity square in the arrow graph category. Given a construction such as G that lifts categories and object functions to graph categories and graph operators, we can ask when a (not necessarily natural) transformation lifts to a parametric transformation.

Proposition 1 *Natural transformations from F to G are identical with parametric transformations from $F^{\downarrow}$ to $G^{\downarrow}$.*

$$
\begin{array}{ccc}
Fs_0 & \xrightarrow{\alpha_{s_0}} & Gs_0 \\
\Big\downarrow{\scriptstyle Fs} & & \Big\downarrow{\scriptstyle Gs} \\
Fs_1 & \xrightarrow{\alpha_{s_1}} & Gs_1
\end{array}
$$

Within this framework of binary relational parametricity, the fundamental notions are functorality: when does an object function lift to a functor? And parametricity: When does a transformation lift to a parametric transformation? As the Proposition shows, naturality is a derived notion. Another derived notion is that of diparametricity.

Definition 1 *A* diparametric transformation *between operators F and G on $S^{\mathrm{op}} \times S$ is a parametric transformation between the operators $F \circ \Delta$ and $G \circ \Delta$ on S.*

$$
S \xrightarrow{\Delta} S^{\mathrm{op}} \times S \underset{F}{\overset{G}{\rightrightarrows}} R
$$

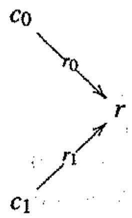
Just as dinaturality weakens naturality, diparametricity weakens parametricity. Note, however, that diparametricity is expressed in terms of parametricity, unlike dinaturality which cannot, in general, be expressed in terms of naturality. Also, because diparametricity squares compose, diparametrics compose.

$$
\begin{array}{ccccc}
F(s_0,s_0) & \xrightarrow{\alpha_{s_0}} & G(s_0,s_0) & \xrightarrow{\alpha'_{s_0}} & H(s_0,s_0) \\
\Big\updownarrow{\scriptstyle F(s,s)} & & \Big\updownarrow{\scriptstyle G(s,s)} & & \Big\updownarrow{\scriptstyle H(s,s)} \\
F(s_1,s_1) & \xrightarrow{\alpha_{s_1}} & G(s_1,s_1) & \xrightarrow{\alpha'_{s_1}} & H(s_1,s_1)
\end{array}
$$

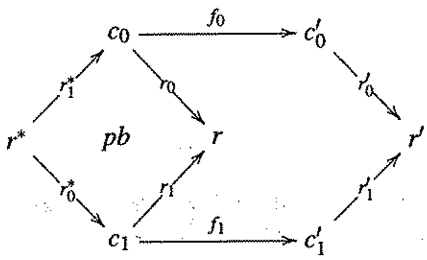
In Section 6 we see that dinaturality can be derived from diparametricity if we weaken the notion of graph category by allowing partial composability in the category of edges, in which case diparametrics are only partially composable.

3 Quotient Relations

Definition 2 *Given a category* C *with pull-backs, the category* QC *of* (binary) quotient relations over C *has cospans* r_i *in* C *for objects*

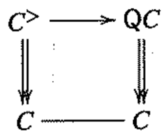


and pairs of maps f_i *such that* $r'_0 \circ f_0 \circ r^*_1 = r'_1 \circ f_1 \circ r^*_0$ *for arrows,*

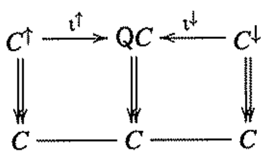


where r^*_1 *is a pull-back of* r_1 *along* r_0 *and* r^*_0 *is the corresponding pull-back of* r_0 *along* r_1. *Composition and identities are given by composition and identities in* $C \times C$.

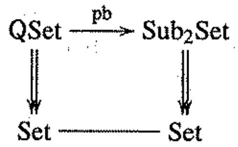
We think of a cospan $r_i : c_i \to r$ as relating those pairs in $c_0 \times c_1$ sent to the same element of r. The definition of arrow does not depend on the choice of pull-back r^*. In fact, any weak pull-back will do, so C need only have weak pull-backs. The properties of (weak) pull-backs ensure that composition in $C \times C$ lifts to QC. The categories QC and C form the edge and vertex categories of a graph category with source and target functors as in $C^{>}$.



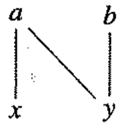
There is an identity-on-objects functor from $C^{>}$ to QC, and although it is not generally full or faithful it extends the full and faithful graph embeddings of $C^{\downarrow}$ and $C^{\uparrow}$ into $C^{>}$ to full and faithful graph embeddings into QC.



While we have presented the objects of Sub_2Set as subobjects of products, these correspond to equivalence classes of jointly monic spans. If we use pull-backs to cast the objects of QSet as jointly monic spans, we obtain a full and faithful embedding of QSet into Sub_2Set.



This means that the notion of arrow in QSet matches the notion of arrow in Sub_2Set which corresponds to the usual logical relations definition. Note, however, that not every binary relation over Set is represented by the pull-back of a cospan. For example, no quotient relation relates



without also relating b and x. This is a generic counter-example in that a binary relation is represented by the pull-back of a cospan *iff* it is zig-zag complete.

Takeyama and Tennent proposed zig-zag completeness [6] as a characterization of relations induced between different concrete data types that represent the same abstract data type.

Definition 3 (Takeyama and Tennent) *An n-ary relation r is* zig-zag complete *if, for each permutation* σ,

$$\sigma r(a,\mathbf{x}) \quad \text{and} \quad a \sim b \quad \text{imply} \quad \sigma r(b,\mathbf{x}),$$

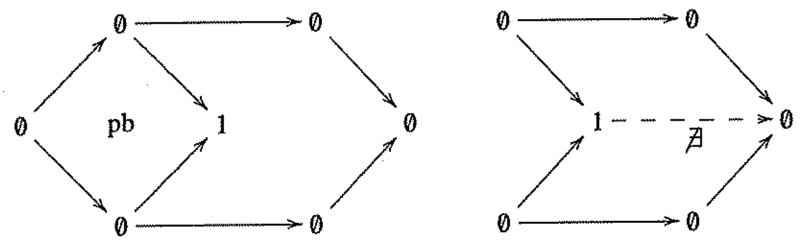
where $a \sim b$ *if there exists y such that* $\sigma r(a,\ldots,y,\ldots)$ *and* $\sigma r(b,\ldots,y,\ldots)$.

Proposition 2 *Over* Set, *an n-ary relation r is represented by the wide pull-back of an n-ary cospan* iff *it is zig-zag complete.*

When $n = 2$, zig-zag completeness can be described in terms of composition of binary relations. In a category R with an involution $(\cdot)^{\circ} : R^{\mathrm{op}} \to R$, an arrow r, is *difunctional* [3] if $r \circ r^{\circ} \circ r = r$. This condition well known in the context of relations between algebras. Ordinarily R is the category of relations Rel$C$ over some regular category C. See Meisen [3] for a study of the relationship between pull-back spans and difunctional relations over categories other than Set.

A regular category C is *Malt'cev* if every arrow in Rel$C$ is difunctional. For example, the category of groups is is Malt'cev. Another example is $\mathrm{Set}^{\mathrm{op}}$ [5]. This is interesting because, while $\mathrm{Rel}(\mathrm{Set}^{\mathrm{op}})$ uses spans in $\mathrm{Set}^{\mathrm{op}}$ to represent arrows, we are using spans in $\mathrm{Set}^{\mathrm{op}}$ to represent objects in Sub_2Set which correspond to arrows in RelSet and, either way, the arrows we get are all difunctional. What about the category we haven't mentioned, $\mathrm{Sub}_2(\mathrm{Set}^{\mathrm{op}})$? The category QSet is equivalent its full subcategory of jointly epic cospans, and this is equivalent to the opposite of $\mathrm{Sub}_2(\mathrm{Set}^{\mathrm{op}})$.

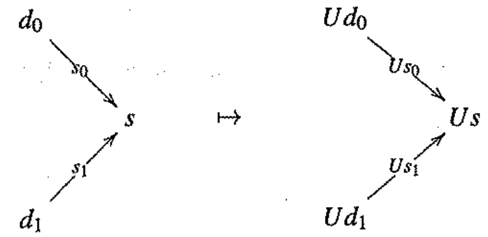
There is nearly an isomorphism between QSet and $\exists_{ver}\mathrm{Set}^{>}$, the category of cospan diagrams with the vertex component of arrows existentially quantified. The exceptions occur with arrows to the identity cospan on empty sets from other cospans on empty sets.



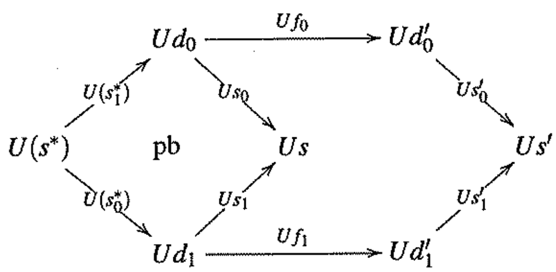
When C is the category of non-empty sets or the category of pointed sets and point preserving functions, the category of quotient relations QC is isomorphic to the cospan category $\exists_{ver}C$ which is equivalent to the opposite of $\mathrm{Sub}_2(C^{\mathrm{op}})$.

4 Componentwise Liftings

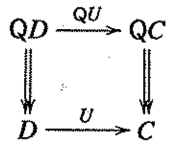
If $U : D \to C$ is a pull-back functor (functor preserving pull-backs between categories with pull-backs), then it lifts componentwise to a functor $QU : QD \to QC$.



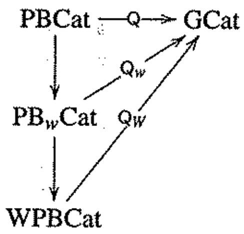
Given an arrow pair f_i in QD, we check that the image Uf_i is an arrow pair in QC by taking, for our pull-back of Us_i, the image of a pull-back of s_i.



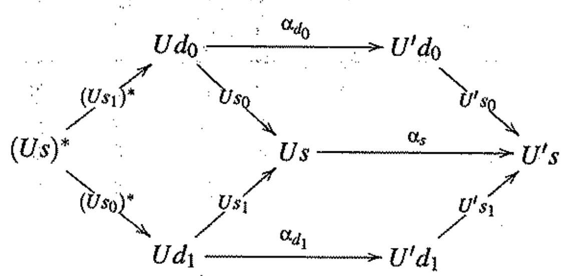
Taken with U, the functor QU gives a graph functor.



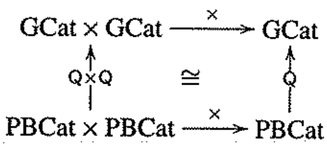
Definition 2 thus gives the object part of a functor $\mathrm{Q} : \mathrm{PBCat} \to \mathrm{GCat}$ from the category of pull-back categories to the category of graph categories. In fact, the construction extends to functors taking pull-backs to weak pull-backs and to functors preserving weak pull-backs between categories with weak pull-backs.



These Q extend to 2-functors: the functors QU act componentwise on the cospans that make up the objects of $\mathrm{Q}D$, so the components of any natural transformation $\alpha : U \Rightarrow U'$ also give a graph transformation from QU to QU'.



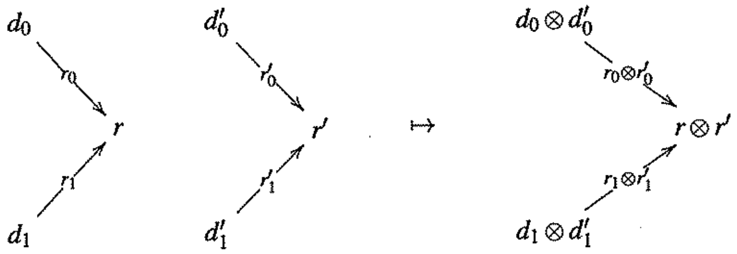
When D has pull-backs, $D \times D$ has pull-backs. These are computed componentwise, so $\mathrm{Q}(D \times D)$ is isomorphic to $\mathrm{Q}D \times \mathrm{Q}D$.



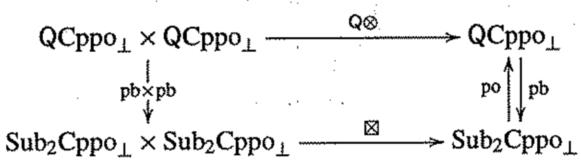
Any binary operation M on $D \times D$ that preserves pull-backs therefore lifts componentwise to a binary operation on $\mathrm{Q}D \times \mathrm{Q}D$.

$$\mathrm{Q}D \times \mathrm{Q}D \cong \mathrm{Q}(D \times D) \xrightarrow{\mathrm{Q}M} \mathrm{Q}C$$

For example, the smash product $\otimes$ on $\mathrm{Cppo}_\perp$ can be expressed as a pull-back (the image under lifting of the product expressed as a pull-back in Cpo) and so preserves pull-backs. It therefore lifts componentwise to $\mathrm{QCppo}_\perp$.



If pull-backs are used to embed $\mathrm{QCppo}_\perp$ into $\mathrm{Sub_2Cppo}_\perp$, then $\mathrm{Q}\otimes$ corresponds the smash product $\boxtimes$ [1] which is adjoint to the parametric function space $[-,+]_{\mathrm{Sub_2Cppo}_\perp}$.



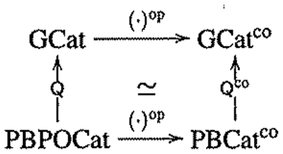
While $\mathrm{Q}\otimes$ is equivalent to the composite $\mathrm{po} \circ \boxtimes \circ (\mathrm{pb} \times \mathrm{pb})$, the functor $\mathrm{Q}\otimes$ is given by a natural, componentwise definition, while $\boxtimes$ is defined using an existential quantification and a completion or, more abstractly, certain coequalizers in the category of lift algebras.

How might the above apply to operations of mixed variance? When D and D^{op} have pull-backs, $D^{\mathrm{op}} \times D$ has pull-backs. If F is a pull-back functor on $D^{\mathrm{op}} \times D$, we obtain a graph functor $\mathrm{Q}F$ on $\mathrm{Q}(D^{\mathrm{op}}) \times \mathrm{Q}D \cong \mathrm{Q}(D^{\mathrm{op}} \times \mathrm{Q}D)$. The following proposition then gives us a graph functor on $(\mathrm{Q}D)^{\mathrm{op}} \times \mathrm{Q}D$.

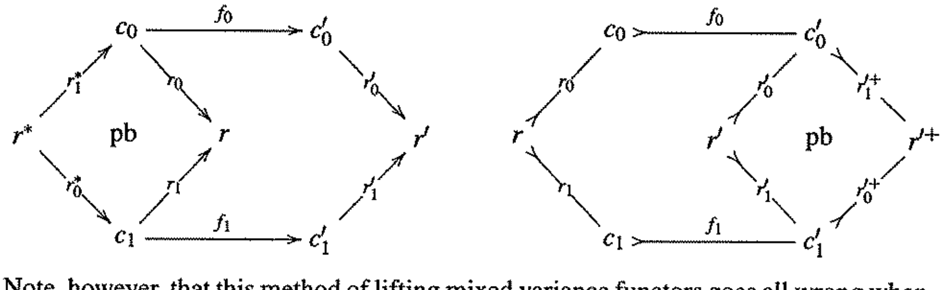
Proposition 3 *When D and D^{op} have pull-backs, $\mathrm{Q}(D^{\mathrm{op}})$ is equivalent to $(\mathrm{Q}D)^{\mathrm{op}}$.*

Proof. The pull-back operations on the objects of $\mathrm{Q}(D^{\mathrm{op}})$ and $(\mathrm{Q}D)^{\mathrm{op}}$ give the equivalence. **QED.**

In other words, on those D that have both pull-backs and push-outs, the 2-functor Q commutes up to an equivalence with the opposite category 2-functor $(\cdot)^{\mathrm{op}} : \mathrm{Cat} \to \mathrm{Cat}^{\mathrm{co}}$.



T his is a little surprising. It means $\mathrm{Q}C$ is equivalent to $(\mathrm{Q}(C^{\mathrm{op}}))^{\mathrm{op}}$ when C has both pull-backs and push-outs. Both $\mathrm{Q}C$ and $(\mathrm{Q}(C^{\mathrm{op}}))^{\mathrm{op}}$ have pairs of maps for arrows, but the objects are different and arrows are tested differently. Take for example QSet. In comparison with the category $\mathrm{Sub_2Set}$ of all binary relations over Set, QSet has the same notion of arrow but only represents some relations (the difunctional ones), while $(\mathsf{Q}(\mathsf{Set}^{\mathrm{op}}))^{\mathrm{op}}$, having spans for objects, represents all relations but has a more liberal notion of arrow tested using a push-out (pull-back in $\mathsf{Set}^{\mathrm{op}}$) of the codomain span.

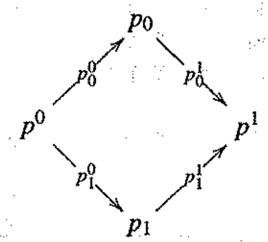


Note, however, that this method of lifting mixed variance functors goes all wrong when applied to closed structure. Assuming C^{op} has pull-backs, these are not generally preserved by the contravariant parts $[-,c]_C : C^{\mathrm{op}} \to C$ of a function space functor $[-,+]_C$. However, given just pull-backs in C, any function space functor $[-,+]_C : C^{\mathrm{op}} \times C \to C$ lifts to a reflexive graph functor $[-,+]_{\mathsf{Q}}C : (\mathsf{Q}C)^{\mathrm{op}} \times \mathsf{Q}C \to \mathsf{Q}C$, with or without pull-backs in C^{op} and whether or not those in C are preserved.

5 Push-me-pull-you's

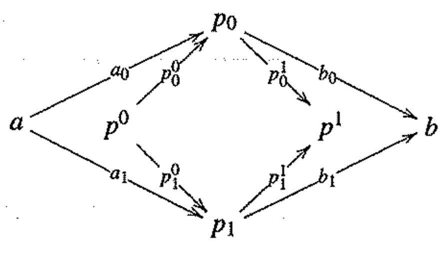
Before we consider function spaces, we introduce a pleasantly symmetrical construction K which is equivalent to Q when applied to pull-back categories.

Definition 4 *Given any category C, the category $\mathsf{K}C$ of* push-me-pull-you's *over C has, for objects, commutative diamonds p_i^j*

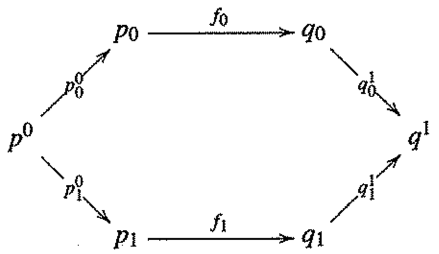


such that any span a_i that commutes with the cospan p_i^1 commutes with any cospan b_i that commutes with the span p_i^0,

$$p_0^1 \circ a_0 = p_1^1 \circ a_1 \quad \text{and} \quad p_0^0 \circ b_0 = p_1^0 \circ b_1 \quad \Rightarrow \quad b_0 \circ a_0 = b_1 \circ a_1,$$



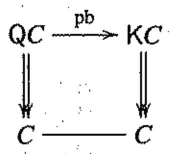
and, for arrows, pairs of maps f_i *such that* $q_0^1 \circ f_0 \circ p_0^0 = q_1^1 \circ f_1 \circ p_1^0$.



Composition and identities are given by composition and identities in $C \times C$.

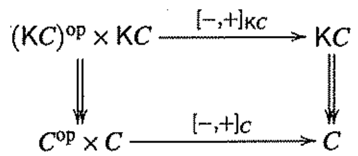
The push-me-pull-you condition abstracts the property of (weak) pull-backs that ensures composition lifts from $C \times C$. Identities lift from $C \times C$ because the diamonds commute. Informally, the span generates pairs, while the cospan tests pairs. Commutativity says every pair generated must test good. The push-me-pull-you condition says every pair that tests good must be generated. The objects of $\mathsf{K}C$ include all (weak) pull-backs and (weak) push-outs in C. Over Set or any C with both pull-backs and push-outs, a diamond is a push-me-pull-you *iff* the pull-back of the cospan commutes with the push-out of the span.

Proposition 4 *A choice of pull-backs in C gives an equivalence between the categories* $\mathsf{Q}C$ *and* $\mathsf{K}C$ *which is a graph equivalence over the identity on C.*



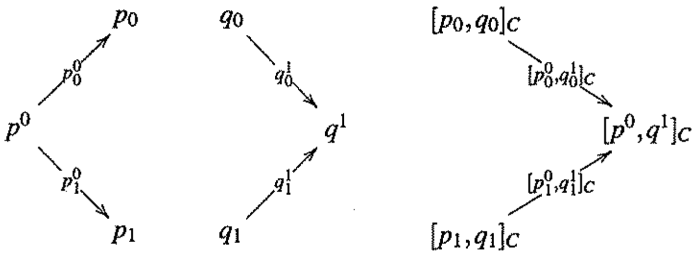
Proof. One direction of the equivalence takes a cospan in $\mathsf{Q}C$ to its chosen pull-back diamond in $\mathsf{K}C$ and arrow pairs to themselves. The other direction takes a diamond in $\mathsf{K}C$ to its cospan part in $\mathsf{Q}C$ and arrows pairs to themselves. These two functors give the identity on $\mathsf{Q}C$ and the endofunctor on $\mathsf{K}C$ that normalizes push-me-pull-you's to pull-back's. Any push-me-pull-you is isomorphic to any (weak) pull-back diamond with the same cospan. **QED.**

Proposition 5 *Given a choice of pull-backs in C, closed structure on C lifts to closed structure on* $\mathsf{K}C$.

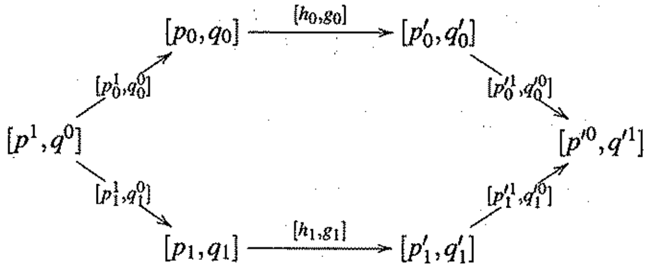


Proof. The cospan part of $[-,+]_{\mathsf{K}C}$ is given by applying $[-,+]_C$ to the cospan of the covariant argument and the span of the contravariant argument. The span part is given by the pull-back of the cospan part.

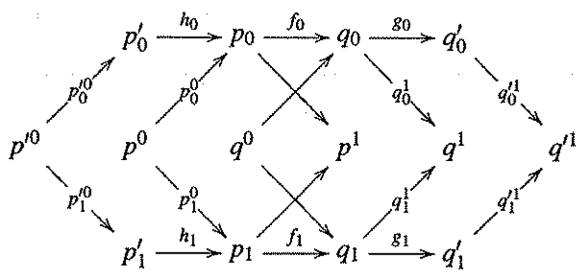
$$p \qquad q \qquad \mapsto \qquad [p,q]_{KC}$$



The cospan $[p_i^0, q_i^1]$ internalizes the test on arrow pairs in the definition of KC. The pull-back span then generates all pairs that test good. We must check that $[h_i, g_i]$ carries these to good pairs according to the cospan $[p_i'^0, q_i'^1]$. Note that the span $[p_i^1, q_i^0]$ does not generally generate all the pairs in $[p_i^0, q_i^1]^*$—the contravariant place in the function space does not generally preserve pull-backs—so we cannot appeal to the diagram



Instead, consider a pair $f_i : p_i \to q_i$ in $[p^0, q^1]^*$. We know this gives an arrow from p to q because $[p_i^0, q_i^1]$ internalises the arrow test.

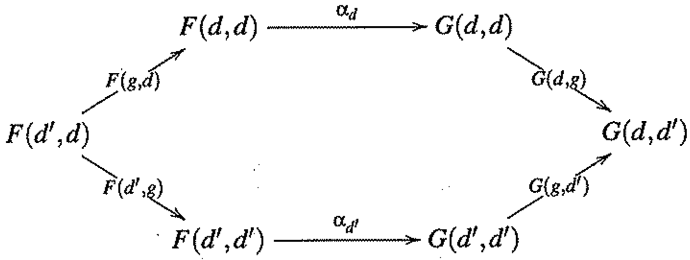


By the push-me-pull-you properties of p_i^j and q_i^j, the composite pair $h_i \circ f_i \circ g_i$, which is $[h_i, g_i]$ applied to f_i, gives an arrow from p' to q', but then this pair tests good according to the cospan $[p_i'^0, q_i'^1]$ which internalizes the arrow test from p' to q'. **QED**

Corollary 1 *Given a choice of pull-backs in C, closed structure on C lifts to QC.*

6 Strong Dinaturality

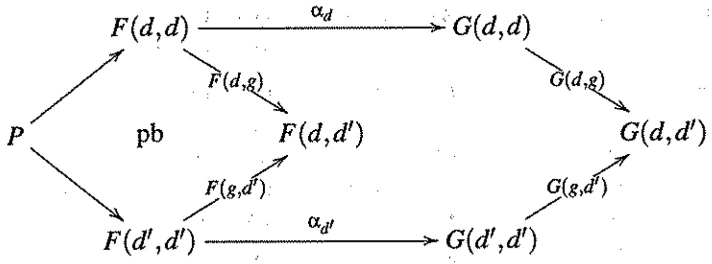
First we consider ordinary dinaturality in terms of diparametricity. Given a category C, the quasi-category[1] $\mathsf{D}C$ of *diamonds* has commutative diamonds for objects and commuting pairs of maps (as in $\mathsf{K}C$) for arrows. Without the push-me-pull-you condition there is no guarantee that componentwise composition of arrows in $\mathsf{D}C$ gives arrows in $\mathsf{D}C$. On the other hand, every functor F lifts componentwise to a quasi-functor $\mathsf{D}F$ which preserves what composites exist. Also, as with push-me-pull-you's, we have a isomorphism between the graph quasi-categories $(\mathsf{D}C)^{\mathrm{op}}$ and $\mathsf{D}(C^{\mathrm{op}})$. Dinaturality is parametricity with respect to certain objects in the quasi-category of diamonds.



Proposition 6 *Dinatural transformations from F to G are identical with diparametric transformations from* $\mathsf{D}F \circ ((\iota^{\uparrow} \circ \Downarrow\!\Uparrow) \times \iota^{\downarrow})$ *to* $\mathsf{D}G \circ ((\iota^{\uparrow} \circ \Downarrow\!\Uparrow) \times \iota^{\downarrow})$.

$$(D^{\downarrow})^{\mathrm{op}} \times D^{\downarrow} \xrightarrow{\Downarrow\!\Uparrow \times D^{\downarrow}} (D^{\mathrm{op}})^{\uparrow} \times D^{\downarrow} \xrightarrow{\iota^{\uparrow} \times \iota^{\downarrow}} \mathsf{D}(D^{\mathrm{op}}) \times \mathsf{D}D \xrightarrow[\text{iso}]{} \mathsf{D}(D^{\mathrm{op}} \times D) \underset{\mathsf{D}F}{\overset{\mathsf{D}G}{\rightrightarrows}} \mathsf{D}C$$

Assuming C has pull-backs, strong dinaturality [4] is parametricity with respect to certain objects in $\mathsf{Q}C$.

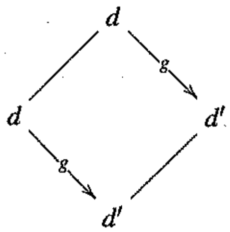


Proposition 7 *Strong dinatural transformations from F to G are identical with diparametric transformations from* $\mathsf{Q}F \circ ((\iota^{\uparrow} \circ \Downarrow\!\Uparrow) \times \iota^{\downarrow})$ *to* $\mathsf{Q}G \circ ((\iota^{\uparrow} \circ \Downarrow\!\Uparrow) \times \iota^{\downarrow})$.

$$(D^{\downarrow})^{\mathrm{op}} \times D^{\downarrow} \xrightarrow{\Downarrow\!\Uparrow \times D^{\downarrow}} (D^{\mathrm{op}})^{\uparrow} \times D^{\downarrow} \xrightarrow{\iota^{\uparrow} \times \iota^{\downarrow}} \mathsf{Q}(D^{\mathrm{op}}) \times \mathsf{Q}D \xrightarrow[\text{iso}]{} \mathsf{Q}(D^{\mathrm{op}} \times D) \underset{\mathsf{Q}F}{\overset{\mathsf{Q}G}{\rightrightarrows}} \mathsf{Q}C$$

[1] By 'quasi' we mean that composition is a partial operation on composable pairs and the equations of category theory hold just where both sides exist. A 'graph quasi-category' has a category of vertices and quasi-category of edges. The definitions of 'graph operator' and 'parametric transformation' are unaffected because they ignore composition.

Diamonds of the form



are pull-backs, This means the above functor $(\iota^{\uparrow} \circ \downharpoonleft\!\upharpoonright)$ can be written as $\iota^{\downarrow}$ followed by a pull-back operation from $(\mathrm{Q}D)^{\mathrm{op}}$ to $\mathrm{Q}(D^{\mathrm{op}})$.

$$\begin{array}{ccc} (D^{\downarrow})^{\mathrm{op}} & \xrightarrow{(\iota_D^{\downarrow})^{\mathrm{op}}} & (\mathrm{Q}D)^{\mathrm{op}} \\ \downarrow{\scriptstyle \downharpoonleft\upharpoonright} & & \downarrow{\scriptstyle \mathrm{pb}} \\ (D^{\mathrm{op}})^{\uparrow} & \xrightarrow{\iota^{\uparrow}_{D^{\mathrm{op}}}} & \mathrm{Q}(D^{\mathrm{op}}) \end{array}$$

The above functors on $(D^{\downarrow})^{\mathrm{op}} \times D^{\downarrow}$ therefore factor through functors on $(\mathrm{Q}D)^{\mathrm{op}} \times \mathrm{Q}D$.

$$(D^{\downarrow})^{\mathrm{op}} \times D^{\downarrow} \xrightarrow{(\iota^{\downarrow})^{\mathrm{op}} \times \iota^{\downarrow}} (\mathrm{Q}D)^{\mathrm{op}} \times \mathrm{Q}D \xrightarrow{\mathrm{pb} \times \mathrm{Q}D} \mathrm{Q}(D^{\mathrm{op}}) \times \mathrm{Q}D \xrightarrow[\mathrm{iso}]{} \mathrm{Q}(D^{\mathrm{op}} \times D) \overset{\mathrm{Q}G}{\underset{\mathrm{Q}F}{\rightrightarrows}} \mathrm{Q}C$$

Corollary 2 *A family of maps is strongly dinatural if it gives a diparametric transformation from* $\mathrm{Q}F \circ (\mathrm{pb} \times \mathrm{Q}D)$ *to* $\mathrm{Q}G \circ (\mathrm{pb} \times \mathrm{Q}D)$.

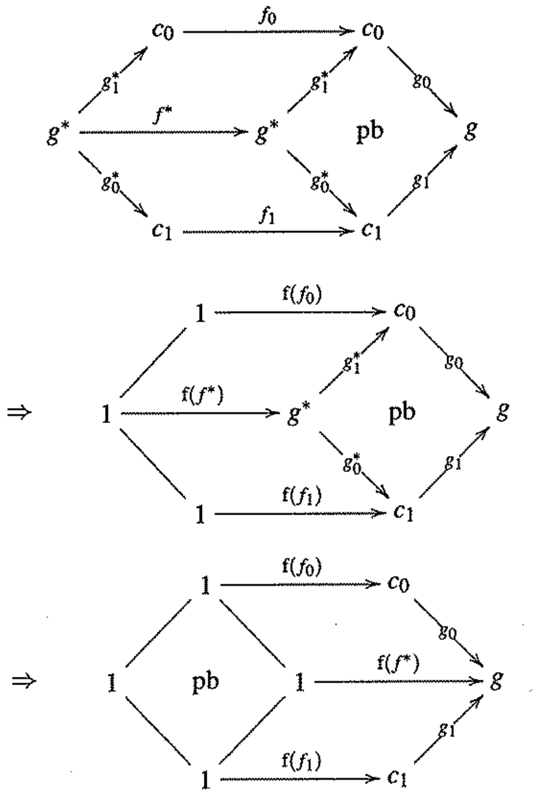
Now suppose α has the type of a fixed point operator. That is, suppose F is the function space $[-,+]$ of some closed category C and G is second projection π_C on $C^{\mathrm{op}} \times C$. The function space lifts to a functor $[+,-]_{\mathrm{Q}C}$ given by $\mathrm{Q}[-,+] \circ (\mathrm{pb} \times \mathrm{Q}C)$. The second projection $\pi_{\mathrm{Q}C}$ on $(\mathrm{Q}C)^{\mathrm{op}} \times \mathrm{Q}C$ throws away its first argument and so equals $\mathrm{Q}\pi_C \circ (\mathrm{pb} \times \mathrm{Q}C)$. Therefore, by the Corollary, a diparametric transformation from $[+,-]_{\mathrm{Q}C}$ to $\pi_{\mathrm{Q}C}$ gives a strong dinatural transformation from the functor $[+,-]$ to the functor π_C. In this case, however, the converse holds.

Proposition 8 *A family of maps* $\mathrm{f}_c : [c,c] \to c$ *is strongly dinatural* iff *it gives a diparametric transformation from* $[-,+]_{\mathrm{Q}C} : (\mathrm{Q}C)^{\mathrm{op}} \times \mathrm{Q}C \to \mathrm{Q}C$ *to* $\pi_{\mathrm{Q}C} : (\mathrm{Q}C)^{\mathrm{op}} \times \mathrm{Q}C \to \mathrm{Q}C$.

Proof. We check that the *a priori* stronger parametricity condition is implied by strong dinaturality. Strong dinaturality for these two functors says, modulo an internalisation, that we have $g(\mathrm{f}(f_0)) = \mathrm{f}(f_1)$, for any f_0, f_1 and g such that $g \circ f_0 = f_1 \circ g$.

$$\begin{array}{ccc} c_0 & \xrightarrow{f_0} & c_0 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ c_1 & \xrightarrow{f_1} & c_1 \end{array} \quad \Rightarrow \quad \begin{array}{ccc} 1 & \xrightarrow{\mathrm{f}(f_0)} & c_0 \\ \Big| & & \downarrow{\scriptstyle g} \\ 1 & \xrightarrow{\mathrm{f}(f_1)} & c_1 \end{array}$$

Diparametricity says, modulo an internalisation, that we have $g_0(\mathrm{f}(f_0)) = g_1(\mathrm{f}(f_1))$, for any f_0, f_1 and cospan (g,h) such that $g \circ f_0 \circ g_1^* = h \circ f_1 \circ g_0^*$.

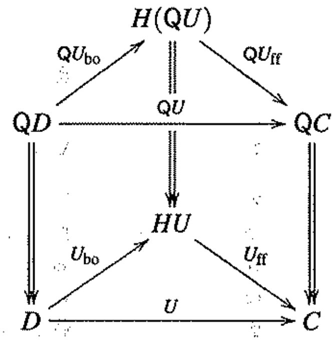


This follows from strong dinaturality applied to f^*, f_0 and g_1^* and to f^*, f_1 and g_0^*, where f^* is the unique span map from $(f_0 \circ g_1^*, f_1 \circ g_0^*)$ to (g_1^*, g_0^*). **QED.**

7 Restricted Variation

For Plotkin's Axiom to characterize least fixed points, it is necessary that g vary over *strict, continuous* maps only. In Mulry's setting, the notion of strong dinaturality must be correspondingly weakened. In that setting, strict maps are algebra morphisms and so Mulry uses the notion of strong dinatural transformation with variation restricted to algebra morphisms [4, Def. 3.11]. In this Section, we use a factorization system in Cat to describe the correspondingly weakened parametricity condition. Strict maps will be those in the image of some functor $U : D \to C$. In Mulry's setting, U is the forgetful functor from the category of LU-algebras.

In Cat every functor F factors as a bijective-on-objects functor F_{bo} followed by a full-and-faithful functor F_{ff}. If D is the domain of F, the the interpolating category HF has the objects of D for objects and has hom sets given by Hom(F(−),F(+)). This factorization system lifts to graph functors, because the operation H is the object part of a functor from Fun, the category of functors and functor squares commuting up to given natural isomorphisms, to Cat. A graph morphism gives a graph in Fun which H then sends to a graph category. Assuming U is a pull-back functor, we therefore obtain a graph $H(QU)$ by factoring the graph functor QU. Note that $H(QU)$ is not $Q(HU)$. Our definition of the latter requires HU to have pull-backs (which it does, by the way, if U creates pull-backs).



When $L \dashv U$ is a closed adjunction, the closed structure on C lifts along U_{ff} to closed structure on HU by defining

$$[-,+]_{HU} \stackrel{\mathrm{def}}{=} [LU-,+]_{D}.$$

This closed structure commutes with U_{ff} up to a natural isomorphism $\tau : U[LU-,+]_D \Rightarrow [U-,U+]_C$ which is the internalisation of transposition. Similarly, closed structure on QC lifts to closed structure on $H(QU)$ and we obtain a graph functor $[-,+]_{H(QU)} : H(QU)^{\mathrm{op}} \times H(QU) \to H(QU)$. This is the image under H of closed structure on the graph QU over U in Fun.

Proposition 9 *A family of maps* $\mathrm{f}_d : [Ud,Ud] \to Ud$ *in* C *indexed by the objects of* D *is strongly dinatural with variation restricted to* D iff *it gives a diparametric transformation from* $[-,+]_{H(QU)}$ *to* $\pi_{H(QU)}$.

Suppose $U : D \to C$ is the inclusion of the category $\mathrm{Cppo}_\perp$ of complete pointed partial orders and strict continuous maps into the category Cpo of complete partial orders and continuous maps. This inclusion has a left adjoint which lifts partial orders to pointed partial orders and so, as a right adjoint, preserves pull-backs. The category $H(QU)$ then has cospans of strict continuous maps for objects and pairs of continuous maps for arrows. The family of the maps producing least fixed points, is characterized by Plotkin's Axiom which is equivalent to strong dinaturality with variation restricted to $\mathrm{Cppo}_\perp$ which is equivalent to diparametricity with respect to this $H(QU)$.

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