Killing boundary data for anti-de Sitter-like spacetimes

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Abstract

Given an initial-boundary value problem for an anti-de Sitter-like spacetime, we analyse conditions on the conformal boundary ensuring the existence of Killing vectors in the spacetime arising from this problem. This analysis makes use of a system of conformal wave equations describing the propagation of the Killing equation first considered by Paetz. We identify an obstruction tensor constructed from Killing vector candidate and the Cotton tensor of the conformal boundary whose vanishing is a necessary condition for the existence of Killing vectors in the spacetime. This obstruction tensor vanishes if the conformal boundary is conformally flat.

1 Introduction

Anti-de Sitter-like spacetimes are solutions to the Einstein field equations with negative Cosmological constant having a global structure similar to that of the anti-de Sitter spacetime. In particular, they can be conformally extended in such a way that the resulting conformal boundary is a timelike hypersurface of the conformal extension. Members of this class of solutions to the Einstein field equations constitute prime examples of spacetimes which are not *globally hyperbolic*. Accordingly, initial data is not enough to reconstruct one of these solutions to the Einstein field equations —one also needs to prescribe some suitable data at the conformal boundary. The construction of anti-de Sitter spacetimes by means of a initial-boundary value problem has been analysed in [6] where a large family of *maximally dissipative boundary conditions* involving incoming and outgoing components of the Weyl tensor have been identified. In this respect, anti-de Sitter spacetimes provide a convenient setting to study initial-boundary value problems for the Einstein equations as the conformal boundary is a hypersurface with a rich structure —despite the use of the conformal Einstein field equations, the formulation of the initial-boundary value problem for anti-de Sitter-like spacetimes as given in [6] is considerably simpler than the analysis of the *general* initial-boundary value problem for the Einstein field equations as given in e.g. [8]. In particular, the anti-de Sitter construction allows to establish *geometric uniqueness* while the analysis in [8] leaves unanswered this question —see [7] for a further discussion on this important issue.

The problem of encoding (continuous) symmetries of a spacetime at the level of initial data is an important classical problem in Relativity —see e.g. [12]. A modern presentation of this issue and the related theory can be found in [3, 1]. The key outcome of this theory is the so-called set of *Killing initial data equations*, a system of overdetermined equations for a scalar field and a spatial vector on a spacelike hypersurface —corresponding, respectively, to the *lapse* and *shift* with respect to the normal of the hypersurface of an hypothetical Killing vector of the spacetime.

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If these Killing equations admit a solution, a so-called *Killing initial data set (KID)*, then the development of the initial data will have a Killing vector. The theory of KID for the Cauchy problem for the Einstein field equations can be also adapted to other settings like the (finite and asymptotic) characteristic initial value problem [4, 13] and, more relevant for the purposes of the present article, to the asymptotic initial value problem for the de Sitter-like spacetimes [15] —i.e. solutions to the vacuum Einstein field equations with positive Cosmological constant.

Main results of the present article

The purpose of the present article is to construct a theory of Killing initial and boundary data in the setting of anti-de Sitter-like spacetimes. Given the nature of the problem, we perform the analysis in a *conformal setting* —that is, we work with a suitable (*unphysical*) conformal representation of the spacetime rather than with the *physical* spacetime itself. As these spacetimes are not globally hyperbolic, in addition to satisfying the KID equations on some initial hypersurface, one also needs to prescribe some *Killing boundary data (KBD)* to ensure the existence of a Killing vector in the spacetime. The use of a conformal setting allows to perform the analysis of the boundary conditions for the Killing equations by means of local (differential geometric) computations. The Killing boundary data restricts, in turn, the structure of the conformal boundary. In addition, the Killing initial and boundary data have to satisfy some compatibility conditions at the corner where the initial hypersurface and the conformal boundary meet.

Our strategy to identify the Killing boundary data is to make use of a system of conformal wave equations describing the propagation of the Killing vector equation first discussed by Paetz in [15] —the *Killing equation conformal propagation system*, see Lemma 3, equations (8a)-(8e). If this system has the trivial (vanishing) solution then a suitably constructed Killing vector candidate is, in fact, a Killing vector of the spacetime. Accordingly, one is naturally lead to consider an initial-boundary value problem with both vanishing initial data and Dirichlet boundary data for the Killing equation conformal propagation system. While the vanishing initial data naturally leads to a conformal version of the Killing initial data equations, the vanishing Dirichlet boundary data give the Killing boundary data conditions —see equations (21a)-(21g). A detailed formulation of this result is given in Proposition 5. The conditions obtained by this approach are, in first instance, restrictions on spacetime tensors. In a second step we analyse the interdependencies between these conditions and express them in terms of objects which are intrinsic to the conformal boundary —a so-called *reduced Killing boundary equations*, equations (22a)-(22e). A key ingredient in this analysis is given by the constraint equations, (12a)-(12j), implied by the conformal Einstein equations on the timelike conformal boundary.

The analysis of the reduced Killing boundary equations shows that a necessary condition for the existence of a Killing vector in the anti-de Sitter-like spacetime is the existence of a conformal Killing vector in the conformal boundary —see equation (22c) in the main text. In order to obtain further insight into the content of the reduced Killing boundary equations we analyse the conditions under which it is possible to ensure the existence of such conformal Killing vector in terms of assumptions on the conformal boundary and initial data at the corner. To this end, we mimic the analysis on the spacetime and consider a conformal Killing equation propagation system intrinsic to the boundary —see the equations in Lemma 6. This systems allows the identification of an *obstruction tensor* $\mathcal{O}_{ab}$, constructed from an intrinsic conformal Killing vector candidate and the Cotton tensor of the conformal boundary, whose vanishing ensures the existence of the required intrinsic conformal Killing vector —see equation (25). In particular, if the conformal boundary is conformally flat (as in the case, for example, of the Kerr-anti de Sitter spacetime) then the obstruction tensor vanishes. The existence of the conformal Killing vector intrinsic to the conformal boundary is formulated in Proposition 6. Finally, our main result concerning the existence of Killing vectors in the development of an initial-boundary value problem for the conformal Einstein equations is given in Theorem 1

An important property of the analysis described in the previous paragraphs which follows from working in an unphysical (i.e. conformally rescaled) spacetime is that the boundary conditions (both at a spacetime and intrinsic level) required for the existence of a Killing vector in the physical spacetime are conformally invariant. Thus, the analysis is independent of the conformal representation one is working with.

An alternative approach to the analysis of continuous symmetries in anti-de Sitter-like spacetimes has been started in [9, 10]. In this work, the objective is to encode the existence of Killing vector solely through conditions on the conformal boundary —in the spirit of the principle of *holography.* The required analysis, thus, leads to the study of ill-posed initial value problems for wave equations which requires the use of methods of the *theory of unique continuation.* Accordingly, their analysis should lead to an infinite hierarchy of conditions on the conformal boundary while the discussion in the present work requires, roughly speaking, the first conditions on the hierarchy. The trade off is that our analysis also requires a solution to the KID equation on a spacelike hypersurface and compatibility conditions between the Killing initial and boundary data.

Conventions

Through out, the term *spacetime* will be used to denote a 4-dimensional Lorentzian manifold which not necessarily satisfies the Einstein field equations. Moreover, $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ will denote a vacuum spacetime satisfying the Einstein equations with anti de Sitter-like cosmological constant λ. The signature of the metric in this article will be $(-,+,+,+)$. It follows that $\lambda < 0$. The lowercase Latin letters a, b, $c, \ldots$ are used as abstract spacetime tensor indices while the indices i, j, $k, \ldots$ are abstract indices on the tensor bundle of hypersurfaces of $\tilde{\mathcal{M}}$. The Greek letters μ, ν, $\lambda, \ldots$ will be used as spacetime coordinate indices while α, β, $\gamma, \ldots$ will serve as spatial coordinate indices.

Our conventions for the curvature are

$$\nabla_c\nabla_d u^a - \nabla_d\nabla_c u^a = R^a{}_{bcd}u^b.$$

2 The metric conformal Einstein field equations

Throughout all this work we will make use of the Einstein equations on a conformal setting; therefore, in this section the properties of this representation will be presented.

Let $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ a 4-dimensional spacetime satisfying the vacuum Einstein field equations

$$\tilde{R}_{ab} = \lambda \tilde{g}_{ab}, \tag{1}$$

where $\tilde{R}_{ab}$ is the Ricci tensor associated to the metric $\tilde{g}_{ab}$ and λ the so-called cosmological constant. Now, consider a conformal embedding consider a spacetime $(\mathcal{M}, g_{ab})$ which is related to $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ via a *conformal embedding*

$$\tilde{\mathcal{M}} \overset{\varphi}{\hookrightarrow} \mathcal{M}, \qquad \tilde{g}_{ab} \overset{\varphi}{\mapsto} g_{ab} \equiv \Xi^2 (\varphi^{-1})^* \tilde{g}_{ab}, \qquad \Xi|_{\varphi(\tilde{\mathcal{M}})} > 0.$$

Slightly abusing of the notation we write

$$g_{ab} = \Xi^2 \tilde{g}_{ab}, \tag{2}$$

where the *conformal factor* Ξ is a non-negative scalar function. The set of points of $\mathcal{M}$ for which Ξ vanishes will be called the *conformal boundary.* We use the notation $\mathscr{I}$ to denote the parts of the conformal boundary which are an hypersurface of $\mathcal{M}$.

2.1 Basic properties

In what follows, let ∇_a denote the Levi-Civita connection of the metric g_{ab}. Let $R^a{}_{bcd}$, R_{ab}, R and $C^a{}_{bcd}$ denote, respectively, the corresponding Riemann tensor, Ricci tensor, Ricci scalar and (conformally invariant) Weyl tensor. In a conformal context it is customary to introduce *Schouten tensor* L_{ab}, defined as

$$L_{ab} \equiv \frac{1}{2}\bigg(R_{ab} - \frac{1}{6}R g_{ab}\bigg).$$

Moreover, it is useful to define the following quantities:

$$s \equiv \frac{1}{4}\nabla^c\nabla_c\Xi + \frac{1}{24}R\Xi, \qquad d^a{}_{bcd} \equiv \Xi^{-1}C^a{}_{bcd},$$

where the former is the so-called *Friedrich scalar* and the latter is the *rescaled Weyl tensor*.

In terms of the objects defined above, and under a conformal transformation, the Einstein equations (1) imply a system of differential equations known as the *metric vacuum conformal Einstein field equations*, given by:

$$\nabla_a\nabla_b\Xi = -\Xi L_{ab} + s g_{ab}, \tag{3a}$$
$$\nabla_a s = -L_{ac}\nabla^c\Xi, \tag{3b}$$
$$\nabla_a L_{bc} - \nabla_b L_{ac} = \nabla_e\Xi d^e{}_{cab}, \tag{3c}$$
$$\nabla_e d^e{}_{cab} = 0, \tag{3d}$$
$$\lambda = 6\Xi s - 3\nabla_c\Xi\nabla^c\Xi. \tag{3e}$$

A detailed derivation of this system for the general case of a non-zero matter component can be found in [16].

Remark 1. Expressions (3a)-(3d) are differential equations for the fields Ξ, s, L_{ab} and $d^a{}_{bcd}$, while equation (3e) will be regarded as a constraint. As shown in Lemma 8.1 in [16], if (3a) and (3b) are satisfied, (3e) will automatically do so as long as it holds at a single point.

By a solution to the metric conformal Einstein field equations it is understood a collection

$$(g_{ab}, \Xi, s, L_{ab}, d^a{}_{bcd})$$

satisfying equations (3a)-(3e). If $\tilde{g}_{ab}$ is a solution to the Einstein equations (1) and it is conformally related to g_{ab}, then the latter is a solution to the conformal Einstein field equations. The converse of this statement is given as follows:

Proposition 1. *Let* $(g_{ab}, \Xi, s, L_{ab}, d^a{}_{bcd})$ *denote a solution to the metric conformal Einstein field equations* (3a)-(3d) *such that* $\Xi \neq 0$ *on an open set* $\mathcal{U} \subset \mathcal{M}$*. If, in addition, equation* (3e) *is satisfied at a point* $p \in \mathcal{U}$*, then the metric*

$$\tilde{g}_{ab} = \Xi^{-2}g_{ab}$$

is a solution to the Einstein field equations (1) *on* $\mathcal{U}$*.*

A proof of this proposition is given in [16] —see Proposition 8.1 in that reference.

The causal character of the conformal boundary $\mathscr{I}$ is determined by the sign of the Cosmological constant. As this will be of key importance in the forthcoming sections, we make this more precise:

Proposition 2. *Suppose that the Friedrich scalar* s *is finite at* $\mathscr{I}$*. Then* $\mathscr{I}$ *is a null, spacelike or timelike hypersurface of* $\mathcal{M}$*, respectively, depending on whether* $\lambda = 0$*,* $\lambda > 0$ *or* $\lambda < 0$*.*

Proof. This result follows directly from evaluating equation (3e) at $\mathscr{I}$ and recalling that $\nabla_a\Xi$ is normal to this hypersurface. □

2.2 Wave equations for the conformal fields

In [14] it has been shown how the conformal Einstein field equations (3a)-(3d) imply a system of *geometric* wave equations for the components of the fields (Ξ, s, L_{ab}, $d^a{}_{bcd}$). Such system takes the form:

Proposition 3. *Any solution* $(\Xi,\, s,\, L_{ab},\, d^a{}_{bcd})$ *to the vacuum conformal Einstein field equations* (3a)-(3d) *satisfies the equations*

$$\Box\Xi = 4s - \frac{1}{6}\Xi R, \tag{4a}$$

$$\Box s = \Xi L_{ab}L^{ab} - \frac{1}{6}sR - \frac{1}{6}\nabla_a R\nabla^a\Xi, \tag{4b}$$

$$\Box L_{ab} = 4L_a{}^c L_{bc} - g_{ab}L_{cd}L^{cd} - 2\Xi d_{acbd}L^{cd} + \frac{1}{6}\nabla_a\nabla_b R, \tag{4c}$$

$$\Box d_{abcd} = 2\Xi d_a{}^e{}_d{}^f d_{becf} - 2\Xi d_a{}^e{}_c{}^f d_{bedf} - 2\Xi d_{ab}{}^{ef}d_{cedf} + \frac{1}{2}d_{abcd}R. \tag{4d}$$

3 Killing vectors in the conformal setting

In this section we briefly review the theory of Killing vectors from a conformal point of view. Our presentation follows that of [15].

3.1 Conformal properties of the Killing vector equation

We begin by recalling the relation between Killing vectors in the physical spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ and conformal Killing vectors in the unphysical spacetime $(\mathcal{M}, g_{ab})$:

Lemma 1. *A vector field* $\tilde{\xi}^a$ *is a Killing vector field of* $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$*, that is*

$$\tilde{\nabla}_a\tilde{\xi}_b + \tilde{\nabla}_b\tilde{\xi}_a = 0,$$

if and only if its push-forward $\xi^a \equiv \varphi_*\tilde{\xi}^a$ *is a conformal Killing vector field in* $(\mathcal{M}, g_{ab})$*, i.e.*

$$\nabla_a\xi_b + \nabla_b\xi_a = \frac{1}{2}\nabla_c\xi^c g_{ab} \tag{5}$$

and, moreover, one has that

$$\xi^a\nabla_a\Xi = \frac{1}{4}\Xi\nabla_a\xi^a. \tag{6}$$

The proof of this result can be found in [15].

Remark 2. In the following we will call equations (5) and (6) the *unphysical Killing equations.* Observe that if g_{ab} extends smoothly across $\mathscr{I}$, then the unphysical Killing equations are well defined at the conformal boundary.

This leads to a natural question about the conditions for the existence of unphysical Killing vectors. This will be addressed in the remaining of this section.

3.2 Necessary conditions

For convenience set

$$\eta \equiv \frac{1}{4}\nabla_a\xi^a.$$

Then one has the following result:

Lemma 2. *Any solution to the unphysical Killing equations satisfies the system*

$$\Box\xi_a + R_a{}^b\xi_b + 2\nabla_a\eta = 0, \tag{7a}$$

$$\Box\eta + \frac{1}{6}\xi^a\nabla_a R + \frac{1}{3}R\eta = 0. \tag{7b}$$

The proof of the above result follows by direct computation from (5) and (6).

Remark 3. The wave equations (7a) and (7b) are necessary conditions for a vector ξ^a to be an *unphysical Killing vector*. However, not every solution to these equations is an unphysical Killing vector. In this sense, a vector field satisfying (7a)-(7b) will be called a *unphysical Killing vector candidate.*

3.3 The unphysical Killing equation propagation system

The sufficient conditions are now discussed. It will be convenient to define the following *zero-quantities*:

$$\begin{aligned}
&S_{ab} \equiv \nabla_a \xi_b + \nabla_b \xi_a - 2\eta g_{ab}, \\
&S_{abc} \equiv \nabla_a S_{ab}, \\
&\phi \equiv \xi^a \nabla_a \Xi - \Xi\eta, \\
&\psi \equiv \eta s + \xi^a \nabla_a s - \nabla_a \eta \nabla^a \Xi, \\
&B_{ab} \equiv \mathcal{L}_\xi L_{ab} + \nabla_a \nabla_b \eta,
\end{aligned}$$

with $\mathcal{L}_\xi$ denoting the *Lie derivative* along the direction of ξ^a. Recall that

$$\mathcal{L}_\xi L_{ab} = \xi^c \nabla_c L_{ab} + L_{cb} \nabla_a \xi^c + L_{ac} \nabla_b \xi^c.$$

In terms of these quantities, a lengthy computation leads to the following result proved in [15]:

Lemma 3. *Let ξ^a and η be a pair of fields satisfying equations* (7a)-(7b). *One then has that the tensor fields*

$$S_{ab}, \quad S_{abc}, \quad \phi, \quad ,\psi, \quad B_{ab},$$

satisfy a closed system of homogeneous wave equations. Schematically one has that

$$\square S_{ab} = H_{ab}(S, B), \tag{8a}$$
$$\square S_{abc} = H_{abc}(S, B, \nabla S, \nabla B), \tag{8b}$$
$$\square \phi = H(\phi, \psi, S), \tag{8c}$$
$$\square \psi = K(\phi, S, B, \psi, \nabla\phi), \tag{8d}$$
$$\square B_{ab} = K_{ab}(S, B, \nabla S, \nabla B, \nabla^2 S). \tag{8e}$$

Remark 4. In what follows the system consisting of equations (7a)-(7b) together with (8a)-(8e) will be called the *unphysical Killing equation propagation system.*

The homogeneity of the unphysical Killing equation evolution system (8a)-(8e) together with the theory of initial-boundary value problems for systems of wave equations (see e.g. [2, 5]) suggests to consider a Dirichlet problem to ensure the existence of a solution to the unphysical Killing vector equations. Let $\mathcal{S}_\star$ be an initial spacelike hypersurface. The conditions for the problem are:

(i) Initial data

$$S_{ab} = 0, \quad S_{abc} = 0, \quad \phi = 0, \quad \psi = 0, \quad B_{ab} = 0, \tag{9a}$$
$$\nabla_e S_{ab} = 0, \quad \nabla_e S_{abc} = 0, \quad \nabla_e \phi = 0, \quad \nabla_e \psi = 0, \quad \nabla_e B_{ab} = 0, \qquad \text{on} \quad \mathcal{S}_\star; \tag{9b}$$

(ii) (Dirichlet) boundary data

$$S_{ab} = 0, \quad S_{abc} = 0, \quad \phi = 0, \quad \psi = 0, \quad B_{ab} = 0, \qquad \text{on} \quad \mathscr{I}.$$

If the above conditions are satisfied, the homogeneity of wave equations (8a)-(8e) guarantees that the only solution of the system is the trivial one. This means, therefore, that the solution to equations (7a)-(7b) will actually be an unphysical Killing vector.

Remark 5. Strictly speaking, the initial conditions require only the vanishing of the zero-quantities and of their normal derivatives to the initial hypersurface. If these conditions hold then the full covariant derivative of the zero-quantities vanish initially and conversely.

4 The conformal constraint equations

In order to investigate conditions for the Dirichlet problem, we recall that the conformal Einstein equations impose some restrictions on the conformal boundary. In this context a $3+1$ decomposition arises as a natural approach to face the problem.

4.1 The 3 + 1 decomposition of the conformal field equations

Let $\mathcal{H} \subset \mathcal{M}$ be a 3-dimensional hypersurface with normal vector n_a. The hypersurface $\mathcal{K}$ is endowed with a metric k_{ab} [1] related to the spacetime one via:

$$k_{ab} = g_{ab} - \epsilon n_a n_b,$$

where $\epsilon \equiv n_a n^a$ take either the value 1 if $\mathcal{K}$ is timelike or -1 if it is spacelike. The nilpotent operator $k_a{}^b$ effectively projects spacetime objects into $\mathcal{K}$. Moreover, it induces a decomposition of the covariant derivative via the relation

$$\nabla_a = k_a{}^b \nabla_b + \epsilon n_a n^b \nabla_b \equiv D_a + \epsilon n_a D.$$

Here, D_a is the covariant derivative intrinsic to $\mathcal{K}$ which satisfies the metric compatibility condition $D_a k_{bc} = 0$, and D corresponds to the derivative in the normal direction. Additionally, the intrinsic curvature associated to $\mathcal{K}$, denoted by K_{ab}, can be conveniently expressed in terms of the acceleration $a_b \equiv n^c \nabla_c n_b$ as

$$\nabla_a n_b = K_{ab} + n_a a_b.$$

The fields appearing in the conformal Einstein field equations can be naturally decomposed using the projector $k_a{}^b$. Relevant for the subsequent work, let

$$\Sigma, \quad s, \quad k_{ab}, \quad \theta_a, \quad \theta_{ab}, \quad d_{ab}, \quad d_{abc}, \quad d_{abcd}$$

denote, respectively, the pull-backs of

$$n^a \nabla_a \Xi, \quad s, \quad g_{ab}, \quad n^c k_a{}^d L_{cd}, \quad k_a{}^c k_b{}^d L_{cd}, \quad n^b n^d k_e{}^a k_f{}^c d_{abcd}, \quad n^b k_e{}^a k_f{}^c k_g{}^d d_{abcd}$$

to $\mathcal{K}$.

Remark 6. The fields d_{ab} and d_{abc} represent, respectively, the *electric* and *magnetic parts* of the rescaled Weyl tensor d_{abcd} with respect to the normal n_a. The following properties can be verified:

$$d_a{}^a = 0, \quad d_{ab} = d_{ba}, \quad d_{abc} = -d_{acb}, \quad d_{[abc]} = 0.$$

4.2 The conformal constraint equations

When the Einstein field equations (3a)-(3e) are projected into an hypersurface $\mathcal{K}$ via $k_a{}^b$ the result is a system known as the *conformal constraint equations.* In terms of the quantities defined above, a long computation results in the system

$$D_i D_j \Omega = -\epsilon \Sigma K_{ij} - \Omega L_{ij} + s k_{ij}, \tag{10a}$$
$$D_i \Sigma = K_i{}^k D_k \Omega - \Omega L_i, \tag{10b}$$
$$D_i s = -\epsilon L_i \Sigma - L_{ik} D^k \Omega, \tag{10c}$$
$$D_i L_{jk} - D_j L_{ik} = -\epsilon \Sigma d_{kij} + D^l \Omega d_{lkij} - \epsilon (K_{ik} L_j - K_{jk} L_i), \tag{10d}$$
$$D_i L_j - D_j L_i = D^l \Omega d_{lij} + K_i{}^k L_{jk} - K_j{}^k L_{ik}, \tag{10e}$$
$$D^k d_{kij} = \epsilon \big(K^k{}_i d_{jk} - K^k{}_j d_{ik} \big), \tag{10f}$$
$$D^i d_{ij} = K^{ik} d_{ijk}, \tag{10g}$$
$$\lambda = 6\Omega s - 3\epsilon \Sigma^2 - 3 D_k \Omega D^k \Omega. \tag{10h}$$

[1] In this work, intrinsic 3-dimensional objects will be regarded as living on the spacetime, so they will be denoted using Latin indices.

Additionally, these are supplemented by the conformal versions of the Codazzi-Mainardi and Gauss-Codazzi equations which. These are, respectively:

$$D_j K_{ki} - D_k K_{ji} = \Omega d_{ijk} + k_{ij} L_k - k_{ik} L_j, \tag{11a}$$

$$l_{ij} = -\epsilon\Omega d_{ij} + L_{ij} + \epsilon\Big(K\big(K_{ij} - \frac{1}{4}K k_{ij}\big) - K_{ki}K_j{}^k + \frac{1}{4}K_{kl}K^{kl}k_{\boldsymbol{ij}} \Big). \tag{11b}$$

Here, l_{ab} is the 3–dimensional Schouten tensor, which is given in terms of the associated Ricci tensor and scalar r_{ab} and r, respectively, by:

$$l_{ab} \equiv r_{ab} - \frac{1}{4} r k_{ab}.$$

A detailed derivation of this equations, as well as a discussion about some of their properties, can be found in [16]. In the following it will be shown that, under a gauge choice, this system enables us to analyse the conformal boundary in a more manageable way.

4.2.1 The conformal constraints on $\mathscr{I}$

Hereafter, $\simeq$ will denote equality at the conformal boundary $\mathscr{I}$. When the constraints (10a)-(10h), along with (11a) and (11b) are evaluated on $\mathscr{I}$ —for which $\epsilon = 1$—, they take a particularly simple form as, by definition, the conformal factor identically vanishes. It follows that the constraint on $\mathscr{I}$ are:

$$s\ell_{ab} \simeq \Sigma K_{ab}, \tag{12a}$$

$$D_a \Sigma \simeq 0, \tag{12b}$$

$$D_a s \simeq -\Sigma\theta_a, \tag{12c}$$

$$D_a\theta_{bc} - D_b\theta_{ac} \simeq -\Sigma d_{cab} + (K_{bc}\theta_a - K_{ac}\theta_b), \tag{12d}$$

$$D_a\theta_b - D_b\theta_a \simeq K_a{}^c\theta_{bc} - K_b{}^c\theta_{ac}, \tag{12e}$$

$$D^c d_{cab} \simeq K^a{}_b d_{ac} - K^c{}_a d_{bc}, \tag{12f}$$

$$D^a d_{ab} \simeq K^{ac} d_{abc}, \tag{12g}$$

$$\lambda \simeq -3\Sigma^2, \tag{12h}$$

$$D_b K_{ac} - D_c K_{ab} \simeq \ell_{ab}\theta_c - \ell_{ac}\theta_b, \tag{12i}$$

$$l_{ab} \simeq \theta_{ab} + K\big(K_{ab} - \frac{1}{4}K\ell_{ab}\big) - K_{ac}K_b{}^c + \frac{1}{4}K_{cd}K^{cd}\ell_{ab}. \tag{12j}$$

In [6], an approach to find a solution of the above system has been given. The main characteristic of this method resides in considering s as a gauge quantity. Such result can be enunciated as follows:

Proposition 4. *Given a 3-dimensional Lorentzian metric ℓ_{ab}, a smooth function $\varkappa$ and a symmetric field d_{ab} satisfying $\ell^{ab}d_{ab} = 0$ and $D^a d_{ab} = 0$, then the following fields are a solution to the conformal constraint equations* (12a)-(12j) *on $\mathscr{I}$:*

$$\Sigma \simeq \sqrt{\frac{|\lambda|}{3}}, \tag{13a}$$

$$s \simeq \varkappa\Sigma, \tag{13b}$$

$$K_{ab} \simeq \varkappa\ell_{ab}, \tag{13c}$$

$$\theta_a \simeq -D_a\varkappa, \tag{13d}$$

$$\theta_{ab} \simeq l_{ab} - \frac{1}{2}\varkappa^2\ell_{ab}, \tag{13e}$$

$$d_{abc} \simeq -\Sigma^{-1} y_{abc}, \tag{13f}$$

where $y_{abc} \equiv (D_b l_{ca} - D_c l_{ba})$ is the Cotton tensor of the metric $\boldsymbol{\ell}$.

It will be seen that, under a particular gauge, this result eases the interpretation of the conditions imposed on the Killing vector candidate by the vanishing boundary data.

5 Decomposition of the zero-quantities

The 3+1 decomposition as described in the previous section if also key to study the zero-quantities associated to the Killing vector equation on a given hypersurface $\mathcal{K}$. In this respect, let define the following relevant quantities:

$$\zeta_a, \quad \zeta, \quad \mathcal{S}_{ab}, \quad \mathcal{S}_a, \quad \mathcal{S}, \quad \mathcal{S}_{abc}, \quad \mathcal{B}_{ab} \quad \mathcal{B}_a, \quad \mathcal{B}$$

as the respective the pull-backs of the following projections of the Killing vector candidate ξ_a and the zero-quantities into $\mathcal{H}$:

$$k_a{}^b\xi_b, \quad n^a\xi_a, \quad k_a{}^ck_b{}^dS_{cd}, \quad n^ck_a{}^bS_{bc}, \quad n^an^bS_{ab}, \quad k_a{}^dk_b{}^ek_c{}^fS_{def},$$
$$k_a{}^ck_b{}^dB_{cd}, \quad n^ck_a{}^bB_{bc}, \quad n^an^bB_{ab}.$$

The latter will allow us to decompose the various zero-quantities. In this way, one can systematically extract the consequences of the vanishing of the zero-quantities at both $\mathcal{S}_\star$ and $\mathscr{I}$.

Remark 7. As mentioned in Section 3.3, the initial data for the wave equations (8a)-(8e) requieres the vanishing of not only the zero-quantities at the initial hypersurface but also the vanishing of their first order covariant derivatives. Given that we can decompose ∇_a in terms of intrinsic and normal operators, then if the zero-quantities vanish initially so will all their intrinsic derivatives. Thus, the subsequent analysis only needs to consider normal derivatives.

5.1 Decomposition of ϕ and ψ

From their definitions, a straightforward decomposition of the zero-quantities ϕ and ψ and their normal derivatives leads to the following expressions:

$$\phi = \zeta^aD_a\Xi + \epsilon\zeta\Sigma - \eta\Xi, \tag{14a}$$
$$n^a\nabla_a\phi = -\eta\Sigma - \Xi D\eta + D\zeta^aD_a\Xi + \zeta^a(D_a\Sigma - K_a{}^cD_c\Xi) + \epsilon(\zeta D\Sigma + \Sigma D\zeta), \tag{14b}$$

and

$$\psi = \eta s + \zeta^aD_as + \epsilon\zeta Ds - D_a\eta D^a\Xi - \epsilon\Sigma D\eta, \tag{15a}$$
$$\begin{aligned} n^a\nabla_a\psi = {} & \eta Ds + sD\eta + D\zeta^aD_as + \zeta^a(D_aDs - K_a{}^bD_bs) - D^a\eta(D_a\Sigma - K_a{}^bD_b\Xi) \\ & -D_a\Xi(D^aD\eta - K^a{}_bD^b\eta) + \epsilon(\zeta D^2s + D\zeta Ds - D\Sigma D\eta - \Sigma D^2\eta). \end{aligned} \tag{15b}$$

5.2 Decomposition of S_{ab} and B_{ab} and their derivatives

Before decomposing the remaining zero-quantities some observations will be made about the redundancy of some of them. For this task their explicit decompositions will not be required but expressions will be given in terms of functions which are homogeneous in some zero quantities; this will be useful when imposing the vanishing initial-boundary data.

Lemma 4. *The components of DS_{ab}, B_{ab} as well as S_{abc} can be written as homogeneous functions of S_{ab}, $D\mathcal{S}_{ab}$, $\mathcal{B}_{ab}$ and $D\mathcal{B}_{ab}$.*

Proof. In the following, for ease of presentation, let f denote a generic homogeneous function of its arguments which may change from line to line. As pointed out in [15], equation (7a) implies the identity

$$\nabla_aS_b{}^a - \frac{1}{2}\nabla_bS_a{}^a = 0. \tag{16}$$

Expressing S_{ab} in terms of its components, a short calculation yields

$$\epsilon D\mathcal{S}_b + \frac{1}{2}n_bD\mathcal{S} = f(\mathcal{S}_{ab},\ D_a\mathcal{S}_{bc},\ D\mathcal{S}_{ab}). \tag{17}$$

Here f is homogeneous in its arguments. Multiplying this equation by $k_a{}^b$, an equation for $D\mathcal{S}_a$ is obtained. Similarly, multiplying equation (17) by n^b we obtain an analogous expression for

$D\mathcal{S}$. Then, all the components of DS_{ab} can be computed. This in turns, enables us to determine S_{abc} as, by definition, can be completely expressed in terms of D_cS_{ab} and DS_{ab}.

In order to analyse the fields derived from B_{ab}, consider equation (8a) which can be written in a slightly more explicit way as:

$$D^2 S_{ab} = -4\epsilon B_{ab} + f(S_{ab},\ \nabla_c S_{ab},\ D_c D_d S_{ab}). \tag{18}$$

As it is assumed that $\mathcal{B}_{ab}$ is known on $\mathcal{H}$, then one can solve for $D^2\mathcal{S}_{ab}$ from this last equation; in particular, $D^2\mathcal{S}_a{}^a$ can be computed. On the other hand, applying ∇_c to (16), a lengthy but direct decomposition leads to the following two relations:

$$D^2\mathcal{S}_a = f(S_{ab},\ \nabla_c S_{ab}), \tag{19a}$$
$$\epsilon D^2\mathcal{S} - D^2\mathcal{S}_a{}^a = f(S_{ab},\ \nabla_c S_{ab}). \tag{19b}$$

so we observe that the data on $\mathcal{H}$ enables us to obtain the components $D^2\mathcal{S}_a$ and $D^2\mathcal{S}$. Then, (18) implies that the components $\mathcal{B}_a$ and $\mathcal{B}$ can be computed.

Regarding the normal derivatives of B_{ab}, we make use of the identity

$$\nabla_a B_b{}^a - \tfrac{1}{2}\nabla_b B_a{}^a = S_{cd}(\nabla^c L_b{}^d - \tfrac{1}{2}\nabla_b L^{cd}),$$

whose validity is guaranteed by equations (7b) and (7a). Observe that its left hand side has the same form as equation (16), while its right hand side is homogeneous on S_{ab} —which is already known. Then we conclude that $D\mathcal{B}_a$ and $D\mathcal{B}$ are computable.

Finally, the normal derivative of S_{abc} can be analysed from its definition. Commuting derivatives, a short calculation yields:

$$DS_{abc} = D_a(DS_{bc}) + \epsilon n_a D^2 S_{bc} + f(S_{ab},\ \nabla_c S_{ab}).$$

Since it has been proved that all the terms are either computable or part of the given data, the proof is complete.

□

Remark 8. Lemma 4 is valid either for a spacelike or timelike hypersurface, but given that it assumes certain normal derivatives, it is naturally adapted to a spacelike one. If $\mathcal{K}$ is timelike, then $D\mathcal{S}_{ab}$ plays the role of the only necessary component of S_{abc}, while $D\mathcal{B}_{ab}$ is not required.

In view of the previous result, the explicit form of the remaining independent data under a decomposition on $\mathcal{K}$ is given by:

$$\mathcal{S}_{ab} = D_a\zeta_b + D_b\zeta_a + 2\epsilon\zeta K_{ab} - 2\eta k_{ab}, \tag{20a}$$
$$\mathcal{S}_a = D\zeta_a + D_a\zeta - \zeta^b K_{ab}, \tag{20b}$$
$$\mathcal{S} = 2D\zeta - 2\epsilon\eta, \tag{20c}$$
$$\begin{aligned} D\mathcal{S}_{ab} = {} & 2D_{(a}D\zeta_{b)} - 2K_{(a}{}^c D_{|c|}\zeta_{b)} + 2\zeta^c D_c K_{ab} - 2\zeta^c D_{(a}K_{b)c} + 2\epsilon\zeta D K_{ab} \\ & +2\epsilon K_{ab} D\zeta - 2k_{ab} D\eta, \end{aligned} \tag{20d}$$
$$\mathcal{B}_{ab} = \zeta^c D_c\theta_{ab} + 2\theta_{c(a}D_{b)}\zeta^c + 2\epsilon\zeta K_{(a}{}^c\theta_{b)c} + 2\epsilon\theta_{(a}D_{b)}\zeta + \epsilon\zeta D\theta_{ab} + D_a D_b\eta, \tag{20e}$$
$$\begin{aligned} D\mathcal{B}_{ab} = {} & D_c\theta_{ab}D\zeta^c + K_c{}^e D_e\theta_{ab} + D_c D\theta_{ab} + 2n^e\theta_{(a}{}^d R_{b)dec} + 2K_{(a}{}^c\theta_{b)c}D\zeta \\ & +2\zeta K_{(a}{}^c D\theta_{b)c} + \zeta\theta_{c(a}DK_{b)}{}^c + 2D_{(b}\zeta^c D\theta_{a)c} + 2\theta_{c(a}(D_b D\zeta^c - K_{b)}{}^e D_e\zeta^c \\ & +n^e R_{b)ed}{}^c\zeta^d) + 2D\theta_{(a}D_{b)}\zeta + 2\theta_{(a}D_{b)}D\zeta - 2\theta_{(a}K_{b)}{}^c D_c\zeta + 2\zeta D^2\theta_{ab} \\ & +2D\zeta D\theta_{ab} + D_a D_b D\eta - 2K_{(a}{}^e D_{b)}D_e\eta - D_e\eta D_a K_b{}^e - n^e R_{aeb}{}^d D_b\eta. \end{aligned} \tag{20f}$$

6 Boundary analysis

The aim of this section is to discuss the explicit requirements a well-posed initial-boundary problem with vanishing Dirichlet data impose on the conformal Killing vector candidate and the related quantities. As a result of this analysis it will be shown that some components cannot be freely chosen either on $\mathscr{I}$.

6.1 Zero-quantities on $\mathscr{I}$

In this subsection we study the decomposition for the zero-quantities associated to the Dirichlet boundary conditions for the Killing vector equation evolution system. As mentioned in Remark 8, the independent data on $\mathscr{I}$ are given by ϕ, ψ, $\mathcal{S}_{ab}$, $D\mathcal{S}_{ab}$ and $\mathcal{B}_{ab}$. Evaluating equations (14a), (15a) and (20a)-(20e) on $\mathscr{I}$ one obtains

$$\phi \simeq \Sigma\zeta, \tag{21a}$$
$$\psi \simeq \eta s + \zeta^a D_a s + \zeta D s - \Sigma D\eta, \tag{21b}$$
$$\mathcal{S}_{ab} \simeq D_a\zeta_b + D_b\zeta_a + 2\varkappa\zeta\ell_{ab} - 2\eta\ell_{ab}, \tag{21c}$$
$$\mathcal{S}_a \simeq D_a\zeta + D\zeta_a - \varkappa\zeta_a, \tag{21d}$$
$$\mathcal{S} \simeq 2D\zeta - 2\eta, \tag{21e}$$
$$\begin{aligned} D\mathcal{S}_{ab} = {} & 2D_{(a}D\zeta_{b)} - 2\varkappa D_{(a}\zeta_{b)} + 2\ell_{ab}\zeta^c D_c\varkappa - 2\zeta_{(a}D_{b)}\varkappa + 2\zeta DK_{ab} \\ & + 2\varkappa\ell_{ab}D\zeta - 2\ell_{ab}D\eta, \end{aligned} \tag{21f}$$
$$\mathcal{B}_{ab} = \zeta^c D_c l_{ab} + 2l_{c(a}D_{b)}\zeta^c + 2\varkappa\zeta l_{ab} - 2D_{(a}\varkappa D_{b)}\zeta + \zeta D\theta_{ab} + D_aD_b\eta. \tag{21g}$$

Imposing Dirichlet vanishing data on $\mathscr{I}$, equations (21a)-(21g) provide a number of conditions for the fields and their derivatives on the conformal boundary. Using the definition of η and the result of Proposition 4 it follows that the set of independent conditions is given by:

$$\zeta \simeq 0, \tag{22a}$$
$$D\zeta_a \simeq \varkappa\zeta_a, \tag{22b}$$
$$D_a\zeta_b + D_b\zeta_a \simeq 2\eta\ell_{ab}, \tag{22c}$$
$$D\eta \simeq \eta\varkappa + \zeta^c D_c\varkappa, \tag{22d}$$
$$\mathcal{L}_\zeta l_{ab} + D_aD_b\eta \simeq 0. \tag{22e}$$

Conversely, it is straightforward to check that equations (22a)-(22e) are sufficient to guarantee the vanishing of the equations (21a)-(21g). The above discussion leads to the following proposition:

Proposition 5. *Let $(\mathcal{M}, g_{ab})$ be a conformal extension of an anti-de Sitter spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ with timelike conformal boundary $\mathscr{I}$. Let ξ^a be a conformal Killing vector field candidate and ϕ, ψ, S_{ab}, B_{ab} and S_{abc} be the corresponding zero-quantities. Then, the equations* (21a)-(21g) *vanish on if and only if the components ζ_a, ζ and η satisfy the conditions* (22a)-(22e).

Remark 9. Equations (22a)-(22e) will be called the *Killing boundary data.* They acquire a simpler form if one makes use of a gauge for which $\varkappa = 0$.

6.2 Existence of the intrinsic conformal Killing vector

As stated in Proposition 5, one of the necessary conditions under which the set of zero-quantities vanish on $\mathscr{I}$ is given by (22c) —i.e. the transversal component ζ_a of the conformal Killing vector candidate has to be a conformal Killing vector with respect to the connection D_a. In order to guarantee the existence of a solution to this equation we consider an initial value problem on $\mathscr{I}$. Following the model of the spacetime problem, we construct a suitable wave equation for ζ_a. More precisely, one has that

Lemma 5. *Let ζ_a and η a pair of fields satisfying the conformal Killing equation* (22c) *and* (22e) *on $\mathscr{I}$. Then, it follows that*

$$\Delta\zeta_a \simeq -r_a{}^b\zeta_b - D_a\eta, \tag{23a}$$
$$\Delta\eta \simeq -\tfrac{1}{2}\eta r - \tfrac{1}{4}\zeta^b D_b r, \tag{23b}$$

where $\Delta \equiv \ell^{ab}D_aD_b$ is the D'Alambertian operator of the metric ℓ_{ab}.

Proof. The result is readily obtained by applying D^a to (22c) and taking the trace of (22e). $\square$

Remark 10. Given that this system of wave equations propagates η and ζ_a along the conformal boundary, it must be provided with initial data at the corner $\partial\mathcal{S} = \mathcal{S}_\star \cup \mathscr{I}$, where $\mathcal{S}_\star \subset \mathcal{M}$ is some initial spacelike hypersurface.

To prove that a solution to these wave equations also solves the conformal Killing equation on the boundary, a suitable system of wave equations for the corresponding 3-dimensional zero-quantities has to be constructed. The desired relations are contained in the following lemma:

Lemma 6. *Let $\mathcal{S}_{ab}$, $\mathcal{S}_{abc}$ and $\mathcal{B}_{ab}$ be the projections of the zero-quantities into $\mathscr{I}$. Assume that there exist fields ζ_a and η on $\mathscr{I}$ satisfying the wave equations* (23a) *and* (23b) *in Lemma 5. Then, one has that*

$$
\begin{aligned}
&\Delta\mathcal{S}_{ab} \simeq l_b{}^c\mathcal{S}_{ac} + l_a{}^c\mathcal{S}_{bc} - 2r_{acbd}\mathcal{S}^{cd} - 2\mathcal{B}_{ab}\\
&\Delta\mathcal{S}_{eab} \simeq r_e{}^c\mathcal{S}_{cab} - 2r_{bdec}\mathcal{S}^c{}_a{}^d - 2r_{adec}\mathcal{S}^c{}_b{}^d - \tfrac{1}{2}r\mathcal{S}_{eab} + r_b{}^c\mathcal{S}_{eac} + r_a{}^c\mathcal{S}_{ebc} - 2r_{acbd}\mathcal{S}_e{}^{cd}\\
&\qquad\qquad +\mathcal{S}_b{}^cD_ar_{ec} + \mathcal{S}_a{}^cD_br_{ec} - \mathcal{S}_b{}^cD_cr_{ae} - \mathcal{S}_a{}^cD_cr_{be} + \mathcal{S}_b{}^cD_er_{ac} + \mathcal{S}_a{}^cD_er_{bc}\\
&\qquad\qquad -\tfrac{1}{2}\mathcal{S}_{ab}D_er - 2\mathcal{S}^{cd}D_er_{acbd} - 2D_e\mathcal{B}_{ab}\\
&\Delta\mathcal{B}_{ab} \simeq \mathcal{O}_{ab} + f(\mathcal{B}_{ab},\ \mathcal{S}_{ab},\ \mathcal{S}_{abc}),
\end{aligned}
$$

where

$$
\mathcal{O}_{ab} \equiv \mathcal{L}_\zeta D_c y_a{}^c{}_b + 2\eta D_c y_a{}^c{}_b + 2D_c\eta y_{(a}{}^c{}_{b)} \tag{25}
$$

and f is a homogeneous function of its arguments.

Proof. The wave equations for $\mathcal{S}_{ab}$ and $\mathcal{S}_{abc}$ are obtained by direct calculation. For the zero–quantity $\mathcal{B}_{ab}$ we have the two following identities:

$$
\begin{aligned}
&D_a\mathcal{B}_b{}^a \simeq \tfrac{1}{2}l^{ac}\mathcal{S}_{bac} + \mathcal{S}^{ac}D_cl_{ba},\\
&D_a\mathcal{B}_{bc} \simeq \tfrac{1}{2}\theta_c{}^d\mathcal{S}_{abd} + \tfrac{1}{2}\theta_c{}^d\mathcal{S}_{bad} - \tfrac{1}{2}\theta_a{}^d\mathcal{S}_{bcd} - \tfrac{1}{2}\theta_a{}^d\mathcal{S}_{cbd} - \tfrac{1}{2}\theta_c{}^d\mathcal{S}_{dab} + \tfrac{1}{2}\theta_a{}^d\mathcal{S}_{dbc} + d_{bcd}\Sigma D_a\zeta^d\\
&\qquad\qquad -d_{dac}\Sigma D_b\zeta^d - d_{bad}\Sigma D_c\zeta^d + D_c\mathcal{B}_{ab} - \zeta^d\Sigma D_ad_{bac}.
\end{aligned}
$$

Applying the D^a operator to the latter expression and then using the former one, as well as using the Bianchi identities, one has that:

$$
\begin{aligned}
\Delta\mathcal{B}_{ab} &\simeq 2\eta\Sigma D_cd_{ab}{}^c + \Sigma\zeta^eD_eD_cd_{ab}{}^c - \Sigma d_{cab}D^c\eta + 2\Sigma d_{bac}D^c\eta + \Sigma D_a\zeta^cD_ed_{cb}{}^e\\
&\quad +\Sigma D_c\zeta^bD_ed_{ac}{}^e + f(\mathcal{B}_{ab},\ \mathcal{S}_{ab},\ \mathcal{S}_{abc})\\
&\simeq -\mathcal{L}_\zeta D_cy_{ab}{}^c - 2\eta D_cy_{ab}{}^c - 2D^c\eta y_{abc} - D^c\eta y_{cab} + f(\mathcal{B}_{ab},\ \mathcal{S}_{ab},\ \mathcal{S}_{abc})\\
&\simeq \mathcal{L}_\zeta D_cy_a{}^c{}_b + 2\eta D_cy_a{}^c{}_b + D_c\eta(y_a{}^c{}_b + y_b{}^c{}_a) + f(\mathcal{B}_{ab},\ \mathcal{S}_{ab},\ \mathcal{S}_{abc}).
\end{aligned}
$$

$\square$

Remark 11. The system of wave equations in the previous lemma is homogeneous in the zero quantities $\mathcal{S}_{ab}, \mathcal{S}_{abc}$ and $\mathcal{B}_{ab}$ as long as the *obstruction tensor* $\mathcal{O}_{ab}$ vanishes identically on $\mathscr{I}$.

Remark 12. *If $\mathscr{I}$ is conformally flat, then the obstruction vanishes identically as $y_{abc} = 0$.*

Lemmas 5 and 6 lead to the following proposition:

Proposition 6. *Let $(\mathcal{M}, g_{ab})$ a conformal extension of an anti-de Sitter-like spacetime with corner $\partial\mathcal{S} \equiv \mathcal{S}_\star \cap \mathscr{I}$. Let ζ_a, η and y_{abc} be vector fields on $\mathscr{I}$ with y_{abc} having the symmetries of the magnetic part of the Weyl tensor. Assume that $\mathcal{S}_{ab}$, $\mathcal{B}_{ab}$ and $\mathcal{S}_{abc}$ vanish identically on $\partial\mathcal{S}$. Then ζ_a satisfies the unphysical conformal Killing equation on $\mathscr{I}$ if and only if $\mathcal{O}_{ab} \simeq 0$.*

Remark 13. It should be stressed that the analysis carried out in the previous sections is conformally invariant. More precisely, if the unphysical Killing vector candidate is such that the zero-quantities associated to the Killing equation conformal evolution system vanish for a particular conformal representation, then it follows that they will also vanish for any other conformal representation. This follows from the conformal transformation properties for the zero-quantities implied by the change of connection transformation formulae. From this observation it follows also that the reduced Killing boundary conditions (22a)-(22e) have similar conformal invariance properties.

7 Initial data at $\partial\mathcal{S}$

As mentioned in Remark 10, the system (23a)-(23b) must be complemented with data at $\partial\mathcal{S}$, that is to say, we have to bring into consideration the conditions implied by the zero quantities on $\mathcal{S}_\star$ and make them consistent with the ones obtained from the boundary analysis in the previous section. The main difference between this section and the preceding ones is the introduction of an adapted system of coordinates suited for studying the corner conditions.

7.1 Set up

For simplicity, let us introduce a system of coordinates $x^\mu = (x^0, x^1, x^\mathcal{A})$ where x^0 and x^1 correspond to the time and radial coordinates, respectively, while the caligraphic index $\mathcal{A}$ represents angular coordinates. This system of coordinates is adapted to our problem in the sense that $\mathcal{S}_\star$ and $\mathscr{I}$ are given by

$$\mathcal{S}_\star = \{p \in \mathcal{M} \mid x^0 = 0\} \quad \text{and} \quad \mathscr{I} = \{p \in \mathcal{M} \mid x^1 = 0\}.$$

The corner is determined then by the condition $x^0 = x^1 = 0$.

Let h_{ab} be the intrinsic metric on $\mathcal{S}_\star$ and t^a be its the normal vector. As the hypersurface $\mathcal{S}_\star$ is spacelike then $t_a t^a = -1$. For convenience, let use the symbol $\hat{}$ to denote quantities defined on this hypersurface. Following this condition $\hat{D}_a$ and $\hat{D}$ denote, respectively, the intrinsic and normal covariant derivatives on $\mathcal{S}_\star$.

Once coordinates have been introduced, the metrics can be written explicitly in terms of the lapse and shift functions. Adopting a Gaussian gauge, the metrics on $\mathcal{S}_\star$ and $\mathscr{I}$ take, respectively, the form

$$\boldsymbol{g}|_{\mathcal{S}_\star} = -\mathbf{d}x^0 \otimes \mathbf{d}x^0 + h_{\gamma\delta}\mathbf{d}x^\gamma \otimes \mathbf{d}x^\delta, \quad (\gamma, \delta = 1, 2, 3) \tag{28a}$$
$$\boldsymbol{g} \simeq \mathbf{d}x^1 \otimes \mathbf{d}x^1 + \ell_{ij}\mathbf{d}x^i \otimes \mathbf{d}x^j \quad (i, j = 0, 2, 3). \tag{28b}$$

From here, we find that the non-zero components of the metric at the corner $\partial\mathcal{S}$ are:

$$g_{00} = \ell_{00} = -1, \quad g_{11} = h_{11} = 1, \quad g_{\mathcal{AB}} = h_{\mathcal{AB}} = \ell_{\mathcal{AB}}.$$

7.2 Corner conditions

As noticed in Remark 10, the wave equations (23a) and (23b) require suitable initial data at $\partial\mathcal{S}$. These are naturally provided by the conditions the initial data impose on η, ζ_a and their first derivatives along the conformal boundary. Here we describe how such conditions can be obtained.

Let $\hat{\zeta}_a$ and $\hat{\zeta}$ denote, respectively, the pull-backs of $h_a{}^b\xi_b$ and $t^a\xi_a$ into $\mathcal{S}_\star$. Although this decomposition with respect to h_{ab} is clearly different from the one performed on the conformal boundary we can observe that, when expressed in the adapted coordinates x^μ, the following identities hold at the corner:

$$\hat{\zeta}_1 = \zeta = 0, \qquad \hat{\zeta} = \zeta_0, \qquad \hat{\zeta}_\mathcal{A} = \zeta_\mathcal{A}.$$

In this way, the angular components $\zeta_\mathcal{A}$ on $\mathscr{I}$ are fixed by the initial data. Similarly, if one requires the conformal factor Ξ to have continuous first derivatives, it follows then that the conditions

$$\hat{\partial}_0\Xi = \partial_0\Xi = 0, \quad \hat{\partial}_1\Xi = \partial_1\Xi = \Sigma, \quad \hat{\partial}_\mathcal{A}\Xi = \partial_\mathcal{A}\Xi = 0.$$

must be satisfied at $\partial\mathcal{S}$.

A straightforward calculation shows that equations (14a)-(15b) provide with no relevant information. On the other hand, expression (20c) takes the form $\eta = -\partial_0\zeta_0$, where η is given once the equations on the initial hypersurface have been solved. An inspection of the zero-quantities on $\mathcal{S}_\star$ suggests that some pieces of the corner data are free. In particular, $\partial_0\eta$ can be solved from (20d) if ζ_0 is specified while equations (20a) and (20b) constraint the angular and time derivatives of ζ_i at $\partial\mathcal{S}$.

8 Conclusions

Once the conditions for the existence of a conformal Killing vector on $\mathscr{I}$ have been established, we can link Proposition 6 to the initial-boundary problem in the spacetime via Lemmas 2 and 3. The main result of this work can be formulated as follows:

Theorem 1. *Let* $(\mathcal{M}, \boldsymbol{g})$ *a conformal extension of an anti de Sitter-like spacetime with conformal boundary* $\mathscr{I}$. *Let* $\mathcal{S}_\star \subset \mathcal{M}$ *be a spacelike hypersurface intersecting* $\mathscr{I}$ *at* $\partial\mathcal{S}$. *Let* $\xi_{a\star}$ *and* $\eta_\star$ *satisfy the conformal KID equations* (9a) *and* (9b) *on* $\mathcal{S}_\star$. *Let* ζ_a *and* η *be the fields obtained from solving the wave equations* (23a) *and* (23b) *with initial data given by the restriction of* $\xi_{a\star}$ *and* $\eta_\star$ *to* $\partial\mathcal{S}$. *Assume further that the obstruction tensor* $\mathcal{O}_{ab}$ *constructed from* ℓ_{ab}, η *and* ζ_a *and defined by equation* (25) *vanishes. Then the Killing vector candidate* ξ_a *obtained from solving equations* (7a) *and* (7b) *with initial data* $\xi_{a\star}$, $\eta_\star$ *and boundary data* ζ_a, η *pull-backs to a Killing vector* $\tilde{\xi}_a$.

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