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Journal of Combinatorial Theory, Series B 00 (2018) 1–22

J. Comb. Theory, Series B

Point-hyperplane frameworks, slider joints, and rigidity preserving transformations

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Abstract

A one-to-one correspondence between the infinitesimal motions of bar-joint frameworks in $\mathbb{R}^d$ and those in $\mathbb{S}^d$ is a classical observation by Pogorelov, and further connections among different rigidity models in various different spaces have been extensively studied. In this paper, we shall extend this line of research to include the infinitesimal rigidity of frameworks consisting of points and hyperplanes. This enables us to understand correspondences between point-hyperplane rigidity, classical bar-joint rigidity, and scene analysis. Among other results, we derive a combinatorial characterization of graphs that can be realized as infinitesimally rigid frameworks in the plane with a given set of points collinear. This extends a result by Jackson and Jordán, which deals with the case when three points are collinear.

Keywords: infinitesimal rigidity, bar-joint framework, point-hyperplane framework, spherical framework, slider constraints.
2000 MSC: 52C25, 05B35, 68R10

1. Introduction

Given a collection of objects in a space satisfying particular geometric constraints, a fundamental question is whether the given constraints uniquely determine the whole configuration up to congruence. The rigidity problem for bar-joint frameworks in $\mathbb{R}^d$, where the objects are points, the constraints are pairwise distances and only local deformations are considered, is a classical example.

Pogorelov [16, Chapter V] observed that the space of infinitesimal motions of a bar-joint framework whose vertices are constrained to lie on a semi-sphere is isomorphic to those of the framework obtained by a central projection to Euclidean space. Since then, connections between various types of rigidity models in different spaces have been extensively studied, see, e.g., [1, 2, 8, 18, 19, 25, 26]. When talking about infinitesimal rigidity, these connections are often just consequences of the fact that infinitesimal rigidity is preserved by projective transformations. A key essence of the research is its geometric and combinatorial interpretations, which sometimes give us unexpected connections between theory and real applications.

[*] Supported by EPSRC First Grant EP/M013642/1.
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[1] Supported in part by NSERC, Canada.
[2] Supported by JSPS Postdoctoral Fellowships for Research Abroad and JSPS Grant-in-Aid for Scientific Research(A)(25240004).

In this paper we shall extend this line of research to include *point-hyperplane rigidity*. A *point-hyperplane framework* consists of points and hyperplanes along with point-point distance constraints, point-hyperplane distance constraints, and hyperplane-hyperplane angle constraints. The 2-dimensional point-line version has been considered, for example in [10, 15, 30], inspired by possible applications to CAD. We will show that the infinitesimal rigidity of a point-hyperplane framework is closely related to that of a bar-joint framework with *nongeneric* positions for some of its joints. Understanding the infinitesimal rigidity of such nongeneric bar-joint frameworks is a classical but still challenging problem, and our results give new insight into this problem.

Specifically, in Section 2 we establish a one-to-one correspondence between the space of infinitesimal motions of a point-hyperplane framework and that of a bar-joint framework with a given set of joints in the same hyperplane by extending the correspondence between Euclidean rigidity and spherical rigidity. Combining this with a result by Jackson and Owen [10] for point-line rigidity, we give a combinatorial characterization of a graph that can be realized as an infinitesimally rigid bar-joint framework in the plane with a given set of points collinear. This extends a result by Jackson and Jordán [9], which deals with the case when three points are collinear.

Let us denote the underlying graph of a point-hyperplane framework in $\mathbb{R}^d$ by $G = (V_P \cup V_L, E_{PP} \cup E_{PL} \cup E_{LL})$, where V_P and V_L represent the sets of points and hyperplanes, respectively. The edge set is partitioned into E_{PP}, E_{PL}, E_{LL} according to the bipartition $\{V_P, V_L\}$ of the vertex set. Each $i \in V_P$ is associated with $p_i \in \mathbb{R}^d$ while each $j \in V_L$ is associated with a hyperplane $\{x \in \mathbb{R}^d : \langle a_j, x\rangle + r_j = 0\}$ for some $a_j \in \mathbb{S}^{d-1}$ and $r_j \in \mathbb{R}$. We will see in Section 2.2 that the infinitesimal motions of the framework are given by the solutions of the following system of linear equations in $\dot{p}_i, \dot{a}_j, \dot{r}_j$:

$$\begin{aligned}
\langle p_i - p_j, \dot{p}_i - \dot{p}_j\rangle &= 0 && (ij \in E_{PP})\\
\langle p_i, \dot{a}_j\rangle + \langle \dot{p}_i, a_j\rangle + \dot{r}_j &= 0 && (ij \in E_{PL})\\
\langle a_i, \dot{a}_j\rangle + \langle \dot{a}_i, a_j\rangle &= 0 && (ij \in E_{LL})\\
\langle a_i, \dot{a}_i\rangle &= 0 && (i \in V_L).
\end{aligned}$$

If $V_L = \emptyset$, the system becomes that of a bar-joint framework on V_P in Euclidean space, while, if $V_P = \emptyset$, the system becomes that of a bar-joint framework on V_L in spherical space (i.e. a bar-joint framework on V_L whose vertices are constrained to lie on the sphere $\mathbb{S}^{d-1}$). See Section 2.1 for more details. Hence the system of point-hyperplane frameworks is a mixture of these two settings. Further detailed restrictions of the system enable us to link various types of rigidity models with point-hyperplane rigidity in Sections 3 and 4 of this paper.

- When $\dot{a}_j = 0$ ($j \in V_L$), the system models the case when the normal of each hyperplane is fixed. Such a rigidity model was investigated by Owen and Power [14] for $d = 2$. We show how to derive their combinatorial characterization in the plane from the result of Jackson and Owen [10].

  If $E_{PP} = E_{LL} = \emptyset$, we further point out a connection to the parallel drawing problem from scene analysis, and we derive a combinatorial characterization of graphs $G = (V_P \cup V_L, E_{PL})$ which can be realized as a fixed-normal rigid point-hyperplane framework in $\mathbb{R}^d$ using a theorem of Whiteley [27].

- When $\dot{r}_j = 0$ ($j \in V_L$), the system can model the case when concurrent hyperplanes can rotate around a common intersection point. We derive a characterization of graphs which can be realized as a rigid point-line framework in the plane in this rigidity model. By using the rigidity transformation established in Section 2, this result is translated to a characterization of the infinitesimal rigidity of bar-joint frameworks in the plane with horizontal slider-joints on a line. Our result allows us to put slider points anywhere on this line.

- When $\dot{a}_j = \dot{r}_j = 0$ ($j \in V_L$), the system models the case when each hyperplane is fixed. A combinatorial characterization of infinitesimal rigidity is derived for $d = 2$ by first transforming the point-line framework to a bar-joint framework (with nongeneric positions for its joints) and then applying a theorem by Servatius et al [20].

Point-line frameworks in the plane with different types of constraints imposed on the lines may be used to model structures in engineering with various types of slider-joints (e.g. linkages with prismatic joints in mechanical engineering), see [11, 13, 17, 22]. Indeed, the use of slider-joints in both mechanical and civil engineering provides a key motivation for our work. We will describe one example from [17] in more detail. Consider the ‘sliding pair chain’ shown in Figure 1(a) consisting of four rigid bodies (labelled B_1, B_2, B_3, B_4) connected at five slider joints (labeled $\ell_1, \ell_2, \ldots, \ell_5$). Each slider joint constrains the relative motion between its two incident bodies to be a translation in a direction determined by the orientation of the slider joint. We may model this system as a point-line framework, with each body represented by a 'bar' i.e. two points joined by a distance constraint, as indicated in Figure 1(b). We will see in Section 2.3 that this framework has at least one degree of freedom.

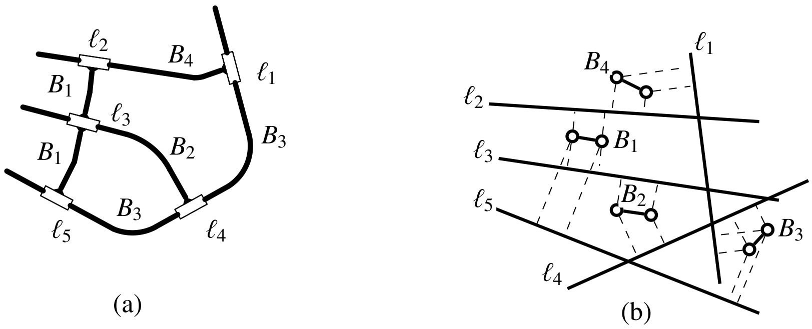


Figure 1: A 4-body sliding pair chain (a) that is modelled as a point-line framework (b). In (a) thick lines represent bodies and rectangles represent slider joints. In (b) a dashed line between a point and a line indicates a point-line distance constraint, and a solid line between two points indicates a point-point distance constraint.

2. Rigidity preserving transformations

In this section we explain how the rigidity of point-hyperplane frameworks is related to the rigidity of bar-joint frameworks on the sphere or in Euclidean space by using a rigidity preserving transformation.

We use $\mathbb{R}^d$ to denote d-dimensional Euclidean space equipped with the standard inner product $\langle\cdot,\cdot\rangle$, $\mathbb{S}^d$ to denote the unit d-dimensional sphere centered at the origin, and consider $\mathbb{S}^d \subset \mathbb{R}^{d+1}$. Let $\mathbf{e} \in \mathbb{R}^{d+1}$ be the vector with a one as its last coordinate and zeros elsewhere, and let $\mathbb{A}^d = \{x \in \mathbb{R}^{d+1} : \langle x, \mathbf{e}\rangle = 1\}$ be the hyperplane of $\mathbb{R}^{d+1}$ with $\mathbf{e} \in \mathbb{A}^d$ and with normal $\mathbf{e}$. We also use $\mathbb{S}^d_{>0} = \{x \in \mathbb{S}^d : \langle x, \mathbf{e}\rangle > 0\}$, $\mathbb{S}^d_{\geq 0} = \{x \in \mathbb{S}^d : \langle x, \mathbf{e}\rangle \geq 0\}$ and put $\mathbb{S}^d_{<0} = \mathbb{S}^d \setminus \mathbb{S}^d_{\geq 0}$. The *equator* of $\mathbb{S}^d$ is defined to be $\mathbb{S}^d_{\geq 0} \setminus \mathbb{S}^d_{>0}$ and is denoted by Q. In the following discussion, the last coordinate in $\mathbb{R}^{d+1}$ will have a special role (as one may expect from the definitions of $\mathbb{A}^d$ and $\mathbb{S}^d_{>0}$). Hence we sometimes refer to a coordinate of a point in $\mathbb{R}^{d+1}$ as a pair $(x, x') \in \mathbb{R}^d \times \mathbb{R}$, where x' denotes the last coordinate. For example, a point in $\mathbb{A}^d$ is denoted by $(x, 1)$ with $x \in \mathbb{R}^d$.

2.1. Euclidean space vs spherical space

It is a classical fact that there is a one-to-one correspondence between frameworks in $\mathbb{R}^d$ and those in $\mathbb{S}^d_{>0}$ at the level of infinitesimal motions. Since the transformation between these two spaces is the starting point of our study, we first give a detailed description of this transformation.

By a *framework in a space M* we mean a pair (G, p) of an undirected finite graph $G = (V, E)$ and a map $p : V \to M$. The most widely studied examples are frameworks (G, p) in $\mathbb{R}^d$, where p is a map from V to $\mathbb{R}^d$. We will assume henceforth that the points $p(V)$ affinely span $\mathbb{R}^d$ to simplify our discussion.

The most fundamental question for a framework (G, p) in $\mathbb{R}^d$ is whether there is a different framework (up to congruences) in some (arbitrarily small) neighborhood of p satisfying the same system of length constraints:

$$\|p_i - p_j\| = \text{const} \qquad (ij \in E).$$

Such a constrained framework is referred to as a *bar-joint framework* since it models the continuous motions of stuctures consisting of rigid bars attached to each other at universal joints.

A common strategy to answer this question is to take the derivative of the square of each length constraint to get the first-order length constraint,

$$\langle p_i - p_j, \dot{p}_i - \dot{p}_j\rangle = 0 \qquad (ij \in E), \tag{1}$$

and then check the dimension of the solution space with variables $\dot{p}$. We say that $\dot{p} : V \to \mathbb{R}^d$ is an *infinitesimal motion* of (G, p) if $\dot{p}$ satisfies (1), and (G, p) is called *infinitesimally rigid* if the dimension of the space of infinitesimal motions of (G, p) is equal to $\binom{d+1}{2}$, the dimension of the space of Euclidean motions in $\mathbb{R}^d$.

Less well-known but still widely appearing models of constrained frameworks are *bar-joint frameworks in* $\mathbb{S}^d$. We assume here that the points $p(V)$ of our framework (G, p) in $\mathbb{S}^d$ linearly span $\mathbb{R}^{d+1}$. In $\mathbb{S}^d$, both the Euclidean distance and the spherical distance between two points are determined by the inner product if the points. Hence we are interested in the solutions to the system of inner product constraints:

$$\langle p_i, p_j \rangle = \text{const} \qquad (ij \in E).$$

Since p_i is constrained to be on $\mathbb{S}^d$, we also have the extra constraints

$$\langle p_i, p_i \rangle = 1 \qquad (i \in V).$$

Again, taking the derivative, we can obtain the system of first-order inner product constraints:

$$\langle p_i, \dot{p}_j \rangle + \langle p_j, \dot{p}_i \rangle = 0 \qquad (ij \in E) \tag{2}$$

$$\langle p_i, \dot{p}_i \rangle = 0 \qquad (i \in V). \tag{3}$$

A map $\dot{p} : V \to \mathbb{R}^{d+1}$ is said to be an *infinitesimal motion* of (G, p) if it satisfies this system of linear constraints, and the framework (G, p) is *infinitesimally rigid* if the dimension of its space of infinitesimal motions is equal to $\binom{d+1}{2}$, the dimension of the space of $\mathrm{SO}(d + 1)$. For each $x \in \mathbb{S}^d$, let

$$T_x\mathbb{S}^d = \{m \in \mathbb{R}^{d+1} \mid \langle x, m \rangle = 0\}$$

be the tangent hyperplane at x. Then we may give an equivalent definition for an infinitesimal motion of (G, p) as a map $i \mapsto \dot{p}_i \in T_{p_i}\mathbb{S}^d$ which satisfies (2) for all $i \in V$.

In order to relate the rigidity models in $\mathbb{R}^d$ and $\mathbb{S}^d$, a key step is to identify $\mathbb{R}^d$ with the hyperplane $\mathbb{A}^d$ in $\mathbb{R}^{d+1}$. For a bar-joint framework (G, p) in $\mathbb{A}^d$, we define an infinitesimal motion as a map $i \mapsto \dot{p}_i \in T\mathbb{A}^d$ satisfying (1), where

$$T\mathbb{A}^d = \{m \in \mathbb{R}^{d+1} \mid \langle \mathbf{e}, m \rangle = 0\}$$

Then the space of infinitesimal motions $\dot{p}$ of a framework (G, p) in $\mathbb{R}^d$ coincides with the space of infinitesimal motions $\dot{\tilde{p}}$ of the framework $(G, \tilde{p})$ in $\mathbb{A}^d$, when we take $\tilde{p}_i = (p_i, 1)$ and $\dot{\tilde{p}}_i = (\dot{p}_i, 0)$ for all $i \in V$. Hence in the subsequent discussion we may consider the infinitesimal rigidity of frameworks in $\mathbb{A}^d$ rather than $\mathbb{R}^d$.

We can now describe the rigidity preserving transformation from $\mathbb{A}^d$ to $\mathbb{S}^d$. Let $\phi : \mathbb{A}^d \to \mathbb{S}^d_{>0}$ be the central projection, that is,

$$\phi(x) = \frac{x}{\|x\|} \qquad (x \in \mathbb{A}^d).$$

For each $x \in \mathbb{A}^d$, define $\psi_x : T\mathbb{A}^d \to T_{\phi(x)}\mathbb{S}^d$ by

$$\psi_x(m) = \frac{m - \langle m, x \rangle \mathbf{e}}{\|x\|} \qquad (m \in T\mathbb{A}^d).$$

The image of ψ_x indeed lies in $T_{\phi(x)}\mathbb{S}^d$ because

$$\langle \phi(x), \psi_x(m) \rangle = \frac{\langle x, m - \langle m, x \rangle \mathbf{e} \rangle}{\|x\|^2} = \frac{\langle x, m \rangle - \langle m, x \rangle}{\|x\|^2} = 0$$

where $\langle x, \langle m, x \rangle \mathbf{e} \rangle = \langle m, x \rangle$ follows from the fact that the last coordinate of $x \in \mathbb{A}^d$ is equal to one.

Given a framework (G, p) in $\mathbb{A}^d$ and an infinitesimal motion $\dot{p}$ of (G, p), a simple calculation shows that

$$\langle \phi(p_i), \psi_{p_j}(\dot{p}_j) \rangle + \langle \phi(p_j), \psi_{p_i}(\dot{p}_i) \rangle = -\frac{\langle p_i - p_j, \dot{p}_i - \dot{p}_j \rangle}{\|p_i\|^2 \|p_j\|^2} = 0$$

for all $ij \in E$, and hence $\psi(\dot{p}) := (\psi_{p_i}(\dot{p}_i))_{i \in V}$ is an infinitesimal motion of $(G, \phi \circ p)$ in $\mathbb{S}^d$. It is known that ψ_x is a bijective linear map for each $x \in \mathbb{A}^d$, and ψ is an isomorphism between the spaces of infinitesimal motions of (G, p) and $(G, \phi \circ p)$, see, e.g., [8, 18, 19]. This isomorphism is illustrated in Fig. 2. In particular, we have the following.

Proposition 2.1. *A bar-joint framework (G, p) is infinitesimally rigid in $\mathbb{A}^d$ if and only if $(G, \phi \circ p)$ is infinitesimally rigid in $\mathbb{S}^d_{>0}$.*

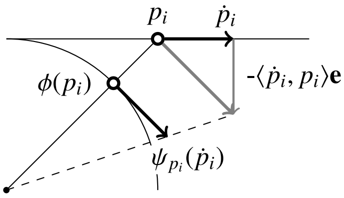


Figure 2: Transfer of infinitesimal motions between $\mathbb{A}^d$ and $\mathbb{S}^d_{>0}$.

In the next subsection, we will extend the correspondence between infinitesimally rigid bar-joint frameworks in $\mathbb{R}^d$ and $\mathbb{S}^d$ given in Proposition 2.1 further by allowing points to lie on the equator of the sphere. Note that, in the transformation described above, a point on the equator of $\mathbb{S}^d$ corresponds to a 'point at infinity' in $\mathbb{A}^d$.

2.2. Point-hyperplane vs bar-joint

The frameworks considered in Section 2.1 were *bar-joint frameworks*. A different kind of framework consisting of points and lines in $\mathbb{R}^2$ mutually linked by distance or angle constraints (see Figure 1(b) for example), usually referred to as a *point-line framework*, was introduced in [15]. A combinatorial characterization for generic rigidity of such frameworks was recently provided in [10]. We will consider the d-dimensional generalisation of these frameworks and refer to them as *point-hyperplane frameworks*. We will use the rigidity preserving transformation given in Section 2.1 to establish an equivalence (at the level of infinitesimal rigidity) between a point-hyperplane framework in $\mathbb{R}^d$ and a bar-joint framework in $\mathbb{R}^d$ in which a given set of joints lie on the same hyperplane. The idea is to first transform the point-hyperplane framework in $\mathbb{R}^d$ to a bar-joint framework in $\mathbb{S}^d$ so that the hyperplanes are mapped to points on the equator of $\mathbb{S}^d$. We then rotate the spherical framework to obtain a congruent spherical framework with no points on the equator. We can then transform the new spherical framework to a bar joint-framework in $\mathbb{R}^d$.

Formally, we define a *point-hyperplane framework* in $\mathbb{R}^d$ to be a triple (G, p, ℓ) where $G = (V_P \cup V_L, E)$ is a *point-hyperplane graph*, i.e. a graph G in which the vertices have been partitioned into two sets V_P, V_L corresponding to points and hyperplanes, respectively, and each edge in E indicates a point-point distance constraint, a point-hyperplane distance constraint, or a hyperplane-hyperplane angle constraint. Thus the edge set E is partitioned into three subsets E_{PP}, E_{PL}, E_{LL} according to the types of end-vertices of each edge. The point-configuration and the hyperplane-configuration are specified by $p : V_P \to \mathbb{R}^d$, and $\ell = (a, r) : V_L \to \mathbb{S}^{d-1} \times \mathbb{R}$, where the hyperplane associated to each $j \in V_L$ is given by $\{x \in \mathbb{R}^d \,:\, \langle x, a_j \rangle + r_j = 0\}$. Moreover, we assume here that the points $p(V_P)$ and hyperplanes $\ell(V_L)$ affinely span $\mathbb{R}^d$. For $i \in V_P$ and $j, k \in V_L$, the distance between the point p_i and the hyperplane ℓ_j is equal to $|\langle p_i, a_j \rangle + r_j|$, and the angle between the two hyperplanes ℓ_j, ℓ_k is determined by $\langle a_j, a_k \rangle$. Hence the system of constraints can be written as

$$\|p_i - p_j\| = \text{const} \qquad (ij \in E_{PP}) \tag{4}$$
$$|\langle p_i, a_j \rangle + r_j| = \text{const} \qquad (ij \in E_{PL}) \tag{5}$$
$$\langle a_i, a_j \rangle = \text{const} \qquad (ij \in E_{LL}). \tag{6}$$

Since $a_j \in \mathbb{S}^{d-1}$, we also have the constraint

$$\langle a_i, a_i \rangle = 1 \qquad (i \in V_L).$$

Taking the derivative we get the system of first order constraints

$$\langle p_i - p_j, \dot{p}_i - \dot{p}_j \rangle = 0 \qquad (ij \in E_{PP}) \tag{7}$$
$$\langle p_i, \dot{a}_j \rangle + \langle \dot{p}_i, a_j \rangle + \dot{r}_j = 0 \qquad (ij \in E_{PL}) \tag{8}$$
$$\langle a_i, \dot{a}_j \rangle + \langle \dot{a}_i, a_j \rangle = 0 \qquad (ij \in E_{LL}) \tag{9}$$
$$\langle a_i, \dot{a}_i \rangle = 0 \qquad (i \in V_L). \tag{10}$$

A map $(\dot{p}, \dot{\ell})$, where $\dot{\ell} = (\dot{a}, \dot{r})$, is said to be an *infinitesimal motion* of (G, p, ℓ) if it satisfies this system of linear constraints, and (G, p, ℓ) is *infinitesimally rigid* if the dimension of the space of its infinitesimal motions is equal to $\binom{d+1}{2}$, the dimension of the space of Euclidean motions in $\mathbb{R}^d$.

In order to use the rigidity preserving transformation from Section 2.1, we will first translate the point-hyperplane framework (G, p, ℓ) to a point-hyperplane framework $(G, \tilde{p}, \ell)$ in the hyperplane $\mathbb{A}^d$ by taking $\tilde{p}_i = (p_i, 1)$ for all $i \in V_p$. The system of constraints (7)-(10) then becomes:

$$\begin{aligned}
\langle \tilde{p}_i - \tilde{p}_j, \dot{\tilde{p}}_i - \dot{\tilde{p}}_j \rangle &= 0 && (ij \in E, i, j \in V_P) && (11)\\
\langle \tilde{p}_i, \dot{\ell}_j \rangle + \langle \dot{\tilde{p}}_i, \ell_j \rangle &= 0 && (ij \in E, i \in V_P, j \in V_L) && (12)\\
\langle a_i, \dot{a}_j \rangle + \langle \dot{a}_i, a_j \rangle &= 0 && (ij \in E, i, j \in V_L) && (13)\\
\langle \dot{\tilde{p}}_i, \mathbf{e} \rangle &= 0 && (i \in V_P) && (14)\\
\langle a_i, \dot{a}_i \rangle &= 0 && (i \in V_L). && (15)
\end{aligned}$$

We now relate this system of linear equations with that for bar-joint frameworks on $\mathbb{S}^d$. We first observe that r_j does not appear in (12) because $\dot{\tilde{p}}_i \in T\mathbb{A}^d$ (and hence the last coordinate of $\dot{\tilde{p}}_i$ is equal to zero). This implies that the last coordinate of ℓ_j is not important when analyzing the infinitesimal rigidity of $(G, \tilde{p}, \ell)$, and we may always assume that ℓ is a map with $\ell : V_L \to \mathbb{S}^{d-1} \times \{0\}$. Under this assumption, we can regard each ℓ_j as a point on the equator Q of $\mathbb{S}^d$ by identifying $\mathbb{S}^{d-1} \times \{0\}$ with Q. Hence (15) can be written as $\langle \ell_j, \dot{\ell}_j \rangle = 0$, i.e. $\dot{\ell}_j \in T_{\ell_j}\mathbb{S}^d$ for all $j \in V_L$, and (13) gives

$$\langle \ell_i, \dot{\ell}_j \rangle + \langle \dot{\ell}_i, \ell_j \rangle = 0$$

for all $ij \in E$ with $i, j \in V_L$. We have already seen that (11) can be rewritten as

$$\langle \phi(\tilde{p}_i), \psi_{\tilde{p}_j}(\dot{\tilde{p}}_j) \rangle + \langle \phi(\tilde{p}_j), \psi_{\tilde{p}_i}(\dot{\tilde{p}}_i) \rangle = \frac{\langle \tilde{p}_i - \tilde{p}_j, \dot{\tilde{p}}_i - \dot{\tilde{p}}_j \rangle}{\|\tilde{p}_i\| \|\tilde{p}_j\|} = 0$$

for all $ij \in E$ with $i, j \in V_P$. A similar calculation shows that (12) can be rewritten as

$$\langle \phi(\tilde{p}_i), \dot{\ell}_j \rangle + \langle \psi_{\tilde{p}_i}(\dot{\tilde{p}}_i), \ell_j \rangle = \frac{\langle \tilde{p}_i, \dot{\ell}_j \rangle + \langle \dot{\tilde{p}}_i, \ell_j \rangle}{\|\tilde{p}_i\|} = 0$$

for all $ij \in E$ with $i \in V_P$ and $j \in V_L$.

These equations imply that $(\dot{\tilde{p}}, \dot{\ell})$ is an infinitesimal motion of $(G, \tilde{p}, \ell)$ if and only if $\dot{q}$ is an infinitesimal motion of (G, q), where (G, q) is the bar-joint framework in $\mathbb{S}^d_{\geq 0}$ given by

$$q_i = \begin{cases} \phi(\tilde{p}_i) & (i \in V_P) \\ (a_i, 0) & (i \in V_L), \end{cases} \tag{16}$$

and $\dot{q}_i \in T_{q_i}\mathbb{S}^d$ is given by

$$\dot{q}_i = \begin{cases} \psi_{\tilde{p}_i}(\dot{\tilde{p}}_i) & (i \in V_P) \\ \dot{\ell}_i & (i \in V_L). \end{cases}$$

Note that equation (14) is needed for $(G, \tilde{p}, \ell)$ in $\mathbb{A}^d$, but it is of course discarded for (G, q) in $\mathbb{S}^d_{\geq 0}$.

Since each ψ_x is bijective and hence invertible, this gives us an isomorphism between the spaces of infinitesimal motions of $(G, \tilde{p}, \ell)$ and (G, q). In particular, if we denote the map q given in (16) by $\phi \circ (\tilde{p}, \ell)$, then we have the following result.

Lemma 2.2. *Let $(G, \tilde{p}, \ell)$ be a point-hyperplane framework with $G = (V_P \cup V_L, E)$, $\tilde{p} : V_P \to \mathbb{A}^d$ and $\ell = (a, r) : V_L \to \mathbb{S}^{d-1} \times \mathbb{R}$. Let $(G, \phi \circ (\tilde{p}, \ell))$ be the bar-joint framework in $\mathbb{S}^d_{\geq 0}$ obtained by central projection of each $\tilde{p}_i$ $(i \in V_P)$ and by regarding each hyperplane $\ell_i = (a_i, r_i)$ $(i \in V_L)$ as the point $(a_i, 0)$ on the equator of $\mathbb{S}^d$. Then $(G, \tilde{p}, \ell)$ is infinitesimally rigid if and only if $(G, \phi \circ (\tilde{p}, \ell))$ is infinitesimally rigid.*

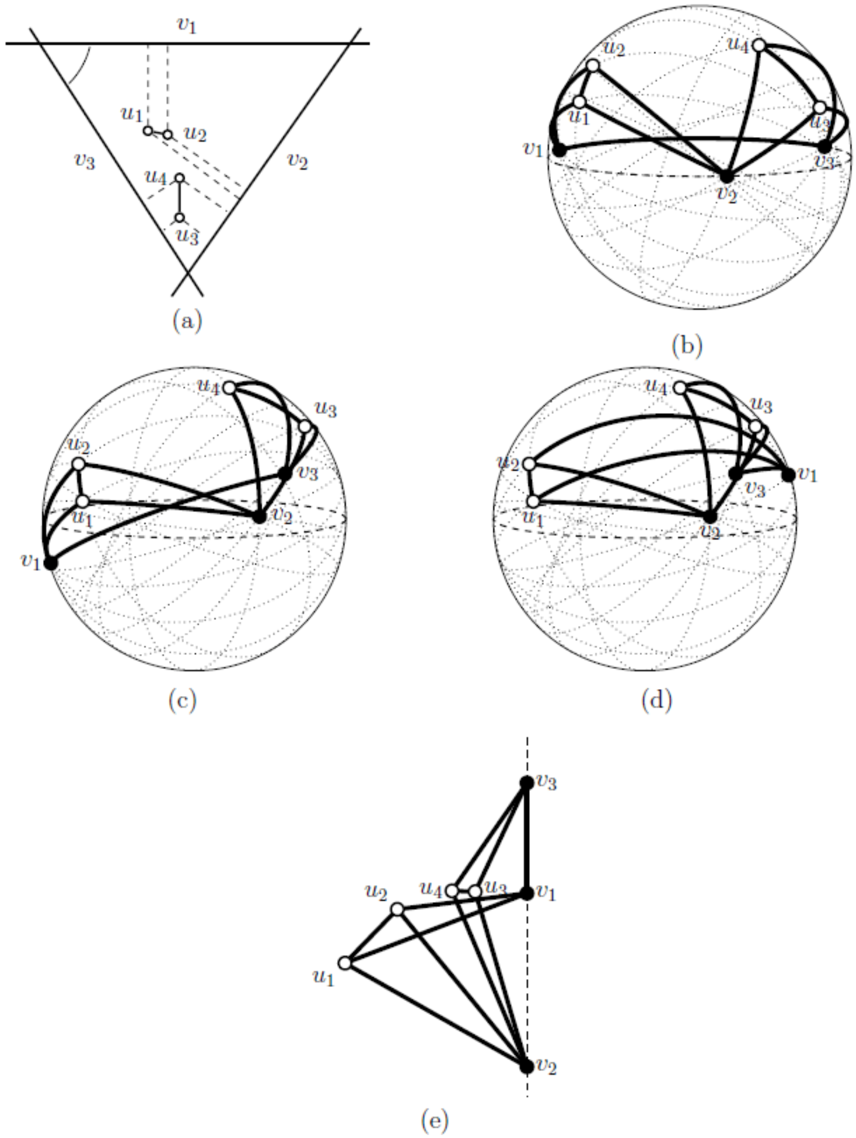


Figure 3: An illustration of the rigidity preserving transformations in Lemmas 2.2 and 2.3. (a) A point-line framework $(G, \tilde{p}, \ell)$. (b) The corresponding spherical framework $(G, \phi \circ (\tilde{p}, \ell))$ in $\mathbb{S}^2_{\geq 0}$ with three points on the equator. The spherical framework in (c) arises from (b) by a small rotation to take points off the equator. An inversion of points in $\mathbb{S}^d_{<0}$ then gives (d). Finally in (e) we have a projection to the plane as a bar-joint framework with three collinear points.

The transformation used in Lemma 2.2 is illustrated in Figure 3(a), (b).

In order to relate $(G, \phi \circ (\tilde{p}, \ell))$ with a bar-joint framework in $\mathbb{A}^d$, we further consider transformations for frameworks in $\mathbb{S}^d$ introduced in [19]. Given a framework (G, q) in $\mathbb{S}^d$, a *rotation* γ is an operator acting on q such that $(\gamma \circ q)_i = Rq_i$, for all $i \in V$, for some orthogonal matrix R. Note that $\dot{q}$ is an infinitesimal motion of (G, q) if and only if the map $\gamma \circ \dot{q}$ defined by $(\gamma \circ \dot{q})_i = R\dot{q}_i$ $(i \in V)$ is an infinitesimal motion of $(G, \gamma \circ q)$. In particular, (G, q) is infinitesimally rigid if and only if $(G, \gamma \circ q)$ is infinitesimally rigid.

Given a framework (G, q) in $\mathbb{S}^d$ and $I \subseteq V$, the *inversion* ι (with respect to I) is an operator acting on q such that $(\iota \circ q)_i = -q_i$ if $i \in I$ and $(\iota \circ q)_i = q_i$ otherwise. Note that $\dot{q}$ is an infinitesimal motion of (G, q) if and only if $\iota \circ \dot{q}$ defined by $(\iota \circ \dot{q})_i = -\dot{q}_i$ $(i \in I)$ and $(\iota \circ \dot{q})_i = \dot{q}_i$ $(i \in V \setminus I)$ is an infinitesimal motion of $(G, \iota \circ q)$, which again means that ι preserves infinitesimal rigidity.

We shall use an inversion to flip points in $\mathbb{S}_{<0}$ to $\mathbb{S}_{>0}$ so that a framework (G, q) in $\mathbb{S}^d$ is transferred to a framework $(G, \iota \circ q)$ in $\mathbb{S}^d_{\geq 0}$. In the following discussion, ι always refers to such an operator. Then a framework (G, q) in $\mathbb{S}^d$ can be transformed to a framework $(G, \iota \circ \gamma \circ q)$ in $\mathbb{S}_{>0}$ by first applying a rotation γ which moves all points off the equator, and then applying ι to flip points to $\mathbb{S}_{>0}$. For a framework in $\mathbb{S}_{>0}$ we can then use the inverse of ϕ to transfer it to $\mathbb{A}^d$. An important property of this sequence of transformations is that point-hyperplane incidence is preserved, i.e. points in (G, q) lie on a hyperplane in $\mathbb{S}^d$ if and only if the corresponding points in $(G, \phi^{-1} \circ \iota \circ \gamma \circ p)$ lie on a hyperplane in $\mathbb{A}^d$. Combining this with Lemma 2.2 we have the following result. (See also Figure 3 for an illustration.)

Lemma 2.3. *Let $(G, \tilde{p}, \ell)$ be a point-hyperplane framework in $\mathbb{A}^d$ with $G = (V_P \cup V_L, E)$, $\tilde{p} : V_P \to \mathbb{A}^d$ and $\ell = (a, r) : V_L \to \mathbb{S}^{d-1} \times \mathbb{R}$. Let $(G, \tilde{q})$ be the bar-joint framework in $\mathbb{A}^d$ with $\tilde{q} = \phi^{-1} \circ \iota \circ \gamma \circ \phi \circ (\tilde{p}, \ell)$. Then the points in $\tilde{q}(V_L)$ all lie on a hyperplane in $\mathbb{A}^d$, and $(G, \tilde{p}, \ell)$ is infinitesimally rigid if and only if $(G, \tilde{q})$ is infinitesimally rigid.*

Note that the above transformation is reversible, i.e., we may start with a framework $(G, \tilde{q})$ in $\mathbb{A}^d$ with points $\tilde{q}(X)$ being on a hyperplane for $X \subset V$, project this framework to $\mathbb{S}_{>0}$, rotate the framework so that the points of X lie on the equator, invert points in $\mathbb{S}_{<0}$ to $\mathbb{S}_{>0}$ (if necessary), and then project the framework back to $\mathbb{A}^d$. For any choice of $r : X \to \mathbb{R}$, this reverse process yields a point-hyperplane framework $(G, \tilde{p}, \ell)$ in $\mathbb{A}^d$ with $\ell = (a, r)$, $V_L = X$ and $V_P = V \setminus X$ such that $(G, \tilde{q})$ is infinitesimally rigid if and only if $(G, \tilde{p}, \ell)$ is infinitesimally rigid. We can now associate $\mathbb{A}^d$ with $\mathbb{R}^d$ to obtain our main geometric result.

Theorem 2.4. *Let $G = (V, E)$ be a graph and $X \subseteq V$. Then the following are equivalent:*

(a) G can be realised as an infinitesimally rigid bar-joint framework in $\mathbb{R}^d$ such that the points assigned to X lie on a hyperplane.

(b) G can be realised as an infinitesimally rigid point-hyperplane framework in $\mathbb{R}^d$ such that each vertex in X is realised as a hyperplane and each vertex in $V \setminus X$ is realised as a point.

2.3. Combinatorial characterization in the plane

To see the power of our main theorem, let us consider the case when $d = 2$. In the plane, Jackson and Owen [10] were able to give a combinatorial characterization of graphs which can be realised as an infinitesimally rigid point-line framework. Combining this with Theorem 2.4 we immediately obtain the following characterization of graphs which can be realised as infinitesimally rigid bar-joint frameworks in the plane with a given set of collinear points. This theorem extends a result by Jackson and Jordán [9], where they give a characterization for the case when three specified points are collinear. We will need the following notation. Given a graph $G = (V, E)$, $X \subseteq V$ and $A \subseteq E$, let $\nu_X(A)$ be the number of vertices of X which are incident to edges in A.

Theorem 2.5. *Let $G = (V, E)$ be a graph and $X \subseteq V$. Then the following are equivalent:*

(a) G can be realised as an infinitesimally rigid bar-joint framework in $\mathbb{R}^2$ such that the points assigned to X lie on a line.

(b) G can be realised as an infinitesimally rigid point-line framework in $\mathbb{R}^2$ such that each vertex in X is realised as a line and each vertex in $V \setminus X$ is realised as a point.

(c) G contains a spanning subgraph $G' = (V, E')$ such that $E' = 2|V| - 3$ and, for all $\emptyset \neq A \subseteq E'$ and all partitions $\{A_1, \dots, A_s\}$ of A,

$$|A| \leq \sum_{i=1}^{s}(2\nu_{V\setminus X}(A_i) + \nu_X(A_i) - 2) + \nu_X(A) - 1.$$

Proof. The equivalence of (a) and (b) follows from Theorem 2.4. The equivalence of (b) and (c) was established in [10]. ∎

We illustrate this result using the underlying point-line graph of the point-line framework in Figure 1(b). This graph is shown as a bar-joint framework with collinear points in Figure 4(a). It has $|V_P| = 8$ and $|V_L| = 5$, and hence we have $2|V_P| + 2|V_L| - 3 = 23$. If we take X to be the set of line vertices V_L and A_i to be the set of edges incident to the body B_i for $i = 1, 2, 3, 4$, then

$$\sum_{i=1}^{4}(2\nu_{V\setminus X}(A_i) + \nu_X(A_i) - 2) + \nu_X(E) - 1 = 22.$$

Since A_1, A_2, A_3, A_4 partition E, no subset A of E with $|A| = 23$ can satisfy Theorem 2.5(c).

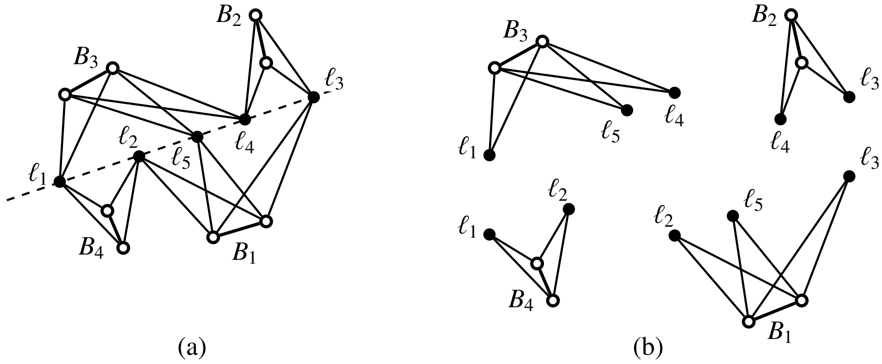


Figure 4: A bar-joint framework with five collinear points corresponding to the point-line framework in Figure 1(b) and a partition of the edge set into A_1, A_2, A_3 and A_4.

3. Connection to scene analysis

In this section we describe a connection between point-hyperplane frameworks and scene analysis.

A *d-scene* is a pair consisting of a set of points and a set of hyperplanes in $\mathbb{R}^d$. We can use a bipartite graph $G = (V_P \cup V_L, E)$ to represent the point-hyperplane incidences (where each vertex in V_P corresponds to a point, each vertex in V_L to a hyperplane, and each edge in E to a point-hyperplane incidence). Then a d-scene can be formally defined as a triple (G, p, ℓ) of a bipartite graph G, and maps $p : V_P \to \mathbb{R}^d$ and $\ell : V_L \to \mathbb{S}^{d-1} \times \mathbb{R}$, satisfying the incidence condition

$$\langle p_i, a_j \rangle + r_j = 0 \qquad (ij \in E, i \in V_P, j \in V_L), \tag{17}$$

where $\ell_j = (a_j, r_j)$ for all $j \in V_L$. Given the hyperplane normals $(a_j)_{j \in V_L}$, we can always construct a d-scene with these normals by choosing the points P_i to be coincident, i.e. choosing a fixed $t \in \mathbb{R}^d$ and putting $p_i = t$ $(i \in V_P)$ and $r_j = -\langle t, a_j \rangle$ $(j \in V_L)$. We will call such a d-scene *trivial*.

In the *d-scene realisation problem* (see [28] for example) we are asked whether there is a *non-trivial d*-scene with a given set of hyperplane normals $(a_j)_{j \in V_L}$. Note that the set of all d-scenes forms a linear space whose dimension is at least d, with equality if and only if every d-scene is trivial. It follows that the existence of a nontrivial d-scene can be checked by determining the dimension of the solution space of the following linear system of equations for the variables $x : V_p \to \mathbb{R}^d$ and $y : V_L \to \mathbb{R}$:

$$\langle x_i, a_j \rangle + y_j = 0 \qquad (ij \in E, i \in V_P, j \in V_L). \tag{18}$$

Now let us return to point-hyperplane frameworks, and consider the restricted rigidity model when the normal of each hyperplane is fixed. We can obtain the first order constraints for a fixed-normal point-hyperplane framework (G, p, ℓ) with $G = (V_P \cup V_L, E)$ by setting $\dot{a}_j = 0$ in the system (11)-(15). This gives

$$\langle p_i - p_j, \dot{p}_i - \dot{p}_j \rangle = 0 \qquad (ij \in E_{PP}) \tag{19}$$

$$\langle \dot{p}_i, a_j \rangle + \dot{r}_j = 0 \qquad (ij \in E_{PL}), \tag{20}$$

where $\dot{p}$ and $\dot{r}$ are variables. We say that the (G, p, ℓ) is *infinitesimally fixed-normal rigid* if the dimension of the space of infinitesimal motions, i.e. the solution space of the system of equations (19) and (20), is equal to d. Note that the system of equations (20) depends only on the normals $(a_j)_{j\in V_L}$. This implies that the infinitesimal fixed-normal rigidity of (G, p, ℓ) depends only on the normals $(a_j)_{j\in V_L}$ when (G, p, ℓ) is *naturally bipartite* i.e. when all constraints are point-hyperplane distance constraints.

Now observe that (18) and (20) represent exactly the same system of equations by identifying x with $\dot{p}$ and y with $\dot{r}$. This means that, for any bipartite graph $G = (V_P \cup V_L, E)$ and any fixed set of hyperplane normals $(a_j)_{j\in V_L}$, every realisation of G as a d-scene with hyperplane normals $(a_j)_{j\in V_L}$ is trivial if and only if every realisation of G as a naturally bipartite point-hyperplane framework with hyperplane normals $(a_j)_{j\in V_L}$ is infinitesimally fixed-normal rigid.

Whiteley [27] gave a combinatorial characterization of graphs that can be realized as d-scenes with generic hyperplane normals , i.e., the set of entries in $(a_j)_{j\in V_L}$ is algebraically independent over $\mathbb{Q}$. By the above discussion, this gives a combinatorial characterization of the infinitesimal fixed-normal rigidity of naturally bipartite point-hyperplane frameworks with generic hyperplane normals.

Theorem 3.1. *Let $G = (V_P \cup V_L, E)$ be a bipartite graph. Then the following are equivalent.*

(a) *The dimension of the solution space of system (18) is equal to d for some $(a_j)_{j\in V_L}$.*
(b) *Every realisation of G as a d-scene with generic hyperplane normals is trivial.*
(c) *Every realisation of G as a point-hyperplane framework in $\mathbb{R}^d$ with generic hyperplane normals is infinitesimally fixed-normal rigid.*
(d) *G contains a spanning subgraph $G' = (V_P \cup V_L, E')$ with $|E'| = d|V_P| + |V_L| - d$ such that $|A| \leq d\,\nu_{V_P}(A) + \nu_{V_L}(A) - d$ for all $\emptyset \neq A \subseteq E'$.*
(e) *For any partition $\{A_1, \ldots, A_s\}$ of E,*

$$\sum_{i=1}^{s} (d\,\nu_{V_P}(A_i) + \nu_{V_L}(A_i) - d) \geq d|V_P| + |V_L| - d.$$

Proof. The equivalence of (a), (b) and (c) follows from the above discussion. The equivalence of (a) and (d) follows from [27, Theorem 4.1]. The equivalence of (d) and (e) follows from a result of Edmonds [4] on matroids induced by submodular functions. ∎

Note that the problem of characterising fixed normal rigidity of generic point-hyperplane frameworks in $\mathbb{R}^d$ which are not naturally bipartite is at least as difficult as that of characterising the rigidity of generic bar-joint frameworks in $\mathbb{R}^d$, which is notoriously difficult when $d \geq 3$. We will solve the non-bipartite fixed normal rigidity problem when $d = 2$ in the next section.

4. Combinatorial characterizations of constrained point-line frameworks in the plane

In this section we investigate point-line frameworks with different types of constraints imposed on the lines.

4.1. Fixed-line rigidity

We begin with the *fixed-line rigidity* of point-line frameworks in $\mathbb{R}^2$. In this rigidity model, each line is fixed and hence has no velocity. More formally, given a point-line framework (G, p, ℓ), we are interested in the following system:

$$\langle p_i - p_j, \dot{p}_i - \dot{p}_j \rangle = 0 \qquad (ij \in E_{PP})$$
$$\langle \dot{p}_i, a_j \rangle = 0 \qquad (ij \in E_{PL})$$

obtained by setting $\dot{a}_j = 0$ and $\dot{r}_j = 0$ in the system (7)-(10). We say that (G, p, ℓ) is *infinitesimally fixed-line rigid* if this system has no nonzero solution.

By the results of Section 2, we know how to convert a point-hyperplane framework (G, p, ℓ) in $\mathbb{R}^d$ to a bar-joint framework (G, q) in $\mathbb{R}^d$ in such a way that infinitesimal rigidity is preserved.[3] From the isomorphism between the spaces of infinitesimal motions of (G, p, ℓ) and (G, q) (given in the proof of Lemma 2.3), it is easy to see that $\dot{\ell}_i = 0$ if and only if the corresponding $\dot{q}_i = 0$ for $i \in V_L$. This implies that (G, p, ℓ) is infinitesimally fixed-line rigid if and only if (G, q) is an infinitesimally rigid bar-joint framework under the constraint that the vertices in V_L are pinned.

The rigidity of *pinned bar-joint frameworks* is a classical concept, and in $\mathbb{R}^2$ several combinatorial characterizations are known. Here we should be careful since, as shown in Lemma 2.3, the points in $q(V_L)$ all lie on a line, and hence (G, q) may not be a generic bar-joint framework. Fortunately, Servatius et al. [20, Theorem 4] (see also [11, Theorem 7.5]) already gave a characterization of the infinitesimal rigidity of pinned bar-joint frameworks in $\mathbb{R}^2$ in which the assumption of genericity is not required for the positions of the pinned vertices. This gives us the following characterization of infinitesimal fixed-line rigidity.

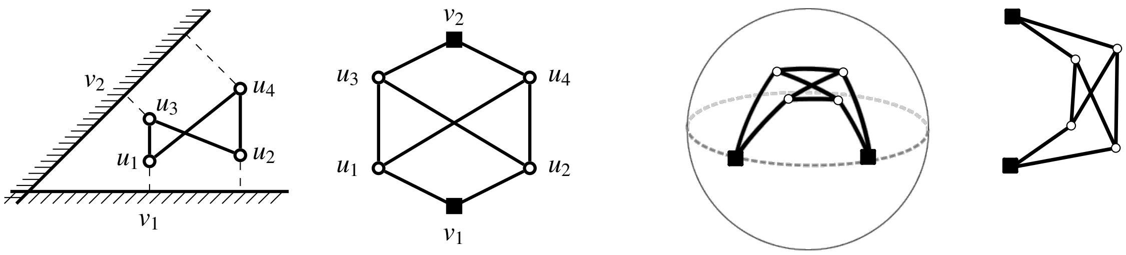
Theorem 4.1. *Let $G = (V_P \cup V_L, E)$ be a point-line graph and let $a_i \in \mathbb{S}^1$ for each $i \in V_L$. Then G can be realised as a minimally infinitesimally fixed-line rigid point-line framework in $\mathbb{R}^2$ such that each $i \in V_L$ is realised as the line with normal a_i if and only if $|E| = 2|V_P|$ and*

$$|F| \leq 2\nu_{V_P}(F) - 3 + \min\{3, 2a(F)\}$$

for all nonempty $F \subseteq E$, where

$$a(F) := \dim\langle a_i : i \in V_L(F)\rangle.$$

An example illustrating Theorem 4.1 is shown in Figure 5.



(a) (b) (c) (d)

Figure 5: A minimally infinitesimally fixed-line rigid point-line framework (a) and its associated point-line graph (b). The corresponding framework on the sphere (c) and its projection to the plane (d).

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[3] Although the transformation was presented in affine space $\mathbb{A}^d$, it can be extended to Euclidean space $\mathbb{R}^d$ by simply first lifting p_i $(i \in V_P)$ to $\tilde{p}_i = \binom{p_i}{1}$, applying the transformation to $(G, \tilde{p}, \ell)$ to obtain $(G, \tilde{q})$, and then projecting $\tilde{q}_i = \binom{q_i}{1}$ $(i \in V)$ to q_i.

4.2. *Fixed-normal rigidity*

We introduced the fixed-normal rigidity of a point-hyperplane framework (G, p, ℓ) in Section 3 and observed that the infinitesimal motions of (G, p, ℓ) which preserve the normals of the hyperplanes are determined by the system of equations

$$\langle p_i - p_j, \dot{p}_i - \dot{p}_j\rangle = 0 \qquad (ij \in E_{PP}) \tag{21}$$

$$\langle \dot{p}_i, a_j\rangle + \dot{r}_j = 0 \qquad (ij \in E_{PL})\,. \tag{22}$$

Owen and Power [14] had previously used a recursive construction to characterise the fixed-normal rigidity of generic point-line frameworks in $\mathbb{R}^2$. We will show that their result can be deduced from Theorem 2.5.

Theorem 4.2. *Let $G = (V_P \cup V_L, E)$ be a point-line graph with $|V_P| \geq 1$ and $|V_L| \geq 2$ and T be the edge set of a tree with vertex set V_L. Then the following statements are equivalent:*
(a) G can be realised as a point-line framework in $\mathbb{R}^2$ which is minimally infinitesimally fixed-normal rigid;
(b) $G + T$ can be realized as an infinitesimally rigid bar-joint framework in $\mathbb{R}^2$ such that the points assigned to V_L are collinear;
(c) $|E| = 2|V_P| + |V_L| - 2$, $|F| \leq 2\nu_{V_P}(F) - 3$ for all $\emptyset \neq F \subseteq E$ with $\nu_{V_L}(F) = 0$, and $|F| \leq 2\nu_{V_P}(F) + \nu_{V_L}(F) - 2$ for all $\emptyset \neq F \subseteq E$.

Proof. It is straightforward to show that (a) implies (c).

Suppose that G satisfies (c). We will show that $G + T$ satisfies (b) by showing it satisfies the conditions of Theorem 2.5(c) with $V \setminus X = V_P$ and $X = V_L$. Since $|E| = 2|V_P| + |V_L| - 2$, $G + T$ has $2|V| - 3$ edges. Choose a nonempty $A \subseteq E \cup T$, let $\mathcal{A} = \{A_1, A_2, \ldots, A_s\}$ be a partition of A and let $\mathcal{A}' = \{A_i \in \mathcal{A} : A_i \setminus T \neq \emptyset\}$. Then

$$\begin{aligned}\sum_{A_i \in \mathcal{A}} (2\nu_{V_P}(A_i) + \nu_{V_L}(A_i) - 2) + \nu_{V_L}(A) - 1 \quad &\geq \quad \sum_{A_i \in \mathcal{A}'} (2\nu_{V_P}(A_i \setminus T) + \nu_{V_L}(A_i \setminus T) - 2) + |A \cap T| \\ &\geq \quad \sum_{A_i \in \mathcal{A}'} |A_i \setminus T| + |A \cap T| = |A|.\end{aligned}$$

Thus $G + T$ satisfies the condition of Theorem 2.5(c). Hence $G + T$ also satisfies Theorem 2.5(a) so (b) holds

Finally we suppose that (b) holds. Then $G + T$ can be realised as an infinitesimally rigid point-line framework $(G + T, p, \ell)$. This implies that the dimension of the solution space of the system (7)-(10) for $(G + T, p, \ell)$ is equal to three. Choose a special vertex $i^* \in V_L$, and add the extra constraint

$$\langle a_{i^*}^{\perp}, \dot{a}_{i^*}\rangle = 0 \tag{23}$$

to the system (7)-(10), where $x^{\perp}$ denotes the $\pi/2$ clockwise rotation of a vector $x \in \mathbb{R}^2$. Since the system (7)-(10) contains a rotation in its solution space, adding the extra equation (23) decreases the dimension of the solution space by one.

Note that, in the system (7)-(10) for $(G + T, p, \ell)$, each edge in T gives the following constraint:

$$\langle a_i, \dot{a}_j\rangle + \langle \dot{a}_i, a_j\rangle = 0 \qquad (ij \in T). \tag{24}$$

A simple inductive argument, starting from i^*, implies that (23), (24), and (10) hold if and only if

$$\dot{a}_j = 0 \qquad (j \in V_L). \tag{25}$$

Since the combination of (11)-(15) with (25) is equivalent to the system (21)-(22) for (G, p, ℓ), we conclude that the dimension of the solution space of the latter system is equal to two. In other words, (G, p, ℓ) admits only trivial infinitesimal motions as a fixed-normal point-line framework and (a) holds. ∎

An example illustrating Theorem 4.2 is shown in Figure 6.

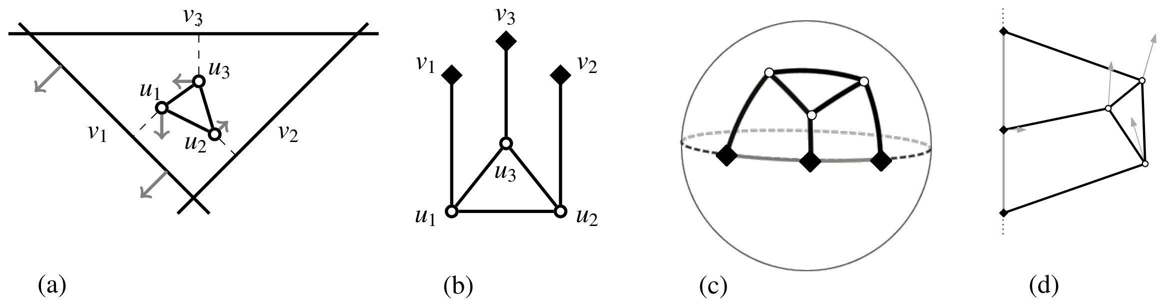


Figure 6: An infinitesimally flexible fixed-normal point-line framework (a) and its associated point-line graph (b). The corresponding framework on the sphere (c) and its projection to the plane (d) in which the line-vertics are collinear and constraints that the lines have fixed normals are modeled by adding a tree of grey edges between the three line-vertices. The point-line framework is fixed-normal flexible since it has only six constraints and $2|V_P| + |V_L| - 2 = 7$.

4.3. *Fixed-intercept rigidity*

We now consider point-line frameworks in which each line is allowed to rotate about some fixed point but cannot translate. Such a framework will have at most one trivial motion (a rotation), and this will exist only when each of the lines are allowed to rotate about the same point. We will focus on the special case when all of the lines are concurrent and are allowed to rotate about their common point of intersection. We will refer to such a point-line framework as a *line-concurrent framework*. See Figure 7(d)(e).

Given a line-concurrent framework (G, p, ℓ), we may always assume that the common intersection point of the lines is the origin, i.e., $r_j = 0$ for all $j \in V_L$, and hence the fixed-intercept constraint implies that $\dot{r}_j = 0$ for all $j \in V_L$. Substituting $\dot{r}_j = 0$ into (7)-(10), we deduce that the infinitesimal motions are determined by the following system:

$$\langle p_i - p_j, \dot{p}_i - \dot{p}_j\rangle = 0 \qquad (ij \in E_{PP}) \tag{26}$$

$$\langle p_i, \dot{a}_j\rangle + \langle \dot{p}_i, a_j\rangle = 0 \qquad (ij \in E_{PL}) \tag{27}$$

$$\langle a_i, \dot{a}_j\rangle + \langle \dot{a}_i, a_j\rangle = 0 \qquad (ij \in E_{LL}) \tag{28}$$

$$\langle a_i, \dot{a}_i\rangle = 0 \qquad (i \in V_L). \tag{29}$$

We say that (G, p, ℓ) is *infinitesimally fixed-intercept rigid* if the above system admits only the trivial infinitesimal motion.

Our theorem gives a characterization of infinitesimal fixed-intercept rigidity even in the case when the normals of the lines are specified as input without assuming genericity. We will see below that allowing arbitrary normals gives potential applications to engineering.

Theorem 4.3. *Let $G = (V_P \cup V_L, E)$ be a point-line graph with $|V_L| \geq 2$ and let $a_i \in \mathbb{S}^1$ for each $i \in V_L$. Suppose that each line has a distinct normal. Then G can be realised as a minimally infinitesimally fixed-intercept rigid line-concurrent framework such that each $i \in V_L$ is realised as the line with normal a_i if and only if $|E| = 2|V_P| + |V_L| - 1$ and*

$$|F| \leq 2\nu_{V_P}(F) + \nu_{V_L}(F) - 3 + \min\{2, \nu_{V_L}(F)\} \tag{30}$$

for all nonempty $F \subseteq E$.

We will in fact prove a stronger statement, in which lines are allowed to have the same normal (as in the setting of Theorem 4.1). To state the result we need the following notation. For a point-line graph $G = (V_P \cup V_L, E)$, let G^P be the graph on V_P obtained from G by removing V_L and regarding each edge ij in E_{PL} with $i \in V_P$ as a loop at i. Similarly, let G^L be the graph on V_L obtained from G by removing V_P and regarding each edge ij in E_{PL} with $j \in V_L$ as a loop at j. For an edge set F of G, let $G[F]$ be the subgraph of G induced by F. Also for a graph H, let $C(H)$ be the set of connected components in H.

Theorem 4.4. *Let $G = (V_P \cup V_L, E)$ be a point-line graph with $|V_L| \geq 2$ and let $a_i \in \mathbb{S}^1$ for each $i \in V_L$. Then G can be realised as a minimally infinitesimally fixed-intercept rigid line-concurrent framework such that each $i \in V_L$ is realised as the line with normal a_i if and only if*

- $|E| = 2|V_P| + |V_L| - 1$,
- $a_i \neq \pm a_j$ *for each* $ij \in E_{LL}$, *and*
- $|F| \leq 2\nu_{V_P}(F) + \nu_{V_L}(F) - 1 - \sum_{H \in C((G[F])^P)}(2 - \dim\langle a_j : ij \in F \cap E_{PL}, i \in V(H)\rangle)$ *for all nonempty* $F \subseteq E$.

Consider the point-line graph G shown in Figure 7(a). Two different realisations as a line-concurrent point-line framework are shown in (d) and (e). The framework in (d) has two lines with the same normal. We can use Theorem 4.4 to show that it is not infinitesimally fixed intercept rigid by taking F to be the edge-set of sugraph of G shown in (b). Then $G[F]^P$ is as shown in (c) and the right hand side of the inequality of Theorem 4.4 is $2 \cdot 4 + 2 - 1 - 2 = 7$, which is less than $|F| = 8$. On the other hand the realisaton shown in (e) is infinitesimally fixed intercept rigid. In particular if we evaluate the right hand side of the inequality of Theorem 4.3 for F, we obtain $2 \cdot 4 + 2 - 1 = 9$ so the inequality holds.

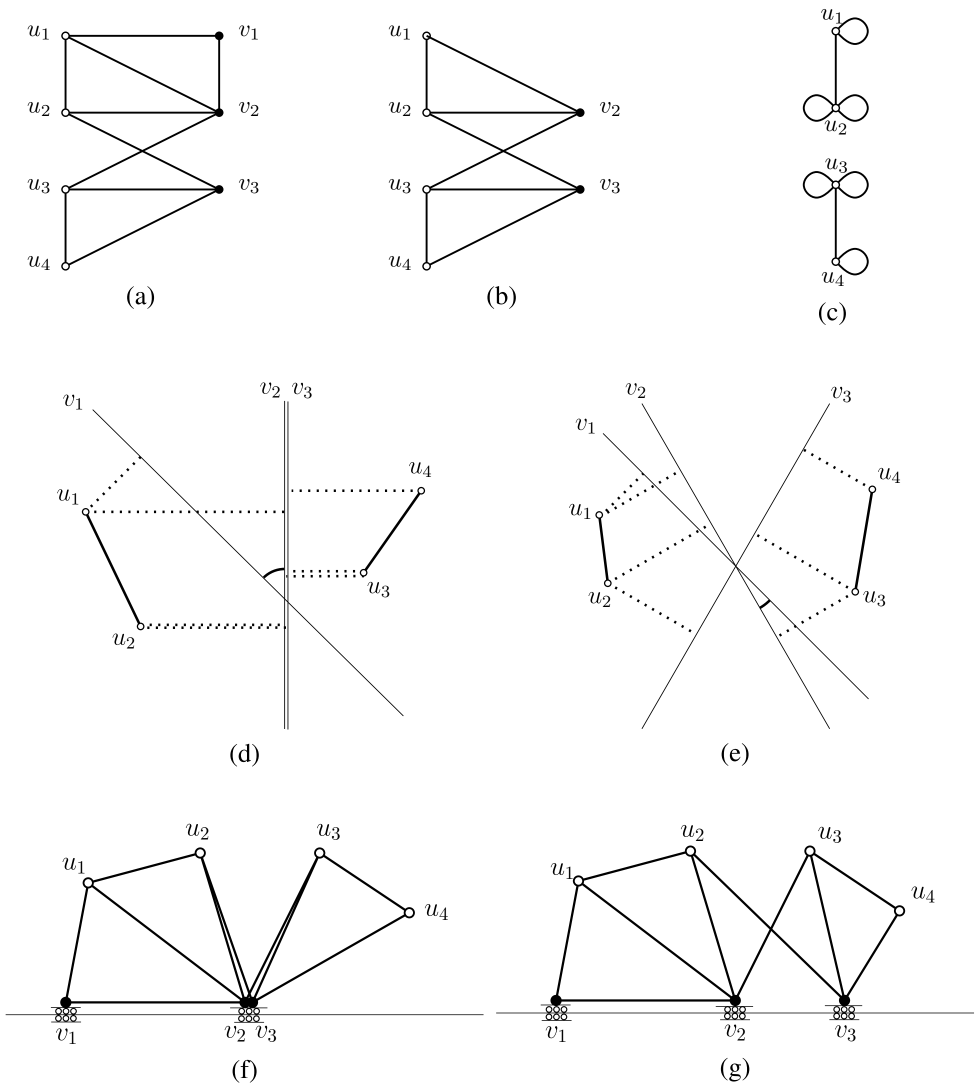
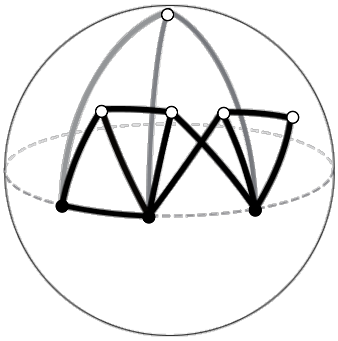
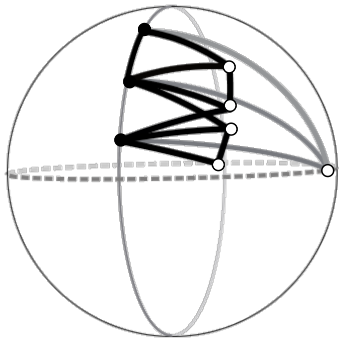


Figure 7: (a) $G = (V_P \cup V_L, E)$, where the left side is V_P and the right side is V_L. (b) an edge set F violating the count in Theorem 4.4 when two normals coincide as in (d). (c) $G[F]^P$. (d) A line-concurrent realization such that the two lines v_2, v_3 have the same normal. (e) A generic line-concurrent realization. (f)(g) Bar-joint frameworks with horizontal slider joints corresponding to (d)(e).

We will see in the next section that the generic version of Theorem 4.3 can be deduced from Theorem 2.5. However there seems to be no such reduction in the nongeneric case, so we provide a direct proof. Since the proof



(a)



(b)

Figure 8: (a) shows the spherical framework corresponding to the point-line framework in Figure 7(e), in which we have added a new point-vertex joined to the three line-vertices by grey edges to model the constraints that the lines must rotate about a fixed point. We rotate this framework to move the line-vertices off the equator and the new point-vertex onto the equator to obtain the spherical framework in (b). This projects to the bar-joint framework in Figure 7(g). The three line-vertices project to three collinear joints and the new point-vertex and its incident grey edges correspond to the constraints that the collinear joints are forced to move along the line.

is rather technical we defer the proof to the end of the paper and instead describe a consequence of the theorem for bar-joint frameworks which may have applications in engineering.

Consider, again, the transformation given in Section 2, which converts a point-line framework (G, p, ℓ) to a bar-joint framework (G, q). Note that a line-concurrent point-line framework (G, p, ℓ) will be mapped to a bar-joint framework (G, q) such that all the points in $q(V_L)$ lie on a line, say a horizontal line. If the rotation on the sphere is done such that the north pole is mapped to a point on the equator (so that the north pole is finally mapped to a point at infinity after the projection to the plane), then in the isomorphism between the spaces of infinitesimal motions of (G, p, ℓ) and (G, q), we have that $\dot{r}_j = 0$ if and only if $\dot{q}_j$ is in the horizontal direction. In other words each point $q(v)$ for $v \in V_L$ can only slide along the horizontal line. Therefore, the question about the fixed-intercept rigidity of (G, p, ℓ) can be rephrased as the rigidity question of bar-joint frameworks with horizontal slider joints on the ground. This transformation is illustrated in Figure 7(d),(e),(f),(g) and Figure 8(a),(b).

More formally, a bar-joint framework with horizontal slider joints is a tuple (G, X, p) of a graph G, $X \subseteq V(G)$, and $p : V \to \mathbb{R}^2$, where X will represent a set of slider joints. An infinitesimal motion $\dot{p}$ of (G, X, p) is an infinitesimal motion of (G, p) with $\dot{p}(v) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$ for all $v \in X$, and (G, X, p) is said to be infinitesimally rigid if horizontal translations are the only possible infinitesimal motions of (G, X, p). By the rigidity transformation explained above, Theorem 4.4 can be restated as follows.

Theorem 4.5. *Let $G = (V_P \cup V_L, E)$ be a point-line graph with $|V_L| \geq 2$ and let $x_i \in \mathbb{R}^1$ for each $i \in V_L$. Then G can be realised as a minimally infinitesimally rigid bar-joint framework in $\mathbb{R}^2$ with V_L as a set of horizontal slider joints such that the coordinate of $i \in V_L$ is $\begin{pmatrix} x_i \\ 0 \end{pmatrix}$ if and only if*

- $|E| = 2|V_P| + |V_L| - 1$,
- $x_i \neq x_j$ *for each* $ij \in E_{LL}$, *and*
- $|F| \leq 2\nu_{V_P}(F) + \nu_{V_L}(F) - 1 - \sum_{H \in C((G[F])^P)} \max\{0, 2 - |\{x_j : ij \in F \cap E_{PL}, i \in V(H)\}|\}$ *for all nonempty* $F \subseteq E$.

Note that Theorem 4.5 has no restriction on the coordinates of the slider joints. This is a much stronger statement than previous results [22, 11], where a certain genericity is assumed for the coordinates of slider joints. Such bar-joint frameworks with horizontal sliders frequently appear in the structural engineering literature, where sliders are often located on the horizontal ground.

We also remark that the combinatorial condition in Theorem 4.5 can be simplified to the form of Theorem 4.3 if $x_i \neq x_j$ for any distinct $i, j \in V_L$.

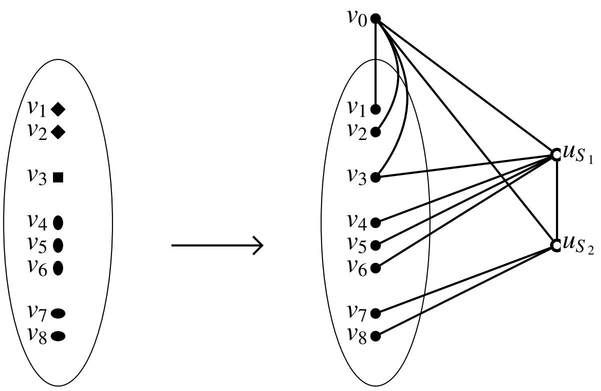


Figure 9: A constrained point-line graph G with eight constrained line vertices: v_1 and v_2 have fixed normals; v_3 is fixed; $S_1 = \{v_4, v_5, v_6\}$ and $S_2 = \{v_7, v_8\}$ have fixed centres of rotation. We transform G to an unconstrained point-line graph G' by adding the rigid graph K with two point-vertices, u_{S_1} and u_{S_2}, and one line-vertex v_0.

4.4. Mixed constraints

A natural question is how to generalise the results of Sections 4.1-4.3 to the case when the lines have a mixture of constraints. That is, some lines are completely fixed, some lines have fixed normals so can translate but not rotate, some lines can rotate about a fixed point but cannot translate, and some are unconstrained. We will extend the construction used in the proof of Theorem 3.1 to show that generic instances of this mixed constraint problem can be transformed to the unconstrained problem and then solved using Lemma 2.3.

Suppose we have a point-line graph G which has various types of line vertices i.e. a set V_L^F of fixed lines, a set V_L^N of lines with fixed normals, a collection $\mathcal{R}$ of pairwise disjoint sets of lines with a fixed centre of rotation, and unconstrained lines. A *realisation* of G in $\mathbb{R}^2$ is a framework (G, p, ℓ) together with a map $c : \mathcal{R} \to \mathbb{R}^2$, where $c(S)$ is the centre of rotation for all lines in S for each $S \in \mathcal{R}$. We say that the constrained framework (G, p, ℓ, c) is *generic* if the set of coordinates $\{p_i, a_j, c_S \ : \ i \in V_P, j \in V_L, S \in \mathcal{R}\}$ are algebraically independent over $\mathbb{Q}$.

We first consider the case when $|V_L^F| + |\mathcal{R}| \geq 1$ and, if equality occurs, then $|\mathcal{R}| = 1$ and $|V_L^N| \geq 1$. (In this case no rotation or translation of $\mathbb{R}^2$ will satisfy the constraints on the lines of any generic realisation of G.) We construct an unconstrained point-line graph G' by first adding a large rigid point-line graph K to G. We then choose a line-vertex v_0 in K and add an edge from v_0 to each $v \in V_L^N$. This corresponds to the operation of adding the 'tree of grey edges' joining the (fixed-normal) line-vertices in Figure 6(d). For each set $S \in \mathcal{R}$, we choose a distinct point-vertex u_S in K and add an edge from u_S to each vertex of S. This corresponds to the operation of adding a new point-vertex joined by 'grey edges' to each of the (fixed-intercept) line-vertices in Figure 8. Finally we join each $v \in V_L^F$ to v_0 and a point-vertex of K. This construction is illustrated in Figure 9.

Let (G, p, ℓ, c) and (G', p', ℓ') be generic realisations of G and G' in $\mathbb{R}^2$ so that $p(u) = p'(u)$ for all $u \in V_P$, $\ell(v) = \ell'(v)$ for all $v \in V_L$, and $c(S) = p'(u_S)$ for all $S \in \mathcal{R}$. Then (G, p, ℓ, c) has a non-zero infinitesimal motion if and only if (G', p', ℓ') has a non-zero infinitesimal motion which keeps K fixed. Hence (G, p, ℓ, c) is infinitesimally rigid as a constrained point-line framework if and only if (G', p', ℓ') is infinitesimally rigid as an unconstrained point-line framework. Thus we may determine whether (G, p, ℓ, c) is infinitesimally rigid by applying Lemma 2.3 to (G', p', ℓ'). Note that our definition of the genericity of (G, p, ℓ, c) is independent of the choice of r_j for $j \in V_L$. (The definition makes sense because r_j does not appear in the system (11)-(15) for describing infinitesimal motions of point-hyperplane framework as discussed in Section 2.2.) This means that we are free to choose the values of the r_j such that, for each $S \in \mathcal{R}$, each of the lines $\ell(v)$ with $v \in S$ passes through the the point $c(S)$, so we can take the lines in each S to be concurrent as in Section 4.3 if we wish.

Similar, but simpler constructions, can be used in the cases: $|\mathcal{R}| = 1$ and $|V_L^F| = 0 = |V_L^N|$; $|\mathcal{R}| = 0$ and $|V_L^F| = 1$; $|\mathcal{R}| = 0 = |V_L^F|$.

5. Further remarks and open problems

The combinatorial conditions in Theorem 2.5(c), Theorem 3.1(d),(e) and Theorem 4.1 can all be checked in polynomial time, see [10], [7, 10, 23] and [20], respectively. We show in the proof of Theorem 4.3 that the right hand side of (30) defines a submodular function. This implies that one can decide whether the condition is satisfied in polynomial time by a general submodular function minimization algorithm. Currently we do not have a more efficient specialized algorithm to check this condition.

Theorems 4.1 and 4.4 characterize the fixed-line rigidity and fixed-intercept rigidity for point-line frameworks with *arbitrary* normals for their lines. The problem of deriving an analogous result for fixed-normal rigidity is open. An important special case is the problem of characterizing fixed-normal rigidity for point-line frameworks in which the lines have been partitioned into parallel classes with generic normals (this was posed by Bill Jackson and John Owen at the rigidity workshop in Banff in 2015). In view of the relationship between fixed normal rigidity and scene analysis described in Section 3, this problem is challenging even when the underlying graph is naturally bipartite (as it is equivalent to understanding when an arbitrary 2-scene has only trivial realisations). We have constructed examples of (nongeneric) 2-dimensional naturally bipartite point-line frameworks with distinct line-normals which satisfy the count condition of Theorem 3.1 but are not fixed-normal rigid.

Results on the transfer of infinitesimal rigidity from d-dimensional Euclidean space to d-dimensional Minkowski space and from d-dimensional Minkowski space to d-dimensional Hyperbolic space (i.e. a sphere in $(d+1)$-dimensional Minkowski space) are given in [8, 19, 18]. It would be an interesting open problem to obtain analogous results for point-hyperplane frameworks in these spaces. Since Minkowski space has the full space of translations for hyperplanes, it is a natural setting to do this.

A classical result of Tay [24] characterizes generic rigidity for *body-bar frameworks* in $\mathbb{R}^d$. These consist of d-dimensional rigid bodies which are linked by rigid bars. His result has recently been extended by allowing more exotic constraints between bodies such as (nongeneric) pinning, slider-joints, and point-line, as well as point-plane, distance constraints, see [6, 11, 5].

Two-dimensional *direction-length frameworks* have also appeared in the literature. These are frameworks in $\mathbb{R}^2$ with a mixture of point-point distance and point-point direction constraints. They can be viewed as fixed-normal point-line frameworks, with exactly two points joined to each line; and with all point-line distances set to 0 i.e. they are point-line *incidences*. Servatius and Whitely characterised generic rigidity for these frameworks in [21]. Their characterization has one more condition than the characterization for generic fixed normal point-line frameworks in Theorem 4.2 (caused by the fact that sub-frameworks with no point-point distance constraints can be dilated). An extension in which fixed normal lines are allowed to contain an arbitrary number of points is given in [14]. Many other extensions are open for exploration. These connections also suggest that prior results on parallel-drawings and direction-length frameworks in higher dimensions, again with two vertices per line, such as [3, 29], may be generalized to combinatorially special fixed-normal point-hyperplane frameworks in higher dimensions.

6. Proofs of Theorems 4.3 and 4.4

We first give several tools from matroid theory in Section 6.1 and then give the proof of Theorem 4.4 in Section 6.2. In Section 6.3 we show that, if each line has a distinct normal, the combinatorial condition of Theorem 4.4 is equivalent to that of Theorem 4.3. Although the combinatorial condition in Theorem 4.3 is much simpler than that in Theorem 4.4, currently we have no direct proof of Theorem 4.3.

6.1. Matroid preliminaries

Let $G = (V, E)$ be a graph which may contain loops and let d be a positive integer. We assign a copy of $\mathbb{R}^d$ to each vertex and let $(\mathbb{R}^d)^V$ be the direct sum of those spaces over all vertices. For $x \in (\mathbb{R}^d)^V$, let $x(i) \in \mathbb{R}^d$ be the restriction of x to the space assigned to $i \in V$. Consider the incidence matrix of an oriented G, that is, the $(|E| \times |V|)$-matrix in which the entries in row $e = ij$ with $i < j$ are 1 in column i, -1 in column j and 0 elsewhere, and the entries in a row corresponding to a loop at i are 1 in column i and 0 elsewhere. This matrix gives a linear representation of a variant of the *cycle matroid* of G. This matroid has rank equal to

$$|V| - \sum_{H \in C(G)} \lambda(H),$$

where $\lambda(H) := 1$ if H has no loop and $\lambda(H) = 0$ otherwise. We can obtain a linear representation of this matroid by assigning a one-dimensional vector space A_e to each $e \in E$, where

$$A_e = \{x \in \mathbb{R}^V : x(i) + x(j) = 0, x(k) = 0\ \forall k \in V \setminus \{i, j\}\} \qquad (e = ij \text{ is not a loop})$$
$$A_e = \{x \in \mathbb{R}^V : x(k) = 0\ \forall k \in V \setminus \{i\}\} \qquad (e \text{ is a loop at } i)$$

with each edge $e \in E$. Then $\dim\langle A_e : e \in E\rangle = |V| - \sum_{H\in C(G)} \lambda(H)$.

Next we take the direct sum of d copies of A_e, which gives a d-dimensional vector space A_e^d for each edge e:

$$A_e^d = \{x \in (\mathbb{R}^d)^V : x(i) + x(j) = 0, x(k) = 0\ \forall k \in V \setminus \{i, j\}\} \qquad (e = ij \text{ is not a loop})$$
$$A_e^d = \{x \in (\mathbb{R}^d)^V : x(k) = 0\ \forall k \in V \setminus \{i\}\} \qquad (e \text{ is a loop at } i).$$

Clearly

$$\dim\langle A_e^d : e \in E\rangle = d|V| - \sum_{H\in C(G)} d\lambda(H)\,. \tag{31}$$

We now establish a variant of equation (31). Suppose that a d-dimensional vector a_e is assigned to each loop e. We then assign a vector space B_e to e by putting

$$B_e = A_e^d \qquad (e = ij \text{ is not a loop})$$
$$B_e = \{x \in (\mathbb{R}^d)^V : x(i) \in \langle a_e\rangle, x(k) = 0\ \forall k \in V \setminus \{i\}\} \qquad (e \text{ is a loop at } i)$$

where $\langle a_e\rangle$ denotes the span of a_e. Let Loop(H) be the set of loops in a graph H.

Lemma 6.1.

$$\dim\langle B_e : e \in E\rangle = d|V| - \sum_{H\in C(G)} (d - \dim\langle a_e : e \in \mathrm{Loop}(H)\rangle). \tag{32}$$

Proof. This is implicit in [11], but we give a direct proof since the claim is easy. A vector $y \in (\mathbb{R}^d)^V$ is in the orthogonal complement of $\langle B_e : e \in E \setminus \mathrm{Loop}(G)\rangle$ if and only if $y(i) = y(j)$ for every $H \in C(G)$ and every $i, j \in V(H)$. Such vectors form a $d|C(G)|$-dimensional space. Among those vectors, a vector y is in the orthogonal complement of $\langle B_e : e \in E\rangle$ if and only if $y(i)$ is in the orthogonal complement of $\langle a_e : e \in \mathrm{Loop}(H)\rangle$ for every $H \in C(G)$ and every $i \in V(H)$. Thus the orthogonal complement of $\langle B_e : e \in E\rangle$ has dimension equal to $\sum_{H\in C(G)}(d - \dim\langle a_e : e \in \mathrm{Loop}(H)\rangle)$. ∎

Another type of subspace which we will associate to each edge is

$$C_e = \{x \in \mathbb{R}^V : x(k) = 0\ \forall k \in V \setminus \{i, j\}\} \qquad (e = ij \text{ is not a loop})$$
$$C_e = \{x \in \mathbb{R}^V : x(k) = 0\ \forall k \in V \setminus \{i\}\} \qquad (e \text{ is a loop at } i).$$

These subspaces give a linear representation of the *bicircular matroid* of G, and we have

$$\dim\langle C_e : e \in E\rangle = |V(E)|. \tag{33}$$

We will also need the following result of Lovász [12] which gives a geometric interpretation of the so-called Dilworth truncation of a matroid. We say that a linear subspace H of codimension one in $\mathbb{R}^d$ intersects a family $\mathcal{U}$ of linear subspaces *transversally* if $\dim(H \cap U) = \dim U - 1$ for every $U \in \mathcal{U}$.

Lemma 6.2. *Let E be a finite set and $\mathcal{U} = \{U_e : e \in E\}$ be a family of linear subspaces of $\mathbb{R}^d$. Then there exists a linear subspace H of codimension one which intersects $\mathcal{U}$ transversally and is such that*

$$\dim\langle U_e \cap H\ : e \in E\rangle = \min\left\{\sum_{i=1}^{k}(\dim\langle U_e\ :\ e \in E_i\rangle - 1)\right\}, \tag{34}$$

where the minimum is taken over all partitions $\{E_1, E_2, \ldots, E_k\}$ of E.

6.2. *Proof of Theorem 4.4*

Proof of Theorem 4.4. Let $\ell_j = (a_j, 0)$ for each $j \in V_L$ and let $R(G, p, \ell)$ be the rigidity matrix of a framework (G, p, ℓ) representing the system (26)–(29). This is a $(|V_L| + |E|) \times (2|V_P| + 2|V_L|)$-matrix whose rows are of one of the following four types:

$$\begin{array}{c} \\ ij \in E_{PP} \\ jk \in E_{PL} \\ kl \in E_{LL} \\ l \in V_L \end{array} \begin{array}{c} \begin{array}{ccccccccc} & i & & j & & k & & l & \end{array} \\ \begin{pmatrix} \dots & p_i - p_j & \dots & p_j - p_i & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & a_k & \dots & p_j & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & a_l & \dots & a_k & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & a_l & \dots \end{pmatrix} \end{array}$$

where i, j denote vertices in V_P while k, l denote vertices in V_L (and unspecified entries are equal to zero). Since the set of the row vectors of $R(G, p, \ell)$ indexed by the vertices in V_L is linearly independent, $R(G, p, \ell)$ is row-independent if and only if the projections of the remaining row vectors of $R(G, p, \ell)$ onto the orthogonal complement of the space spanned by the row vectors indexed by V_L form a linearly independent set. In other words, $R(G, p, \ell)$ is row-independent if and only if the following $|E| \times 2(|V_P| + |V_L|)$-matrix is row-independent:

$$\begin{array}{c} \\ ij \in E_{PP} \\ jk \in E_{PL} \\ kl \in E_{LL} \end{array} \begin{array}{c} \begin{array}{ccccccccc} & i & & j & & k & & l & \end{array} \\ \begin{pmatrix} \dots & p_i - p_j & \dots & p_j - p_i & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & a_k & \dots & \langle p_j, a_k^\perp\rangle a_k^\perp & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \langle a_l, a_k^\perp\rangle a_k^\perp & \dots & \langle a_k, a_l^\perp\rangle a_l^\perp & \dots \end{pmatrix} \end{array}$$

Note that, if $a_k = \pm a_l$ for $k, l \in V_L$ with $kl \in E_{LL}$, then the corresponding row in the above matrix becomes zero, and hence $a_k \neq \pm a_l$ is necessary for (G, p, ℓ) to be minimally infinitesimally rigid. Thus in the following discussion we assume $a_k \neq \pm a_l$ for all $kl \in E_{LL}$.

By taking a suitable linear combination of the two columns indexed by each $k \in V_L$ to convert one of these columns to a zero column and then deleting this zero column, and using the fact that $\langle a_l, a_k^\perp\rangle = -\langle a_k, a_l^\perp\rangle$ for all pairs $k, l \in V_L$, we may deduce that $R(G, p, \ell)$ is row-independent if and only if the following $|E| \times (2|V_P| + |V_L|)$-matrix $R'(G, p, \ell)$ is row-independent:

$$\begin{array}{c} \\ ij \in E_{PP} \\ jk \in E_{PL} \\ kl \in E_{LL} \end{array} \begin{array}{c} \begin{array}{ccccccccc} & i & & j & & k & & l & \end{array} \\ \begin{pmatrix} \dots & p_i - p_j & \dots & p_j - p_i & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & a_k & \dots & \langle p_j, a_k^\perp\rangle & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 1 & \dots & -1 & \dots \end{pmatrix} \end{array} \tag{35}$$

We will show that there is an injective map $p : V_P \to \mathbb{R}^2$ such that $R'(G, p, \ell)$ is row-independent if and only if

$$|F| \leq 2\nu_{V_P}(F) + \nu_{V_L}(F) - 1 - \sum_{H \in C((G[F])^P)} (2 - \dim\langle a_j : ij \in F \cap E_{PL}, i \in V(H)\rangle)$$

for all nonempty $F \subseteq E$, implying the theorem.

To this end, we define the following linear subspace U_e^P in $(\mathbb{R}^2)^{V_P}$ for each $e \in E$:

$$\begin{aligned} U_e^P &= \{x \in (\mathbb{R}^2)^{V_P} : x(i) + x(j) = 0, x(k) = 0\ \forall k \in V_P \setminus \{i, j\}\} && (ik \in E_{PP}) \\ U_e^P &= \{x \in (\mathbb{R}^2)^{V_P} : x(i) \in \langle a_j\rangle, x(k) = 0\ \forall k \in V_P \setminus \{i\}\} && (ij \in E_{PL}, j \in V_L) \\ U_e^P &= \{0\} && (ij \in E_{LL}) \end{aligned}$$

Note that the linear subspaces are in the form of B_e given in Section 6.1 with the underlying graph G^P. Moreover, for $H \in C(G^P)$, there is a correspondence between a loop in H and an edge $ij \in E_{PL}$ with $i \in V(H)$. Therefore Lemma 6.1 gives

$$\dim\langle U_e^P : e \in E\rangle = 2\nu_{V_P}(E) - \sum_{H \in C((G[E])^P)} (2 - \dim\langle a_j : ij \in E_{PL}, i \in V(H)\rangle). \tag{36}$$

For each $e \in E$, we also define the following linear subspace U_e^L in $\mathbb{R}^{V_L}$:

$$\begin{aligned} U_e^L &= \{0\} && (ik \in E_{PP}) \\ U_e^L &= \{x \in \mathbb{R}^{V_L} : x(k) = 0\ \forall k \in V_L \setminus \{j\}\} && (ij \in E_{PL}, j \in V_L) \\ U_e^L &= \{x \in \mathbb{R}^{V_L} : x(k) = 0\ \forall k \in V_L \setminus \{i, j\}\} && (ij \in E_{LL}) \end{aligned}$$

Note that the linear subspaces are in the form of C_e given in Section 6.1 with the underlying graph G^L obtained from G by removing V_P and regarding each edge ij in E_{PL} with $j \in V_L$ as a loop at j. Hence by (33)

$$\dim\langle U_e^L : e \in E\rangle = \nu_{V_L}(E). \tag{37}$$

Now we consider the direct sum of $(\mathbb{R}^2)^{V_P}$ and $\mathbb{R}^{V_L}$, and let U_e be the direct sum of U_e^P and U_e^L for each edge e. Combining (36) and (37),

$$\dim\langle U_e : e \in E\rangle = 2\nu_{V_P}(E) + \nu_{V_L}(E) - \sum_{H \in C((G[F])^P)} (2 - \dim\langle a_j : ij \in E_{PL}, i \in V(H)). \tag{38}$$

By Lemma 6.2, there is a linear subspace H of codimension one in $(\mathbb{R}^2)^{V_P} \times \mathbb{R}^{V_L}$ intersecting $\{U_e : e \in E\}$ transversally and satisfying (34). Denote a normal vector of H by $s \in (\mathbb{R}^2)^{V_P} \times \mathbb{R}^{V_L}$. Since H intersects $\{U_e : e \in E\}$ transversally, we may assume $s(i) \neq s(j)$ for $i, j \in V_P$ with $i \neq j$ and $s(k) \neq 0$ for $k \in V_L$ (since a small perturbation of s will not change the property (34).)

We define $p : V_P \to \mathbb{R}^2$ by $p(i) = s(i)^\perp$ and show that $\dim\langle U_e \cap H : e \in E\rangle$ is equal to the rank of $R'(G, p, \ell)$ given in (35). We will use the following claim, which directly follows from the definition of U_e and the fact that $x \in U_e \cap H$ if and only if $x \in U_e$ and $\langle x, s\rangle = 0$.

Claim 1. *A vector $x \in U_e$ lies in H if and only if:*

- *for $e = ij \in E_{PP}$, $x(i) = -x(j)$ and $x(i)$ is proportional to $p(i) - p(j)$;*
- *for $e = ij \in E_{PL}$ with $j \in V_L$, $\frac{x(i)}{x(j)} = -\frac{s(j)a_j}{\langle p_i, a_j^\perp\rangle}$;*
- *for $e = ij \in E_{LL}$, $\frac{x(i)}{x(j)} = -\frac{s(j)}{s(i)}$.*

Since each $U_e \cap H$ is one-dimensional, Claim 1 implies that $\langle U_e \cap H : e \in E\rangle$ is equal to the row space of the $|E| \times (2|V_P| + |V_L|)$-matrix having the following form:

$$\begin{array}{c} \\ ij \in E_{PP} \\ jk \in E_{PL} \\ kl \in E_{LL} \end{array} \begin{array}{c} \begin{array}{cccccccc} & i & & j & & k & & l \end{array} \\ \begin{pmatrix} \dots & p_i - p_j & \dots & p_j - p_i & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & a_k & \dots & \langle p_j, a_k^\perp\rangle / s(k) & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 1/s(k) & \dots & -1/s(l) & \dots \end{pmatrix} \end{array}.$$

By scaling each column indexed by a vertex in V_L, this matrix is transformed to $R'(G, p, \ell)$ (as defined in (35)). In other words,

$$\operatorname{rank} R'(G, p, \ell) = \dim\langle U_e \cap H : e \in E\rangle. \tag{39}$$

By (34), (38), and (39), we get $\operatorname{rank} R'(G, p, \ell) = \min\{\sum_{F \in \mathcal{E}} f(F)\}$ where

$$f(F) = 2\nu_{V_P}(F) + \nu_{V_L}(F) - 1 - \sum_{H \in C((G[F])^P)} \Big(2 - \dim\langle a_j : ij \in F \cap E_{PL}, i \in V(H)\rangle\Big)$$

and the minimum is taken over all partitions $\mathcal{E}$ of E into nonempty subsets. The function $f : 2^E \to \mathbb{Z}$ is submodular, nondecreasing and non-negative, since it determines the dimension of $\langle\{U_e : e \in F\}\rangle$ by (38). Hence f induces the row matroid of $R'(G, p, \ell)$ by [4]. This implies that $\operatorname{rank} R'(G, p, \ell) = |E|$ if and only if $|F| \leq f(F)$ for all nonempty $F \subseteq E$. ∎

6.3. Proof of Theorem 4.3

Proof of Theorem 4.3. By Theorem 4.4 it suffices to prove that the two combinatorial conditions in Theorem 4.3 and Theorem 4.4 are equivalent under the assumption that the normals are distinct.

For each edge set F and each $H \in C(G[F]^P)$, recall that $V(H)$ is a subset of $V(G)$. We let $F(H)$ be the set of edges in F incident to $V(H)$ in G. Then the counts of Theorem 4.3 and Theorem 4.4 can be written as

$$|F| \leq 2\nu_{V_P}(F) + \nu_{V_L}(F) - 1 - \max\{0, 2 - |V_L(F)|\} \tag{40}$$

$$|F| \leq 2\nu_{V_P}(F) + \nu_{V_L}(F) - 1 - \sum_{H \in C(G[F]^P)} \max\{0, 2 - |V_L(F(H))|\}, \tag{41}$$

respectively, where the count of Theorem 4.4 is simplified to (41) due to the assumption that the normals are distinct.

Since $F(H) \subseteq F$, $\max\{0, 2 - |V_L(F(H))|\} \geq \max\{0, 2 - |V_L(F)|\}$ for each $H \in C(G[F]^P)$. Thus (41) implies (40) if $C(G[F]^P) \neq \emptyset$. If $C(G[F]^P) = \emptyset$, then $F \subseteq E_{LL}$ holds, and hence $|V_L(F)| \geq 2$. Thus the right hand side of (40) and (41) coincide. Hence, (41) always implies (40).

To complete the proof, we show that F satisfies (41) if each nonempty subset of F satisfies (40). Let $H_1, \ldots, H_k$ be all the components in $C(G[F]^P)$ with $|V_L(F(H_i))| \leq 1$. Let $F' = F \setminus \bigcup_{i=1}^k F(H_i)$. Then by (40) we have

$$\begin{aligned} |F(H_i)| &\leq 2\nu_{V_P}(F(H_i)) + \nu_{V_L}(F(H_i)) - 1 - \max\{0, 2 - |V_L(F(H_i))|\} \\ |F'| &\leq 2\nu_{V_P}(F') + \nu_{V_L}(F') - 1. \end{aligned}$$

Since $H_i \in C(G[F]^P)$, $V(F(H_i)) \cap V(F \setminus F(H_i)) \subseteq V_L$ holds, implying

$$\nu_{V_P}(F') + \sum_{i=1}^k \nu_{V_P}(F(H_i)) = \nu_{V_P}(F).$$

Moreover, by $|V_L(F(H_i))| \leq 1$ for $1 \leq i \leq k$, $|V(F(H_i)) \cap V(F \setminus F(H_i))| \leq 1$ holds, which implies

$$\nu_{V_L}(F') + \sum_{i=1}^k \nu_{V_L}(F(H_i)) \leq \nu_{V_L}(F) + k.$$

Therefore,

$$\begin{aligned} |F| &= |F'| + \sum_{i=1}^k |F(H_i)| \\ &\leq 2\nu_{V_P}(F') + \nu_{V_L}(F')) - 1 + \sum_{i=1}^k (2\nu_{V_P}(F(H_i)) + \nu_{V_L}(F(H_i)) - 1 - \max\{0, 2 - |V_L(F(H_i))|\}) \\ &\leq 2\nu_{V_P}(F) + \nu_{V_L}(F) - 1 + \sum_{i=1}^k \max\{0, 2 - |V_L(F(H_i))|\} \\ &= 2\nu_{V_P}(F) + \nu_{V_L}(F) - 1 + \sum_{H \in C(G[F]^P)} \max\{0, 2 - |V_L(F(H_i))|\}, \end{aligned}$$

and F satisfies (40). This completes the proof. ∎

Acknowledgements

We thank Banff International Research Station for hosting the 2015 workshop on 'Advances in Combinatorial and Geometric Rigidity' during which this work was started. We also thank HIM (Bonn) and ICMS (Edinburgh) for hosting further rigidity workshops in October 2015 and May 2016, respectively, and we thank DIMACS (Rutgers) for hosting the workshop on 'Distance Geometry' in July 2016. The discussions we had during these workshops played a crucial role in the writing of this paper.

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