

A Flexible Galerkin Scheme for Option Pricing in Lévy Models*

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Abstract. One popular approach to option pricing in Lévy models is through solving the related partial integro differential equation (PIDE). For the numerical solution of such equations powerful Galerkin methods have been put forward e.g. by Hilber, Reichmann, Schwab, Winter (2013). As in practice large classes of models are maintained simultaneously, flexibility in the driving Lévy model is crucial for the implementation of these powerful tools. In this article we provide a tool that enables the implementation of finite element Galerkin methods *flexibly in the model*. To this end we exploit the Fourier representation of the infinitesimal generator, i.e. the related symbol, which is explicitly available for the most relevant Lévy models. Empirical studies for the Merton, NIG and CGMY model confirm the numerical feasibility of the tool.

Key words. Lévy processes, partial integro differential equations, pseudo-differential operators, symbol, option pricing, Galerkin approach, finite element method

AMS subject classifications. 91G80, 60G51, 35S10, 65M60

1. Introduction. In computational finance, methods to solve partial differential equations come into play, when both run-time and accuracy matter. In contrast to Monte Carlo simulation for example, run-time is very appealing and a deterministic and conservative error analysis is established and well understood. In addition, compared to Fourier methods, the possibility to capture path-dependent features like early exercise and barriers is naturally built in. Within these appealing features lies the capacity to attract interest from academia and satisfy the needs of the financial industry alike.

In academia, a series of publications by Cont and Voltchkova in 2005 [10], Hilber, Reich, Schwab and Winter in 2009 [17], Jackson, Jaimungal and Surkov in 2012 [21] Salmi, Toivanen and Sydow in 2014 [24], Itkin in 2015 [19], Glau in 2016 [16], and the monograph of Hilber, Reichmann, Schwab and Winter in 2013 [18] have opened the theory to include even more sophisticated models of Lévy type, resulting in Partial *Integro* Differential Equations (PIDEs). The theoretical results have been validated by sophisticated numerical studies. In this context, Schwab and his working group in particular have taken the lead and unveiled the potential of PIDE theory in high generality and for practical purposes in the financial industry. Combining state of the art compression techniques with a wavelet finite element setup has resulted in a numerical framework for option pricing in advanced and multivariate jump models, which thereby moved academic boundaries.

Two standard methods are available for solving PIDEs, that is the finite difference approach and the finite element method (FEM). More recently, also radial basis methods have been pushed forward to solve pricing PIDEs. For all of these concepts implementations for a variety of models and option types have already been developed: Finite difference schemes

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38 solving PIDEs for pricing European and barrier options with an implementation for Merton
 39 and Variance Gamma are provided by Cont and Voltchkova in 2015 [10], [9]. The method has
 40 been further developed in different directions, we mention one example, by Itkin and Carr
 41 in 2012 [20], who exploit a special representation of the equation tailored to jump diffusions
 42 with jump intensity of tempered stable type. Wavelet-Galerkin methods for PIDEs related
 43 to a class generalizing tempered stable Lévy processes are derived by Matache, Nitsche and
 44 Schwab in 2005 [23] for American options and e.g. by Marazzani, Reichmann and Schwab in
 45 2012 [22] for a high-dimensional extension. A Fourier time stepping scheme combining PIDE
 46 with fast Fourier transform methods has been proposed in Jackson, Jaimungal and Surkov in
 47 2012 [21]. Radial basis approaches for the Merton and Kou model, American and European
 48 options are provided by Chan and Hubbert in 2014 [7] and further developed for CGMY
 49 models by Brummelhuis and Chan in 2014 [4].

50 In the financial industry an awareness of the full potential of these tools is yet to be
 51 developed. Advocating the advancement of numerical methods one must acknowledge what
 52 practice cherishes most. Due to model uncertainty and behavioral characteristics of different
 53 portfolios, financial institutions need to deal with a number of different pricing models in
 54 parallel. Or, in the words of Föllmer in [13]: "In any case, the signal towards the practitioners
 55 of risk management is clear: Do not commit yourself to a single model, remain flexible, vary
 56 the models in accordance with the problem at hand, always keeping in mind the worst case
 57 scenario."¹ Desirable features that the numerical environment must offer include

- 58 (1) a degree of accuracy reaching levels relevant to practical applications that can be
 59 measured and controlled by a theoretical error analysis,
- 60 (2) fast run times,
- 61 (3) low and feasible implementational and maintenance cost,
- 62 (4) a flexibility of the toolbox towards different options and models.

63 An implementation that is flexible in the driving model as well as in the option type first of
 64 all requires a problem formulation covering the collectivity of envisaged models and options.
 65 In view of feature (1), a unified approach to the error analysis of the resulting schemes is
 66 of equal importance. Galerkin methods, accruing from the Hilbert space formulation of the
 67 Kolmogorov equation, seem to be predestined to deliver the adequate level of abstraction
 68 for this task. It is precisely this abstract level that makes Galerkin methods flexible in the
 69 option type and the dimension of the underlying driving process. Consequently, even though
 70 Galerkin methods seem to be more involved at first glance in comparison to finite difference
 71 schemes, they still promise to lead to a lucid code that is easy to maintain and to extend, and
 72 that allows clear an extensive convergence and error analysis. This is of great importance for
 73 implementation and controlling methodological risk in finance. Moreover, Galerkin methods
 74 allow for efficient compression techniques such as wavelet-compressions, see [18], and reduced
 75 order modeling, see e.g. [8], [5]. We therefore consider the finite element, or more general
 76 Galerkin methods, worth exploring further for financial applications.

77 Unfortunately, although flexibility towards models goes well with the abstract formulation,
 78 the finite element method faces numerical challenges when implementing Lévy model based
 79 pricing tools. More precisely, the Lévy operator that determines the stiffness matrix is of

¹Translated from German.

80 integro differential type. Firstly, the resulting matrix is densely populated and in general not
 81 symmetric. Secondly, and even more severe, the matrix entries typically are not explicitly
 82 available. Instead, they require the evaluation of double integral terms possibly involving a
 83 numerically inaccessible Lévy measure. In these cases, a thorough analysis of the respective
 84 integrals may lead to approximation schemes deriving the stiffness matrix entries with the
 85 required precision. Pursuing this way, however, most likely results in a model specific scheme,
 86 contradicting requirement (4).

87 In this paper we develop a new methodology for option pricing in Lévy models using finite
 88 elements which is flexible in the choice of model. We address this goal by expressing the
 89 operator in the Fourier space. This means accessing the model specific information via the
 90 symbol, and we call the resulting tool the *symbol method*. In contrast to the operator, the
 91 symbol is explicitly available for a variety of models and is thus numerically superior. Further
 92 advantages will be highlighted in subsequent sections. It is worth mentioning a conceptual
 93 relation of this new approach to the Fourier time stepping scheme of [21]. Both methods
 94 result in PIDE discretizations that rely on the symbol of the driving Lévy process. While
 95 we propose to express the bilinear form in the Galerkin representation via the symbol, the
 96 methods of [21] are based on applying the Fourier transform to the pricing PIDEs and is not
 97 related to Galerkin approximations.

98 Section 2 introduces the theoretical framework for our PIDEs of interest and their weak
 99 formulation. The next section describes the solution scheme, that is the Galerkin approxi-
 100 mation in space. We investigate the scheme with regard to the numerical challenges arising
 101 during its implementation. Section 4 introduces the symbol method itself. All components
 102 of the FEM solver are expressed in Fourier space. The subsequent numerical evaluation of
 103 the stiffness matrix entries is supported by an elementary approximation result. Several ex-
 104 amples of symbols for well-known Lévy models confirm the wide applicability of the method
 105 and its numerical advantages. The actual implementation of the symbol method poses new
 106 challenges. We propose two different ways to tackle these challenges and to obtain a conver-
 107 gent and flexible scheme. As first approach, we propose to mollify the classic hat functions
 108 in Section 5. We analyse the error in detail and under standard conditions, obtain the same
 109 rate of convergence as for the case without mollification. Section 6 introduces an alternative
 110 approach by choosing splines as basis functions. The numerical studies in Section 7 confirm
 111 theoretically prescribed rates of convergence and validate the claim of numerical feasibility.

112 **2. Kolmogorov equations for option pricing in Lévy models.** We first introduce the
 113 underlying stochastic processes, the Kolmogorov equation, its weak formulation as well as the
 114 solution spaces of our choice.

115 **2.1. Lévy processes.** Let a stochastic basis $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ be given and let L be
 116 an \mathbb{R}^d -valued Lévy process with characteristics $(b, \sigma, F; h)$, i.e. for fixed $t \geq 0$ its characteristic
 117 function is given by

$$118 \quad (1) \quad E e^{i\langle \xi, L_t \rangle} = e^{-tA(-\xi)} \quad \text{for every } \xi \in \mathbb{R}^d,$$

120 where the *symbol of the process* is defined as

$$121 \quad (2) \quad A(\xi) := \frac{1}{2} \langle \xi, \sigma \xi \rangle + i \langle \xi, b \rangle - \int_{\mathbb{R}^d} \left(e^{-i\langle \xi, y \rangle} - 1 + i \langle \xi, h(y) \rangle \right) F(dy).$$

122 Here, σ is a symmetric, positive semi-definite $d \times d$ -matrix, $b \in \mathbb{R}^d$, and F is a Lévy measure,
 123 i.e. a positive Borel measure on \mathbb{R}^d with $F(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) F(dx) < \infty$. Moreover,
 124 h is a truncation function i.e. $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $h(x) = x$ in a neighborhood of 0 and
 125 $\int_{\{|x|>1\}} h_j(x) F(dx) < \infty$, where h_j denotes the j -th component of the truncation function
 126 h for all $j = 1, \dots, d$. The Kolmogorov operator of a Lévy process L with characteristics
 127 $(b, \sigma, F; h)$ is given by

$$\begin{aligned} \mathcal{A}\varphi(x) := & -\frac{1}{2} \sum_{j,k=1}^d \sigma^{j,k} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(x) - \sum_{j=1}^d b^j \frac{\partial \varphi}{\partial x_j}(x) \\ & - \int_{\mathbb{R}^d} \left(\varphi(x+y) - \varphi(x) - \sum_{j=1}^d \frac{\partial \varphi}{\partial x_j}(x) h_j(y) \right) F(dy) \end{aligned}$$

130 for every $\varphi \in C_0^\infty(\mathbb{R}^d)$.

131 **2.2. Kolmogorov equation in variational form.** Key for the variational formulation of
 132 the Kolmogorov equation

$$133 \quad (4) \quad \partial_t u + \mathcal{A}u = f$$

$$134 \quad (5) \quad u(0) = g$$

136 is the definition of the bilinear form

$$137 \quad (6) \quad a(\varphi, \psi) := \int_{\mathbb{R}^d} (\mathcal{A}\varphi)(x) \psi(x) dx \quad \text{for all } \varphi, \psi \in C_0^\infty(\mathbb{R}^d).$$

138 It is one of the major advantages of variational formulations of evolution equations that
 139 solution spaces of low regularity, as compared to the space C^2 for example, are incorporated
 140 in an elegant way. Departing from the space $C_0^\infty(\mathbb{R}^d)$ of smooth functions with compact
 141 support, we can select from a large variety of function spaces V that are characterized by the
 142 following assumption.

143 (A1) V and H are Hilbert spaces such that $C_0^\infty(\mathbb{R}^d)$ is dense in V and there exists a
 144 continuous embedding from V into H .

145 Existence and uniqueness of a variational solution critically hinges on the following two prop-
 146 erties of the bilinear form:

147 (A2) *Continuity*: There exists a constant $C > 0$ such that

$$148 \quad |a(\varphi, \psi)| \leq C \|\varphi\|_V \|\psi\|_V \quad \text{for all } \varphi, \psi \in C_0^\infty(\mathbb{R}^d).$$

149 (A3) *Gårding inequality*: There exists constants $G > 0$ and $G' \geq 0$ such that

$$150 \quad a(\varphi, \varphi) \geq G \|\varphi\|_V^2 - G' \|\varphi\|_H^2 \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d).$$

151 We observe that due to (A1) and (A2), the bilinear form a possesses a unique continuous
 152 bilinear extension $a : V \times V$ that is continuous, i.e. for a constant $C > 0$ we have $|a(\varphi, \psi)| \leq$
 153 $C \|\varphi\|_V \|\psi\|_V$ for all $\varphi, \psi \in V$. Also (A3) holds for all $\varphi \in V$.

154 As V is separable, this is also true for H and one can find a continuous embedding from H
 155 to the dual space V^* of V , i.e. (V, H, V^*) is a Gelfand triplet. We then denote by $L^2(0, T; H)$
 156 the space of all functions $u : [0, T] \rightarrow H$ such that for every $h \in H$ the map $s \mapsto \langle u(s), h \rangle$
 157 is Borel measurable and $\int_0^T \|u(t)\|_H^2 dt < \infty$. Moreover, we denote by $\partial_t u$ the derivative of u
 158 with respect to time in the distributional sense. For a detailed definition, which relies on the
 159 Bochner integral, we refer to Section 24.2 in [29]. The Sobolev space

$$160 \quad (7) \quad W^1(0, T; V, H) := \left\{ u \in L^2(0, T; V) \mid \partial_t u \in L^2(0, T; V^*) \right\}$$

161 will play the role of the solution space in the variational formulation of the Kolmogorov
 162 equation (4), (5).

163 **Definition 1.** Let $f \in L^2(0, T; V^*)$ and $g \in H$. Then $u \in W^1(0, T; V, H)$ is a variational
 164 solution of Kolmogorov equation (4), if for almost every $t \in (0, T)$,

$$165 \quad (8) \quad \langle \partial_t u(t), v \rangle_H + a(u(t), v) = \langle f(t) | v \rangle_{V^* \times V} \quad \text{for all } v \in V$$

166 and $u(t)$ converges to g for $t \downarrow 0$ in the norm of H .

167 **Remark 2.** Assumptions (A1)–(A3) guarantee the existence and uniqueness of a variational
 168 solution $u \in W^1(0, T; V, H)$ of (8), see for instance Theorem 23.A in [30].

169 **2.3. Solution spaces.** Expression (6) is based on the L^2 -scalar product and is appropri-
 170 ate for variational equations in Sobolev spaces. Then, typically $H = L^2$. For Kolmogorov
 171 equations for option prices the initial condition g in (5) plays the role of the (logarithmically
 172 transformed) payoff function of the option. For a call option with strike K it is of the form
 173 $x \mapsto (S_0 e^x - K)^+$, for a digital up and out option it is given by $x \mapsto \mathbb{1}_{e^x < b}$ for some $b \in \mathbb{R}$. We
 174 thus have to observe that the initial condition g is not square integrable for most of the typical
 175 cases of interest. Therefore, we base our analysis more generally on exponentially weighted
 176 L^2 spaces: For $\eta \in \mathbb{R}^d$ let

$$177 \quad L_\eta^2(\mathbb{R}^d) := \left\{ u \in L_{loc}^1(\mathbb{R}^d) \mid u e^{\langle \eta, \cdot \rangle} \in L^2(\mathbb{R}^d) \right\}, \quad \|u\|_{L_\eta^2} := \left(\int_{\mathbb{R}^d} |u(x)|^2 e^{2\langle \eta, x \rangle} dx \right)^{1/2}$$

178 and

$$179 \quad (9) \quad a(\varphi, \psi) := \langle \mathcal{A}\varphi, \psi \rangle_{L_\eta^2} = \int_{\mathbb{R}^d} (\mathcal{A}\varphi)(x) \psi(x) e^{2\langle \eta, x \rangle} dx \quad \text{for all } \varphi, \psi \in C_0^\infty(\mathbb{R}^d).$$

180 We notice that all assertions of the precedent section, concerning assumptions (A1)–(A3) and
 181 variational equations hold for bilinear form a defined by (9) instead of a from (6) as well.

182 As solution spaces V we consider weighted Sobolev-Slobodeckii spaces. These have proven
 183 to apply to a large set of option types and models. We refer to [12] and [16], where particularly
 184 Feynman-Kac type formulas have been derived linking European and path-dependent options
 185 to weak solutions of Kolmogorov equations in Sobolev-Slobodeckii spaces.

186 To introduce the spaces, we denote by $C_0^\infty(\mathbb{R}^d)$ the set of smooth real-valued functions
 187 with compact support in \mathbb{R}^d and let

$$188 \quad (10) \quad \widehat{\varphi}(\xi) = \mathcal{F}(\varphi)(\xi) := \int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} \varphi(x) dx$$

189 be the Fourier transform of $\varphi \in C_0^\infty(\mathbb{R}^d)$ and \mathcal{F}^{-1} be its inverse. We define the *exponentially*
 190 *weighted Sobolev-Slobodeckii space* $H_\eta^\alpha(\mathbb{R}^d)$ with index $\alpha \geq 0$ and weight $\eta \in \mathbb{R}^d$ as the
 191 completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_{H_\eta^\alpha}$ given by

$$192 \quad (11) \quad \|\varphi\|_{H_\eta^\alpha}^2 := \int_{\mathbb{R}^d} (1 + |\xi|)^{2\alpha} |\mathcal{F}(\varphi)(\xi - i\eta)|^2 d\xi.$$

193 By construction $H_\eta^\alpha(\mathbb{R}^d)$ is a separable Hilbert space and we denote its dual space by $(H_\eta^\alpha(\mathbb{R}^d))^*$. ■

194 **3. Implementational Challenges.** Based on this theoretical introduction we are now in
 195 the position to focus on its implementation and related numerical questions.

196 **3.1. Abstract Galerkin approximation in space.** For a countable Riesz basis $\{\varphi_1, \varphi_2, \dots\}$
 197 of V we define

$$198 \quad V_N := \text{span}\{\varphi_1, \dots, \varphi_N\} \quad \text{for all } N \in \mathbb{N}.$$

199 Since V is dense in H , we may further choose g_N in V_N such that $g_N \rightarrow u(0)$ in H . For each
 200 fixed $N \in \mathbb{N}$ the semidiscrete problem is defined by restricting (8) to the finite dimensional
 201 space: *Find a function* $u_N \in W^1(0, T; V_N; H \cap V_N)$ *that satisfies for all* $\chi \in C_0^\infty(0, T)$ *and*
 202 $\varphi \in V_N$,

$$203 \quad (12) \quad - \int_0^T \langle u_N(t), \varphi \rangle_{L^2} \dot{\chi}(t) dt + \int_0^T a(u_N(t), \varphi) \chi(t) dt = \int_0^T \langle f(t) | \varphi \rangle_{V^* \times V} \chi(t) dt$$

$$u_N(0) = g_N.$$

204 As a result of the elegant Hilbert space formulation, the semidiscrete problem (12) is uniquely
 205 solvable and the convergence of the sequence u_N to u is guaranteed, see Theorem 23.A and
 206 Remark 23.25 in [30].

207 The major advantage of equation (12) in regard to implementation is that it suffices to
 208 insert the basis functions as test functions. Thus, denoting $g_N = \sum_{j=1}^N \alpha_j \varphi_j$ and $u_N(t) :=$
 209 $\sum_{j=1}^N U_j(t) \varphi_j$ we arrive at

$$210 \quad \sum_{l=1}^N \dot{U}_l(t) \langle \varphi_l, \varphi_k \rangle_{L^2} + \sum_{l=1}^N U_l(t) a(\varphi_l, \varphi_k) = \langle f(t) | \varphi_k \rangle_{V^* \times V}$$

$$211 \quad U_j(0) = \alpha_j \quad \text{for all } j = 1, \dots, N.$$

213 Written in matrix form the problem is to find $U : [0, T] \rightarrow \mathbb{R}^N$ such that

$$214 \quad (13) \quad \mathbf{M} \dot{U}(t) + \mathbf{A} U(t) = \mathbf{F}(t)$$

$$215 \quad (14) \quad U(0) = \alpha,$$

217 where the *right hand side (vector)* \mathbf{F} is given by $\mathbf{F} = (F_1, \dots, F_N)^\top$ with $F_j(t) = \langle f(t) | \varphi_j \rangle_{V^* \times V}$
 218 for $j = 1, \dots, N$, $\alpha = (\alpha_1, \dots, \alpha_N)^\top$, and the *mass matrix* \mathbf{M} and *stiffness matrix* \mathbf{A} are
 219 given by

$$220 \quad (15) \quad M_{kl} = \langle \varphi_l, \varphi_k \rangle_{L^2}, \quad A_{kl} = a(\varphi_l, \varphi_k) \quad \text{for all } k, l = 1, \dots, N.$$

221 **3.2. Fully discrete scheme.** As fully discrete scheme, we approximate (13) with a θ
 222 scheme in time, namely

$$223 \quad (16) \quad \mathbf{M} \frac{U^{m+1} - U^m}{\Delta t} + \mathbf{A}U^{m+\theta}(t) = \mathbf{F}^{m+\theta}(t)$$

$$224 \quad (17) \quad U(0) = \alpha,$$

226 where $U^{m+\theta} = \theta U^{m+1} + (1 - \theta)U^m$, $F^{m+\theta}$ accordingly, and $\theta \in [0, 1]$.

227 **3.3. Flexible implementation for different driving Lévy processes.** We inspect equations
 228 (13) and (14) in regard to flexibility towards different options as well as models. All ingredients
 229 depend on the choice of the basis. While M is independent of the specific problem at hand, F
 230 and α represent the input data and therefore may vary for different option types. The stiffness
 231 matrix A carries the information of the driving process. So in order to obtain flexibility
 232 towards model types, we need a *generic way to compute the entries of the stiffness matrix*.
 233 For smooth basis functions with compact support and solution spaces without weighting, i.e.
 234 $\eta = 0$, according to (3) and (6), the stiffness matrix entries are given by

$$235 \quad a(\varphi_l, \varphi_k) = \sum_{i,j=1}^d \frac{\sigma^{i,j}}{2} \int_{\mathbb{R}^d} \frac{\partial}{\partial x_j} \varphi_l(x) \frac{\partial}{\partial x_i} \varphi_k(x) dx - \sum_{i=1}^d b^i \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} \varphi_l(x) \varphi_k(x) dx$$

$$236 \quad (18) \quad - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\varphi_l(x+y) - \varphi_l(x) - \sum_{i=1}^d \frac{\partial}{\partial x_i} \varphi_l(x) h_i(y) \right) F(dy) \varphi_k(x) dx.$$

238 Typical basis functions are not smooth. Therefore it is not a priori clear if the integral
 239 representation (18) extends to the usual basis functions. Observe that an extension of this
 240 representation requires some care: For a large and important class of pure jump Lévy pro-
 241 cesses, the solution spaces are Sobolev-Slobodeckii spaces of fractional order, i.e. H^α with
 242 some $0 < \alpha < 1$. For functions in H^α with $\alpha < 1$, however, the first order weak derivative
 243 in (18) is not defined and therewith this integral representation of the bilinear form is not
 244 well-defined. Understanding that the basis functions are usually in H^1 , we derive the validity
 245 of the representation under appropriate assumptions that also include the more challenging
 246 case of solution spaces with fractional order derivatives.

247 **Lemma 3.** *Let $d = 1$. Let a be defined by (9). Assume (A1)–(A3) for a , V and H and*
 248 *denote by $a : V \times V$ its unique bilinear continuous extension. If $H_\eta^1(\mathbb{R}) \subset V$, we have for*
 249 *every $\varphi, \psi \in H_\eta^1(\mathbb{R})$,*

$$250 \quad a(\varphi, \psi) = \frac{\sigma}{2} \int_{\mathbb{R}} \varphi'(x) \psi'(x) e^{2\eta x} dx - b(\eta, \sigma, F) \int_{\mathbb{R}} \varphi'(x) \psi(x) e^{2\eta x} dx$$

$$251 \quad (19) \quad - \int_{\mathbb{R}} \int_{|y| < 1} \int_0^y \int_0^z \varphi'(x+v) dv dz F(dy) (\psi'(x) + 2\eta\psi(x)) e^{2\langle \eta, x \rangle} dx$$

$$252 \quad - \int_{\mathbb{R}} \int_{|y| > 1} (\varphi(x+y) - \varphi(x)) F(dy) \psi(x) e^{2\langle \eta, x \rangle} dx$$

253

254 *with*

$$255 \quad b(\eta, \sigma, F) = b - 2\sigma\eta + \int_{|y|<1} (y - h(y))F(dy) - \int_{|y|>1} h(y)F(dy).$$

256 The proof of the Lemma is provided in Section A.1.

257 Inspecting the expression for the bilinear form, we encounter several numerical challenges
258 due to the integral part—stemming from the jumps of the process:

- 259 1. The appealing tridiagonal structure of the stiffness matrix for classic hat functions
260 related to the Black-Scholes equation does not extend to the general Lévy setting.
261 Instead, the stiffness matrix is densely populated. Pleasantly, it is still a Toeplitz
262 matrix.
- 263 2. For some choices of Lévy measures and bases the stiffness matrix entries may be derived
264 in closed form. This is for instance the case for the Merton model and piecewise linear
265 basis functions when $\eta = 0$. Following Section 10.6.2 in [18], the stiffness matrix
266 entries may be derived in semi-closed form expressions for a further group of jump
267 intensities including tempered stable, CGMY and KoBoL processes and the choice of
268 piecewise linear basis functions. In general, however, closed form expressions for the
269 stiffness matrix entries, when arbitrary models and basis functions are considered, are
270 not available.

271 An implementation that is flexible in the driving Lévy process therefore has to rely on
272 numerical approximations of the entries of the stiffness matrix. These approximations in-
273 evitably affect the accuracy of the solution to the scheme (13)–(14). The following question
274 arises: *How accurate does the integration routine have to be chosen in order to meet a desired*
275 *accuracy of the solution V ?*

276 In order to gain a first practical insight in the magnitude of the error resulting from
277 an inaccuracy in the stiffness matrix entries, consider Section 3.4.2 in [14]. The numerical
278 investigations presented therein reveal that an impressively high precision of the computation
279 of the entries of the stiffness matrix is required.

280 **4. Fourier approach to the Kolmogorov equation.** In regard to the high accuracy the
281 approximation of the stiffness matrix entries needs to achieve, we would like to avoid numerical
282 evaluations of the stiffness matrix entries on the basis of representation (??). Seeking for
283 alternative representations, let us point out that the symbol A of the Lévy process is always
284 available. Even more, it is an explicit function of the parameters of the process and thus can be
285 seen as the modelling quantity of the process as the Examples 9–12 show below. We therefore
286 take a Fourier perspective on the variational formulation of the Kolmogorov equation. This is
287 especially promising since the Kolmogorov operator \mathcal{A} of a Lévy process is a pseudo differential
288 operator with symbol A ,

$$289 \quad (20) \quad \mathcal{A}\varphi = \mathcal{F}^{-1}(A\mathcal{F}(\varphi)) \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d),$$

290 as elementary manipulations show. Now Parseval's identity yields

$$291 \quad (21) \quad a(\varphi, \psi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}(\mathcal{A}\varphi)(\xi) \overline{\mathcal{F}(\psi)(\xi)} \, d\xi$$

292 for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^d)$, respectively,

$$293 \quad (22) \quad a(\varphi, \psi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} A(\xi) \mathcal{F}(\varphi)(\xi) \overline{\mathcal{F}(\psi)(\xi)} \, d\xi.$$

294 This well-known identity has already proved to be highly beneficial for the analysis of the
 295 variational solutions of the Komogorov equations, compare e.g. [18], [15] and [16]. Let us
 296 point out the transition from the operator to the symbol from (21) to (22) in the bilinear form
 297 and recall its role for the derivation of the stiffness matrix in (15). The resulting alternative
 298 representation is key for the flexibility of our numerical approach. Exploiting the symbol will
 299 facilitate the numerical implementation considerably.

300 **Lemma 4 (Continuous extension of bilinear forms).** *Let A be the symbol of a Lévy process*
 301 *given by the characteristic triplet (b, σ, F) . Denote by $\mathcal{A} : C_0^\infty(\mathbb{R}^d, \mathbb{C}) \rightarrow C^\infty(\mathbb{R}^d, \mathbb{C})$ the pseu-*
 302 *do-differential operator associated with symbol A . Furthermore, denote by $a : C_0^\infty \times C_0^\infty \rightarrow \mathbb{C}$*
 303 *the bilinear form associated with the operator \mathcal{A} . Let $\eta \in \mathbb{R}^d$. If*
 304 *the exponential moment condition*

$$305 \quad (23) \quad \int_{|x|>1} e^{-\langle \eta', x \rangle} F(dx) < \infty$$

306 *holds for all $\eta' \in \text{sgn}(\eta^1)[0, |\eta^1|] \times \cdots \times \text{sgn}(\eta^d)[0, |\eta^d|]$ and*
 307 *there exists a constant $C_1 > 0$ with*

$$308 \quad (24) \quad |A(z)| \leq C_1(1 + \|z\|)^\alpha$$

309 *for all $z \in U_{-\eta} := U_{-\eta^1} \times \cdots \times U_{-\eta^d}$ with $U_{-\eta^j} = \mathbb{R} - i \text{sgn}(\eta^j)[0, |\eta^j|]$,*
 310 *then $a(\cdot, \cdot)$ possesses a unique linear extension $a : H_\eta^{\alpha/2} \times H_\eta^{\alpha/2} \rightarrow \mathbb{R}$ that can be written as*

$$311 \quad (25) \quad a(\varphi, \psi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} A(\xi - i\eta) \widehat{\varphi}(\xi - i\eta) \overline{\widehat{\psi}(\xi - i\eta)} \, d\xi$$

312 *for all $\varphi, \psi \in H_\eta^{\alpha/2}(\mathbb{R}^d)$.*

313 *Proof.* The proof can be found in [11] using Theorem 4.1 therein and Parseval's identity. ■

314 In order to gain first insight in the convergence analysis, we fix a level N in the Galerkin
 315 scheme and derive conditions for the convergence of the sequence of weak solutions that we
 316 obtain by approximating the stiffness matrix entries. In the implementation in Section 7 below
 317 we will also approximately compute the right hand side F of the equation. We therefore more
 318 generally consider sequences of stiffness matrices, right hand sides and initial conditions.

319 As usual, we denote for a given bilinear form $a : V \times V \rightarrow \mathbb{R}$ the associated operator
 320 $\mathcal{A} : V \rightarrow V^*$ defined by $\mathcal{A}(u)(v) := a(u, v)$ for all $u, v \in V$.

321 **Lemma 5.** *Let V, H and $a : V \times V \rightarrow \mathbb{R}$ satisfy (A1)–(A3). Let $X := \text{span}\{\varphi_1, \dots, \varphi_N\} \subset$
 322 V and for each $n \in \mathbb{N}$ let*

323 *(An1) $f_n, f \in L^2(0, T; H)$ with $f_n \rightarrow f$ in $L^2(0, T; X^*)$,*

324 *(An2) $g_n, g \in H$ with $g_n \rightarrow g$ in H ,*

325 (An3) $a_n : V \times V \rightarrow \mathbb{R}$ be a bilinear form such that for all $l, k \leq N$,

326 (26)
$$|(a_n - a)(\varphi_l, \varphi_k)| \rightarrow 0.$$

328 Then the sequence of unique weak solutions $u_n \in W^1(0, T; X, H)$ of

329 (27)
$$\dot{u}_n + \mathcal{A}_n u_n = f_n, \quad u_n(0) = g_n$$

330 converges strongly² in $L^2(0, T; X) \cap C(0, T; H)$ to the unique weak solution $u \in W^1(0, T; X, H)$
331 of

332 (28)
$$\dot{u} + \mathcal{A}u = f, \quad u(0) = g.$$

333 The proof is provided in Section A.2.

334 Next we introduce our approach to approximate the stiffness matrix entries.

335 **4.1. The symbol method.** The key component of a Galerkin FEM solver is the model
336 dependent stiffness matrix $\mathbf{A} \in \mathbb{R}^{N \times N}$. Using expression (18) of Section 3.3 above, the entries
337 of that matrix can be derived. The way the Lévy measure F enters that expression, however,
338 renders the numerical derivation of the matrix rather cumbersome. Additionally, the empirical
339 accuracy study of Section 3.4.2 in [14] emphasizes that utmost care must be taken when the
340 stiffness matrix entries are numerically derived. Consequently, in this section we approach
341 the calculation of the FEM solver components differently. The Fourier approach indicated by
342 Lemma 4 will allow us to access the model information required for the stiffness matrix and
343 all other FEM solver components via the symbol that is associated with the operator. In stark
344 contrast to the operator, the symbol of a Lévy model is numerically accessible in a unified
345 way for a large set of underlying models and we will present several examples highlighting this
346 feature.

347 Let us state the core lemma of this section. Here we concentrate on basis functions obeying
348 a simple nodal translation property, which is in particular satisfied for classical piecewise
349 polynomial basis functions.

350 **Lemma 6 (Symbol method for bilinear forms).** *Let the assumptions of Lemma 4 be satisfied*
351 *with $\eta = 0$. Assume further for $N \in \mathbb{N}$ a set of functions $\varphi_0, \varphi_1, \dots, \varphi_N \in H_0^{\alpha/2}(\mathbb{R})$ and nodes*
352 *$x_1, \dots, x_N \in \mathbb{R}$, such that for all $j = 1, \dots, N$*

353
$$\varphi_j(x) = \varphi_0(x - x_j) \quad \forall x \in \mathbb{R}.$$

354 Then we have

355 (29)
$$a(\varphi_l, \varphi_k) = \frac{1}{2\pi} \int_{\mathbb{R}} A(\xi) e^{i\xi(x_l - x_k)} |\widehat{\varphi_0}(\xi)|^2 d\xi.$$

356 for all $k, l = 1, \dots, N$. If additionally

357 (30)
$$\Re(A(\xi)) = \Re(A(-\xi)) \quad \text{and} \quad \Im(A(\xi)) = -\Im(A(-\xi)),$$

²Strong convergence in the Hilbert space $L^2(0, T; X)$ means $\|u_n - u\|_{L^2(0, T; X)} \rightarrow 0$.

358 then

$$359 \quad (31) \quad a(\varphi_l, \varphi_k) = \frac{1}{\pi} \int_0^\infty \Re \left(A(\xi) e^{i\xi(x_l - x_k)} \right) |\widehat{\varphi}_0(\xi)|^2 d\xi$$

360 for all $k, l = 1, \dots, N$.

361 *Proof.* Elementary properties of the Fourier transform yield

$$362 \quad (32) \quad \widehat{\varphi}_j(\xi) = e^{i\xi x_j} \widehat{\varphi}_0(\xi) \quad \forall \xi \in \mathbb{R}.$$

363 Since $\varphi_j \in H_0^{\alpha/2}(\mathbb{R})$ for all $j = 1, \dots, N$, the identity (29) follows from identity (25) with
 364 $\eta = 0$ above. The second claim (31) is then elementary. ■

365 When classic hat functions on an equidistant grid with mesh size $h \in \mathbb{R}$ are chosen as
 366 basis functions with

$$367 \quad (33) \quad \varphi_0(x) = (1 - |x|/h) \mathbb{1}_{|x| \leq h} \quad \forall x \in \mathbb{R}$$

368 we have

$$369 \quad (34) \quad \widehat{\varphi}_0(\xi) = \frac{2}{\xi^2 h} (1 - \cos(\xi h)) \quad \forall \xi \in \mathbb{R}.$$

370 **Corollary 7 (Symbol method for stiffness matrices).** *Let A be a univariate symbol with*
 371 *associated operator \mathcal{A} satisfying (24) with $\eta = 0$. Denote by $\varphi_j \in L^1(\mathbb{R})$, $j \in 1, \dots, N$ the basis*
 372 *functions of a Galerkin scheme associated with an equidistantly spaced grid $\Omega = \{x_1, \dots, x_N\}$*
 373 *possessing the property*

$$374 \quad (35) \quad \varphi_j(x) = \varphi_0(x - x_j) \quad \forall x \in \mathbb{R},$$

375 for some $\varphi_0 : \mathbb{R} \rightarrow \mathbb{R}$ with $\varphi_0 \in H_0^{\alpha/2}(\mathbb{R})$. Then, the stiffness matrix $\mathbf{A} \in \mathbb{R}^{N \times N}$ of the
 376 scheme can be computed by

$$377 \quad (36) \quad A_{kl} = \frac{1}{2\pi} \int_{\mathbb{R}} A(\xi) e^{i\xi(x_l - x_k)} |\widehat{\varphi}_0(\xi)|^2 d\xi$$

378 for all $k, l = 1, \dots, N$.

379 *Proof.* The proof is an immediate consequence of Lemma 6. ■

380 **Remark 8 (On the symbol method for bilinear forms).** *From a numerical perspective, the*
 381 *representations of the stiffness matrix entries provided in Lemma 6 and Corollary 7 are highly*
 382 *promising:*

383 1. *Instead of the double integrals appearing in (18), only one dimensional integrals need to be*
 384 *computed.*

385 2. *The model specific information is expressed via the symbol $\xi \mapsto A(\xi)$, which for a large set of*
 386 *models is available in form of an explicit function of ξ and the model parameters, a feature*
 387 *that we now can exploit numerically. We give a short list of examples below. For further*
 388 *examples we refer to [15] and [16].*

383. Representation (36) displays the entries of the stiffness matrix as Fourier integrals. Moreover,
 390 the nodes appear as Fourier variables. As a consequence, Fast Fourier Transform (FFT)
 391 methods can be used to accelerate their simultaneous computation.

394. The essential assumption of Lemma 6 and Corollary 7 is that the basis functions are obtained
 393 by shifting (and possibly scaling) a "mother" basis function. This is the case for a large and
 394 interesting class of bases, including the wavelet bases, and in particular extends to the multi-
 395 variate case. Therefore the methods we propose and analyse in this article in the univariate
 396 setting naturally extend to the multivariate case.

397 Expression (3) introduced operators \mathcal{A} for Lévy processes L in terms of the characteristic
 398 triplet (b, σ, F) . The following examples present the respective symbols for some well known
 399 Lévy models, where the asset price follows $S_t = S_0 e^{L_t}$ for every $t \geq 0$ and r is the deterministic
 400 continuously compounding interest rate.

401 **Example 9 (Symbol in the Black-Scholes (BS) model).** In the Black-Scholes model, deter-
 402 mined by the Brownian volatility $\sigma^2 > 0$, the symbol is given by

$$403 \quad (37) \quad A(\xi) = A^{bs}(\xi) = i\xi b + \frac{1}{2}\sigma^2\xi^2,$$

404 with drift b set to

$$405 \quad (38) \quad b = r - \frac{1}{2}\sigma^2$$

406 as required by the no-arbitrage condition.

407 **Example 10 (Symbol in the Merton model).** In the Merton model where $\sigma > 0$, $\lambda > 0$,
 408 $\alpha \in \mathbb{R}$ and $\beta > 0$, the symbol computes to

$$409 \quad (39) \quad A(\xi) = A^{merton}(\xi) = i\xi b + \frac{1}{2}\sigma^2\xi^2 - \lambda \left(e^{-i\alpha\xi - \frac{1}{2}\beta^2\xi^2} - 1 \right)$$

410 with drift set to

$$411 \quad (40) \quad b = r - \frac{1}{2}\sigma^2 - \lambda \left(e^{\alpha + \frac{\beta^2}{2}} - 1 \right),$$

412 as required by the no-arbitrage condition.

413 **Example 11 (Symbol in the CGMY model).** In the CGMY model of [6] with $\sigma > 0$, $C > 0$,
 414 $G \geq 0$, $M \geq 0$ and $Y \in (1, 2)$, the symbol computes to

$$415 \quad (41) \quad A(\xi) = A^{cgmy}(\xi) = i\xi b + \frac{1}{2}\sigma^2\xi^2 - C\Gamma(-Y) \left[(M + i\xi)^Y - M^Y + (G - i\xi)^Y - G^Y \right],$$

416 for all $\xi \in \mathbb{R}$, with drift b set to

$$417 \quad (42) \quad b = r - \frac{1}{2}\sigma^2 - C\Gamma(-Y) \left[(M - 1)^Y - M^Y + (G + 1)^Y - G^Y \right]$$

418 for martingale pricing. This class is a special case of the classes referred to as Koponen and
 419 KoBoL in the literature, see e.g. [3] and as tempered stable processes.

420 **Example 12 (Symbol in the NIG model).** With $\sigma > 0$, $\alpha > 0$, $\beta \in \mathbb{R}$ and $\delta > 0$ such that
 421 $\alpha^2 > \beta^2$, the symbol of the NIG model is given by

422 (43)
$$A(\xi) = A^{nig}(\xi) = i\xi b + \frac{1}{2}\sigma^2\xi^2 - \delta \left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta - i\xi)^2} \right)$$

423 for all $\xi \in \mathbb{R}$ with drift given by

424 (44)
$$b = r - \frac{1}{2}\sigma^2 - \delta \left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2} \right)$$

425 as required by the no-arbitrage condition.

426 Implementing (36), we encounter new numerical challenges: From the perturbation study
 427 in Section 3.4.2 in [14] we conclude that we need to evaluate the integrals at high precision.
 428 Consider first the Black-Scholes model and choose the piecewise linear hat functions as basis
 429 elements as a toy example. Applying a standard Matlab integration routine will lead to
 430 considerable errors. To understand the effect, let us first consider the oscillatory contribution
 431 by the hat functions stemming from the Fourier transform in expression (34) to the integrands
 432 in (36). We depict $\widehat{\varphi}_0$ in Figure 1.

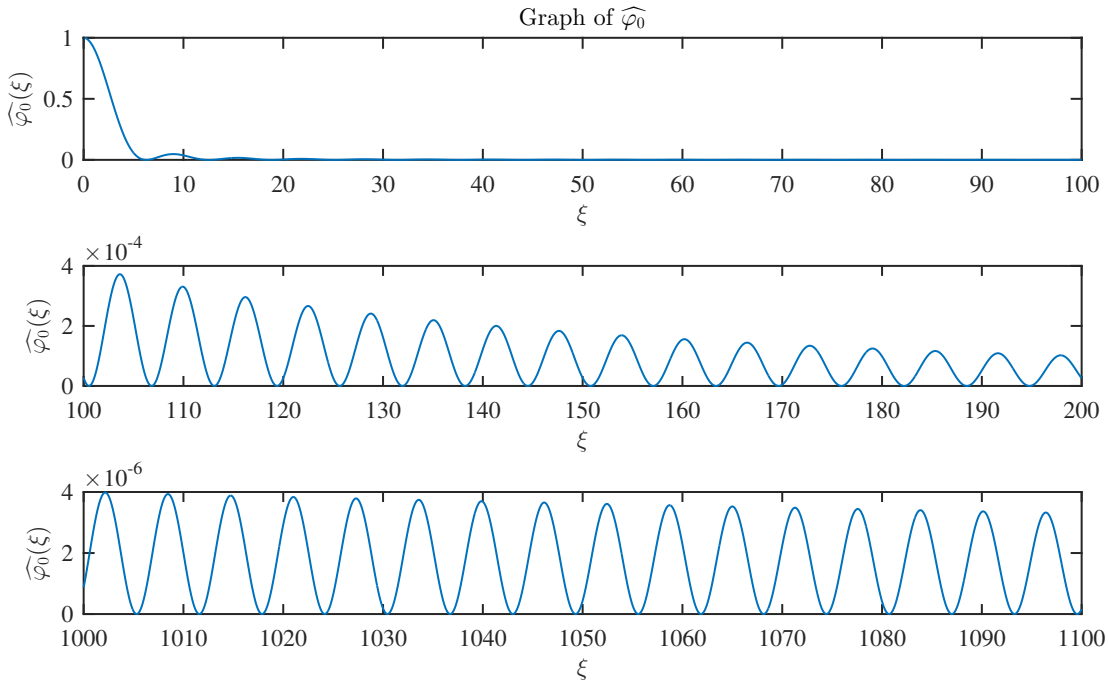


Figure 1. Consider the hat function φ_0 of expression (33) with $h = 1$. The graph depicts its Fourier transform $\widehat{\varphi}_0$ which is evaluated over three subintervals of \mathbb{R}^+ . The oscillations and the rather slow decay to zero complicate numerical integration with high accuracy requirements considerably.

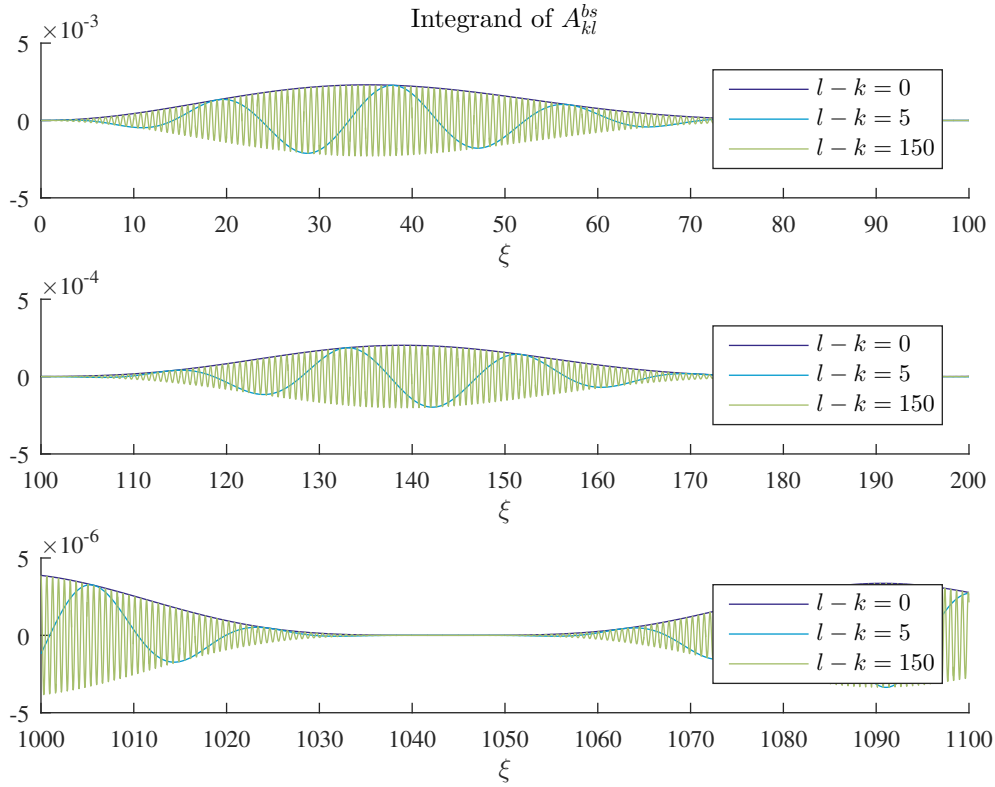


Figure 2. The integrand for the Black-Scholes stiffness matrix A_{kl} for several values of $l-k$. The grid of the hat functions spans the interval $[-5, 5]$ with 150 equidistantly spaced inner nodes and grid fineness $h = 0.0662$. A Black-Scholes solution on this grid would thus be represented by the weighted sum of 150 hat functions. We observe that oscillations of the integrand increase in the value of $|l-k|$ and so does the number of supporting points for naive numerical integration.

433 Furthermore, Figure 2 shows several integrands of $\mathbf{A} \in \mathbb{R}^{N \times N}$ in the representation pro-
 434 vided by (36) of Corollary 7 with the Black-Scholes symbol of Example 9. Therein, each
 435 integrand is evaluated for a different value of $l-k$ over three different subintervals taken
 436 from the unbounded integration range. Here, the integrands of A_{kl} , $1 \leq k, l \leq N$, have to be
 437 numerically integrated for all $l-k \in \{-(N-1), \dots, -1, 0, 1, \dots, N-1\}$.

438 The larger $|l-k|$, however, the more severe the numerical challenges for evaluating the
 439 integrand, as Figure 2 demonstrates. All integrands illustrated therein decay rather slowly.
 440 Additionally, oscillations increase in $|l-k|$. In combination, these two observations seriously
 441 threaten a numerically reliable evaluation of the integral. With increasing values of $|l-k|$,
 442 the oscillations of the integrand accelerate and the number of necessary supporting points for
 443 accurate integration increases. Computation of the stiffness matrix entries along these lines by
 444 invoking standard integration routines e.g. based on Matlab's `quadgk` demands considerable
 445 run times for accurate results.

446 These findings show that we need to further investigate the problem to obtain a flexible
 447 method to compute the stiffness matrix reliably and with low computational cost. The path
 448 that we propose here is to modify the problem in such a way that the resulting integrands
 449 decay much faster so that the domain of integration can be chosen considerably smaller and
 450 a usual integration routine such as Matlab's function `quadgk` is sufficient. To do so, we first
 451 observe that the hat functions, which we used in our toy example, are piecewise linear. While
 452 being continuous they are not continuously differentiable everywhere and thus lack smoothness
 453 on an elementary level already. This lack of smoothness translates into a slow decay of their
 454 Fourier transform or $\widehat{\varphi}_0$, respectively.

455 Therefore, we propose to replace the piecewise linear basis functions by basis functions that
 456 display considerably higher regularity leading to appealing decay properties of the integrands
 457 in (36). In the following two sections, we present two different approaches to implement such
 458 a problem modification.

459 **5. From classic hat functions to mollified hats.** It is well known that convolution with
 460 a smooth function has a smoothing effect on the function that the convolution is applied to.
 461 Functions that qualify for this smoothing by convolution are called mollifiers. In order to
 462 choose an appropriate mollifier for our purposes—the fast and accurate computation of the
 463 integrals in (36), the mollifiers need to display two essential features:

- 464 (1) The Fourier transform of the modified basis function needs to be available.
 465 (2) The smoothing effect needs to be steerable through a parameter.

466 As the Fourier transform of the convolution of two functions is the product of the two Fourier
 467 transformed functions, (1) boils down to the availability of the Fourier transform of the mol-
 468 lifier. Since the Fourier transform of standard mollifiers is not available in closed form, we
 469 widen the range of the standard mollifiers and allow for non-compact support. More precisely,
 470 we call the sequence $m = (m_k)_{k \in \mathbb{N}}$, $m_k \in L^1(\mathbb{R})$ for all $k \in \mathbb{N}$, a *mollifier*, if

- 471 1. $m_k \geq 0$, for all $k \in \mathbb{N}$,
 472 2. $\int_{\mathbb{R}} m_k(x) dx = 1$, and
 473 3. for all $\varrho > 0$ we have the convergence $\int_{[-\varrho, \varrho]^c} m_k(x) dx \rightarrow 0$ for $k \rightarrow \infty$.

474 Feature (2) is often required and we follow the usual construction here. By Proposition
 475 and Definition 2.14 in [1] we can adjust the influence of mollification by a parameter ε . To
 476 this end let $m \in L^1(\mathbb{R})$ with

477 (45)
$$m \geq 0, \quad \text{and} \quad \int_{\mathbb{R}} m(x) dx = 1.$$

478 Define

479 (46)
$$m^\varepsilon = \frac{1}{\varepsilon} m \left(\frac{\cdot}{\varepsilon} \right).$$

480 Then for each $\varrho > 0$ we have $\int_{\mathbb{R}} m^\varepsilon(x) dx = 1$ and $\int_{[-\varrho, \varrho]^c} m^\varepsilon(x) dx \rightarrow 0$ for $\varepsilon \rightarrow 0$. Conse-
 481 quently, for each null sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ the sequence $(m^{\varepsilon_k})_{k \in \mathbb{N}}$ is a mollifier in the sense of
 482 our definition.

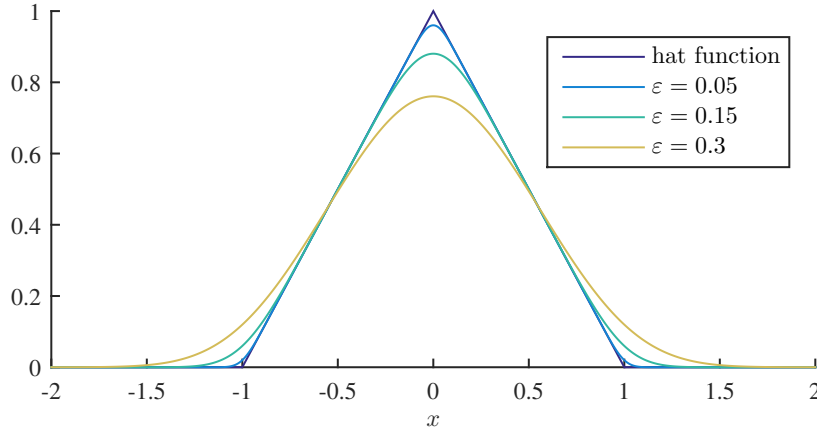


Figure 3. A comparison between the classic hat function φ_0 on a grid with $h = 1$ and the mollified hat function $\varphi_0^\varepsilon = \varphi_0 * m_{\text{Gaussian}}^\varepsilon$ for several values of $\varepsilon \in \{0.05, 0.15, 0.3\}$ using the Gaussian mollifier of Example 13.

483 **Example 13 (A mollifier based on the Normal distribution).** We present an example for a
484 mollifier. Define

$$485 \quad (47) \quad m_{\text{Gaussian}}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

486 Then we call $(m_{\text{Gaussian}}^{\varepsilon_k})_{k \in \mathbb{N}}$ defined according to (46) a Gaussian mollifier. The characteristic
487 function of the Gaussian mollifier is known in closed form,

$$488 \quad (48) \quad \widehat{m_{\text{Gaussian}}^\varepsilon}(\xi) = \exp\left(-\frac{1}{2}\varepsilon^2\xi^2\right),$$

489 thus exhibiting exponential decay.

490 It is a well known result, that mollified functions $f * m_k$ converge to f in $L^p(\mathbb{R})$, $1 \leq p < \infty$
491 when k tends to infinity, see for example Satz 2.15 in [1].

492 Figure 4 displays the decay of the Fourier transform of the mollified hat function in
493 comparison with the behaviour of the hat.

494 **5.1. Convergent Scheme based on mollified hats.** In this section we propose and analyse
495 a convergent fully discrete scheme based on the symbol method of Section 4 and mollified hats.
496 We also analyze the rate of convergence of the scheme. We introduce stronger assumptions that
497 allow us to use the result of [27]. Namely, we assume ellipticity of the bilinear form instead of
498 the weaker assumption that a Gårding inequality.

499 According to the symbol method introduced in Corollary 7, we solve the θ scheme (16)–
500 (17) with stiffness matrix \mathbf{A} given by equation (36),

$$501 \quad (49) \quad \mathbf{A}_{kl} = a(\varphi_l, \varphi_k) = \frac{1}{2\pi} \int_{\mathbb{R}} A(\xi) e^{i\xi(x_l - x_k)} |\widehat{\varphi_0}(\xi)|^2 d\xi$$

502 for all $k, l = 1, \dots, N$, where φ_l are the hat functions and φ_0 is the hat function at the origin
 503 given by (35).

504 For a light notation let $m_\varepsilon := m_{\text{Gaussian}}^\varepsilon$. Following the approach we introduced in Section
 505 5 to use mollified hats, we replace the stiffness matrix of (33) by

$$506 \quad (50) \quad \mathbf{A}_{kl}^\varepsilon := \frac{1}{2\pi} \int_{\mathbb{R}} A(\xi) e^{i\xi(x_l - x_k)} |\widehat{\varphi_0}(\xi)|^2 |\widehat{m}_\varepsilon(\xi)|^2 d\xi$$

507 On the level of the bilinear form this means we replace the bilinear form a by

$$508 \quad (51) \quad a^\varepsilon(u, v) := \frac{1}{2\pi} \int_{\mathbb{R}} A(\xi) \widehat{u}(\xi) \overline{\widehat{v}(\xi)} |\widehat{m}_\varepsilon(\xi)|^2 d\xi.$$

509 In order to achieve the optimal order of convergence of the thus perturbed θ scheme, we need
 510 to choose ε dependent on h , i.e. $\varepsilon = \varepsilon(h)$. Moreover, in an actual implementation, we will
 511 need to truncate the range of integration. In order to preserve the two fundamental properties,
 512 Gårding inequality and continuity with respect to the solution space V of the original equation,
 513 we incorporate here the *asymptotic behaviour of the symbol*. The asymptotic behaviour of the
 514 symbol \widetilde{A} plays a decisive role in the determination of the solution space. To this aim,

515 let $\widetilde{A} : \mathbb{R} \rightarrow \mathbb{R}$ be such that there exists $N > 0$ such that

$$516 \quad (52) \quad |A(\xi) - \widetilde{A}(\xi)| \leq |A(\xi)|/2 \quad \text{for all } |\xi| > N$$

517 To illustrate what form \widetilde{A} can take in practice, let us briefly consider a simple example. \widetilde{A}
 518 carries the asymptotic behaviour of A and the convergence needs to be fast enough. This
 519 is for instance satisfied if we take for A the symbol in Merton's model from Example 10,
 520 $A(\xi) = A^{\text{merton}}(\xi) = i\xi b + \frac{1}{2}\sigma^2\xi^2 - \lambda(e^{-i\alpha\xi - \frac{1}{2}\beta^2\xi^2} - 1)$ and for \widetilde{A} we use its Brownian part,
 521 $\widetilde{A}(\xi) = \frac{1}{2}\sigma^2\xi^2$.

522 Now let

$$523 \quad (53) \quad \widetilde{a}^\varepsilon(u, v) := \frac{1}{2\pi} \int_{-N(\varepsilon)}^{N(\varepsilon)} A(\xi) \widehat{u}(\xi) \overline{\widehat{v}(\xi)} |\widehat{m}_\varepsilon(\xi)|^2 d\xi + \frac{1}{2\pi} \int_{[-N(\varepsilon), N(\varepsilon)]^c} \widetilde{A}(\xi) \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi$$

524 Now choose $N(\varepsilon) := \frac{\delta}{\varepsilon}$ and $\varepsilon(h) := \delta h$ for some $0 < \delta < 1$, $0 < \delta < \min\{\frac{1}{2\delta^2}, \frac{1}{\sqrt{2}}\}$. Then

$$525 \quad (54) \quad N^2(\varepsilon)\varepsilon^2 < 1/2, \text{ and } N(\varepsilon(h))(\varepsilon(h))^2 \leq \delta\delta h \quad \text{for all } h.$$

526 Under standard conditions convergence of the fully discrete version of the (13)–(15) with
 527 a θ -scheme in time has been provided in [27]. Assuming the same standard conditions, we
 528 show that the resulting fully discrete scheme when replacing in (13)–(15) the bilinear form a
 529 by $\widetilde{a}^{\varepsilon(h)}$ still leads to a convergent scheme of the same rate.

530 While the asymptotic behaviour of A is used in the theoretical analysis, numerically the
 531 same error behaviour is already achieved when neglecting the second term in (54), compare
 532 Section 7.4. This shows the potential of the approach even beyond the cases where the
 533 asymptotic behaviour of A is accessible in a simple form that allows to compute the second
 534 term in (53).

535 **5.2. Convergence analysis.** General assumptions and notation: $I = (a, b) \subset \mathbb{R}$, $H :=$
536 $L^2(I)$, $V^s := H^s(I)$, let V_h^s be a Galerkin space, e.g. the linear space spanned by the hat
537 functions with mesh fineness h . For $\varepsilon > 0$ consider the Gauss kernel $m_{\text{Gaussian}}^\varepsilon$ from (46), (13).
538 Now let $\varepsilon : (0, \infty) \rightarrow (0, \infty)$ and define $\widetilde{V}_h^s := \{(m_{\text{Gaussian}}^{\varepsilon(h)} * u_h)|_I \mid u_h \in V_h^s\}$, where with a
539 slight abuse of notation, we denote by u_h the extension of u_h by zero outside of I in order
540 to define the convolution with $m_{\text{Gaussian}}^{\varepsilon(h)}$. We notice that this extension is not necessarily in
541 $H^s(\mathbb{R})$. We also denote $\widetilde{u}_h := (m_{\text{Gaussian}}^{\varepsilon(h)} * u_h)|_I$.
542 We denote by $u_h^0 = g_h$ the initial condition of the θ scheme.
543 Furthermore we set

$$544 \quad V^t := \begin{cases} \widetilde{H}^s(I) & \text{if } s = \alpha, \\ H^{s+1}(I) \cap \widetilde{H}^s(I) & \text{if } t = \alpha + 1. \end{cases}$$

546 Finally, set $a(u, v) = \int_{\mathbb{R}^d} A(\xi) \hat{u}(\xi) \overline{\hat{v}(\xi)} \, d\xi$, $\|u\|_a := \sqrt{a(u, u)}$ and $\|f\|_* := \frac{f(v_h)}{\|v_h\|_a}$.

547 We consider the following set of conditions that form the basis of the perturbation analysis
548 in [27]:

549 **Conditions 14.** Fix index $\alpha \in [0, 1]$.

550 (A1) (Continuity and coercivity) There exist constants $0 < \beta, \gamma$ such that for all $\xi \in \mathbb{R}$,

$$551 \quad \beta |\xi|^{2\alpha} \leq A(\xi) \leq \gamma |\xi|^{2\alpha}.$$

553 (A2) (Approximation property of the Galerkin space) There exists a family of bounded linear
554 projectors $P_h : V^\alpha \rightarrow V_h^\alpha$ and a constant $C_1 > 0$ such that for all $u \in V^{\alpha+1}$

$$555 \quad (55) \quad \|u - P_h u\|_{V^\alpha} \leq C_1 h \|u\|_{V^{\alpha+1}}.$$

556 (A3) (Inverse property) There is a constant $C_{\text{IP}} > 0$ independent of $h > 0$ such that with
557 $0 \leq s \leq \alpha$ we have for all $u_h \in V_h^s$

$$558 \quad (56) \quad \|u_h\|_{V^s} \leq C_{\text{IP}} h^{-s} \|u_h\|_H.$$

559 (A4) (Quasi-optimality of the initial condition) There is a constant $C_I > 0$ independent of
560 $h > 0$ such that

$$561 \quad (57) \quad \|g - g_h\|_H \leq C_I \inf_{v_h \in V_h^s} \|g - v_h\|_H.$$

562 Condition (A1) is equivalent to the continuity and ellipticity of the bilinear form a with
563 respect to V^α . Conditions (A2)–(A4) are basic approximation conditions on the Galerkin
564 spaces. They are not only satisfied for V_h^s being the linear space spanned by the hat functions
565 with mesh fineness h , but also for wavelet approximation spaces, see [27].

566 We consider an equidistant time grid, $t_m = m * T / (M - 1)$, $m = 0, \dots, M$ and denote
567 $u^m = u(t_m)$, $u^{m+\theta} = \theta u^{m+1} + (1 - \theta) u^m$, $u_h^\kappa = \sum_{j=1}^{\dim(V_h^\alpha)} U_j^\kappa \varphi_j$ for $\kappa = m$ or $\kappa = m + \theta$.

568 Let us first consider the rate of convergence of the θ scheme without perturbation that we
569 directly obtain from Theorem 5.4 of [27], by choosing $\tilde{a} = a$ and $\nu = 0$ $p = \alpha$ and $\alpha = \varrho/2$ in
570 their setting:

571 **Lemma 15 (Convergence rate of the θ scheme).** Assume Conditions 14 and let $u \in$
 572 $W^1(0, T; V^\alpha, H)$ be the weak solution to problem (4)–(5). Then there exists a constant $\bar{C} > 0$
 573 such that

$$\begin{aligned}
 & \|u^M - u_h^M\|_H^2 + \Delta t \sum_{m=0}^{M-1} \|u^{m+\theta} - u_h^{m+\theta}\|_a^2 \leq \bar{C} h^2 \max_{0 \leq \tau \leq T} \|u(\tau)\|_{V^{\alpha+1}}^2 \\
 & + \bar{C} h^2 \int_0^T \|u(\tau)\|_{V^{\alpha+1}}^2 d\tau \\
 & + \bar{C} \begin{cases} (\Delta t)^2 \int_0^T \|\ddot{u}(s)\|_*^2 ds, & \forall \theta \in [0, 1] \\ (\Delta t)^4 \int_0^T \|\ddot{u}(s)\|_*^2 ds, & \theta = \frac{1}{2}. \end{cases}
 \end{aligned}
 \tag{58}$$

575 Notice that the assertion of the lemma is only meaningful if the regularity of u implies finiteness
 576 of the right-hand-side of the equation. In other words, the assertion on the convergence rate
 577 implicitly comes with regularity assumptions on the solution u .

578 **5.2.1. Convergence rate for θ scheme, mollified hat.** We denote by $(\widetilde{u}_h^m)_{m=1, \dots, M}$ the
 579 interpolated solution of the θ scheme induced by $\widetilde{a}^{\varepsilon(h)}$.

580 **Proposition 16.** The assertion of Lemma 15 also holds for the solution $(\widetilde{u}_h^m)_{m=1, \dots, M}$ of the
 581 perturbed θ scheme instead of $(u_h^m)_{m=1, \dots, M}$.

582 *Proof.* In view of Conditions 14, in order to apply Theorem 5.4 of [27], it is enough to
 583 verify two conditions for the perturbation of the bilinear form a , namely

584 (i) There exists a constant $\eta < 1$ independent of h such that

$$|a(u, v) - \widetilde{a}^{\varepsilon(h)}(u, v)| \leq \eta \|u\|_a \|v\|_a \quad \text{for all } u, v \in V^\alpha.
 \tag{59}$$

585 (ii) For the family of projectors P_h of Condition (A2) there exists a constant $C > 0$ independent
 586 of h such that

$$|a(P_h u, v_h) - \widetilde{a}^{\varepsilon(h)}(P_h u, v_h)| < Ch \|u\|_{V^{\alpha+1}} \|v_h\|_{V^\alpha} \quad \text{for all } u \in V^{\alpha+1}, v_h \in V_h^\alpha.
 \tag{60}$$

587 These two conditions are inequalities (3.8) and (3.9) of [27].

588 Verify (i): Inserting the definition, we see, denoting $N = N(\varepsilon(h))$ that

$$\begin{aligned}
 |a(u, v) - \widetilde{a}^{\varepsilon(h)}(u, v)| & \leq \frac{1}{2\pi} \left| \int_{-N}^N A(\xi) \widehat{u}(\xi) \overline{\widehat{v}(\xi)} (1 - |\widehat{m}_\varepsilon(\xi)|^2) d\xi \right| \\
 & + \frac{1}{2\pi} \left| \int_{[-N, N]^c} (A - \widetilde{A})(\xi) \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi \right|,
 \end{aligned}$$

589 where $|\widehat{m}_\varepsilon(\xi)|^2 = e^{-\varepsilon^2 \xi^2}$ and $0 \leq 1 - |\widehat{m}_\varepsilon(\xi)|^2 = 1 - e^{-\varepsilon^2 \xi^2} \leq \varepsilon^2 \xi^2$, and hence

$$\frac{1}{2\pi} \left| \int_{-N}^N A(\xi) \widehat{u}(\xi) \overline{\widehat{v}(\xi)} (1 - |\widehat{m}_\varepsilon(\xi)|^2) d\xi \right| \leq \varepsilon^2 N^2 \|u\|_a \|v\|_a.$$

597 Moreover,

$$598 \quad \frac{1}{2\pi} \left| \int_{[-N,N]^c} (A - \tilde{A})(\xi) \widehat{u}(\xi) \overline{\widehat{v}(\xi)} \, d\xi \right| \leq 1/2 \int_{[-N,N]^c} |A(\xi)| |\widehat{u}(\xi)| |\widehat{v}(\xi)| \, d\xi$$

$$600 \quad \leq 1/2 \|u\|_a \|v\|_a.$$

601 Summarizing, since $\varepsilon(h)^2 N(h)^2 < 1/2$ for h small enough, we have

$$602 \quad |a(u, v) - \widetilde{a^{\varepsilon(h)}}(u, v)| \leq \eta \|u\|_a \|v\|_a$$

604 for some $\eta < 1$.

605 Verify (ii): We first show the assertion when we replace $P_h u$ by u . We observe that

$$606 \quad |a(u, v_h) - \widetilde{a^{\varepsilon(h)}}(u, v_h)| \leq \frac{1}{2\pi} \int_{-N}^N |A(\xi) \widehat{u}(\xi) \overline{\widehat{v}(\xi)}| \left(1 - |\widehat{m}_\varepsilon(\xi)|^2\right) \, d\xi$$

$$607 \quad + \frac{1}{2\pi} \int_{[-N,N]^c} |A - \tilde{A}(\xi)| |\widehat{u}(\xi) \overline{\widehat{v}(\xi)}| \, d\xi.$$

609 Using Hölder's inequality, inserting again $1 - |\widehat{m}_\varepsilon(\xi)|^2 = 1 - e^{-\varepsilon^2 \xi^2} \leq \varepsilon^2 \xi^2$, the continuity
610 condition from (A1) and inequality (54) we get

$$611 \quad \frac{1}{2\pi} \int_{-N}^N |A(\xi) \widehat{u}(\xi) \overline{\widehat{v}_h(\xi)}| \left(1 - |\widehat{m}_\varepsilon(\xi)|^2\right) \, d\xi$$

$$612 \quad \leq \frac{1}{2\pi} \left(\int_{\mathbb{R}} |A(\xi)| |\widehat{v}_h(\xi)|^2 \, d\xi \right)^{1/2} \left(\int_{-N}^N |A(\xi)| |\widehat{u}(\xi)|^2 \left(1 - |\widehat{m}_\varepsilon(\xi)|^2\right)^2 \, d\xi \right)^{1/2}$$

$$613 \quad \leq \sqrt{1/(2\pi)} \|v_h\|_a \left(\int_{-N}^N |A(\xi)| |\widehat{u}(\xi)|^2 \varepsilon^4 \xi^4 \, d\xi \right)^{1/2}$$

$$614 \quad \leq \varepsilon^2 N \sqrt{1/(2\pi)} \|v_h\|_a \left(\int_{\mathbb{R}} |A(\xi)| \xi^2 |\widehat{u}(\xi)|^2 \, d\xi \right)^{1/2}$$

$$615 \quad \leq \varepsilon^2 N \sqrt{\gamma/(2\pi)} \|v_h\|_a \|u\|_{V^{\alpha+1}}$$

$$616 \quad \leq h/(2\delta) \sqrt{\gamma/(2\pi)} \|v_h\|_a \|u\|_{V^{\alpha+1}}.$$

618 Finally,

$$619 \quad \frac{1}{2\pi} \int_{[-N,N]^c} |A(\xi) - \tilde{A}(\xi)| |\widehat{u}(\xi) \widehat{v}_h(\xi)| \, d\xi$$

$$620 \quad \leq \frac{1}{4\pi} \int_{[-N,N]^c} |A(\xi)| |\widehat{u}(\xi)| |\widehat{v}_h(\xi)| \, d\xi$$

$$621 \quad \leq \frac{1}{4\pi} \int_{[-N,N]^c} |A(\xi)| |\widehat{u}(\xi)| \frac{|\xi|}{N} |\widehat{v}_h(\xi)| \, d\xi$$

$$622 \quad \leq \frac{1}{4\pi N} \left(\int_{\mathbb{R}} |A(\xi)| |\widehat{v}_h(\xi)|^2 \, d\xi \right)^{1/2} \left(\int_{\mathbb{R}} |A(\xi)| \xi^2 |u(\xi)|^2 \, d\xi \right)^{1/2}$$

$$623 \quad \leq h\delta/(2\delta) \sqrt{\gamma/(2\pi)} \|v_h\|_a \|u\|_{V^{\alpha+1}}.$$

625 Now we are in a position to derive assertion (ii): By the triangle inequality we have

$$626 \quad |a(P_h u, v_h) - \widetilde{a^{\varepsilon(h)}}(P_h u, v_h)| \leq |a(u, v_h) - \widetilde{a^{\varepsilon(h)}}(u, v_h)| + |(a - \widetilde{a^{\varepsilon(h)}})(P_h u - u, v_h)|.$$

628 Invoking inequality (59), (60) for u instead of $P_h u$ and approximation property (A2) of Con-
629 ditions 14 show the existence of a constant $C > 0$ such that

$$630 \quad (61) \quad |a(P_h u, v_h) - \widetilde{a^{\varepsilon(h)}}(P_h u, v_h)| < Ch \|u\|_{V^{\alpha+1}} \|v_h\|_{V^\alpha} \quad \text{for all } u \in V^{\alpha+1}, v_h \in V_h^\alpha. \quad \blacksquare$$

631 Before we test the numerical performance of this approach to modify the Galerkin scheme
632 in Section 7 below, we introduce an alternative approach based on splines. We keep the
633 presentation of the second approach shorter since the numerical results are more promising
634 for the mollified hat approach.

635 **6. Splines as basis functions.** Instead of mollification of piecewise linear basis functions,
636 we can alternatively choose basis functions that display higher regularity itself. We therefore
637 investigate a well-established class of finite element basis functions as candidates for our
638 purposes, namely cubic splines. Spline theory applies to a very broad context and we refer
639 the reader to [26] for an introduction and overview. From our perspective, splines are smooth
640 basis functions. Their Fourier transform is accessible and the theory of function spaces they
641 span is well-established. We give the definition of the Irwin-Hall cubic spline that inherits
642 its name from the related probability distribution. We define the univariate *Irwin-Hall spline*
643 $\varphi_0 : \mathbb{R} \rightarrow \mathbb{R}^+$ by

$$644 \quad (62) \quad \varphi_0(x) = \frac{1}{4} \begin{cases} (x+2)^3 & , -2 \leq x < -1 \\ 3|x|^3 - 6x^2 + 4 & , -1 \leq x < 1 \\ (2-x)^3 & , 1 \leq x \leq 2 \\ 0 & , \text{elsewhere} \end{cases}$$

645 for all $x \in \mathbb{R}$. The spline φ_0 has compact support on $[-2, 2]$ and is a cubic spline. We use it
646 to define a spline basis:

647 **Definition 17 (Spline basis functions on an equidistant grid).** Choose $N \in \mathbb{N}$. Assume an
648 equidistant grid $\Omega = \{x_1, \dots, x_N\}$, $x_j \in \mathbb{R}$ for all $j = 1, \dots, N$, with mesh fineness $h > 0$. Let
649 φ_0 be the Irwin-Hall spline of (62). For $j = 1, \dots, N$ define

$$650 \quad \varphi_j(x) = \varphi_0((x - x_j)/h) \quad \forall x \in \mathbb{R}.$$

651 We call φ_j the spline basis function associated to node j .

652 For a given grid $\Omega = \{x_1, \dots, x_N\}$, $x_j \in \mathbb{R}$, Definition 17 provides the set of spline basis
653 functions that we also use in our numerical implementation, later. In standard literature, the
654 Irwin-Hall basis is usually enriched with additional splines associated with the first and the last
655 node of the grid that provide further flexibility in terms of Dirichlet and Neumann boundary
656 conditions, see for example [26]. We omit the three Irwin-Hall basis functions associated with
657 either of the first and the last grid nodes thus implicitly prescribing Dirichlet, Neumann and
658 second order derivative zero boundary conditions.

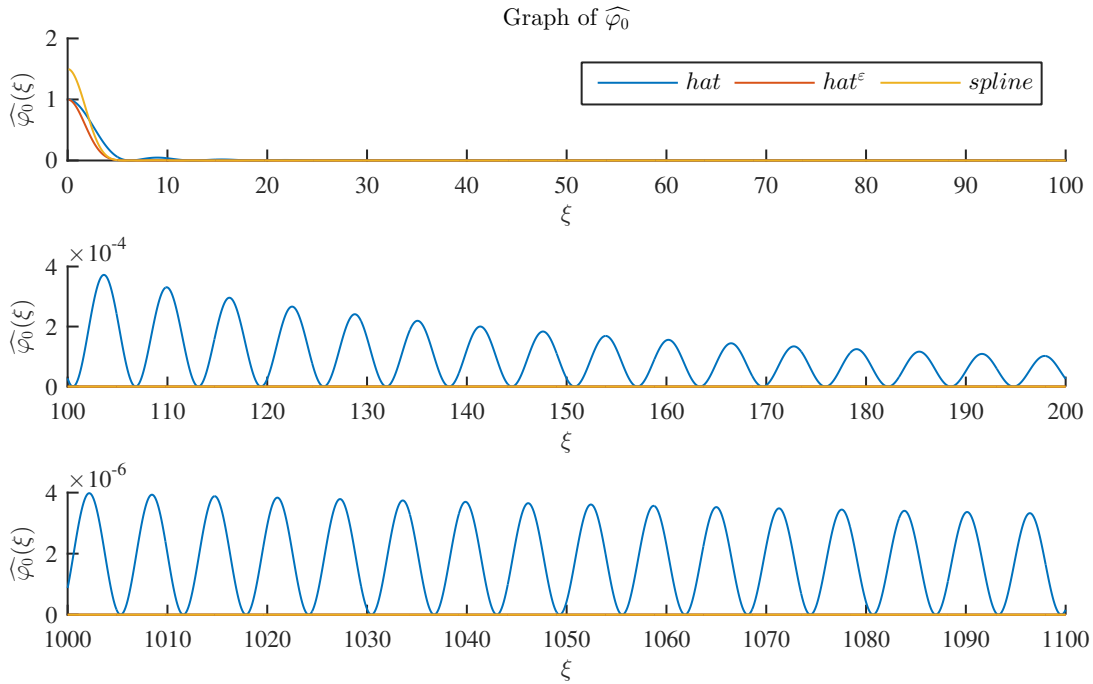


Figure 4. Graphs of the Fourier transforms of all basis function candidates presented in this section, evaluated over three subintervals of \mathbb{R}^+ . The mesh is chosen with $h = 1$, the mollification parameter is set to $\varepsilon = 0.3h$.

659 **Lemma 18 (Fourier transform of the Irwin-Hall spline).** Let φ_0 be the Irwin-Hall cubic spline
 660 of (62). Then its Fourier transform $\widehat{\varphi}_0$ is given by

$$661 \quad (63) \quad \widehat{\varphi}_0(\xi) = \frac{3}{\xi^4} (\cos(2\xi) - 4 \cos(\xi) + 3)$$

662 for all $\xi \in \mathbb{R}$.

663 The proof of the Lemma follows by elementary calculation. This immediately gives the fol-
 664 lowing corollary.

665 **Corollary 19 (Fourier transform of spline basis functions on an equidistant grid).** Choose
 666 $N \in \mathbb{N}$. Assume an equidistant grid $\Omega = \{x_1, \dots, x_N\}$, $x_j \in \mathbb{R}$ for all $j = 1, \dots, N$, with
 667 mesh fineness $h > 0$ and let φ_j be the spline basis function associated with node j as defined
 668 in Definition 17. Its Fourier transform is given by

$$669 \quad \widehat{\varphi}_j(\xi) = e^{i\xi x_j} \frac{3}{h^3 \xi^4} (\cos(2\xi h) - 4 \cos(\xi h) + 3)$$

670 for all $\xi \in \mathbb{R}$.

671 Figure 4 compares the decay behaviour of the Fourier transforms of all three basis pre-
 672 sented function types. As Figure 1 already illustrated, the Fourier transform of the classic

673 hat functions exhibits both slow decay rates and oscillatory behaviour. In stark contrast the
 674 Fourier transforms of the mollified hats as well as the Fourier transform of Irwin-Hall splines
 675 visually decay to zero instantly. In case of the mollified hat functions this is due to the expo-
 676 nential decay of the Fourier transform of the Gaussian mollifier while for splines Corollary 19
 677 displays a polynomial decay of order 4. In this regard, both alternatives to the classic hat
 678 functions are promising candidates for the implementation of the symbol method of Corol-
 679 lary 7. In Section 7 we put that promise to the test. Before that we briefly discuss the error
 680 analysis for the symbol method via spline basis functions as presented.

681 *Convergence rate for θ scheme, splines.* The spline approximation we consider falls into the
 682 framework of approximation with NURBS (non-uniform rational B-splines) of [2]. Since the
 683 geometry of our domain is the simplest possible one, namely an interval, large part of the
 684 analysis from [2] is not required in our case. Nevertheless, working with splines, we need to
 685 replace the standard Sobolev space \tilde{H}^1 by a so-called "bent" Sobolev space \mathcal{H}^1 , where the
 686 Sobolev spaces on the individual elements (subintervals in our case), on which the splines
 687 are cubic polynomials, are "bent" together by the corresponding regularity conditions at
 688 the interfaces, see equation (8) in [2]. Ignoring the boundary conditions we will impose,
 689 Lemma 3.3 in [2] provides the approximation property of the spline Galerkin space, (A2) from
 690 Conditions 14, and the inverse property, (A3) from Conditions 14, follows from Theorem 4.2
 691 in [2]. Now, since the proofs in [27] do not hinge on the specific properties of the space
 692 \tilde{H}^1 (also consult Section 3.6.2 of [14]) Lemma 15 extends to the setting with splines. As
 693 one might expect, both the approximation property (A2) from Conditions 14 and the inverse
 694 property (A3) are satisfied with a higher order in h , i.e. for h^4 . Hence Theorem 5.4 of [27]
 695 is valid with an order of h^4 . However, all terms on the right-hand side of the estimate in
 696 this theorem need to be finite, in particular $\max_{0 \leq t \leq T} \|u\|_{\mathcal{H}^4}$, and therewith the respective
 697 regularity for the initial value g . As this is not given in our implementation we cannot hope
 698 for the order h^4 .³ To summarize we can expect a convergence rate of h^2 as in the case of the
 699 approach with mollified hats.

700 **7. Numerical Implementation.** In this section we implement the pricing PIDEs for plain
 701 vanilla call and put options and test the two approaches to the symbol method experimentally.

702 **Theorem 20 (Feynman-Kac).** *Let $(L_t)_{t \geq 0}$ be a (time-homogeneous) Lévy process. Consider*
 703 *the PIDE*

$$704 \quad (64) \quad \begin{aligned} \partial_t U^{C,P} + \mathcal{A}U^{C,P} + rU^{C,P} &= 0, & \text{for almost all } t \in (0, T) \\ U^{C,P}(0) &= g^{C,P}, \end{aligned}$$

705 *where \mathcal{A} is the operator associated with the symbol of $(L_t)_{t \geq 0}$ and $g^{C,P} \in L^2_\eta(\mathbb{R})$. Assume*
 706 *further the assumptions (A1)–(A3) of [11] to hold. Then (64) possesses a unique weak solution*

$$707 \quad (65) \quad U^{C,P} \in W^1(0, T; H_\eta^{\alpha/2}(\mathbb{R}), L^2_\eta(\mathbb{R}))$$

³Additional numerical experiments with smooth initial conditions, performed within in a master thesis in the working group, that we do not report in this article in more detail showed the convergence rate of h^4 thus confirming the theoretical discussion from this section.

708 where $\alpha > 0$ is the Sobolev index of the symbol of $(L_t)_{t \geq 0}$ and $\eta \in \mathbb{R}$ is chosen according to
 709 Theorem 6.1 in [11]. If additionally $g_\eta^{C,P} \in L^1(\mathbb{R})$ then the relation

$$710 \quad (66) \quad U^{C,P}(T-t, x) = \mathbb{E} [g^{C,P}(L_{T-t} + x)]$$

711 holds for all $t \in [0, T]$, $x \in \mathbb{R}$.

712 *Proof.* For $r = 0$, the result is proved in [11] and follows from Theorem 6.1 therein. For
 713 general $r \geq 0$, that proof is easily adapted.

714 **Remark 21.** Setting $g^{C,P} = g^C$ in (64), the payoff profile of a European call option, results
 715 in U^C being the price of a European call option. Analogously, setting $g^{C,P} = g^P$, the payoff
 716 profile of a European put option, results in U^P being the price of a European call option.

717 **7.1. Truncation to zero boundary conditions.** As we derive prices of plain vanilla Eu-
 718 ropean call and put options, the solution to the respective pricing PIDE is defined on the
 719 whole real line. As a first step towards a discretization, we want to truncate the domain to
 720 bounded interval (a, b) and we choose to implement zero boundary conditions. Under further
 721 assumptions, exponential convergence of the truncation error has been shown in [9, Section
 722 4.1]. Here, we follow the standard procedure to subtract an appropriate auxiliary function
 723 ψ that matches the asymptotic behavior of $U^{C,P}$. Having chosen ψ , the resulting modified
 724 problem for $\phi = U^{C,P} - \psi$ is

$$725 \quad (67) \quad \begin{aligned} \partial_t \phi(t, x) + (\mathcal{A}\phi)(t, x) + r\phi(t, x) &= f(t, x) & \forall (t, x) \in (0, T) \times \mathbb{R}, \\ \phi(0, x) &= g_\Psi(x) & \forall x \in \mathbb{R}, \end{aligned}$$

726 where $g_\Psi(x) = g(x) - \psi(0, x)$ for all $x \in \mathbb{R}$ and the right hand side f is given by

$$727 \quad f(t, x) := -(\partial_t \psi(t, x) + (\mathcal{A}\psi)(t, x) + r\psi(t, x)).$$

728 The solution $U^{C,P}$ to the original problem (64) can easily be restored by $U^{C,P} = \phi + \psi$.
 729 Examples for ψ will be presented, later.

730 The right hand side in vector notation is given by $\mathbf{F}(t^k) = (F_1(t^k), \dots, F_N(t^k)) \in \mathbb{R}^N$ for
 731 each t^k on the time grid with $F_j(\cdot)$, $j = 1, \dots, N$, given by

$$732 \quad (68) \quad F_j(t) = - \int_{\mathbb{R}} (\partial_t \psi(t, x) + (\mathcal{A}\psi)(t, x) + r\psi(t, x)) \varphi_j(x) dx$$

733 for all $j = 1, \dots, N$.

734 We observe that the operator \mathcal{A} applied to the auxiliary function ψ appears in the integral
 735 of expression (68). For the same reasons as in the computation of the stiffness matrix entries,
 736 we decide to apply the symbol method for the computation of the entries of the right hand
 737 side $\mathbf{F} \in \mathbb{R}^N$. We pursue these considerations in the following section.

738 **7.2. Computation of the right hand side \mathbf{F} .** First, we need to choose an appropriate
 739 auxiliary function ψ . As its purpose is to enable us to truncate the domain and insert zero

740 boundary conditions, we need to inspect the limit behaviour of the price value

$$741 \quad (69) \quad \begin{aligned} U^C(x, t) &\rightarrow 0, & x &\rightarrow -\infty, t \in [0, T] \\ U^C(x, t) &\rightarrow e^x - Ke^{-rt}, & x &\rightarrow +\infty, t \in [0, T] \end{aligned}$$

742 for call options and

$$743 \quad (70) \quad \begin{aligned} U^P(x, t) &\rightarrow Ke^{-rt} - e^x, & x &\rightarrow -\infty, t \in [0, T] \\ U^P(x, t) &\rightarrow 0, & x &\rightarrow +\infty, t \in [0, T] \end{aligned}$$

744 for put options. This is the usual way to obtain the auxiliary function. Now, in regard
745 to our specific approach, relying on the Fourier transforms, we identify additional desirable
746 features for the auxiliary function. We denote $\widehat{\psi}(t, z) := \widehat{\psi}(t, \cdot)(z)$. Consider a smooth function
747 $\psi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that $\psi(t) \in H_\eta^{\alpha/2}(\mathbb{R})$ for all $t \in [0, T]$ for some $\eta \in \mathbb{R}$. Then, for the
748 second summand in (68) we have by applying the symbol method of Lemma 4 that

$$749 \quad (71) \quad \int_{\mathbb{R}} (\mathcal{A}\psi)(t, x) \varphi_j(x) dx = \frac{1}{2\pi} \int_{\mathbb{R}} A(\xi - i\eta) \widehat{\psi}(t, \xi - i\eta) \overline{\widehat{\varphi}_j(\xi + \eta)} d\xi,$$

750 where A denotes the symbol of the model. With the above identity, we are able to derive the
751 right hand side $(F_j)_{j=1, \dots, N}$ of the PIDE in vector notation as introduced by (68) in terms of
752 Fourier transforms by

$$753 \quad (72) \quad F_j = -\frac{1}{2\pi} \int_{\mathbb{R}} (\partial_t \widehat{\psi}(t, \xi - i\eta) + (A(\xi - i\eta) + r) \widehat{\psi}(t, \xi - i\eta)) \overline{\widehat{\varphi}_j(\xi + \eta)} d\xi.$$

754 This shows that ψ is numerically suitable for the purpose of localizing the pricing PIDE if
755 ψ is quickly evaluable on the region $[a, b] \times [0, T]$ and the integrals determining F_j can be
756 numerically evaluated fast for all $j = 1, \dots, N$. The first feature allows retransforming the
757 solution of the localized problem into the solution of the original pricing PIDE, while the
758 second grants the fast numerical derivation of the right hand side in equation (67). These
759 considerations lead us to the following list of desirable features for the auxiliary function ψ
760 that is required to obey the respective limit conditions (69), (70):

- 761 1. a (semi-)closed expression of the function ψ ,
- 762 2. a (semi-)closed expression of its Fourier transform $\widehat{\psi}$
- 763 3. and fast decay of $|\widehat{\psi}(\xi)|$ and $|\partial_t \widehat{\psi}(\xi)|$ for $|\xi| \rightarrow \infty$.

764 The smoother ψ , the faster $|\widehat{\psi}|$ decays. In the following two subsections we analyze two
765 candidates for ψ that display these desired features.

766 A first suggestion for ψ consists in using Black-Scholes prices as functions in $x = \log(S_0) \in$
767 $[a, b]$ and time to maturity $t \in [0, T]$ for localization of the pricing PIDE. We express the
768 price of a European option with payoff profile $g^{C,P}$ in the Black-Scholes model in terms of
769 (generalized) Fourier transforms and define ψ accordingly, as the following Lemma explains.

770 **Lemma 22 (Subtracting Black-Scholes prices).** *Choose a Black-Scholes volatility $\sigma^2 > 0$,*
771 *let $r \geq 0$ be the prevailing risk-free interest rate and set $\eta < -1$ in the case of a call option*
772 *and $\eta > 0$ for the put. Define ψ to be the associated Black-Scholes price,*

$$773 \quad (73) \quad \psi(t, x) = \psi^{bs}(t, x) := e^{-\eta x} e^{-rt} \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} g^{C,P}(-(\xi + i\eta)) \varphi_{t,\sigma}^{bs}(\xi + i\eta) d\xi,$$

774 with $\varphi_{t,\sigma}^{\text{bs}}(z) = e^{tA^{\text{bs}}(z)}$. We denote by A the symbol of the associated operator \mathcal{A} . Then the
 775 right hand side $\mathbf{F}: [0, T] \rightarrow \mathbb{R}^N$ can be written in the form
 776

(74)

$$777 \quad F_j(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \left((A^{\text{bs}} - A)(\xi - i\eta) \widehat{g}^{\mathcal{C},P}(\xi - i\eta) \exp\left(-t(r + A^{\text{bs}}(\xi - i\eta))\right) \overline{\widehat{\varphi}_j(\xi + i\eta)} \right) d\xi$$

778

779 for all $j = 1, \dots, N$.

780 *Proof.* In order to derive the right hand side, we need to represent ψ in Fourier terms.
 781 Since for call and put options, $\psi \notin L^1(\mathbb{R})$, we compute the (generalized) Fourier transform of
 782 ψ or the Fourier transform of $\psi_\eta = e^\eta g^{\mathcal{C},P}$, respectively. We get

$$783 \quad (75) \quad \psi_\eta(t, x) = e^{-rt} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi x} \widehat{g}^{\mathcal{C},P}(\xi - i\eta) \varphi_{t,\sigma}^{\text{bs}}(-(\xi - i\eta)) d\xi.$$

784 The integral in (75) is a Fourier (inversion) integral. We read off

$$785 \quad (76) \quad \widehat{\psi}_\eta(t, \xi) = \widehat{g}^{\mathcal{C},P}(\xi - i\eta) \exp\left(-t\left(r + A^{\text{bs}}(\xi - i\eta)\right)\right),$$

786 where we used the relation between the characteristic function and the symbol of a process.
 787 Now,

$$788 \quad (77) \quad \widehat{\frac{\partial}{\partial t} \psi}_\eta(t, \xi) = -\left(r + A^{\text{bs}}(\xi - i\eta)\right) \widehat{\psi}_\eta(t, \xi).$$

789 Finally, since $\psi^{\text{bs}} \in H_\eta^{\alpha/2}(\mathbb{R})$, we have that

$$790 \quad (78) \quad \int_{\mathbb{R}} (\mathcal{A}\psi^{\text{bs}})(t, x) \varphi_j(x) dx = \frac{1}{2\pi} \int_{\mathbb{R}} A(\xi - i\eta) \widehat{\psi}^{\text{bs}}(t, \cdot)(\xi - i\eta) \overline{\widehat{\varphi}_{j-\eta}(\xi)} d\xi.$$

791 Collecting our results proves the claim. ■

792 The candidate $\psi = \psi^{\text{bs}}$ matches the desired criteria. It is quickly evaluable, since Black-
 793 Scholes prices are implemented in many code libraries. Also, the integral in (74) is numerically
 794 accessible, since the integrand decays fast. Observe that FFT techniques could be employed
 795 to compute $F_j(t)$ for all $j = 1, \dots, N$ simultaneously. A major disadvantage of choosing
 796 $\psi = \psi^{\text{bs}}$, however, lies in the fact that $t \in [0, T]$ can not be separated from the integrand
 797 in (74). Consequently, $F_j(t^k)$, must be numerically evaluated for each $j = 1, \dots, N$ and
 798 $k = 1, \dots, M$, individually. This results in significant numerical cost. We therefore present a
 799 second candidate for ψ .

800 **Lemma 23 (Subtracting quasi-hockey stick).** Let $\sigma_\psi > 0$. Define $\psi^{\mathcal{C}}$ in the call option and
 801 $\psi^{\mathcal{P}}$ in the put option case by

$$802 \quad (79) \quad \begin{aligned} \psi^{\mathcal{C}}(t, x) &= (e^x - Ke^{-rt}) \Phi(x), & (t, x) &\in [0, T] \times [a, b], \\ \psi^{\mathcal{P}}(t, x) &= (Ke^{-rt} - e^x) (1 - \Phi(x)), & (t, x) &\in [0, T] \times [a, b], \end{aligned}$$

803 where Φ denotes the cumulative distribution function of the normal $\mathcal{N}(0, \sigma_\psi^2)$ distribution.
 804 Furthermore, in the call option case choose $\eta < -1$ and $\eta > 0$ in the put option case. Then,
 805 the right hand side $\mathbf{F}: [0, T] \rightarrow \mathbb{R}^N$ is given by

$$806$$

$$807 \quad (80) \quad F_j(t) = \frac{1}{2\pi} \left(\int_{\mathbb{R}} (A(\xi - i\eta) + r) \frac{\widehat{f^{\mathcal{N}}}(\xi - i(\eta + 1))}{i\xi + (\eta + 1)} \overline{\widehat{\varphi}_j(\xi + i\eta)} d\xi \right.$$

$$808 \quad \left. - e^{-rt} K \int_{\mathbb{R}} A(\xi - i\eta) \frac{\widehat{f^{\mathcal{N}}}(\xi - i\eta)}{i\xi + \eta} \overline{\widehat{\varphi}_j(\xi + i\eta)} d\xi \right),$$

$$809$$

810 for all $j = 1, \dots, N$ with $t \in [0, T]$, where A is the symbol of the associated operator \mathcal{A} and
 811 where

$$812 \quad \widehat{f^{\mathcal{N}}}(\xi) = \exp\left(-\frac{1}{2}\sigma_\psi^2\xi^2\right),$$

813 the Fourier transform of the normal $\mathcal{N}(0, \sigma_\psi^2)$ density.

814 *Proof.* We consider the call option case first. To derive the expression for F_j in (80) we
 815 need to compute the Fourier transform of (the appropriately weighted) ψ^C . We choose $\eta < -1$
 816 and $t \in [0, T]$ arbitrarily and compute for $K = 1$,

$$817 \quad (81) \quad \widehat{\psi_\eta^C(t, \cdot)}(\xi) = \int_{\mathbb{R}} e^{i\xi x} e^{\eta x} (e^x - e^{-rt}) \Phi(x) dx$$

$$= \int_{\mathbb{R}} e^{i\xi x} e^{(\eta+1)x} \Phi(x) dx - e^{-rt} \int_{\mathbb{R}} e^{i\xi x} e^{\eta x} \Phi(x) dx.$$

818 Integration by parts and l'Hôpital's rule yield that

$$819 \quad (82) \quad \int_{\mathbb{R}} e^{i\xi x} e^{(\eta+1)x} \Phi(x) dx = -\frac{1}{i\xi + (\eta + 1)} \int_{\mathbb{R}} e^{i(\xi - i(\eta+1))x} f^{\mathcal{N}}(x) dx,$$

820 which can be expressed in terms of the Fourier transform of the normal distribution yielding

$$821 \quad (83) \quad \int_{\mathbb{R}} e^{i\xi x} e^{(\eta+1)x} \Phi(x) dx = -\frac{\widehat{f^{\mathcal{N}}}(\xi - i(\eta + 1))}{i\xi + (\eta + 1)}.$$

822 Equivalently, we obtain for the second integral in (81) that

$$823 \quad (84) \quad \int_{\mathbb{R}} e^{i\xi x} e^{\eta x} \Phi(x) dx = -\frac{\widehat{f^{\mathcal{N}}}(\xi - i\eta)}{i\xi + \eta}.$$

824 Assembling these results we find

$$825 \quad (85) \quad \widehat{\psi_\eta^C(t, \cdot)}(\xi) = -\frac{\widehat{f^{\mathcal{N}}}(\xi - i(\eta + 1))}{i\xi + (\eta + 1)} + e^{-rt} \frac{\widehat{f^{\mathcal{N}}}(\xi - i\eta)}{i\xi + \eta}.$$

826 We deduce from (72) that

827

$$828 \quad (86) \quad F_j(t) = \frac{1}{2\pi} \left(\int_{\mathbb{R}} (A(\xi - i\eta) + r) \frac{\widehat{f}^{\mathcal{N}}(\xi - i(\eta + 1))}{i\xi + (\eta + 1)} \overline{\widehat{\varphi}_j(\xi + i\eta)} d\xi \right. \\ 829 \qquad \qquad \qquad \left. - e^{-rt} \int_{\mathbb{R}} A(\xi - i\eta) \frac{\widehat{f}^{\mathcal{N}}(\xi - i\eta)}{i\xi + \eta} \overline{\widehat{\varphi}_j(\xi + i\eta)} d\xi \right). \\ 830$$

831 For the put option case we choose ψ^P as defined in (79). The computations for $\widehat{\psi}_\eta^P$ follow
832 along the same lines as for the call and we get the relation

$$833 \quad (87) \quad \widehat{\psi}_\eta^P(t, \cdot)(\xi) = \widehat{\psi}_\eta^C(t, \cdot)(\xi) \quad \forall (t, \xi) \in [0, T] \times \mathbb{R},$$

834 for η set to some $\eta > 0$, which proves the claim. ■

835 **Remark 24 (Computational features of ψ^C and ψ^P).** While ψ^C serves as localizing function
836 for the call option case, ψ^P can be used in the put option case. Both candidates are based on
837 the payoff functions of call and put options but avoid the lack of differentiability with respect
838 to x in $x = \log(Ke^{-rt})$ for $t \in [0, T]$. As a consequence, both ψ^C and ψ^P are smooth functions
839 and thus fulfill the requirements collected above when σ_ψ is chosen small enough. Additionally,
840 the two integrals in (80) do not depend on the time variable $t \in [0, T]$ and thus need to be
841 computed only once for each basis function φ_j . This results in a significant acceleration in
842 computational time compared to the suggestion $\psi = \psi^{bs}$ of Lemma 22.

843 Algorithm 1 summarizes the abstract structure of a general FEM solver based on the sym-
844 bol method. By plugging the symbol associated to the model of choice into the computation
845 of line 9 of the algorithm, the solver instantly adapts to that model. In other words, only one
846 line needs to be specified to obtain a model specific solver for option pricing. As Examples 9,
847 10, 11, 12 and others emphasize, the symbol exists in analytically (semi-)closed form for many
848 models, indeed. Algorithm 1 thus provides a very appealing tool for FEM pricing in practice.

849 **7.3. Implementation of the symbol method.** As outlined in sections 5 and 6, we im-
850 plement two versions of the symbol method. On the one hand, we approximate the entries
851 of the stiffness matrix according to the approach of mollified hats, on the other hand we use
852 Irwin-Hall cubic splines as basic functions. For the mollified hats, we simplify the scheme
853 proposed in Section 5 further. Namely, we omit the second term in the defining equation
854 (53) for the approximate bilinear form and we truncate the first integral at a fixed level. The
855 numerical results already show the convergence rate of h^2 for this simplified version, thanks
856 to the small magnitude of the tail integral.

857 **7.4. Empirical Convergence Results.** Now we implement the symbol method for both
858 mollified hats and splines. Finally, we conduct an empirical order of convergence study. We
859 consider the univariate Merton, CGMY and NIG model and investigate the empirical rates of
860 convergence for the different implementations as Table 1 summarizes. For each model and
861 each implemented basis function type enlisted in the table we consider the payoff function

$$862 \quad (88) \quad g(x) = \max(e^x - 1, 0).$$

Algorithm 1 A symbol method based FEM solver

-
- 1: Choose equidistant space grid $x_i, i = 1, \dots, N$
 - 2: Choose basis functions $\varphi_i, i = 1, \dots, N$, with $\varphi_i(x) = \varphi_0(x - x_i)$ for some φ_0
 - 3: Choose equidistant time grid $T_j, j = 0, \dots, M$
 - 4: **Procedure** COMPUTE MASS MATRIX **M**
 - 5: Derive the mass matrix $\mathbf{M} \in \mathbb{R}^{N \times N}$ by
 - 6: $M_{kl} = \int_{\mathbb{R}} \varphi_l(x) \varphi_k(x) dx \quad \forall k, l = 1, \dots, N$
 - 7: **Procedure** COMPUTE STIFFNESS MATRIX **A**
 - 8: Derive the stiffness matrix $\mathbf{A} \in \mathbb{R}^{N \times N}$ by plugging the symbol A of the chosen model into the following formula and computing
 - 9: $A_{kl} = \frac{1}{2\pi} \int_{\mathbb{R}} A(\xi) e^{i\xi(x_k - x_l)} |\widehat{\varphi_0}(\xi)|^2 d\xi + rM_{kl} \quad \forall k, l = 1, \dots, N$
 - 10: using numerical integration
 - 11: **Procedure** RUN THETA SCHEME
 - 12: Choose a function ψ to subtract from the original pricing problem to obtain a zero boundary problem and retrieve the respective basis function coefficient vectors $\overline{\psi}^k \in \mathbb{R}^N, k = 1, \dots, M$. Consider the suggestions by Lemma 22 or Lemma 23 for plain vanilla European options above.
 - 13: Choose an appropriate basis function coefficient vector $U^1 \in \mathbb{R}^N$ matching the initial condition of the transformed problem
 - 14: Derive the right hand side vectors $\mathbf{F}^k \in \mathbb{R}^N, k = 0, \dots, M$. Consult Lemma 22 or Lemma 23 matching the choice of ψ .
 - 15: Choose $\theta \in [0, 1]$ and run the iterative scheme
 - 16: **for** $k = 0 : (M - 1)$
 - 17: $U^{k+1} = (\mathbf{M} + \Delta t \theta \mathbf{A})^{-1} ((\mathbf{M} - \Delta t (1 - \theta) \mathbf{A}) U^k + \theta \mathbf{F}^{k+1} + (1 - \theta) \mathbf{F}^k)$
 - 18: **end**
 - 19: **Procedure** RECONSTRUCT SOLUTION TO ORIGINAL PROBLEM
 - 20: Add previously subtracted right hand side ψ to the solution of the transformed problem by computing
 - 21: $\tilde{U}^k = U^k + \overline{\psi}^k, \quad k = 0, \dots, M$
 - 22: to retrieve the basis function coefficient vectors $\tilde{U}^k, k = 0, \dots, M$, to the original pricing problem
-

863 of a call option with strike $K = 1$. In each study we compute FEM prices for N_k basis
 864 functions with

$$865 \quad (89) \quad N_k = 1 + 2^k, \quad k = 4, \dots, 9,$$

866 resulting in $N_4 = 17$ basis functions in the most coarse and $N_9 = 513$ basis functions in
 867 the most granular case. On each grid, the nodes that basis functions are associated with
 868 are equidistantly spaced and the supports of the basis functions cover the space interval
 869 $\Omega = [-5, 5]$. The time discretization is kept constant with $N_{\text{time}} = 2000$ equidistantly spaced

Model	Symbol	Parameter choices	Implemented basis functions		
			Mollified hats	Splines	
Merton	Example 10	$\sigma = 0.15,$ $\beta = 0.2,$	$\alpha = -0.04,$ $\lambda = 3$	✓	✓
CGMY	Example 11	$C = 0.5,$ $M = 27.24,$	$G = 23.78,$ $Y = 1.1$	✓	✓
NIG	Example 12	$\alpha = 12.26,$ $\delta = 0.52$	$\beta = -5.77,$	✓	✓

Table 1

An overview of the models considered in the empirical order of convergence analysis and their parametrization. For these models, the symbol method is implemented and tested for both mollified hat functions and splines. In all models, the constant risk-less interest rate has been set to $r = 0.03$.

870 time nodes spanning a grid range of two years up until maturity, thus covering a time to
871 maturity interval of

$$872 \quad (90) \quad [T_1, T_{N_{\text{time}}}], \quad \text{with } T_1 = 0 \text{ and } T_{N_{\text{time}}} = 2.$$

873 For each $k = 4, \dots, 9$, the resulting price surface constructed by N_k basis functions in space
874 and $N_{\text{time}} = 2000$ grid points in time is computed. A comparison of these surfaces is drawn to
875 a price surface of most granular structure based on the same type of basis functions. We call
876 this most granular surface *true* price surface. It rests on $N_{\text{true}} = N_{11} = 1 + 2^{11} = 2049$ basis
877 functions in space and N_{time} grid points in time covering the same grid intervals as above, that
878 is $\Omega = [-5, 5]$ in space and $[0, 2]$ in time, respectively. The underlying FEM implementation
879 is thus based on distances h_{true} between grid nodes that basis functions are associated with of

$$880 \quad (91) \quad \begin{aligned} h_{\text{true}}^{\text{mollified hat}} &= (5 - (-5))/(2 + 2^{11}) \approx 0.0049, \\ h_{\text{true}}^{\text{splines}} &= (5 - (-5))/(4 + 2^{11}) \approx 0.0049, \\ \Delta t_{\text{true}} &= 2/(2000 - 1) \approx 0.001 \end{aligned}$$

881 in space and time, respectively. Note that all space grids are designed in such a way that
882 the log-strike $\log(K) = 0$ is one of the space nodes. For each model and method and each
883 $k = 4, \dots, 9$ the (discrete) L^2 error ε_{L^2} is calculated as

$$884 \quad \varepsilon_{L^2}(k) = \sqrt{\Delta t_{\text{true}} \cdot h_{\text{true}} \cdot \sum_{i=1}^{N_{\text{time}}} \sum_{j=1}^{N_{\text{true}}} \left(Price_{\text{true}}(i, j) - Price_k(i, j) \right)^2},$$

885 wherein $Price_{\text{true}}(i, j)$ is the value of the true pricing surface at space node $j \in \{1, \dots, 1 + 2^{11}\}$
886 and time node $i = 1, \dots, 2000$ and $Price_k(i, j)$ is the respective, linearly interpolated value of

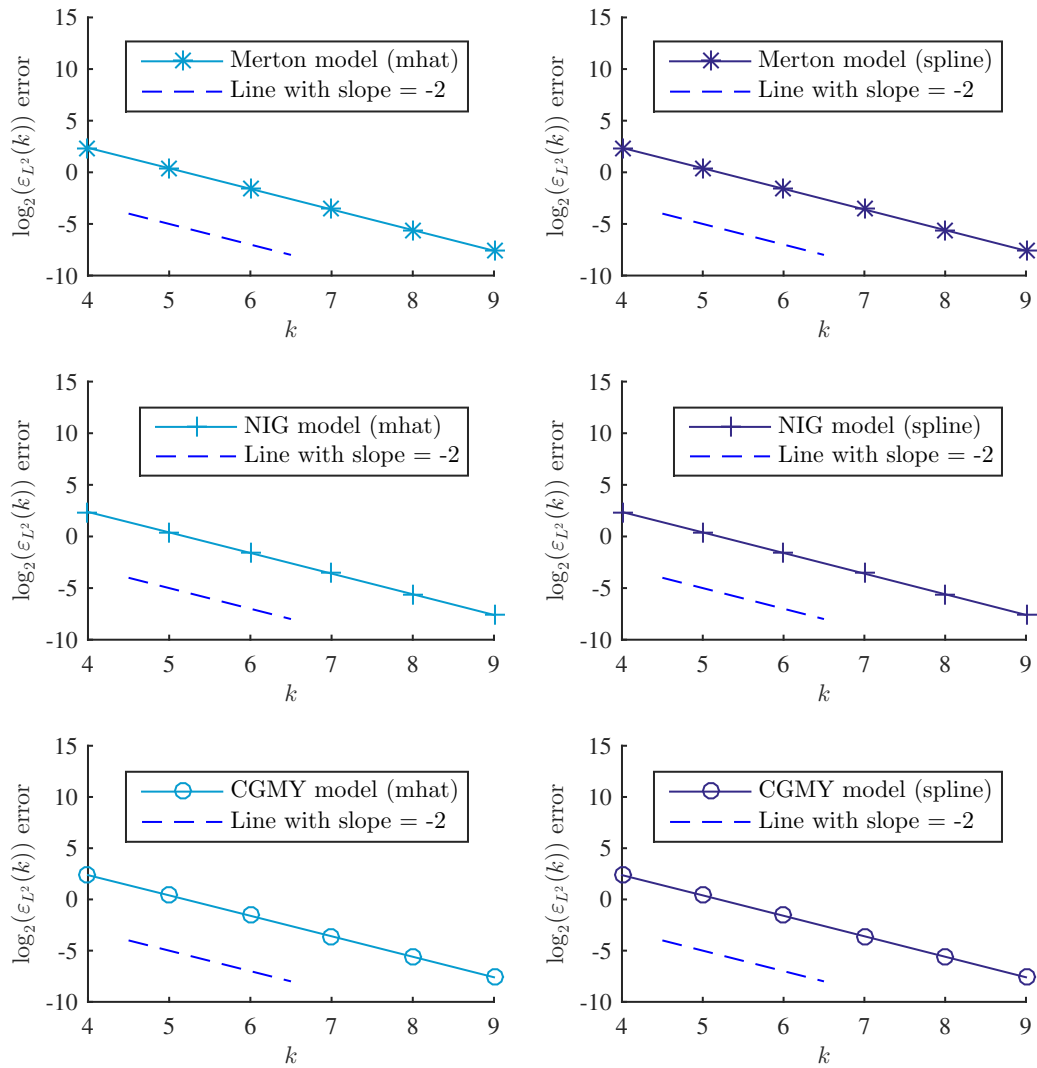


Figure 5. Results of the empirical order of convergence study for the Merton, the NIG and the CGMY model using mollified hats (left pictures) and splines (right pictures) as basis functions. All models are parametrized as stated in Table 1. Additionally, part of a straight line with (absolute) slope of 2 is depicted in each figure serving as a comparison.

887 the coarser pricing surface supported by only N_k basis functions.

888 Figure 5 summarizes the results of the six studies of empirical order of convergence in
 889 the Merton, the NIG and the CGMY model in a symbol based implementation once using
 890 mollified hats and once using splines as basis functions. In each implementation and for all

891 considered models, the (discrete) L^2 error decays exponentially with rate 2. The convergence
 892 result of Theorem 5.4 by [28] suggest that this is the best possible rate we can hope for, which
 893 yields the experimental validation of both approaches.

894 **8. Conclusion and outlook.** We have presented a tool for finite element solvers that allows
 895 for an implementation that is highly flexible in the model choice and that maintains numerical
 896 feasibility. Invoking the symbol was key. The transition into Fourier space has introduced
 897 smoothness as a new requirement to the basis functions. We have presented mollified hats
 898 and splines as compatible basis functions in our approach. Several numerical examples have
 899 confirmed the convergence rates expected by the theoretical considerations in both cases.

900 Let us mention several possible extensions of the approach. Firstly, the implementation
 901 naturally extends to time-inhomogeneous Lévy models that we neglected here for notational
 902 convenience. Secondly, combining the symbol method with wavelet basis functions allows
 903 for compression techniques that might further improve the overall numerical performance, as
 904 Hilber, Reichmann, Schwab and Winter in [18] outline. Thirdly, the polynomial decay that
 905 we observe in our numerical experiments can possibly be improved to exponential rates by
 906 invoking an hp -discontinuous Galerkin scheme, see e.g. Schötzau and Schwab in [25]. Fourthly,
 907 the method can be extended to multivariate settings. In particular, tensor-based multivariate
 908 extensions are conceptually straightforward. Since the domain for financial applications typi-
 909 cally is a (hyper)rectangular, tensorized extensions of the basis functions are a natural choice.
 910 Both the mollified hats and the splines have natural tensorized generalizations.

911 Appendix A. Proofs.

912 A.1. Proof of a more general version of Lemma 3.

913 *Proof.* We first consider $\varphi, \psi \in C_0^\infty(\mathbb{R})$.

914 For $F \equiv 0$ the assertion follows directly from partial integration. Since the Lévy measure
 915 may be unbounded around the origin, the representation of the jump part of the bilinear form,

$$916 \quad a^{jump}(\varphi, \psi) := - \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\varphi(x+y) - \varphi(x) - \varphi'(x)h(y) \right) F(dy) \psi(x) e^{2\langle \eta, x \rangle} dx,$$

917
 918 needs to be carefully derived. In order to exploit the identity

$$919 \quad \varphi(x+y) - \varphi(x) - y\varphi'(x) = \int_0^y \int_0^z \varphi''(v) dv dz$$

921 we split the integral with respect to the Lévy measure in three parts, set $c(F) := \int_{|y|<1} (y -$
 922 $h(y)) F(dy) - \int_{|y|>1} h(y) F(dy)$ and obtain

$$923 \quad a^{jump}(\varphi, \psi) := - \int_{\mathbb{R}} \int_{|y|<1} \int_0^y \int_0^z \varphi''(x+v) dv dz F(dy) \psi(x) e^{2\langle \eta, x \rangle} dx$$

$$924 \quad - c(F) \int_{\mathbb{R}} \varphi'(x) \psi(x) e^{2\langle \eta, x \rangle} dx$$

$$925 \quad - \int_{\mathbb{R}} \int_{|y|>1} (\varphi(x+y) - \varphi(x)) F(dy) \psi(x) e^{2\langle \eta, x \rangle} dx.$$

926

927 Thanks to $\int_0^y \int_0^z |\varphi''(v)| dv dz \leq cy^2$ with some constant $c > 0$ for all $y \in [-1, 1]$ and

$$\begin{aligned}
 928 \quad (92) \quad & \int_{\mathbb{R}} \int_{|y|<1} \int_0^y \int_0^z |\varphi'(x+v)| dv dz F(dy) |\psi'(x) + 2\eta\psi(x)| e^{2\langle\eta, x\rangle} dx \\
 929 & \leq (1 + 2\eta) \|\varphi\|_{H_\eta^1} \|\psi\|_{H_\eta^1} \int_{|y|<1} y^2 F(dy)
 \end{aligned}$$

930 we can apply the theorem of Fubini and partial integration to obtain

$$\begin{aligned}
 931 \quad & - \int_{\mathbb{R}} \int_{|y|<1} \int_0^y \int_0^z \varphi''(x+v) dv dz F(dy) \psi(x) e^{2\langle\eta, x\rangle} dx \\
 932 \quad & = \int_{\mathbb{R}} \int_{|y|<1} \int_0^y \int_0^z \varphi'(x+v) dv F(dy) (\psi'(x) + 2\eta\psi(x)) e^{2\langle\eta, x\rangle} dx. \\
 933
 \end{aligned}$$

934 This yields the assertion for $\varphi, \psi \in C_0^\infty(\mathbb{R})$.

935 Next, we verify that the bilinear form as stated in Lemma 3 is well defined for $\varphi, \psi \in H_\eta^1(\mathbb{R})$
 936 and is continuous with respect to the norm of $H_\eta^1(\mathbb{R})$. For $F \equiv 0$ this is obvious. The assertion
 937 follows for the jump part from inequality (92) and

$$\begin{aligned}
 938 \quad & \int_{\mathbb{R}} \int_{|y|>1} |\varphi(x+y) - \varphi(x)| F(dy) |\psi(x)| e^{2\langle\eta, x\rangle} dx \leq 2F(\mathbb{R} \setminus [-1, 1]) \|\varphi\|_{L_\eta^2} \|\psi\|_{L_\eta^2}. \\
 939
 \end{aligned}$$

940 Thus a from Lemma 3 is a continuous bilinear form on $H_\eta^1(\mathbb{R}) \times H_\eta^1(\mathbb{R})$ that coincides with
 941 (9) on the dense subset $C_0^\infty(\mathbb{R}) \times C_0^\infty(\mathbb{R})$. This proves the assertion. \blacksquare

942 A.2. Proof of Lemma 5.

943 *Proof.* To prove the assertion, we verify the conditions of Lemma 7.1 in [16], which provides
 944 an abstract robustness result for weak solutions. We first observe that the conditions for
 945 f_n, f, g_n, g coincide with those of Lemma 7.1 in [16]. Second, we verify conditions (An1)–(An3)
 946 of Lemma 7.1 in [16]. Therefore we assign to each $u, v \in X$ the coefficients $\alpha_k(u), \alpha_k(v) \in \mathbb{R}$
 947 for $k \leq N$ such that $u = \sum_{k=1}^N \alpha_k(u) \varphi_k$ and $v = \sum_{k=1}^N \alpha_k(v) \varphi_k$. Thanks to the finite
 948 dimensionality of X , there exists a constant $\tilde{C} > 0$ such that for all $u \in X$,

$$\begin{aligned}
 949 \quad (93) \quad & \|u\|_V \leq \sum_{k=1}^N |\alpha_k(u)| \|u\|_V \leq C' \|u\|_V.
 \end{aligned}$$

950 Thanks to (26) there exists a sequence $0 < c_n \rightarrow 0$ such that for all $j, k \leq N$,

$$\begin{aligned}
 951 \quad (94) \quad & |(a_n - a)(\varphi_j, \varphi_k)| \leq c_n \|\varphi_j\|_V \|\varphi_k\|_V.
 \end{aligned}$$

953 Together with assumption (A2) this yields for all $j, k \leq N$,

$$\begin{aligned}
 954 \quad (95) \quad & |a_n(\varphi_j, \varphi_k)| \leq C_1 \|\varphi_j\|_V \|\varphi_k\|_V.
 \end{aligned}$$

956 Inequalities (95) and (93) together yield for all $u, v \in X$,

$$\begin{aligned}
 957 \quad |a_n(u, v)| &\leq \sum_{j=1}^N \sum_{k=1}^N |\alpha_j(u) \alpha_k(v)| |a_n(\varphi_j, \varphi_k)| \\
 958 \quad &\leq C_1 \sum_{j=1}^N \sum_{k=1}^N |\alpha_j(u) \alpha_k(u)| \|\varphi_j\|_V \|\varphi_k\|_V \\
 959 \quad &\leq C_1 \tilde{C}^2 \|u\|_V \|v\|_V,
 \end{aligned}$$

961 which shows that condition (An1) of Lemma 7.1 in [16] is satisfied. Due to inequalities (94)
 962 and (93), we have for all $u \in X$,

$$\begin{aligned}
 963 \quad |(a - a_n)(u, u)| &\leq \sum_{j=1}^N \sum_{k=1}^N |\alpha_j(u) \alpha_k(u)| |a_n(\varphi_j, \varphi_k)| \\
 964 \quad &\leq c_n \sum_{j=1}^N \sum_{k=1}^N |\alpha_j(u) \alpha_k(u)| \|\varphi_j\|_V \|\varphi_k\|_V \\
 965 \quad &\leq c_n \tilde{C}^2 \|u\|_V^2,
 \end{aligned}$$

967 which shows assumption (An3) of Lemma 7.1 in [16]. Finally, from assumption (A1) and the
 968 last inequality for all $u \in X$ we obtain

$$\begin{aligned}
 969 \quad a_n(u, u) &\geq a(u, u) - |(a - a_n)(u, u)| \\
 970 \quad &\geq G \|u\|_V^2 - G' \|u\|_H^2 - c_n \tilde{C}^2 \|u\|_V^2,
 \end{aligned}$$

972 which shows that there exists $N_0 \in \mathbb{N}$ such that condition (An2) of Lemma 7.1 in [16] is
 973 satisfied for all $n > N_0$. This shows the assertion of Lemma 5. ■

974

REFERENCES

- 975 [1] H. ALT, *Lineare Funktionalanalysis: Eine anwendungsorientierte Einführung*, Springer Lehrbuch,
 976 Springer, 6th ed., 2011.
- 977 [2] Y. BAZILEVS, L. BEIRÃO DA VEIGA, J. COTTRELL, T. HUGHES, AND G. SANGALLI, *Isogeometric Anal-*
 978 *ysis: Approximation, stability and error estimates for h-refined meshes*, Mathematical Models and
 979 Methods in Applied Sciences, 16 (2006), pp. 1031–1090.
- 980 [3] S. I. BOYARCHENKO AND S. Z. LEVENDORSKIĬ, *Barrier options and touch-and-out options under regular*
 981 *Lévy processes of exponential type*, Annals of Applied Probability, 12 (2002), pp. 1261–1298.
- 982 [4] R. BRUMMELHUIS AND R. CHAN, *An RBF scheme for option pricing in exponential Lévy models*, Applied
 983 Mathematical Finance, 21 (2014), pp. 238–269.
- 984 [5] O. BURKOVSKA, B. HAASDONK, J. SALOMON, AND B. WOHLMUTH, *Reduced basis methods for pricing*
 985 *options with the Black-Scholes and Heston models*, SIAM Journal of Financial Mathematics, 6 (2015),
 986 pp. 685–712.
- 987 [6] P. CARR, H. GEMAN, D. B. MADAN, AND M. YOR, *The fine structure of asset returns: An empirical*
 988 *investigation*, Journal of Business, 75 (2002), pp. 305–332.

- 989 [7] R. CHAN AND S. HUBBERT, *Options pricing under the one-dimensional jump-diffusion model using the*
990 *radial basis function interpolation scheme*, Review of Derivatives Research, 17 (2014), pp. 161–189.
- 991 [8] R. CONT, N. LANTOS, AND O. PIRONNEAU, *A reduced basis for option pricing*, SIAM Journal on Financial
992 Mathematics, 2 (2011), pp. 287–316.
- 993 [9] R. CONT AND E. VOLTCHKOVA, *A finite difference scheme for option pricing in jump diffusion and*
994 *exponential Lévy models*, SIAM Journal on Numerical Analysis, 43 (2005), pp. 1596–1626.
- 995 [10] R. CONT AND E. VOLTCHKOVA, *Integro-differential equations for option prices in exponential Lévy models*,
996 Finance and Stochastics, 9 (2005), pp. 299–325.
- 997 [11] E. EBERLEIN AND K. GLAU, *PIDEs for pricing European options in Lévy models – a Fourier approach*.
998 Technische Universität München, 2011.
- 999 [12] E. EBERLEIN AND K. GLAU, *Variational solutions of the pricing PIDEs for European options in Lévy*
1000 *models*, Applied Mathematical Finance, 21 (2014), pp. 417–450.
- 1001 [13] H. FÖLLMER, *Alles richtig und trotzdem falsch?*, Mitteilungen der DMV, 17 (2009), pp. 148–154.
- 1002 [14] M. GASS, *PIDE methods and concepts for parametric option pricing*, PhD thesis, Technical University
1003 Munich, 2016. online version forthcoming.
- 1004 [15] K. GLAU, *Classification of Lévy processes with parabolic Kolmogorov backward equations*. forthcoming in
1005 SIAM Journal Theory of Probability and its Applications, 2016.
- 1006 [16] K. GLAU, *Feynman-Kac type formula for Lévy processes with discontinuous killing rate*. accepted for
1007 publication in Finance and Stochastics, 2016.
- 1008 [17] N. HILBER, N. REICH, C. WINTER, AND C. SCHWAB, *Numerical methods for Lévy processes*, Finance
1009 and Stochastics, 13 (2009), pp. 471–500.
- 1010 [18] N. HILBER, O. REICHMANN, C. SCHWAB, AND C. WINTER, *Computational Methods for Quantitative*
1011 *Finance*, Springer, 2013.
- 1012 [19] A. ITKIN, *Efficient solution of backward jump-diffusion PIDEs with splitting and matrix exponentials*,
1013 Journal of Computational Finance, 19 (2016), pp. 29–70.
- 1014 [20] A. ITKIN AND P. CARR, *Using pseudo-parabolic and fractional equations for option pricing in jump*
1015 *diffusion models*, Computational Economics, 40 (2012), pp. 63–104.
- 1016 [21] K. JACKSON, S. JAIMUNGAL, AND V. SURKOV, *Fourier space time stepping for option pricing with Lévy*
1017 *Models*, Journal of Computational Finance, 12 (2012), pp. 1–29.
- 1018 [22] D. MARAZZINA, O. REICHMANN, AND C. SCHWAB, *hp-DGFEM for Kolmogorov-Fokker-Planck equations*
1019 *of multivariate Lévy processes*, Mathematical Models and Methods in Applied Sciences (M3AS), 22
1020 (2012).
- 1021 [23] A.-M. MATACHE, P.-A. NITSCHKE, AND C. SCHWAB, *Wavelet Galerkin pricing of American options on*
1022 *Lévy driven assets*, Quantitative Finance, 5 (2005), pp. 403–424.
- 1023 [24] S. SALMI, J. TOIVANEN, AND L. VON SYDOW, *An IMEX-scheme for pricing options under stochastic*
1024 *volatility models with jumps*, SIAM Journal on Scientific Computing, 36 (2014), pp. B817–B834.
- 1025 [25] D. SCHÖTZAU AND C. SCHWAB, *Time discretization of parabolic problems by the hp-version of the discon-*
1026 *tinuous Galerkin finite element method*, SIAM Journal of Numerical Analysis, 38 (2006), pp. 837–875.
- 1027 [26] L. SCHUMAKER, *Spline Functions: Basic Theory*, Cambridge Mathematical Library, Cambridge Univer-
1028 sity Press, 3rd ed., 2007.
- 1029 [27] T. VON PETERSDORFF AND C. SCHWAB, *Wavelet discretizations of parabolic integrodifferential equations*,
1030 SIAM Journal on Numerical Analysis, 41 (2003), pp. 159–180 (electronic).
- 1031 [28] T. VON PETERSDORFF AND C. SCHWAB, *Wavelet discretizations of parabolic integrodifferential equations*,
1032 SIAM Journal on Numerical Analysis, 41 (2003), pp. 159–180.
- 1033 [29] J. WLOKA, *Partial Differential Equations*, Cambridge University Press, 1987.
- 1034 [30] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications*, vol. II/A, Springer, 1990. Linear Mono-
1035 tone Operators.