MATRIX ORBIT CLOSURES

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Abstract. Let G be the group $\mathrm{GL}_r(\mathbf{C}) \times (\mathbf{C}^\times)^n$. We conjecture that the finely-graded Hilbert series of a G orbit closure in the space of r-by-n matrices is wholly determined by the associated matroid. In support of this, we prove that the coefficients of this Hilbert series corresponding to certain hook-shaped Schur functions in the $\mathrm{GL}_r(\mathbf{C})$ variables are determined by the matroid, and that the orbit closure has a set-theoretic system of ideal generators whose combinatorics are also so determined. We also discuss relations between these Hilbert series for related matrices, including their stabilizing behaviour as r increases.

1. Introduction

In this paper we study a collection of affine varieties that we call *matrix orbit closures*, obtained as follows. Let $\mathbf{k}$ be an algebraically closed field of characteristic zero (this assumption can be relaxed in many but not all of our results). Pick an r-by-n matrix v and consider all matrices which define a configuration of n points in $\mathbb{P}^{r-1}$ equivalent to v, that is, which differ from v only by row operations and rescalings of columns. The resulting collection is an orbit for the group $G = \mathrm{GL}_r(\mathbf{k}) \times (\mathbf{k}^\times)^n$ the former factor acting on the left of the matrix space $\mathbf{A}^{r\times n}$ and the latter as diagonal matrices on the right. This orbit is denoted X_v°. The Zariski closure of X_v° is an (irreducible) affine variety denoted by X_v.

The principal question we seek to address is to what extent the matroid of v controls algebraic and geometric properties of X_v. It is reasonable to expect some control, on the account that there is a quotient of $X_v \subseteq \mathbf{A}^{r\times n}$ for which the control is very strong. Let us assume that v has the maximum possible rank r; this is inessential but makes the statement cleaner. Let $G(r, n)$ be the Grassmannian of r dimensional subspaces of $\mathbf{k}^n$, which is the target of the rational map $\pi : \mathbf{A}^{r\times n} \to G(r, n)$ sending a matrix to the span of its rows, and inherits the action of the algebraic torus $T = (\mathbf{k}^\times)^n$. Then X_v is the closure of the preimage of (the closure of) the torus orbit containing $\pi(v)$. These torus orbit closures in $G(r, n)$ are classified up to isomorphism by the rank r matroids on a fixed n element set that are realizable over $\mathbf{k}$. The class of $\pi(X_v)$ within the zeroth T-equivariant K-theory of $G(r, n)$ is also a function of the associated matroid, as shown by Speyer [35, Proposition 12.5]. Interest in these torus orbit closures antedates the above results: Klyachko gave a formula for their equivariant cohomology classes in a special case [25], and Kapranov undertook a thorough study of the Chow quotient whose points represent them [24].

We prove in a companion paper [7] that the G-equivariant Chow class of X_v is a function of the matroid of v. In this paper our specific interest is in a finer invariant,

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its class in the G-equivariant K-theory group, which contains the same information as the multigraded Hilbert series of the coordinate ring of X_v. Our main conjecture is that this refinement adds no distinguishing power:

Conjecture 5.1. The G-equivariant K-class of X_v is determined by the matroid of v.

We fully resolve this conjecture here only for rank 2 uniform matroids.

The relation of X_v to its quotient $G(r, n)$ is analogous to that between *matrix Schubert varieties* and (classical) Schubert varieties: matrix Schubert varieties are closures of $B \times B$-orbits of square matrices, B being a Borel group, whereas Schubert varieties are closures of B-orbits in the quotient of this space by B. The techniques of combinatorial commutative algebra have traction in the setting of matrix Schubert varieties, which have been used to great advantage by Fulton [18], Knutson and Miller [28], and others since. The techniques have also been adapted for other varieties arising in Schubert calculus, such as the Richardson varieties [26].

Further motivation for introducing this K-class comes from studying the general linear group representation generated by the tensor product of the columns of v, which we call the *tensor module* of v. The tensor module appears as a multigraded component of the coordinate ring of X_v; indeed, all other multigraded components are tensor modules of configurations obtained from v. Tensor modules have previously been an object of study in the guise of the question whether symmetrizations of decomposable tensors are zero, which attracted the interest of Gamas [20], Dias da Silva [13], and others.

One of our two main results, Theorem 8.1, uses the representation-theoretic perspective on the tensor module to describe certain coefficients in terms of matroidal combinatorics, namely non-broken circuits. The coefficients in question are those which are Schur functions of hook shape, s_λ where $\lambda = (n-k+1, 1^{k-1})$, in the GL_n variables and squarefree in the torus variables. Its translation to the setting of the equivariant K-class of X_v, Theorem 8.6, is particularly pleasant: the corresponding terms are a multigraded enumerator of dependent sets of the matroid. This provides an explicit affirmation of part of Conjecture 5.1: all hook-shape coefficients in the K-class are matroid invariants.

Our other main result, Theorem 4.2, gives a generating set for the ideal of X_v up to radical, the construction of whose generators involves only matroid combinatorics. When n has a uniform matroid of rank 2 (Proposition 4.7) or corank 2 (Proposition 4.6) we prove that our ideal is reduced, i.e. is the ideal of X_v.

The structure of our paper is as follows. In Section 2 we recall some background on matroid theory. In Section 3 we classify the points of X_v. This enters into the proof of Theorem 4.2 in the next section. Section 5 is dedicated to Conjecture 5.1, on the relationship between the matroid of v and the K-class of X_v, and its affirmative resolution in the rank 2 uniform case, Proposition 5.2.

In Section 6 we consider the problem of studying X_v when v is of some rank $r' < r$. Letting v' denote a matrix whose rows are a basis for the row span of v, we relate the Hilbert series and K-polynomials of X_v and $X_{v'}$. We call X_v the *stabilization* of $X_{v'}$, since X_v is obtained by embedding $X_{v'}$ in $\mathbf{A}^{r \times n}$, by adding $r - r'$ rows equal to zero, and then taking the $\mathrm{GL}_r(\mathbf{k})$ orbit of $X_{v'}$. A similar operator in cohomology is called a "raising" operator in [16]. Stabilization has a trivial effect on Hilbert series (Lemma 6.4); its translation to K-classes (Lemma 6.6) has a less transparent appearance. We turn to the tensor module in Section 7, introducing several fundamental properties of this module including a Schur–Weyl dual representation. These are used in the next section to prove Theorems 8.1 and 8.6.

Finally, Section 9 discusses operations on v whose effect on the K-class we can describe. One of these is the direct sum of matrices, for which stabilization plays a central role; another is duplicating a column of v.

Conventions. A *variety* is taken to be an integral scheme of finite type over $\mathbf{k}$.

2. Matroid theory background

White's *Theory of Matroids* [38] serves as an excellent reference for the matroid theory needed here. For the convenience of the reader, we gather the required notions in this section.

A **matroid** is a simplicial complex M on a finite *ground set* E whose faces satisfy the following *exchange axiom*: for faces I and I' of M, if $|I| < |I'|$ then there is some $e \in I' \setminus I$ such that $I \cup \{e\}$ is a face of M. Two matroids are isomorphic if they are isomorphic as simplicial complexes: that is, if there is a bijection between their ground sets inducing a bijection between their faces. We will refer to the isomorphism type of a matroid as an **unlabeled matroid**.

For any matrix $v \in \mathbf{A}^{r\times n}$ the matroid of v, denoted $M(v)$, is the simplicial complex whose faces are those $I \subset [n]$ such that the columns of v indexed by I are linearly independent. Any matrix in the orbit $G \cdot v$ has the same matroid as v. The set of matrices in $\mathbf{A}^{r\times n}$ with a prescribed matroid is a subscheme of $\mathbf{A}^{r\times n}$ called a **matroid stratum** or a **matroid realization space**. It is a result of Sturmfels [36] that this is not a stratification in any nice sense (particularly that of Whitney). Worse, a matroid stratum can contain arbitrarily complicated singularities, a result referred to as Mnëv–Sturmfels universality [31].

Matroids that can be written as $M(v)$ for some $v \in \mathbf{A}^{r\times n}(\mathbf{k})$ are said to be **realizable** over $\mathbf{k}$. The faces and non-faces of M are called *independent* and *dependent* sets, respectively. The minimal dependent sets are called *circuits* and the maximal independent sets are called *bases*.

The **uniform matroid** of rank r on n elements, $U_{r,n}$, is the matroid with ground set $[n]$ whose bases are all r element subsets of $[n]$. It is the matroid of a generic element of $\mathbf{A}^{r\times n}$.

We denote the rank of a matrix v by $\mathrm{rk}(v)$. The **rank** $\mathrm{rk}(M)$ of a matroid M is the cardinality of a maximal independent set. In particular, $\mathrm{rk}(M(v)) = \mathrm{rk}(v)$. On many occasions we will assume that the rank of matroids we deal with is full, i.e., equals r. In particular, when we state the hypothesis "v has a uniform matroid", we mean uniform of rank r.

For any $v \in \mathbf{A}^{r\times n}$, its **Gale dual** is any $v^\perp \in \mathbf{A}^{(n-\mathrm{rk}(v))\times n}$ whose rows form a basis for the (right) kernel of v. Thus, the Gale dual is determined up to the action of $\mathrm{GL}_{n-\mathrm{rk}(v)}(\mathbf{k})$ on $\mathbf{A}^{(n-\mathrm{rk}(v))\times n}$. If v has full rank then Gale duality really is a duality, $\mathrm{GL}_r(\mathbf{k})(v^\perp)^\perp = \mathrm{GL}_r(\mathbf{k})v$. To a matroid M we associate a **dual matroid** M^* whose bases are complements of bases of M. If $M(v)$ is the matroid of a matrix v, then $M(v)^*$ is the matroid $M(v^\perp)$ of the Gale dual of v.

The **direct sum** of two matroids on disjoint sets is the join of the two simplicial complexes. A matroid is said to be **connected** if it is indecomposable with respect to this operation. Any matroid M can be written uniquely as a direct sum of connected matroids, the constituents of which are called the **connected components** of M. A **coloop** of M is an element of E in every base of M and a **loop** of M is an element of E in no base of M.

The **rank partition** of M is the sequence of numbers $\lambda(M) = (\lambda_1, \lambda_2, \dots)$ determined by the condition that for all $k \geq 1$, its kth partial sum is the size of the largest union of k independent sets of M. It is a theorem of Dias da Silva [13] that $\lambda(M)$ is a partition (i.e., it is weakly decreasing). If M is loop-free then $\lambda(M)$ is the maximum partition λ in dominance order such that M can be E can be partitioned into independent sets of sizes $\lambda_1, \lambda_2, \dots$.

The **restriction** of M to a subset $J \subset E$, denoted $M|J$, consists of those independent sets belonging to J. The **contraction** of M by J is $(M^*|J^c)^*$, where $J^c = E \setminus J$, and is denoted M/J. If $M = M(v)$ is realizable then M/J is obtained as follows. Let $A \in \mathrm{End}(\mathbf{k}^r)$ be a matrix whose kernel is spanned by $\{v_j : j \in J\}$ and is generic with respect to this property. Then M/J is the matroid of Av, with columns J deleted.

If there is a matroid M' with ground set $E' \supset E$ such that $M = M'|E$ then $M'/(E' \setminus E)$ is said to be a **quotient** of M. It follows that every quotient of a realizable matroid is again realizable. The **truncation** of M to rank $r' < \mathrm{rk}(M)$ is the matrioid of rank r' whose bases are the independent sets of M of size r'; it can be realized as a quotient of M by adding an element to the ground set of M that is in no circuit of size less than $\mathrm{rk}(M)+1$ and then contracting this element. If M is realized by a matrix v then the truncation is realized by applying a generic matrix with a one dimensional kernel to the left of v.

Let the indicator vector of a subset B of $[n]$ be $e_B = \sum_{i \in B} e_i$. The **matroid (base) polytope** $P(M)$ of a matroid M with ground set $[n]$, essentially due to Edmonds [11], is the convex hull of the indicator vectors of the bases of M in $\mathbf{R}^n$. It is a theorem of Gel'fand, Goresky, MacPherson and Serganova [21] that, among non-empty polytopes P with vertices chosen from the set $\{e_B : B \subset [n]\}$, matroid polytopes are exactly those that lie in a plane where the coordinates sum to a positive integer and every edge of P has the form $\mathrm{conv}\{e_B, e_{B\cup j \setminus i}\}$ for some $B \subset [n]$ and $i \in B$, $j \notin B$.

3. The points of a matrix orbit closure

In this section we discuss the geometry of the matrix orbit closures X_v with respect to the G orbits they comprise.

Proposition 3.1. *The closure of a G-orbit in $\mathbf{A}^{r\times n}$ is an irreducible affine variety. If v has a matroid of rank r with c connected components, then*

$$\dim(X_v) = r^2 + n - c.$$

Proof. Since G is a connected linear algebraic group the first claim follows. The second follows since the stabilizer of v is seen to be a c-dimensional torus inside the diagonal torus of G. $\square$

Let $(\mathbf{A}^{r\times n})^{\mathrm{fr}}$ denote the open subvariety of full rank matrices in $\mathbf{A}^{r\times n}$. There is a GL_r bundle $\pi : (\mathbf{A}^{r\times n})^{\mathrm{fr}} \to G(r,n)$, which takes a matrix to its row span. Consider the case that $v \in (\mathbf{A}^{r\times n})^{\mathrm{fr}}$. Then $\overline{\pi(v)T} \subset G(r,n)$ is the (normal) toric variety associated to the matroid polytope of $M(v)$. The T-orbits in $\overline{\pi(v)T}$ are in bijection with the faces of the matroid base polytope $P(M(v))$. One can give a combinatorial description of the faces of the matroid polytope as follows [1, Proposition 2]. Let $S_\bullet$ be a flag of subsets

$$\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} = [n].$$

Every face of $P(M(v))$ is of the form $P(M(v)_{S_\bullet})$ where

$$M(v)_{S_\bullet} = \bigoplus_{i=1}^{k+1} (M(v)|S_i)/S_{i-1}.$$

Two different flags can produce the same matroid, but there is only one T-orbit in $\overline{\pi(v)T}$ with a given matroid. A realization of this result in terms of torus orbit closures is obtained as follows. Rescale column $i \in S_j \setminus S_{j-1}$ of v by s^{j-1}. Projecting this matrix into $G(r,n)$ we obtain a subspace $\pi(v)\lambda(s)$, where $\lambda(s)$ is a one-parameter subgroup of T, i.e., an element of $T(\mathbf{k}(\!(s)\!))$. Here $\mathbf{k}(\!(s)\!)$ is the field of Laurent series in s over $\mathbf{k}$. Taking the limit $\lim_{s\to 0} \pi(v)\lambda(s)$ yields a point of $\overline{\pi(v)T}$ with matroid $M(v)_{S_\bullet}$. Every T-orbit in $\overline{\pi(v)T}$ is reached in this way and so our argument is complete.

The pullback $\pi^{-1}(\lim_{s\to 0} \pi(v)\lambda(s))$ is the G-orbit of a full rank matrix in X_v whose matroid is $M(v)_{S_\bullet}$. We call any such matrix a **projection of** v **along the flag** $S_\bullet$. As before, there is only one G-orbit in X_v whose points have a prescibed matroid of the form $\bigoplus_{i=1}^{k} M(v)|S_i/S_{i-1}$.

The next result shows that all elements of X_v are obtained by projecting v along some flag and applying some element $g \in \mathrm{End}(\mathbf{k}^r)$ on the left.

Proposition 3.2. *Suppose that v has rank r and $w \in X_v$ is a matrix of rank less than r. Then there is a matrix $w' \in X_v$ whose rank is that of v, and $w = gw'$ for some singular $g \in \mathrm{End}(\mathbf{k}^r)$.*

Proof. Let $V = \mathbf{A}^{r\times n}$ and suppose that w has rank ℓ. After applying an element of GL_r, and relabeling the columns of our matrices, we may assume that w has a row equal to zero and its first ℓ columns are the first ℓ standard basis vectors. By the valuative criterion for properness, there is an element $(g(s), t(s))$ of $G(\mathbf{k}(\!(s)\!)) = \mathrm{GL}_r(\mathbf{k}(\!(s)\!)) \times T(\mathbf{k}(\!(s)\!))$ such that $g(s)vt(s) \in \mathbf{k}[[s]] \otimes_{\mathbf{k}} V$ and

$$g(s)vt(s) \equiv w \mod s.$$

Applying an element of $\mathrm{GL}_r(\mathbf{k}[[s]])$ we may assume that the first ℓ columns of $g(s)vt(s)$ are the first ℓ standard basis vectors. Let ν_i be the least of the non-negative integers that appears as an exponent in row i of $g(s)vt(s)$. The limit of

$$\mathrm{diag}(s^{-\nu_1}, \ldots, s^{-\nu_r})(g(s)vt(s))$$

as $s \to 0$ gives an element w' that has rank strictly larger than w. If the rank of w' is not the rank of v then, by induction, there is some $w'' \in X_v$ of rank r and $g \in \mathrm{End}(\mathbf{k}^r)$ such that $gw'' = w'$. Applying an element of $\mathrm{End}(\mathbf{k}^r)$ that zeros out the appropriate rows, we bring w' to w. $\square$

Corollary 3.3. *If $w \in X_v$ then there is a flag of sets $S_\bullet$ such that the matroid of w is a quotient of*

$$\bigoplus_{i=1}^{k+1}(M(v)|S_i)/S_{i-1}.$$

Conversely, every quotient of such a matroid occurs as the matroid of some $w \in X_v$.

Proof. Combining the remarks above about faces of the matroid polytope $P(M(v))$ with Proposition 3.2, we obtain the first claim. The converse follows since the contraction of a matroid by a set of elements is realized by applying an element of $\mathrm{End}(\mathbf{k}^r)$ to a vector configuration in $\mathbf{k}^r$. This means that every quotient of $\bigoplus_{i=1}^{k+1}(M(v)|S_i)/S_{i-1}$ is the matroid of a point in the $\mathrm{End}(\mathbf{k}^r)$-orbit of $\pi^{-1}(\overline{\pi(v)T})$. $\square$

Example 3.4. The correspondence between matroids and orbits in X_v is not in general bijective as the following example shows. If

$$v = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Then for every $\mu \in \mathbf{k}$,

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & \mu - 1 & 1 & \mu \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \in X_v^\circ \implies \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & \mu \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in X_v.$$

The matrices of the latter form are all projectively inequivalent. This stands in contrast to the situation with $\overline{\pi(v)T}$, where orbits are in bijection with the matroids of the points in the orbit closure.

4. The ideal of a matrix orbit closure

Let $R = \mathbf{k}[x_{ij} : i \in [r], j \in [n]]$, and regard $\mathrm{Spec}(R) = \mathbf{A}^{r \times n}$. Since a matrix orbit closure X_v is irreducible in $\mathbf{A}^{r\times n}$, it is the vanishing locus of a prime ideal $I_v \subset R$. In this section we discuss this ideal. Our main result is Theorem 4.3, which gives a finite generating set for an ideal I'_v given by minors of certain matrices, for which $\sqrt{I'_v} = I_v$.

4.1. The ideal I'_v.

We now give the polynomial conditions for a matrix to lie in X_v. Recall from Section 2 the notion of Gale duality. For $v \in \mathbf{A}^{r\times n}$, its Gale dual is any $v^\perp \in \mathbf{A}^{(n-\mathrm{rk}(v))\times n}$ whose rows form a basis for the kernel of v. For any $w = (w_1, \dots, w_n) \in X_v^\circ$, the vectors

$$w_1 \otimes v_1^\perp, \quad w_2 \otimes v_2^\perp, \quad \dots, \quad w_n \otimes v_n^\perp$$

are linearly dependent. This can be seen by expanding a linear combination in the standard basis of $\mathbf{k}^r \otimes \mathbf{k}^{n-r}$. By continuity this holds for any $u \in X_v$. More is true:

Proposition 4.1 (Kapranov [24])**.** *Suppose that $w \in \mathbf{A}^{r\times n}$ has a connected matroid of full rank. If the collection of tensors*

$$w_1 \otimes v_1^\perp, \quad w_2 \otimes v_2^\perp, \quad \ldots, \quad w_n \otimes v_n^\perp$$

forms a circuit in $\mathbf{k}^r \otimes \mathbf{k}^{n-r}$ then $w \in X_v^\circ$.

For a subset J of $[n]$, let v_J be the submatrix of v on the columns indexed by J, so that the rank $\mathrm{rk}(M|J)$ in the matroid of v is the dimension of the span of these columns in $\mathbf{k}^r$. The Gale dual of v_J is not $(v^\perp)_J$, but it is a projection of this configuration. This fact is matroidally manifested by the equality $(M|J)^* = M^*/J^{\mathrm{c}}$ where J^{c} is the complement of J in the ground set of M.

Theorem 4.2. *For any $v \in \mathbf{A}^{r\times n}$, a matrix w is in X_v if and only if for every $J = \{j_1, \ldots, j_\ell\} \subset [n]$, the tensors*

$$\{w_{j_i} \otimes (v_J^\perp)_i : i = 1, \ldots, \ell\}, \tag{4.1}$$

are linearly dependent.

The proof of the theorem can be found in Section 4.3 below.

An immediate consequence of Theorem 4.2 is the next theorem, giving set-theoretic equations for X_v. Let x denote the matrix of variables $x_{i,j}$, and let x_j denote the j-th column $(x_{1,j}, \ldots, x_{r,j})^{\mathrm{t}}$ of x. For each subset $J = \{j_1, \ldots, j_\ell\} \subset [n]$ we form the matrix $x_J \odot v_J^\perp$, whose columns are the tensors $x_{j_i} \otimes (v_J^\perp)_i \in R^r \otimes \mathbf{k}^{n-\mathrm{rk}(v_J)}$. There exists a linear dependence among the columns of $x_J \odot v_J^\perp$ if and only if all its size $|J|$ minors vanish. As such:

Theorem 4.3. *Let the size $|J|$ minors of the matrices $x_J \odot v_J^\perp$, $J \subset [n]$, generate the ideal $I_v' \subset R$. Then $\sqrt{I_v'} = I_v$.*

Remark 4.4. There are two special cases that occur when applying this result. The first occurs when a subconfiguration v_J consists of linearly independent vectors. In this case $v_J^\perp$ is a configuration of n null vectors. We interpret $x_J \otimes v_J^\perp$ to be the zero matrix in this case. The second special case is when v_J has $U_{|J|-1,|J|}$ as its matroid. In this case the dimensions of the matrix $x_J \odot v_J^\perp$ are $(|J|-1)$-by-$|J|$, and hence all its size $|J|$ minors vanish. The point here is that the conditions given by subsets J where $\mathrm{rk}(v_J) \geq |J| - 1$ are vacuous.

4.2. **On the primality of I_v'.** We first make a conjecture.

Conjecture 4.5. *The ideal I_v' is equal to I_v.*

We can prove this conjecture in two special cases.

Proposition 4.6. *Suppose $v \in \mathbf{A}^{(n-2)\times n}$ and that v has a uniform matroid of rank $n-2$. Then $I_v' = I_v$ and R/I_v' is a Cohen–Macaulay ring.*

Proof. The hypothesis on v ensures that it has a full dimensional orbit. The codimension of the orbit closure is thus $n-3$. It follows that the codimension of I_v' is $n-3$.

If $v \in \mathbf{A}^{(n-2)\times n}$ then Remark 4.4 implies that I_v' is generated by the size $(n-2)$ minors of $x \odot v^\perp$ — it is a determinantal ideal. Since $x \odot v^\perp$ has dimension $2(n-2)$-by-n, I_v' has the expected codimension. We apply [14, Corollary 4] to conclude that

I'_v is prime and hence $I'_v = I_v$. The cited result also implies that R/I'_v is a Cohen–Macaulay ring. $\square$

The second case of primality is Gale dual to the first.

Proposition 4.7. *Suppose that $v \in \mathbf{A}^{2\times n}$ and that v has a uniform matroid of rank 2. Then $I'_v = I_v$ and R/I'_v is a Cohen–Macaulay ring.*

The proof of this result follows by constructing a third ideal from v with the desired properties. This ideal is contained in I'_v and we will show that the former ideal cuts out X_v. Specifically, let I''_v denote the ideal generated by the size 4 minors of the 4-by-n matrix

$$x \odot v = \left[\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} \otimes v_1 \quad \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} \otimes v_2 \quad \cdots \quad \begin{bmatrix} x_{1n} \\ x_{2n} \end{bmatrix} \otimes v_n \right].$$

Given two integers $a < b \in [n]$, we let $p_{ab}(v)$ denote the determinant of the 2-by-2 submatrix of v with columns a and b. Similarly define $p_{ab}(x)$. It is an immediate calculation that the minors of $x \odot v$ are all of the form

$$p_{ab}(v)p_{cd}(v)p_{ac}(x)p_{bd}(x) - p_{ac}(v)p_{bd}(v)p_{ab}(x)p_{cd}(x).$$

This polynomial is obtained from the equality of the cross ratio

$$\frac{p_{ab}(v)p_{cd}(v)}{p_{ac}(v)p_{bd}(v)} = \frac{p_{ab}(gvt)p_{cd}(gvt)}{p_{ac}(gvt)p_{bd}(gvt)},$$

which holds on the orbit X_v°, which is open in its closure.

Proposition 4.8. *The vanishing locus of I''_v is X_v.*

Proof. The result follows by induction on n. If $n = 4$ then $\operatorname{codim} X_v = 1$, and so I_v must be principal. Since I''_v is principal and I_v cannot be generated by a linear form or a constant, we must have equality.

Suppose that $n > 4$ and let $w = (w_1, \dots, w_n)$ be a 2-by-n matrix in the vanishing locus of I''_v. If w has a zero column then $w \in X_v$ by induction on n. Hence, we assume that no column of w is zero. Assume that w has a pair of parallel columns. The fact that w vanishes at all the generators of I''_v implies that at least $n-1$ of the columns of w are parallel. A simple calculation proves that $w \in X_v$.

Finally, assume that w has no parallel columns. By induction we can bring the first $n-1$ columns of w to those of v by a projective transformation. Since the first, second, third and last column of w have a presribed cross ratio, we see that w_n must be a non-zero scalar multiple of v_n. $\square$

Proof of Proposition 4.7. By [14, Corollary 4] we know that I''_v is prime. A short calculation gives $I''_v \subset I'_v$ and we conclude that $I''_v = I'_v = I_v$. The cited result also yields the fact that R/I''_v is a Cohen–Macaulay ring. $\square$

4.3. **Proof of Theorem 4.2.** The "only if" direction of the theorem is true by the discussion proceeding Proposition 4.1. Our proof of the "if" direction is by induction.

Lemma 4.9. *Suppose that for every $r' < r$ Theorem 4.2 is true for $\mathbf{A}^{r'\times n}$. Then to prove the theorem for $\mathbf{A}^{r\times n}$, we may assume that $v \in (\mathbf{A}^{r\times n})^{\mathrm{fr}}$.*

Proof. Suppose that $v \in \mathbf{A}^{r\times n}$ has rank $r' < r$ and that $u \in \mathbf{A}^{r\times n}$ has rank larger than r'. Replacing v with a matrix in $\mathrm{GL}_r v$ we may assume that the last $r - r'$ rows of v are zero. Let $J = \{j_1, \dots, j_{r'+1}\} \subset [n]$ denote a set of indices of size $r'+1$ such that u_J has rank $r'+1$. Not every column of v_J can be a coloop of the matroid $M(v)|J$, so by throwing away elements of J, assume that v_J is coloop free. Consider the tensors

$$u_{j_1} \otimes (v_J^{\perp})_1, \quad u_{j_2} \otimes (v_J^{\perp})_2, \dots, \quad u_{j_{r'+1}} \otimes (v_J^{\perp})_{r'+1},$$

These are linearly independent, since the columns of u_J are linearly independent, except in the case that one of the columns of $v_J^{\perp}$ is zero. However, every column of $v_J^{\perp}$ is non-zero since $M(v)|J$ is coloop free.

Suppose that for every $J \subset [n]$, the tensors in (4.1), with u taking the place of w, are linearly dependent. Then u has rank at most r'. Replacing u with a GL_r-translate, we may assume that the latter $r - r'$ rows of u *and* v are zero. Ignoring the latter $r - r'$ rows of u and v, we can appeal to the truth of Theorem 4.2 for $\mathbf{A}^{r'\times n}$, thus proving the lemma. □

Lemma 4.10. *Suppose that for every $n' < n$ Theorem 4.2 is true for matrices in $\mathbf{A}^{r\times n'}$. Then, to prove the theorem for $v \in \mathbf{A}^{r\times n}$, we may assume that v has a connected matroid.*

Proof. Suppose that v has a disconnected matroid and, after permuting columns of v, write $v = [v'\ v'']$ where $M(v) = M(v') \oplus M(v'')$. After applying an element of GL_r, we may assume that the first r' rows of v'' are zero, and the latter $r'' = r - r'$ rows of v' are zero, so that v is a direct sum of matrices. From this we see that $v^{\perp}$ is also a direct sum of matrices.

Pick $w \in \mathbf{A}^{r\times n}$ satisfying conditions (4.1) and write $w = [w'\ w'']$, where w' and w'' have the same numbers of columns as v' and v''. It follows that $w' \in X_{v'}$ and $w'' \in X_{v''}$ by our induction hypothesis.

By Proposition 3.2 there are configurations $\tilde{w}' \in X_{v'}$, $\tilde{w}'' \in X_{v''}$, of rank r' and r'' respectively, and matrices $g, h \in \mathrm{End}(\mathbf{k}^r)$ such that $g\tilde{w}' = w'$ and $h\tilde{w}'' = w''$. We may assume that $\tilde{w}''$ has its first r' rows equal to zero, and that the latter r'' rows of $\tilde{w}'$ are zero. Thus, $[\tilde{w}'\ \tilde{w}''] \in X_v$. Taking the first r' columns of g and the latter r'' columns of h and forming a new matrix A from these, we have that $A[\tilde{w}'\ \tilde{w}''] = w$, which is thus in X_v. □

We are now in a good place to prove the theorem.

Proof of Theorem 4.2. We have reduced to the case that v has full rank and a connected matroid. This implies that the orbit X_v° has dimension $r^2 + n - 1$. By Remark 4.4 we may assume $n > r + 1$.

Assume that we have $u \in \mathbf{A}^{r\times n}$ such that for all $J \subset [n]$, the tensors in (4.1) are linearly dependent. We prove that $u \in X_v$ by induction on n.

We start with the case when the rank of u is less than r. Assume that the last column of u is non-zero, since we are done if it is. Applying elements of GL_r we may assume that the rth row of u is all zeros and the last entry of v_n is non-zero.

Let $u' = (u_1, \dots, u_{n-1})$ and likewise for v. By induction on n, we know that there is some element $(g(s), t(s)) \in \mathrm{GL}_r(\mathbf{k}(\!(s)\!)) \times (\mathbf{k}(\!(s)\!)^{\times})^{n-1}$ such that $g(s)\, v'\, t(s)$ has coordinates in $\mathbf{k}[[s]]$ and

$$g(s)\, v'\, t(s) \equiv u' \mod s.$$

Since the bottom row of u' is all zeros we can replace the bottom row of $g(s)$ with $(0,\dots,0,s^m)$, $m \gg 0$, and obtain the same reduction modulo s. Now let $t'(s) = (t(s), s^{-m}) \in (\mathbf{k}(\!(s)\!)^\times)^n$ and consider the matrix

$$g(s)\, v\, t'(s).$$

Setting $s = 0$ yields a matrix whose first $n-1$ columns agree with those of u and whose last column is the rth standard basis vector of $\mathbf{k}^r$. Applying the element of $\mathrm{End}(\mathbf{k}^r)$ that fixes the first $r-1$ basis vectors and sends the last to u_n, we bring this matrix to u. We conclude that $u \in X_v$.

Suppose that u has rank r. If u has a connected matroid then Proposition 4.1 shows that $u \in X_v^\circ$. We thus reduce to the case that u has a disconnected matroid. Our goal is to show that $M(u)$ has a connected component K such that the orbit of u_K equals the orbit of v_K, and without loss of generality u_K is equal to v_K. If this happens we will be able to apply the induction hypothesis to the columns *not* indexed by K, and use the matrices thus produced to find $(g(s), t(s)) \in \mathrm{GL}_r(\mathbf{k}(\!(s)\!)) \times (\mathbf{k}(\!(s)\!)^\times)^n$ with $g(s)vt(s) = u$ modulo s.

For any $J \subset [n]$, the rank of $(v_J)^\perp$ is $\dim\ker(v_J) = |J| - \mathrm{rk}_{M(v)}(J)$. If $K \subset J$, then the restriction of $(v_J)^\perp$ to the columns indexed by K has rank $\dim\ker(v_J) - \dim\ker(v_{J\setminus K})$.

Since the tensors in (4.1) are dependent, it follows that for any $J \subset [n]$ there is a connected component K of $M(u)$ with $J \cap K$ non-empty and $\dim\ker(v_J) - \dim\ker(v_{J\setminus K})$ linearly independent dependences among $u_{J\cap K}$. That is,

$$\dim\ker(u_{J\cap K}) \geq \dim\ker(v_J) - \dim\ker(v_{J\setminus K}),$$

and hence

$$\mathrm{rk}\, u_{J\cap K} \leq \mathrm{rk}\, v_J - \mathrm{rk}\, v_{J\setminus K}. \tag{4.2}$$

for some connected component K of $M(u)$.

Applying (4.2) with $J = [n]$ we obtain a component K_1 of $M(u)$. Apply (4.2) again with $J = [n] \setminus K_1$ and obtain a connected component K_2 of $M(u)$. Continue in this way to obtain $K_1, \dots, K_\ell$, an ordering of the components of $M(u)$. Summing the inequalities obtained from (4.2) yields,

$$\begin{aligned}\mathrm{rk}(u_{K_1}) + \mathrm{rk}(u_{K_2}) + \cdots + \mathrm{rk}(u_{K_\ell}) &\leq (\mathrm{rk}\, v - \mathrm{rk}\, v_{[n]\setminus K_1}) \\ &\quad + (\mathrm{rk}\, v_{[n]\setminus K_1} - \mathrm{rk}\, v_{[n]\setminus K_1 \cup K_2}) + \cdots + (\mathrm{rk}\, v_{K_\ell} - \mathrm{rk}\, v_\emptyset)\end{aligned}$$

The left and right sides of this are both r and hence all the inequalities above are all equalities. It follows that $\mathrm{rk}(u_{K_\ell}) = \mathrm{rk}(v_{K_\ell})$. We know that $u_{K_\ell} \in X_{v_{K_\ell}}$ by the induction hypothesis, and thus v_{K_ℓ} is connected because u_{K_ℓ} is. We conclude from Proposition 4.1 that the orbit of v_{K_ℓ} equals the orbit of u_{K_ℓ}. We thus take $u_{K_\ell} = v_{K_\ell}$.

Setting $K_\ell^{\mathrm{c}} = [n] \setminus K_\ell$, there is some $g(s), t(s)$ such that $g(s)\, v_{K_\ell^{\mathrm{c}}}\, t(s) \equiv u_{J^{\mathrm{c}}}$ mod s. Since the first $\mathrm{rk}(u_{K_\ell})$ rows of $u_{K_\ell{}^{\mathrm{c}}}$ can be taken to be zero, we replace the first $\mathrm{rk}(u_{K_\ell})$ rows of $g(s)$ with the corresponding rows of $s^m \mathrm{Id}_r$ for $m \gg 0$, and apply $g(s), (s^{-m}, \dots, s^{-m}, t(s))$ to v. The result is u, modulo s. □

5. Multigraded Hilbert series and K-polynomials

In this section we define the multigraded Hilbert series and K-polynomial of a G-equivariant R-module. We then propose the fundamental question of our work, which is on the matroid invariance of the K-polynomial of the coordinate ring of a matrix orbit closure. We then give a formula for this K-polynomial when $v \in \mathbf{A}^{2\times n}$ has a uniform rank 2 matroid.

5.1. **Background on Hilbert series.** Let R denote the polynomial ring

$$\mathbf{k}[x_{ij} : i \in [r], j \in [n]]$$

and regard $\mathbf{A}^{r\times n}$ as $\operatorname{Spec} R$. R is graded by $\mathbb{Z}^r \times \mathbb{Z}^n$, the degree of x_{ij} being $a_i + b_j$, where $a_1, \dots, a_r, b_1, \dots, b_n$ are the standard basis vectors of $\mathbb{Z}^r \times \mathbb{Z}^n$. The grading group should be thought of as the weight lattice of the maximal torus in $G = \mathrm{GL}_r(\mathbf{k}) \times T$ obtained as (the diagonal torus of $\mathrm{GL}_r(\mathbf{k})$) $\times\, T$.

Any finitely generated graded R-module $M = \bigoplus_{(\mathbf{a},\mathbf{b})\in\mathbb{Z}^r\times\mathbb{Z}^n} M_{(\mathbf{a},\mathbf{b})}$ has **Hilbert series**

$$\mathrm{Hilb}(M) = \sum_{(\mathbf{a},\mathbf{b})\in\mathbb{Z}^r\times\mathbb{Z}^n} \dim_{\mathbf{k}}(M_{(\mathbf{a},\mathbf{b})}) u^{\mathbf{a}} t^{\mathbf{b}} \in \mathbb{Z}[[u_1^{\pm 1}, \dots, u_r^{\pm 1}, t_1^{\pm 1}, \dots, t_n^{\pm 1}]].$$

By [33, Theorem 8.20], there is a Laurent polynomial $\mathcal{K}(M; u, t)$ such that

$$\mathrm{Hilb}(M) = \frac{\mathcal{K}(M; u, t)}{\prod_{i=1}^r \prod_{j=1}^n (1 - u_i t_j)},$$

and we refer to this polynomial as the **K-polynomial** of M. In particular, if $X \subset \mathbf{A}^{r\times n}$ is a closed subvariety with defining ideal I, we write $\mathcal{K}(X; u, t)$ for the K-polynomial of R/I. We will often write the K-polynomials as $\mathcal{K}(M)$, $\mathcal{K}(X)$, etc..

The ring R has the action of G given by $((g, t) \cdot f)(v) = f(g^{-1} v t)$. The decomposition of R into its various graded pieces $R_{(\mathbf{a},\mathbf{b})}$ is a refinement of the irreducible decomposition of R as a G-module; it is precisely the refinement into weight spaces. It is important to take care that the gradation and weight space decompositions have the property that if $f \in R_{(\mathbf{a},\mathbf{b})}$, then f has $\mathrm{GL}_r(\mathbf{k})$-weight $-\mathbf{a}$. The arguably more natural convention of setting $\deg(x_{ij}) = b_j - a_i$ results in ugly formulas and does not agree with the standard grading of R. Thus, given a G-equivariant graded module M we pass back and forth between its character, as a G-module, and its Hilbert series by inverting all the u variables.

Let $K_G^0(\mathbf{A}^{r\times n})$ denote the Grothendieck group of G-equivariant coherent R-modules. Since $\mathbf{A}^{r\times n}$ is a trivial vector bundle over a point, there is a natural identification of $K_G^0(\mathbf{A}^{r\times n})$ with the Grothendieck group of rational representations of G. Indeed, we have

$$\begin{aligned} K_G^0(\mathbf{A}^{r\times n}) &= \mathbb{Z}[u_1, \dots, u_r, t_1, \dots, t_n][u_1^{-1}, \dots, u_r^{-1}, t_1^{-1}, \dots, t_n^{-1}]^{\mathfrak{S}_r} \\ &= \mathbb{Z}[e_1(u), \dots, e_r(u), t_1, \dots, t_n][e_r(u)^{-1}, t_1^{-1}, \dots, t_n^{-1}], \end{aligned}$$

where the symmetric group $\mathfrak{S}_r$ acts on the u variables. Here we have written $e_i(-)$ for the ith elementary symmetric polynomial in its arguments. Under this identification, the K-polynomials $\mathcal{K}(X)$ and $\mathcal{K}(\mathcal{E})$ represent the equivariant K-classes of the structure sheaf of X and the global sections of $\mathcal{E}$, respectively.

In what follows, we will employ the following standard notation.

- Par_r is the set of partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq 0)$ of length at most r.
- $s_\lambda(x)$ is a Schur polynomial in the list of variables $x = (x_1, \ldots, x_k)$.
- $e_k(x) = s_{1^k}(x)$ is an elementary symmetric polynomial.
- $h_k(x) = s_k(x)$ is a complete homogeneous symmetric polynomial.
- If λ is a partition, its transpose λ^{t} is the partition whose ith part is the number of parts of λ which are at least i.

It will also be useful to give meaning to $s_\lambda(x_1, \ldots, x_k)$ when λ is a k-tuple of integers, possibly negative, that need not be a partition. We do this using the determinantal formula

$$s_\lambda(x_1, \ldots, x_k) = \frac{\det((x_i^{\lambda_j + k - j})_{i,j})}{\det((x_i^{k-j})_{i,j})}.$$

In particular, the set

$$\{s_\lambda(x_1, \ldots, x_k) : \lambda \in \mathbb{Z}^k \text{ is non-increasing}\}$$

is a basis for the symmetric Laurent polynomials in $x_1, \ldots, x_k$.

5.2. **Two fundamental problems.** The main motivation behind our work is

(1) to determine if $\mathcal{K}(X_v)$ is determined by $M(v)$ alone, and
(2) when it is, determine exact formulae for $\mathcal{K}(X_v)$ in terms of $M(v)$.

The first problem is opposed to whether $\mathcal{K}(X_v)$ is determined by "higher order" geometric properties of the configuration v. In all computed cases, $\mathcal{K}(X_v)$ only varies with $M(v)$. For example, consider the configuration of 6 points in $\mathbf{P}^2$ coming from $v \in \mathbf{A}^{3\times 6}$ with matroid $U_{3,6}$. Then, it appears that $\mathcal{K}(X_v)$ cannot detect if the 6 points in $\mathbf{P}^2$ lay on a conic or not. We conjecture that the answer to question (1) is positive:

Conjecture 5.1. *$\mathcal{K}(X_v)$ is determined by $M(v)$.*

One result motivating the conjecture is that if we replace X_v with its Grassmannian analog $\overline{\pi(v)T}$ then the result is true. Specifically, a result of Speyer [35, Proposition 12.5] says that the T-equivariant K-theory class of the structure sheaf of $\overline{\pi(v)T}$ is determined by the matroid $M(v)$. This result follows by equivariant localization, a tool which is difficult to bring to bear on the equivariant K-theory of $\mathbf{A}^{r\times n}$.

The second motivating problem appears to be very difficult to answer for arbitrary v. In the rest of this section we give an answer when $v \in \mathbf{A}^{2\times n}$ has matroid $U_{2,n}$. This follows since we know that the ideal of X_v is determinantal, and so we can compute a free resolution of its coordinate ring using known results. In Section 8 we determine the coefficient of $s_\lambda(u)t^{\mathbf{b}}$ when $\mathbf{b}$ is a $\{0,1\}$-vector and λ is a single column. The answer is always ± 1, and the proof of this relies on results about broken circuit complexes and multivariate Tutte polynomials.

5.3. **The K-polynomial of X_v when $M(v) = U_{2,n}$.** Proposition 4.7 allows us to explicitly determine the K-polynomial $\mathcal{K}(X_v)$ when $M(v) = U_{2,n}$.

Proposition 5.2. *Let $v \in \mathbf{A}^{2\times n}$ have a uniform matroid. The K-polynomial of X_v is*

$$\mathcal{K}(X_v) = 1 - \sum_{\substack{\lambda=(\lambda_1\geq\lambda_2)\\ 2\leq\lambda_2,\ \lambda_1+\lambda_2\leq n}} (-1)^{|\lambda|} s_\lambda(1,1) s_\lambda(u) e_{|\lambda|}(t).$$

Note that $s_\lambda(1,1) = \lambda_1 - \lambda_2 + 1$.

Proof. The degeneracy locus of the map ψ_v defined by the matrix $x \odot v$ is X_v by Proposition 4.7. Therefore, the ideal I_v of X_v is resolved G-equivariantly by the Eagon-Northcott complex $\mathcal{C}_\bullet(\psi_v) \to I_v \to 0$, wherein

$$\mathcal{C}_m(\psi_v) = \mathrm{Sym}^{m-4}\left(\mathrm{End}_R(R^2)\right) \otimes \bigwedge^m R^n, \quad m = 4, 5, \ldots, n.$$

This is a minimal resolution since $\mathrm{depth}(I_v) = \mathrm{codim}(I_v) = n - 3$ [15, Theorem A2.10], as we have shown above.

With respect to the $\mathbb{Z}^r$ part of the grading of Section 5.1, all the maps in $(\bigwedge^2 R^2) \otimes_R \mathcal{C}_\bullet \to I_v \to 0$ are linear maps. To compute the K-polynomial of the terms of the resolution it suffices to compute the character of the G-module

$$\left(\bigwedge^2 \mathbf{k}^2 \otimes \mathrm{Sym}^m \mathrm{End}(\mathbf{k}^2)\right) \otimes \bigwedge^m \mathbf{k}^n.$$

The character of the $\mathrm{GL}_2(\mathbf{k})$-module $\mathrm{Sym}^m \mathrm{End}(\mathbf{k}^2)$ has been computed by Désarménien, Kung and Rota [12] as

$$\sum_{\lambda=(\lambda_1\geq\lambda_2)\vdash m} s_\lambda(1,1) s_\lambda(u_1, u_2).$$

The proposition follows. $\square$

Example 5.3. For a matrix $v \in \mathbf{A}^{2\times 4}$ with matroid $U_{2,4}$, we have

$$\mathcal{K}(X_v) = 1 - s_{(2,2)}(u) e_4(t) = 1 - u_1^2 u_2^2 t_1 t_2 t_3 t_4.$$

The case when $v \in \mathbf{A}^{r\times n}$, $r > 2$, has matroid $U_{2,n}$ is dealt with in the next section.

6. Stabilization

The basic operation we consider in this section is embedding a G-invariant subvariety $X \subset \mathbf{A}^{r\times n}$ in a matrix space with more rows, and stabilizing it under the larger general linear group action. We extend this notion to certain equivariant coherent modules, and describe what it does at the level of K-polynomials.

6.1. **The structure of R as a G-module.** We need to recall the decomposition of the ring R as a module for the group G. This decomposition can be gleaned from the Cauchy identity,

$$\mathrm{Hilb}(R) = \prod_{\substack{1\leq i\leq r\\ 1\leq j\leq n}} \frac{1}{1 - u_i t_j} = \sum_\lambda s_\lambda(u_1, \ldots, u_r) s_\lambda(t_1, \ldots, t_n),$$

which decomposes R as module for $\mathrm{GL}_r(\mathbf{k}) \times \mathrm{GL}_n(\mathbf{k})$. It gives the irreducible decomposition,

$$R \approx \bigoplus_{\lambda} \mathbf{S}^{\lambda}(\mathbf{k}^r)^{\vee} \otimes \mathbf{S}^{\lambda}(\mathbf{k}^n),$$

the direct sum running over partitions λ with at most $\min\{r, n\}$ parts. Here, $\mathbf{S}^{\lambda}$ is the Schur functor indexed by λ, which has the property that when applied to a vector space E (over a field of characteristic zero), $\mathbf{S}^{\lambda}(E)$ is the irreducible representation of $\mathrm{GL}(E)$ with highest weight λ [19, Section 15.5]. To obtain the irreducible decomposition as a G-module one takes the weight space decomposition of $\mathbf{S}^{\lambda}(\mathbf{k}^n)$ and obtains

$$R \approx \bigoplus_{\lambda,\tau} \mathbf{S}^{\lambda}(\mathbf{k}^r)^{\vee} \otimes \mathbf{k}_{\mathrm{cont}(\tau)},$$

where the sum is over partitions λ with at most r parts and semistandard Young tableaux $\tau : \lambda \to \{1, 2, \dots, n\}$. Here $\mathbf{k}_{\mathrm{cont}(\tau)}$ is the one-dimensional representation of T with action $t \cdot 1 = \prod_{i=1}^{n} t_i^{\mathrm{cont}(\tau)_i}$.

The ring R has the $\mathbf{k}$-linear basis of standard bitableaux, which we now describe. If the shapes of σ and τ are both (1^{ℓ}), $\ell \leq \min\{r, n\}$, then the **bitableau** (σ, τ) is the determinant of the square submatrix of $x = (x_{ij})$ whose rows are selected by the entries of σ and whose columns are selected by the entries of τ, both in order. When σ and τ have the same shape and more than one column, one takes the product of the bitableaux obtained from corresponding pairs of columns of σ and τ. A **standard bitableau** is a bitableaux (σ, τ) where both σ and τ are semistandard Young tableaux.

A G **lowest weight vector** of R is a $\mathbb{Z}^r \times \mathbb{Z}^n$-graded homogeneous polynomial that is fixed by the subgroup of $\mathrm{GL}_r(\mathbf{k})$ consisting of upper-triangular matrices with 1's on the diagonal. Explicitly, the standard bitableaux (σ, τ) where row i of σ is filled with the number i form a basis for the space of lowest weight vectors of R. Every irreducible G-module in R possesses a unique lowest weight vector which is a linear combination of these special bitableaux. It is important in what follows that every lowest weight vector in R whose weight is λ only involves those variables x_{ij} with $i \leq \ell(\lambda)$.

6.2. **Stabilization of modules.** For the rest of this section we need to emphasize the number of rows of the matrices we are working with. As such, we write R_r for $\mathbf{k}[x_{ij} : i \in [r], j \in [n]]$ and G_r for $\mathrm{GL}_r(\mathbf{k}) \times T$. We have a tower of $\mathbf{k}$-algebras

$$R_1 \subset R_2 \subset \cdots \subset R_r \subset R_{r+1} \subset \cdots$$

given by adding a row of indeterminates.

We also have a tower of groups

$$G_1 \subset G_2 \subset \cdots \subset G_r \subset G_{r+1} \subset \cdots$$

given by taking the direct sum with a 1-by-1 identity matrix. The two towers are compatible in the sense that the G_r lowest weight vectors of R_r whose weight λ has s non-zero parts are contained in R_s, and are G_s lowest weight vectors therein. It follows that if we take the smallest G_{r+1} representation in R_{r+1} containing a fixed G_r representation within R_r, the characters are the same in the sense that one is obtained from the other by replacing the Schur polynomial $s_{\lambda}(u_1^{-1}, \dots, u_r^{-1})$ with $s_{\lambda}(u_1^{-1}, \dots, u_{r+1}^{-1})$.

Let V be a finite dimensional representation of the torus T. We can view V as a representation of G_r by letting the $\mathrm{GL}_r(\mathbf{k})$ factor of G_r act trivially. Consider the equivariant finite free module $R_r \otimes V$ and a G_r-equivariant submodule N_r thereof. There are inclusions

$$N_r \subset R_r \otimes V \subset R_{r+1} \otimes V,$$

and we define N_{r+1} to be the smallest G_{r+1}-equivariant R_{r+1}-module satisfying $N_r \subset N_{r+1} \subset R_{r+1} \otimes V$. Let J_r be the ideal in R_r generated by the size r minors of the coordinate matrix $[x_{ij}]$.

Proposition 6.1. *Suppose that the character of N_r as a G_r module is written uniquely as linear combination of irreducible characters,*

$$\sum_{\lambda \in \mathrm{Par}_r, \mathbf{a} \in \mathbb{Z}^n} d_{\lambda,\mathbf{a}} s_\lambda(u_1^{-1}, \ldots, u_r^{-1}) t^{\mathbf{a}},$$

for some integers $d_{\lambda,\mathbf{a}}$. Then, the character of $N_{r+1} + (J_{r+1} \otimes V)$ is

$$\sum_{\lambda \in \mathrm{Par}_r, \mathbf{a} \in \mathbb{Z}^n} d_{\lambda,\mathbf{a}} s_\lambda(u_1^{-1}, \ldots, u_r^{-1}, u_{r+1}^{-1}) t^{\mathbf{a}}.$$

plus the character of $J_{r+1} \otimes V$.

Proof. Since V is a sum of trivial representations of $\mathrm{GL}_r(\mathbf{k}) \subset G_r$, every lowest weight vector of $R_{r+1} \otimes V$ whose weight λ satisfies $\lambda_{r+1} \neq 0$ is contained in $J_{r+1} \otimes V$. All of these lowest weight vectors are contained in $N_{r+1} + (J_{r+1} \otimes V)$. Any other lowest weight vector of $N_{r+1} + (J_{r+1} \otimes V)$ has a weight λ which satisfies $\lambda_{r+1} = 0$. Such lowest weight vectors are contained in $R_r \otimes V$. Intersecting $N_{r+1} + (J_{r+1} \otimes V)$ with $R_r \otimes V$ returns N_r, and thus every such lowest weight vector must be a lowest weight vector of N_r. □

Let M be a G_r-equivariant R_r-module with an equivariant presentation of the form

$$0 \to N_r \to R_r \otimes V \to M \to 0. \tag{6.1}$$

Define a **stabilization** of M to be G_{r+1}-equivariant R_{r+1}-module $\rho(M)$ making the following sequence exact:

$$0 \to N_{r+1} + (J_{r+1} \otimes V) \to R_{r+1} \otimes V \to \rho(M) \to 0.$$

Example 6.2. A stabilization of R_r is R_{r+1}/J_{r+1}. More generally, a stabilization of a quotient R_r/I_r is $R_{r+1}/(I_{r+1} + J_{r+1})$, where I_{r+1} is the smallest G_{r+1} invariant ideal in R_{r+1} containing I_r.

Given a G_r-stable closed subvariety $X \subset \mathbf{A}^{r \times n}$, we let $\rho(X)$ denote the smallest G_{r+1}-stable closed subvariety of $\mathbf{A}^{(r+1) \times n}$ that contains X, namely $\overline{G_{r+1} X}$.

Lemma 6.3. *Let X be a G_r-stable closed subvariety of $\mathbf{A}^{r \times n}$. Then, the coordinate ring of $\rho(X)$ is a stabilization of the coordinate ring of X.*

Proof. Denote the ideal defining X by $I_r \subset R_r$. Suppose we have a lowest weight vector of R_{r+1} that vanishes on $\rho(X)$. If it has weight λ satisfying $\lambda_{r+1} = 0$ then it must lie in $R_r \subset R_{r+1}$. It follows that this lowest weight vector is in I_r, since $\rho(X) \cap \mathbf{A}^{r \times n} = X$. If the weight λ has $\lambda_{r+1} \neq 0$ then it is in J_{r+1}. Since $\rho(X)$ contains no rank $r+1$ matrices, its ideal contains J_{r+1} which contains all lowest weight vectors whose weight λ satisfies $\lambda_{r+1} \neq 0$. It follows that the ideal defining $\rho(X)$ is $I_{r+1} + J_{r+1}$. □

6.3. **Stabilization of K-polynomials.** Although stabilization of modules is not unique, its effect on K-classes is. This is most easily understood at the level of Hilbert series.

Lemma 6.4. *Let M be a G_r-equivariant R_r-module, with a presentation as in* (6.1)*. Let $\rho(M)$ be any stabilization of M. Write*

$$\operatorname{Hilb}(M) = \sum_{\lambda \in \operatorname{Par}_r, \mathbf{a} \in \mathbb{N}^n} d_{\lambda,\mathbf{a}} s_\lambda(u_1, \ldots, u_r) t^{\mathbf{a}}. \tag{6.2}$$

Then,

$$\operatorname{Hilb}(\rho(M)) = \sum_{\lambda \in \operatorname{Par}_r, \mathbf{a} \in \mathbb{N}^n} d_{\lambda,\mathbf{a}} s_\lambda(u_1, \ldots, u_r, u_{r+1}) t^{\mathbf{a}}.$$

Proof. The character of R_{r+1} is the character of J_{r+1} plus the character of R_{r+1}/J_{r+1}. The latter is obtained from that of R_r by replacing each $s_\lambda(u_1^{-1}, \ldots, u_r^{-1})$ with $s_\lambda(u_1^{-1}, \ldots, u_{r+1}^{-1})$. Now appeal to the additivity of characters along exact sequences and Proposition 6.1. □

We proceed to define a collection of linear operators

$$\rho_k : \mathbb{Z}[u_1, \ldots, u_r]^{\mathfrak{S}_r} \to \mathbb{Z}[u_1, \ldots, u_r, u_{r+1}]^{\mathfrak{S}_{r+1}},$$

which we extend to maps $\mathbb{Z}[u, t^{\pm 1}]^{\mathfrak{S}_r} \to \mathbb{Z}[u, u_{r+1}, t^{\pm 1}]^{\mathfrak{S}_r}$ by letting them act linearly on the t variables. In the basis of Schur functions ρ_k is defined by

$$\rho_k \, s_\lambda(u_1, \ldots, u_r) := s_{(\lambda,k)}(u_1, \ldots, u_r, u_{r+1})$$

when λ is a partition with r parts, possibly including zero parts, and (λ, k) denotes the sequence $(\lambda_1, \ldots, \lambda_r, k)$. Since (λ, k) is not necessarily a partition we use the determinantal formula to compute this Schur function, so that,

$$(\rho_k s_\lambda)(u_1, \ldots, u_{r+1}) = \frac{\det(u_i^{\lambda_j + r + 1 - j})_{i,j=1,\ldots,r+1}}{\det(u_i^{r+1-j})_{i,j=1,\ldots,r+1}}$$

where $\lambda_{r+1} = k$. Alternatively, $\rho_k s_\lambda(u)$ equals $(-1)^\ell s_\mu(u, u_{r+1})$ if μ is a partition containing λ such that the skew shape $\mu \setminus \lambda$ is a ribbon with k boxes and $\ell + 1$ non-empty rows whose lowest box is in row $r + 1$; and $\rho_k s_\lambda(u) = 0$ if there is no such μ.

Example 6.5. We see that $\rho_k 1 = 0$ for $0 < k < r$ and $\rho_r 1 = (-1)^r s_{(1^{r+1})}(u_1, \ldots, u_{r+1})$. In general $\rho_0 s_\lambda(u_1, \ldots, u_r) = s_\lambda(u_1, \ldots, u_r, u_{r+1})$.

We also collect these operators ρ_k into a sum

$$\rho = \sum_{k=0}^{n} (-1)^k e_k(t) \, \rho_k : \mathbb{Z}[u, t^{\pm 1}]^{\mathfrak{S}_r} \to \mathbb{Z}[u, u_{r+1}, t^{\pm 1}]^{\mathfrak{S}_{r+1}}.$$

We will sometimes abuse notation and allow ρ to denote the analogous operator on the ring of symmetric polynomials in $r - 1$ variables. The argument of ρ makes clear which operator is being referred to.

Proposition 6.6. *If M has a presentation as in* (6.1)*, then,*

$$\mathcal{K}(\rho(M)) = \rho \, \mathcal{K}(M). \tag{6.3}$$

In particular, if X is a closed subvariety of $\mathbf{A}^{r \times n}$, then $\mathcal{K}(\rho(X)) = \rho \, \mathcal{K}(X)$.

Proof. Consider the following sum, corresponding to a single Schur polynomial in $\mathcal{K}(M)$ within the right side of (6.3):

$$\sum_{k=0}^{n} e_k(-t)\rho_k s_\lambda(u_1,\ldots,u_r).$$

We expand along the last row the numerator in our determinantal definition of ρ_k, corresponding to the introduced $\lambda_{r+1}=k$. This turns the displayed sum above into

$$\sum_{k=0}^{n} e_k(-t)\frac{\sum_{\ell=1}^{r+1}(-1)^{\ell-1}u_\ell^k \det(u_i^{\lambda_j+r+1-j})_{i=1,\ldots,\widehat{\ell},\ldots,r+1}^{j=1,\ldots,r}}{\det(u_i^{r+1-j})_{i,j=1}^{r+1}}.$$

Moving the inner summation outside, we may rewrite the sum over k as a product, yielding

$$\begin{aligned}
&\sum_{\ell=1}^{r+1}\prod_{j=1}^{n}(1-u_\ell t_j)\frac{(-1)^{\ell-1}\det(u_i^{\lambda_j+r+1-j})_{i=1,\ldots,\widehat{\ell},\ldots,r+1}^{j=1,\ldots,r}}{\det(u_i^{r+1-j})_{i,j=1}^{r+1}}\\
&=\sum_{\ell=1}^{r+1}\prod_{j=1}^{n}(1-u_\ell t_j)\frac{\left(\prod_{i\neq\ell}u_i\right)(-1)^{\ell-1}\det(u_i^{\lambda_j+r-j})_{i=1,\ldots,\widehat{\ell},\ldots,r+1}^{j=1,\ldots,r}}{\prod_{k<i}(u_k-u_i)}\\
&=\sum_{\ell=1}^{r+1}\prod_{j=1}^{n}(1-u_\ell t_j)\frac{\left(\prod_{i\neq\ell}u_i\right)s_\lambda(u_1,\ldots,\widehat{u_\ell},\ldots,u_{r+1})}{\prod_{i\neq\ell}(u_\ell-u_i)}
\end{aligned}$$

using the Vandermonde identity. Since λ only appears in this expression in a single s_λ, in which the rest of the expression is linear, and since the ρ are linear in the t variables as well, the whole right side of (6.3) equals

$$\begin{aligned}
&\sum_{\ell=1}^{r+1}\prod_{j=1}^{n}(1-u_\ell t_j)\frac{\left(\prod_{i\neq\ell}u_i\right)\operatorname{Hilb}(M;u_1,\ldots,\widehat{u_\ell},\ldots,u_{r+1},t)\prod_{i\neq\ell}\prod_j(1-u_it_j)}{\prod_{i\neq\ell}(u_\ell-u_i)}\\
&=\left(\prod_{i=1}^{r+1}\prod_{j=1}^{n}(1-u_it_j)\right)\sum_{\ell=1}^{r+1}\frac{\left(\prod_{i\neq\ell}u_i\right)\operatorname{Hilb}(M;u_1,\ldots,\widehat{u_\ell},\ldots,u_{r+1},t)}{\prod_{i\neq\ell}(u_\ell-u_i)},
\end{aligned}$$

where we have also used the definition of the K-polynomial.

By the same determinantal manipulations used above, expanding along the last row, this is equal to

$$\left(\prod_{i=1}^{r+1}\prod_{j=1}^{n}(1-u_it_j)\right)\sum_{\lambda,\mathbf{a}} d_{\lambda,\mathbf{a}}s_\lambda(u_1,\ldots,u_r,u_{r+1})t^{\mathbf{a}}$$

if $\operatorname{Hilb}(M)$ is given the expansion in coefficients from (6.2). By Lemma 6.4, this is

$$\left(\prod_{i=1}^{r+1}\prod_{j=1}^{n}(1-u_it_j)\right)\operatorname{Hilb}(\rho(M);u_1,\ldots,u_{r+1},t),$$

which equals $\mathcal{K}(\rho(M))$. This proves (6.3). □

Example 6.7. We will use the previous result and Example 5.3 to compute $K(X_v)$ when $v \in \mathbf{A}^{3\times 4}$ has matroid $U_{2,4}$. To this end, we compute $\rho(1 - s_{(2,2)}(u)e_4(t))$. We thus need to find $\rho(1)$ and $\rho(s_{(2,2)}(u))$, which are respectively

$$\rho(1) = 1 - s_{(1,1,1)}(u)e_3(t) + s_{(2,1,1)}(u)e_4(t),$$
$$\rho(s_{(2,2)}(u)) = s_{(2,2,0)}(u) - s_{(2,2,1)}(u)e_1(t) + s_{(2,2,2)}(u)e_2(t).$$

It follows that

$$\mathcal{K}(X_v) = 1 - s_{(1,1,1)}(u)e_3(t) + s_{(2,1,1)}(u)e_4(t) \\ - (s_{(2,2,0)}(u) - s_{(2,2,1)}(u)e_1(t) + s_{(2,2,2)}(u)e_2(t))e_4(t).$$

When expanding this as a polynomial in the u_i and t_j, note that all the monomials occuring in $s_{(2,1,1)}(u)e_4(t)$ cancel with monomials occuring in the expansion of $s_{(2,2,0)}(u)e_4(t)$.

7. The tensor module

The **tensor module** of $v \in \mathbf{A}^{r\times n}$ is the cyclic $\mathrm{GL}_r(\mathbf{k})$-module in $(\mathbf{k}^r)^{\otimes n}$ generated by

$$v_1 \otimes \cdots \otimes v_n.$$

We denote the tensor module of v by $G(v)$. In this section we consider the connection between the tensor module, the K-class, and the matroid of v.

7.1. $\mathcal{O}(1,\ldots,1)$ and the tensor module.

Let $(\mathbf{A}^{r\times n})^{\mathrm{nz}}$ denote the space of matrices in $\mathbf{A}^{r\times n}$ with no column equal to zero. This is a principal T-bundle over the product of projective spaces $(\mathbb{P}^{r-1})^n$. We let $j : (\mathbf{A}^{r\times n})^{\mathrm{nz}} \to \mathbf{A}^{r\times n}$ denote the inclusion, and $p : (\mathbf{A}^{r\times n})^{\mathrm{nz}} \to (\mathbb{P}^{r-1})^n$ the projection. The tensor module $G(v)$ can be constructed from the line bundle $\mathcal{O}(1,\ldots,1)$ on $(\mathbb{P}^{r-1})^n$, which is the external tensor product of the $\mathcal{O}(1)$'s on each factor.

The inverse image $j^{-1}X_v$ is the intersection of X_v with $(\mathbf{A}^{r\times n})^{\mathrm{nz}}$ and the projection of this to $(\mathbb{P}^{r-1})^n$ is the $\mathrm{GL}_r(\mathbf{k})$-orbit closure of $p(v)$.

Proposition 7.1. *For $v \in (\mathbf{A}^{r\times n})^{\mathrm{nz}}$, $G(v)$ is dual as a $\mathrm{GL}_r(\mathbf{k})$-module to the global sections of $\mathcal{O}(1,\ldots,1)|_{\overline{\mathrm{GL}_r(\mathbf{k})p(v)}}$. The character of $G(v)$, as a $\mathrm{GL}_r(\mathbf{k})$-module, is the coefficient of $t_1 \cdots t_n$ in $\mathrm{Hilb}(X_v)$.*

Proof. The dual of $G(v)$ consists of those multilinear polynomials defined on $\mathrm{GL}_r(\mathbf{k})p(v)$. This proves the first claim. The second follows since the character of $\mathcal{O}(1,\ldots,1)|_{\overline{\mathrm{GL}_r(\mathbf{k})p(v)}}$ is obtained from the coefficient in $\mathbb{Z}[u_1,\ldots,u_r]$ of $t_1 \cdots t_n$ in the multigraded Hilbert series of X_v by replacing u with $1/u$. Taking the dual representation at the level of characters corresponds to replacing each u_i with $1/u_i$. The second claim follows. □

We will occasionally need to use facts about the tensor module of v and all of its parallel extensions. As such, we set up the notation for this now. Given $\mathbf{b} \in \mathbb{N}^n$, we let $v_{\mathbf{b}}$ denote the vector configuration obtained from v by duplicating the ith column $\mathbf{b}_i$ times (or omitting it if $\mathbf{b}_i = 0$). Let $v_{\mathbf{b}}^{\otimes}$ be tensor product of the vectors in the configuration $v_{\mathbf{b}}$, in order, and define $G(v_{\mathbf{b}})$ to be the cyclic $\mathrm{GL}_r(\mathbf{k})$-module in $(\mathbf{k}^r)^{\otimes|\mathbf{b}|}$ generated by $v_{\mathbf{b}}^{\otimes}$. The obvious generalization of Proposition 7.1 is true for $v_{\mathbf{b}}$.

Proposition 7.2. *The character of $G(v_{\mathbf{b}})$ is the coefficient of $t^{\mathbf{b}}$ in* $\mathrm{Hilb}(X_v)$.

7.2. **Support of the tensor module.** The **support** of the tensor module is the collection of partitions of n that index the irreducible representations appearing in the irreducible decomposition of $G(v)$. The **rank partition** of a matroid M is the sequence $\lambda(M(v)) = (\lambda_1, \lambda_2, \lambda_3, \dots)$ determined by the condition that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k$$

is the size of the largest union of k independent sets in M.

Theorem 7.3 (Dias da Silva [13])**.** *The rank partition of M is a partition. If M has no loops then there is a set partition of the ground set of M into independent sets of sizes $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell) \vdash n$ if and only if $\mu \leq \lambda(M)$ in dominance order.*

The following result is related to a generalization of Gamas's theorem on the vanishing of symmetrized tensors (see [3, 13]).

Proposition 7.4. *The tensor module $G(v)$ has an irreducible submodule of lowest weight μ if and only if $\mu \geq \lambda(M(v))^{\mathrm{t}}$. Further, $\mu \geq \lambda(M(v))^{\mathrm{t}}$ if and only if there is a standard Young tableau of shape μ whose columns index independent sets of $M(v)$.*

As an immediate corollary we have the following result.

Corollary 7.5. *Given $\mathbf{b} \in \mathbb{N}^n$, the coefficient of $s_\mu(u)t^{\mathbf{b}}$ in the multigraded Hilbert series of X_v is positive if and only if $\mu \geq \lambda(M(v_{\mathbf{b}}))^{\mathrm{t}}$. The latter condition happens if and only if there is a semi-standard Young tableaux of shape μ and content* $\mathbf{b}$ *whose columns index independent sets of $M(v)$.*

7.3. **Schur-Weyl duality.** One can study the irreducible decomposition of the tensor module using the representation theory of the symmetric group.

We will denote the cyclic $\mathfrak{S}_n$-module in $(\mathbf{k}^r)^{\otimes n}$ generated by $v_1 \otimes \cdots \otimes v_n$ by $\mathfrak{S}(v)$.

Proposition 7.6. *The tensor module $G(v)$ is Schur–Weyl dual to $\mathfrak{S}(v)$. That is, there are isomorphisms,*

$$\mathrm{Hom}_{\mathrm{GL}_r(\mathbf{k})}(G(v), (\mathbf{k}^r)^{\otimes n}) \cong \mathfrak{S}(v) \qquad \mathrm{Hom}_{\mathfrak{S}_n}(\mathfrak{S}(v), (\mathbf{k}^r)^{\otimes n}) \cong G(v),$$

of $\mathfrak{S}_n$-modules and $\mathrm{GL}_r(\mathbf{k})$-modules, respectively.

Proof. Either of the above isomorphisms is defined as

$$\varphi \mapsto \varphi(v_1 \otimes \cdots \otimes v_n).$$

Such a homomorphism, say $\varphi : G(v) \to (\mathbf{k}^r)^{\otimes n}$, extends to a map of $\mathrm{GL}_r(\mathbf{k})$-modules $\tilde{\varphi} \in \mathrm{End}_{\mathrm{GL}_r(\mathbf{k})}((\mathbf{k}^r)^{\otimes n})$. Schur–Weyl duality [19, Section 6.2] asserts that $\tilde{\varphi} \in \mathbf{k}\mathfrak{S}_n$. It follows that

$$\varphi(v_1 \otimes \cdots \otimes v_n) \in \mathfrak{S}(v).$$

The other isomorphism is proved similarly. $\square$

Combining Propositions 7.2 and 7.6 we obtain a second proof of Lemma 6.4. Indeed the isomorphism type of $\mathfrak{S}(v_{\mathbf{b}})$ visibly does not change when we embed v in a matrix space with more rows.

The irreducible representations of $\mathfrak{S}_n$ that can appear in $(\mathbf{k}^r)^{\otimes n}$, and hence $\mathfrak{S}(v)$, are indexed by partitions of n with at most r parts. The irreducible representations of $\mathrm{GL}_r(\mathbf{k})$ that can appear in $G(v)$ are indexed by the exact same set of partitions, as we have discussed.

Corollary 7.7. *For any partition λ of n, the multiplicity of λ in $\mathfrak{S}(v)$ is equal to the multiplicity of λ in $G(v)$.*

Proof. By another formulation of Schur–Weyl duality [19, Section 6.1], the functors $\mathrm{Hom}_{\mathfrak{S}_n}(-,(\mathbf{k}^r)^{\otimes n})$ and $\mathrm{Hom}_{\mathrm{GL}_r(\mathbf{k})}(-,(\mathbf{k}^r)^{\otimes n})$ take an irreducible indexed by λ to an irreducible indexed by λ. Since these functors commute with direct sums we are done. $\square$

As a first application of Proposition 7.1 and Corollary 7.7, we extract from Proposition 5.2 the character of the tensor module for the uniform matroid in rank 2.

Corollary 7.8. *Let $v \in \mathbf{A}^{2\times n}$ have uniform matroid. The character of the tensor module $G(v)$ is*

$$s_{(n,0)}(u) + \sum_{\ell=1}^{n/2}(n-2\ell+1)s_{(n-\ell,\ell)}(u).$$

Proof. To see this we take the coefficient of $e_n(t) = t_1\cdots t_n$ in the product of the K-polynomial $\mathcal{K}(X_v;u,t)$ from Proposition 5.2 with $1/\prod_{i=1}^n(1-u_1t_i)(1-u_2t_i)$. Writing

$$\mathcal{K}(X_v;u,t) = 1 + p_4(u)e_4(t) - p_5(u)e_5(t) - \cdots + (-1)^n p_n(u)e_n(t),$$

we see that the coefficient of $e_n(t)$ in the product in question is

$$(u_1+u_2)^n + \binom{n}{4}(u_1+u_2)^{n-4}p_4(u) - \binom{n}{5}(u_1+u_2)^{n-5}p_5(u) + \cdots + (-1)^n p_n(u).$$

Setting $u_1 = u_2 = 1$ in this formula tells us the dimension of $G(v)$. We use the fact that $p_i(1,1) = (-1)^{i+1}\sum_{k=2}^{i/2}(i-2k+1)^2 = (-1)^{i+1}\binom{i-1}{3}$. From this we obtain

$$\dim G(v) = 2^n + \sum_{i=4}^{n}(-1)^{i+1}\binom{n}{i}\binom{i-1}{3}2^{n-i} = (n^3+5n+6)/6.$$

Since the multiplicity of the Specht module indexed by $(n-k,k)$ in $(\mathbf{k}^2)^{\otimes n}$ is at most $(n-2k+1)$, by Schur–Weyl duality, we see that the multiplicity of $(n-k,k)$ in $G(v)$ is likewise bounded. Also, the multiplicity of (n) in $G(v)$ is at most one since $\mathrm{Sym}^n(\mathbf{k}^2)$ is an irreducible $\mathrm{GL}_2(\mathbf{k})$ module. If any of these multiplicities were less than these trivial upper bounds we would have

$$(n^3+5n+6)/6 < (n+1) + \sum_{i=1}^{n/2}(n-2k+1)^2 = (n^3+5n+6)/6.$$

It follows that each multiplicity is as large as possible in $G(v)$, thus proving the proposition. $\square$

Example 7.9. The tensor module of $v \in \mathbf{A}^{2\times 4}$ with matroid U_{24} has character,

$$s_{(4,0)}(u) + 3s_{(3,1)}(u) + s_{(2,2)}(u).$$

Using Proposition 9.5, which explains how duplicating a column effects the K-polynomial of X_v, it is possible to extract the isomorphism type of the tensor module of any rank two configuration $v \in \mathbf{A}^{2\times n}$ with no zero columns. In practice, the computation becomes an endless checking of cases. We state the result here, referring the reader to [6, Theorem 3.5.1] for a proof avoiding the technology of K-polynomials, and relying further on Schur–Weyl duality.

Proposition 7.10. *Let $v \in \mathbf{A}^{2\times n}$ have rank two and no columns equal to zero. Let $\mu = (\mu_1 \geq \mu_2 \geq \dots)$ denote the sizes of the equivilance classes of the relation on columns of v that puts two columns in relation when they are parallel. Then, the character of the tensor module of v is*

$$s_{(n,0)}(u) + \sum_{k=1}^{n/2} \max(\mu_1^{\mathrm{t}} + \dots + \mu_k^{\mathrm{t}} - 2k + 1, 0) s_{(n-k,k)}(u).$$

Here μ^{t} is the transposed partition.

8. Hook shapes

A partition is called a **hook** if it has at most one part that is not equal to one. The multiplicity of a hook shape λ in $\mathfrak{S}(v)$ (equivalently, $G(v)$) is determined by the subcomplex of non-broken circuit sets of $M(v)$. This is used to determine the certain coefficients of the K-polynomial $\mathcal{K}(X_v)$.

8.1. **Hooks in the Schur-Weyl dual of the tensor module.** For a matroid M with ground set contained in a totally ordered set, a **broken circuit** of M is a containment minimal dependent set with its smallest ordered element removed. A set is said to be an **nbc set** if does not contain any of the broken circuits of M. The collection of nbc sets of M forms a subcomplex of M whose structure is well studied. The nbc sets of $M(v)$ are known to be intimately related to the cohomology ring of the complement in $(\mathbf{k}^r)^*$ of the hyperplanes given by the vanishing of the linear functionals $v_1, \dots v_n$ on $(\mathbf{k}^r)^*$. In particular, enumerating the nbc sets by corank yields the coefficients of the Poincaré polynomial of this variety.

Here is our main result relating hook shapes and non-broken circuits.

Theorem 8.1. *The multiplicity of $(n-k+1, 1^{k-1})$ in $\mathfrak{S}(v)$ is the number of nbc bases of the truncation of $M(v)$ to rank k, if $k \leq \mathrm{rk}(M(v))$. It is zero otherwise.*

We will let $\lambda_{n,k}$ denote the hook shape that is a partition of n with length k, i.e., $\lambda_{n,k} = (n-k+1, 1^{k-1})$.

Proof. The element

$$\sum_{\substack{\sigma \in \mathfrak{S}_{[k]} \\ \tau \in \mathfrak{S}_{[n]\setminus[k]}}} (-1)^{\ell(\sigma)} \sigma\tau \in \mathbf{k}\mathfrak{S}_n$$

acts as a projector from $\mathbf{k}\mathfrak{S}_n$ to the sum of two irreducible Specht modules, one of shape $\lambda_{n,k+1}$, the other of shape $\lambda_{n,k}$. This follows from the Pieri rule and the fact that the above element is a product of a row symmetrizer and a column anti-symmetrizer.

Since $\mathfrak{S}(v)$ is a cyclic module, it follows that the sum of the multiplicities of $\lambda_{n,k+1}$ and $\lambda_{n,k}$ in it is the dimension of the vector space

$$\mathfrak{S}(v) \sum_{\substack{\sigma\in\mathfrak{S}_{[k]} \\ \tau\in\mathfrak{S}_{[n]\setminus[k]}}} (-1)^{\ell(\sigma)}\sigma\tau \subset \bigwedge^k(\mathbf{k}^r) \otimes \mathrm{Sym}^{n-k}(\mathbf{k}^r) \subset (\mathbf{k}^r)^{\otimes n}.$$

The image of this space is spanned by the tensors

$$\left\{ \bigwedge_{i\in I} v_i \otimes \prod_{j\notin I} v_j : I \in \binom{[n]}{k} \right\}$$

If $k = r$ then the wedges simply record whether I is a basis of $M(v)$. In this case we can forget the wedges and simply look at the dimension of the vector space spanned by

$$\left\{ \prod_{j\notin B} v_j : B \in \mathcal{B}(M(v)) \right\} \subset \mathrm{Sym}^{n-r}(\mathbf{k}^r),$$

where $\mathcal{B}(M(v))$ denotes the bases of $M(v)$. By a result of Orlik and Terao (reproved as [4, Corollary 2.3]) the dimension of this vector space is the number of nbc bases of $M(v)$, which agrees with the statement of the theorem, since the hook $\lambda_{n,r+1}$ does not appear in $\mathfrak{S}(v)$ (even in $(\mathbf{k}^r)^{\otimes n}$).

In case $k < r$ we project v onto a generic k-dimensional subspace through the origin, to obtain a new configuration v'. The multiplicity of $\lambda_{n,k}$ in $\mathfrak{S}(v')$ is the number of nbc bases of the truncation of M to rank k. Since $\mathfrak{S}(v')$ is a homomorphic image of $\mathfrak{S}(v)$ this gives a lower bound for the multiplicity of the length k hook in $\mathfrak{S}(v)$. It follows from [5, Theorem 5.4] that this multiplicity in $\mathfrak{S}(v)$ is at most the number of nbc bases of the truncation of $M(v)$ to rank k and from this the theorem follows. $\square$

The Tutte polynomial of a matroid M is the unique polynomial $T_M = T_M(x, y) \in \mathbb{Z}[x, y]$ satisfying the conditions:

- T1. $T_M(x, y) = x$ if M is rank zero on one element and $T_M(x, y) = y$ if T is rank one on one element.
- T2. $T_{M\oplus N} = T_M T_N$.
- T3. If e is neither a loop nor a coloop of M, then $T_M = T_{M\setminus e} + T_{M/e}$.

It is well known that the Tutte evaluation $q^{\mathrm{rk}(M)}T_M(1 + 1/q, 0)$ is the generating function for the nbc sets of M by their rank, as is seen by appealing to the deletion contraction recurrence (T3).

Corollary 8.2. *Let $d_{\lambda_{n,k},\mathbf{1}}$ denote the multiplicity of $\lambda_{n,k}$ in $G(v)$. Then*

$$\sum_{k=0}^{r} d_{\lambda_{n,k},\mathbf{1}} q^{k-1}(q + 1) = q^{\mathrm{rk}(M(v))}T_{M(v)}(1 + 1/q, 0).$$

Proof. First, $q^{\mathrm{rk}(M(v))}T_{M(v)}(1 + 1/q, 0)$ is the generating function for nbc sets of $M(v)$ by their rank. Next, the number of nbc bases of the truncation of $M(v)$ to rank k plus the number of nbc bases of the truncation of $M(v)$ to rank $k+1$ is the number of nbc sets of $M(v)$ of size k. To see this add the element 1 to each size k nbc set of $M(v)$ that does not already contain it. The sets that already contained 1 were the nbc bases of the truncation to rank k, the other sets correspond to the nbc bases of the truncation to rank $k+1$. $\square$

Reformulating the above result in terms of the multigraded Hilbert series of X_v yields the following result.

Corollary 8.3. *Write*

$$\mathrm{Hilb}(X_v) = \sum_{\lambda \in \mathrm{Par}_r, \mathbf{b} \in \mathbb{N}^n} d_{\lambda,\mathbf{b}} s_\lambda(u) t^{\mathbf{b}}.$$

Then,

$$\sum_{k=0}^{r} d_{\lambda_{n,k},\mathbf{b}} q^{k-1}(q+1) = q^{\mathrm{rk}(M(v_{\mathbf{b}}))} T_{M(v_{\mathbf{b}})}(1+1/q, 0).$$

Proof. The coefficient in question is the multiplicity of $\lambda_{|\mathbf{b}|,k}$ in $G(v_{\mathbf{b}})$. Since this is the multiplicity of $\lambda_{|\mathbf{b}|,k}$ in $\mathfrak{S}(v)$, by Corollary 7.7, the result follows from Proposition 7.2 and Corollary 8.2. $\square$

8.2. **Hooks and the multivariate Tutte polynomial.** In this section we use the multivariate Tutte polynomial of Sokal to describe the hook shapes that appear in the K-polynomial of a matrix orbit closure.

We start with a definition, given a matroid M with ground set $[n]$ we define

$$\widetilde{Z}_M(q; t_1, \ldots, t_n) = \sum_{\mathbf{b} \in \{0,1\}^n} q^{-\mathrm{rk}(M|\mathbf{b})} t^b.$$

This is the **multivariate Tutte polynomial** of M, due to Sokal [34], which is also known to statistical physicists as the q-state Potts model partition function.

Lemma 8.4 (Ardila–Postnikov [2, Lemma 6.6])**.** *The generating function for the Tutte polynomials of $M|\mathbf{b}$, as $\mathbf{b}$ ranges over $\mathbb{N}^n$, can be expressed as*

$$\sum_{\mathbf{b} \in \mathbb{N}^n} (x-1)^{-\mathrm{rk}(M|\mathbf{b})} T_{M|\mathbf{b}}(x,y) t^{\mathbf{b}}$$
$$= \frac{1}{\prod_{j=1}^n (1-t_j)} \widetilde{Z}_M\left((x-1)(y-1); \frac{(y-1)t_1}{1-yt_1}, \ldots, \frac{(y-1)t_n}{1-yt_n}\right).$$

In what follows we will work in the subring S of $\mathbb{Z}[u_1, \ldots, u_r, t_1, \ldots, t_n]^{\mathfrak{S}_r}$ where the u-degree of a polynomial equals its t-degree.

Using Lemma 6.4 we can unambiguously extend the Hilbert series of X_v to a *symmetric function* in the infinitely many variables $u_1, u_2, \ldots,$ with coefficients in $\mathbb{Z}[[t_1, \ldots, t_n]]$. From this symmetric function we mod out those Schur functions in u that are not hook shapes. It is a consequence of the Littlewood–Richardson rule that this quotient of S is isomorphic, as a ring, to $\mathbb{Z}[[q, t_1, \ldots, t_n]]$. The image of $s_{\lambda_{|\mathbf{b}|,k}}(u) t^{\mathbf{b}}$ under this isomorphism is $q^{k-1}(q+1)t^{\mathbf{b}}$.

We take the image of the Hilbert series $\mathrm{Hilb}(X_v)$ in $\mathbb{Z}[[q, t_1, \ldots, t_n]]$, which has the form

$$\sum_{\mathbf{b}\in\mathbb{N}^n} \sum_{k=0}^{\infty} d_{\lambda_{|b|,k}} q^{k-1}(q+1)t^{\mathbf{b}} = \sum_{\mathbf{b}\in\mathbb{N}^n} q^{\mathrm{rk}(M(v)|\mathbf{b})} T_{M(v)|\mathbf{b}}(1+1/q, 0)t^{\mathbf{b}},$$

the second equality being Corollary 8.3. Lemma 8.4 then condenses the sum to

$$\frac{1}{\prod_{j=1}^{n}(1-t_j)} \widetilde{Z}_{M(v)}(-q^{-1}; -t_1, \ldots, -t_n).$$

We can now state our result on hook shapes in terms of the K-polynomial of X_v.

Proposition 8.5. *The enumerator of hook shapes in the K-polynomial of X_v is*

$$\prod_{j=1}^{n} \frac{1-(q+1)t_j + q(q+1)t_j^2 - q^2(q+1)t_j^3 + \cdots}{(1-t_j)} \widetilde{Z}_{M(v)}(-q^{-1}; -t_1, \ldots, -t_n).$$

in the following sense: The coefficient of $q^{k-1}(q+1)t^{\mathbf{b}}$, $k \leq r$, is equal to the coefficient of $s_{\lambda_{|\mathbf{b}|,k}}(u_1, \ldots, u_r)t^{\mathbf{b}}$ in the K-polynomial.

Proof. This follows from the definition of K-polynomial, after we push the denominator of the Hilbert series $\prod_{j=1}^{n}\prod_{i=1}^{\infty}(1-u_i t_j)$ into $\mathbb{Z}[[q, t_1, \ldots, t_n]]$. $\square$

Theorem 8.6. *Take $\mathbf{b} \in \{0,1\}^n$. The coefficient of $s_{\lambda_{|\mathbf{b}|,k}}(u)t^{\mathbf{b}}$ in the K-polynomial of X_v is $(-1)^k$ if $\mathbf{b}$ indexes a rank $k-1$ dependent set of $M(v)$. It is zero otherwise.*

Proof. Since we are only interested in the square-free monomials in the t variables, we work modulo $\langle t_1^2, \ldots, t_n^2\rangle$. By Proposition 8.5, the image of the K-polynomial in $\mathbb{Z}[[q, t_1, \ldots, t_n]]/\langle t_1^2, \ldots, t_n^2\rangle$ is

$$\prod_{j=1}^{n} \frac{1-t_j(1+q)}{(1-t_j)} \widetilde{Z}_{M(v)}(-q^{-1}; -t_1, \ldots, -t_n) \equiv \prod_{j=1}^{n}(1-t_j q)\widetilde{Z}_{M(v)}(-q^{-1}; -t_1, \ldots, -t_n).$$

Give the right-hand side of this equality the name $\mathrm{FakeDep}_{M(v)}(q; t_1, \ldots, t_n)$. By [34, Eq. (4.18a)] we see that FakeDep satisfies the recurrence

$$\mathrm{FakeDep}_{M(v)} = (1-t_i q)\mathrm{FakeDep}_{M(v-v_i)} + t_i q \mathrm{FakeDep}_{M(v/v_i)}, \quad (v_i \neq 0).$$

We define the multivariate polynomial

$$\mathrm{Dep}_{M(v)}(q; t_1, \ldots, t_n) = 1 + \sum_{\substack{\mathbf{b}\in\{0,1\}^n, \mathbf{b}\neq(0,\ldots,0) \\ \mathrm{rk}\, M(v_{\mathbf{b}})<|\mathbf{b}|}} (-1)^{\mathrm{rk}\, M(v_{\mathbf{b}})} q^{\mathrm{rk}\, M(v_{\mathbf{b}})-1}(q+1)t^{\mathbf{b}}.$$

The sum is over the dependent sets of $M(v)$. It is straight forward that Dep satisfies the same recurrence as FakeDep. Further, if v_i is the zero vector then

$$\mathrm{Dep}_{M(v)}(q; t_1, \ldots, t_n) = (1-qt_i)\mathrm{Dep}_{M(v-v_i)}(q; t_1, \ldots, t_n),$$

and likewise for FakeDep. Since both FakeDep and Dep evaluate to 1 when $v = (v_1)$, $v_1 \neq 0$, and to $(1-t_1 q)$ when $v = (0)$ it follows that they are equal in general. The theorem follows by taking the coefficient of $q^{\mathrm{rk}(M(v_{\mathbf{b}}))-1}(q+1)t^{\mathbf{b}}$ in $\mathrm{Dep}_{M(v)}$. $\square$

9. Direct sum, parallel extension and K-polynomials

In this section we consider some common matroid operations as operations on vector configurations, and see how these manifest themselves at the level of K-classes.

9.1. Direct sum.

In order to understand how direct sums interact with K-classes we first consider a concatenation operation.

Proposition 9.1. *Suppose that $v^1 \in \mathbf{A}^{r\times n_1}$ has its last r' rows equal to zero and $v^2 \in \mathbf{A}^{r\times n_2}$ has its first $r-r'$ rows equal to zero. Let $v = (v^1, v^2)$ be the concatenation of v^1 and v^2. Then, the K-polynomial of v is the product of the K-polynomials of v^1 and v^2.*

Combining this result with Proposition 6.6 allows us to see how the direct sum of vector configurations manifests itself at the level of K-polynomials.

Corollary 9.2. *Let v^1 and v^2 be vector configurations in $\mathbf{A}^{r_1\times n_1}$ and $\mathbf{A}^{r_2\times n_2}$, and $v^1 \oplus v^2$ their direct sum in $\mathbf{A}^{r\times n}$. Then,*

$$\mathcal{K}(X_{v^1\oplus v^2}) = \rho^{r_2}\mathcal{K}(X_{v^1}) \cdot \rho^{r_1}\mathcal{K}(X_{v^2})$$

Proof of Proposition 9.1. There is a projection of $\mathbf{A}^{r\times n}$ onto the first n_1 columns and the last n_2 columns. The orbit of v is the intersection of the pullbacks of the orbits of v^1 and v^2 under these projections. It follows that the ideal of the orbit of $v^1 \oplus v^2$ is the sum of the inclusions of the ideals of v^1 and v^2 into R. Denote these ideals and their inclusion by I_{v^1} and I_{v^2}. Since these ideals are in different variables we conclude that

$$\mathrm{Hilb}(R/I_{v^1\oplus v^2}) = \mathrm{Hilb}(R/(I_{v^1} + I_{v^2})) = \mathrm{Hilb}(R/I_{v^1} \otimes R/I_{v^2})$$

and that the K-polynomial of $R/I_{v^1\oplus v^2}$ is

$$\mathrm{Hilb}(R/I_{v^1})\,\mathrm{Hilb}(R/I_{v^2}) \prod_{i=1}^{r}\prod_{j=1}^{n}(1-u_it_j)$$

The K-polynomial of R/I_{v^1} is

$$\mathcal{K}(R_1/I_{v^1}) \prod_{i=1}^{r}\prod_{j=n_1+1}^{n}(1-u_it_j),$$

where R_1 is the coordinate ring of $\mathbf{A}^{r\times n_1}$. Similarly, the K-polynomial of R/I_{v^2} is

$$\mathcal{K}(R_2/I_{v^2}) \prod_{i=1}^{r}\prod_{j=1}^{n_1}(1-u_it_j),$$

and from this the result follows. □

9.2. Parallel extension.

Here we are concerned with the effect on the K-polynomial of duplicating a column of $v \in \mathbf{A}^{r\times(n-1)}$, which corresponds to a parallel extension of the underlying matroid. In view of Section 9.1, it is just as informative to compare the matrix with duplicated column to a matrix of the same size with one of the duplicated columns replaced by zero. This gives the next theorem a particularly nice form.

Let δ_{n-1} be the $(n-1)$th Demazure operator on the t variables, given by

$$\delta_{n-1}(f) = \frac{t_{n-1}f - t_n\sigma_{n-1}f}{t_{n-1} - t_n}$$

for $f \in \mathbb{Z}[t_1, \ldots, t_n, u_1, \ldots, u_r]$, where $\sigma_{n-1} \in \mathfrak{S}_n$ is the transposition $(n-1\ n)$, and $\mathfrak{S}_n$ acts by permuting the t variables.

Theorem 9.3. *Suppose that the last two columns of $v^{\|} \in \mathbf{A}^{r\times n}$ are nonzero and equal, and $v \in \mathbf{A}^{r\times n}$ is obtained from $v^{\|}$ by changing the last column to zero. Then*

$$\mathcal{K}(X_{v^{\|}}) = \delta_{n-1}\mathcal{K}(X_v)$$

This theorem comes quickly from a $\mathbb{P}^1$-bundle construction like the one used for Schubert varieties [8, 10]. To extend the parallelism of these two situations, Schubert varieties Ω_λ in $G(r,n)$ are in bijection with Schubert matroids M_λ, in such a way that a generic point of Ω_λ has matroid M_λ. If the indexing partition λ of one Schubert matroid satisfies $\lambda_1 = n-r-1$, and $\lambda' = (n-r, \lambda_2, \ldots, \lambda_r)$ is obtained from it by adding a box, then n is a loop in M_λ and $M_{\lambda'}$ is a parallel extension of $M_\lambda \setminus \{n\}$, while $\mathcal{K}(X_{\lambda'}) = \delta_{n-1}\mathcal{K}(X_\lambda) \in K^0(G(r,n))$.

The precise statement we will use is the following "sweeping lemma". It is a K-theoretic analogue of the cohomological lemma [27, Lemma 2.2.1], whose statement we have mimicked closely.

Lemma 9.4. *Let $T \subseteq B \subseteq P$ be a triple of Lie groups such that $P/B \cong \mathbb{P}^1$ and T is a torus acting with weight μ on $\mathfrak{p}/\mathfrak{b}$. Let $r \in N_P(T)$ be an element of the normalizer of T in P, inducing an automorphism r of the weight lattice T^* such that $r \cdot \mu = -\mu$.*

Let V be a P-representation and $X \subseteq V$ a B-invariant subvariety. Then in $K_0^T(V)$ we have the equality of K-classes

$$d\mathcal{K}(P \cdot X) = \frac{\mathcal{K}(X) - \mu r\mathcal{K}(X)}{1-\mu}$$

where d is the degree of the map $P \times^B X \to P \cdot X$ (or 0 if Y is P-invariant).

The proof goes through as in [27], *mutatis mutandis*, except that for a torus T acting on $\mathbb{P}^1$ via the weight μ, the relation that one attains in $K_0^T(\mathbb{P}^1)$ is

$$[\{0\}] - \mu[\{\infty\}] = 1 - \mu,$$

as can still be checked by equivariant localization.

Proof of Theorem 9.3. In any matrix $w \in X_{v^{\|}}$, the last two columns w_{n-1} and w_n are parallel. Let $a \in \mathbf{k}^n$ be one of w_{n-1} and w_n which is nonzero, if either is, or 0 if $w_{n-1} = w_n = 0$. Then $(w_1, \ldots, w_{n-2}, a, 0)$ is in X_v, and we can write $w = (w_1, \ldots, w_{n-2}, y_1 a, y_2 a)$ for some generically unique choice of $(y_1 : y_2) \in \mathbb{P}^1(\mathbf{k})$.

Let the variety $X \subseteq \mathbf{A}^{r\times n}$ be X_v. Since $v_n = 0$, this orbit is in fact B-equivariant where

$$B = \mathrm{GL}_r \times (\mathbf{k}^\times)^{n-2} \times \left\{ \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \right\} \supseteq \mathrm{GL}_r \times T = G$$

and the $\{\begin{bmatrix} * & 0 \\ * & * \end{bmatrix}\}$ factor acts on the last two columns. Let $P = \mathrm{GL}_r \times (\mathbf{k}^\times)^{n-2} \times \mathrm{GL}_2$, so that as above $P \cdot X = X_{v^{\|}}$, and the map $P \times^B X \to P \cdot X$ is degree 1. Take the

T of the lemma to be the maximal torus in G. Then $\mu = t_n/t_{n-1}$, and we will take r to be $(1, 1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$, whose action is the same as that of σ_{n-1}. Then the theorem is immediate from Lemma 9.4. $\square$

We can now combine Proposition 5.2 and Theorem 9.3 to obtain an explicit formula for the K-polynomial of an arbitrary $v \in \mathbf{A}^{2\times n}$. We will only formulate our result for those v of full rank with no columns equal to zero. If the jth column of t is zero, then $\mathcal{K}(X_v)$ is simply the K-class where this column is deleted multiplied by $(1 - u_1 t_j)(1 - u_2 t_j)$.

If $v \in \mathbf{A}^{2\times n}$ has no zero columns then we define its **parallelism partition** to be the decreasing sequence of sizes of its rank one flats. For example, a matrix with uniform matroid $U_{2,n}$ has parallelism partition $(1, 1, \ldots, 1) = (1^n)$.

Proposition 9.5. *Suppose that $v \in \mathbf{A}^{2\times n}$ has rank two and no zero columns. Write*

$$\mathcal{K}(X_v; u, t) = \sum_{\substack{\mathbf{b}\in\{0,1\}^n \\ 0\le k\le|\mathbf{b}|/2}} d_{k,\mathbf{b}}(v) s_{(|\mathbf{b}|-k,k)}(u) t^{\mathbf{b}},$$

as we may. Then,

1. $d_{0,\mathbf{0}}(v) = 1$, *and* $d_{0,\mathbf{b}}(v) = 0$ *for all other* $\mathbf{b} \in \{0, 1\}^n$.
2. *If* $k = 1$ *and* $v_{\mathbf{b}}$ *has rank one then* $d_{k,\mathbf{b}}(v) = (-1)^{|\mathbf{b}|+1}$.
3. *If* $k = 1$ *and the rank of* $v_{\mathbf{b}}$ *is two then* $d_{k,\mathbf{b}}(v) = 0$.
4. *If* $k \ge 2$ *and* $v_{\mathbf{b}}$ *has parallelism partition* $\mu = (\mu_1 \ge \cdots \ge \mu_\ell)$, $\ell \ge 4$ *and* $\mu_1^{\mathrm{t}} + \cdots + \mu_{k-1}^{\mathrm{t}} \ge 2k-1$ *then* $d_{k,\mathbf{b}}(v) = (-1)^{|\mathbf{b}|+1}(\mu_1^{\mathrm{t}} + \cdots + \mu_{k-1}^{\mathrm{t}} - 2k+1)$. *Otherwise,* $d_{k,\mathbf{b}}(v) = 0$.

Proof. We express Theorem 9.3 in the following way: If $v' \in \mathbf{A}^{2\times(n+1)}$ is obtained by duplicating the last column of $v \in \mathbf{A}^{2\times n}$ then

$$K(\overline{\mathrm{GL}_2 v' T^{n+1}}) = \delta_n \left(K(\overline{\mathrm{GL}_2 v T^n})(1 - u_1 t_{n+1})(1 - u_2 t_{n+1}) \right).$$

Temporarily write $K(v)$ and $K(v')$ for the K-polynomials of the orbit closures of v and v'.

Since, by induction, $K(v)$ is square free in the t-variables, we may uniquely write $K(v) = K(v)_0 + K(v)_1 t_n$, where t_n does not appear in $K(v)_0$. A simple computation yields

$$\delta_n(1 - u_1 t_{n+1})(1 - u_2 t_{n+1}) = 1 - u_1 u_2 t_n t_{n+1},$$
$$\delta_n t_n(1 - u_1 t_{n+1})(1 - u_2 t_{n+1}) = t_n + t_{n+1} - (u_1 + u_2) t_n t_{n+1},$$

and it follows that

$$K(v') = K(v)_0 - s_{(1,1)}(u) t_n t_{n+1} K(v)_0 + K(v)_1 t_n + K(v)_1 t_{n+1} - s_{(1)}(u) t_n t_{n+1} K(v)_1. \tag{9.1}$$

We conclude that $K(v')$ is square free in the t-variables, does not contain any Schur polynomials of partitions of length 1, and the coefficient of any $t^{\mathbf{b}}$ in $K(v')$ that does not contain *both* t_n and t_{n+1} is as described in the proposition.

Suppose that $t^{\mathbf{b}} = t^{\mathbf{a}} t_n t_{n+1}$, $k \geq 1$, and write $(\mathbf{a}, 1)$ for the exponent vector of $t^{\mathbf{a}} t_n$. Then (9.1) implies that

$$d_{k,\mathbf{b}}(v') = -\left(d_{k-1,\mathbf{a}}(v) + d_{k-1,(\mathbf{a},1)}(v) + d_{k,(\mathbf{a},1)}(v)\right).$$

Since $d_{k,(\mathbf{a},1)}(v) = d_{k,(\mathbf{a},1)}(v')$, by our computation above, this yields

$$d_{k,\mathbf{b}}(v') + d_{k,(\mathbf{a},1)}(v') = -\left(d_{k-1,\mathbf{a}}(v) + d_{k-1,(\mathbf{a},1)}(v)\right). \tag{9.2}$$

What follows from here is a tedious check that the coefficients described in the proposition obey (9.2). To preserve the reader's patience, we will not provide the details of all possible cases, which are many. This will be forgiven since the case $k = 1$ is addressed in much greater generality by Theorem 8.6.

We will focus on the least degenerate case, when $k \geq 3$ and $d_{k-1,\mathbf{a}}(v)$, $d_{k-1,(\mathbf{a},1)}(v)$ and $d_{k,(\mathbf{a},1)}(v) = d_{k,(\mathbf{a},1)}(v')$ are all non-zero. In this case, induction yields

$$\begin{aligned} d_{k-1,\mathbf{a}}(v) &= (-1)^{|\mathbf{a}|+1}\left(\mu_1^{\mathrm{t}}(\mathbf{a}) + \cdots + \mu_{k-2}^{\mathrm{t}}(\mathbf{a}) - 2k + 3\right), \\ d_{k-1,(\mathbf{a},1)}(v) &= (-1)^{|\mathbf{a}|+2}\left(\mu_1^{\mathrm{t}}(\mathbf{a},1) + \cdots + \mu_{k-2}^{\mathrm{t}}(\mathbf{a},1) - 2k + 3\right), \\ d_{k,(\mathbf{a},1)}(v') &= (-1)^{|\mathbf{a}|+2}\left(\mu_1^{\mathrm{t}}(\mathbf{a},1) + \cdots + \mu_{k-2}^{\mathrm{t}}(\mathbf{a},1) + \mu_{k-1}^{\mathrm{t}}(\mathbf{a},1) - 2k + 1\right). \end{aligned}$$

Here, $\mu(\mathbf{a})$ and $\mu(\mathbf{a}, 1)$ are the parallelism partitions of $v_{\mathbf{a}}$ and $v_{(\mathbf{a},1)}$. Now, $\mu(\mathbf{a})$ and $\mu(\mathbf{a}, 1)$ differ in exactly one position. Hence, the sum of the first two terms is $(-1)^{|\mathbf{a}|+2}$, unless the number of vectors parallel to v_n in $v_{\mathbf{a}}$ is $k-1$ or larger. If the latter happens then the $d_{k-1,\mathbf{a}}(v) + d_{k-1,(\mathbf{a},1)}(v) = 0$. This implies that $d_{k,\mathbf{b}}(v')$ differs from $d_{k,(\mathbf{a},1)}(v')$ by ± 1 or 0, according to whether the number of vectors parallel to v_n in $v_{(\mathbf{a},1)}$ is larger than $k-1$, or not. Hence $d_{k,\mathbf{b}}(v')$ is given by the given by the formula

$$\mu_1^{\mathrm{t}}(\mathbf{b}) + \cdots + \mu_{k-1}^{\mathrm{t}}(\mathbf{b}) - 2k + 1,$$

where $\mu(\mathbf{b})$ is the parallelism partition of $v_{\mathbf{b}}$. The remaining cases are left to the reader. □

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References

[1] F. Ardila, C. Klivans. The Bergman complex of a matroid and phylogenetic trees. *J. Comb. Theory, Ser. B* 96(1):38–49, 2006.

[2] F. Ardila, A. Postnikov. Combinatorics and geometry of power ideals. *Trans. Amer. Math. Soc.* 362 (2010), no. 8, 4357–4384, 2010.

[3] A. Berget. Equality of symmetrized tensors and the flag variety. *Linear Algebra Appl.*, 438(2):658–656, 2013.

[4] A. Berget. Products of linear forms and Tutte polynomials. *European J. of Combin.*, 31(7):1924–1935, 2010.

[5] A. Berget. Tableaux in the Whitney module of a matroid. *Sém. Lothar. Combin.* 63, Art. B63f, 17 pp, 2010.

[6] A. Berget. Symmetries of tensors. PhD Thesis, University of Minnesota, 2009.

[7] A. Berget, A. Fink. Equivariant Chow classes of matrix orbit closures. *Transformation Groups.* 22(3): 631–643, 2017.

[8] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand. Schubert cells and cohomology of the spaces G/P. *Russian Mathematical Surveys* 28:1–26, 1973.

[9] C. Chindris. On the geometry of orbit closures for representation-infinite algebras. *Gaslg. Math. J.*, 54(3):629–636.
[10] M. Demazure. Désingularization des variétés de Schubert généralisées. *Ann. Sc. ENS sér. 4*, 7:53-88, 1974.
[11] J. Edmonds. Submodular functions, matroids, and certain polyhedra. in *Comb. Structures and their App.*, eds. Guy, Hanani, Sauer, Schonheim. Gordon and Breach, New York, 69–87, 1970.
[12] J. Désarménien and J. Kung and G.C. Rota. Invariant theory, Young bitableaux and combinatorics, *Adv. Math.* 27, 1978.
[13] J. A. Dias da Silva. On the μ-colorings of a matroid. *Linear and Multilinear Algebra*, 27(1):25–32, 1990.
[14] J. Eagon, M. Hochster. Cohen-Macaulay Rings, Invariant Theory, and the Generic Perfection of Determinantal Loci. *American J. Math.*, 93(4):1020–1058, 1971.
[15] D. Eisenbud. *Commutative algebra with a view toward algebraic geometry.* Springer-Verlag, 1995.
[16] L. Fehér, A. Némethi, R. Rimányi. Equivariant classes of matrix matroid varieties. *Comment. Math. Helv.* 87:861–889, 2012.
[17] A. Fink and D. Speyer. K-classes of matroids and equivariant localization. *Duke Math. J.*, 161(14):2699–2723, 2012.
[18] William Fulton. Flags, Schubert polynomials, degeneracy loci, and determinantal formulas. *Duke Math. J.*, 65(3):381–420, 1992.
[19] W. Fulton, J Harris. *Representation theory. A first course.* Springer-Verlag, 1991.
[20] C. Gamas. Conditions for a symmetrized decomposable tensor to be zero. *Linear Algebra Appl.*, 108:83–119, 1988.
[21] I. M. Gel′fand, R. M. Goresky, R. D. MacPherson, V. V. Serganova. Combinatorial geometries, convex polyhedra, and Schubert cells. *Adv. in Math.*, 63(3):301–316, 1987.
[22] H. W. Gould. Tables of combinatorial identities. Available from http://www.math.wvu.edu/~gould/.
[23] D. Grayson, M. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/
[24] M. Kapranov. Chow quotients of Grassmannians. I. In *I. M. Gel′fand Seminar*, volume 16 of *Adv. Soviet Math.*, pages 29–110. Amer. Math. Soc., Providence, RI, 1993.
[25] A. Klyachko. Orbits of a maximal torus on a flag space. *Funktsional. Anal. i Prilozhen.*, 19(1):77–78, 1985.
[26] Allen Knutson. Puzzles, positroid varieties, and equivariant k-theory of Grassmannians. Preprint, arXiv:1008.4302, 2010.
[27] A. Knutson. Introduction to geometric representation theory. Course notes, available from www.math.cornell.edu/~allenk/courses/10fall/notes.pdf
[28] A. Knutson and E. Miller. Gröbner geometry of Schubert polynomials. *Ann. of Math. (2)*, 161:1245–1318, 2005.
[29] A. Knutson and E. Miller and A. Yong. Gröbner geometry of vertex decompositions and of flagged tableaux. *J. Reine Angew. Math.*, 630:1–31, 2009.
[30] A. Knutson and T. Tao. Puzzles and (equivariant) cohomology of Grassmannians. *Duke Math. J.* 119(2):221–260, 2003.
[31] S.H. Lee, R. Vakil. Mnëv–Sturmfels universality for schemes. arXiv:1202.3934, 2012.
[32] I. G. MacDonald. *Symmetric functions and Hall polynomials.* Oxford University Press, New York, 1995.
[33] E. Miller and B. Sturmfels. *Combinatorial commutative algebra.* Springer-Verlag, 2005.
[34] A. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. *Surveys in combinatorics 2005*, London Math. Soc. Lecture Notes 327, 2005.
[35] D. Speyer. A matroid invariant via the K-theory of the Grassmannian. *Adv. Math.*, 221(3):882–913, 2009.
[36] B. Sturmfels. On the matroid stratification of Grassmann varieties, specialization of coordinates, and a problem of N. White. *Adv. Math.*, 75:202–211, 1989.
[37] N. White. The basis monomial ring of a matroid. *Adv. Math.*, 24(3):292–297, 1977.
[38] N. White, editor. *Theory of matroids*, volume 26 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1986.

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