**Noname manuscript No.**
(will be inserted by the editor)

# Cartan Invariants and Event Horizon Detection

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March 15, 2018

**Abstract** We show that it is possible to locate the event horizon of a black hole (in arbitrary dimensions) by the zeros of certain Cartan invariants. This approach accounts for the recent results on the detection of stationary horizons using scalar polynomial curvature invariants, and improves upon them since the proposed method is computationally less expensive. As an application, we produce Cartan invariants that locate the event horizons for various exact four-dimensional and five-dimensional stationary, asymptotically flat (or (anti) de Sitter), black hole solutions and compare the Cartan invariants with the corresponding scalar curvature invariants that detect the event horizon.

## 1 Introduction

General Relativity predicts the existence of singularities hidden by a horizon (Misner et al. 1973), which are commonly called black holes. Naively speaking one can regard a black hole as a region of spacetime from which nothing can escape; i.e., after crossing the horizon towards the singularity, a photon can never escape to asymptotic infinity. While this captures the basic property of black holes it is unsatisfactory in applications. As a theory, general relativity is entirely local, and the definition of an event horizon requires global information on the entire spacetime (Choquet-Bruhat et al. 1982; Choquet-Bruhat 2000).

Therefore, it is desirable to find alternative definitions or characterizations of horizons that are quasi-local. For example, a local characterization of the horizon of a black hole is necessary in the numerical study of the evolution of configurations of many black holes. At this time, only approximate localizations are possible, such as considering the horizon as a marginally outer trapped surface, a minimal surface, a Killing horizon or an apparent horizon (Ashtekar and Krishnan 2004; Booth 2005) which are foliation dependent. Recently it was shown that specific combinations of the scalar polynomial invariants (SPIs) vanish on the horizon of a stationary black hole. This provides a local technique for the localization of the event horizon, and a generalization of the method of Paiva et al. (1993) where information is extracted from invariants regarding the mass, angular momentum and electric charge of a black hole (Abdelqader and Lake 2015; Page and Shoom 2015).

In this paper we shall examine stationary asymptotically flat (or (anti) de Sitter) black hole solutions in four (4D) and five (5D) dimensions (Polchinski 2005; Zwiebach 2009) and apply the Cartan-Karlhede algorithm (in 4D this is termed the Karlhede algorithm (Collins et al. 1990; Collins and d'Inverno 1993; McNutt et al. 2017)) to compute Cartan invariants that detect the horizon. While both the SPIs and Cartan invariants are foliation independent, the Cartan invariants have two advantages over SPIs: they are linear in terms of the components of the curvature tensor instead of quadratic or higher degree, and it is possible to construct from the Cartan invariants suitable invariants that only vanish on the horizon. Finally, we compute SPIs for the 4D and 5D examples using the results of Page and Shoom (2015), and in 4D we show how the Cartan invariants are related to the SPIs: thereby the rather complicated expressions used for the SPIs in previous work are shown to have simpler forms.

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### 1.1 Horizon Detection with Scalar Polynomial Curvature invariants

The scalar polynomial invariants (SPI) of a given spacetime metric, $g_{ab}$, form the set of functions generated by contractions of copies of the curvature tensors and their covariant derivatives, such as

$$R_{ab}R^{ab}, C_{abcd}C^{abef}C_{ef}{}^{cd}, R_{ab;c}R^{ab;c}, C_{abcd;e}C^{abcd;e}. \tag{1.1}$$

We shall denote by $\mathcal{I} = \{R, R_{ab}R^{ab}, C_{abcd}C^{abcd}, \dots\}$ the set of SPIs of a spacetime $(\mathbf{M}, g)$. In general the set $\mathcal{I}$ is not sufficient to locally characterize a spacetime as it is possible that two different metrics can have the same set $\mathcal{I}$ (MacCallum 2015). In the particular case in which a spacetime *is* fully characterized by its SPIs the spacetime is said to be $\mathcal{I}$-non-degenerate (Coley et al. 2009). Using the alignment classification (Milson et al. 2005), if a spacetime metric is of Ricci type **I/G**, Weyl type **I**, or Riemann type **I/G** or its covariant derivative is of type **I** (relative to the alignment classification, which is reviewed in section 2) then the metric is $\mathcal{I}$-non-degenerate. Moreover, in the case that the metric is not $\mathcal{I}$-non-degenerate, then it is necessarily contained in the Kundt class or is locally homogeneous (Coley et al. 2009). We note that the black hole solutions are $\mathcal{I}$-non-degenerate.

Some examples of SPIs, denoted, following Abdelqader and Lake (2015) and Page and Shoom (2015), by $I_1, \dots, I_7$, are:

$$\begin{aligned} &I_1 = C^{abcd}C_{abcd}, \quad I_2 = C^{*abcd}C_{abcd}, \quad I_3 = C^{abcd;e}C_{abcd;e}, \\ &I_4 = C^{*abcd;e}C_{abcd;e}, \quad I_5 = (I_1)_{;a}(I_1)^{;a}, \quad I_6 = (I_2)_{;a}(I_2)^{;a}, \quad I_7 = (I_1)_{;a}(I_2)^{;a}, \end{aligned} \tag{1.2}$$

where $C_{abcd}$ is the Weyl tensor and $C^*_{abcd}$ is its dual, and a semicolon denotes a covariant derivative.

Horizon detection was remarked upon by Karlhede et al. (1982) where the invariant $R_{abcd;e}R^{abcd;e}$ was shown to detect horizons for several type **D** solutions. However in the case of the Kerr horizon, it detected the stationary limit, and not the outer horizon itself. This was first noted by Skea in his doctoral thesis (Skea 1986), where it was observed that $R_{abcd;e}R^{abcd;e}$ did not provide an adequate test for horizons. More recently in Abdelqader and Lake (2015) a collection of invariants were examined from which the physical properties of spacetimes around rotating black holes were determined, including the detection of the horizons. These invariants are constructed from SPIs (note that being in vacuum we do not distinguish between the Riemann and Weyl tensors):

$$Q_1 = \frac{(I_1^2 - I_2^2)(I_5 - I_6) + 4I_1I_2I_7}{3\sqrt{3}(I_1^2 + I_2^2)^{\frac{9}{4}}}, \quad Q_2 = \frac{I_5I_6 - I_7^2}{27(I_1^2 + I_2^2)^{\frac{5}{2}}}, \quad Q_3 = \frac{I_5 + I_6}{6\sqrt{3}(I_1^2 + I_2^2)^{\frac{5}{4}}} \tag{1.3}$$

where $I_1$ to $I_7$ are given by (1.2). From the dimensionless invariants $Q_1$, $Q_2$ and $Q_3$ one can read off the physical properties of the Kerr metric since they locate the horizon and ergosurface in an algebraic manner. With this information, there are two approaches to computing the angular momentum and mass of the black hole, one global and the other local. To determine the mass and angular momentum in the global approach, the area of the horizon and of the ergosurface must be calculated, requiring that these two surfaces must be located. The local method, which makes use of (1.2) alone, does not require knowledge of the location of the black hole or its event horizon. By knowing the forms of the invariants $I_1, ..., I_7$ it is possible to express the mass and angular momentum as functions in terms of these invariants.

The relationship between (1.3) and (1.2) has been expanded in Page and Shoom (2015) and a general approach was introduced to determine the location of the event horizon and ergosurface for the Kerr metric. More generally, this method will give the exact location of the horizon for a stationary black hole, although it is believed that it will be able to determine the approximate location for any nearly stationary horizon. This technique relies on the fact that the squared norm of the wedge product of $n$ gradients of functionally independent local smooth curvature invariants will always vanish on the horizon of any stationary black hole, where $n$ is the local *cohomogeneity* of the metric, which is defined as the codimension of the maximal dimensional orbits of the isometry group of the local metric. Their results can be summarized by the following theorem (Page and Shoom 2015):

**Theorem 1.** *For a spacetime of local cohomogeneity $n$ that contains a stationary horizon (a null hypersurface that is orthogonal to a Killing vector field that is null there and hence lies within the hypersurface and is its null generator) and which has $n$ independent scalar polynomial curvature invariants $S^{(i)}$ whose gradients are well-defined there, the $n$-form wedge product*

$$W = dS^{(1)} \wedge ... \wedge dS^{(n)}$$

*has zero squared norm on the horizon,*

$$||W||^2 = \frac{1}{n!}\delta^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_n} g^{\beta_1\gamma_1}...g^{\beta_n\gamma_n} \times S^{(1)}_{;\alpha_1}...S^{(n)}_{;\alpha_n} S^{(1)}_{;\gamma_1}...S^{(n)}_{;\gamma_n} = 0,$$

*where the permutation tensor $\delta^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_n}$ is $+1$ or $-1$ if $\alpha_1, ..., \alpha_n$ is an even or odd permutation of $\beta_1, ..., \beta_n$ respectively, and is zero otherwise.*

## 2 The Cartan-Karlhede method for determining local equivalence of spacetimes

The main idea of this method is to reduce the frame bundle to the smallest possible dimension at each step by casting the curvature and its covariant derivatives into a canonical form and only permitting those frame changes which preserve that canonical form. In 4D, we will work with null tetrads, $\{l^a, n^a, m^a \bar{m}^a\}$ such that $l^a l_a = n^a n_a = m^a m_a = \bar{m}^a \bar{m}_a = 0$ and $-l_a n^a = 1 = m^a \bar{m}_a$ and where a bar denotes a complex conjugate. In terms of this complex null tetrad the metric is

$$ds^2 = -2l_{(a}n_{b)} + 2m_{(a}\bar{m}_{b)}, \tag{2.1}$$

where round parentheses denote symmetrization.

The Cartan-Karlhede algorithm in any dimension may be summarized as (MacCallum 1986):

1. Set the order of differentiation $q$ to 0.
2. Calculate the derivatives of the Riemann tensor up to the $q$th order.
3. Find the canonical form of the Riemann tensor and its covariant derivatives.
4. Fix the frame as much as possible using this canonical form, and note the residual frame freedom (the group of allowed transformations is the *linear isotropy group* $H_q$). The dimension of $H_q$ is the dimension of the remaining *vertical* freedom of the frame bundle.
5. Find the number $t_q$ of independent functions of spacetime position in the components of the Riemann tensor and its covariant derivatives, in the canonical form. This tells us the remaining *horizontal* freedom.
6. If the isotropy group and number of independent functions are the same as in the previous step, let $p + 1 = q$, and the algorithm terminates; if they differ (or if $q = 0$), increase $q$ by 1 and go to step 2.

The nonzero components of $R_{abcd}$ and its covariant derivatives are referred to as *Cartan invariants*: a statement of the minimal set required, taking Bianchi and Ricci identities into account, was given by MacCallum and Åman (1986). We will refer to the invariants constructed from, or equal to, Cartan invariants of any order as *extended invariants.* Thus for sufficiently smooth metrics, a result of the test of equivalence gives sets of scalars providing a unique local geometric characterization, as the $D$-dimensional spacetime is then characterized by the canonical form used, the two discrete sequences arising from the successive isotropy groups and the independent function counts, and the values of the (nonzero) Cartan invariants. As there are $t_p$ essential spacetime coordinates, the remaining $D - t_p$ are ignorable, and so the dimension of the isotropy group of the spacetime will be $s = \dim(H_p)$, and the isometry group has dimension $r = s + D - t_p$.

Theorem 1 can be readily generalized to the set of Cartan invariants arising from the equivalence algorithm:

**Theorem 2.** *For a spacetime of local cohomogeneity $n$ that contains a stationary horizon and which has $n$ independent Cartan invariants $C^{(i)}$ whose gradients are well-defined there, the $n$-form wedge product*

$$W = dC^{(1)} \wedge ... \wedge dC^{(n)}$$

*has zero squared norm on the horizon,*

$$||W||^2 = \frac{1}{n!}\delta^{\alpha_1,...,\alpha_n}_{\beta_1,...,\beta_n} g^{\beta_1\gamma_1}...g^{\beta_n\gamma_n} \times C^{(1)}_{;\alpha_1}...C^{(n)}_{;\alpha_n} C^{(1)}_{;\gamma_1}...C^{(n)}_{;\gamma_n} = 0.$$

*Proof.* The number of functionally independent invariants at the end of the algorithm, $t_p$, is directly related to the dimension of the local cohomogeneity. To see this, we note that the dimension of the isometry group is given by $r = D - t_p + dim(H_p)$ where $H_p$ is the dimension of the isotropy group of the curvature tensor and all its covariant derivatives. However, the maximal dimensional orbits of the isometry group will be given by $r - dim(H_p)$ $= D - t_p$, since this is the quotient of the Lie group of Killing vectors by the isotropy group, and therefore $n =$ $D - r + dim(H_p) = t_p$. Using $n$ functionally independent Cartan invariants, the proof carries forward in a similar manner to the proof of theorem 1 in Page and Shoom (2015). □

Alternatively, we can use the first order Cartan invariants (those arising from the covariant derivative of the Riemann tensor) to produce new invariants that detect the stationary horizons. These invariants will be much simpler than the corresponding scalar polynomial invariants.

### 2.1 The Cartan-Karlhede algorithm in Five Dimensions

We would like to apply the Cartan-Karlhede algorithm to determine a set of Cartan invariants which detect the stationary horizon for 5D black hole metrics. This can be achieved in arbitrary dimension by examining the $q$th covariant derivative of the Weyl and Ricci tensor at iteration, $q$, and using frame transformations to transform the $q$th covariant derivative of the Weyl tensor and Ricci tensor into some canonical form.

In 5D, relative to the frame[1] with $n_a l^a = 1$, $n_a n^a = l_a l^a = 0$ and $m^{(i)}_a m^{(j)}_b = \delta^{ij}$ in terms of which the metric can be written as $g_{ab} = 2l_{(a}n_{b)} + \delta^j_i m^{(i)}_a m^{(j)}_b$, any element of the group of orthochronous local Lorentz transformations is generated by combining the following frame transformations (Coley et al. 2004; Milson et al. 2005):

[1] Because only the first two vectors are null we call this “half-null”.

1. Null rotations about $l$:

$$\hat{l} = l, \quad \hat{n} = n + z_i m^i - \frac{1}{2} z_i z^i l, \quad \hat{m}_i = m_i - z_i l \tag{2.2a}$$

2. Null rotations about $n$:

$$\hat{l} = l + y_i m^i - \frac{1}{2} y_i y^i n, \quad \hat{n} = n, \quad \hat{m}_i = m_i - y_i n \tag{2.2b}$$

3. Spins:

$$\hat{l} = l, \quad \hat{n} = n, \quad \hat{m}_i = X_i^j m_j \tag{2.2c}$$

4. Boost:

$$\hat{l} = \lambda l, \quad \hat{n} = \lambda^{-1} n, \quad \hat{m}_i = m_i \,, \tag{2.2d}$$

where $X_i^j$ denotes the usual rotation matrices for rotations about the axes $m_2$, $m_3$, $m_4$ respectively. We stress that the quantities $z_i = z_i(x^a)$, $\theta = \theta(x^a)$ and $\lambda = \lambda(x^a)$ depend on the coordinates. We also note that the Lorentz transformations in 5D have 10 parameters.

For dimension $D > 4$, we no longer have the usual spinor approach to simplify calculations, and the 4D algebraic classifications of the Weyl and Ricci tensors are no longer applicable. Instead, we consider the boost weight decomposition (Coley and Hervik 2010; Ortaggio et al. 2011; Coley et al. 2012) to classify the curvature tensor. Relative to the basis $\{\theta^a\} = \{n, \ell, m^i\}$, the components of an arbitrary tensor of rank $p$ transform under the boost (2.2d) by:

$$T'_{a_1 a_2 \dots a_p} = \lambda^{b_{a_1 a_2 \dots a_p}} T_{a_1 a_2 \dots a_p}, \quad b_{a_1 a_2 \dots a_p} = \sum_{i=1}^{p} (\delta_{a_i 0} - \delta_{a_i 1}) \tag{2.3}$$

where $\delta_{ab}$ denotes the Kronecker delta symbol. This quantity is called the *boost weight* (b.w) of the frame component $T_{a_1 a_2 \dots a_p}$. This approach, called the *alignment classification*, relies on the fact that the frame basis written as a null basis transforms in a simple manner under a boost given by (2.2d) and that this identifies null directions relative to which the Weyl tensor has components of a particular b.w. configuration, called Weyl aligned null directions (WANDs). Typically, we must use null rotations to identify the WANDs for a given tensor.

We define the *boost order* of $T_{a_1 a_2 \dots a_p}$ as the maximal b.w. of its non-vanishing components relative to the frame. As this integer is invariant under the group of Lorentz transformations that fix the null direction $[\ell]$, it is a function of $[\ell]$ only, and will be denoted by $b_T([\ell])$. We introduce another integer, $B_T = \max_\ell b_T([\ell])$, which is entirely dependent on the form of the tensor. For a generic $\ell$ the Weyl and Ricci tensors have boost order $b_R([\ell]) = b_C([\ell]) = 2$, and so $B_R = B_C = 2$. If a null direction $[\ell]$ exists for which $b_T([\ell]) \leq B_T - 1$, it is said to be a $T$ *aligned null direction* of *alignment order*: $B_T - 1 - b_T([\ell])$. As an example, for a WAND, the alignment order can be $0, 1, 2, 3$. The alignment order can be related to another integer invariant,

$$\zeta \equiv \min_\ell b_C([\ell]),$$

which is a pointwise invariant of the spacetime defining the (Weyl) primary or principal alignment type $2 - \zeta$ at p. If $\zeta = 2, 1, 0, 1$ or $-2$ this type is denoted by **G**, **I**, **II**, **III** or **N** respectively. If there is more than one WAND in the type **II** case, then this is denoted as **D**. This classification can also be applied to the Ricci tensor since $B_C = B_R = 2$.

This classification reproduces the Petrov and Segre classifications in 4D, and also leads to a coarse classification in higher dimensions. In 5D this classification can be made finer by considering the spin group which is isomorphic to $O(3)$ and acts on the null frame according to (2.2). The details of this approach are expanded upon in Coley et al. (2012). There is a fundamental difficulty with applying the alignment classification, as it may depend on solving degree five polynomials to determine the WANDs. The solutions to these polynomials may not be expressible in terms of algebraic functions, and instead require transcendental functions, which are often too complex to implement in practice. Thus, unlike the case in 4D, the ability to explicitly determine the WANDs is not guaranteed. Assuming a theory of approximate equivalence could be developed, numerical root solving could be implemented to resolve this issue.

## 3 Applications in 4D

In this section we use the Cartan-Karlhede equivalence method for the classification of 4D solutions of the Einstein equations describing black holes to compute Cartan invariants that are capable of identifying the horizons. Since a Killing horizon is a special case of a weakly isolated horizon, in 4D these Cartan invariants correspond to the set of components of the covariant derivatives of the Weyl and Ricci tensors with positive b.w. which vanish on this surface (Coley et al. 2017; Coley and McNutt 2017a). To show this, we will apply the algorithm to the Kerr-Newman-NUT-(Anti)-de Sitter solution which contains the various Kerr solutions and Reissner-Nordström-(Anti)-de Sitter as special cases. We will then relate these Cartan invariants to the SPIs using the Newman-Penrose (NP) formalism.

## 3.1 Kerr-Newman-NUT-(Anti)-de Sitter metric

The 4D Kerr-Newman-NUT-(Anti)-de Sitter metric is given by (Plebański and Demiański 1976; Griffiths and Podolský 2006, 2007)[2]:

$$ds^2 = -\frac{Q}{R^2}\left[dt - \left(a\sin^2\theta + 4l\sin^2\frac{\theta}{2}\right)d\phi\right]^2 + \frac{R^2}{Q}dr^2 + \frac{P}{R^2}\left[adt - \left(r^2 + (a+l)^2\right)d\phi\right]^2 + \frac{R^2}{P}\sin^2\theta d\theta^2 \tag{3.1}$$

where $R \equiv R(r,\theta)$, $P \equiv P(\theta)$ and $Q \equiv Q(r)$ are functions of $\cos\theta$ and $r$, containing the parameters $m, e, g, a, l$, and $\Lambda$ which are, respectively, mass, the electric and magnetic charges, a rotation parameter, a NUT parameter in a de Sitter or anti-de Sitter background, and the cosmological constant:

$$R^2 = r^2 + (l + a\cos\theta)^2 \tag{3.2}$$
$$P = \sin^2\theta(1 + (3l + a\cos\theta)(l + a\cos\theta)\Lambda/3), \tag{3.3}$$
$$Q = (a^2 - l^2 + e^2 + g^2) - 2mr + r^2 - \Lambda[(3l^2 + a^2)r^2 + r^4]/3. \tag{3.4}$$

In the case that $\Lambda$ or $a$ vanishes, the Rainich conditions are satisfied and it is possible to apply a duality rotation to set the magnetic charge $g$ to zero. The locations of the event horizon for this solution are given by the roots of $Q(r)$. The expressions for the $Q_i$ invariants of (1.3) require computation of very large polynomials in $\cos\theta$ and $r$: we show below that they can be more simply expressed using Cartan invariants. Cartan invariants consequently directly allow the construction of simpler candidates for detection of the horizon.

Defining the vectors:

$$t^0 = \frac{\sqrt{Q}}{R}\left[dt - \left(a\sin^2\theta + 4l\sin^2\tfrac{\theta}{2}\right)d\phi\right], \quad t^1 = \frac{R}{\sqrt{Q}}dr, \quad t^2 = \frac{\sqrt{P}}{R}\left[adt - \left(r^2 + (a+l)^2\right)d\phi\right], \quad t^3 = \frac{R}{\sqrt{P}}\sin\theta d\theta,$$

the null frame we will work with is given by

$$\ell = \frac{t^0 - t^1}{\sqrt{2}}, \quad n = \frac{t^0 + t^1}{\sqrt{2}}, \quad m = \frac{t^2 - it^3}{\sqrt{2}}, \quad \bar{m} = \frac{t^2 + it^3}{\sqrt{2}}. \tag{3.5}$$

Note that although at the horizon $Q = 0$ there is a coordinate singularity in (3.1) and our choice of frame, smoothness implies invariants will still be correct at the horizon.

The only nonzero NP curvature scalars are $\Lambda_{NP} = \frac{1}{6}\Lambda$,

$$\Psi_2 = -\left(m + il\left(1 + \frac{1}{3}(a^2 - l^2)\Lambda\right)\right)\left(\frac{1}{r + i(l + a\cos\theta)}\right)^3 + (e^2 + g^2)\left(\frac{1}{R^2(r + i(l + a\cos\theta))^2}\right), \tag{3.6}$$

and

$$\Phi_{11} = \frac{1}{2}\frac{e^2 + g^2}{R^4}. \tag{3.7}$$

This implies that the Weyl and Ricci tensors are both of type **D**; i.e., the Weyl tensor is of Petrov type **D** and the Ricci tensor is of Segre type $\{(11)(1,1)\}$. It is worth noting that the presence of $\Lambda$ modifies the NUT term in $\Psi_2$, which is just $(m + il)/(r + i(l + a\cos\theta))^3$ when $\Lambda = 0$.

At zeroth order of the Cartan-Karlhede algorithm, we obtain as Cartan invariants the real and imaginary parts of $\Psi_2$, which are functionally independent. The other nonzero Cartan invariants at this order are $\Lambda$ and $\Phi_{11'}$, both real, neither of which gives a further functionally independent quantity, and so $t_0 = 2$. The zeroth order isotropy group consists of boosts and spins, and so dim $(H_0) = 2$. At the first iteration of the algorithm, using the notation introduced in MacCallum and Åman (1986), we find that the nonzero components of the symmetrized first covariant derivative of the Weyl tensor in any Petrov type D metric are:

$$\nabla\Psi_{20'} = 3(D\Psi_2 + 2\rho\Psi_2)/5, \ \nabla\Psi_{21'} = 3(\delta\Psi_2 + 2\tau\Psi_2)/5, \ \nabla\Psi_{30'} = 3(\bar{\delta}\Psi_2 - 2\pi\Psi_2)/5, \ \nabla\Psi_{31'} = 3(\Delta\Psi_2 - 2\mu\Psi_2)/5. \tag{3.8}$$

The boosts and spins are no longer in the isotropy group. We have dim $(H_1) = 0$. The canonical choice for such a Petrov type D metric is a boost and spin such that $\nabla\Psi_{31'} = -\nabla\Psi_{20'}$. For the metric (3.1), the tetrad of (3.5) is this canonical choice, and in that frame we have

$$\rho = \mu, \quad \pi = \tau.$$

Since partial and covariant derivatives agree for scalars and the frame is fixed, $D\Psi_2$, $\Delta\Psi_2$, $\delta\Psi_2$, and $\bar{\delta}\Psi_2$ are Cartan invariants (though not members of the minimal set defined by MacCallum and Åman (1986)) and hence $\rho$, $\tau$, $\pi$ and $\mu$ in this frame are extended Cartan invariants. No new functionally independent invariants appear at first order, and so $t_1 = 2$. It is known already for the Kerr-NUT case (Åman 1984) that the Cartan-Karlhede algorithm concludes at the second iteration, since no new functionally independent invariants appear; $t_1 = t_2 = 2$ and dim $(H_1) =$ dim $(H_2) = 0$. This remains true with nonzero $e$, $g$ and $\Lambda$.

[2] Here the constant $a_0$ of Griffiths and Podolský (2007) has been set to $1 + l^2\Lambda$, rather than unity.

For Petrov type D metrics in which $\Phi_{11}$ is the only nonzero matter term, the Bianchi identities give

$$\begin{gathered} D\Psi_2 = 3\rho\Psi_2 + 2\rho\Phi_{11}, \quad \Delta\Psi_2 = -3\mu\Psi_2 - 2\mu\Phi_{11}, \\ \delta\Psi_2 = 3\tau\Psi_2 - 2\tau\Phi_{11}, \quad \bar{\delta}\Psi_2 = -3\pi\Psi_2 + 2\pi\Phi_{11}, \\ D\Phi_{11} = 2(\rho + \bar{\rho})\Phi_{11}, \quad \delta\Phi_{11} = 2(\tau - \bar{\pi})\Phi_{11}, \quad \Delta\Phi_{11} = 2(\mu + \bar{\mu})\Phi_{11}. \end{gathered} \tag{3.9}$$

Like those of $\Psi_2$, the frame derivatives of $\Phi_{11}$ in the canonical frame are Cartan invariants. The corresponding elements of the minimal set are

$$\begin{aligned} \nabla\Phi_{11'} &= 4(D\Phi_{11} + (\rho + \bar{\rho})\Phi_{11})/9, \\ \nabla\Phi_{12'} &= 4(\delta\Phi_{11} + (\tau - \bar{\pi})\Phi_{11})/9, \\ \nabla\Phi_{22'} &= 4(\Delta\Phi_{11} - (\mu + \bar{\mu})\Phi_{11})/9. \end{aligned}$$

We would like to have an extended Cartan invariant that detects the event horizon. Looking at $\nabla\Psi_{20'}$ we find that

$$\rho = \mu = -\frac{1}{\sqrt{2}}\frac{\sqrt{Q}[r - i(a\cos\theta + l)]}{R^3}. \tag{3.10}$$

This extended Cartan invariant has a well-understood geometric interpretation: since the horizon is a marginally trapped surface, the outgoing null vector $\ell$ must be surface-forming and non-expanding implying that $\rho = 0$ there (MacCallum 2006). Computing the roots of $Q(r)$ for arbitrary $a, l, m, e, g$ and $\Lambda$ is not a pleasant task. However, for this extended Cartan invariant we do not need to compute them, as it is clear that the zeros of $\rho$ are exactly the zeros of $Q(r)$.

The ergosurface can be detected by combining $\rho$ with another Cartan invariant,

$$\tau = \pi = \frac{1}{\sqrt{2}}\frac{a\sqrt{P}[r - i(a\cos\theta + l)]}{R^3}, \tag{3.11}$$

to produce the following extended Cartan invariant:

$$\rho^2 - \tau^2 = \frac{(Q - a^2P)[r - i(a\cos\theta + l)]^2}{2R^6}. \tag{3.12}$$

Comparing with the component $g_{tt}$ in the metric (3.1), the invariant $\rho^2 - \tau^2$ vanishes when $g_{tt} = 0$. It is important to stress that this approach requires a particular invariantly defined choice of coframe, and that $\rho$ and $\tau$ can be regarded as invariants only in the form they take relative to the canonical frame.

It is possible to implement Theorem 2 to generate an extended Cartan invariant that detects the event horizon and make the choice of frame irrelevant. Working with $\Psi_2$ and its complex conjugate, we find the following invariant which detects the horizons:

$$||d\Psi_2 \wedge d\bar{\Psi}_2||^2 = \frac{a^2PQ(9F - G^4)^2}{2R^{24}}, \tag{3.13}$$

where

$$G = e^2 + g^2, \quad F = |m + iL|^2R^2 + 2G^2(mr + (L(l + a\cos\theta)) + G^4 \text{ and } L = l\left(1 + \frac{\Lambda}{3}(a^2 - l^2)\right). \tag{3.14}$$

Due to the relationship between the Cartan invariant $\Psi_2$ and the pair of SPIs $I_1$ and $I_2$ (Abdelqader and Lake 2015), $||d\Psi_2 \wedge d\bar{\Psi}_2||^2$ is also a SPI. In fact, we will show that it is equivalent to $Q_2$. This will vanish on the axis in Kerr, and it may vanish at some other points in the Kerr-Newman-NUT-(Anti)-de Sitter solution.

## 3.2 Scalar Polynomial Invariants in Terms of Cartan Invariants in 4D

We first note that (Abdelqader and Lake 2015)

$$I_1 + iI_2 = 48{\Psi_2}^2.$$

Since $\Psi_2$ is therefore expressible in terms of SPIs, its gradient $\nabla\Psi_2$ (which for scalars is the same as a covariant derivative) can also be used to form SPIs. From the definitions (1.2), it is immediately obvious that $I_5$, $I_6$ and $I_7$ can be expressed using $\Psi_2$ and $\nabla\Psi_2$ and their complex conjugates, and the same follows for $I_3$ and $I_4$ using Page and Shoom's equation (9). Writing $\nabla A.\nabla B$ for $A_{,\mu}B^{,\mu}$, we find that

$$(96\Psi_2)^2(\nabla\Psi_2.\nabla\Psi_2) = 12 \cdot 48(\Psi_2)^2(I_3 + iI_4)/5, \tag{3.15}$$

so $I_3$ and $I_4$ are the real and imaginary parts of $160(\nabla\Psi_2.\nabla\Psi_2)$. We also find that

$$I_5 = (96)^2[(\Psi_2)^2(\nabla\Psi_2.\nabla\Psi_2) + cc + 2\Psi_2\bar{\Psi}_2(\nabla\Psi_2.\nabla\bar{\Psi}_2)]/4, \tag{3.16}$$

$$I_6 = (96)^2[-(\Psi_2)^2(\nabla\Psi_2.\nabla\Psi_2) - cc + 2\Psi_2\bar{\Psi}_2(\nabla\Psi_2.\nabla\bar{\Psi}_2)]/4, \tag{3.17}$$

$$I_7 = (96)^2[(\Psi_2)^2(\nabla\Psi_2.\nabla\Psi_2) - cc]/4, \tag{3.18}$$

where cc means the complex conjugate of the preceding expression. For Kerr-NUT-(Anti)-de Sitter we have, using (3.9) and evaluating in the canonical frame,

$$\nabla\Psi_2.\nabla\Psi_2 = 18{\Psi_2}^2(\rho^2 - \tau^2), \tag{3.19}$$

$$\nabla\Psi_2.\nabla\bar{\Psi}_2 = 18\Psi_2\bar{\Psi}_2(|\rho|^2 + |\tau|^2). \tag{3.20}$$

We can now easily compute the $Q_i$ which are

$$Q_1 = \frac{2\mathcal{R}[(\bar{\Psi}_2^2(\nabla\Psi_2.\nabla\Psi_2)]}{9(\Psi_2\bar{\Psi}_2)^{5/2}}, \tag{3.21}$$

$$Q_2 = \frac{-2||\nabla\bar{\Psi}_2 \wedge \nabla\Psi_2||^2}{18^2(\Psi_2\bar{\Psi}_2)^3}, \tag{3.22}$$

$$Q_3 = \frac{\nabla\Psi_2.\nabla\bar{\Psi}_2}{18(\Psi_2\bar{\Psi}_2)^{3/2}}, \tag{3.23}$$

where $\mathcal{R}$ denotes the real part.

While the original formula (1.3) for $Q_2$ is more complicated to compute, it is in fact a dimensionless version of our proposed invariant (3.13). We can easily compute the $Q_i$ for the Kerr case in the canonical tetrad, using (3.19) and (3.20), to obtain

$$Q_1 = \frac{(\rho^2 - \tau^2) + cc}{(\Psi_2\bar{\Psi}_2)^{1/2}}$$
$$= (r^2 - 2mr + a^2\cos^2\theta)(r^2 - a^2\cos^2\theta)/mR^3 \tag{3.24}$$

$$Q_2 = \frac{QPa^2}{R^8|\Psi_2|^2} = \frac{QPa^2}{m^2R^2} \tag{3.25}$$

$$Q_3 = \frac{(Q + a^2\sin^2\theta)}{2mR} \tag{3.26}$$

agreeing with Abdelqader and Lake's formulae (10), (11) and (A8).

Note that (3.25), using (3.3), includes the effects of the NUT parameter and $\Lambda$, and so it vanishes at the horizon, on the axis and at points where $1 + (3l + a\cos\theta)(l + a\cos\theta)\Lambda/3 = 0$. (However, from (3.10), $\rho$ vanishes only on the horizon and at the origin even when there is a Maxwell field.) Since the numerator of $Q_1$ equals (3.12), $Q_1$ will detect the ergosurface for the Kerr solution. In fact, $Q_1$ will detect the ergosurface for the Kerr-NUT-(Anti)-de Sitter solution, but not for the Kerr-Newman-NUT-(Anti)-de Sitter solution (unlike the extended Cartan invariant $\rho^2 - \tau^2$). For this general case, the coordinate expressions for $Q_1$ $Q_2$, and $Q_3$ can be written as polynomials in terms of $G = e^2 + g^2$ with coefficients expressed in terms of $\rho, \tau$, $R$ and $\Psi_2$ and their complex conjugates. The invariant $Q_2$ will still detect the horizon in the Kerr-Newman-NUT-(Anti)-de Sitter solution as it takes the form:

$$Q_2 = \frac{a^2PQ(9F - G^4)^2}{81F^3}, \tag{3.27}$$

where $G$ and $F$ are defined in (3.14).

## 4 Examples in 5D

In this section we will apply the Cartan-Karlhede algorithm to 5D analogues of the black hole metrics studied in the previous section. In particular, we will show how the Cartan invariants provide a simpler approach for locating the horizons than the corresponding SPIs.

### 4.1 5D Reissner-Nordström-(Anti)-de Sitter

From Konoplya and Zhidenko (2008), the metric is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2dS_3, \qquad f(r) = 1 - \frac{2M}{r^2} - \frac{\Lambda r^2}{6} + \frac{Q^2}{r^4}, \tag{4.1}$$

where $dS_3$ is the line element for the unit 3-sphere. We use the following orthonormal frame:

$$e_0 = \sqrt{f(r)}dt, \quad e_1 = \sqrt{\frac{dr}{f(r)}}, \quad e_2 = rd\theta, \quad e_3 = r\sin(\theta)d\phi, \quad e_4 = r\sin(\theta)\sin(\phi)d\omega; \tag{4.2}$$

from which we build the frame:

$$l = \frac{1}{\sqrt{2}}(e_1 - e_0), \quad n = \frac{1}{\sqrt{2}}(e_0 + e_1), \quad m_2 = e_2, \quad m_3 = e_3, \quad m_4 = e_4. \tag{4.3}$$

In this frame, $l$ and $n$ are WANDs; to see this we compute the components of the Weyl and Ricci tensor[3]:

$$R_{01} = \frac{2(\Lambda r^6 - 6Q^2)}{3r^6}, \quad R_{ii} = \frac{2(\Lambda r^6 + 3Q^2)}{3r^6}, \quad i \in [2,4]\ , \tag{4.4}$$

$$C_{0101} = \tfrac{1}{3}C_{0i1i} = \tfrac{3}{2}\frac{4Mr^2 - 5Q^2}{r^6}, \tag{4.5}$$

with the remaining nonzero components $C_{ijij},\ i,j \in [2,4], i \neq j$ algebraically dependent on $C_{0101}$. That is, relative to this frame, the only nonzero components are the b.w. zero terms.

At zeroth order, it can be shown that the isotropy group of the Weyl and Ricci tensor consists of boosts and any spatial rotation; hence dim $(H_0) = 4$. The number of functionally independent invariants is $t_0 = 1$. Continuing the Cartan-Karlhede algorithm, we compute the covariant derivative of the Weyl and Ricci tensors:

$$R_{01;1} = -2R_{1i;i} = -\frac{1}{2}R_{jj;1} = \frac{8Q^2}{8Mr^2 - 15Q^2}, \quad R_{01;0} = -4R_{0i;i} = -\frac{1}{2}R_{jj;0} = -\frac{36(8Mr^2 - 15Q^2)f(r)Q^2}{r^{14}} \tag{4.6}$$

$$C_{0101;1} = -3C_{0i1i;1} = 3C_{ijij;1} = 1, \tag{4.7}$$

$$C_{0101;0} = 3C_{0i1i;0} = -3C_{ijij;0} = -\frac{18}{4}\frac{(8Mr^2 - 15Q^2)^2 f(r)}{r^{14}}, \tag{4.8}$$

$$-C_{011i;i} = -2C_{1iij;j} = -\frac{2}{3}\frac{4Mr^2 - 5Q^2}{8Mr^2 - 15Q^2} \tag{4.9}$$

$$-C_{010i;i} = 2C_{0iij;j} = \frac{3(8Mr^2 - 15Q^2)f(r)(4Mr^2 - 5Q^2)}{r^{14}} \tag{4.10}$$

Here we have fixed the boosts by setting the component $C_{0101;1} = 1$. Through direct inspection, it is clear that spatial rotation but not translation of the 3-sphere has no effect on the first order Cartan invariants; hence dim $(H_1) = 3$. The number of functionally independent invariants remains $t_1 = 1$. The Cartan-Karlhede algorithm continues for one more iteration since $t_1 = t_2 = 1$ and dim $(H_1) =$ dim $(H_2) = 3$.

Notice that all positive b.w. terms of the covariant derivative of the Weyl and Ricci tensors detect the horizon at first order. In fact, for all higher order derivatives of the Weyl and Ricci tensors, the positive b.w. terms will vanish on the horizon, suggesting that the geometric horizon conjecture for weakly isolated horizons is valid in higher dimensions (Coley et al. 2017; Coley and McNutt 2017a). Since the cohomogeneity is $n = 1$, Theorem 1 ensures that we may produce a SPI that detects the horizon using $I_1$:

$$||\mathrm{d}I_1||^2 = \frac{2^6 3^4 (4Mr^2 - 5Q^2)^2 (8Mr^2 - 15Q^2)^2 f(r)}{r^{26}}. \tag{4.11}$$

In contrast, applying Theorem 2 and taking the norm of the exterior derivative of a non-constant zeroth order Cartan invariant will give an extended Cartan invariant of lower order that will also detect the horizon.

### 4.2 5D Kerr-NUT-(Anti)-de Sitter

For the 5D Kerr-NUT-(Anti)-de Sitter solution, we will use the metric relative to the coordinate system given by equations (22)-(23) in Chen et al. (2006):

$$\mathrm{d}s^2 = \frac{\mathrm{d}x_1^2}{Q_1} + \frac{\mathrm{d}x_2^2}{Q_2} + Q_1\left(\mathrm{d}\psi_0 + x_2^2\mathrm{d}\psi_1\right)^2 + Q_2\left(\mathrm{d}\psi_0 + x_1^2\mathrm{d}\psi_1\right)^2 - \frac{c_0}{x_1^2x_2^2}\left(\mathrm{d}\psi_0 + \left(x_1^2 + x_2^2\right)\mathrm{d}\psi_1 + x_1^2x_2^2\mathrm{d}\psi_2\right)^2 \tag{4.12}$$

where

$$Q_1 = \frac{X_1}{U}, Q_2 = -\frac{X_2}{U}, U = x_2^2 - x_1^2, \quad X_1 = c_1x_1^2 + c_2x_1^4 + \frac{c_0}{x_1^2} - 2b_1, \text{ and } X_2 = c_1x_2^2 + c_2x_2^4 + \frac{c_0}{x_2^2} - 2b_2. \tag{4.13}$$

The constants $c_0, c_1, c_2, b_1, b_2$ are free parameters, which are related to the rotation parameters $a_1$, $a_2$, the mass and NUT charge $M_1, M_2$, and a cosmological constant $\Lambda$ as follows:

$$c_0 = a_1^2a_2^2, \quad c_1 = 1 - \frac{\Lambda}{4}(a_1^2 + a_2^2), \quad c_2 = \frac{\Lambda}{4}, \quad b_\mu = \frac{1}{2}(a_1^2 + a_2^2 - a_1^2a_2^2\frac{\Lambda}{4}) - M_\mu, \quad \mu = 1, 2. \tag{4.14}$$

[3] To display components here and in the next subsection we will use repeated indices: this will not indicate summation, unless indicated by a repeated index being raised.

This metric has been Wick rotated and so it no longer has a Lorentzian signature. This will lead to complex null vectors relative to this coordinate system. However, relative to the original coordinates in Chen et al. (2006) they will be real.

We first define an orthonormal frame:

$$e_0 = \frac{\mathrm{d}x_1}{\sqrt{Q_1}}, \quad e_1 = \frac{\mathrm{d}x_2}{\sqrt{Q_2}}, \tag{4.15}$$

$$e_2 = \sqrt{Q_1}\left(\mathrm{d}\psi_0 + x_2^2\mathrm{d}\psi_1\right), \quad e_3 = \sqrt{Q_2}\left(\mathrm{d}\psi_0 + x_1^2\mathrm{d}\psi_1\right), \quad e_4 = \frac{\sqrt{-c_0}}{x_1x_2}\left(\mathrm{d}\psi_0 + \left(x_1^2+x_2^2\right)\mathrm{d}\psi_1 + x_1^2x_2^2\mathrm{d}\psi_2\right). \tag{4.16}$$

Then, according to Hamamotoa et al. (2007) and Pravda et al. (2007), the WANDs are simply the null vectors $n$ and $\ell$ in the following half-null frame:

$$l = \frac{i}{\sqrt{2Q_2}}(e_1 + ie_3), \quad n = -i\sqrt{\frac{Q_2}{2}}(e_1 - ie_3), \quad m_2 = e_0, \quad m_3 = e_2, \quad m_4 = e_4. \tag{4.17}$$

Using the WANDs in this frame, it may be shown that any of the components are functionally dependent on the choice of two independent components at zeroth order. Thus $t_0 = 2$. All components are of b.w. zero and they do not change under a rotation about $m_4$. To see why, we express the Weyl tensor components as the following matrices as defined by Table 1 in (Coley et al. 2012):

$$C_{0101} = -\frac{2(x_1^2+3x_2^2)(b_1-b_2)}{U^3} \tag{4.18}$$

$$M_{ij} = C_{0i1j} = \begin{pmatrix} -\frac{2(x_1^2+x_2^2)(b_1-b_2)}{U^3} & \frac{4ix_1x_2(b_1-b_2)}{U^3} & 0 \\ -\frac{4ix_1x_2(b_1-b_2)}{U^3} & -\frac{2(x_1^2+x_2^2)(b_1-b_2)}{U^3} & 0 \\ 0 & 0 & -\frac{-2(b_1-b_2)}{U^2} \end{pmatrix} \tag{4.19}$$

$$A_{ij} = C_{01ij} = \begin{pmatrix} 0 & \frac{8ix_1x_2(b_1-b_2)}{U^3} & 0 \\ -\frac{8ix_1x_2(b_1-b_2)}{U^3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.20}$$

We note that this is a vacuum solution and so $R_{ij} = \Lambda g_{ij}$. Since $A_{ij} = \epsilon_{ijk}w^k$, rotations about $m_4$ do not change $A_{ij}$. And from the form of $M_{ij}$, it follows that $M_{ij}$ is unaffected by spatial rotations about $m_4$. Thus $\dim(H_0) = 2$.

At first iteration, we have several non-trivial components, but they are all functionally dependent on the two functionally independent invariants at zeroth order, $t_1 = 2$. We can fix the remaining isotropy by applying a boost to set $C_{0101;0} = -C_{0101;1}$: a rotation about $m_4$ is not needed as our frame already gives the canonical choice $C_{0101;3} = 0$. Therefore, $\dim(H_1) = 0$. The algorithm would carry on for one more iteration, since $t_1 = t_2 = 2$ and dim $(H_1)$ = dim $(H_2) = 0$; however, we will omit these details. Instead of listing components of the covariant derivative of the Weyl tensor, we note that the following components at first order detect the horizon, which occurs when the function $X_2 = 0$:

$$C_{0101;0} = -C_{0101;1} = \frac{12\sqrt{2}(x_1^2+x_2^2)x_2\sqrt{X_2}(b_1-b_2)}{U^{9/2}} \tag{4.21}$$

To determine the location of the event horizon, we may compute the expansion of the boosted $\ell$ (Pravda et al. 2007), $\theta_{(\ell)} = \frac{1}{3}h^{ab}\ell_{(a;b)}$, where $h_{ab} = g_{ab} - \ell_{(a}n_{b)}$:

$$\theta_{(\ell)} = -\frac{i}{3\sqrt{2}}\frac{(x_1^2-3x_2^2)\sqrt{Q_2}}{Ux_2}. \tag{4.22}$$

As in the previous example, all of the positive b.w. components of the covariant derivative of the Weyl tensor vanish on the horizon, and similarly for all higher order derivatives of the Weyl tensor. Applying Theorem 1, we may produce a SPI that detects the horizon using $I_1 = C_{abcd}C^{abdc}$ and $J_1 = C_{abcd}C^{abef}C^{cd}{}_{ef}$:

$$||\mathrm{d}I_1 \wedge \mathrm{d}J_1||^2 = \frac{2^{37}3^4(3x_1^4+2x_1^2x_2^2+3x_2^4)^2x_1^2x_2^2X_1X_2(b_1-b_2)^{10}}{(x_1-x_2)^{30}(x_1+x_2)^{30}}. \tag{4.23}$$

Alternatively, we could use Theorem 2 to produce an extended Cartan invariant from the non-constant zeroth order Cartan invariants that will detect the horizon and will be of lower order than the above SPI.

## 5 Conclusion

We have shown that it is possible to locate the event horizon of any stationary asymptotically flat (or (anti) de Sitter) black hole from the zeros of Cartan invariants. Our work complements the related results on the detection of stationary horizons using SPIs (Abdelqader and Lake 2015; Page and Shoom 2015). Our approach has a notable advantage in that it is computationally less expensive compared to the approach using the related SPIs. In the reviewed examples we have also computed extended Cartan invariants whose zeros only occur on the surface of the stationary horizons, and the related SPIs (Page and Shoom 2015) are computed for each solution as a comparison. In 4D, we employ the NP formalism relative to the frame arising from the Karlhede algorithm to demonstrate the relationship between the SPIs and the Cartan invariants.

While we have only considered stationary horizons with spherical topology, in higher dimensions other topologies are permitted for the horizon. For example, the 5D black rings have horizon topology $S^1 \times S^2$. For the rotating and supersymmetric black rings, it has been shown that the approach employing Cartan invariants will detect the horizon (Coley and McNutt 2017b). Furthermore, the results of Coley and McNutt (2017b) show that the Cartan-Karlhede algorithm can be implemented to produce Cartan invariants that detect the horizon even when WANDs are not known. This indicates that the Cartan-Karlhede algorithm can be implemented in dimensions $D \geq 5$ and that the resulting invariants will be easier to compute than the related SPIs.

In future work we will consider the horizons of solutions containing more than one black hole, including the analytical example of the Kastor-Traschen solution (Kastor and Traschen 1993). This dynamical extension may allow us to follow the formation of the event horizon during the merger of two black holes, during the phase of collapse of a star into a single black hole (Penrose 1969), and perhaps even the disappearance of the horizon during the evaporation of a single black hole (Hawking 1974). We will also extend our method to the study of evolving event horizons for time dependent metrics, including metrics currently used for cosmological modelling. We hope that these results will play an important role in numerical relativity in which configurations of many black holes are evolved in time (Baumgarte and Shapiro 2010), and a sharp localization of the event horizon is required.

## Acknowledgements

The work was supported by NSERC of Canada (A.A.C, A.F., D.B.) and AARMS (D.G.) and through the Research Council of Norway, Toppforsk grant no. 250367: Pseudo-Riemannian Geometry and Polynomial Curvature Invariants: Classification, Characterisation and Applications (D.M.). We are also grateful to the authors of the free software GRtensorII, Reduce and Sheep used in checking our calculations.

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