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Segmented strings coupled to a B-field

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Abstract: In this paper we study segmented strings in AdS_3 coupled to a background two-form whose field strength is proportional to the volume form. By changing the coupling, the theory interpolates between the Nambu-Goto string and the $SL(2,\mathbb{R})$ Wess-Zumino-Witten model. In terms of the kink momentum vectors, the action is independent of the coupling and the classical theory reduces to a single discrete-time Toda-type theory. The WZW model is a singular point in coupling space where the map into Toda variables degenerates.

Keywords: Integrable Field Theories, Long strings, AdS-CFT Correspondence, Bosonic Strings

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1 Introduction

Segmented strings in flat space are piecewise linear classical string solutions: at any given time the string embedding is a union of straight lines. Kinks between the segments move with the speed of light and their worldlines form a lattice on the worldsheet. The constraint on the kink velocity is necessary, otherwise the shape of the string would deform due to the non-zero string tension, and would not stay piecewise linear. Segmented strings generalize the string constructions in [1, 2] which play a role in the Lund model of hadronization. Furthermore, the kinks can be regarded as a toy model for gravitational shockwaves in the two-dimensional "gravity theory" on the string worldsheet [3]. Finally, we note that any smooth string can be approximated by segmented strings to arbitrary accuracy (although segmented strings involve no approximations).

The idea of segmented strings can be generalized to AdS_3 target space (or more generally to $\mathrm{(A)dS}_n$ and their orbifolds) where the embedding is built from AdS_2 patches [4, 5]. The construction provides an exact discretization of the non-linear string equations of motion. Since the string is discrete in both space and time, the time-evolution equations are reduced to purely algebraic operations on the initial data. Solving discrete equations has numerous advantages over approximate numerical solutions of partial differential equations. Most importantly, there are no numerical errors that would otherwise accumulate over a long period of time.

For recent developments, the reader is referred to [6–9].

In [10], the area of segmented strings has been computed using cross-ratios constructed from the kink momentum vectors. The cross-ratios were expressed in terms of purely left-handed (or right-handed) Toda variables. In this way, classical Nambu-Goto string theory in AdS_3 could be reduced to an integrable time-discretized relativistic Toda-type lattice.

In [7], Gubser pointed out that the segmented string evolution equations simplify if they are derived from the $SL(2,\mathbb{R})$ Wess-Zumino-Witten action. In this theory, strings couple to the NSNS three-form field strength which supports the AdS_3 geometry. (For the quantum theory see [11].) AdS_3 backgrounds are special since they can be supported by a combination of NSNS and RR fluxes. String motion on such mixed backgrounds is still integrable [12–16].

In this paper we study classical segmented strings on mixed backgrounds. This generalizes our earlier results in [10]. Our starting point is the action [8]

$$S = -\frac{\tau_1}{2}\int d^2\sigma\, \partial_a Y^M \partial_b Y^N (\sqrt{-h}h^{ab}G_{MN} + \kappa\epsilon^{ab}B_{MN}) \tag{1.1}$$

where τ_1 is the tension. In order to simplify the formulas, we set the prefactor to one ($\tau_1 = -2$). Y^M are coordinates on AdS_3, h and G are the worldsheet and background metrics, respectively. B is the background two-form with field strength proportional to the volume form of AdS_3. Finally, κ is the coupling of the two-form to the worldsheet. On causal grounds [8], its value is restricted to $\kappa \in [-1,1]$. We can without loss of generality restrict κ to be non-negative.

The equations of motion derived from (1.1) are

$$\partial_a\sqrt{-h}h^{ab}\partial_b Y^M + \sqrt{-h}h^{ab}\Gamma^M_{NL}\partial_a Y^N \partial_b Y^L - \frac{\kappa}{2}\epsilon^{ab}H^M{}_{NL}\partial_a Y^N \partial_b Y^L = 0. \tag{1.2}$$

Following [8], we will choose coordinates (τ,σ) on the worldsheet such that $\sqrt{-h}h^{ab} = \eta^{ab} = \mathrm{diag}\{-1,1\}$, and $\epsilon^{01} = -1$. The canonical embedding of AdS_3 into $\mathbb{R}^{2,2}$ is given by the universal covering space of the surface

$$\vec{X}\cdot\vec{X} \equiv -X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 = -1. \tag{1.3}$$

In terms of these ambient coordinates, and if we pick the gauge in which $H_{\mu\nu\lambda} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}X^\rho$, the equation of motion becomes

$$\partial_+\partial_- X_\mu - (\partial_+ X^\rho \partial_- X_\rho)X_\mu - \kappa\epsilon_{\mu\nu\lambda\rho}X^\nu\partial_+ X^\lambda \partial_- X^\rho = 0. \tag{1.4}$$

The second term comes from a Lagrange multiplier that keeps the string on the AdS_3 hyperboloid. Here we have used lightcone coordinates

$$\sigma^\pm = \frac{1}{2}(\tau \pm \sigma), \qquad \partial_\pm = \partial_\tau \pm \partial_\sigma. \tag{1.5}$$

The equations are supplemented by the Virasoro constraints

$$\partial_+\vec{X}\cdot\partial_+\vec{X} = \partial_-\vec{X}\cdot\partial_-\vec{X} = 0.$$

In terms of the Y variables, these constraints can be derived by varying (1.1) w.r.t. h_{ab}.

The equations of motion are exactly solved by segmented strings. These can be built by gluing diamond-shaped worldsheet patches. Each patch borders four others along null kink lines. Let us consider the patch in figure 1. The four vertices in the $\mathbb{R}^{2,2}$ embedding

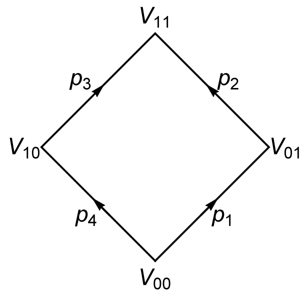

Figure 1. A single patch of the worldsheet. The four edges are the kink worldlines where the normal vector jumps. In $\mathbb{R}^{2,2}$ these are straight null lines with direction vectors p_i.

space of AdS_3 are labeled by V_{ij}. We have $V_{ij}^2 = -1$. The boundary of the worldsheet patch consists of four null kink lines. Let us define the following kink momentum vectors

$$\begin{aligned} p_1 &= V_{01} - V_{00} \qquad & p_2 &= V_{11} - V_{01} \\ p_3 &= V_{11} - V_{10} \qquad & p_4 &= V_{10} - V_{00} \end{aligned} \tag{1.6}$$

These vectors satisfy

$$p_i^2 = 0 \qquad \text{and} \qquad p_1 + p_2 = p_3 + p_4$$

The latter equation can be interpreted as "momentum conservation" during the scattering of two massless scalar particles with initial and final momenta $p_{1,2}$ and $p_{3,4}$, respectively.

Let $X(\sigma^-, \sigma^+) \in \mathbb{R}^{2,2}$ denote the embedding function of the string into spacetime where $\sigma^\pm$ are lightcone coordinates on the worldsheet. The patch is bounded by

$$\begin{aligned} X(\sigma^-, 0) &= V_{00} + \sigma^- p_4 \\ X(0, \sigma^+) &= V_{00} + \sigma^+ p_1 \end{aligned}$$

for $\sigma^\pm \in (0, 1)$.

Points on the surface are given by the *interpolation ansatz* [8] which solves the equation of motion (1.4)

$$\begin{aligned} X(\sigma^-, \sigma^+) = &\frac{1 + (1+\kappa^2)\sigma^-\sigma^+ p_4 \cdot p_1/2}{1 - (1-\kappa^2)\sigma^-\sigma^+ p_4 \cdot p_1/2} V_{00} + \\ &+ \frac{\sigma^- p_4 + \sigma^+ p_1 + \kappa \sigma^+ \sigma^- N}{1 - (1-\kappa^2)\sigma^+\sigma^- p_4 \cdot p_1/2} \end{aligned} \tag{1.7}$$

where

$$N^\mu = \epsilon^\mu_{\nu\lambda\rho} V_{00}^\nu p_4^\lambda p_1^\rho$$

From the interpolation ansatz, we have

$$V_{11} = X(1, 1). \tag{1.8}$$

This equality constitutes a discrete evolution equation for segmented strings. The set of points in AdS_3 null separated from both V_{10} and V_{01} is a one-dimensional locus, conveniently parametrized by $V_{11}(\kappa)$. Note that at $\kappa = 0$, the interpolation ansatz gives an AdS_2 patch embedded into AdS_3.

In the next section, we evaluate the action for a single patch. The action can be written in terms of Mandelstam variables corresponding to kink momentum vectors. Section III computes the Euler character for the string. Section IV discusses the singular $\kappa = 1$ case where the map into Toda variables degenerates. We finish with a short discussion of the results.

2 The action of segmented strings

The value of the action evaluated on the patch is analogous to a scattering amplitude in four dimensional Minkowski spacetime. Instead of the Lorentz symmetry, however, the action is invariant under the $SO(2,2)$ isometry group of AdS_3. The only independent invariants are the Mandelstam variables $s = (p_1 + p_2)^2$ and $u = (p_1 - p_4)^2$ where the $p_i \in \mathbb{R}^{2,2}$ are the difference vectors in (1.6), see also figure 1. The patch action then takes the form

$$S_{\text{patch}} = L^2 \mathcal{F}\left(\frac{u}{s}\right)$$

where L is the AdS_3 radius (henceforth set to one) and $\mathcal{F}(x)$ is a dimensionless function. In the following, we will determine this function by evaluating the action for certain symmetrical patches.

2.1 Fixing the patch

Using the $SO(2,2)$ symmetry we can rotate and boost any patch such that V_{00}, V_{10}, V_{01} are parametrized by two numbers $c, \tilde{c} \in \mathbb{R}$ as follows

$$\begin{aligned} V_{00} &= (1,\, 0,\,\ \ 0,\, 0)^T \\ V_{10} &= (1,\, c, -c,\, 0)^T \\ V_{01} &= (1,\, \tilde{c},\,\ \ \tilde{c},\, 0)^T. \end{aligned}$$

These points satisfy $V_{ij}^2 = -1$. It is easy to check that the corresponding difference vectors from (1.6) indeed satisfy $p_i^2 = 0$.

The interpolation ansatz in (1.7) gives the patch surface which stretches between the vertices V_{00}, V_{10}, V_{01}:

$$X(\sigma^-, \sigma^+) = \frac{1}{1 + c\tilde{c}(1-\kappa^2)\sigma^-\sigma^+} \begin{pmatrix} 1 - c\tilde{c}\left(1+\kappa^2\right)\sigma^-\sigma^+ \\ c\sigma^- + \tilde{c}\sigma^+ \\ -c\sigma^- + \tilde{c}\sigma^+ \\ -2c\tilde{c}\kappa\sigma^-\sigma^+ \end{pmatrix} \tag{2.1}$$

where $\sigma^\pm \in (0,1)$. The fourth vertex is computed from $V_{11} \equiv X(1,1)$.

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The Mandelstam variables are found to be

$$s = -\frac{4c\tilde{c}}{1 + c\tilde{c}(1-\kappa^2)} \qquad \text{and} \qquad u = 4c\tilde{c}$$

from which

$$\frac{u}{-s} = 1 + c\tilde{c}(1-\kappa^2) \tag{2.2}$$

for the argument of $\mathcal{F}(x)$. Note that $\kappa = 1$ is a special point, since at this value the ratio is independent of $c\tilde{c}$. This is precisely the Wess-Zumino-Witten theory.

2.2 Evaluating the action

Let us now evaluate action (1.1) for our worldsheet patch (2.1). For simplicity, we will use Poincaré coordinates (using ambient coordinates would not be particularly beneficial for this calculation). The metric and a canonical B-field are

$$ds^2 = \frac{-dt^2 + dx^2 + dz^2}{z^2} \qquad B_0 = \frac{dx \wedge dt}{2z^2}\,.$$

The field strength three-form is

$$H = dB_0 = -\mathrm{Vol}_{\mathrm{AdS}_3}$$

Since H and the background metric are both $\mathrm{SO}(2,2)$ invariant, we expect that the patch action will also be invariant and therefore it can be expressed in terms of Mandelstam variables. A coordinate transformation between the ambient and Poincaré coordinates is given by

$$(t,\,x,\,z) = \left(\frac{X_0}{X_{-1}-X_1},\,\frac{-X_2}{X_{-1}-X_1},\,\frac{1}{X_{-1}-X_1}\right).$$

In terms of these coordinates, the interpolation ansatz (1.7) is given by

$$\begin{aligned}
t(\sigma^-,\sigma^+) &= \frac{c\sigma^- + \tilde{c}\sigma^+}{1 - \tilde{c}\sigma^+ - c\sigma^-(\tilde{c}(1+\kappa^2)\sigma^+ - 1)} \\
x(\sigma^-,\sigma^+) &= \frac{2c\tilde{c}\kappa\sigma^-\sigma^+}{1 - \tilde{c}\sigma^+ - c\sigma^-(\tilde{c}(1+\kappa^2)\sigma^+ - 1)} \\
z(\sigma^-,\sigma^+) &= -\frac{1 + c\tilde{c}(1-\kappa^2)\sigma^-\sigma^+}{1 - \tilde{c}\sigma^+ - c\sigma^-(\tilde{c}(1+\kappa^2)\sigma^+ - 1)}
\end{aligned} \tag{2.3}$$

The $\kappa = 0$ case is straightforward and the results for evaluating the action were presented in [10]. At $\kappa \neq 0$, however, the action would depend separately on c and $\tilde{c}$ and thus it cannot be expressed in terms of Mandelstam variables which are functions of the product $c\tilde{c}$ only (see eq. (2.2)). This issue is due to the fact that the bulk action alone is not gauge-invariant if the worldsheet has boundaries. In order to preserve gauge invariance, point particles with opposite charges must be attached to the string endpoints. These particles are charged under a one-form gauge field A. The minimal coupling is described by

$$S_A = \int_{\partial\Sigma} A$$

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where $\partial\Sigma$ is the worldsheet boundary. The variation of the bulk worldsheet action under a gauge transformation Λ can be canceled if we let A transform according to

$$
\begin{aligned}
B &\to B + d\Lambda \\
A &\to A + \Lambda
\end{aligned}
$$

Our strategy will be the following. For a given B, we choose A such that $S + S_A$ is $\mathrm{SO}(2,2)$ invariant (and thus a function of s, t, u). Clearly, A does not affect closed string motion because those worldsheets have no boundaries. Using the gauge transformation above, we set $A = 0$. As a result, S_A vanishes and the entire patch action will come from S. When the dust settles, all we have done was a gauge transformation on B (and we can forget about S_A).

Consider the following B-field

$$
\begin{aligned}
B &= \frac{b_+}{2z^2 b_-} dx \wedge dt + \frac{x}{z b_-} dt \wedge dz + \frac{1-t}{z b_-} dx \wedge dz \\
\text{with} \quad b_\pm &\equiv 1 + t^2 - 2t - x^2 \pm z^2
\end{aligned}
$$

It is gauge-equivalent to B_0 and — as we will see — gives an $\mathrm{SO}(2,2)$ invariant result.

Let us now evaluate (1.1) on the patch given by the interpolation ansatz (2.3) using the expression above for the B-field. The explicit form of the Lagrangian density is somewhat complicated

$$
\begin{aligned}
\mathcal{L} = & \frac{-(\partial_\sigma t)^2 + (\partial_\sigma x)^2 + (\partial_\sigma z)^2 + (\partial_\tau t)^2 - (\partial_\tau x)^2 - (\partial_\tau z)^2}{z^2} + \\
& + \frac{\kappa}{z^2\left(-(t-2)t + x^2 + z^2 - 1\right)} \Big\{ 4xz[(\partial_\sigma t)(\partial_\tau z) - (\partial_\sigma z)(\partial_\tau t)] + \\
& \quad + 2\left((\partial_\sigma t)(\partial_\tau x)\, x^2 - (\partial_\tau x)\left((\partial_\sigma t)\left((t-1)^2 + z^2\right) - 2(\partial_\sigma z)(t-1)z\right)\right) + \\
& \quad + 2(\partial_\sigma x)\left((\partial_\tau t)\left((t-2)t - x^2 + z^2 + 1\right) - 2(\partial_\tau z)(t-1)z\right) \Big\}
\end{aligned} \tag{2.4}
$$

where $t = t(\tau,\sigma)$, $x = x(\tau,\sigma)$, and $z = z(\tau,\sigma)$ are the embedding coordinates. By plugging in (2.3), the purely geometrical part (i.e. the first line in (2.4)) gives (using worldsheet lightcone coordinates)

$$
\mathcal{L}_1 = \frac{2c\tilde{c}}{\left(c\tilde{c}\left(\kappa^2 - 1\right)\sigma^- \sigma^+ - 1\right)^2} \tag{2.5}
$$

The remaining terms in the Lagrangian are proportional to κ. They give

$$
\mathcal{L}_2 = -\kappa^2 \mathcal{L}_1 \tag{2.6}
$$

Integrating $\mathcal{L}_1 + \mathcal{L}_2$ over the patch gives the value of the action. We get

$$
S_{\text{patch}} = \int_0^1 d\sigma^- \int_0^1 d\sigma^+ (\mathcal{L}_1 + \mathcal{L}_2) = 2\log[1 + c\tilde{c}(1-\kappa^2)]
$$

Combining these results with (2.2) results in the covariant formula

$$
S_{\text{patch}} = 2\log\frac{u}{-s} \tag{2.7}
$$

Note that the expression is independent of κ.

2.3 Toda variables

The Mandelstam variables may be expressed in terms of Toda variables as in [10]. In order to do this, the lightlike kink momenta p are written as products of helicity spinors. We define

$$\sigma^\mu = (1, -i\sigma_2, \sigma_1, \sigma_3)$$
$$p_{a\dot{a}} = \sigma^\mu_{a\dot{a}} p_\mu$$

Since $p^2 = \det(p_{a\dot{a}}) = 0$, we can write

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$$

We will call λ left-handed spinors and $\tilde{\lambda}$ right-handed spinors. In terms of these two-component variables, the patch action can be written as

$$S_{\text{patch}} = 2\log\left|\frac{\langle\lambda_1,\lambda_4\rangle\langle\lambda_2,\lambda_3\rangle}{\langle\lambda_1,\lambda_2\rangle\langle\lambda_3,\lambda_4\rangle}\right|$$

There is a similar formula in terms of right-handed spinors. The spinor modulus drops out of the action. Thus, by defining the angles α_i via

$$|\lambda_i| e^{i\alpha_i} := \lambda_i^1 + i\lambda_i^2$$

one can write

$$S_{\text{patch}} = 2\log\left|\frac{\sin(\alpha_1-\alpha_4)\sin(\alpha_2-\alpha_3)}{\sin(\alpha_1-\alpha_2)\sin(\alpha_3-\alpha_4)}\right|$$

The α angles are the *global* Toda variables. Let us further define the left-handed *Poincaré* Toda variables by

$$a_i := \tan\alpha_i\,.$$

For a kink momentum vector $p \in \mathbb{R}^{2,2}$, the left-handed and right-handed Poincaré Toda variables are simply given by

$$a(p) = \frac{p_{-1}+p_2}{p_0+p_1}, \qquad \tilde{a}(p) = \frac{p_{-1}+p_2}{-p_0+p_1} \tag{2.8}$$

In terms of these fields the patch action becomes

$$S_{\text{patch}} = 2\log\left|\frac{(a_1-a_4)(a_2-a_3)}{(a_1-a_2)(a_3-a_4)}\right|\,. \tag{2.9}$$

The total action is then the sum of all patch contributions

$$S = 2\sum_{i,j}\log\left|\frac{a_{i,j}-a_{i+1,j}}{a_{i,j}-a_{i,j+1}}\right| \tag{2.10}$$

where the i,j indices indicate the position of the kink edge in the two-dimensional lattice (see figure 3). The previously used one-index a_k are

$$a_1 \to a_{ij} \qquad a_2 \to a_{i,j+1}$$
$$a_3 \to a_{i-1,j+1} \qquad a_4 \to a_{i-1,j}$$

and a_{ij} sits on a white dot in the lattice, see figure 3.

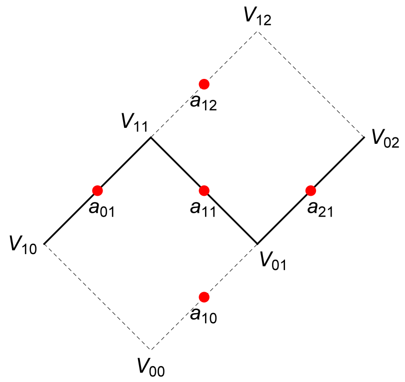

Figure 2. Two adjacent patches on the worldsheet. V_{00} and V_{12} can be computed using the interpolation ansatz. Then, the five left-handed Toda variables a_{ij} computed from the difference vectors will satisfy the Toda-type equation of motion.

2.4 Equation of motion

The equation of motion computed from (2.10) is [10]

$$\frac{1}{a_{i,j}-a_{i,j+1}}+\frac{1}{a_{i,j}-a_{i,j-1}}=\frac{1}{a_{i,j}-a_{i+1,j}}+\frac{1}{a_{i,j}-a_{i-1,j}} \tag{2.11}$$

This equation is independent of κ. It has been obtained in [17] as the equation of motion of a time discretization of a relativistic Toda-type lattice.

The validity of the above equation of motion can be checked directly as follows. Let us consider two adjacent worldsheet patches as in figure 2. The solid and dashed lines are the kink worldlines. We will pick four vertices V_{10}, V_{11}, V_{01}, $V_{02} \in \mathbb{R}^{2,2}$ as initial data. Since the kink worldlines are null, these vertices must be lightlike separated,

$$(V_{10}-V_{11})^2=(V_{11}-V_{01})^2=(V_{01}-V_{02})^2=0$$

In order to simplify the calculation, one can pick a frame in which V_{01} is moved into a fixed location

$$\begin{aligned} V_{01} &= (1,\, 0,\, 0,\, 0)^T \\ V_{11} &= \left(1,\, \sqrt{c_1^2+c_2^2},\, c_1,\, c_2\right)^T \\ V_{02} &= \left(1,\, \sqrt{c_3^2+c_4^2},\, c_3,\, c_4\right)^T. \end{aligned}$$

V_{11} and V_{02} are specified by four parameters c_i. We now have to choose V_{10} such that

$$(V_{10}-V_{11})^2=0 \quad \text{and} \quad V_{10}^2=-1.$$

We can take

$$V_{10}=(x,\, y,\, c_5,\, c_6)^T,$$

and then determine x and y from the two equations.

There are two solutions and either one can be picked for V_{10}:

$$V_{10} = \left(\frac{1 + c_1 c_5 + c_2 c_6 \mp D\sqrt{c_1^2 + c_2^2}}{1 + c_1^2 + c_2^2}, \frac{\sqrt{c_1^2 + c_2^2}(1 + c_1 c_5 + c_2 c_6) \pm D}{1 + c_1^2 + c_2^2}, c_5, c_6 \right)^T$$

where $D = \sqrt{\left(1 + c_6^2\right) c_1^2 - 2c_1 c_5 \left(1 + c_2 c_6\right) + c_5^2 + c_6^2 + c_2^2 \left(1 + c_5^2\right) - 2c_2 c_6}$. Let us pick the first solution.

Altogether there are six real constants $c_1 \dots c_6$ parametrizing the vertices in the initial data. Using the interpolation ansatz (1.7) we now compute[1] V_{00} and V_{12}. We get

$$V_{12} = \frac{1}{C} \begin{pmatrix} 1 + \frac{1+\kappa^2}{2} \left(c_1 c_3 + c_2 c_4 - \sqrt{c_1^2 + c_2^2}\sqrt{c_3^2 + c_4^2} \right) \\ \sqrt{c_1^2 + c_2^2} + \sqrt{c_3^2 + c_4^2} + \kappa(c_1 c_4 - c_2 c_3) \\ c_1 + c_3 + \kappa c_4 \sqrt{c_1^2 + c_2^2} - \kappa c_2 \sqrt{c_3^2 + c_4^2} \\ c_2 + c_4 - \kappa c_3 \sqrt{c_1^2 + c_2^2} + \kappa c_1 \sqrt{c_3^2 + c_4^2} \end{pmatrix}$$

with

$$C = 1 - \frac{1 - \kappa^2}{2} \left(c_1 c_3 + c_2 c_4 - \sqrt{c_1^2 + c_2^2}\sqrt{c_3^2 + c_4^2} \right).$$

and

$$V_{00} = \frac{1}{2 - \lambda_2(1 - \kappa^2)} \begin{pmatrix} 2 - \lambda_2(1 - \kappa^2) \\ \sqrt{c_1^2 + c_2^2} \left(\kappa^2 + 1\right) \lambda_2 + 2c_2 c_5 \kappa - 2c_1 c_6 \kappa - 2\sqrt{c_1^2 + c_2^2} + 2\lambda_3 \\ c_1 \left(\left(\kappa^2 + 1\right) \lambda_2 - 2 \right) + 2 \left(c_2 \kappa \lambda_3 - \sqrt{c_1^2 + c_2^2} c_6 \kappa + c_5 \right) \\ c_2 \left(\left(\kappa^2 + 1\right) \lambda_2 - 2 \right) + 2 \left(-c_1 \kappa \lambda_3 + \sqrt{c_1^2 + c_2^2} c_5 \kappa + c_6 \right) \end{pmatrix}$$

with

$$\begin{aligned} (\lambda_1)^2 &= \left(c_6^2 + 1\right) c_1^2 - 2c_5 \left(c_2 c_6 + 1\right) c_1 + c_5^2 + c_6^2 + c_2^2 \left(c_5^2 + 1\right) - 2c_2 c_6 \\ \lambda_2 &= \frac{\sqrt{c_1^2 + c_2^2}\lambda_1 + c_1^2 - c_5 c_1 + c_2^2 - c_2 c_6}{1 + c_1^2 + c_2^2} \\ \lambda_3 &= \frac{\sqrt{c_1^2 + c_2^2} \left(c_1 c_5 + c_2 c_6 + 1\right) + \lambda_1}{1 + c_1^2 + c_2^2} \end{aligned}$$

Now that we have all six vertices, from the difference vectors (e.g. $p_{01} \equiv V_{11} - V_{10}$) the five a_{ij} variables can be computed using eq. (2.8). For instance we get

$$\begin{aligned} a_{11} &= \frac{c_2}{\sqrt{c_1^2 + c_2^2} + c_1} \\ a_{21} &= \frac{c_4}{\sqrt{c_3^2 + c_4^2} + c_3} \\ a_{01} &= \frac{-\frac{-\sqrt{c_1^2+c_2^2}\lambda_4 + c_1 c_5 + c_2 c_6 + 1}{c_1^2+c_2^2+1} + c_2 - c_6 + 1}{-\frac{\sqrt{c_1^2+c_2^2}(c_1 c_5 + c_2 c_6 + 1) + \lambda_4}{c_1^2+c_2^2+1} + \sqrt{c_1^2 + c_2^2} + c_1 - c_5} \end{aligned}$$

where $(\lambda_4)^2 = \left(c_6^2 + 1\right) c_1^2 - 2c_5 \left(c_2 c_6 + 1\right) c_1 + c_5^2 + c_6^2 + c_2^2 \left(c_5^2 + 1\right) - 2c_2 c_6$. The formulas for a_{12} and a_{10} are too large to present here but can be computed in a straightforward way.

These variables can be plugged into (2.11) and they satisfy the equation.

[1]Note that in order to get V_{00} from V_{10}, V_{11}, V_{01}, the sign of κ must be reversed in the formula.

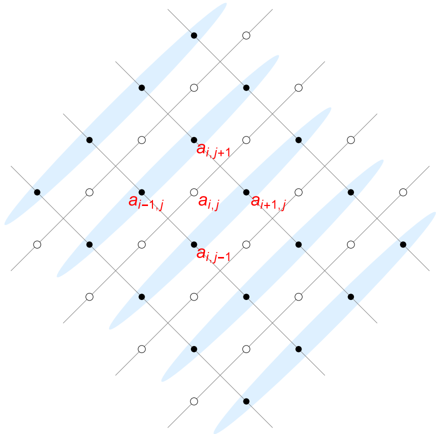

Figure 3. Kink worldlines form a rectangular lattice on the string worldsheet. The field a_{ij} lives on the edges (black or white dots depending on edge orientation). For $\kappa = 1$, the variables grouped together are equal (blue shading).

3 Degenerate WZW limit

Classical solutions for the SL(2) Wess-Zumino-Witten model are given by

$$g = g_+(\sigma^+)g_-(\sigma^-) \in \mathrm{SL}(2, \mathbb{R})$$

How can such a (classically) trivial theory be mapped into the non-trivial Toda-type theory? The answer is that at $\kappa = 1$ the map is not surjective: segmented strings are mapped into a smaller subspace of the Toda phase space.[2] The fact that this point in coupling space is singular can already be seen from (2.2) that gives $u/s = -1$ which is independent of the c and $\tilde{c}$ patch parameters.

Figure 3 shows the trivial subspace of the left-handed Toda phase space. The a_{ij} variables sitting on the black dots depend only on $i + j$ and they are independent of $i - j$. This means that the black dots grouped together (blue shading) have the same values. This is clearly a lower dimensional subspace. We have not included a separate (mirror) figure, but the right-handed variables $\tilde{a}_{ij}$ are similarly degenerate: the ones sitting on white dots depend only on $i - j$ (and not on $i + j$).

Let us sketch the proof of degeneracy discussed above. The calculation is similar to the one in section 2.4.

Consider the following forward null triple

$$\begin{aligned}
V_{00} &= (1,\ 0,\ 0,\ 0)^T \\
V_{10} &= \left(1,\ c_1, -c_2,\ \sqrt{c_1^2 - c_2^2}\right)^T \\
V_{01} &= \left(1,\ \tilde{c}_1,\quad \tilde{c}_2,\ \sqrt{\tilde{c}_1^2 - \tilde{c}_2^2}\right)^T
\end{aligned}$$

[2]Note that for strings with $|\kappa| < 1$, only *positive* Toda solutions play a role. These are the field configurations for which elementary patch areas (2.9) are non-negative.

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The interpolation ansatz (1.7) then gives the fourth vertex,

$$
\begin{aligned}
V_{11} &= X(1,1) = \frac{1}{C}\begin{pmatrix} \frac{\kappa^2+1}{2}\left(\sqrt{(c_1^2-c_2^2)(\tilde{c}_1^2-\tilde{c}_2^2)}-c_1\tilde{c}_1-c_2\tilde{c}_2\right)+1 \\ -\kappa\left(\tilde{c}_2\sqrt{c_1^2-c_2^2}+c_2\sqrt{\tilde{c}_1^2-\tilde{c}_2^2}\right)+c_1+\tilde{c}_1 \\ -\tilde{c}_1\kappa\sqrt{c_1^2-c_2^2}+c_1\kappa\sqrt{\tilde{c}_1^2-\tilde{c}_2^2}-c_2+\tilde{c}_2 \\ \sqrt{c_1^2-c_2^2}+\sqrt{\tilde{c}_1^2-\tilde{c}_2^2}-\kappa(c_1\tilde{c}_2+\tilde{c}_1c_2) \end{pmatrix} \\
C &= \frac{\kappa^2-1}{2}\left(\sqrt{(c_1^2-c_2^2)(\tilde{c}_1^2-\tilde{c}_2^2)}-c_1\tilde{c}_1-c_2\tilde{c}_2\right)+1
\end{aligned}
$$

We now perform a global

$$
\mathcal{R}\in \mathrm{SO}(2,2)=\mathrm{SL}(2)_L\times \mathrm{SL}(2)_R
$$

transformation on the kink momentum vectors which transforms V_{00} into a generic position. Left-handed variables are invariant under $\mathrm{SL}(2)_R$ and thus it is enough to consider $w\in \mathrm{SL}(2)_L$ rotations. The left-handed Toda variables computed from the difference vectors are

$$
\begin{aligned}
a_1 \equiv a(\mathcal{R}(V_{01}-V_{00})) &= \frac{w_{11}\sqrt{\tilde{c}_1-\tilde{c}_2}+w_{12}\sqrt{\tilde{c}_1+\tilde{c}_2}}{w_{21}\sqrt{\tilde{c}_1-\tilde{c}_2}+w_{22}\sqrt{\tilde{c}_1+\tilde{c}_2}} \\
a_4 \equiv a(\mathcal{R}(V_{10}-V_{00})) &= \frac{w_{11}\sqrt{c_1+c_2}+w_{12}\sqrt{c_1-c_2}}{w_{21}\sqrt{c_1+c_2}+w_{22}\sqrt{c_1-c_2}}
\end{aligned}
$$

where $a(p)$ denotes the Poincaré Toda variable corresponding to a kink momentum vector p, see eq. (2.8).

The other two variables a_2 and a_3 depend on κ. They are easy to compute, but the formulas are too large to present here. They satisfy

$$
\begin{aligned}
a_2 &\equiv a(\mathcal{R}(V_{11}-V_{01})) \xrightarrow{\kappa\to 1} a_4 \\
a_3 &\equiv a(\mathcal{R}(V_{11}-V_{10})) \xrightarrow{\kappa\to -1} a_1
\end{aligned}
$$

There are degeneracies in the right-handed variables which can be proven in a similar fashion.

We finish this section with the following observation. Let us exchange black and white dots in the lattice. This duality exchanges patches and kink collision vertices and changes the string embedding. After the transformation, left-handed variables sitting on white dots along the same kink line will be equal. This configuration correspond to trivial left-moving kinks (edges with black dots), since they do not cause a time delay when they cross a right-moving kink. Thus, this string embedding only contains right-moving shockwaves and in the $\kappa=0$ case it is equivalent to Mikhailov's construction [18].

4 Discussion

In this paper, we have computed the action of segmented strings in AdS_3. The worldsheet is coupled to a background two-form whose field strength is proportional to the volume form.

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We have used the interpolation ansatz of [8] to parametrize elementary patches. Segmented string solutions are obtained by gluing the patches along null boundaries (kink lines).

The null kink momentum vectors in the embedding space $\mathbb{R}^{2,2}$ can be decomposed using helicity spinors. Then, the action can be expressed in terms of cross-ratios of the spinor angles. We have called both these angles and their tangents "Toda variables". Time evolution of segmented strings can be described by the evolution equation of a discrete-time Toda-type lattice. This equation was presented in section 2.4.

Interestingly, the final form of the action does not depend on the two-form coupling κ. Thus, the theory in terms of Toda variables treats the classical Nambu-Goto theory ($\kappa = 0$) and the SL(2) Wess-Zumino-Witten model ($\kappa = 1$) on the same footing. However, the latter theory is a special one, because the map from the segmented string into Toda variables degenerates as $\kappa \to 1$. This is seen in figure 3: the variables grouped together will become equal at $\kappa = 1$. By performing a duality that exchanges black and white dots in the lattice, WZW solutions can be mapped to string solutions with purely left- or purely right-moving kinks.

The results generalize those in [10] which can be obtained by setting $\kappa = 0$. We have not discussed the reconstruction of string embeddings from solutions of the Toda-like lattice. The procedure should be analogous to the $\kappa = 0$ case.

An interesting question is how these ideas generalize to other spacetimes (e.g. dS_n or AdS_n) and what kind of Toda-like theories one would get by a similar reduction. We leave this for future work.

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A Euler character

In this appendix, we compute the Euler character

$$\chi = \frac{1}{4\pi} \int d^2\sigma \sqrt{-g}\, R\,.$$

The main reason for this is to provide an independent check on previous calculations.

Let us consider a worldsheet with torus topology.[3] Such worldsheets are known to give $\chi = 0$. In the following, we will check this result for segmented strings.

Using the interpolation ansatz (1.7) and $\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)$, the induced metric is diagonal with components

$$g_{\sigma\sigma} = -g_{\tau\tau} = \frac{c\tilde{c}}{\left(1 + \frac{1-\kappa^2}{4}\,(\tau^2 - \sigma^2)\, c\tilde{c}\right)^2}\,.$$

[3] This is possible if the target space is the surface $X^2 = -1$, without going to the covering space. Equivalently, one may consider time-periodic solutions in AdS_3.

The Ricci scalar computed from g is constant away from kink collision vertices

$$R_0 = -2(1-\kappa^2) \tag{A.1}$$

Let us decompose the Euler character as an integral away from the vertices plus integrals at the vertices

$$4\pi\chi = R_0 \int d^2\sigma\sqrt{-g} + \sum_i \int_{\mathcal{V}_i} d^2\sigma\sqrt{-g}\,R\,. \tag{A.2}$$

Here the sum is over kink vertices and $\mathcal{V}_i$ labels an infinitesimally small area around the i^{th} vertex.

Let us momentarily set $\kappa = 0$. From the appendix of [10], we have

$$\int_{\mathcal{V}_i} d^2\sigma\sqrt{-g}\,R \overset{\kappa=0}{=} 8\log\cos\frac{\phi_i}{2}$$

where ϕ_i is defined such that $\cos\phi_i$ is the scalar product of the normal vectors of the two space-like separated patches around the i^{th} collision point.

Denote the kink collision point by $X_M \in \mathbb{R}^{2,2}$. Then, two kink momentum vectors emanating from this point span an AdS_2 patch with normal vector [10]

$$\begin{aligned} N(\alpha,\beta) = {}& \frac{1}{\sin(\alpha-\beta)} \times \\ & \times \begin{pmatrix} X_0\cos(\alpha-\beta) - X_1\cos(\alpha+\beta) - X_2\sin(\alpha+\beta) \\ -X_{-1}\cos(\alpha-\beta) - X_2\cos(\alpha+\beta) + X_1\sin(\alpha+\beta) \\ -X_2\cos(\alpha-\beta) - X_{-1}\cos(\alpha+\beta) + X_0\sin(\alpha+\beta) \\ X_1\cos(\alpha-\beta) - X_0\cos(\alpha+\beta) - X_{-1}\sin(\alpha+\beta) \end{pmatrix} \end{aligned}$$

Here α and β are the left-handed global Toda variables corresponding to the two kink momentum vectors. Using this formula, ϕ_i can be expressed and we arrive at

$$\int_{\mathcal{V}_i} d^2\sigma\sqrt{-g}\,R \overset{\kappa=0}{=} 4\log\left|\frac{(a_1^{(i)}-a_4^{(i)})(a_2^{(i)}-a_3^{(i)})}{(a_1^{(i)}-a_2^{(i)})(a_3^{(i)}-a_4^{(i)})}\right| \tag{A.3}$$

where $a_k^{(i)}$ are the four left-handed Poincaré Toda variables around the i^{th} collision point on the worldsheet.

The result (A.3) holds even for $\kappa \neq 0$. Recall that in [10], the integrated Ricci scalar was computed in flat background space and then the result was expressed in terms of AdS_3 quantities (i.e. $\mathbb{R}^{2,2}$ normal vectors). The curvature of the target space did not matter since the collision of kinks was instantaneous. The Christoffel symbols can therefore be neglected in a limit where we zoom in on the collision point. Similary, we can argue that the three-form field strength can also be neglected in this limit and thus (A.3) should be independent of κ.

By performing the sum over kink collisions we get

$$\sum_i \int_{\mathcal{V}_i} d^2\sigma\sqrt{-g}\,R = 4\sum_{i,j}\log\left|\frac{a_{i,j}-a_{i+1,j}}{a_{i,j}-a_{i,j+1}}\right| \tag{A.4}$$

where now the indices of a_{ij} label positions in the kink lattice, see figure 3. In the r.h.s., using the results of the previous section, we recognize twice the total action (2.10).

Plugging (A.4) into (A.2) and using (A.1) we get

$$4\pi\chi = -2(1-\kappa^2)A + 2S_{\text{total}} \tag{A.5}$$

where A is the worldsheet area. From (2.5) and (2.6) the integrand in the bulk action is

$$\mathcal{L}_{\text{total}} = \mathcal{L}_1 + \mathcal{L}_2 = (1-\kappa^2)\mathcal{L}_1$$

which, after integration, yields $S_{\text{total}} = (1-\kappa^2)A$. Plugging this result back into (A.5) finally gives $\chi = 0$.

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