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Applications of Conformal Methods to Relativistic Trace-free Matter Models

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Abstract

Conformal methods have proven to be very useful in the analysis global properties and stability of vacuum spacetimes in general relativity. These methods transform the physical spacetime into a different Lorentzian manifold known as the unphysical spacetime where the ideal points at infinity are located at a finite position. This thesis makes use of conformal methods and applies them to various problems involving trace-free matter models. In particular, it makes progress towards the understanding of the evolution of unphysical spacetimes perturbed by trace-free matter as well as the behaviour of the the matter itself. To this end, evolution equations (wave equations) are derived and analyzed for both the unphysical spacetime and the matter. To investigate the relation between solutions of these wave equations to the Einstein field equations, a suitable system of subsidiary evolution equations is also derived. Furthermore, this thesis looks in detail at the behaviour of an unphysical spacetime coupled to the simplest matter trace free model: the confomally invariant scalar field. Finally, the system of conformal wave equations is used to show that the deSitter spacetime is non-linearly stable under perturbations by trace-free matter.

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Contents

1 Introduction

1.1 General Relativity and the Einstein equations

The theory of General Relativity (GR), first developed in 1915 by Einstein, is one of the greatest achievements in modern physics. In essence, it postulates that gravity is not a force, but is the result of spacetime curvature [5]. A spacetime is a pair $(\mathcal{M}, g_{ab})$, where g_{ab} is the metric and $\mathcal{M}$ is a 4-dimensional manifold. A manifold is an object which on a small, local scale looks different than on a global scale. An example of this is a sphere; on a small enough scale of distance a sphere looks like a flat plane, but if you look at it from a large enough scale of distance you find that this is not the case. It is for this reason that the Earth appears flat from the point of view of anyone standing on its surface. The metric is an object that describes how things are measured on the manifold; together these two objects form a spacetime. The universe that we live in is a 4-dimensional spacetime, since we need four components to describe the behaviour of objects living in the spacetime (three spatial coordinates and one time coordinate). This 4-dimensional spacetime becomes curved in the presence of matter. This curvature is the very source of what we perceive as the force of gravity. The metric is the main object of interest in GR since, as mentioned beforehand, it describes the measurement of distances on a manifold, however the measurement of distances on the manifold can be used to quantify curvature. This means that the metric can be thought of as the relativistic analogue of the gravitational potential, as in classical mechanics one obtains a force by differentiating the necessary potential; likewise in GR, one obtains the curvature (analogous to the gravitational tidal forces) by differentiating the metric. Precisely how the curvature of spacetime is related to the presence of matter is given by the Einstein field equations (EFEs)

$$R_{ab} - \frac{1}{2}g_{ab}R + \lambda g_{ab} = \frac{8\pi G}{c^4}T_{ab}, \tag{1.1}$$

where R_{ab} is the Ricci curvature tensor, which describes the curvature of the spacetime, R is the Ricci scalar curvature that is derived from R_{ab}, λ is the cosmological constant and T_{ab} is the energy-momentum tensor (also known as the stress-energy tensor), whose individual components describe the matter fields. Hence, the left hand side of the EFEs completely describes the curvature and the right hand side describes the distribution of matter. One can obtain the equations that describe the vacuum of space simply by setting T_{ab} in (1.1) to zero and then taking the trace of the resulting equation, which gives

$$R_{ab} = \lambda g_{ab}. \tag{1.2}$$

Despite their relatively simple appearance, the EFEs are extremely complicated and very difficult to solve, so much that Einstein himself predicted that no one would ever be able to find a solution to the equations. However, less than a year after the EFEs were introduced to the scientific community, Karl Schwarzchild successfully derived the simplest solution of the EFEs [32]. Since then a large number of exact solutions have been found; much research has been devoted to the understanding of these exact solutions and how they can be applied to various problems in physics.

1.1.1 Conformal methods in GR

Despite the success of GR in describing gravity and the nature of the universe, there are noticeable limitations to the theory when one wants to extract useful information. Two major problems were apparent when the theory was still considered new. The first problem was related to the uniqueness of solutions. One of the main guiding points that was used when developing the theory was that it should be invariant under coordinate transformations. Physically, this means that any law in nature should not depend on a particular choice of coordinates, since physical quantities do not change based on our point of view. GR is indeed invariant under a coordinate transformation, as a result it becomes possible to work in any system of coordinates and produce correct results. However, this strength of the theory is at the same time one of its biggest weaknesses. As a result of this invariance it becomes impossible to decide whether a solution is unique i.e. to decide which solution is truly a new result and not just an existing solution in a different coordinate system; we will return to this problem later when we discuss the process of solving the necessary equations.

The second problem is related to the concept of infinity. An important concept in physics is the study of objects at infinitely large distances. The inherent problem with this idea is how does one define infinity? Let us suppose that we have a system that is written in terms of polar coordinates (t, r, ϕ, θ). From this it is possible to define infinity in a number of different ways. For example: we could set r to be arbitrarily large and then analyse the behaviour of quantities like gravitational radiation in this region over infinitely large regions of time, however it is not entirely clear whether we will obtain the same result if we set first set t to be arbitrarily large then move to the same value of r. Here is an alternative way of looking at this problem, which is related to the so called *background independence problem*. In many undergraduate physics courses, all of the equations that one studies are all assumed to be in flat space. Additionally, when one wishes to solve these equations it is necessary to use boundary conditions. So in this sense, one usually analyses some field equations with respect to the Minkowski metric as a background metric. However, in GR the metric is no longer known a priori, since it is the very object that one is trying to solve for. Furthermore, the metric is not a static object, it changes in time, which means that the very arena in which one analyses the physical quantities also changes. This means it becomes difficult to define precisely the exact location of infinity and therefore to define the boundary conditions that one requires to solve the problem.

In the 1960s Penrose proposed a solution to this problem. The idea was to combine GR with a branch of mathematics called *conformal geometry*, which is the study of transformations that preserve angles. The key idea of conformal geometry is a conformal rescaling, which transforms the geometric object in question into another shape [22, 23]. The most basic type of conformal rescaling is the multiplication of all sides of a 2-dimensional Euclidean shape by a number. Doing this causes the size of the shape to increase, but crucially the angles do not change. In GR however, the objects that we analyse are not 2-dimensional; they are 4-dimensional objects whose properties are described by the metric tensor. The conformal transformation in this case has the form

$$g_{ab} = \Omega^2 \tilde{g}_{ab}, \tag{1.3}$$

where Ω is a positive scalar function called the *conformal factor*, $\tilde{g}$ is the original metric prior to the rescaling and is called the *physical metric* and g is the metric after the rescaling and is called the *unphysical metric*. There is also the equivalent contravariant metric conformal transformation

$$g^{ab} = \Omega^{-2} \tilde{g}^{ab}.$$

Performing a conformal transformation on the metric transforms the spacetime into a completely different spacetime known as the unphysical spacetime. In this fully geometric point of view, there is only one single definition of infinity, namely at the point where the conformal factor vanishes; in this sense only the conformal factor defines infinity. In spite of the usefulness of this idea, there are some difficulties that one encounters when using conformal methods. Firstly, it is not obvious whether or not these conformal methods are compatible with the EFEs; we will return to this particular challenge very shortly. Secondly, the equations that one needs to solve and analyse increase considerably from ten (using the standard EFEs) to thirty-two; this number increases further when working with spinors or tetrads. Additionally, the use of conformal methods is not feasible if one wishes to discuss a theory with higher dimensions, at least for the approach used in this thesis.

1.1.2 Global properties of solutions

Despite the complexity of the EFEs, a large number of exact solutions have been found and studied [17, 33]. Usually, when constructing an exact solution in GR, one has to make some simplifying assumptions. The most common type of simplification is to consider situations with very high degrees of symmetry.

Typically, when one constructs an exact solution it is usually done in a coordinate system adapted to the assumptions being made. Very often these coordinates only cover and describe a portion of the entire spacetime. Thus, one needs to find new coordinate systems that enable us to analyse the global properties of the spacetime. This process of attempting to identify a coordinate system that covers and completely describes the entirety of the spacetime is referred to as trying to find a *maximal analytic extension* of the spacetime [36].

Unfortunately, this process of trying to find a maximum analytic extension is not physically feasible due to the fact that there are only certain portions of a spacetime that can be described from an initial position in a spacetime, since the only things that a single observer will be able to measure and influence is that which is contained within their own light cone. In spite of this limitation, one would still like to be able to construct global solutions and analyse global properties of spacetimes even if said global properties might be naturally limited. A noteworthy theory that is relevant to this point is the *cosmic censorship conjecture* (CCC) [24, 38]. The CCC can be subdivided into two sub-categories: the weak CCC and the strong CCC. The weak CCC roughly states that other than the big bang, no singularities can exist outside of the event horizon of a black hole. The strong CCC, which is the more relevant version of the CCC for this thesis, roughly states that it is always possible to predict the fate of all observers, at least in the classical sense. This is relevant due to the fact that we are trying to analyse the spacetime globally which, at least according to the strong CCC, should be possible since the initial data enables one to predict the worldline for observers given some initial data.

In order to properly analyse the global properties of spacetimes it is convenient to formulate an *initial value problem* [4, 20, 37]. This problem involves taking a differential equation that describes a certain system together with some initial conditions in order to find solutions of the equation. The modelling of physical systems frequently leads to the solving of an initial value problem, which in turn helps us to understand how a system will evolve in time. However, in the case of GR, this is not so easy. As mentioned previously, it is difficult to find unique solutions in GR due to coordinate independence. This means that is is not clear whether or not the EFEs give rise to a system of differential equations that can be solved. Despite this, there are several principles in physics that can serve as guidelines to the nature of the differential equations that we should obtain. One such principle is the *causality principle*, which says that the cause of something must always precede the effect; as a result of this nothing can travel faster than the speed of light. Amongst the various classes of partial differential equations, the only type that are compatible with the causality principle are hyperbolic equations, since they allow for finite propagation speed. This means that is should be possible to construct some wave equations for the EFEs. This result was proven in 1952 by Choquet Bruhat [6]. In essence, Choquet-Bruhat showed that it was possible to write the EFEs as a system of wave equations by picking a particular choice of coordinates. This procedure of extracting evolution equations from the EFEs is called *hyperbolic reduction* [15, 29]. Hence, it becomes possible to formulate an initial value problem for GR.

1.2 The initial value problem in GR

Let us elaborate on the procedure of formulating an initial value problem in more detail. For the following analysis we will adopt a certain notation: any quantities written with Latin indices a, b, c refer to objects that are tensorial and objects written with Greek indices refer to any object that are evaluated with respect to a specific coordinate system (even if the object in question happens to be a tensor). The Ricci tensor can be written in terms of the metric tensor and its derivatives in the following manner

$$R_{\mu\nu} = -\frac{1}{2}g^{\sigma\rho}\partial_\sigma\partial_\rho g_{\mu\nu} + \nabla_{(\mu}\Gamma_{\nu)} + g_{\sigma\rho}g^{\alpha\beta}\Gamma^\sigma_{\alpha\mu}\Gamma^\rho_{\beta\nu} + 2\Gamma^\alpha_{\sigma\beta}g^{\sigma\beta}g_{\alpha(\mu}\Gamma^\rho{}_{\nu)\beta}, \tag{1.4}$$

where $\Gamma^\beta_{\alpha\gamma}$ are the *Christoffel symbols*, which can be written in terms of derivatives of the metric tensor

$$\Gamma^\beta_{\alpha\gamma} = \frac{1}{2}g^{\beta\sigma}(\partial_\gamma g_{\sigma\alpha} + \partial_\alpha g_{\gamma\sigma} - \partial_\sigma g_{\alpha\gamma}), \tag{1.5}$$

and we have defined Γ^β as

$$\Gamma^\beta \equiv g^{\alpha\gamma}\Gamma^\beta_{\alpha\gamma}, \tag{1.6}$$

the so called *reduced Christoffel symbols*. When examining equation (1.4), we notice that the principal part (i.e. the highest order derivatives) governing the evolution of the equations are the first two terms

$$-\frac{1}{2}g^{\sigma\rho}\partial_\sigma\partial_\rho g_{\mu\nu} + \nabla_{(\mu}\Gamma_{\nu)}. \tag{1.7}$$

The goal of this whole procedure is to recast the EFEs in a form that is *hyperbolic*, but what do we mean by hyperbolic? A hyperbolic equation is one which possess the same basic properties as wave equations. Recall that the wave equation is written as

$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\phi = \nabla^a\nabla_a\phi, \tag{1.8}$$

where ϕ is some arbitrary variable and the differential operator $\Box$ is known as the wave operator or the *d'Alembertian operator*. The principal part of a hyperbolic equation is usually this operator, though there are exceptions to this rule, most noticeably when the physical variables described by ϕ are not massless; we will return to this point when we discuss fluids. This equation has the following properties: first, the initial value problem is *well-posed*, which means that given some initial data that describes both the initial position and the initial velocity of the variable $(\phi, \partial_t\phi)$ then it is possible to find a solution to (1.8) for at least a small amount of time [29]. The next property of hyperbolic equations is that of *Cauchy stability*, which, in essence, means that any solution that one obtains for some data set should not look too different for any other solutions that are obtained from perturbations of this same data set. It should be noted that this property of Cauchy stability is not exclusive to hyperbolic equations. For example, it is possible to obtain stability for parabolic equations, but crucially this is only possible for one direction in time. For this reason, one can use parabolic equations for stability analysis of thermal systems since thermodynamics is not time-reversible due to entropy; time flows in one direction and parabolic equations can be used. However, since GR is a time symmetric theory one must use other equations, which is one reason why the use of hyperbolic equations is desirable. We will return to stability in much greater detail in the final chapter of this thesis. Finally, there is the fact that the physical quantities described by ϕ should propagate at a finite speed; this is especially important for the causality principle since SR postulates that nothing should travel faster than the speed of light. The important point is that any equation whose principal part is the d'Alembertian operator is a hyperbolic equation and can be used to perform an analysis of how a certain system will evolve.

Let us now return to (1.4) and its principal part (1.7). We can see if not for the second term in (1.7) then (1.4) would be hyperbolic, which in turn would allow one to formulate the EFEs as an initial value problem. The key to overcoming this hurdle is as follows; we define introduce some quantities known as the *coordinate gauge source functions* $Ł^\mu(x)$ and then choose a coordinate system that satisfies the following condition

$$\Box x^\mu = -Ł^\mu(x). \tag{1.9}$$

If this condition is satisfied it causes the second term of (1.7) to vanish as required. These are the so called *generalised wave coordinates* [12], which enables one to construct a system of non-linear wave equations for g_{ab}. Given a choice of initial data one then attempts to solve these wave equations in order to determine how a particular spacetime will evolve.

The process of solving these equations in order to find global solutions is accomplished with a process known as the 3+1 decomposition [2, 3]. The 3+1 decomposition is an approach to GR that involves slicing the 4-dimensional spacetime by a large number of spacelike (meaning that the normal describing the flow of time is always tangential to the surface) 3-dimensional surfaces called *hypersurfaces*. An alternative way of thinking about the 3+1 decomposition is as follows; consider only the spatial parts of the manifold at a specific moment in time. At this point, the spacetime looks like a flat sheet that stretches out to infinity in the spatial direction. We then consider this hypersurface at an infinitesimal moment later in time, and then at another later point in time and so on. Then we build up a picture of spacetime as being a collection of spacelike hypersurfaces stacked on top of one another, almost like leaves.

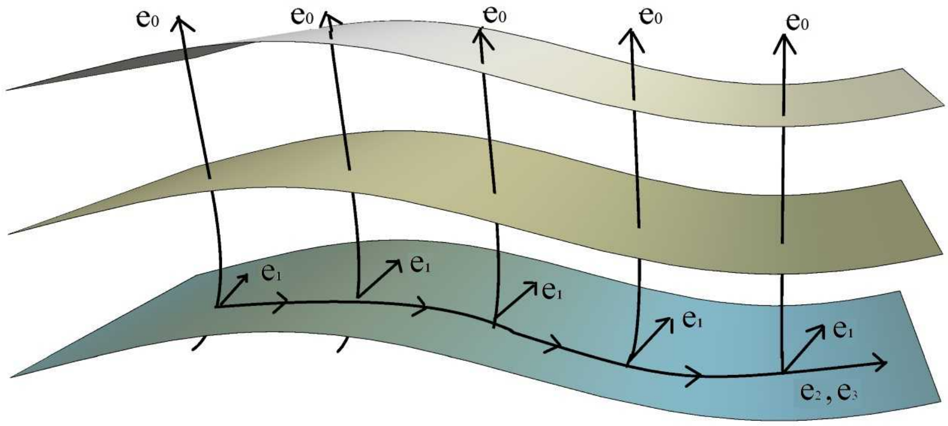

Figure 1.1: *A visual representation of the 3+1 Decomposition*

An important point is that this choice of 3+1 decomposition is not unique but is based on an arbitrary choice of coordinate system. This is somewhat against the spirit of GR since, in this sense, we are breaking the covariance and introducing a privileged time direction. There are other types of decompositions that can be achieved with other choices of coordinates, such as the 2+2 decomposition [19]. Nonetheless, the 3+1 decomposition is a very useful method for constructing solutions to the Einstein equations on a global scale. In order to form an initial value problem using the 3+1 decomposition we follow this procedure. First, we take a single initial hypersurface at a specific instant in time and prescribe some initial conditions for the evolution equations; next we solve the equations in order to determine exactly how the metric (and hence the entire gravitational system) evolves over time. Unfortunately, not only is solving the equations a complex process, but even the process of choosing initial data is highly non-trivial. It is easy to simply pick some initial conditions and solve the equations, however not every choice of initial data corresponds to a solution of the EFEs. To pick initial data, one needs to check that they satisfy the constraint equations of GR. We will consider this whole process in more detail later on in this thesis.

1.2.1 Asymptotics

One of the biggest applications of conformal methods in GR is to the field of asymptotics, namely the behaviour of the gravitational field at infinity. Central to this field is the notion of *asymptotic simplicity* [22, 28, 34]. Physically, an asymptotically simple spacetime is one which, far away from the source of the gravitational field, looks like either Minkowski space, de Sitter or anti de Sitter space depending on whether the universe has a cosmological constant of zero, plus one or minus one, respectively. This is intuitive, since the matter is the source of gravity. Therefore, far away from any matter, you expect the spacetime to appear like a vacuum. A more specific version of asymptotic simplicity is that of *asymptotic flatness*, where the spacetime looks exclusively like Minkowski in the asymptotic region.

Another concept that is vitally important to the study of asymptotics is that of an isolated system. An isolated system is an idealisation that is often used to model physical systems. The idea is as follows: consider a single gravitating body located in the middle of flat space, then one makes the assumption that this is the only gravitating body in existence. Next, one proceeds to analyse the behaviour of the gravitational field of this system as one moves farther and farther away from the source. While this may be an idealisation, it is not an unreasonable one, since gravity is a very weak force that decays very quickly via the inverse square law. Also, within the fields of astrophysics and cosmology, one typically works on very large scales of distance where gravitational interactions are negligible. Hence, it is natural to model various systems as isolated systems. Furthermore, in this model, we ignore the effects of the expansion of the universe. This approach enables us to define concepts of physical interest, such as: the total energy of the system, or the mass lost due to gravitational radiation. Spacetimes with a vanishing cosmological constant and matter with a suitable decay rate play an important role in the analysis of these isolated systems.

From the point of view of conformal methods asymptotic simplicity is defined as follows. First, we consider a spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$; such a spacetime is said to be asymptotically simple if it is possible to extend said spacetime to an unphysical spacetime $(\mathcal{M}, g_{ab})$ with a transformation of the form (1.3). This unphysical spacetime $(\mathcal{M}, g_{ab})$ has the following properties: firstly, it has a boundary, which we will denote $\mathscr{I}$. At all points on $(\mathcal{M}, g_{ab})$ the conformal factor Ω does not vanish, except for $\mathscr{I}$ where $\Omega = 0$, but importantly its derivative, $d\Omega$, is non-zero. Finally, all null like geodesics on $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ acquire a distinct endpoint on $\mathscr{I}$. It should be noted that this whole picture excludes black holes and singularities .

This is an intuitive definition, since we expect certain conditions to be satisfied at infinity. First of all, we expect all gravitational interactions to vanish at infinity; this turns out to be the case in the physical spacetime, however, when transforming into the unphysical spacetime then not all of the relevant quantities vanish. Consequently, this means that one can still extract useful information, furthermore, the point where $\Omega = 0$, which represents infinity is the simplest case, as one would expect. Finally, $d\Omega$ does not vanish because infinity is represented by a hypersurface and this hypersurface must have a distinct direction, which is defined by the normal.

1.3 Outlook of the thesis

In chapter 2 of this thesis, we will take a look at the various mathematical tools needed for the derivation of results in this thesis. We will look at general definitions of curvature, tensors and connections before moving on to the conformal versions of these quantities including: the Schouten tensor, the Weyl tensor and conformal transformations. From there, we will combine the two in order to show that simply applying a transformation law of the form (1.3) leads to equations that are singular; we will then give a derivation of a conformally regular version of the EFEs.

In chapter 3, we will turn our attention to the derivation of wave equations describing the evolution of conformal spacetimes perturbed by trace-free matter. We will include an overview of the vacuum case before giving a derivation of the wave equations with trace-free matter. Furthermore, we will show that any solution to the wave equations also implies a solution to the corresponding field equations.

Chapter 4 will be concerned with the derivation of evolution equations for the trace-free matter. Essentially, this will follow a very similar procedure to chapter 3; namely, deriving wave equations describing the evolution of trace-free matter models and then showing that any solution the wave equation implies a solution to the corresponding field equations. We will analyse four different matter models: the conformally invariant scalar field, electromagnetism, classical Yang Mills and radiation fluids.

Chapter 5 will step back from wave equations and instead look at the coupling of a simple scalar field matter model to the conformally regular field equations. From then, we will derive the field equations for this particular spacetime and then proceed to predict how this particular spacetime will evolve. Finally, we will look at conformal geodesics and the conformal geodesics for this spacetime.

Chapter 6 will be concerned with initial data and the procedure for solving the wave equations derived in chapter 4. We will take a more in depth look at the

3+1 decomposition and how all the required equations can be reformulated in this picture. From here, we will proceed to prove a result that tells us what variables we need to completely determine the initial data.

Finally, in Chapter 7, we will look at stability. We will give an overview of stability and explain how this is related to the work done in prior chapters. We will also use this to prove the global stability of the de Sitter spacetime.

2 Mathematical prerequisites of GR

2.1 Differential Geometry, Manifolds and Tensors

2.1.1 Manifolds and tangent spaces

The mathematical language of GR is differential geometry and differential geometry is the study of manifolds [36, 37]. Recall that we gave a brief, non-rigorous definition of what a manifold is in the introduction; we will now give a more detailed definition of a manifold. A manifold is a combination of a *topological space* $\mathcal{M}$ (i.e. a collection of objects that have some notion of distance) and a *maximal atlas*. An atlas is a collection of charts; a chart is a pair $(\mathcal{U};\phi)$ where $\mathcal{U} \subset \mathcal{M}$ and $\phi : \mathcal{U} \to \mathbb{R}^n$ (i.e. $\mathcal{U}$ is a small part of the space $\mathcal{M}$ and ϕ is a coordinate system that represents the position of any item in $\mathcal{U}$ as a collection of numbers). More precisely, $\mathcal{U}$ is what is called an *open set*, namely a section of a space for which it is always possible to draw a ball (i.e. a circular shape of any arbitrary dimension) that is contained within said section, provided we make the radius of the ball small enough. This of course means that we must exclude the boundaries of the section, since if we draw a ball with its centre on the boundary itself, then it will be impossible to completely contain this ball within $\mathcal{U}$ regardless of how small we make our ball. This idea of using an open set is important since, from a classical point of view, the spacetime must be continuous and open sets provide a natural way of defining continuous objects.

Let us put this idea of an atlas into context by doing a side-by-side comparison with a real life example, namely a world atlas. An atlas of the world can be thought of as an atlas in the mathematical sense. Each page of the atlas representing a country or continent is a subspace U and the markers on the side of the page are ϕ; both together are a chart. The collection of all possible charts covering every section of the space is called a maximal atlas.

An important idea in differential geometry is the manipulation of geometric objects, such as vectors, on the surface of a manifold. Unlike in Euclidean geometry, one cannot simply add or subtract two vectors together and get another vector. Thus, one needs a new system for algebraically manipulating vectors on a differentiable manifold; this leads to the idea of a *tangent space*. It is possible to recover the properties of Euclidean geometry on a differential manifold since on a small enough distance scale, all surfaces appear flat. This property of local flatness enables us to define at every point on the manifold a tangent space. The tangent space at an arbitrary point p on a manifold $\mathcal{M}$ (denoted by $T_p(\mathcal{M})$), is a vector space, which contains all possible vectors that are locally tangential to the surface. As is the case with all vector spaces, it is crucial to define a basis, i.e. a set of unit vectors, of $T_p(\mathcal{M})$. This basis is characterized by a set of partial derivative operators $\partial/\partial x^\mu$. This is intuitive because partial derivatives mean only one variable is changed, whilst all others remain constant (analogous to moving in one direction). Analogous to standard vectors there also exists a dual vector space $T_p^*(M)$ such that when an element of this dual space is multiplied with an element of the standard tangent space, then the result is a number. $T_p^*(\mathcal{M})$ has the same dimension as $T_p(\mathcal{M})$ and its elements are covectors.

2.1.2 Tensors

Using these tangents spaces as basic building blocks, we can define objects called tensors. Tensors are higher rank extensions of scalars and vectors; they can be expressed as a multilinear map (i.e. they map the product of several vector spaces to a single vector space). A contravariant tensor of rank k is defined as a multilinear map

$$M : T^*(\mathcal{M}) \times T^*(\mathcal{M}) \times ... \times \; T^*(\mathcal{M}) \to \mathbb{R},$$

where there are k number of dual tangent spaces. Put in other words, if we have a contravariant tensor of some rank and we act on it with a covariant tensor of the same rank then the end result is just a scalar.

Conversely, the definition of a covariant tensor is one that maps contravariant objects to real numbers. A covariant tensor of rank l is then defined as

$$M : T(\mathcal{M}) \times T(\mathcal{M}) \times ... \times \; T(\mathcal{M}) \to \mathbb{R},$$

where the total number of tangent spaces is given by l. Similarly, a mixed tensor of rank m is defined as

$$M : T(M) \times T(M) \times ... \times \; T(M) \times T^*(M) \times T^*(M) \times ... \; T^*(M) \to \mathbb{R},$$

where there are k dual tangent spaces and l tangent spaces and $m = k + l$. An example that many will likely be familiar with is the scalar product. In terms of vectors and covectors, the scalar product is defined as

$$v_i v^i = v_1 v^1 + v_2 v^2 + ... + v_n v^n,$$

we can see that upon multiplying a contravariant vector v^i (represented by the upper index) with a covariant vector v_i (represented by the lower index) we end up with a scalar. So, by acting on a covector with a contravariant vector of the same rank, we have obtained a number, as the above definition has stated. Similarly if one acts on a contravariant tensor of rank two T^{ab} with a covariant tensor of rank two T_{ab} one would once again end up with a number.

The most important property of these quantities is how they behave under a coordinate transformation. Tensors are constructed from multiple vector spaces, whose elements are vectors. The crucial point of vectors is that equations expressed in terms of vector quantities do not depend on a particular choice of coordinates. This property of vectors masks an extremely important property of physics, namely the fact that physical quantities should be independent of an arbitrary choice of coordinates. This is intuitive, since physical quantities should not change depending on how you choose to measure them. For example, let us suppose that we have some vector and we measure it in both a Cartesian coordinate system and a Polar coordinate system. Regardless of which coordinate system we use to measure the vector, the value of the magnitude of the vector that we measure remains the same. Crucially, as tensors are built up from vectors, this means that the same property carries over to tensors, meaning that tensors themselves are independent of any coordinate transformation. This property of tensors being coordinate independent and its connection to physics is summed up nicely in the *Covariance Principle.* The covariance principle states that all physical equations should contain tensors only, since only tensors have this property of transforming properly under a coordinate transformation.

Let us consider the transformation of coordinates from one frame of reference (x^0, x^1, x^2, x^3) to another (x'^0, x'^1, x'^2, x'^3). Let us now consider the functions that express the variables of the x coordinate system in terms of the variables of the x' coordinate system i.e. $x^n = f^0(x'^0, x'^1, x'^2, x'^3)$, then the general transformation law for a tensor of an arbitrary rank is

$$T^{\alpha 1 \alpha 2 ... \alpha N}_{\beta 1 \beta 2 ... \beta M} = \Lambda^{\alpha 1}_{\mu 1} \Lambda^{\alpha 2}_{\mu 2} \dots \Lambda^{\alpha M}_{\mu M} \tilde{\Lambda}^{\nu 1}_{\beta 1} \tilde{\Lambda}^{\nu 2}_{\beta 2} \dots \tilde{\Lambda}^{\nu N}_{\beta N} T'^{\mu 1 \mu 2 ... \mu N}_{\nu 1 \nu 2 ... \nu M}, \tag{2.1}$$

where Λ^i_j and $\tilde{\Lambda}^i_j$ are transformation matrices, which are defined as

$$\Lambda^\mu_\nu \equiv \frac{\partial x^\mu}{\partial x'^\nu}, \qquad \tilde{\Lambda}^\mu_\nu \equiv \frac{\partial x'^\mu}{\partial x^\nu}.$$

Any tensor of any arbitrary rank follows this transformation law and any physical laws are therefore invariant under an arbitrary coordinate transformation as a result.

The metric tensor

By far the most important tensor in GR, is the metric tensor g_{ab}. As mentioned in the introduction, the metric tensor is the very source of gravity, as it enables one to define concepts like distance and curvature [37]. It is possible to represent the metric tensor as a 4×4 matrix with 16 components

$$g_{\mu\nu} = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{23} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}. \tag{2.2}$$

The metric tensor is a symmetric, non-degenerate, rank 2 covariant tensor, meaning that $g_{ab} = g_{ba}$ and that it is possible to construct an inverse of the metric g^{ab}. The metric is used in calculations to transform contravariant tensors into covariant tensors and vice versa

$$g^{ab}A_b = A^a, \qquad A_a = g_{ab}A^b.$$

Therefore, using the metric we can raise and lower the indices as we please, almost like juggling indices. Consequently, we can think of covariant, contravariant and mixed tensors as being different representations of the same geometric quantity. There is also an expression for the contraction of two different metric tensors

$$g_{ab}g^{bc} \equiv \delta_a^c,$$

where δ_a^c is the Kronecker delta symbol, which is nothing more than the unit matrix

$$\delta_\mu^\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.3}$$

Building on this, we can deduce the equation for when two metrics are contracted on both their indices $g_{ab}g^{ab} = d$, where d is the dimension of the manifold that we are working with; in our case since we are working with 4-dimensional spacetimes $d = 4$. In addition to transforming vectors and tensors, the metric is also used to measure distances in spacetime, with said spacetime distance being given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu.$$

This equation can be used to show that all distances in spacetime are completely invariant. Even though in SR both time and distance separately are relative to an observer, together as a unified quantity they become an invariant quantity. Another important characteristic of the metric is the *metric signature*, which is simply the number of positive, negative and zero eigenvalues of the matrix g_{ab}. A metric that has three eigenvalues of one sign and one of the opposite sign is known as a *Lorentzian* metric. In this thesis we will be working with the $(-1, 1, 1, 1)$ signature, meaning the metric has three positive eigenvalues and one negative.

These are some of the mathematical properties of the metric tensor, but just what is the metric and why does it represent the gravitational field? The metric tensor is a generalization of the dot product between two tangent vectors. The dot product is used to measure the angle and lengths between two tangent vectors, which in turn can tell us how space is curved. If we have two vectors separated by a certain angle and we transport them along a flat surface then this angle will not change, however, this will not be the case for a curved surface. Furthermore, how quickly the angles change will tell us precisely how much the surface is curved. GR tells us that this curvature is the source of gravity, hence the metric tensor is the source of gravity itself.

This particular point also enables one to deduce a link between coordinate transformations and the gravitational field. As we mentioned earlier, on a local scale all surfaces appear flat. The metric that describes a flat surface is the Minkowski metric

$$\eta_{\mu\nu} \equiv \mathrm{diag}(-1, 1, 1, 1),$$

as a result of local flatness it is possible to define a coordinate system where $g_{ab} \to \eta_{ab}$ at a certain point. We may define this transformation law using (2.1)

$$g^{\mu\nu} = \Lambda^{\mu}_{(0)\alpha}\Lambda^{\nu}_{(0)\beta}\eta^{\alpha\beta}.$$

This means that if we know the transformation law from the local inertial frame of reference to an arbitrary frame of reference, we know the metric in this arbitrary frame of reference and hence we know the gravitational field, which is encoded in the metric.

2.2 Connections and curvature

2.2.1 Riemannian curvature and the Levi-Civita connection

Although we have defined the notion of a tensor and its properties, we have not yet specified any rules or laws concerning the motion of tensors or vectors from one point to another on a manifold. To do so, we introduce the concept of a *connection* [36, 37]. A connection describes the way in which vectors may be transported from one point to another on a manifold. Just as there are many different ways of moving from one point on the manifold to another, there exist an equal number of connections.

The most common connection in GR and certainly the most familiar to undergraduate students is the *Levi-Civita* connection. A Levi-Civita connection ∇, is a torsion-free metric connection, meaning that it satisfies the following two properties:

$$\nabla_a g_{bc} = 0, \tag{2.4a}$$
$$\nabla_a \nabla_b \phi - \nabla_b \nabla_a \phi = 0. \tag{2.4b}$$

If any connection satisfies (2.4a) then we say that the connection is *metric compatible*. The physical interpretation of this property is that angles between vectors are preserved when they are transported along the paths of geodesics; since the metric tensor is used to measure the length and angles between pairs of tangent vectors and (2.4a) is the form of the geodesic equation applied to a rank-2 tensor. If any connection satisfies (2.4b) then we say that the connection is *torsion free*. Physically, a torsion free metric means that any vectors or physical quantities have no tendency to twist when parallel transported along a manifold. The fundamental result of Riemannian geometry states that there is a unique connection that satisfies these two properties; such a connection is the Levi-Civita connection. A spacetime manifold is a Lorentzian manifold, however the fundamental result of Riemannian geometry works equally well in the Lorentzian case, meaning it is always possible to specify a Levi-Civita connection for an arbitrary spacetime manifold. In the theory of Riemannian manifolds, and in all first courses in GR, the Levi-Civita connection is called the covariant derivative. The components of a connection with respect to some local coordinates are called the *connection coefficients*; for the Levi-Civita connection, the connection coefficients are the Christoffel symbols, first given by equation (1.5).

A more intuitive and in-depth explanation for the Christoffel symbols is as follows. Recall that all laws of physics should remain invariant under a change of coordinates; consequently all equations should contain tensors only. The main problem with several fundamental laws in physics is that they contain partial derivatives, which do not transform properly under a coordinate transformation. This is a problem that is not readily apparent for equations in flat space, since most of the additional terms that are created when one transforms from one frame to another vanish. However, for curved space the additional terms that get created from the transformation remain, which spoils the tensorial nature of the equations. Hence, to ensure physical quantities remain invariant under an arbitrary coordinate transformation, one redefines the very notion of a derivative; this leads to the notion of a covariant derivative. As was discussed previously, there are a multitude of different ways of defining a derivative, however, courtesy of the fundamental result of Riemannian geometry, the most natural choice is the Levi-Civita connection. The precise form of this derivative depends on the rank of the tensor that it is applied to. For a covariant vector the Levi-Civita connection is

$$\nabla_\alpha v_\beta = \partial_\alpha v_\beta - \Gamma^\sigma_{\alpha\beta} v_\sigma, \tag{2.5}$$

when applied to a contravariant vector, the covariant derivative has the form

$$\nabla_\alpha v^\beta = \partial_\alpha v^\beta + \Gamma^\alpha_{\beta\sigma} v^\sigma. \tag{2.6}$$

From the above, we can see an intuitive meaning behind the Levi-Civita connection. The Levi-Civita connection is equal to the original partial derivative plus some additional terms involving the Christoffel symbols. These symbols are essentially corrections that cancel out the terms that are created whenever a partial derivative is transformed, hence the tensorial nature of the equations is preserved. Although equation (1.5) is not tensorial, when combined with quantities that are also not tensorial, they give rise to an object which is a tensor.

This idea of there being various different connections also extends to the standard tensors and equations used in GR. Traditionally, most of the tensors and equations are given in terms of the Levi-Civita connection. For example, the Riemann tensor, which is given in most standard textbooks as

$$R^\kappa{}_{\lambda\mu\nu} = -\partial_\lambda \Gamma^\kappa_{\mu\nu} + \partial_\mu \Gamma^\kappa_{\lambda\nu} - \Gamma^\iota_{\mu\nu}\Gamma^\kappa_{\iota\lambda} + \Gamma^\iota_{\lambda\nu}\Gamma^\kappa_{\iota\mu}, \tag{2.7}$$

is derived by applying the Levi-Civita connection to the equation

$$\nabla_a \nabla_b v^c - \nabla_b \nabla_a v^c = R^c{}_{dab} v^d. \tag{2.8}$$

However, it is possible to derive an alternative equation for the curvature simply by applying a different connection to (2.8). Let us now obtain a more general expression for the curvature. By definition, the Levi-Civita connection is one that is torsion free, thus one needs to specify a completely general curvature equation that does take into account the presence of torsion. To this end, we define a new tensor called the *torsion tensor*. Torsion arises as a result of the commutator of connections acting on scalar fields. If the two connections commute, as specified by (2.4b), then there is no torsion. If, however, the two connections do not commute, then we can write

$$[\nabla_a, \nabla_b] f = \Sigma_a{}^c{}_b \nabla_c f,$$

where f is a scalar field on the manifold and $\Sigma_a{}^b{}_c$ is the torsion tensor which, as the name suggests, specifies torsion; we can see that this tensor is antisymmetric with respect to the a and b indices.

With this torsion tensor, it becomes possible to define a more general relation for the curvature. The general relation for the curvature is given by the difference of the commutator and the torsion:

$$([\nabla_a, \nabla_b] - \Sigma_a{}^c{}_b \nabla_c) u^d = R^d{}_{cab} u^c.$$

The equation for the curvature changes depending on what rank tensor the commutator bracket is acting on. For example: acting on a covector gives the equation

$$[\nabla_a, \nabla_b]\omega_d = -R^c{}_{dab}\omega_c + \Sigma_a{}^c{}_b \nabla_c \omega_d,$$

or if we apply it to a rank two tensor $A^e{}_f$ the result is

$$[\nabla_a, \nabla_b] A^e{}_f = R^e{}_{dab} A^d{}_f - R^d{}_{fab} A^e{}_d + \Sigma_a{}^c{}_b \nabla_c A^e{}_f.$$

Whilst the predominant connection in GR is the Levi-Civita connection there are certain cases where it might be more desirable to use alternative connections.

The Weyl and Schouten Tensor

Two tensors that are related to the curvature, and that are used very frequently in conformal methods, are the Weyl and Schouten tensors [13, 36]. The Schouten tensor is given by the equation

$$L_{ab} \equiv \frac{1}{2} R_{ab} - \frac{1}{12} g_{ab} R. \tag{2.9}$$

The definition of the Schouten tensor is dimension dependent; the above equation is valid for four dimensions. It is also useful to mention the trace of the Schouten tensor, which can be verified after contracting the above definition with the metric to be

$$L_a{}^a = \frac{1}{6} R. \tag{2.10}$$

Another important tensor is the Weyl tensor C_{abcd}; this is the fully trace-free part of the Riemann tensor meaning $g^{ac} C_{abcd} = C^c{}_{bcd} = 0$. The precise form of the Weyl tensor depends on the dimension of the spacetime under consideration, for four dimensions the Weyl tensor is defined as

$$C_{abcd} = R_{abcd} - (g_{a[c} R_{d]b} - g_{b[c} R_{d]a}) + \frac{1}{3} R g_{a[c} g_{d]b}. \tag{2.11}$$

This tensor possesses the same symmetries as the Riemann tensor with the extra condition of trace-freeness, in other words a metric contraction on any pair of indices of the Weyl tensor is zero. Physically, the Weyl tensor represents the tidal forces that a body experiences as it moves along a path in spacetime. More precisely, the Weyl tensor conveys the information about how the shape of a body is distorted by tidal forces, since the Ricci curvature, which is the trace component of the Riemann tensor, contains the information of how volumes change in response to tidal forces. This tensor has a number of interesting properties that make it a useful object of research in GR. Firstly, it is invaluable in the study of gravitational waves, since the Weyl curvature is the only component of the curvature that does not vanish in vacuum spacetimes. Meaning, it is used to describe the propagation of gravitational waves through regions devoid of matter and energy. The Weyl tensor also has a number of properties that are very interesting from the point of view of conformal methods. Firstly, when written as a rank (1,3) mixed tensor ${C^a}_{bcd}$, it is completely invariant under any and all conformal rescalings for any arbitrary dimension greater than or equal to four and secondly, there is a linkage between the Weyl tensor and a property of certain manifolds called *conformal flatness*. Conformal flatness means that a particular class of conformal metrics contains the Minkowski metric, consequently every single metric within said class can be written as a conformal rescaling of the Minkowski metric η_{ab}. To understand why this is a useful property, consider the definition of a conformal rescaling $g_{ab} = \Omega^2 \tilde{g}_{ab}$; in the case of a conformally flat manifold $g_{ab} = \Omega^2 \tilde{\eta}_{ab}$. As a result of this, any derivatives that one tries to compute involving the unphysical metric will depend only on the conformal factor, since all derivatives of the Minkowski metric vanish. The same is true if one tries to compute derivatives of various curvature quantities such the Schouten tensor, since all objects related to curvature depend fundamentally on the metric. This helps to simplify certain problems tremendously and is therefore a very useful idea in conformal methods.

For reasons that will become apparent shortly it is convenient to express the Riemann tensor in terms of the Weyl and Schouten tensors

$$
{R^a}_{bcd} = {C^a}_{bcd} + 2{S_{d[a}}^{cf} L_{b]f}, \tag{2.12}
$$

where

$$
{S_{ab}}^{cd} \equiv {\delta_a}^c {\delta_b}^d + {\delta_a}^d {\delta_b}^c - g_{ab} g^{cd}. \tag{2.13}
$$

2.2.2 The Commutator Bracket Notation

Many of the tensorial equations that we will be dealing with in this thesis are extremely long and complicated. A notation used for writing multiple tensor terms that possess certain symmetries as a single term, which will be used multiple times for this point on, is the commutator bracket notation [36, 37]. The commutator bracket is defined as follows

$$
T_{[ab]} \equiv \frac{1}{2} T_{ab} - \frac{1}{2} T_{ba}, \tag{2.14}
$$

where n is an arbitrary integer. If we notice a pair of tensorial terms with indices akin to the RHS of (2.14) then we say that these two terms have an antisymmetric pair of indices and we can write them in a form akin to the LHS of (2.14). This same idea applies to any pair of tensors of any rank, provided that the two tensors have the same factor in front of them and have opposite signs, for example

$$2L_{a[b}d_{c]def} = L_{ab}d_{cdef} - L_{ac}d_{bdef}.$$

It is also possible to make use of the commutator bracket notation when the indices are not adjacent to each other, for example

$$2d_{ab[c|d|}d_{e]fgh} = d_{abcd}d_{efgh} - d_{abed}d_{cfgh},$$

where the $|d|$ indicates that the d index remains in a fixed position whilst the other are permuted in the standard fashion. In analogy to the square bracket notation used to group terms with antisymmetric pairs of indices there is also the round bracket notation used to denote pairs of indices that are symmetric i.e.

$$T_{(ab)} = \frac{1}{2}T_{ab} + \frac{1}{2}T_{ba}. \tag{2.15}$$

The same rules as described for the square bracket notation still apply. It is also possible to stack multiple brackets on top of each other when more than one pair of indices are symmetric or antisymmetric. For example, consider the tensor expression

$$T_{(a[b}Y_{c]|d|e)},$$

as we can see, this expression has two pairs of brackets. This means that it is a shorthand notation, not for two terms with certain symmetries, but four. The rule for expanding terms with multiple commutator brackets is to expand from the outer most pair of brackets, in the case of the above the expression is expanded to

$$T_{(a[b}Y_{c]|d|e)} = \frac{1}{2}T_{a[b}Y_{c]de} + \frac{1}{2}T_{e[b}Y_{c]da},$$

which is then expanded again to give

$$T_{(a[b}Y_{c]|d|e)} = \frac{1}{4}T_{ab}Y_{cde} - \frac{1}{4}T_{ac}Y_{bde} + \frac{1}{4}T_{eb}Y_{cde} - \frac{1}{4}T_{ec}Y_{bda}.$$

Whilst not as common for the purposes of this thesis, it is possible to have commutator brackets on more than two indices, in that case the commutator bracket gives

$$T_{[abc]} \equiv \frac{1}{3!}(T_{abc} + T_{bca} + T_{cab} - T_{acb} - T_{bac} - T_{cba}). \tag{2.16}$$

Whilst there are expressions for four or more indices, none of them are used in this thesis and will therefore be omitted. There are also a couple of useful identities involving these brackets to consider; the first one involves brackets applied to indices that are summation indices

$$T_{[ab]}Z^{abc} = T_{ab}Z^{[ab]c}.$$

Another identity involves contractions between symmetric and antisymmetric tensors. Consider a general symmetric tensor S_{ab} and a general antisymmetric tensor S_{ab} then

$$A_{[ab]}S^{ab} = 0.$$

This same identity also works if one has a contraction between any symmetric pair of indices and any antisymmetric pair of indices

2.3 Bianchi Identities

The idea of different connections yielding different equations also applies to the Bianchi identities [27, 36]. Any first course in GR gives the Bianchi identities as

$$R_{abcd} + R_{acdb} + R_{adbc} = 0,$$

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0. \tag{2.17}$$

However, as was the case with both the equation for the Riemann tensor (2.7) and the equation for the Christoffel symbols (1.5), these equations only apply to the Levi-Civita connection and again do not take into account the effects of torsion. The first Bianchi identity in its most general form is given by

$$R^d{}_{[cab]} + \nabla_{[a}\Sigma_b{}^d{}_{c]} + \Sigma_{[a}{}^e{}_b\Sigma_{c]}{}^d{}_e = 0.$$

For the case when $\Sigma_a{}^b{}_c = 0$, i.e. when the Levi-Civita connection is valid, then we end up with the familiar equation

$$R^d{}_{[cab]} = 0.$$

There also exists an equation for the second Bianchi identity in a more general form, which again takes into account the presence of torsion

$$\nabla_{[a}R^d{}_{|e|bc]} + \Sigma_{[a}{}^h{}_b R^d{}_{|e|c]f} = 0.$$

For the often used Levi-Civita connection, one obtains yet another familiar equation

$$\nabla_{[a}R^{d}{}_{|e|bc]} = 0.$$

A very important relation between the derivatives of the Schouten tensor and the Ricci scalar, which will be used frequently in later calculations, can be obtained from the so called contracted Bianchi identities

$$\nabla_a R^{ab} - \frac{1}{2}g^{ab}\nabla_a R = 0.$$

Using (2.9) to eliminate the Ricci tensor in the above

$$\nabla_a(2L^{ab} + \frac{1}{6}Rg^{ab}) - \frac{1}{2}g^{ab}\nabla_a R = 0,$$

and contracting all the metric terms and rearranging gives the reduced Bianchi identity in terms of the Schouten tensor

$$\nabla_a L^{ab} = \frac{1}{6}\nabla^b R. \tag{2.18}$$

2.3.1 The Index Free Notation

An alternative method of expressing tensors is the *Index Free Notation* which, as the name suggests, represents tensors without any indices [34, 36]. In geometry, objects are naturally contravariant or naturally covariant, as such we should find a natural representation for these two basic objects. For both an arbitrary vector v^a and an arbitrary covector ω_a, these are simply represented by boldface versions of the letters without the indices

$$v^a \leftrightarrow \mathbf{v}, \qquad \omega_a \leftrightarrow \boldsymbol{\omega}.$$

With these two objects represented in the index free notation, we should now find a way to defines certain operations between them. The first one is contraction between contravariant and covariant vectors

$$v^a\omega_a \equiv < \boldsymbol{\omega}, \mathbf{v} > . \tag{2.19}$$

In the framework of GR, one introduces the metric tensor which enables one to represent a contravariant vector as a covariant vector or vice versa. Hence, one needs to find a way of representing metric operations in the index free notation. First, the metric itself is represented as just a boldface letter minus the indices and the inverse of the metric is represented by a boldface letter with a sharp symbol i.e.

$$g_{ab} \leftrightarrow \mathbf{g}, \qquad g^{ab} \leftrightarrow \mathbf{g}^{\sharp}. \tag{2.20}$$

Next, we consider the contraction of vectors and covectors with the metric tensor.

One of the most common operations involving the metric is the scalar product. When two vectors are contracted with the metric then this is denoted as

$$g_{ab}v^a v^b \equiv \mathbf{g}(\mathbf{v}, \mathbf{v}). \tag{2.21}$$

Conversely, if two covectors are contracted with the contravariant metric then the scalar product is denoted as

$$g^{ab}\omega_a\omega_b \equiv \mathbf{g}^\sharp(\omega, \omega). \tag{2.22}$$

The next operation to consider is the transformation of vectors into covectors using the metric and vice versa. First, we consider an object that is naturally a vector being contracted with the metric; in index free notation this can be expressed as

$$g_{ab}v^b \leftrightarrow \mathbf{v}^\flat \equiv \mathbf{g}(\mathbf{v}, \cdot). \tag{2.23}$$

Here we have introduced the notation where any object that is naturally a vector, which has been transformed into a covector, is represented with a $\flat$ symbol. Likewise, if an object that is naturally a covector is transformed into vector, then this is represented as

$$g^{ba}\omega_a \leftrightarrow \omega^\sharp \equiv \mathbf{g}^\sharp(\omega, \cdot). \tag{2.24}$$

In the case where both a vector and a covector are involved in a contraction, then there are a few different ways depending on what objects are involved. For example: if a natural covector and a contracted vector are contracted with a contravariant metric, then it is written as

$$g^{ab}\omega_b v_a \equiv \mathbf{g}^\sharp(\omega, \mathbf{v}^\flat), \tag{2.25}$$

however, if a natural vector and a contracted covector are multiplied by a metric then this is represented as

$$g_{ab}\omega^b v^a \equiv \mathbf{g}(\omega^\sharp, \mathbf{v}). \tag{2.26}$$

By comparing (2.19), (2.25) and (2.26), we can deduce the following identity

$$\mathbf{g}^\sharp(\omega, \mathbf{v}^\flat) \equiv < \omega, \mathbf{v} > . \tag{2.27}$$

The other main type of operation in tensor calculus is the application of covariant derivatives. A very common operation is a covariant derivative applied to a vector and multiplied by another vector contracted with the index of a derivative; such an object is expressed in index free notation as

$$v^b \nabla_b u^a \equiv \nabla_{\mathbf{v}} \mathbf{u} = 0. \tag{2.28}$$

A similar identity exists for a vector contracted with a derivative acting on a scalar field

$$v^a \nabla_a f \equiv < df, \mathbf{v} > \equiv \nabla_{\mathbf{v}} f. \tag{2.29}$$

There are a multitude of other identities involving both curvature or derivatives of more complicated objects that one can look into, however for the purposes of this thesis, these will be the only ones that we will use. This notation shall be used when we proceed to discuss conformal geodesics.

2.3.2 Young Projectors

An identity of extreme importance and that will be used multiple times in the derivation of results in this thesis is the Young projector [1]. In essence the Young Projector is a relation in the same vein as the Bianchi identity, which makes full use of the symmetries of certain tensors. The derivation of this particular relation requires an advanced understanding of group theory. As such we will not give a full detailed derivation of the Young Projector but will try to motivate it. To start with we recall the standard symmetries of the Riemann tensor, firstly the anti-symmetries of pairs of indices

$$C_{abcd} = -C_{bacd} = -C_{abdc} = C_{badc},$$

the symmetries of two pairs of indices

$$C_{abcd} = C_{cdab},$$

and the first Bianchi identity

$$C_{a[bcd]} = 0.$$

The symmetries of the two above relations can be summed up in the following equation

$$C_{abcd} = \tfrac{2}{3}C_{abcd} + \tfrac{1}{3}C_{acbd} - \tfrac{1}{3}C_{adbc}. \tag{2.30}$$

Given (2.30) one can verify via direct computation that the tensor C_{abcd} satisfies the first Bianchi identity and all the other symmetries. This means that the Young projector encodes all the same information about the symmetries and can therefore be used for simplification whenever we have groups of Weyl tensor terms with certain symmetries on their indices.

2.4 Conformal transformations of connections

Conformal geometry is the study of mathematical transformations that leave the angles of geometric objects unchanged. The most basic type of conformal transformation is a rescaling of a simple 2-dimensional object; if we increase the length of each side of a 2-dimensional shape by the same amount, the angles of the shape do not change. However, in our case things are more complicated, since we are not working with simple 2-dimensional Euclidean objects, but 4-dimensional spacetime manifolds. Recall in section 1.1.1, we defined a conformal transformation $g_{ab} = \Omega^2 \tilde{g}_{ab}$; this is the type of transformation of greatest importance when using conformal methods in GR. In the standard convention $\tilde{g}$ is the original metric before any rescaling takes place, and is called the physical metric. You can have a large collection of metrics simply through an arbitrary choice of the conformal factor. This collection of metrics conformally related to the physical metric $\tilde{g}_{ab}$ is called the *conformal class* of a metric, denoted by $[\tilde{g}_{ab}]$. As mentioned previously, the act of performing such a transformation on the metric gives rise to an extension of the original spacetime, known as the unphysical spacetime. The main advantage of using this method is that the conformal factor Ω is completely arbitrary and is entirely dependent on the choice of the user. However, just as the shape of the spacetime changes so does the notion of distance on the surface. Hence, when rescaling an equation that describes a physical system one also needs to change the notion of how objects move on the surface; consequently one needs to transform the connections as well.

We can accomplish the task of transforming between connections using an object known as the transition tensor. This tensor is defined as

$$Q_a{}^c{}_b = S_{ab}{}^{cd}(\Omega^{-1}\nabla_d\Omega), \tag{2.31}$$

where $S_{ab}{}^{cd}$ is the same quantity defined using equation (2.13). Such a tensor enables us to transform freely between different connections

$$(\tilde{\nabla}_a - \nabla_a)v^b = Q_a{}^b{}_c v^c.$$

In fact the transition tensor is nothing more than a generalization of the Christoffel symbols, which are specific to the Levi-Civita connection. The above equation is not a general relation and depends on the exact tensor that the connection is acting on, for example when acting on a rank two contravariant tensor the transformation law is

$$(\tilde{\nabla}_a - \nabla_a)T^{bc} = Q_a{}^b{}_s T^{sc} + Q_a{}^c{}_s T^{bs}.$$

For a rank three tensor there will be a total of three transition tensors, for a rank four tensor there will be four transition tensors and so on. The most general definition of the transformation law for a tensor of rank (m, n) is

$$
\begin{aligned}
(\tilde{\nabla}_a - \nabla_a) T^{bc...m}_{ef...n} = {} & Q_a{}^b{}_s T^{sc...m}_{ef...n} + Q_a{}^c{}_s T^{bs...m}_{ef...n} + ... + Q_a{}^m{}_s T^{bc...s}_{ef...n} \\
& - Q_a{}^s{}_e T^{bc...m}_{sf...n} - Q_a{}^s{}_f T^{bc...m}_{es...n} - ... - Q_a{}^s{}_n T^{bc...m}_{ef...s}
\end{aligned}
\tag{2.32}
$$

3 The Conformal Einstein Field Equations

The entirety of this chapter shall be devoted to the derivation of the regular version of the conformal field equations. As was shown in the previous section the simplest course of action when performing a conformal rescaling leads to terms that are singular, until Helmut Friedrich derived a regular version of said equations. We will go through the exact same procedure as Helmut Friedrich did in 1980; it is especially important since several results make use of these equations, including the results derived in this thesis.

3.1 Derivation of the Conformal Field Equations with Matter

The idea of using conformal geometry to resolve certain issues in GR is indeed an interesting one, the next logical step is to see if the EFEs are compatible with conformal rescalings. Unfortunately, at first glance, the answer seems to be no. To understand why this is the case, consider the conformal transformation for the Ricci curvature tensor; we will examine the derivation of the conformal transformation to give a concrete example of the use of conformal methods and see why the result is unsatisfactory. The starting point of the derivation of the conformal transformation of the Ricci tensor is the definition of the curvature given by (2.8). We will obtain the transformation law of the Ricci tensor by first obtaining the conformal transformation law of the Riemann tensor and then contracting with the metric. In order to transform (2.8) we must transform the covariant derivatives in the expression using the techniques first described in section 2.4. First we apply (2.32) to the innermost derivatives of (2.8); upon doing so and re-arranging one ends up with

$$\nabla_a v^c = \tilde{\nabla}_a v^c - Q_a{}^c{}_d v^d. \tag{3.1}$$

Substituting (3.1) into (2.8) and the result is

$$\nabla_a \tilde{\nabla}_b v^c = \nabla_a Q_b{}^c{}_d v^d - \nabla_b \tilde{\nabla}_a v^c + \nabla_b Q_a{}^c{}_d v^d. \tag{3.2}$$

The next step is to transform the other covariant derivatives, to do so it becomes useful to define a substitution

$$\tilde{\nabla}_b v^c = S_b{}^c. \tag{3.3}$$

If we apply a derivative to the above and then apply (2.32) we get

$$\nabla_a S_b{}^c = \tilde{\nabla}_a S_b{}^c - Q_a{}^c{}_d S_b{}^d + Q_a{}^d{}_b S_d{}^c. \tag{3.4}$$

Re-expressing (3.4) in terms of vectors and derivatives through the application of (3.3) yields

$$\nabla_a \tilde{\nabla}_b v^c = \tilde{\nabla}_a \tilde{\nabla}_b v^c - Q_a{}^c{}_d \tilde{\nabla}_b v^d + Q_a{}^d{}_b \tilde{\nabla}_d v^c, \tag{3.5}$$

substituting (3.5) into (3.2) and the resulting expression upon simplification is

$$\begin{aligned} R^c{}_{dab} v^d - \bar{R}^c{}_{dab} v^d = {} & Q_a{}^d{}_b \tilde{\nabla}_d v^c - Q_a{}^d{}_b \tilde{\nabla}_b v^d - \nabla_c Q_b{}^c{}_d v^d \\ & + Q_b{}^c{}_d \tilde{\nabla}_c v^d - Q_b{}^d{}_c \tilde{\nabla}_d v^c + \nabla_b Q_c{}^c{}_d v^d. \end{aligned} \tag{3.6}$$

This is the transformation law for the Riemann tensor; as we can see it is an equation that contains both the physical Riemann tensor $\tilde{R}^a{}_{dab}$ and the unphysical Riemann tensor $R^a{}_{dab}$, plus several terms that can be determined from the conformal factor. Consequently, it enables us to transform freely between the curvature of both the physical and unphysical spacetimes and to determine one from the other. A more compact form of the transformation law of the Riemann tensor is given as follows

$$R^c{}_{dab} - \tilde{R}^c{}_{dab} = 2(\nabla_{[a} \Upsilon_{b]}{}^c{}_d + \Upsilon_{[a}{}^c{}_{|e|} \Upsilon_{b]}{}^e{}_d), \tag{3.7}$$

where $\Upsilon_a{}^c{}_b$ and is a quantity that is derived from the quantity (2.13), previously given in section 2.4, and is defined as

$$\Upsilon_a{}^c{}_b \equiv S_{ab}{}^{cd} \Upsilon_d, \tag{3.8}$$

where we have defined a vector quantity

$$\Upsilon_a \equiv \Omega^{-1} \nabla_a \Omega. \tag{3.9}$$

Upon substituting in (2.31) to (3.6) and contracting with the metric tensor and after a considerably long simplification then one ends up with the transformation law for the Ricci tensor

$$R_{ab} = \tilde{R}_{ab} - 2\Omega^{-1} \nabla_a \nabla_b \Omega - g_{ab} g^{cd} (\Omega^{-1} \nabla_c \nabla_d \Omega - 3\Omega^{-2} \nabla_c \Omega \nabla_d \Omega). \tag{3.10}$$

By contracting (3.10) with the metric tensor then one obtains the following equation for the Ricci scalar

$$R - \frac{1}{\Omega^2}\tilde{R} = -\frac{6}{\Omega}\nabla_c\nabla^c\Omega + \frac{12}{\Omega^2}\nabla_c\Omega\nabla^c\Omega. \tag{3.11}$$

Through a similar technique that was used to derive the Riemann tensor transformation law then one can derive the following transformation law for the Schouten tensor

$$L_{ab} - \tilde{L}_{ab} = -\frac{6}{\Omega}\nabla_c\nabla^c\Omega + \frac{12}{\Omega^2}\nabla_c\Omega\nabla^c\Omega. \tag{3.12}$$

A more compact form of the transformation law of the Schouten tensor exists

$$L_{ab} - \tilde{L}_{ab} = \nabla_a\Upsilon_b + \frac{1}{2}S_{ab}{}^{cd}\Upsilon_c\Upsilon_d. \tag{3.13}$$

It can be verified by directly substituting (3.9) into (3.13) that (3.13) and (3.12) are the same equation. With these transformation laws, we may now obtain the conformal vacuum EFE. Combining (3.10) with (1.2) and one finally arrives with the result

$$R_{ab} - \frac{1}{2}Rg_{ab} = -2\Omega^{-1}(\nabla_a\nabla_b\Omega - \nabla_c\nabla^c\Omega g_{ab}) - 3\Omega^{-2}\nabla_c\Omega\nabla^c\Omega g_{ab} \tag{3.14}$$

This equation is not particularly desirable from the point of view of conformal methods since it contains Ω^{-1} terms that blow up where $\Omega = 0$. The point where $\Omega = 0$ is the boundary of the unphysical spacetime and at this particular point one would like to make statements about solutions to the conformal EFEs at the conformal boundary, which is not possible. Thus, it seems that the idea of using conformal methods in GR is a dead end, however, in 1981 Friedrich successfully recast the EFEs in a manner that were regular at the conformal boundary [7–9].

3.1.1 The equation for the conformal factor

The starting point for the derivation of these equations is of course the EFEs which, in the physical spacetime upon setting $c = 1$, take the form

$$\tilde{R}_{ab} = \frac{1}{2}\tilde{g}_{ab}\tilde{R} - \lambda\tilde{g}_{ab} + \tilde{T}_{ab}. \tag{3.15}$$

Contracting both sides of the EFEs with the metric $\tilde{g}^{ab}$ and one ends up with the following equation for the Ricci scalar

$$\tilde{R} = 4\lambda - \tilde{T}, \tag{3.16}$$

where we have used the fact that $\tilde{g}_{ab}\tilde{g}^{ab} = 4$. The next step is to re-express the

Schouten tensor in terms of the energy-momentum tensor. To do so one starts by substituting (2.9) and (3.16) into (3.15). Upon doing so and rearranging we end up with

$$\tilde{L}_{ab} = \frac{1}{2}\tilde{T}_{ab} + \frac{1}{4}\tilde{g}_{ab}\tilde{R} - \frac{5}{6}\lambda\tilde{g}_{ab} + \frac{1}{12}\tilde{g}_{ab}\tilde{T}. \tag{3.17}$$

The next step is again to substitute (3.16) into (3.17). After doing this and rearranging the end result is

$$\tilde{L}_{ab} = \frac{1}{2}\tilde{T}_{ab} + \frac{1}{6}(\lambda - \tilde{T})\tilde{g}_{ab}. \tag{3.18}$$

For the next part of the calculation we consider the previously obtained conformal transformation laws of both the Ricci scalar and Schouten tensor. Dividing (3.11) through by twelve and rearranging and one obtains

$$\frac{1}{\Omega^2}\nabla_c\Omega\nabla^c\Omega = \frac{1}{12}\big(R - \frac{1}{\Omega^2}\tilde{R}\big) + \frac{1}{2\Omega}\nabla_c\nabla^c\Omega, \tag{3.19}$$

substituting (3.19) into (3.12) and the end result after multiplying out is

$$L_{ab} = \tilde{L}_{ab} - \frac{1}{\Omega}\nabla_a\nabla_b\Omega + \frac{1}{24}g_{ab}R - \frac{1}{24\Omega^2}\tilde{R}g_{ab} + \frac{1}{4\Omega}g_{ab}\nabla_c\nabla^c\Omega. \tag{3.20}$$

Substituting (3.18) into (3.20) and using the fact that $g_{ab} = \Omega^2\tilde{g}_{ab}$ gives

$$L_{ab} = \frac{1}{2}\tilde{T}_{ab} + \frac{1}{6}(\lambda - \tilde{T})\tilde{g}_{ab} - \frac{1}{\Omega}\nabla_a\nabla_b\Omega + \frac{1}{24}(R - \frac{1}{24\Omega^2}\tilde{R})g_{ab} + \frac{1}{4\Omega}\nabla^c\nabla_c\Omega g_{ab}. \tag{3.21}$$

Now, for the next step in the calculation, we define a new quantity in order to simplify our expressions. We shall call this quantity the *Friedrich scalar*, denoted s, which is defined as

$$s \equiv \frac{1}{4}\nabla_c\nabla^c\Omega + \frac{1}{24}R\Omega, \tag{3.22}$$

doing so enables us to rewrite (3.21) as

$$L_{ab} = \frac{1}{2}\tilde{T}_{ab} + (\frac{1}{6}\lambda - \frac{1}{6}\tilde{T} - \frac{1}{24}\tilde{R})\tilde{g}_{ab} + \frac{1}{\Omega}(sg_{ab} - \nabla_a\nabla_b\Omega).$$

Then, using (3.16) we obtain

$$L_{ab} = \frac{1}{2}\tilde{T}_{ab} - \frac{1}{8}\tilde{g}_{ab}\tilde{T} + \frac{1}{\Omega}(sg_{ab} - \nabla_a\nabla_b\Omega). \tag{3.23}$$

As $\tilde{T}_{ab}$ is not a geometric object derived from the metric we are free to choose a transformation law that best fits the analysis. It is convenient to choose, for reasons that will be justified later, the transformation law for the unphysical energy-momentum tensor to be

$$T_{ab} = \Omega^{-2}\tilde{T}_{ab}. \tag{3.24}$$

It then follows that

$$\frac{1}{2}\tilde{T}_{ab} - \frac{1}{8}\tilde{g}_{ab}\tilde{T} = \Omega^{-2}(\frac{1}{2}T_{ab} - \frac{1}{8}g_{ab}T) = \frac{1}{2}\Omega^2 T_{\{ab\}}, \tag{3.25}$$

where $T \equiv g^{ab}T_{ab}$, so that $\tilde{T} = \Omega^4 T$ and $T_{\{ab\}}$ denotes the trace-free part of T_{ab}. Putting (3.25) into (3.23) gives

$$L_{ab} = \frac{1}{2}\Omega^2 T_{\{ab\}} + \frac{1}{\Omega}(sg_{ab} - \nabla_a\nabla_b\Omega).$$

At first glance, this seems like a singular equation (as the conformal factor goes to zero, the equation blows up because of the $1/\Omega$ term), however, we can solve this problem by viewing this equation as an equation determining the second derivative of Ω. So, multiplying both sides by Ω and rearranging and we get

$$\nabla_a\nabla_b\Omega = \frac{1}{2}\Omega^3 T_{\{ab\}} - \Omega L_{ab} + sg_{ab},$$

which is the first of the CFEs; we can obtain the equivalent vacuum equation simply by setting $T_{\{ab\}} = 0$,

$$\nabla_a\nabla_b\Omega = -\Omega L_{ab} + sg_{ab}.$$

3.1.2 The equation for the Friedrich scalar

The next equation that one needs to derive is a differential equation for the Friedrich scalar. The starting point for this calculation is the first CFE that we just derived. We begin by applying a derivative to (3.42a) to obtain

$$\nabla_c\nabla_a\nabla_b\Omega = \frac{1}{2}(3\Omega^2)\nabla_c\Omega T_{\{ab\}} + \frac{1}{2}\Omega^3\nabla_c T_{\{ab\}} - L_{ab}\nabla_c\Omega - \Omega\nabla_c L_{ab} + \nabla_c s g_{ab}. \tag{3.26}$$

Remembering (2.8) and applying it to (3.26) and we get

$$-R^d{}_{bca}\nabla_d\Omega + \nabla_a\nabla_c\nabla_b\Omega = \frac{3}{2}\nabla_c\Omega T_{\{ab\}} + \frac{1}{2}\Omega^3\nabla_c T_{\{ab\}} - L_{ab}\nabla_c\Omega - \Omega\nabla_c L_{ab} + \nabla_c s g_{ab}. \tag{3.27}$$

Multiplying both sides of (3.27) by g^{bc} and contracting the indices gives

$$R_a{}^c\nabla_c\Omega + \nabla_a\Box\Omega = \frac{3}{2}\Omega^2\nabla^c\Omega T_{\{ac\}} + \frac{1}{3}\Omega^3\nabla^c T_{\{ac\}} - L_{ac}\nabla^c\Omega - \Omega\nabla^c L_{ac} + \nabla_a s. \tag{3.28}$$

We would also like to express the equations in terms of the Schouten tensor only;

to do so we make use of (2.9) with the quantities rearranged to

$$R_{ab} = 2L_{ab} + \frac{1}{6}g_{ab}R. \tag{3.29}$$

There is also a second order derivative in (3.28) applied to the conformal factor, which must be eliminated as the principal part of this differential equation must be applied to the Friedrich scalar. We may accomplish this by making use of the equation for the Friedrich scalar, since said equation contains a second order derivative of the conformal factor. Therefore, combining (3.29) and (3.22) with (3.28) and the result is

$$\frac{1}{6}R\nabla_a\Omega + \frac{1}{6}\Omega\nabla_a R + \frac{3}{2}\Omega^2\nabla^c\Omega T_{\{ac\}} + \frac{1}{2}\Omega^3\nabla^c\Omega T_{\{ac\}} - L_{ac}\nabla^c\Omega - \Omega\nabla^c L_{ac} - 3\nabla_a s = 0. \tag{3.30}$$

Making use of (2.18) to eliminate the derivatives of the Schouten tensor in (3.30), then one finds

$$\nabla_a s = \frac{1}{6}\Omega^3\nabla^c\Omega T_{\{ac\}} + \frac{1}{2}\Omega^2\nabla^c\Omega T_{\{ac\}} - L_{ac}\nabla^c\Omega,$$

which is the second of the CFEs; the vacuum equation is then

$$\nabla_a s = -L_{ac}\nabla^c\Omega.$$

3.1.3 The equations for the curvature

Previously, in section 2.2.1, we introduced both the Schouten tensor L_{ab}, and the Weyl tensor $C^a{}_{bcd}$, both of which are alternative descriptions of the curvature that possess nice conformal properties. We should therefore proceed to construct a differential condition of the curvature with respect to the Weyl and Schouten tensors. It can be shown that the Weyl and Schouten tensors can be related to each other using the second Bianchi identity, substituting (2.12) into (2.17) gives

$$\tilde{\nabla}_a\tilde{L}_{bc} - \tilde{\nabla}_b\tilde{L}_{ac} = \tilde{\nabla}_f C^f{}_{cab}. \tag{3.31}$$

However, this is not a satisfactory differential equation for L_{ab} because it contains the divergence of the Weyl tensor, one needs to find an equation for the latter in terms of the undifferentiated fields. To this end, notice that the RHS of the above can be expanded in terms of the physical energy-momentum tensor $\tilde{T}_{ab}$ using (3.18), but we will not do this yet. First, we express the LHS of the above in terms of the physical Cotton tensor

$$\tilde{Y}_{abc} = \tilde{\nabla}_a\tilde{L}_{bc} - \tilde{\nabla}_b\tilde{L}_{ac} = 2\tilde{\nabla}_{[a}\tilde{L}_{b]c}. \tag{3.32}$$

Following the standard procedure of writing equations out in terms of objects of the unphysical spacetime $(\tilde{\mathcal{M}}, g)$, one would like to express the divergence $\tilde{\nabla}_f C^f{}_{cab}$ in terms of an equivalent expression involving the unphysical connection ∇. For this we make use of the identity

$$\nabla_d(\Omega^{-1} C^d{}_{abc}) = \Omega^{-1} \tilde{\nabla}_d C^d{}_{abc}. \tag{3.33}$$

Making use of (3.33) in (3.32) then we obtain

$$\nabla_d(\Omega^{-1} C^d{}_{abc}) = \Omega^{-1} \tilde{Y}_{bca}. \tag{3.34}$$

This seems to be a dead end because of the Ω^{-1} on the both sides that can be cancelled out. However, if we define the rescaled Weyl tensor

$$d^a{}_{bcd} \equiv \Omega^{-1} C^a{}_{bcd}, \tag{3.35}$$

and the rescaled Cotton tensor

$$T_{abc} = \Omega^{-1} \tilde{Y}_{abc}, \tag{3.36}$$

and then combining (3.34), (3.35) and (3.36) we get the formally regular equation

$$\nabla_f d^f{}_{abc} = T_{abc}.$$

This is the third CFE. Obtaining the next equation is relatively straightforward; we simply need to write out (3.31) in terms of $d^a{}_{bcd}$:

$$\nabla_a L_{bc} - \nabla_b L_{ac} = \nabla_d \Omega d^d{}_{cab} + \Omega T_{abc},$$

which is the fourth CFE; again it is regular at $\Omega = 0$.

3.1.4 The equation for λ

The final CFE is relatively easy to obtain and follows from the transformation equation for the Ricci curvature (3.11), multiplying said equation through by Ω^2 and then rearranging with respect to $\tilde{R}$ gives

$$\tilde{R} = \Omega^2 R + 6\Omega \nabla_a \nabla^a \Omega - 12 \nabla_a \Omega \nabla^a \Omega. \tag{3.37}$$

Now, making use of both (3.16) and (3.22) then it is possible to rewrite (3.37) as

$$4\lambda - \tilde{T} = 24\Omega s - 12 \nabla_a \Omega \nabla^a \Omega. \tag{3.38}$$

Next, recall that the preferred transformation law for the energy momentum tensor is given by (3.24); the trace-free part of this is obtained simply by contracting with the metric which gives

$$\tilde{T} = \Omega^4 T. \tag{3.39}$$

So, substituting (3.39) into (3.38) and then rearranging we then obtain the fifth and final CFE

$$\lambda = 6\Omega s - 3\nabla_a\Omega\nabla^a\Omega + \frac{1}{4}\Omega^4 T,$$

which is a conformally regular equation for the cosmological constant. An important piece of information about the cosmological constant can be obtained via directly differentiating this equation, which gives

$$\nabla_b\lambda = 6s\nabla_b\Omega + 6\Omega\nabla_b s - 6\nabla^a\Omega\nabla_b\nabla_a\Omega. \tag{3.40}$$

If we then substitute (3.42a) and (3.42b) into (3.40) then the end result is trivial i.e.

$$\nabla_b\lambda = 0. \tag{3.41}$$

Physically, this means that the cosmological constant is always a constant, at least from a purely classical point of view.

Summary

So in summary, we have obtained a set of equations that are conformally regular and that enable us to analyse the behaviour of conformally rescaled spacetimes

$$\nabla_a\nabla_b\Omega = \frac{1}{2}\Omega^3 T_{\{ab\}} - \Omega L_{ab} + s g_{ab}, \tag{3.42a}$$
$$\nabla_a s = \frac{1}{6}\Omega^3\nabla^c T_{\{ac\}} + \frac{1}{2}\Omega^2\nabla^c\Omega T_{\{ac\}} - L_{ac}\nabla^c\Omega, \tag{3.42b}$$
$$\nabla_a L_{bc} - \nabla_b L_{ac} = \nabla_d\Omega d^d{}_{cab} + \Omega T_{abc}, \tag{3.42c}$$
$$\nabla_f d^f{}_{abc} = T_{abc}, \tag{3.42d}$$
$$\lambda = 6\Omega s - 3\nabla_a\Omega\nabla^a\Omega + \frac{1}{4}\Omega^4 T. \tag{3.42e}$$

The CFEs that describe the behaviour of conformally rescaled vacuum spacetimes are obtain simply by setting all of the energy momentum tensor components in (3.42a)-(3.42e) to be zero, which gives

$$\nabla_a\nabla_b\Omega = -\Omega L_{ab} + s g_{ab}, \tag{3.43a}$$
$$\nabla_a s = -L_{ac}\nabla^c\Omega, \tag{3.43b}$$

$$\nabla_a L_{bc} - \nabla_b L_{ac} = \nabla_d \Omega d^d{}_{cab}, \tag{3.43c}$$

$$\nabla_f d^f{}_{abc} = 0, \tag{3.43d}$$

$$\lambda = 6\Omega s - 3\nabla_a \Omega \nabla^a \Omega. \tag{3.43e}$$

Especially appropriate for this thesis are the trace-free matter versions of (3.42a)-(3.42e), since we will deal extensively with various trace-free matter models. These equations are obtained simply by setting $T_{\{ab\}} = T_{ab}$ so that $T = 0$. The CFEs then have the form

$$\nabla_a \nabla_b \Omega = \frac{1}{2}\Omega^3 T_{ab} - \Omega L_{ab} + s g_{ab}, \tag{3.44a}$$

$$\nabla_a s = \frac{1}{6}\nabla^c T_{ac} + \frac{1}{2}\Omega^2 \nabla^c \Omega T_{ac} - L_{ac}\nabla^c \Omega, \tag{3.44b}$$

$$\nabla_a L_{bc} - \nabla_b L_{ac} = \nabla_d \Omega d^d{}_{cab} + \Omega T_{abc}, \tag{3.44c}$$

$$\nabla_f d^f{}_{abc} = T_{abc}, \tag{3.44d}$$

$$\lambda = 6\Omega s - 3\nabla_a \Omega \nabla^a \Omega. \tag{3.44e}$$

An important fact about the CFEs is their relationship to the EFEs. A result first obtained by Friedrich states that whenever the conformal factor does not vanish then a solution to the CFEs implies a solution to the EFEs [10]. This means that the CFEs are a useful tool in research in GR as they can be used to gain more information in physics because the result of Friedrich enables one to indirectly obtain results for the EFEs through solving the CFEs.

4 The Conformal Wave Equations

As mentioned previously, global properties of spacetimes can be analysed effectively with the use of conformal methods. This is all well and good, however they do not tell us anything about how the conformal spacetimes will evolve. Recall in section 1.1.2, we stated that in order to properly analyse the evolution of any spacetime, one needs to formulate an initial value problem. The correct way to formulate an initial value problem is to construct a system of wave equations for the fields that you wish to analyse; Yvonne Choquet Bruhat showed that this was possible through a specific choice of coordinates. With this in mind, one then asks the following question: is it possible to formulate the CFEs as an initial value problem? Given the connection between the EFEs and the CFEs, the logical choice is to try and express the CFEs as a system of wave equations. This process has been done in the case of vacuum spacetimes [21], with the vacuum conformal wave equations (CWEs) taking the following form

$$\square\Omega = 4s - \tfrac{1}{6}\Omega R, \tag{4.1a}$$

$$\square s = -\tfrac{1}{6}Rs + \Omega L_{ab}L^{ab} - \tfrac{1}{6}\nabla_a R \nabla^a \Omega, \tag{4.1b}$$

$$\square L_{bd} = -2\Omega d_{badc}L^{ac} - g_{bd}L_{ac}L^{ac} + 4L_b{}^a L_{da} + \tfrac{1}{6}\nabla_b\nabla_d R, \tag{4.1c}$$

$$\square d_{fhqr} = 4\Omega d_f{}^b{}_{[r}{}^{|p|}d_{q]phb} - 2\Omega d_{fh}{}^{bp}d_{qbrp} + \tfrac{1}{2}d_{fhqr}R, \tag{4.1d}$$

which is obtained via a systematic application of derivatives to the CFEs. The question of whether one can obtain an analogous system of equations for matter models is still an open problem. We will now show that it is possible to derive a system of wave equations in the case where the CFEs are coupled to trace-free matter systems.

4.1 Auxiliary results

Before beginning with the actual derivation of the wave equations, we will derive a set of relationships that will be instrumental in virtually every single derivation from this point on.

4.1.1 The rescaled Cotton tensor

We previously looked at the rescaled Cotton-York tensor in section 3.1.3, now we consider the definition of said equation in terms of the energy-momentum tensor

$$T_{abc} = \tfrac{1}{2}(3T_{bc}\nabla_a\Omega + \Omega\nabla_a T_{bc} - 3T_{ac}\nabla_b\Omega - \Omega\nabla_b T_{ac} - T_{bd}g_{ca}\nabla^d\Omega + T_{ad}g_{cb}\nabla^d\Omega). \quad (4.2)$$

We now shall derive some relations from this equation that will be used later on in certain derivations. Begin by contracting (4.2) with the metric tensor

$$T_a{}^b{}_b = -\tfrac{3}{2}T_a{}^b\nabla_b\Omega - \tfrac{1}{2}\Omega\nabla_b T_a{}^b + \tfrac{3}{2}T_{ad}\nabla^d\Omega, \quad (4.3)$$

where we have used the fact that $g_{ab}T^{ab} = 0$, and since by definition the energy-momentum tensor is divergence free (due to conservation of energy) this means that

$$T_a{}^b{}_b = 0. \quad (4.4)$$

The second relation is obtained by applying a contracted derivative to the third index of (4.2). Upon doing so, and making use of the fact the the energy-momentum tensor is divergence free, one obtains

$$\nabla^c T_{abc} = 3T_{bc}\nabla^c\nabla_a\Omega + \Omega\nabla^c\nabla_a T_{bc} - 3T_{ac}\nabla^c\nabla_b\Omega - \Omega\nabla^c\nabla_b T_{ac} + T_{ad}\nabla_b\nabla^d\Omega. \quad (4.5)$$

Making use of (3.42a) and (2.8) then using (3.29) it can be shown that (4.5) simplifies to

$$\nabla^c T_{abc} = 0. \quad (4.6)$$

The next required relation is a direct consequence of the one that was just derived. We begin by applying a second contracted derivative to (4.6), which gives

$$\nabla^b\nabla^c T_{abc} = 0. \quad (4.7)$$

Now using (2.8) to re-write the term on the left hand side of (4.7) then using (3.29) and (2.12) to eliminate the curvature components them it can be shown that (4.7) reduces to

$$\nabla_c\nabla_b T_a{}^{bc} = -\Omega T^{bcd}d_{abcd} + T_a{}^{bc}L_{bc}. \quad (4.8)$$

The third relation is obtained by applying a derivative to the second index of (4.2), upon doing so we get

$$\nabla_b T_a{}^b{}_c = \tfrac{3}{2}T_c{}^b\nabla_b\nabla_a\Omega + \tfrac{1}{2}\Omega\nabla_b\nabla_a T_c{}^b - \tfrac{3}{2}T_{ac}\Box\Omega$$

$$-\tfrac{1}{2}\Omega\nabla_b\nabla^bT_{ac}+\tfrac{1}{2}\nabla_aT_{cb}\nabla^b\Omega-2\nabla_bT_{ac}\nabla^b\Omega$$
$$+\tfrac{1}{2}\nabla^b\Omega\nabla_cT_{ab}+\tfrac{1}{2}T_a{}^b\nabla_c\nabla_b\Omega-\tfrac{1}{2}T^{bd}g_{ac}\nabla_d\nabla_b\Omega. \tag{4.9}$$

As before, using (3.42a), (4.1a) to eliminate the second order derivatives, commuting covariant derivatives with (2.8) and eliminating the Riemann tensor components with (2.12) and (3.29) then (4.9) becomes

$$\nabla_bT_a{}^b{}_c=\Omega^3T_a{}^bT_{cb}-4T_{ac}s-\tfrac{1}{4}\Omega^3T_{bd}T^{bd}g_{ac}-\tfrac{1}{2}\Omega^2T^{bd}d_{abcd}+\tfrac{1}{3}\Omega T_{ac}R$$
$$-\tfrac{1}{2}\Omega\nabla_b\nabla^bT_{ac}+\tfrac{1}{2}\nabla_aT_{cb}\nabla^b\Omega-2\nabla_bT_{ac}\nabla^b\Omega+\tfrac{1}{2}\nabla^b\Omega\nabla_cT_{ab}. \tag{4.10}$$

The last required relation does not involve derivatives of the rescaled Cotton-York Tensor; it is essentially a version of the Bianchi identity involving the rescaled Cotton-York tensor in place of the Riemann tensor

$$T_{bcd}-T_{bdc}+T_{cdb}=0, \tag{4.11}$$

which can be verified via anti-symmetrizing the Cotton-York tensor and through direct substitution into (4.2).

4.1.2 Hodge duals

Another important quantity that is useful in dealing with antisymmetric tensorial equations, is the Hodge dual of a tensor. For the Cotton-York tensor, which is antisymmetric on the first and second indices, the Hodge dual is defined as

$$^*T_{abc}\equiv\frac{1}{2}\epsilon_{ab}{}^{de}T_{dec}, \tag{4.12}$$

where ϵ_{bqrj} is the totally antisymmetric Levi-Civita alternating tensor, which is equal

$$\sqrt{-g}\varepsilon_{abcd}, \tag{4.13}$$

where ε_{abcd} is the totally antisymmetric alternating symbol, which is equal to 1, -1 or zero (depending on whether the indices are an even or odd permutation of the first index set at the start of the calculation).

There are three important points that we should mention about both of these quantities. With regards to the Levi-Civita alternating tensor, it has the property that $\nabla_f\epsilon_{abcd}=0$, in the same manner as the metric tensor. A second important fact about the Levi-Civita tensor is the product of two different Levi-Civita tensors; in this case the precise form of the product depends on how many index contractions occur. For the results we are interested in deriving, we will consider the case where only one index is contracted, in that case the product of two different Levi-Civita tensors is equal to

$$\epsilon_{abcd}\epsilon^{pqrd} = -6\delta_a{}^{[p}\delta_b{}^{q}\delta_c{}^{r]}. \tag{4.14}$$

Before proceeding any further, it is useful to make note of a couple of properties of the Weyl tensor with respect to the Hodge dual. The first is that for a rank four tensor, like the Weyl tensor, there are in fact two different versions of the Hodge dual. The first version is the left Hodge dual of the Weyl tensor which is defined as

$$^{*}d_{abcd} = \tfrac{1}{2}\epsilon_{ab}{}^{fh}d_{fhcd}. \tag{4.15}$$

The second form of the Hodge dual is the right Hodge dual, which is defined as

$$d^{*}_{abcd} = \tfrac{1}{2}\epsilon_{cd}{}^{fh}d_{abfh}, \tag{4.16}$$

The equations (4.15) and (4.16) are in fact equivalent to one another, i.e.

$$d^{*}_{abcd} =^{*} d_{abcd},$$

this property is a unique to the Weyl tensor; it is not true for a general rank 4 tensor. Using both definitions of the dual, it is possible to obtain a second dual by performing a contraction on (4.15) and (4.16), a dual of a dual if you will. The second duals are defined as

$$^{**}d_{abcd} = \tfrac{1}{2}\epsilon_{ab}{}^{fh}\,{}^{*}d_{fhcd}, \quad d^{**}_{abcd} = \tfrac{1}{2}\epsilon_{cd}{}^{fh}d^{*}_{abfh}. \tag{4.17}$$

It can be readily verified from (4.15) that

$$^{**}d_{abcd} =^{*} d^{*}_{abcd} = d^{**}_{abcd} = -d_{abcd}, \tag{4.18}$$

which enables us to write out equivalent identities using the dual rather easily. From the above relation it follows that

$$\nabla^{a}\,{}^{*}d_{abcd} =^{*} T_{cdb}. \tag{4.19}$$

With regards to the Hodge dual itself, it encodes the same information as the tensor it was derived from, despite being a different object. This third point is important as the Hodge dual shall be used several times later on to derive key results.

4.2 Derivation of the conformal wave equations

4.2.1 Wave equation for the conformal factor

Let us now begin to derive the CWEs. First, the wave equation for the conformal factor, which can be inferred from (3.44a). Contracting said equation with the metric gives

$$\Box\Omega = -\Omega L_a{}^a + 4s + \tfrac{1}{2}\Omega^3 T_a{}^a, \tag{4.20}$$

where we have used the fact that $\delta^a_a = 4$ due to the dimension of the spacetime. Upon simplification using (2.10), (4.20) becomes an equation identical to (4.1a). So the CWE of the conformal factor with trace-free matter is identical to the CWE of conformal vacuum spacetimes. Alternatively, one could have arrived at the same equation via simple rearrangement of (3.22).

4.2.2 Wave equation for the Friedrich scalar

The next CWE is obtained by differentiating (3.44b), so that

$$\Box s = \tfrac{1}{2}\Omega^2 T_{ab}\nabla^a\nabla^b\Omega - L_{ab}\nabla^a\nabla^b\Omega + \Omega T_{ab}\nabla^a\Omega\nabla^b\Omega - \nabla^a L_{ab}\nabla^b\Omega,$$

where we have made note of the fact that T_{ab} is divergence free. Then, after applying (3.44a) and (2.18), one can verify that the above equation reduces to

$$\Box s = \tfrac{1}{4}\Omega^5 T_{ab}T^{ab} - \tfrac{1}{6}sR - \Omega^3 T^{ab}L_{ab} + \Omega L_{ab}L^{ab} - \tfrac{1}{6}\nabla_a R\nabla^a\Omega + \Omega T_{ab}\nabla^a\Omega\nabla^b\Omega. \tag{4.21}$$

4.2.3 Wave equation for the Schouten tensor

With this task completed we have now completed half of our objectives; the next step is to derive a wave equation for the Schouten tensor. This derivation is a bit trickier than the previous two as it requires slightly more sophisticated techniques. To do so we start by applying a covariant derivative to (3.44c), which gives

$$\Box L_{db} - \nabla^a\nabla_d L_{ab} = T_{cdb}\nabla^c\Omega + \Omega\nabla^c T_{cdb} + \nabla_a\Omega\nabla^c d^a{}_{bcd} + d^a{}_{bcd}\nabla^c\nabla_a\Omega. \tag{4.22}$$

Now, in order to eliminate the second order derivatives in (4.22) we make use of a particular technique. Recall the general definition of the curvature tensor for torsion free connections

$$\begin{aligned}[\nabla_a, \nabla_b]T^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_n} = {} & R^{i_1}{}_{sab}T^{s i_2 \ldots i_m}_{j_1 j_2 \ldots j_n} + R^{i_2}{}_{sab}T^{i_1 s \ldots i_m}_{j_1 j_2 \ldots j_n} + \ldots + R^{i_m}{}_{sab}T^{i_1 i_2 \ldots s}_{j_1 j_2 \ldots j_n} \\ & - R^{s}_{j_1 ab}T^{i_1 i_2 \ldots i_m}_{s j_2 \ldots j_n} - R^{s}{}_{j_2 ab}T^{i_1 i_2 \ldots i_m}_{j_1 s \ldots j_n} - \ldots - R^{s}{}_{j_n ab}T^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots s}.\end{aligned} \tag{4.23}$$

This is the most general form of the curvature when covariant derivatives are applied to a tensor of an arbitrary rank (m, n); through direct substitution this allows us to eliminate the second order derivative terms, but at the expense of introducing several curvature tensor components into the equation. As we mentioned previously in section 3.1, the Riemann tensor (and by extension the Ricci tensor) is not desirable from a conformal point of view as it is creates singular terms when transformed. However, it is possible to write out the Riemann tensor in terms of the conformally invariant Weyl and Schouten tensors, using (2.12).

Making use of (4.23) to deal with the second term in (4.22) followed by the application of (3.44a) and (2.10) to eliminate the second order derivatives and finally using (2.12) and (3.29) to eliminate the Riemann and Ricci tensor components and we get

$$\begin{aligned}\Box L_{bd} = \tfrac{1}{2}\Omega^3 T^{ac} d_{badc} - 2\Omega d_{badc} L^{ac} - g_{bd} L_{ac} L^{ac} + 4 L_b{}^a L_{da} \\ - \Omega \nabla_a T_d{}^a{}_b + \nabla_c d_{bad}{}^c + \tfrac{1}{6}\nabla_d \nabla_b R,\end{aligned} \tag{4.24}$$

or, equivalently by using (3.44d)

$$\begin{aligned}\Box L_{bd} = \tfrac{1}{2}\Omega^3 T^{ac} d_{badc} - 2\Omega d_{badc} L^{ac} - g_{bd} L_{ac} L^{ac} + 4 L_b{}^a L_{da} \\ - \Omega \nabla_a T_d{}^a{}_b - T_{bad} \nabla_c \Omega + \tfrac{1}{6}\nabla_d \nabla_b R.\end{aligned} \tag{4.25}$$

At first glance it appears that this is not a wave equation, as the last term in the equation involves a second order derivative and consequently the principal part of the differential equation is not the d'Almbertian; a necessary condition for a wave equation. However, this second order derivative is applied to the Ricci scalar, which is a special case from the point of view of conformal methods. When we examine the CFEs we notice that the Ricci scalar does not appear in any of the equations, meaning that the Ricci scalar has the form of a gauge. As a result of this the Ricci scalar does not affect the equations in any way and (4.25) is indeed a valid wave equation.

4.2.4 Wave equation for the rescaled Weyl tensor

We are almost done, the last equation that we need to compute is the wave equation for the Weyl tensor, which is significantly more complicated than any of the derivations thus far. The starting point of this highly non-trivial calculation is the second Bianchi identity written in terms of the Weyl tensor

$$\nabla_b d_{fhpr} - \nabla_q d_{fhbr} + \nabla_r d_{fhbq} = -\epsilon_{bqra}{}^* T_{fh}{}^a. \tag{4.26}$$

As in the previous derivations we start by applying a covariant derivative to (4.26), which gives

$$\Box d_{fhqr} - \nabla^b \nabla_q d_{fhbr} + \nabla^b \nabla_r d_{fhbq} = -\epsilon_{bqra} \nabla^{b*} T_{fh}{}^a. \tag{4.27}$$

Re-arranging (4.27) and making use of (4.23) in the same manner as in the derivation of the wave equation for the Schouten tensor and one ends up with

$$\begin{aligned} \Box d_{fhqr} = &-d_{fhrj} R_q{}^j + d_{fhqj} R_r{}^j - d_h{}^b{}_r{}^j d_{fbqj} + d_h{}^b{}_q{}^j + d_f{}^b{}_r{}^j R_{hbrj} \\ &-d_f{}^b{}_q{}^j R_{hbrj} - 2 d_{fh}{}^{bj} R_{qbrj} + \epsilon_{qrbj} \nabla^{j*} T_{fh}{}^b + \nabla_q T_{fhr} - \nabla_r T_{fhq}, \end{aligned} \tag{4.28}$$

we then apply (2.12), which produces an equation containing several terms involving products of the Weyl and Schouten tensor:

$$\begin{aligned} \Box d_{fhqr} = &\, 2\Omega d_f{}^b{}_r{}^p d_{hbqp} - 2\Omega d_f{}^b{}_q{}^p d_{hbrp} - 2\Omega d_{fh}{}^{bp} d_{qbrp} + \tfrac{1}{3} d_{fhqr} R - g_{hr} d_{fbqp} L^{bp} \\ &+ g_{hq} d_{fbrp} L^{bp} + g_{fr} d_{hbqp} L^{bp} - g_{fq} d_{hbrp} L^{bp} + d_{hqrb} L_f{}^b - d_{hrqb} L_f{}^b - d_{fqrb} L_h{}^b \\ &+ d_{frqb} L_h{}^b + d_{fbhr} L_q{}^b - d_{frhb} L_q{}^b - d_{fbhq} L_r{}^b + d_{fqhb} L_r{}^b + \nabla_q T_{fhr} - \nabla_r T_{fhq}. \end{aligned} \tag{4.29}$$

In the derivation of the vacuum CWEs it was possible to eliminate any terms containing the Schouten tensor. Since any result we obtain should reduce to the equivalent vacuum equation upon setting both the energy-momentum and rescaled Cotton terms to zero, one should logically be able to do the same with (4.29). The starting point of this simplification is the second Bianchi identity written in terms of the Riemann tensor which, upon substituting (2.12) into (2.17) becomes

$$\begin{aligned} &d_{bcdf} \nabla_a \Omega + \Omega \nabla_a d_{bcdf} + g_{fc} \nabla_a L_{db} - g_{fb} \nabla_a L_{dc} - g_{dc} \nabla_a L_{fb} + g_{db} \nabla_a L_{fc} \\ &- d_{acdf} \nabla_b \Omega - \Omega \nabla_b d_{acdf} - g_{fc} \nabla_b L_{da} + g_{fa} \nabla_b L_{dc} + g_{dc} \nabla_b L_{fa} - g_{da} \nabla_b L_{fc} \\ &+ d_{abdf} \nabla_c \Omega + \Omega \nabla_c d_{abdf} + g_{fb} \nabla_c L_{da} - g_{fa} \nabla_c L_{db} - g_{db} \nabla_c L_{fa} + g_{da} \nabla_c L_{fb} = 0. \end{aligned} \tag{4.30}$$

We will make use of this equation to eliminate the dependence of the Schouten tensor in (4.29). Before that, however, we will express (4.30) in a more convenient form. We notice that (4.30) contains derivatives of the Schouten tensor, all of which occur in pairs that are antisymmetric. We can therefore make use of (3.44c) to eliminate the derivatives of the Schouten tensor, at the expense of introducing several Cotton-York and Weyl tensor terms. Upon multiple applications of (3.44c), (4.30) becomes

$$\begin{aligned} &\Omega g_{cf} T_{abd} - \Omega g_{cd} T_{abf} + \Omega g_{af} T_{bcd} - \Omega g_{ad} T_{bcf} - \Omega \epsilon_{abch} {}^* T_{df}{}^h + g_{ad} d_{bcfh} \nabla^h \Omega \\ &- \Omega g_{bf} T_{acd} + d_{bcdf} \nabla_a \Omega - d_{acdf} \nabla_b \Omega + d_{abdf} \nabla_c \Omega - g_{cf} d_{abdh} \nabla^h \Omega + g_{cd} d_{abfh} \nabla^h \Omega \end{aligned}$$

$$+\Omega g_{bd}T_{acf} + g_{bf}d_{acdh}\nabla^h\Omega - g_{bd}d_{acfh}\nabla^h\Omega - g_{af}d_{bcdh}\nabla^h\Omega = 0. \quad (4.31)$$

At this point, we examine (4.29) and notice that it contains derivatives of the rescaled Cotton tensor; we therefore make an educated guess and differentiate (4.31). This gives a long and complicated expression containing several second order derivatives of the conformal factor and derivatives of the Weyl tensor, which may be simplified with the use of (3.44d) and (4.1a). Following this, we express the Hodge dual of the rescaled Cotton tensor in terms of the standard rescaled Cotton Tensor. Upon doing so and making use of (4.4) we get

$$\begin{aligned} g_{cf}d_{badh}L^{ah} &= -\tfrac{1}{2}\Omega^2 T_c{}^a d_{badf} + \tfrac{1}{2}\Omega^2 T^{ah} g_{cf} d_{badh} - \tfrac{1}{2}\Omega^2 T^{ah} g_{cd} d_{bafh} - \tfrac{1}{2}\Omega^2 T_f{}^a d_{bcda} \\ &+\tfrac{1}{2}\Omega^2 T_d{}^a d_{bcfa} + \tfrac{1}{2}\Omega^2 T_b{}^a d_{cadf} - \tfrac{1}{2}\Omega^2 T^{ah} g_{bf} d_{cadh} + \tfrac{1}{2}\Omega^2 T^{ah} g_{bd} d_{cafh} + g_{cd} d_{bafh} L^{ah} \\ &+g_{bf} d_{cadh} L^{ah} - g_{bd} d_{cafh} L^{ah} - d_{cadf} L_b{}^a + d_{badf} L_c{}^a - d_{bcfa} L_d{}^a + d_{bcda} L_f{}^a - \tfrac{1}{6} d_{bcdf} R. \end{aligned} \quad (4.32)$$

Substituting (4.32) into (4.29) and making use of the first Bianchi identity to eliminate terms yields the desired wave equation for the Weyl tensor

$$\begin{aligned} \Box d_{fhqr} &= 2\Omega^2 T^{bp} g_{(r[f} d_{h]|b|q)r} + \Omega^2 T^b{}_{[r} d_{q]bfh} + \Omega^2 T^b{}_{[h} d_{f]bqr} + 4\Omega d_f{}^b{}_{[r}{}^{|p|} d_{q]phb} \\ &-2\Omega d_{fh}{}^{bp} d_{qbrp} + \tfrac{1}{2} d_{fhqr} R + \epsilon g_{qrbp} \nabla^p\, {}^*T_{fh}{}^b + \nabla_q T_{fhr} - \nabla_r T_{fhq}, \end{aligned} \quad (4.33)$$

or in a more compact form,

$$\begin{aligned} \Box d_{fhqr} &= \zeta_{fhqr} + 2\Omega^2 T^{bp} g_{(r[f} d_{h]|b|q)r} + \Omega^2 T^b{}_{[r} d_{q]bfh} + \Omega^2 T^b{}_{[h} d_{f]bqr} \\ &+\epsilon g_{qrbp} \nabla^p\, {}^*T_{fh}{}^b + \nabla_q T_{fhr} - \nabla_r T_{fhq}, \end{aligned} \quad (4.34)$$

where ζ is the equivalent vacuum CWE term written as a single tensor, i.e.

$$\zeta_{fhqr} \equiv 4\Omega d_f{}^b{}_{[r}{}^{|p|} d_{q]phb} - 2\Omega d_{fh}{}^{bp} d_{qbrp} + \tfrac{1}{2} d_{fhqr} R.$$

Written in the form given by (4.34) one can clearly see that this reduces to the equivalent vacuum equation given in [21].

4.2.5 The wave equation for λ

One thing that has not been mentioned thus far is the wave equation for the cosmological constant in the fifth CFE. It turns out that the wave equation for the cosmological constant is in fact trivial. This can be easily verified from section 3.1 where we showed that in fact $\nabla_a \lambda = 0$, hence when one applies a derivative to (3.41) the result is

$$\Box\lambda = 0. \tag{4.35}$$

Summary

We have successfully derived a set of wave equations that described the evolution of the variables of the CFEs; our analysis is completely general with the only assumption being that the matter content of the spacetime is trace-free and that the CFEs hold.

4.3 The subsidiary equations

We have derived a set of wave equations that describe the evolution of conformal fields coupled to trace-free matter models. As mentioned beforehand, we have assumed only that the matter is trace-free and that the CFEs are true. However, this is not the end of the analysis, since it is not clear whether any solution that one may obtain for the wave equations is also a solution to the field equations. In order to verify that this is indeed the case, we must construct a system of subsidiary equations. In essence, what this means is that instead of assuming that the field equations are true and the wave equations are not, one assumes the wave equations are true and the field equations are not true. To that end one uses (3.44a)-(3.44e) to define

$$Z_{ab} \equiv \nabla_a\nabla_b\Omega + \Omega L_{ab} - sg_{ab} - \frac{1}{2}\Omega^3 T_{ab}, \tag{4.36a}$$

$$Z_a \equiv \nabla_a s + L_{ac}\nabla^c\Omega - \frac{1}{2}\Omega^2\nabla^c\Omega T_{ac}, \tag{4.36b}$$

$$\Delta_{cdb} \equiv \nabla_c L_{db} - \nabla_d L_{cb} - \nabla_a\Omega d^a{}_{bcd} - \Omega T_{cdb}, \tag{4.36c}$$

$$\Lambda_{cdb} \equiv \nabla_a d^a{}_{bcd} - T_{cdb}, \tag{4.36d}$$

$$\Upsilon \equiv 6\Omega s - 3\nabla_c\Omega\nabla^c\Omega - \frac{1}{4}\Omega^4 T - \lambda. \tag{4.36e}$$

One then proceeds to compute a series of wave equations for the *subsidiary fields* $Z_{ab}, Z_a, \Delta_{cdb}, \Lambda_{cdb}, \Upsilon$. In order for any solution of the wave equations to solve the field equations, then any initial data chosen has to be special. So to see that this is the case we make use of the subsidiary system. The main point of the wave equations is that they have to be homogeneous in the subsidiary fields. This is due to a well known property of wave equations that if one has a homogeneous wave equation and one chooses the homogeneous variable to be zero initially, then that solution vanishes at all times. In fact, not only does it vanish, but it is the only possible solution. There is a multi-variable version of this theory that states if you have a system that is homogeneous in a certain set of variables and one chooses all of these variables to be zero initially then this choice of initial conditions is valid at all times. Let us specify what we mean by homogeneous, from our point of view homogeneous means that each term in the RHS of the equation is a product of one of the zero quantities defined in equations (4.36a)-(4.36e).

So let us consider the hypothetical case where one has successfully derived a set of homogeneous wave equations for the subsidiary variables, what does this imply? Recall the definitions of the subsidiary variables, if we say that the only possible solution to this system of wave equations is that the subsidiary variables are zero at all times, then we are saying that the CFEs are satisfied at all times. Furthermore, if one makes use of the CWEs in the derivation of these wave equations, this proves that any solution to the CWEs satisfies the CFEs; this technique is known as the propagation of the constraints. More precisely, the constraint equations only need to be solved initially as long as the CWEs hold. Let us now show that this is indeed the case.

4.3.1 Wave equation for Υ

We will start with equation (4.36a) as that is the simplest case and it helps to illustrate the techniques used in the later, less elegant derivations. We begin by applying the d'Alembertian operator to (4.36a), recalling that the wave equation for λ is trivial then one obtains

$$\square\Upsilon = 6\square(\Omega s) - 3\square(\nabla_c\Omega\nabla^c\Omega). \tag{4.37}$$

Expanding with the Leibnitz rule and (4.37) becomes

$$\square\Upsilon = 6s\square\Omega + 6\Omega\square s + 12\nabla_c s\nabla^c\Omega - 6\nabla^c\Omega\square\nabla_c\Omega - 6\nabla_h\nabla_c\Omega\nabla^h\nabla^c\Omega \tag{4.38}$$

Upon using (4.1a) and (4.21), then (4.38) becomes

$$\begin{aligned}\square\Upsilon = {}& \frac{3}{2}\Omega^6 T_{ch}T^{ch} + 24s^2 - 2\Omega sR - 6\Omega^4 T^{ch}L_{ch} + 6\Omega^2 L_{ch}L^{ch} + 12\nabla_c s\nabla^c\Omega \\ & -\Omega\nabla_c R\nabla^c\Omega - 6\nabla^c\Omega\nabla_h\nabla^h\nabla_c\Omega + 6\Omega^2 T_{ch}\nabla^c\Omega\nabla^h\Omega - 6\nabla_h\nabla_c\Omega\nabla^h\nabla^c\Omega.\end{aligned} \tag{4.39}$$

Then commuting the covariant derivatives with (4.23) as before and using (3.29) to remove the curvature terms, then (4.39) becomes

$$\begin{aligned}\square\Upsilon = {}& \frac{3}{2}\Omega^6 T_{ch}T^{ch} + 24s^2 - 2\Omega sR - 6\Omega^4 T^{ch}L_{ch} + 6\Omega^2 L_{ch}L^{ch} - R\nabla_c\Omega\nabla^c\Omega + 12\nabla_c s\nabla^c\Omega \\ & -\Omega\nabla_c R\nabla^c\Omega - 6\nabla_c\nabla_h\nabla^h\Omega\nabla^c\Omega + 6\Omega^2 T_{ch}\nabla^c\Omega\nabla^h\Omega - 12L_{ch}\nabla^c\Omega\nabla^h\Omega - 6\nabla_h\nabla_c\Omega\nabla^h\nabla^c\Omega.\end{aligned} \tag{4.40}$$

Once again, there is the problem of second order derivatives. However, we cannot make use of (3.44a) to eliminate these derivatives since one of our prior assumptions was that the CFEs are not satisfied. Instead, we make use of the subsidiary system to eliminate the higher order derivatives. Rearranging (4.36a) and (4.36b) gives

$$\nabla_a \nabla_b \Omega = Z_{ab} - \Omega L_{ab} + s g_{ab} + \frac{1}{2}\Omega^2 T_{ab}, \tag{4.41}$$

$$\nabla_a s = Z_a - L_{ac}\nabla^c \Omega + \frac{1}{2}\Omega^2 \nabla^c \Omega T_{ac}. \tag{4.42}$$

Upon substituting both (4.41) and (4.42) along with (2.10) into (4.40) yields

$$\Box \Upsilon = -6\Omega^3 T^{ch} Z_{ch} + 12\Omega L^{ch} Z_{ch} - 12 s Z^c{}_c - 6 Z_{ch} Z^{ch} - 12 Z^c \nabla_c \Omega - 6 \nabla_c Z^h{}_h \nabla^c \Omega. \tag{4.43}$$

A relatively straightforward calculation shows that upon contracting with the metric, equation (4.41) becomes

$$Z_c{}^c = 0. \tag{4.44}$$

Making use of the above and we obtain

$$\Box \Upsilon = -6\Omega^3 T^{ch} Z_{ch} + 12\Omega L^{ch} Z_{ch} - 6 Z_{ch} Z^{ch} - 12 Z^c \nabla_c \Omega, \tag{4.45}$$

which is homogeneous in the subsidiary fields, as required.

4.3.2 Wave equation for Z_{ab}

It can be shown through a similar, but more lengthy calculation, that the wave equations for both Z_{ab} and Z_a are homogeneous in the subsidiary variables. We will derive the wave equation for Z_{ab} first as it is the simpler of the two. Applying the d'Alembertian operator to (4.36a) and we get

$$\begin{aligned}\Box Z_{ab} = \frac{3}{2}\Omega^2 T_{ab}\Box\Omega + L_{ab}\Box\Omega - \frac{1}{2}\Omega^3 \Box T_{ab} - g_{ab}\Box s + \Omega \Box L_{ab} + \Box \nabla_a \nabla_b \Omega \\ -3\Omega T_{ab}\nabla_c \Omega \nabla^c \Omega - 3\Omega^2 \nabla_c T_{ab} \nabla^c \Omega + 2\nabla_c L_{ab} \nabla^c \Omega.\end{aligned} \tag{4.46}$$

To simplify this, we need to make use of some additional pieces of information. First of all, just like in the calculation of (4.45), there are several terms that can be eliminated by rearranging (4.36c) and (4.36d) into

$$\nabla_c L_{db} = \Delta_{cdb} + \nabla_d L_{cd} + \nabla_a d^a{}_{bcd} + \Omega T_{cdb}, \tag{4.47}$$

$$\nabla_a d^a{}_{bcd} = \Lambda_{cdb} + T_{cdb}. \tag{4.48}$$

Another useful identity, which is essentially an alternative form of (2.18), comes from performing a contraction on (4.36c) with the metric tensor. Upon doing so, one can verify, using (2.10) and (4.4), that

$$\nabla_c L_d{}^c = \Delta_{cd}{}^d + \tfrac{1}{6}\nabla_d R. \tag{4.49}$$

Making use of (4.23) to commute the covariant derivatives and using (4.1a), (2.10), (2.12), (3.29), (4.21), (4.25), (4.41), (4.42) and (4.49) to eliminate terms accordingly and one ends with

$$\begin{aligned}\Box Z_{ab} = \frac{1}{6}RZ_{ab} + 4L_b{}^c Z_{ac} - 2g_{ab}L_{cd}Z^{cd} - 2\Omega d_{acbd}Z^d - \Delta_a{}^c{}_c\nabla_b\Omega \\ +4\nabla_b Z_a + \Omega\Lambda_{bac}\nabla^c\Omega + \Delta_{bca}\nabla^c\Omega + g_{ab}\Delta_c{}^d{}_d\nabla^c\Omega + M_{ab},\end{aligned} \tag{4.50}$$

where M_{ab} represents the matter terms. Naturally $M_{ab} = 0$ when working with vacuum spacetimes, which shows that the wave equation for vacuum spacetimes is homogeneous and the propagation of the constraints is satisfied, as one would expect. M_{ab} is defined as

$$\begin{aligned}M_{ab} \equiv \Omega^2 T_a{}^c T_{bc} - 4\Omega^2 T_{ab}s - \frac{1}{4}\Omega^5 T_{cd}T^{cd}g_{ab} - \frac{1}{2}\Omega^4 T^{cd}d_{abcd} + \frac{1}{3}\Omega^3 T_{ab}R - \Omega^2\nabla_c T_b{}^c{}_a \\ -\frac{1}{2}\Omega^3\Box T_{ab} + \Omega T_{abc}\nabla^c\Omega - 2\Omega T_{bca}\nabla^c\Omega + 4\Omega T_{ac}\nabla_b\Omega\nabla^c\Omega + 2\Omega^2\nabla_b T_{ac}\nabla^c\Omega \\ -3\Omega T_{ab}\nabla_c\Omega\nabla^c\Omega - 3\Omega^2\nabla_c T_{ab}\nabla^c\Omega - 3\Omega^2\nabla_c T_{ab}\nabla^c\Omega - \Omega T_{cd}g_{ab}\nabla^c\Omega\nabla^d\Omega,\end{aligned} \tag{4.51}$$

so we need to show that M_{ab} is homogeneous in the subsidiary variables. Immediately we notice a problem, there is a term in (4.51) containing a d'Alembertian applied to the energy-momentum tensor. In order to eliminate this we recall the definition of the rescaled Cotton tensor. We first notice that (4.2) is written in terms of first order derivatives of the energy-momentum tensor, and that (4.51) contains a contracted first order of the rescaled Cotton tensor. This suggests that it might be possible to use (4.2) to cancel out the second order derivatives of the energy-momentum tensor, which does indeed turn out to be the case. Substituting (4.2) into (4.51) and simplifying then one obtains

$$\begin{aligned}M_{ab} = \Omega^5 T_a{}^c T_{bc} - 4\Omega T_{ab}s - \frac{1}{4}\Omega^5 T_{cd}T^{cd}g_{ab} - \frac{1}{2}\Omega^4 T^{cd}d_{acbd} + \frac{1}{3}\Omega^3 T_{ab}R \\ -\frac{1}{2}\Omega^2 T_b{}^c\nabla_c\nabla_b\Omega - \frac{1}{2}\Omega^3\nabla_c\nabla_b T_a{}^c + \frac{1}{2}\Omega^2 T^{cd}g_{ab}\nabla_d\nabla_c\Omega + \frac{3}{2}\Omega^2 T_{ab}\Box\Omega.\end{aligned} \tag{4.52}$$

Using both (4.1a) and (4.41), commuting the covariant derivatives as before then eliminating the Ricci and Riemann tensor components and the matter terms simplify to

$$M_{ab} = -\frac{1}{2}\Omega^2 T_b{}^c Z_{ac} - \frac{3}{2}\Omega^2 T_a{}^c Z_{bc}. \tag{4.53}$$

Combining (4.50) and (4.53) and one obtains

$$\Box Z_{ab} = \frac{1}{6}RZ_{ab} - \frac{1}{2}\Omega^2 T^c_b Z_{ac} + 4L^c_b Z_{ac} + \frac{1}{2}\Omega^2 T^{cd} g_{ab} Z_{cd} - 2g_{ab}L_{cd}Z^{cd} - 2\Omega d_{acbd}Z^{cd}$$
$$-\Delta_a{}^c{}_c \nabla_b \Omega + 4\nabla_b Z_a + \Omega \Lambda_{bac}\nabla^c\Omega + 2\Delta_{abc}\nabla^c\Omega - \Delta_{acb}\nabla^c\Omega + g_{ab}\Delta_c{}^d{}_d\nabla^d\Omega, \tag{4.54}$$

and the wave equation for Z_{ab} with trace-free matter is homogeneous.

4.3.3 Wave Equation for Z_a

Next, we derive the wave equation for Z_a, so start by applying the d'Alembertian to (4.36b), which gives

$$\Box Z_a = \Box\nabla_a s - 2\Omega T_a{}^c \nabla^b\Omega\nabla_c\nabla_b\Omega - \Omega T_{ab}\nabla^b\Box\Omega - \frac{1}{2}\Omega^2\nabla^b\Box T_{ab}$$
$$+\nabla^b\Omega\Box L_{ab} - \frac{1}{2}\Omega^2 T_a{}^b\Box\nabla_b\Omega + L_a{}^b\Box\nabla_b\Omega - T_{ac}\nabla_b\Omega\nabla^b\Omega\nabla^c\Omega$$
$$-2\Omega\nabla^b\nabla_c T_{ab}\nabla^c\Omega - \Omega^2\nabla_c T_{ab}\nabla^c\nabla^b\Omega + 2\nabla_c L_{ab}\nabla^c\nabla^b\Omega, \tag{4.55}$$

and using exactly the same method as in the derivation of (4.54) to simplify (4.55) then one obtains

$$\Box Z_a = -2\Omega^2 T_{ab}Z^b + 6L_{ab}Z^b - \Omega^3 T^{bc}\Delta_{abc} + 2\Omega L^{bc}\Delta_{abc} - 2s\Delta_a{}^b{}_b + \frac{1}{2}\Omega^2 T_{bc}Z^{bc}\nabla_a\Omega$$
$$+\frac{3}{2}\Omega T_b{}^c Z_{ac}\nabla^b\Omega - \frac{7}{2}\Omega T_a{}^c Z_{bc}\nabla^b\Omega - \frac{1}{6}Z_{ab}\nabla^b R - \Omega^2 Z^{bc}\nabla_c T_{ab} + 2Z^{bc}\nabla_c L_{ab}. \tag{4.56}$$

So the wave equation for Z_a turns out to be homogeneous as required.

4.3.4 Wave Equation for Λ_{bcd}

The next wave equation is the the one for Λ_{bcd}; this is considerably more difficult to derive. We begin by differentiating (4.36d)

$$\Box\Lambda_{bcd} = -\Box T_{cbd} + \Box\nabla_a d^a{}_{bcd}, \tag{4.57}$$

then upon commuting the covariant derivatives with (4.23), one can verify that (4.57) becomes

$$\Box\Lambda_{bcd} = d_{bjcd}\nabla_a R^{aj} - \Box T_{cdb} - \nabla_a \Box d_b{}^a{}_{cd} + 2R_d{}^{ajf}\nabla_f d_{bjca} - 2R_c{}^{ajf}\nabla_f d_{bjda}$$
$$+2R_b{}^{ajf}\nabla_f d_{cdaj} + d_{cd}{}^{aj}\nabla_f R_{baj}{}^f - d_b{}^a{}_d{}^j\nabla_f R_{cja}{}^f + d_b{}^a{}_c{}^j\nabla_f R_{dja}{}^f + R^{aj}\nabla_j d_{bacd}. \quad (4.58)$$

In order to proceed further one needs to make use of several relationships that were derived in section 4.1.1; specifically the equations relating derivatives of the rescaled Cotton tensor. If one substitutes (2.12), (3.29), (4.33), (4.49), along with (4.47) to eliminate the derivatives of the Schouten tensor along with (2.30) to eliminate terms quadratic in the Weyl tensor, then one obtains

$$\Box\Lambda_{bcd} = 2L_b{}^a\Lambda_{acd} + \Omega d_{cdah}\Lambda^a{}_b{}^h - \Omega d_{badh}\Lambda^a{}_c{}^h + \Omega d_{bach}\Lambda^a{}_d{}^h - 2\Omega d_{cadh}\Lambda_b{}^{ah}$$
$$+\tfrac{1}{3}R\Lambda_{bcd} - 2L_d{}^a\Lambda_{cba} + 2L_c{}^a\Lambda_{dba} - 2\epsilon_{cdhf}L^{ah}\,{}^*\Lambda^f{}_{ba} - d_{bcdh}\Delta^{ah}{}_a + d_{bdch}\Delta^{ah}{}_a$$
$$+d_{bhcd}\Delta^{ah}{}_a - d_{cdah}\Delta_b{}^{ah} + d_{bhda}\Delta_c{}^{ah} - d_{bhca}\Delta_d{}^{ah} + M_{bcd}, \quad (4.59)$$

where M_{bcd} is shorthand for the matter terms in the equation. Naturally, this tensor is zero in the vacuum case, which means (4.59) reduces to

$$\Box\Lambda_{bcd} = V_{bcd}, \quad (4.60)$$

where V_{bcd} is a tensor that is a shorthand notation for the all vacuum terms of the equation which, as we can see from (4.59), is equal to

$$V_{bcd} \equiv 2L_b{}^a\Lambda_{acd} + \Omega d_{cdah}\Lambda^a{}_b{}^h - \Omega d_{badh}\Lambda^a{}_c{}^h + \Omega d_{bach}\Lambda^a{}_d{}^h$$
$$-2\Omega d_{cadh}\Lambda_b{}^{ah} + \tfrac{1}{3}R\Lambda_{bcd} + 4L_{[c}{}^{|a|}\Lambda_{d]ba} - 2\epsilon_{cdhf}L^{ah}\,{}^*\Lambda^f{}_{ba}$$
$$+2d_{bhcd}\Delta^{ah}{}_a - d_{cdah}\Delta_b{}^{ah} + d_{bhda}\Delta_c{}^{ah} - d_{bhca}\Delta_d{}^{ah}. \quad (4.61)$$

We can see that for vacuum spacetimes the wave equation for Λ_{bcd} is homogeneous. With this part of the job done, we now need to check if the matter tensor M_{bcd} is homogeneous in the subsidiary variables. To start we notice that M_{bcd} contains derivatives of the rescaled Cotton tensor, however these cannot be simplified using (4.8) and (4.6) since both of those relations were derived using the CFEs and one of our prior assumptions for this analysis is that the CFEs are not automatically satisfied. Hence, we need to derive an alternative form of all the equations in section 4.1.1. Essentially, we need to repeat the derivations of these equations using (4.41) in place of (3.42a) to simplify the equations where necessary. Upon doing so we obtain

$$\nabla^c T_{abc} = T_b{}^c Z_{ac} - T_a{}^c Z_{bc}, \tag{4.62a}$$

$$\nabla_c \nabla_b T_a{}^{bc} = -\Omega T^{bcd} d_{abcd} + T_a{}^{bc} L_{bc}$$
$$-Z^{bc} \nabla_c T_{ab} + T^{bc} \nabla_c Z_{ab} - T_a{}^b \nabla_c Z_b{}^c, \tag{4.62b}$$

$$\nabla_b T_a{}^b{}_c = \Omega^3 T_a{}^b T_{cb} - 4T_{ac} s - \tfrac{1}{4}\Omega^3 T_{bd} T^{bd} g_{ac} - \tfrac{1}{2}\Omega^2 T^{bd} d_{abcd}$$
$$+\tfrac{1}{3}\Omega T_{ac} R + \tfrac{3}{2} T_c{}^b Z_{ab} - \tfrac{1}{2} T^{bd} g_{ac} Z_{bd} + \tfrac{1}{2} T_a{}^b Z_{cb}$$
$$-\tfrac{1}{2}\Omega \Box T_{ac} + \tfrac{1}{2}\nabla_a T_{cb} \nabla^b \Omega - 2\nabla_b T_{ac} \nabla^b \Omega + \tfrac{1}{2}\nabla^b \Omega \nabla_c T_{ab}. \tag{4.62c}$$

We then apply (4.62c) and (4.62a) to the derivatives of the rescaled Cotton tensor in M_{bcd}; (4.14), to eliminate the dual terms of the rescaled Cotton Tensor; (4.11), to eliminate products of terms that are created as a result removing the dual terms; and the second Bianchi identity, to eliminate derivatives of the Weyl tensor. Upon doing so, one ends up with

$$M_{bcd} = \Omega T_{[d}{}^{|h|} d_{c]hba} \nabla^a \Omega + \Omega T_{[a}{}^{|h|} d_{b]hcd} \nabla^a \Omega + \Omega T^{hf} g_{b[d} d_{c]haf} \nabla^a \Omega + \Omega T^{ah} d_{ba[d|h|} \nabla_{c]} \Omega, \tag{4.63}$$

where we have ignored any terms containing the subsidiary variables. We next notice that all terms in (4.63) are antisymmetric on two of the indices, which we can take advantage of by using the Hodge duals. Multiplying (4.63) by a Levi-Civita alternating tensor with a contraction on the second and third free indices of the matter tensor, using (4.18) to re-express the Weyl tensor in terms of the duals, expanding said dual terms and simplifying using (2.30) and it can be verified that the matter terms are homogeneous in the subsidiary variables. Combining these terms with (4.61) and the full wave equation for Λ_{bjl} with trace-free matter can be verified to be

$$\Box \Lambda_{bjl} = V_{bcd} + \tfrac{1}{2}\Omega^2 T^{ac} \varepsilon_{jlcd}\, {}^*\Lambda^d{}_{ba} + 6T_{[l|a|} g_{j]b} Z^a + 2\Omega T_{[l}{}^{|a|} g_{j]b} \Delta_a{}^c{}_c + 2Z_{[l}{}^{|a|} \nabla_{|b|} T_{j]a}$$
$$+2T_{[j}{}^{|a|} \nabla_{|b|} Z_{l]a} + 2Z^{ac} g_{b[j} \nabla_{|c|} T_{l]a} + 2T^{ac} g_{b[l} \nabla_{|c|} Z_{j]a} + 2Z_{[l}{}^{|a|} \nabla_{|j|} T_{b]a} + 2T_{[b}{}^{|a|} \nabla_{|j|} Z_{l]a}$$
$$+2Z_{[b}{}^{|a|} \nabla_{|l|} T_{j]a} + 2T_{[j}{}^{|a|} \nabla_{|l|} Z_{b]a}. \tag{4.64}$$

We can see that every single term whose commutator brackets are expanded will be multiplied by one of the subsidiary variables and hence the wave equation is homogeneous as required.

4.3.5 Wave Equation for Δ_{bcd}

The last thing that we need to do is the propagation of the constraints for the third subsidiary variable Δ_{bcd}, which is the most difficult calculation to do by far. Begin, by differentiating (4.36c), which yields

$$\begin{aligned}\Box\Delta_{cdb} = -T_{cdb}\Box\Omega - \Omega\Box T_{cdb} - \nabla_a\Omega\Box d^a{}_{bcd} - d^a{}_{bcd}\Box\nabla_a\Omega + \Box\nabla_c L_{db} \\ -\Box\nabla_d L_{cb} - \nabla_j T_{cdb}\nabla^j\Omega - \nabla_j\Omega\nabla^j T_{cdb} - \nabla_j\nabla_a\Omega\nabla^j d^a{}_{bcd} - \nabla_j d^a{}_{bcd}\nabla^j\nabla_a\Omega.\end{aligned} \quad (4.65)$$

From here we proceed in a similar manner to the derivation for the wave equation for the fourth Subsidiary variable, with the main difference being a much larger number of terms to deal with plus one additional relation that is required to eliminate some very specific terms. First, applying (4.23), (2.12), (3.29), (4.1a), (4.25), (4.33), (4.41), (4.47), (4.42), (4.49), (4.26), (2.10) to start with, then dealing with all the matter terms by using (4.14) to expand the Hodge dual terms, (4.4), (4.62c), (4.2) and using (4.23) to deal with terms that have multiple derivatives applied to the energy-momentum tensor (which causes the cancellation of several derivatives at the expense of introducing several curvature components that must be eliminated with (2.12)) and finally both the Bianchi identities to cancel out several groups of terms, then one ends up with an equation that, whilst still relatively unappealing, is significantly less complex than the sum of its parts as several cancellations occur. This suggests that we are on the correct path of this derivation. When examining the remaining terms, we notice that there are terms that contain products of the metric tensor, the Weyl tensor and derivatives of the conformal factor, which we can eliminate by rearranging (4.30) and direct substitution. This action decreases the number of terms even further, ignoring the subsidiary variable terms we are left with

$$\begin{aligned}2\Omega^2 T_a{}^j d_{[c|bj|d]}\nabla^a\Omega + 6\Omega^2 T_b{}^j d_{c[a|d|j]}\nabla^a\Omega - 2\Omega d_{bajf} d_c{}^j{}_d{}^f\nabla^a\Omega \\ +\tfrac{1}{3}(d_{bacd} + d_{bcda} - d_{bdca})R\nabla^a\Omega + 2L_a{}^j(d_{bdcj} - d_{bcdj} - d_{bjcd})\nabla^a\Omega \\ +3L_b{}^j(d_{cdaj} - d_{cadj} + d_{cjda})\nabla^a\Omega + 2\Omega^3 d_{c[j|d|a]}\nabla^j T_b{}^a + 4\Omega d_{c[a|d|j]}\nabla^j L_b{}^a.\end{aligned} \quad (4.66)$$

We can immediately see that there are several terms that can be eliminated using the Bianchi identity; upon doing so (4.66) becomes

$$\begin{aligned}\Omega^2 T_a{}^j d_{bcdj}\nabla^a\Omega - \Omega^2 T_a{}^j d_{bdcj}\nabla^a\Omega + 3\Omega^2 T_b{}^j d_{cadj}\nabla^a\Omega - 3\Omega^2 T_b{}^j d_{cjda}\nabla^a\Omega \\ +\Omega^3 d_{cjda}\nabla^j T_b{}^a - 2\Omega d_{bajf} d_c{}^j{}_d{}^f\nabla^a\Omega - \Omega^3 d_{cadj}\nabla^j T_b{}^a + 4\Omega d_{c[a|d|j]}\nabla^j L_b{}^a.\end{aligned} \quad (4.67)$$

Finally, by substituting (4.47) into the derivatives of the Schouten tensor and then substituting (4.2) into the single rescaled Cotton tensor term that follows from the previous action and those remaining terms in (4.67) vanish. Hence, the only terms left are those containing the subsidiary variables and the equation is homogeneous as required.

Summary

We have successfully derived a system of quasilinear wave equations that describe the evolution of conformal fields of 4-dimensional conformally rescaled spacetimes. Our analysis is completely general with the only assumptions being of the nature of the matter content, namely that the matter content is trace-free. Furthermore, we have shown that assuming that the wave equations are satisfied leads to the conclusion that the field equations are true. As the field equations are already a known result, this proves that the CWEs and the CFES are not independent of one another. Consequently, this means that any solution of the CWEs must be a solution to the CFEs and, by extension, the EFEs as well.

4.3.6 The Bach Tensor

Alternative form of the Third CFE

An alternative way of looking at the third CWE is to make use of a tensor called the Bach tensor. The Bach tensor is a trace-free, conformally invariant (in four dimensions), rank 2 tensor, whose form is given by

$$B_{ab} = -L^{cd}C_{abcd} + \nabla^c\nabla_a L_{bc} - \Box L_{ab}. \tag{4.68}$$

A property of the Bach tensor is that when $B_{ab} = 0$ one has a solution to the vacuum EFEs, where $R_{ab} = \lambda g_{ab}$. Additionally, it can be verified that the Bach tensor is divergence free. Since the definition of the Bach tensor contains a wave operator applied to the Schouten tensor, it is relatively straightforward to derive a wave equation for the Schouten tensor in terms of the Bach tensor

$$\Box L_{ab} = 2L^{cd}C_{acbd} + g_{ab}L_{cd}L^{cd} + \tfrac{1}{6}\nabla_b\nabla_a R - 4L_a{}^c L_{bc} - B_{ab}. \tag{4.69}$$

We have just shown that for trace-free matter it is possible to formulate the CFEs as a system of wave equations. We also just derived a wave equation for the Schouten tensor. We will now show the equivalence between the wave equation of the Schouten tensor as given by (4.25) and the wave equation written in terms of the Bach tensor given by (4.69). First, we recall the third CFE, differentiating (3.42c) and one obtains

$$\nabla^c \nabla_a L_{cb} - \Box L_{ab} = T_{acb} \nabla^c \Omega + \Omega \nabla^c T_{acb} + d^f{}_{bac} \nabla^c \nabla_f \Omega + \nabla^c d^f{}_{bac} \nabla_f \Omega. \tag{4.70}$$

We notice that it is possible rearrange the definition of the Bach tensor such that the LHS of the Bach tensor definition is equal to the LHS of (4.70). Equating the RHS of (4.68) and the RHS of (4.70) gives

$$B_{ab} - L^{cd} C_{acbd} = T_{acb} \nabla^c \Omega + \Omega \nabla^c T_{acb} + d^f{}_{bac} \nabla^c \nabla_f \Omega + \nabla^c d^f{}_{bac} \nabla_f \Omega. \tag{4.71}$$

The goal is to simplify this equation. First, applying both (3.42a) and (3.42d) to (4.71) gives

$$B_{ab} - L^{cd} C_{acbd} = -\tfrac{1}{2} \Omega^3 T^{cd} d_{acbd} + \Omega d_{acbd} L^{cd} + \Omega \nabla_c T_a{}^c{}_b + T_{acb} \nabla^c \Omega + T_{bca} \nabla^c \Omega. \tag{4.72}$$

Then applying (4.68) and rearranging all terms on the LHS of (4.72) yields

$$-\tfrac{1}{2} \Omega^3 T^{cd} d_{acbd} + \Omega d_{acbd} L^{cd} + \Omega \nabla_c T_a{}^c{}_b + T_{acb} \nabla^c \Omega + T_{bca} \nabla^c \Omega - \nabla^c \nabla_a L_{bc} + \nabla^c \nabla_c L_{ab} = 0. \tag{4.73}$$

Substituting in (4.25) into (4.73) gives

$$4 L_a{}^c L_{bc} - g_{ba} L_{cd} L^{cd} - L^{cd} C_{acbd} + \frac{\nabla_b \nabla_a R}{6} + \Omega \nabla_c T_a{}^c{}_b - \Omega \nabla_c T_b{}^c{}_a - \nabla^c \nabla_a L_{bc} = 0, \tag{4.74}$$

then making use of (4.23) to reorder the covariant derivatives as usual, and applying both (2.12) and (3.29) transforms (4.74) to

$$-\tfrac{1}{6} R L_{ab} + L_{ab} L_c{}^c - \nabla_a \nabla_c L_b{}^c + \tfrac{1}{6} \nabla_b \nabla_a R + \Omega \nabla_c T_a{}^c{}_b - \Omega \nabla_c T_b{}^c{}_a = 0. \tag{4.75}$$

Making use of (2.18) to get rid of the second order derivatives of the Schouten tensor and (2.10) then one ends up with

$$\Omega \nabla_c T_a{}^c{}_b - \Omega \nabla_c T_b{}^c{}_a = 0. \tag{4.76}$$

Applying (4.2) to (4.76) and one obtains

$$\Omega T_b{}^c \nabla_c \nabla_a \Omega + \tfrac{1}{2} \Omega^2 \nabla_c \nabla_a T_b{}^c - \Omega T_a{}^c \nabla_c \nabla_b \Omega - \tfrac{1}{2} \Omega^2 \nabla_c \nabla_b T_a{}^c = 0. \tag{4.77}$$

Making use of (4.23) and one gets

$$\tfrac{1}{2}\Omega^2 T_b{}^c R_{ac} - \tfrac{1}{2}\Omega^2 T_a{}^c R_{bc} + \Omega T_b{}^c \nabla_c \nabla_a \Omega - \Omega T_a{}^c \nabla_c \nabla_b \Omega = 0. \tag{4.78}$$

Then using both (3.29) and (3.42a) and it can be verified that the LHS of (4.78) vanishes and hence the two equations are equivalent. This is a very good thing, because the two wave equations of the Schouten tensor are equivalent, this means that it is possible to use either equation depending on the situation. Furthermore, because propagation of the constraints has already been performed for the wave equation for the Schouten tensor (not written in terms of Bach), this means that any solution to (4.69) is a solution to the EFEs.

Conformal properties of the Bach tensor

Despite the success of showing the equivalence of the two different wave equations for the Schouten tensor, there is one final check that needs to be performed. It must be shown that the Bach tensor is regular under a conformal rescaling. If that turns out to be the case then it is possible to apply the techniques of wave equations to cases wherever the Bach tensor is involved. Showing that the Bach tensor is indeed regular under such a transformation is a lengthy task. We begin first by considering the exact conformal transformation of the Bach tensor

$$B_{ab} = \Omega^{-2}\tilde{B}_{ab}. \tag{4.79}$$

As we are working with trace-free matter, it becomes necessary to express $\tilde{B}_{ab}$ in terms of the physical energy-momentum tensor $\tilde{T}_{ab}$. In terms of trace-free matter, the Schouten tensor can be expressed in terms of the energy-momentum tensor as

$$L_{ab} = \tfrac{1}{2}(T_{ab} + \lambda g_{ab}). \tag{4.80}$$

We begin with the definition of the physical Bach tensor. Commuting the covariant derivatives and substituting in the definition of the Schouten tensor in terms of the energy-momentum tensor and one obtains

$$\tilde{B}_{ab} = \lambda\tilde{T}_{ab} + \tilde{T}_a{}^c\tilde{T}_{bc} - \tfrac{1}{4}\tilde{T}_{cd}\tilde{T}^{cd}\tilde{g}_{ab} + \tfrac{1}{12}\tilde{T}_{ab}\tilde{R} - \tfrac{1}{2}\tilde{\nabla}_c\tilde{\nabla}^c\tilde{T}_{ab}. \tag{4.81}$$

We must now check to see if this equation transforms properly under a conformal transformation. To start with, we consider the conformal transformation of the energy-momentum tensor given in (3.24), the conformal transformation law of the metric given by (1.3) and the conformal transformation law of the Ricci scalar given by (3.11). Because of these laws, the first four terms are satisfactory from the unphysical point of view since they do not create terms that are singular at the boundary. The only difficult term to deal with is the last term that involves derivatives of the energy-momentum tensor. We need to make use of the law for transforming derivatives (2.32); applying this to a derivative of the energy-momentum tensor gives

$$\tilde{\nabla}_c \tilde{T}_{ab} = \nabla_a \tilde{T}_{bc} + Q_a{}^s{}_b \tilde{T}_{sc} + Q_a{}^s{}_c \tilde{T}_{bs}. \tag{4.82}$$

Applying, (2.13) to (4.82) and expanding gives

$$\tilde{\nabla}^c \tilde{T}_{ab} = \nabla_a \tilde{T}_b{}^c + 2\Upsilon_a \tilde{T}_b{}^c + \Upsilon^c \tilde{T}_{ba} + \Upsilon_b \tilde{T}_a{}^c - \Upsilon^s \tilde{T}_{bs} \delta_a{}^c - \Upsilon^s \tilde{T}_s{}^c g_{ab}, \tag{4.83}$$

where Υ_a is a vector quantity first defined in (3.9). The next step is to see how a second order derivative of the energy-momentum tensor transforms under a conformal rescaling. To make this job easier we redefine the derivatives of the energy-momentum tensor in (4.83) as

$$\tilde{\sigma}^c{}_{ab} \equiv \tilde{\nabla}^c \tilde{T}_{ab}. \tag{4.84}$$

We then apply (2.32) to (4.84), which gives

$$\tilde{\nabla}_a \tilde{\sigma}^c{}_{bd} = \nabla_a \tilde{\sigma}^c{}_{bd} + Q_a{}^c{}_s \tilde{\sigma}^s{}_{bd} - Q_a{}^s{}_b \tilde{\sigma}^c{}_{sd} - Q_a{}^s{}_d \tilde{\sigma}^c{}_{bs}, \tag{4.85}$$

then expanding as before using (2.13) and then (4.85) becomes

$$\begin{aligned}\tilde{\nabla}_c \tilde{\sigma}^c{}_{bd} = \nabla_c \tilde{\sigma}^c{}_{bd} + 4\Upsilon_c \tilde{\sigma}^c{}_{bd} + \Upsilon_c \tilde{\sigma}^c{}_{bd} - \Upsilon^c \tilde{\sigma}_{cbd} - \Upsilon_b \tilde{\sigma}^c{}_{cd} \\ -\Upsilon_c \tilde{\sigma}^c{}_{bd} - \Upsilon^c \tilde{\sigma}_{bcd} - \Upsilon_d \tilde{\sigma}^c{}_{bc} + \Upsilon_c \tilde{\sigma}^c{}_{bd} - \Upsilon^c \tilde{\sigma}_{dbc}.\end{aligned} \tag{4.86}$$

So, we can make a decision about whether or not the Bach tensor transforms properly by closely examining (4.86), upon doing so we notice that virtually every single term is of the form $\Upsilon_a \tilde{\sigma}_{bcd}$. So we simply need to take note of the fact that each term in (4.83) is comprised of terms that do not contain singular terms at the boundary; to see that this is indeed the case let us see what form this equation has when all the necessary substitutions have been made

$$\tilde{\sigma}^c{}_{ab} = 2\Omega \nabla_a \Omega \tilde{T}_b{}^c + 2\Omega \nabla_a \Omega T_b{}^c + \Omega \nabla^c \Omega T_{ba} + \Omega \nabla_b \Omega T_a{}^c - \Omega \nabla^s \Omega T_{bs} \delta_a{}^c - \Omega \nabla^s \Omega T_s{}^c g_{ab}. \tag{4.87}$$

We can see that there are no terms in (4.87) that are irregular as the conformal factor tends to zero. If we examine the individual terms we notice that the majority of them are products of the vector Υ and the tensor $\tilde{\sigma}^a{}_{bc}$. However, if we look at the full version of the tensor $\tilde{\sigma}^a{}_{bc}$ given by (4.83) it can be inferred that multiplying each individual term of this equation by the vector Υ_a only increases the power of the conformal factor by one. This means that every such term is regular at the boundary, since any term containing a positive value of the conformal factor vanishes at the boundary. The only other term of concern is the derivative of $\tilde{\sigma}^a{}_{bc}$, however if we differentiate (4.83) then this only reduces the power of the conformal factor by one, however this does not create any terms that are singular where $\Omega = 0$. Combining all this information means that every term contained in (4.86) is regular at the boundary, hence the Bach tensor is also regular at the boundary. This means that we can use the version of the third CFE given by (4.69) when using conformal methods.

4.3.7 The reduced wave operator

We have derived a system of wave equations for conformal spacetimes containing trace-free matter, however, there is something else that needs to be mentioned when one proceeds to solve these equations. Whilst (3.42a) and (4.21) provide satisfactory wave equations that are independent of any arbitrary choice of coordinates, this is not the case for (4.25) and (4.33). The reason for this is that the d'Alembertian operator $\square$ acts on tensors (as opposed to the first two CWEs where the operator is applied to scalars), meaning that they involve derivatives of the Christoffel symbols, which ruins the hyperbolic nature of the equations. Thankfully, there is a procedure which enables us to convert the d'Alembertian operator to an alternative form that when acting on any tensor in any arbitrary coordinate system preserves the hyperbolic nature of the terms.

Recall the condition for a system of wave equations (1.9), we will now show how this procedure enables us to preserve the hyperbolic nature of the equations. Let us illustrate this procedure with an example: consider an arbitrary covector ω_a with components (ω_λ) that satisfy (1.9) with respect to some coordinate system $x = (x^\mu)$ and for some choice of coordinate gauge source functions $Ł^\mu(x)$. A direct computation using (2.5) yields

$$\square\omega_\lambda = g^{\mu\nu}\partial_\mu\partial_\nu\omega_\lambda - g^{\mu\nu}\partial_\mu\Gamma^\sigma{}_{\mu\lambda}\omega_\sigma + f_\lambda(g, \partial g, \omega, \partial\omega), \tag{4.88}$$

where $f_\lambda(g, \partial g, \omega, \partial\omega)$ denotes an expression depending on the components $g_{\mu\nu}$, ω_μ and their first order partial derivatives. Now, recall the equation for the Riemann tensor written in terms of the Levi-Civita connection (2.7); if we contract this equation with the metric then we end up with

$$R^\sigma{}_\lambda = g^{\mu\nu}\partial_\lambda\Gamma^\sigma{}_{\nu\mu} - g^{\mu\nu}\partial_\nu\Gamma^\sigma{}_{\lambda\mu} + g^{\mu\nu}\Gamma^\sigma{}_{\lambda\tau}\Gamma^\tau{}_{\nu\mu} - g^{\mu\nu}\Gamma^\sigma{}_{\nu\tau}\Gamma^\tau{}_{\lambda\mu}. \tag{4.89}$$

If we rearrange (4.89), then we notice that one of the terms is equal to the second term in (4.88), so making a substitution yields

$$\square\omega_\lambda = g^{\mu\nu}\partial_\mu\partial_\nu\omega_\lambda + (R^\sigma\lambda - g^{\mu\nu}\partial_\lambda\Gamma^\sigma)\omega_\sigma + f_\lambda(g, \partial g, \omega, \partial\omega). \tag{4.90}$$

Then, applying (1.6) and using some terms contained within the function f_λ to transform the partial derivative in (4.90) into a covariant derivative, one obtains

$$\Box\omega_\lambda = g^{\mu\nu}\partial_\mu\partial_\nu\omega_\lambda + (R_{\tau\lambda} - g_{\sigma\tau}\nabla_\lambda\Gamma^\sigma)\omega^\tau + f_\lambda(g, \partial g, \omega, \partial\omega). \tag{4.91}$$

Now, if we apply (3.29) along with the replacement $\Gamma^\mu \mapsto \text{Ł}^\mu(x)$ then this allows us to define the *reduced wave operator* $\blacksquare$ acting on the components ω_μ as

$$\blacksquare\omega_\lambda \equiv g^{\mu\nu}\partial_\mu\partial_\nu\omega_\lambda + \big(2L_{\tau\lambda} + \tfrac{1}{6}\mathcal{R}(x)g_{\tau\lambda} - g_{\sigma\tau}\nabla_\lambda\text{Ł}^\sigma(x)\big)\omega^\tau + f_\lambda(g, \partial g, \omega, \partial\omega), \tag{4.92}$$

where we have chosen to write out the Ricci scalar R as $\mathcal{R}(x)$ to emphasize that this is a form of the equations that depends on the coordinate system $x = (x^\mu)$. The more general form of this operator, which acts on a tensor of arbitrary rank, is given as

$$\begin{aligned}\blacksquare T_{\lambda\ldots\rho} \equiv \Box T_{\lambda\ldots\rho} &+ \big((2L_{\tau\lambda} + \tfrac{1}{6}\mathcal{R}(x)g_{\tau\lambda} - R_{\tau\lambda}) - g_{\sigma\tau}\nabla_\lambda(\text{Ł}^\sigma(x) - \Gamma^\sigma)\big)T^\tau{}_{\ldots\rho} + \ldots \\ &\ldots + \big((2L_{\tau\rho} + \tfrac{1}{6}\mathcal{R}(x)g_{\tau\rho} - R_{\tau\rho} - R_{\tau\rho}) - g_{\sigma\tau}\nabla_\rho(\text{Ł}^\sigma(x) - \Gamma^\sigma)\big)T_{\lambda\ldots}{}^\tau.\end{aligned} \tag{4.93}$$

Applying this operator to a tensor of arbitrary rank cancels out all the terms that ruin the hyperbolic nature of the equations and enables one to formulate an initial value problem for the equations as per usual.

Summary

We have shown that it is possible to write out the conformal metric and the conformal fields as a system of wave equations. Furthermore, we have shown that any solution of these wave equations implies a solution to the corresponding field equations.

5 Analysis of Trace-Free Matter Models

In the previous chapter we successfully showed, through the propagation of the constraints, that any solution to the CWEs is also a solution to the CFEs. These equations describe the behaviour of conformal fields sourced by trace-free matter. It is a fundamental fact of GR that the very structure of spacetime is influenced by the presence of matter. Matter curves the spacetime, but the spacetime itself then effects how the matter behaves. Whilst the equations that we have derived thus far describe the spacetime, they do not say anything about how the matter evolves. We would therefore like to understand how the matter content of the conformal spacetimes changes over time. All the work that we have done so far suggests that in trying to describe how any system evolves in the framework of GR that we should always proceed to construct equations that are hyperbolic. However, it is not always obvious if it is possible to construct hyperbolic equations for any matter model. This means that when combining a certain matter model with the CWEs, one needs to check that the matter itself is "well behaved". This means doing a similar analysis to the previous section for the equations describing the evolution of the matter fields.

We will now proceed to derive a set of wave equations that describe the evolution of certain trace-free matter fields. We will then proceed to construct subsidiary equations for each of the equations describing the matter content in order to show that the evolution equations propagate in the correct manner.

5.1 Conformally coupled scalar field

As mentioned in the introduction, the EFEs are notoriously difficult to solve unless one makes some simplifying assumptions about the nature of the problem that you wish to solve. However, one would still like to solve more complicated problems, analytically if possible. To this end one takes an Occum's razor approach to the situation, namely we start with the simplest possible solution then gradually make our solution more complex until we get the correct result. The simplest possible solution from the point of view of GR is the Minkowski metric, which describes vacuum spacetimes. The next solution in order of increasing difficulty is the Schwarzchild metric, which describes vacuum spacetimes containing a single spherically symmetric mass. When trying to introduce matter into this picture ideally we would like to preserve spherical symmetry while having a dynamic situation; one of the simplest matter models that allows us to do this is a scalar field ϕ. Given all that we know about how one obtains solutions of evolving systems in GR suggests that one construct a wave equation for the scalar field. The simplest possible wave equation that one can construct is

$$\tilde{\nabla}^a \tilde{\nabla}_a \tilde{\phi} = 0. \tag{5.1}$$

This provides a very convenient way of incorporating degrees of freedom into existing solutions of the EFEs whilst simultaneously preserving spherical symmetry. However, from the point of view of conformal methods, this is not very appealing since it is not invariant under conformal transformations. A useful model for exploring the properties of scalar fields and conformal invariance is given by a conformally invariant wave equation, which arises from the addition of a Ricci scalar curvature term to (5.1):

$$\tilde{\Box}\tilde{\phi} - \tfrac{1}{6}\tilde{R}\tilde{\phi} = 0. \tag{5.2}$$

Indeed if $g_{ab} = \Omega^2 \tilde{g}_{ab}$ and we choose the transformation law of the scalar field to be $\phi = \Omega^{-1}\tilde{\phi}$, then (5.2) implies that

$$\Box\phi - \tfrac{1}{6}R\phi = 0. \tag{5.3}$$

It is possible to derive this equation from the following Lagrangian

$$L = \tfrac{1}{2\kappa}R - \tfrac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi - \tfrac{1}{12}R\phi^2, \tag{5.4}$$

the action for this Lagrangian is given by

$$S = \int (\frac{1}{2\kappa}R - \frac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi - \frac{1}{12}R\phi^2)\sqrt{g}d^4x. \tag{5.5}$$

Varying the action with respect to the scalar field gives

$$\frac{\delta S}{\delta\phi} = \int (-g^{ab}\nabla_a\phi\nabla_b(\delta\phi) - \frac{1}{6}R\phi\delta\phi)\sqrt{-g}d^4x, \tag{5.6}$$

then integrating (5.6) by parts gives

$$\frac{\delta S}{\delta\phi} = \int (g^{ab}\nabla_a\nabla_b\phi\delta\phi - \frac{1}{6}R\phi\delta\phi)\sqrt{-g}d^4x. \tag{5.7}$$

Applying Lagrange's lemma to (5.7) gives the equation of motion (5.3). The energy-momentum tensor for such a scalar field can also be derived by varying (5.5) with respect to the metric, which yields the result

$$T_{ab} = \nabla_a\phi\nabla_b\phi - \tfrac{1}{4}g_{ab}\nabla_c\phi\nabla^c\phi - \tfrac{1}{2}\nabla_a\nabla_b\phi + \tfrac{1}{2}\phi^2 L_{ab}. \tag{5.8}$$

It can be verified that this matter tensor is both trace and divergence free (i.e. $g^{ab}T_{ab} = 0$ and $\nabla^a T_{ab} = 0$). Despite the relative simplicity of (5.2), one runs into a problem when trying to couple this matter model with the CWEs, namely that the act of substituting (5.8) into (4.2) creates second and third order derivatives of the scalar field, which ruins the hyperbolic nature of the equations. To resolve this difficultly we construct field equations for derivatives of the scalar field,

$$\nabla_a\phi \equiv \phi_a, \qquad \nabla_a\nabla_b\phi \equiv \phi_{ab}. \tag{5.9}$$

However, in order to see whether or not it is possible to use these definitions to simplify calculations, we must check that the above definitions possess the correct evolution properties. To this end, one proceeds to compute evolution equations for these variables, which can be done by direct differentiation and making use of (4.23). Doing so gives

$$\square\phi_b = \tfrac{1}{3}\phi_b R + 2\phi^a L_{ba} + \tfrac{1}{6}\phi\nabla_b R, \tag{5.10a}$$

$$\begin{aligned}\square\phi_{cd} &= \Omega\phi^a T_{dac} - 2\Omega\phi^{ab}d_{cadb} + \tfrac{1}{2}\phi_{cd}R - 2\phi^{ab}g_{cd}L_{ab} + 4\phi_d{}^a L_{ca}\\ &-\tfrac{1}{3}\phi R L_{cd} + 4\phi_c{}^a L_{da} - \tfrac{1}{6}\phi^a g_{cd}\nabla_a R - \phi^a\nabla_a L_{cd} - \phi^a d_{cbda}\nabla^b\Omega\\ &+\tfrac{1}{3}\phi_d\nabla_c R + 2\phi^a\nabla_c L_{da} + \tfrac{1}{3}\phi_c\nabla_d R + \phi^a\nabla_d L_{ca} + \tfrac{1}{6}\phi\nabla_d\nabla_c R.\end{aligned} \tag{5.10b}$$

The introduction of these equations allows one to construct an initial value problem for the CFEs coupled with a conformally invariant scalar field since it enables one to simplify derivative terms.

5.1.1 Subsidiary equations for the scalar field system

Again, to verify that these evolution equations are valid one must proceed with the propagation of the constraints. To this end we define the following subsidiary variables

$$Q_a = \phi_a - \nabla_a\phi, \tag{5.11a}$$

$$Q_{ab} = \phi_{ab} - \nabla_a\nabla_b\phi, \tag{5.11b}$$

and check if the wave equations that one can construct from these variables are homogeneous. Differentiating (5.11a) to start with, we obtain

$$\square Q_a = \square\phi_a - \square\nabla_a\phi, \tag{5.12}$$

then applying (4.23) gives

$$\Box Q_a = \nabla_a \Box \phi - \Box \phi_a + R_{ab} \nabla^b \phi. \tag{5.13}$$

Using (5.2) and (5.10a) to eliminate the derivatives and we are left with

$$\Box Q_a = -\phi^b R_{ab} - \tfrac{1}{6}\phi_a R + \tfrac{1}{6} R \nabla_a \phi + R_{ab} \nabla^b \phi. \tag{5.14}$$

Finally, using (5.11a), (5.11b) and (3.29), then we get

$$\Box Q_a = \tfrac{1}{3} Q_a R + 2Q^b L_{ab}, \tag{5.15}$$

which is a homogeneous wave equation as required. Next, we proceed to do the same with the Q_{ab} variable. Begin by differentiating (5.11b) and one ends up with

$$\Box Q_{ab} = -\Box \phi_{ab} + \Box \nabla_b \nabla_a \phi. \tag{5.16}$$

Applying (4.23) to (5.16) then gives

$$\begin{aligned} \Box Q_{ab} = \nabla_b \nabla_a \Box \phi + R_{ac} \nabla_b \nabla^c \phi - \Box \phi_{ab} + \nabla_b R_{ac} \nabla^c \phi \\ + R_{bc} \nabla^c \nabla_a \phi - R^d{}_{acb} \nabla^c \nabla_d \phi - \nabla_c R^d{}_a{}^c{}_b \nabla_d \phi - R^d{}_a{}^c{}_b \nabla_d \nabla_c \phi. \end{aligned} \tag{5.17}$$

Using (2.12), (3.29), (5.11a), (5.11b), (4.48), (4.47), (5.10b), (2.30), (4.49) and (4.11) to simplify and one can verify that

$$\begin{aligned} \Box Q_{ab} = -2\Omega Q^{cd} d_{acbd} + \tfrac{1}{2} Q_{ab} R - 2Q^c{}_c L_{ab} + 4Q_b{}^c L_{ac} + 4Q_a{}^c L_{bc} \\ -2g_{ab} Q^{cd} L_{cd} + \Omega \phi^c \Lambda_{bac} - \phi^c \Delta_{abc} - \phi_b \Delta_a{}^c{}_c + \phi^c g_{ab} \Delta_c{}^d{}_d. \end{aligned} \tag{5.18}$$

5.1.2 Summary

Hence, the wave equations for the subsidiary equations for the scalar field system are homogeneous and the relations in (5.9) are valid.

5.2 Electromagnetic field

The electromagnetic (EM) or Maxwell field is the model that is used to describe the properties of electromagnetic matter and radiation. In the classical sense the EM field is described mathematically by the Maxwell equations, however, it is not apparent if the Maxwell equations transform correctly under a coordinate transformation. To perform a coordinate invariant analysis of gravitational systems coupled with EM sources one needs to write out the Maxwell equations in tensorial form

$$\tilde{\nabla}^a \tilde{F}_{ab} = 0, \tag{5.19a}$$

$$\tilde{\nabla}_{[a}\tilde{F}_{bc]} = 0, \tag{5.19b}$$

where $\tilde{F}_{ab}$ is an antisymmetric tensor called the Faraday tensor, the energy-momentum tensor of the Maxwell field is given by

$$\tilde{T}_{ab} = \tilde{F}_{ac}\tilde{F}_b{}^c - \frac{1}{4}\tilde{g}_{ab}\tilde{F}_{cd}\tilde{F}^{cd}, \tag{5.20}$$

which can be verified to be trace and divergence free. The Maxwell equations are extremely appealing from the point of view of conformal transformations, since they are conformally invariant [27, 36], i.e.

$$\nabla^a F_{ab} = 0, \qquad F_{ab} \equiv \tilde{F}_{ab}, \qquad \nabla_{[a}F_{bc]} = 0.$$

In order to see if the Maxwell field propagates in the correct way, one needs to show that it is possible to construct a homogeneous wave equation for the Faraday tensor. To accomplish this task we expand and differentiate (5.19b), which gives

$$\Box F_{bc} - \nabla^a\nabla_b F_{ac} + \nabla^a\nabla_c F_{ab} = 0. \tag{5.21}$$

Then, upon applying (4.23) to (5.21), one obtains

$$F_c{}^a R_{ba} - F_b{}^a R_{ca} + 2F^{ad}R_{bacd} + \Box F_{bc} + \nabla_b\nabla_a F_c{}^a - \nabla_c\nabla^a F_{ab} = 0, \tag{5.22}$$

which, upon applying (5.19a), simplifies to

$$F_c{}^a R_{ba} - F_b{}^a R_{ca} + 2F^{ad}R_{bacd} + \Box F_{bc} = 0. \tag{5.23}$$

Then, making use of (2.12) and (3.29), one obtains

$$\Box F_{bc} = \tfrac{1}{3}F_{bc}R - 2\Omega F^{ad}d_{bacd}, \tag{5.24}$$

which is the wave equation for the Faraday tensor. Now, in order to show that any solution to the Faraday wave equation also solves the Maxwell equations and that the Maxwell fields behave correctly when coupled to the CFEs, one again needs to make use of the propagation of the constraints. So to that end, one constructs a system of subsidiary equations for the Maxwell equations

$$\nabla^a F_{ab} \equiv M_b, \tag{5.25a}$$

$$\nabla_{[a}F_{bc]} \equiv M_{abc}. \tag{5.25b}$$

It can be verified that the propagation of the constraints is indeed valid for vacuum spacetimes. We will now show that this result is valid for spacetimes perturbed with trace-free matter. Begin by applying a d'Almbertian to (5.25a)

$$\Box M_a = \Box \nabla^b F_{ba}. \tag{5.26}$$

Commuting the covariant derivatives with (4.23), then applying (5.24), (3.29), (2.12), (4.47), (4.48), (4.49) and (5.25a) gives

$$\Box M_a = -\Omega F^{bf}\Lambda_{baf} + \tfrac{1}{6}RM_a + 2L_a{}^bM_b - F^{bf}\Delta_{abf} - F_a{}^b\Delta_b{}^f{}_f. \tag{5.27}$$

The next step is to show that the wave equation for M_{abc} is homogeneous, this can be done, albeit rather cumbersomely, by directly differentiating (5.25b), however it is far simpler to make use of the Hodge dual. Multiplying both sides of (5.27) by Levi-Civita alternating tensor whilst contracting on the three free indices of M_{abc} and multiplying out and differentiating then one obtains,

$$\varepsilon_{dabc}\Box M^{abc} = \varepsilon_{dabc}\Box\nabla^c F^{ab}. \tag{5.28}$$

Commuting the covariant derivatives and using (5.24), (3.29), (2.12), (4.49) to simplify, then rewriting the Weyl tensor terms using (4.15) and making use of both (5.25a) and (5.25b) to deal with the derivatives of the Faraday, along with the fact that the dual of the Weyl tensor is trace-free then one obtains

$$\begin{aligned}\epsilon_{dabc}\Box M^{abc} &= 2\Omega\epsilon_{dbcf}F^{ab}T_a{}^{cf} + 2\Omega F^{ab}\,{}^*T_{abd} + 2\Omega F_d{}^a\,{}^*T_a{}^b{}_b + 2\Omega F_d{}^a\,{}^*\Lambda^b{}_{ab}\,{}^*T_{dab} \\ &-6\Omega\,{}^*d_{dabc}M^{abc} + 2F^{ab}(\epsilon_{dbcf}\Delta_a{}^{cf} - \epsilon_{dabf}\Delta^{cf}{}_c) - 2\epsilon_{dbcf}L^{ab}\nabla_aF^{cf} + \tfrac{1}{2}\epsilon_{dabc}R\nabla^cF^{ab} \\ &+2\Omega F^{ab}({}^*\Lambda_{dab} - {}^*\Lambda_{adb}) - 2F^{bc}\nabla^a\Omega(\epsilon_{dcef}d_a{}^e{}_b{}^f + 2\,{}^*d_{[b|dc|a]}) - 4\epsilon_{dbcf}L^{ab}\nabla^fF_a{}^c.\end{aligned} \tag{5.29}$$

It is at this point that we rewrite this equation in terms of the original variables as opposed to the Hodge duals. Multiplying both sides of (5.29) by a second Levi-Civita tensor with contractions on the one remaining free index, expanding the dual terms and eliminating the derivatives of the Faraday tensor with (5.25b) and the equation transforms to

$$\begin{aligned}\Box M_{hlm} &= 2\Omega d_{lmab}M_h{}^{ab} - 2L_m{}^aM_{hla} + 2L_l{}^aM_{hma} - 2\Omega d_{hmab}M_l{}^{ab} + 2\Omega d_{hlab}M_m{}^{ab} \\ &+\tfrac{1}{3}F_m{}^b(d_{halb} - d_{hamb} - d_{hmab})\nabla^a\Omega + \tfrac{1}{3}F_l{}^b(d_{hbma} - d_{hamb} + d_{hmab})\nabla^a\Omega \\ &-2L_h{}^aM_{lma} + \tfrac{1}{2}RM_{hlm} + \tfrac{1}{3}F_h{}^b(d_{lamb} + d_{lbma} - d_{lmab})\nabla^a\Omega,\end{aligned} \tag{5.30}$$

where we have ignored all the other subsidiary variables except for the Maxwell variables for convenience. Finally, making use of the Bianchi identity to eliminate all the remaining terms containing the Weyl tensor and we are left with, including all the subsidiary variables

$$\begin{aligned}\Box M_{hlm} &= \tfrac{1}{3}\Omega\epsilon_{hlmc}F^{ab}\,{}^{*}\Lambda_{ab}{}^{c} + \tfrac{1}{3}\Omega\epsilon_{hlmc}F^{ab}\,{}^{*}\Lambda^{c}{}_{ab} - \tfrac{1}{3}\Omega\epsilon_{hlmb}F^{ab}\,{}^{*}\Lambda^{c}{}_{ac} + \tfrac{2}{3}F_{m}{}^{a}\Delta_{[h|a|l]}\\ &-\tfrac{1}{3}F_{l}{}^{a}\Delta_{ham} + \tfrac{2}{3}F_{lm}\Delta_{h}{}^{a}{}_{a} + 2\Omega d_{lmab}M_{h}{}^{ab} + \tfrac{1}{2}RM_{hlm} - 2\Omega d_{hmab}M_{l}{}^{ab} - 2L_{h}{}^{a}M_{lma}\\ &+4L_{[l}{}^{a}M_{|h|m]a} + 2\Omega d_{hlab}M_{m}{}^{ab} + \tfrac{1}{3}F_{h}{}^{a}\Delta_{lam} - \tfrac{2}{3}F_{hm}\Delta_{l}{}^{a}{}_{a} + \tfrac{2}{3}F_{[l}{}^{a}\Delta_{|ma|h]} + \tfrac{2}{3}F_{hl}\Delta_{m}{}^{a}{}_{a}. \quad (5.31)\end{aligned}$$

Therefore, the wave equation is homogeneous as required. Despite the propagation of the constraints being satisfied for the Einstein-Maxwell system, one runs into a similar problem when trying to combine the matter model with the CWEs, namely that doing so creates second order derivatives that ruins the hyperbolicity of the equations. In order to solve this problem we use the same method that was used to resolve the scalar field system. Namely, we create a new field,

$$F_{abc} = \nabla_{a}F_{bc}, \quad (5.32)$$

and proceed to see if this variable propagates correctly. We begin by differentiating (5.32) twice and using both (4.23) and (5.24), which gives

$$\begin{aligned}\Box F_{abc} &= -2F^{jd}d_{bjcd}\nabla_{a}\Omega + \tfrac{1}{3}R\nabla_{a}F_{bc} - 2\Omega d_{bjcd}\nabla_{a}F^{fd} - 2\Omega F^{fd}\nabla_{a}d_{bfcd} + R_{a}{}^{f}\nabla_{f}F_{bc}\\ &+\tfrac{1}{3}F_{bc}\nabla_{a}R + F_{c}{}^{f}\nabla_{d}R_{a}{}^{d}{}_{bf} - F_{b}{}^{f}\nabla_{d}R_{a}{}^{d}{}_{cf} - 2R_{adcf}\nabla^{d}F_{b}{}^{f} + 2R_{adbf}\nabla^{d}F_{c}{}^{f}, \quad (5.33)\end{aligned}$$

then using (2.12), (3.29), (4.47), (4.48) and (5.19b), then one obtains,

$$\begin{aligned}\Box F_{abc} &= 4\Omega F^{f}{}_{[c}T_{b]fa} + 4\Omega F^{fd}{}_{[b}d_{c]daf} + 4F^{f}{}_{a[c}L_{b]f} + 2\Omega F_{a}{}^{fd}d_{bfcd}\\ &+4g_{a[b}F^{|f|}{}_{c]}{}^{d}L_{fd} - 2F^{fd}d_{bfcd}\nabla_{a}\Omega + 4F^{f}{}_{bc}L_{af} + \tfrac{1}{2}F_{abc}R - 2\Omega F^{fd}\nabla_{a}d_{bfcd}\\ &+4F^{f}{}_{a[c}L_{b]j} + \tfrac{1}{3}F_{bc}\nabla_{a}R + \tfrac{1}{3}F_{a[c}\nabla_{b]}R + 4F^{d}{}_{[b}d_{c]daf}\nabla^{f}\Omega + \tfrac{1}{3}F_{j[b}g_{c]a}\nabla^{j}R. \quad (5.34)\end{aligned}$$

With the derivation of the wave equation completed, one now needs to do the propagation of the constraints. First we start by defining a new subsidiary variable

$$Q_{abc} = F_{abc} - \nabla_{a}F_{bc}. \quad (5.35)$$

Both this and (5.19b) imply the identity

$$F_{abc} = F_{bac} - F_{cab} + Q_{abc} - Q_{bac} + Q_{cab}. \quad (5.36)$$

As we have done many times before, we now need to show that the wave equation that is computed from (5.35), is homogeneous. Differentiating (5.35) twice then applying both (4.23) and (5.24) gives

$$\Box Q_{abc} = 2F^{fd}d_{bfcd}\nabla_a\Omega - \tfrac{1}{3}R\nabla_a F_{bc} + 2\Omega d_{bfcd}\nabla_a F^{fd} + 2\Omega F^{fd}\nabla_a d_{bfcd} - \tfrac{1}{3}F_{bc}\nabla_a R$$
$$-F_c{}^f\nabla_d R_a{}^d{}_{bf} + F_b{}^f\nabla_d R_a{}^d{}_{cf} + 2R_{adcf}\nabla^d F_b{}^f - 2R_{adbf}\nabla^d F_c{}^f - R_a{}^f\nabla_f F_{bc} + \Box F_{abc}. \quad (5.37)$$

Then, using (2.12) and (3.29) followed by (4.48), (4.49), (4.47), (5.35) and (5.36) to simplify and we end up with

$$\Box Q_{abc} = \Omega F_c{}^f\Lambda_{abf} - \Omega F_b{}^f\Lambda_{acf} - 2\Omega d_{bfcd}Q_a{}^{fd} + \tfrac{1}{2}RQ_{bac} - \tfrac{1}{2}RQ_{cab} - 2L_c{}^f Q_{fab}$$
$$+2L_b{}^f Q_{fac} + 4L_a{}^f Q_{fbc} - 2g_{ac}L^{fd}Q_{fbd} + 2g_{ab}L^{fd}Q_{fcd} - 2\Omega d_{afcd}Q^f{}_b{}^d + 2\Omega d_{afbd}Q^f{}_c{}^d$$
$$+F_c{}^f\Delta_{bfa} + F_{ac}\Delta_b{}^f{}_f - F_b{}^f\Delta_{cfa} - F_{ab}\Delta_c{}^f{}_f + F_c{}^f g_{ab}\Delta_f{}^d{}_d - F_b{}^f g_{ac}\Delta_f{}^d{}_d. \quad (5.38)$$

Thus the wave equation for Q_{abc} is homogeneous and (5.32) is indeed a valid relation.

5.3 Yang-Mills fields

Another trace-free matter model is the Yang-Mills field [4, 11]. From the classical point of view, this matter model is a more complex version of the the electromagnetic theory. Upon quantizing this particular theory, one obtains the theory of Quantum Chromodynamics responsible for the description of the strong nuclear force.

The classical Yang-Mills equations are as follows

$$F^{\mathfrak{c}}{}_{ab} + \nabla^f F^{\mathfrak{a}}{}_{fb} = 0, \quad (5.39a)$$
$$A^{\mathfrak{b}}{}_a A^{\mathfrak{c}}{}_b C^{\mathfrak{a}}{}_{\mathfrak{bc}} + \nabla_a A^{\mathfrak{a}}{}_b - \nabla_b A^{\mathfrak{a}}{}_a - F^{\mathfrak{a}}{}_{ab} = 0, \quad (5.39b)$$

where $F^{\mathfrak{a}}{}_{bc}$ is the Yang-Mills version of the Faraday tensor, $A^{\mathfrak{a}}{}_a$ is analogous to the gauge potential in electromagnetism and $C^{\mathfrak{a}}{}_{\mathfrak{bc}}$ are a set of constants known as the *structure constants* that take into account the non-linearity of Yang-Mills and allow one to fix the particular Yang-Mills field under consideration. Also the indices $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are a set of indices that obey the standard summation convention, but which can have a different total number of terms as the spacetime indices when expanded. The structure constants satisfy the following identities,

$$C^{\mathfrak{a}}{}_{\mathfrak{bc}}C^{\mathfrak{d}}{}_{\mathfrak{ae}} + C^{\mathfrak{a}}{}_{\mathfrak{ec}}C^{\mathfrak{d}}{}_{\mathfrak{ab}} + C^{\mathfrak{a}}{}_{\mathfrak{eb}}C^{\mathfrak{d}}{}_{\mathfrak{ac}} = 0, \quad (5.40)$$

which is the Jacobi identity, as well as an identity analogous to the Young Projector

$$C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{d}}C^{\mathfrak{b}}{}_{\mathfrak{a}\mathfrak{e}} = \tfrac{1}{3}(C^{\mathfrak{a}}{}_{\mathfrak{e}\mathfrak{d}}C^{\mathfrak{b}}{}_{\mathfrak{a}\mathfrak{c}} - C^{\mathfrak{a}}{}_{\mathfrak{e}\mathfrak{c}}C^{\mathfrak{b}}{}_{\mathfrak{a}\mathfrak{d}} + C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{d}}C^{\mathfrak{b}}{}_{\mathfrak{a}\mathfrak{e}} - C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{c}}C^{\mathfrak{b}}{}_{\mathfrak{a}\mathfrak{e}}). \tag{5.41}$$

Another important quantity is the *gauge source function* analogous to the electromagnetic 4-potential in classical electromagnetism. Recall that the 4-potential in electromagnetism is a vector that combines the electric potential and the magnetic potential into a single 4-vector; naturally, the gauge source function is a non-linear version of the this 4-potential. Specifically this function is defined as

$$f^{\mathfrak{a}} = \nabla_a A^{\mathfrak{a}a}. \tag{5.42}$$

An important fact about this function is that, like the corresponding 4-potential, it is Lorentz covariant, which means that it is completely invariant in all frames of reference, rather much like the spacetime interval. Furthermore, this particular quantity possesses what is known as a gauge invariance. In the case of the 4-potential this means that the Faraday tensor does not change when one performs a transformation of the form $A_\mu \to A_\mu - \partial_\mu \Lambda$, where Λ is some arbitrary function. This means that there are infinitely many choices of the 4-potential that give identical results when measuring physical quantities. The gauge source function (5.42) inherits these properties and therefore it is a quantity that is dependent on the choice of the user.

5.3.1 Wave equations for the Yang-Mills fields

We will now proceed to derive the evolution equations for the Yang-Mills variables. We will start by deriving a wave equation for the gauge potential. Differentiating (5.39b) once and one ends up with

$$A^{\mathfrak{c}}{}_b C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} \nabla^a A^{\mathfrak{b}}{}_a + A^{\mathfrak{b}}{}_a C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} \nabla^a A^{\mathfrak{c}}{}_b - \nabla^a F^{\mathfrak{a}}{}_{ab} + \Box A^{\mathfrak{a}}{}_b - \nabla^a \nabla_b A^{\mathfrak{a}}{}_a = 0. \tag{5.43}$$

Using (4.23), (2.12), (3.29) and (5.42) and we obtain from

$$\Box A^{\mathfrak{a}}{}_b + A^{\mathfrak{b}a} C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} \nabla_a A^{\mathfrak{c}}{}_b - \tfrac{1}{6} A^{\mathfrak{a}}{}_b R - 2A^{\mathfrak{a}a} L_{ba} - A^{\mathfrak{b}}{}_b f^{\mathfrak{c}} C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} + \nabla_a F^{\mathfrak{a}}{}_b{}^a - \nabla_a f^{\mathfrak{a}} = 0. \tag{5.44}$$

Finally, applying (5.39a), rearranging terms and it can be verified that (5.44) reduces to

$$\Box A^{\mathfrak{a}}{}_b = \tfrac{1}{6} A^{\mathfrak{a}}{}_b R + 2A^{\mathfrak{a}a} L_{ba} + F^{\mathfrak{c}}{}_{ab} A^{\mathfrak{b}a} C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} + A^{\mathfrak{b}}{}_b f^{\mathfrak{c}} C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} - A^{\mathfrak{b}a} C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} \nabla_a A^{\mathfrak{c}}{}_b + \nabla_b f^{\mathfrak{a}}. \tag{5.45}$$

Now we turn our attention to the Faraday tensor. The wave equation for the

Faraday tensor is unique in the sense that there are two different wave equations that one can derive. The first comes applying derivatives to an identity that is obtained from anti-symmetrising a derivative of the Faraday tensor and the other via directly differentiating the second Yang-Mills equation.

$$F^{\mathfrak{b}}{}_{bc}A^{\mathfrak{b}}{}_{a}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} - F^{\mathfrak{c}}{}_{ac}A^{\mathfrak{b}}{}_{b}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} + F^{\mathfrak{c}}{}_{ab}A^{\mathfrak{b}}{}_{c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} + \nabla_{a}F^{\mathfrak{a}}{}_{bc} - \nabla_{b}F^{\mathfrak{a}}{}_{ac} + \nabla_{c}F^{\mathfrak{a}}{}_{ab} = 0. \quad (5.46)$$

Crucially, since we have obtained (5.46) via anti-symmetrising (5.39a), these two equations are not independent, which means that any wave equation that we obtain from (5.46) will be equivalent to (5.45). This means that we have obtained two equations that we can use to best fit the problem that suits us; it will also be invaluable when constructing a subsidiary system for the Yang Mills fields. Now, let us return to the process of deriving a wave equation from (5.46). Differentiating said equation and applying (4.23), (5.39a), (2.12), (3.29) and (5.42) then (5.46) becomes,

$$\begin{aligned}\Box F^{\mathfrak{a}}{}_{bc} = &-2\Omega F^{\mathfrak{a}ad}d_{bacd} + \tfrac{1}{3}F^{\mathfrak{a}}{}_{bc}R - F^{\mathfrak{c}}{}_{bc}f^{\mathfrak{b}}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} - F^{\mathfrak{d}}{}_{ba}A^{\mathfrak{b}}{}_{c}A^{\mathfrak{c}a}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\\ &-A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{a}F^{\mathfrak{c}}{}_{bc} - F^{\mathfrak{c}}{}_{ca}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^{a}A^{\mathfrak{b}}{}_{b} + F^{\mathfrak{d}}{}_{ca}A^{\mathfrak{b}}{}_{b}A^{\mathfrak{c}a}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}} + F^{\mathfrak{c}}{}_{ba}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^{a}A^{\mathfrak{b}}{}_{c}\\ &+A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{b}F^{\mathfrak{c}}{}_{ca} + F^{\mathfrak{c}}{}_{ca}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{b}A^{\mathfrak{b}a} - A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{c}F^{\mathfrak{c}}{}_{ba} - F^{\mathfrak{c}}{}_{ba}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{c}A^{\mathfrak{b}a}. \quad (5.47)\end{aligned}$$

It is possible to simplify the above even further, first by applying (5.46) to the derivatives of the Faraday tensor, then applying (5.39a) to the derivatives of the gauge potential and (5.41) to the terms that are produced as a result of these substitutions and (5.53) can be re-written as

$$\begin{aligned}\Box F^{\mathfrak{a}}{}_{bc} = &-2\Omega F^{\mathfrak{a}ad}d_{bacd} + \tfrac{1}{3}F^{\mathfrak{a}}{}_{bc}R + 2F^{\mathfrak{b}}{}_{b}{}^{a}F^{\mathfrak{c}}{}_{ca}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} - F^{\mathfrak{c}}{}_{bc}f^{\mathfrak{b}}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\\ &-F^{\mathfrak{d}}{}_{dc}A^{\mathfrak{b}a}A^{\mathfrak{c}}{}_{a}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}} - 2A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{a}F^{\mathfrak{c}}{}_{bc}. \quad (5.48)\end{aligned}$$

The second equation for the Faraday tensor is obtained by differentiating (5.39b), albeit simplifying this equation is a much lengthier and more complicated process. Commuting the covariant derivatives with (4.23), then applying (3.29), (2.12), (5.45), (2.18), (4.48), (4.47), (4.11), (2.30) and (5.39b) to eliminate certain terms and then using (5.41) to eliminate groups of terms containing two structure constants and one obtains

$$\begin{aligned}\Box F^{\mathfrak{a}}{}_{ab} = &-F^{\mathfrak{b}}{}_{a}{}^{c}F^{\mathfrak{c}}{}_{bc}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} - F^{\mathfrak{d}}{}_{bc}A^{\mathfrak{b}}{}_{a}A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}} - 2F^{\mathfrak{c}}{}_{bc}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^{c}A^{\mathfrak{b}}{}_{a} + F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^{c}A^{\mathfrak{b}}{}_{b}\\ &+\tfrac{1}{3}F^{\mathfrak{a}}{}_{ab}R - 2\Omega A^{\mathfrak{b}c}A^{\mathfrak{c}d}d_{acbd}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} - F^{\mathfrak{d}}{}_{bc}A^{\mathfrak{b}}{}_{a}A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}} + F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_{b}A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\\ &-3A^{\mathfrak{b}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{[a}F^{|\mathfrak{c}|}{}_{bc]} + F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{b}A^{\mathfrak{b}c} + 2\Omega d_{acbd}\nabla^{d}A^{\mathfrak{a}c} - 2\Omega d_{adbc}\nabla^{d}A^{\mathfrak{a}c}. \quad (5.49)\end{aligned}$$

The next step to simplifying the above is to isolate and re-write the derivatives of the Faraday tensor. Isolating the three terms containing the derivatives of interest and applying (5.39b) to the derivatives and we end up with

$$\begin{aligned} &A^{\mathfrak{b}c}A^{\mathfrak{c}}{}_{c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\nabla_{a}A^{\mathfrak{d}}{}_{b} - A^{\mathfrak{c}c}A^{\mathfrak{c}}{}_{b}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}}\nabla_{a}A^{\mathfrak{d}}{}_{c} + 2A^{\mathfrak{b}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{a}\nabla_{[c}A^{|\mathfrak{c}|}{}_{b]} \\ &-A^{\mathfrak{b}c}A^{\mathfrak{c}}{}_{c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\nabla_{b}A^{\mathfrak{d}}{}_{a} + A^{\mathfrak{c}c}A^{\mathfrak{b}}{}_{a}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}}\nabla_{b}A^{\mathfrak{d}}{}_{c} + 2A^{\mathfrak{b}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{b}\nabla_{[a}A^{|\mathfrak{c}|}{}_{c]} \\ &+A^{\mathfrak{c}c}A^{\mathfrak{b}}{}_{b}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}}\nabla_{c}A^{\mathfrak{d}}{}_{a} - A^{\mathfrak{c}c}A^{\mathfrak{b}}{}_{a}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}}\nabla_{c}A^{\mathfrak{d}}{}_{b} + 2A^{\mathfrak{b}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{c}\nabla_{[b}A^{|\mathfrak{b}|}{}_{a]}. \end{aligned} \tag{5.50}$$

Applying (4.23) to remove the second order derivatives of the gauge potential, followed by (2.12) to eliminate the Riemann curvature terms and the result is

$$\begin{aligned} &-\Omega A^{\mathfrak{b}c}A^{\mathfrak{c}d}d_{abcd}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} + 2\Omega A^{\mathfrak{b}c}A^{\mathfrak{c}d}d_{acbd}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} + A^{\mathfrak{b}c}A^{\mathfrak{c}}{}_{c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\nabla_{a}A^{\mathfrak{d}}{}_{b} \\ &-A^{\mathfrak{c}c}A^{\mathfrak{b}}{}_{b}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}}\nabla_{a}A^{\mathfrak{d}}{}_{c} - A^{\mathfrak{b}c}A^{\mathfrak{c}}{}_{c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\nabla_{b}A^{\mathfrak{d}}{}_{a} + A^{\mathfrak{c}c}A^{\mathfrak{b}}{}_{a}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}}\nabla_{b}A^{\mathfrak{d}}{}_{c} \\ &+A^{\mathfrak{c}c}A^{\mathfrak{b}}{}_{b}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}}\nabla_{c}A^{\mathfrak{d}}{}_{a} - A^{\mathfrak{c}c}A^{\mathfrak{b}}{}_{a}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}}\nabla_{c}A^{\mathfrak{d}}{}_{b}. \end{aligned} \tag{5.51}$$

Then making use of (2.30) to get rid of the Weyl tensor terms and then (5.39b) to remove the derivatives of the gauge potential and (5.51) becomes

$$\begin{aligned} &F^{\mathfrak{d}}{}_{bc}A^{\mathfrak{b}}{}_{b}A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}} - F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_{b}A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}} \\ &+F^{\mathfrak{d}}{}_{ab}A^{\mathfrak{c}}{}_{c}A^{\mathfrak{b}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}} - A^{\mathfrak{b}}{}_{a}A^{\mathfrak{c}}{}_{b}A^{\mathfrak{d}c}A^{\mathfrak{e}}{}_{e}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{f}}C^{\mathfrak{f}}{}_{\mathfrak{e}\mathfrak{g}}C^{\mathfrak{g}}{}_{\mathfrak{b}\mathfrak{c}} \\ &+A^{\mathfrak{b}}{}_{a}A^{\mathfrak{c}}{}_{b}A^{\mathfrak{d}c}A^{\mathfrak{e}}{}_{e}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{f}}C^{\mathfrak{f}}{}_{\mathfrak{c}\mathfrak{g}}C^{\mathfrak{g}}{}_{\mathfrak{b}\mathfrak{e}} - A^{\mathfrak{b}}{}_{a}A^{\mathfrak{c}}{}_{b}A^{\mathfrak{d}c}A^{\mathfrak{e}}{}_{e}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{f}}C^{\mathfrak{f}}{}_{\mathfrak{b}\mathfrak{g}}C^{\mathfrak{g}}{}_{\mathfrak{c}\mathfrak{e}}. \end{aligned} \tag{5.52}$$

If we replace the terms in (5.49) containing derivatives of the Faraday tensor with the terms in (5.52) and we notice that multiple terms can be cancelled out using (5.41). Upon doing so the equation simplifies to

$$\begin{aligned} \Box F^{\mathfrak{a}}{}_{ab} = &-2\Omega F^{\mathfrak{a}cd}d_{acbd} + \tfrac{1}{3}F^{\mathfrak{a}}{}_{ab}R - 2F^{\mathfrak{b}}{}_{a}{}^{c}F^{\mathfrak{c}}{}_{bc}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} - F^{\mathfrak{c}}{}_{ab}f^{b}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} \\ &+F^{\mathfrak{d}}{}_{ab}A^{\mathfrak{b}c}A^{\mathfrak{c}}{}_{c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}} - 2F^{\mathfrak{c}}{}_{bc}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^{c}A^{\mathfrak{b}}{}_{a} + 2F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^{c}A^{\mathfrak{b}}{}_{b}. \end{aligned} \tag{5.53}$$

We see that both of these wave equations are appealing from a purely theoretical point of view since, upon setting the structure constants to zero, they both reduce to the equivalent wave equation for the Faraday tensor in electromagnetism. This is intuitive since the only thing that separates Yang-Mills from electromagnetism is the addition of non-linear terms that are represented by the inclusion of structure constants.

5.3.2 Subsidiary equations for Yang-Mills fields

As with the previous results we must check that the propagation of the constraints is satisfied. To this end we define some subsidiary variables for the Yang-Mills system.

$$\Pi^{\mathfrak{a}} = -f^{\mathfrak{a}} + \nabla_a A^{\mathfrak{a}a}, \tag{5.54a}$$

$$\Upsilon^{\mathfrak{a}}{}_b = F^{\mathfrak{c}}{}_{ab} A^{\mathfrak{b}a} C^{\mathfrak{a}}{}_{\mathfrak{bc}} + \nabla^f F^{\mathfrak{a}}{}_{fb}, \tag{5.54b}$$

$$\mathcal{F}^{\mathfrak{a}}{}_{ab} = -F^{\mathfrak{a}}{}_{ab} + A^{\mathfrak{b}}{}_a A^{\mathfrak{c}}{}_b C^{\mathfrak{a}}{}_{\mathfrak{bc}} + \nabla_a A^{\mathfrak{a}}{}_b - \nabla_b A^{\mathfrak{a}}{}_a. \tag{5.54c}$$

The subsidiary equation for the gauge source function

We will start by constructing a wave equation for the subsidiary variable for the gauge source function. Applying the d'Alambertian to (5.54a) and making use of (4.23) followed by (3.29) and (5.42) then the end result is

$$\Box \Pi^{\mathfrak{a}} = -f^{\mathfrak{a}} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \Pi^{\mathfrak{c}} - A^{\mathfrak{b}a} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \nabla_a \Pi^{\mathfrak{c}} + A^{\mathfrak{b}a} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \nabla_b F^{\mathfrak{c}}{}_a{}^b + F^{\mathfrak{a}}{}_{ab} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \nabla^b A^{\mathfrak{b}a}. \tag{5.55}$$

Now, an alternate form of (5.54c) is

$$\nabla_a A^{\mathfrak{a}}{}_b = \tfrac{1}{2}(F^{\mathfrak{a}}{}_{ab} - A^{b}{}_a A^{\mathfrak{c}}{}_b C^{\mathfrak{a}}{}_{\mathfrak{bc}}) + \tfrac{1}{2}(\nabla_a A^{\mathfrak{a}}{}_b + \nabla_b A^{\mathfrak{a}}{}_a), \tag{5.56}$$

which is obtained by taking into account the antisymmetry of (5.54c). Substituting (5.56) into (5.55) and making use of (5.54b) to remove the derivative of the Faraday tensor and one obtains

$$\begin{aligned}\Box \Pi^{\mathfrak{a}} = & -\tfrac{1}{2} F^{\mathfrak{d}}{}_{ab} A^{\mathfrak{b}a} A^{\mathfrak{c}b} C^{\mathfrak{a}}{}_{\mathfrak{de}} C^{\mathfrak{e}}{}_{\mathfrak{bc}} - F^{\mathfrak{d}}{}_{ab} A^{\mathfrak{b}a} A^{\mathfrak{c}b} C^{\mathfrak{a}}{}_{\mathfrak{be}} C^{\mathfrak{e}}{}_{\mathfrak{cd}} \\ & - f^{\mathfrak{b}} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \Pi^{\mathfrak{c}} - A^{\mathfrak{b}a} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \Upsilon^{\mathfrak{c}}{}_a - \tfrac{1}{2} F^{\mathfrak{c}}{}_{ab} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \mathcal{F}^{\mathfrak{b}ab} - A^{\mathfrak{b}a} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \nabla_a \Pi^{\mathfrak{c}}.\end{aligned} \tag{5.57}$$

The first two terms of (5.57) can be eliminated using the Young projector of the Jacobi identity. Upon substituting (5.41) into said terms and the end result can be verified to be

$$\Box \Pi = -f^{\mathfrak{b}} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \Pi^{\mathfrak{c}} - A^{\mathfrak{b}a} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \Upsilon^{\mathfrak{c}}{}_a - \tfrac{1}{2} F^{\mathfrak{c}}{}_{ab} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \mathcal{F}^{\mathfrak{b}ab} - A^{\mathfrak{b}a} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \nabla_a \Pi^{\mathfrak{c}}, \tag{5.58}$$

which is a homogeneous wave equation in the subsidiary variables as required.

The subsidiary equation for the second Yang-Mills equation

Next we will proceed to show that the wave equation for (5.54c) is homogeneous in the subsidiary variables. Beginning by applying the d'Alembertian operator to

(5.54c), then using (4.23) followed by (2.12), (3.29), (4.47), (4.48), (5.54c), (4.11), (2.30) and (5.53) then one ends up with

$$\begin{aligned}\Box\mathcal{F}^{\mathfrak{a}}{}_{ab} &= \mathfrak{F}^{\mathfrak{a}}{}_{ab} + \mathfrak{A}^{\mathfrak{a}}{}_{ab} + \Omega A^{\mathfrak{a}c}\Lambda_{abc} - \Omega A^{\mathfrak{a}c}\Lambda_{bac} + A^{\mathfrak{a}c}\Delta_{abc} \\ &-2\Omega d_{acbd}\mathcal{F}^{\mathfrak{a}cd} + \tfrac{1}{3}R\mathcal{F}^{\mathfrak{a}}{}_{ab} - F^{\mathfrak{c}}{}_{bc}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_a\mathcal{F}^{\mathfrak{b}}{}_a{}^c + f^b C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\mathcal{F}^{\mathfrak{c}}{}_{bc} \\ &-A^{\mathfrak{b}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_a\mathcal{F}^{\mathfrak{c}}{}_{cb} + A^{\mathfrak{b}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_b\mathcal{F}^{\mathfrak{c}}{}_{ca} + \tfrac{1}{3}R\mathcal{F}^{\mathfrak{a}}{}_{ab} - 2\Omega d_{acbd}\mathcal{F}^{\mathfrak{a}cd},\end{aligned} \tag{5.59}$$

where, for the sake of making the analysis more manageable and clear, we have chosen to define the quantities

$$\begin{aligned}\mathfrak{F}^{\mathfrak{a}}{}_{ab} &\equiv F^{\mathfrak{b}}{}_a{}^c F^{\mathfrak{c}}{}_{bc}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} - F^{\mathfrak{d}}{}_{bc}A^{\mathfrak{b}}{}_a A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}} - F^{\mathfrak{d}}{}_{ab}A^{\mathfrak{b}c}f^{\mathfrak{d}}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}} \\ &+F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_b A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}} + A^{\mathfrak{b}}{}_a A^{\mathfrak{c}}{}_b f^{\mathfrak{d}}(C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}} - C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}} + C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}),\end{aligned} \tag{5.60}$$

$$\begin{aligned}\mathfrak{A}^{\mathfrak{a}}{}_{ba} &\equiv 2A^{\mathfrak{b}c}A^{\mathfrak{c}}{}_c C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\nabla_{[a}A^{\mathfrak{d}}{}_{b]} + 2A^{\mathfrak{b}}{}_b A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}}\nabla_{[c}A^{\mathfrak{d}}{}_{a]} \\ &+2A^{\mathfrak{b}b}A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}}\nabla_{[b}A^{\mathfrak{d}}{}_{c]} + F^{\mathfrak{c}}{}_{bc}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^c A^{\mathfrak{b}}{}_a + F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_b A^{\mathfrak{b}c} \\ &-C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_b A^{\mathfrak{c}}{}_c\nabla^c A^{\mathfrak{b}}{}_a + 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_c A^{\mathfrak{c}}{}_b\nabla^c A^{\mathfrak{b}}{}_a + C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_a A^{\mathfrak{c}}{}_c\nabla^c A^{\mathfrak{b}}{}_b.\end{aligned} \tag{5.61}$$

Let us now proceed to re-express the remaining variables in a more symmetric manner. First, we notice that the final terms in (5.60) cancel out due to the Jacobi identity. Next, we notice that it is possible to rewrite the last three terms in (5.61) such that all the derivatives of the gauge potential have the same indices by a careful application of (5.54c). When we do this, we find that all of the aforementioned terms are equal to

$$\begin{aligned}&C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\mathcal{F}^{\mathfrak{b}}{}_b{}^c\nabla_c A^{\mathfrak{c}}{}_a - C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_c\mathcal{F}^{\mathfrak{b}}{}_a{}^c A^{\mathfrak{c}}{}_b + A^{\mathfrak{b}}{}_b A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_c A^{\mathfrak{d}}{}_a \\ &-A^{\mathfrak{b}}{}_a A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_c A^{\mathfrak{d}}{}_b - F^{\mathfrak{c}}{}_{bc}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^c A^{\mathfrak{b}}{}_a + F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^c A^{\mathfrak{b}}{}_b.\end{aligned} \tag{5.62}$$

Rewriting (5.61) using (5.62) and then combining all these obtained relations with (5.59) then making use of (5.54c) to again ensure that all the derivatives of the gauge potential have the same indices, then it can be verified that all the obtained terms form groups that can be cancelled out using the Jacobi identity. Grouping all said terms and applying (5.41) accordingly then the end result is

$$\begin{aligned}\Box\mathcal{F}^{\mathfrak{a}}{}_{ab} &= \Omega A^{\mathfrak{a}c}\Lambda_{abc} - \Omega A^{\mathfrak{a}c}\Lambda_{bac} + A^{\mathfrak{a}c}\Delta_{abc} + \tfrac{1}{3}R\mathcal{F}^{\mathfrak{a}}{}_{ab} - 2\Omega d_{acbd}\mathcal{F}^{\mathfrak{a}cd} - F^{\mathfrak{c}}{}_{bc}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\mathcal{F}^{\mathfrak{b}}{}_a{}^c \\ &-F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\mathcal{F}^{\mathfrak{b}c}{}_b + f^{\mathfrak{b}}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\mathcal{F}^{\mathfrak{c}}{}_{ba} - A^{\mathfrak{b}}{}_b A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}}\mathcal{F}^{\mathfrak{d}}{}_{ac} - A^{\mathfrak{b}c}A^{\mathfrak{c}}{}_c C^{\mathfrak{a}}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\mathcal{F}^{\mathfrak{d}}{}_{ba}\end{aligned}$$

$$+A^{\mathfrak{b}}{}_{a}A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}}\mathcal{F}^{\mathfrak{d}}{}_{bc} - A^{\mathfrak{b}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{a}\mathcal{F}^{\mathfrak{c}}{}_{cb} + A^{\mathfrak{b}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{b}\mathcal{F}^{\mathfrak{c}}{}_{ca} + 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\mathcal{F}^{\mathfrak{b}}{}_{[b}{}^{|c|}\nabla_{|c|}A^{|\mathfrak{c}|}{}_{a]}, \quad (5.63)$$

which is a homogeneous wave equation as required.

The subsidiary equation for the first Yang-Mills equation

Finally, we proceed to construct a homogeneous wave equation for (5.54b), which is the most challenging of all the Yang-Mills subsidiary equations to construct. It should be expressed that the derivation of this wave equation relies on a subtlety, which was mentioned at the start of this section. In the derivation of the wave equation for (5.54c) we made use of the wave equation for the Faraday tensor that was constructed from the first Yang-Mills equation i.e. (5.53). However, for the construction of the wave equation for (5.54b), it is much more convenient to use the wave equation for the Faraday tensor that is constructed from the Yang-Mills Bianchi identity to eliminate certain terms. Furthermore, rather non-intuitively, the form of the equation that is used in the analysis is in fact (5.47), not the simpler version of the equation (5.48). As mentioned beforehand, the equations that the two wave equations are derived from are not independent of each other, meaning that the two wave equations are equivalent to each other. This means that using (5.47) in the analysis is a perfectly acceptable course of action. Begin by applying the d'Alembertian to (5.54b), which gives

$$\square\Upsilon^{\mathfrak{a}}{}_{b} = A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\square F^{\mathfrak{c}}{}_{ab} + F^{\mathfrak{c}}{}_{ab}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\square A^{\mathfrak{b}a} + \square\nabla^{f}F^{\mathfrak{a}}{}_{fb} + C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{c}A^{\mathfrak{b}a}\nabla^{c}F^{\mathfrak{c}}{}_{ab}\nabla^{c}A^{\mathfrak{b}a}, \quad (5.64)$$

then using (4.23) followed by (2.12), (3.29) and (2.18) to remove certain curvature relations and (5.64) becomes

$$\begin{aligned}\square\Upsilon^{\mathfrak{a}}{}_{b} = {}& \tfrac{1}{6}R\nabla_{a}F^{\mathfrak{a}}{}_{b}{}^{a} + \tfrac{1}{6}F^{\mathfrak{a}}{}_{b}{}^{a}\nabla_{a}R - \nabla_{a}\square F^{\mathfrak{a}}{}_{b}{}^{a} - F^{\mathfrak{a}cd}d_{bcad}\nabla^{a}\Omega + \tfrac{1}{6}F^{\mathfrak{a}}{}_{ba}\nabla^{a}R \\ & -2L_{b}{}^{a}\nabla_{c}F^{\mathfrak{a}}{}_{a}{}^{c} - F^{\mathfrak{a}ac}\nabla_{c}L_{ba} - A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\square F^{\mathfrak{c}}{}_{ba} - F^{\mathfrak{c}}{}_{ba}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\square A^{\mathfrak{b}a} \\ & -2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{c}F^{\mathfrak{a}}{}_{ba}\nabla^{c}A^{\mathfrak{b}a} + \Omega F^{\mathfrak{a}ac}\nabla_{d}d_{bac}{}^{d} + 2\Omega d_{bacd}\nabla^{d}F^{\mathfrak{a}ac}. \end{aligned} \quad (5.65)$$

Next, we make use of (5.24) and (5.45) in order to remove the d'Alembertian operators in (5.65), followed by (4.23) to arrange the second order derivatives that get produced, then (5.54b), (5.54c) and (5.54a) to introduce the subsidiary variables into the equation. After doing all this (5.65) becomes

$$\begin{aligned}\square\Upsilon^{\mathfrak{a}}{}_{b} = {}& \mathfrak{S}^{\mathfrak{a}}{}_{b} - F^{\mathfrak{e}}{}_{ac}A^{\mathfrak{b}}{}_{b}A^{\mathfrak{c}a}A^{\mathfrak{d}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{f}}C^{\mathfrak{f}}{}_{\mathfrak{c}\mathfrak{g}}C^{\mathfrak{g}}{}_{\mathfrak{d}\mathfrak{e}} + F^{\mathfrak{a}cd}d_{bcad}\nabla^{a}\Omega - \Omega F^{\mathfrak{a}ac}\nabla_{d}d_{bac}{}^{d} \\ & -A^{\mathfrak{b}a}A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{d}}C^{\mathfrak{d}}{}_{\mathfrak{c}\mathfrak{e}}\nabla_{b}F^{\mathfrak{e}}{}_{ac} + 2F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{[\mathfrak{c}|\mathfrak{e}|}C^{\mathfrak{e}}{}_{b]d}\nabla_{b}A^{\mathfrak{c}c} + C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{b}F^{\mathfrak{c}}{}_{ac}\nabla^{c}A^{\mathfrak{b}a}\end{aligned}$$

$$-F^{\mathfrak{a}ac}\nabla_c L_{ba} + F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_b C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\nabla^c A^{\mathfrak{c}a} - F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^c\nabla^a A^{\mathfrak{b}}{}_b + F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^c\nabla_b A^{\mathfrak{b}a}, \quad (5.66)$$

where $\mathfrak{S}^{\mathfrak{a}}{}_b$ is a shorthand notation for all the subsidiary symbols that have been created as a result of all the substitutions, which is specifically equal to

$$\mathfrak{S}^{\mathfrak{a}}{}_b \equiv F^{\mathfrak{d}}{}_{ba}A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}}\Pi^{\mathfrak{c}} + 2L_{ba}\Upsilon^{\mathfrak{a}a} + \tfrac{1}{6}R\Upsilon^{\mathfrak{a}}{}_b - f^{\mathfrak{b}}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\Upsilon^{\mathfrak{c}}{}_b$$
$$-A^{\mathfrak{b}}{}_b A^{\mathfrak{c}a}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\Upsilon^{\mathfrak{d}}{}_a + 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{[b}A^{|\mathfrak{c}|}{}_{a]}\Upsilon^{\mathfrak{b}a} - A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}(\nabla_a\Upsilon^{\mathfrak{c}}{}_b + \nabla_b\Upsilon^{\mathfrak{c}}{}_a). \quad (5.67)$$

Now, we may eliminate the derivative of the Schouten tensor in (5.66) using (4.47), due to the fact that the Schouten tensor is symmetric and the Faraday tensor is antisymmetric and hence a contraction of the two is zero. The derivative of the Weyl tensor is eliminated using (4.48) and the second order derivatives of the gauge potential can be eliminated using (5.54c), which gives

$$\Box\Upsilon^{\mathfrak{a}}{}_b = \check{\mathfrak{S}}^{\mathfrak{a}}{}_b - F^{\mathfrak{e}}{}_{ac}A^{\mathfrak{b}}{}_b A^{\mathfrak{c}a}A^{\mathfrak{d}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{f}}C^{\mathfrak{f}}{}_{\mathfrak{c}\mathfrak{g}}C^{\mathfrak{g}}{}_{\mathfrak{d}\mathfrak{e}} - A^{\mathfrak{b}a}A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{d}}C^{\mathfrak{d}}{}_{\mathfrak{c}\mathfrak{e}}\nabla_b F^{\mathfrak{e}}{}_{ac} - F^{\mathfrak{b}ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_c F^{\mathfrak{c}}{}_{ba}$$
$$+F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}}\nabla_b A^{\mathfrak{c}c} - F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\nabla_b A^{\mathfrak{c}c} + C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_b F^{\mathfrak{c}}{}_{ac}\nabla^c A^{\mathfrak{b}a}$$
$$+F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_b C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^c A^{\mathfrak{c}a} + F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_b C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\nabla^c A^{\mathfrak{c}a} - F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^c A^{\mathfrak{c}}{}_b. \quad (5.68)$$

Where $\check{\mathfrak{S}}^{\mathfrak{a}}{}_b$ is simply (5.67) plus any additional subsidiary variables that have been created as a result of our substitutions, specifically

$$\check{\mathfrak{S}}^{\mathfrak{a}}{}_b \equiv \mathfrak{S}^{\mathfrak{a}}{}_b - \Omega F^{\mathfrak{a}ac}\Lambda_{abc} - F^{\mathfrak{a}ac}\Delta_{bac} - F^{\mathfrak{c}}{}_{bc}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^a\Pi^{\mathfrak{b}} + F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^c\mathcal{F}^{\mathfrak{b}}{}_b{}^c. \quad (5.69)$$

As before we proceed to systematically eliminate the remaining troublesome terms in a methodical step-by-step process. The key to solving this problem is to rewrite the first two terms of the last line of (5.68) in a more appropriate form. Once again, we make use of (5.56) in order to express the terms in a more symmetric fashion, applying this equation to the terms of interest and we find that said terms are equal to

$$F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_b C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^c A^{\mathfrak{c}a} + F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_b C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\nabla^c A^{\mathfrak{c}a} = -\tfrac{1}{2}F^{\mathfrak{c}ac}F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_b C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}}$$
$$+\tfrac{1}{2}F^{\mathfrak{e}}{}_{ac}A^{\mathfrak{b}}{}_b A^{\mathfrak{c}a}A^{\mathfrak{d}c}C^{\mathfrak{a}}{}_{\mathfrak{e}\mathfrak{f}}C^{\mathfrak{f}}{}_{\mathfrak{b}\mathfrak{g}}C^{\mathfrak{g}}{}_{\mathfrak{c}\mathfrak{d}} - \tfrac{1}{2}F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_b C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\mathcal{F}^{\mathfrak{c}ac}$$
$$-\tfrac{1}{2}F^{\mathfrak{e}}{}_{ac}A^{\mathfrak{b}}{}_b A^{\mathfrak{c}a}A^{\mathfrak{d}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{f}}C^{\mathfrak{f}}{}_{\mathfrak{e}\mathfrak{g}}C^{\mathfrak{g}}{}_{\mathfrak{c}\mathfrak{d}} - \tfrac{1}{2}F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_b C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}}\mathcal{F}^{\mathfrak{c}ac}. \quad (5.70)$$

Substituting (5.70) into (5.68) and we can see that multiple terms can be cancelled out with the Jacobi identity. So, grouping together the necessary terms and applying (5.41) and the wave equation for the subsidiary variable reduces to

$$
\begin{aligned}
\Box\Upsilon^{\mathfrak{a}}{}_{b} = \check{\mathfrak{S}}^{\mathfrak{a}}{}_{b} + F^{\mathfrak{c}}{}_{b}{}^{c}F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}} - \tfrac{1}{2}F^{\mathfrak{c}ac}F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_{b}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}} \\
-A^{\mathfrak{b}a}A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{d}}C^{\mathfrak{d}}{}_{\mathfrak{c}\mathfrak{e}}\nabla_{b}F^{\mathfrak{e}}{}_{ac} - F^{\mathfrak{b}ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{c}F^{\mathfrak{c}}{}_{ba} + C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{b}F^{\mathfrak{c}}{}_{ac}\nabla^{c}A^{\mathfrak{b}a} \\
-\tfrac{1}{2}F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_{b}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\mathcal{F}^{\mathfrak{c}ac} - \tfrac{1}{2}F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_{b}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}}\mathcal{F}^{\mathfrak{c}ac}.
\end{aligned}
\tag{5.71}
$$

As we can see, the wave equation has simplified greatly, all that is left to do is to deal with the remaining terms that involve derivatives of the Faraday tensor. Let us isolate the terms involving derivatives of the Faraday tensor to show that they indeed cancel out. First, we start by removing the derivative of the gauge potential by applying (5.56), then the derivatives of the Faraday tensor using (5.54c), and cancelling out multiple terms with (5.41) and (5.71) becomes

$$
\begin{aligned}
\Box\Upsilon^{\mathfrak{a}}{}_{b} = \hat{\mathfrak{S}}^{\mathfrak{a}}{}_{b} + F^{\mathfrak{c}}{}_{b}{}^{c}F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}} - \tfrac{1}{2}F^{\mathfrak{c}ac}F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_{b}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}} \\
-F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{b}A^{\mathfrak{c}c} - F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{b}\nabla^{c}A^{\mathfrak{b}a} - F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_{b}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^{c}A^{\mathfrak{c}a} \\
+F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^{c}A^{\mathfrak{c}}{}_{b} - F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^{c}\nabla^{a}A^{\mathfrak{b}}{}_{b} + F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^{c}\nabla_{b}A^{\mathfrak{b}a},
\end{aligned}
\tag{5.72}
$$

where, as before we have defined

$$
\begin{aligned}
\hat{\mathfrak{S}}^{\mathfrak{a}}{}_{b} \equiv \tfrac{1}{2}A^{\mathfrak{b}a}A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{b}\mathcal{F}^{\mathfrak{d}}{}_{ac} - \tfrac{1}{2}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\mathcal{F}^{\mathfrak{b}ac}\nabla_{b}F^{\mathfrak{c}}{}_{ac} - \tfrac{1}{2}F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{b}\mathcal{F}^{\mathfrak{b}ac} \\
+A^{\mathfrak{b}a}A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\nabla_{b}\mathcal{F}^{\mathfrak{d}}{}_{ac} - F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^{c}\mathcal{F}^{\mathfrak{b}}{}_{b}{}^{a} - \tfrac{1}{2}F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_{b}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}\mathcal{F}^{\mathfrak{c}ac} \\
+\check{\mathfrak{S}}^{\mathfrak{a}}{}_{b} - \tfrac{1}{2}F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_{b}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}}\mathcal{F}^{\mathfrak{c}ac},
\end{aligned}
\tag{5.73}
$$

i.e. we have chosen to collect all the subsidiary variables into a single term to make our equation easier to analyse. Now, returning to (5.72), we first eliminate two of the second order derivatives using (4.23); the third is eliminated using an equation which, like (5.56), is based on symmetry

$$
\nabla_{a}\nabla_{b}A^{\mathfrak{a}}{}_{c} = \tfrac{1}{2}A^{\mathfrak{a}d}R_{abcd} + \tfrac{1}{2}\nabla_{a}\nabla_{b}A^{\mathfrak{a}}{}_{c} + \tfrac{1}{2}\nabla_{b}\nabla_{a}A^{\mathfrak{a}}{}_{c}, \tag{5.74}
$$

it can be verified by applying (4.23) that (5.74) is indeed true. Upon performing all of these tasks, it can be verified that (5.72) reduces to

$$
\begin{aligned}
\Box\Upsilon^{\mathfrak{a}}{}_{b} = \hat{\mathfrak{S}}^{\mathfrak{a}}{}_{b} + \tfrac{1}{2}F^{\mathfrak{c}cd}A^{\mathfrak{b}a}R_{bacd}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} - F^{\mathfrak{c}cd}A^{\mathfrak{b}a}R_{bcad}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} - \tfrac{1}{2}F^{\mathfrak{c}ac}F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_{b}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{d}} \\
+F^{\mathfrak{e}}{}_{ac}A^{\mathfrak{b}}{}_{b}A^{\mathfrak{c}a}A^{\mathfrak{d}c}C^{\mathfrak{a}}{}_{\mathfrak{e}\mathfrak{f}}C^{\mathfrak{f}}{}_{\mathfrak{c}\mathfrak{g}}C^{\mathfrak{g}}{}_{\mathfrak{b}\mathfrak{d}} - F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}}{}_{b}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}}\nabla^{c}A^{\mathfrak{c}a} + F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{b}\mathfrak{c}}\mathcal{F}^{\mathfrak{c}c}{}_{b}.
\end{aligned}
\tag{5.75}
$$

The two terms in (5.75) containing the Riemann tensor can be eliminated using the Young projector, then using (5.56) to remove the term containing a derivative of the gauge potential and (5.75) becomes

$$\Box\Upsilon^{\mathfrak{a}}{}_{b} = \hat{\mathfrak{S}}^{\mathfrak{a}}{}_{b} + F^{\mathfrak{e}}{}_{ac}A^{\mathfrak{b}}{}_{b}A^{\mathfrak{c}a}A^{\mathfrak{d}c}C^{\mathfrak{a}}{}_{\mathfrak{ef}}C^{\mathfrak{f}}{}_{\mathfrak{cg}}C^{\mathfrak{g}}{}_{\mathfrak{bd}} - \tfrac{1}{2}F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{bc}}\nabla_{b}\mathcal{F}^{\mathfrak{b}ac}$$
$$-\tfrac{1}{2}F^{\mathfrak{a}}{}_{ac}A^{\mathfrak{b}}{}_{b}A^{\mathfrak{c}a}A^{\mathfrak{d}c}C^{\mathfrak{a}}{}_{\mathfrak{ef}}C^{\mathfrak{f}}{}_{\mathfrak{bg}}C^{\mathfrak{g}}{}_{\mathfrak{cd}} + F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{de}}C^{\mathfrak{e}}{}_{\mathfrak{bc}}\mathcal{F}^{\mathfrak{c}c}{}_{b}. \qquad (5.76)$$

Finally, the only remaining two non-subsidiary terms can be cancelled out with (5.41). So, replacing all the terms that we started out with in (5.71) with (5.76) and finally one obtains

$$\Box\Upsilon^{\mathfrak{a}}{}_{b} = \hat{\mathfrak{S}}^{\mathfrak{a}}{}_{b} - \tfrac{1}{2}F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{bc}}\nabla_{b}\mathcal{F}^{\mathfrak{b}ac} + F^{\mathfrak{d}}{}_{ac}A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{de}}C^{\mathfrak{e}}{}_{\mathfrak{bc}}\mathcal{F}^{\mathfrak{c}c}{}_{b}, \qquad (5.77)$$

or, equivalently by "unwrapping" the $\mathfrak{S}^{\mathfrak{a}}{}_{b}$ terms using (5.73), (5.69), (5.67) then (5.77) can be written as

$$\Box\Upsilon^{\mathfrak{a}}{}_{b} = F^{\mathfrak{d}}{}_{ba}A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{ce}}C^{\mathfrak{e}}{}_{\mathfrak{bd}}\Pi^{\mathfrak{c}} - \Omega F^{\mathfrak{a}ac}\Lambda_{abc} - f^{\mathfrak{b}}C^{\mathfrak{a}}{}_{\mathfrak{bc}}\Upsilon^{\mathfrak{c}}{}_{b} - \tfrac{1}{2}F^{\mathfrak{d}}{}_{ac}A_{b}C^{\mathfrak{a}}{}_{\mathfrak{be}}C^{\mathfrak{e}}{}_{\mathfrak{cd}}\mathcal{F}^{\mathfrak{c}ac}$$
$$-F^{\mathfrak{a}ac}\Delta_{bac} + 2L_{ba}\Upsilon^{\mathfrak{a}a} - A^{\mathfrak{b}}{}_{b}A^{\mathfrak{c}a}C^{\mathfrak{a}}{}_{\mathfrak{be}}C^{\mathfrak{e}}{}_{\mathfrak{cd}}\Upsilon^{\mathfrak{d}}{}_{a} + 2C^{\mathfrak{a}}{}_{\mathfrak{bc}}\nabla_{[b}A^{|\mathfrak{c}|}{}_{a]} - A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{bc}}(\nabla_{a}\Upsilon^{\mathfrak{c}}{}_{b} + \nabla_{b}\Upsilon^{\mathfrak{c}}{}_{a})$$
$$+\tfrac{1}{6}R\Upsilon^{\mathfrak{a}}{}_{b} - F^{\mathfrak{c}}{}_{bc}C^{\mathfrak{a}}{}_{\mathfrak{bc}}\nabla^{a}\Pi^{\mathfrak{b}} + \tfrac{1}{2}F^{\mathfrak{c}}{}_{ac}C^{\mathfrak{a}}{}_{\mathfrak{bc}}\nabla_{b}\mathcal{F}^{\mathfrak{b}ac} + A^{\mathfrak{b}a}A^{\mathfrak{c}c}(\tfrac{1}{2}C^{\mathfrak{a}}{}_{\mathfrak{de}}C^{\mathfrak{e}}{}_{\mathfrak{bc}} + C^{\mathfrak{a}}{}_{\mathfrak{be}}C^{\mathfrak{e}}{}_{\mathfrak{cd}})\nabla_{b}\mathcal{F}^{\mathfrak{d}}{}_{ac}, \qquad (5.78)$$

which is a homogeneous wave equation in the subsidiary variables, as required.

Summary

We have derived a set of evolution equations for the classical Yang-Mills fields coupled to 4-dimensional conformal spacetimes. Furthermore, we have shown through the propagation of the constraints that any solution to the evolution equations implies a solution to the Yang-Mills equations.

5.4 Perfect Fluids

5.4.1 The Euler equations

The last type of trace-free matter that we will analyse is that of a perfect, irrotational, pure radiation fluid [4]. In the context of GR this fluid satisfies the relativistic Euler equations. The relativistic Euler equations follow from the physical energy-momentum tensor of a fluid

$$\tilde{T}_{ab} = (\tilde{\rho} + \tilde{p})\tilde{u}_{a}\tilde{u}_{b} - \tilde{p}\tilde{g}_{ab}, \qquad (5.79)$$

where $\tilde{p}$ is the physical pressure of the fluid, $\tilde{\rho}$ is the physical density of the fluid and $\tilde{u}_a$ is the physical 4-velocity of the fluid. An important property of the 4-velocity, which is dependent on both the metric signature and the pressure, is that when we contract a 4-velocity vector with itself the end result is unity, i.e.

$$\tilde{u}_a \tilde{u}^a = 1. \tag{5.80}$$

Additionally, even though the energy-momentum tensor is written in terms of the pressure, density and the 4-velocity, the Euler equations are expressed only in terms of the density and the 4-velocity. To this end one needs an equation that relates the variables of the Euler equations to the pressure, which is the *equation of state*. The precise form of the equation of state depends on the specific fluid under consideration; for a perfect radiation fluid, the equation of state is

$$\tilde{p} = \tfrac{1}{3}\tilde{\rho}. \tag{5.81}$$

Now, using the fact that the energy-momentum tensor is divergence free (due to energy conservation), we can deduce the Euler equations by differentiating (5.80), which gives

$$\tilde{u}^a \tilde{\nabla}_a \tilde{\rho} \tilde{u}_b + \tilde{u}^a \tilde{\nabla}_a \tilde{p} \tilde{u}_b + (\tilde{p} + \tilde{\rho}) \tilde{u}_b \tilde{\nabla}^a \tilde{u}_a + (\tilde{\rho} + \tilde{p}) \tilde{u}^a \tilde{\nabla}_a \tilde{u}_b - \tilde{\nabla}_b \tilde{p} = 0. \tag{5.82}$$

Contracting (5.82) with u^b gives

$$\tilde{u}^b \tilde{\nabla}^a \tilde{T}_{ab} = \tilde{u}^a \tilde{\nabla}_a \tilde{\rho} + (\tilde{p} + \tilde{\rho}) \tilde{\nabla}^a \tilde{u}_a + (\tilde{p} + \tilde{\rho}) \tilde{u}^a \tilde{u}^b \tilde{\nabla}_a \tilde{u}_b - \tilde{u}^b = 0. \tag{5.83}$$

However, using (5.80) we can infer that

$$\tilde{\nabla}_b (\tilde{u}_a \tilde{u}^a) = 2\tilde{u}^a \tilde{\nabla}_b \tilde{u}_a = 0,$$

so we can clearly see that

$$\tilde{u}^a \tilde{\nabla}_b \tilde{u}_a = 0. \tag{5.84}$$

Therefore, making use of (5.84) and (5.83) becomes

$$\tilde{u}^a \tilde{\nabla}_a \tilde{\rho} + (\tilde{p} + \tilde{\rho}) \tilde{\nabla}^a \tilde{u}_a = 0, \tag{5.85}$$

which is the first relativistic Euler equation. To obtain the next relativistic Euler equation we need to make use of the following tensor, which decomposes a tensor into its spatial part and its timelike part

$$\tilde{h}_a{}^b = \delta_a{}^b - \tilde{u}_a \tilde{u}^b. \tag{5.86}$$

If we contract (5.86) with a 4-velocity vector then the end result is trivial i.e.

$$\tilde{h}_a{}^b \tilde{u}_b = 0. \tag{5.87}$$

We are now in a position to obtain the second Euler equation; multiplying (5.82) by $h_c{}^b$ yields

$$\tilde{h}_c{}^b (\tilde{\rho} + \tilde{p}) \tilde{u}^a \tilde{\nabla}_a \tilde{u}_b - \tilde{h}_c{}^b \tilde{\nabla}_b \tilde{p} = 0. \tag{5.88}$$

Finally, substituting (5.86) into (5.88), then multiplying out and using (5.84) to simplify gives the result

$$(\tilde{\rho} + \tilde{p}) \tilde{u}^a \tilde{\nabla}_a \tilde{u}_c + \tilde{u}_c \tilde{u}^b \tilde{\nabla}_b \tilde{p} - \tilde{\nabla}_c \tilde{p} = 0, \tag{5.89}$$

which is the second physical Euler equation. As it turns out, both (5.85) and (5.89) are invariant under a specific conformal transformation. Thanks to the freedom offered by the use of conformal transformations, we are able to choose the forms of the unphysical variables. The transformations required to preserve the forms of the Euler equations are

$$T_{ab} \equiv \Omega^{-2} \tilde{T}_{ab}, \qquad u_a \equiv \Omega \tilde{u}_a, \qquad \rho \equiv \Omega^{-4} \tilde{\rho}, \qquad p \equiv \Omega^{-4} \tilde{p}, \tag{5.90}$$

upon substituting (5.90) into (5.85) and (5.89) it can be verified that the unphysical Euler equations have the form

$$u^a \nabla_a \rho + \tfrac{4}{3} \rho \nabla_a u^a = 0, \tag{5.91a}$$

$$\tfrac{4}{3} \rho u^c \nabla_c u_a + \tfrac{1}{3} u_a u^c \nabla_c \rho - \tfrac{1}{3} \nabla_a \rho = 0, \tag{5.91b}$$

where u^a, p and ρ are the unphysical four velocity, unphysical pressure and the unphysical density of the fluid, respectively. As we can see the unphysical Euler equations have precisely the same form as the physical Euler equations and are conformally invariant as a result.

5.4.2 Alternative form of the Euler Equations

Up until this point we have been deriving wave equations via the act of directly differentiating the respective field equations. However, trying to obtain evolution equations by direct differentiation of (5.91a) and (5.91b) is more difficult since doing so produces second order derivatives of both ρ and u^a and, unlike the previous cases, we do not have any means of simplifying these second order derivatives. However, there is a way that one can construct evolution equations for fluids. It turns out that one can construct wave equations for fluids if one re-expresses the Euler equations in terms of a quantity known as the *fluid index*. The fluid index is defined as

$$f(p) \equiv \exp \int_0^p \frac{ds}{\rho(s) + s}, \tag{5.92}$$

where $\rho(s)$ is the density of the fluid written in terms of the equation of state. Physically, this fluid index provides a way of characterising the density of the fluid. From this fluid index, we can define the *dynamic 4-velocity*

$$C_a = f u_a. \tag{5.93}$$

Using the fact that $u^a u_a = 1$ then we obtain an important relation from (5.93)

$$C^a C_a = f^2. \tag{5.94}$$

We now proceed to derive an alternative form of the Euler equations. Before we analyse the Euler equations themselves, we first need to derive some equations for the derivative of the fluid index. Begin by differentiating (5.92), using the chain rule and also taking into account that for our purposes the equations are expressed in terms of pressure (meaning that $s = \rho$ in our analysis), and one obtains

$$\nabla_a f = \frac{f(p)}{\rho(p) + p} \nabla_a p. \tag{5.95}$$

We may also derive an alternative equation for the derivative for the fluid index by differentiating (5.94), which gives

$$\nabla_b f = f^{-1} C^a \nabla_b C_a. \tag{5.96}$$

We are now in the position to derive an alternate form of the Euler equations, beginning with the second equation. First we write out the second Euler equation in a more convenient form by analysing the last two terms of said equation

$$u_c u^b \nabla_b p - \nabla_c p = (u_c u^b - \delta_c{}^b) \nabla_b p, \tag{5.97}$$

then dividing through by $\rho + p$ and substituting (5.97) into (5.91b) and the second Euler equation has the form

$$u^a \nabla_a u_c = \frac{u_c u^b}{\rho + p} \nabla_b p - \frac{1}{\rho + p} \nabla_c p. \tag{5.98}$$

Now, we rewrite all of the terms in (5.98) in terms of the dynamic 4-velocity, using (5.93) and the first term becomes

$$u^a \nabla_a u_b = f^{-2}(C^a \nabla_a C_b - f^{-1} C^a \nabla_a f C_b), \tag{5.99}$$

and by using (5.95) we can rewrite the last two terms of (5.98) as

$$\frac{u_c u^b}{\rho + p}\nabla_b p - \frac{1}{\rho + p}\nabla_c p = f^{-3}C^b C_c \nabla_b f - f^{-1}\nabla_c f. \tag{5.100}$$

So, by substituting in (5.99) and (5.100) into (5.98) and using (5.96) to eliminate the derivative of the fluid index, it can be verified that the second Euler equation has the form

$$C^a(\nabla_a C_b - \nabla_b C_a) = 0.$$

The first Euler equation can also be re-expressed in a form that depends on the dynamic 4-velocity. As before we proceed to rewrite all of the individual terms in the equation. Begin by applying (5.93) and (5.96) to the first term in (5.91a), which gives

$$\nabla^a u_a = -f^{-3}C^a C^c \nabla_a C_c + f^{-1}\nabla C_a. \tag{5.101}$$

To rewrite the second term in the first Euler equation $\nabla_a \rho$, we make use of the rule for the derivative of inverse functions $\frac{d\rho}{dp} = \frac{1}{dp/d\rho}$, as well as the chain rule. Upon doing so it can be verified that the second term in (5.91a) becomes

$$u^a \nabla_a \rho = u^a \frac{d\rho}{dp}\frac{(\rho + p)}{f}\nabla_a f. \tag{5.102}$$

Thus, if we substitute (5.101) and (5.102) into (5.91a) and divide through by a factor of $\rho + p$, the equation takes the form

$$-f^{-3}C^a C^c \nabla_a C_c + f^{-1}\nabla^a C_a + f^{-1}\frac{d\rho}{dp}u^a \nabla_a f = 0. \tag{5.103}$$

Now letting $d\rho/dp = \rho'$ and using (5.96) to remove the derivative of the fluid index and (5.103) becomes

$$\nabla^a C_a + f^{-2}C^a C^c(\rho' - 1)\nabla_a C_c = 0. \tag{5.104}$$

We then proceed to express everything in terms of C^a. Recall that $C^a C_a = f^2$, so substituting into (5.104) and the end result is

$$\nabla^a C_a + \frac{(\rho' - 1)}{C_b C^b}C^a C^c \nabla_a C_c = 0.$$

Now, in order to make future calculations more manageable we can chose to define a new variable

$$P \equiv \frac{1}{C^b C_b}, \tag{5.105}$$

we can apply this to the alternate form of the first Euler equation. So in summary the alternate form of the Euler equations in the case where the equation of state is given by (5.81) are

$$\nabla^a C_a + P(\rho' - 1) C^a C^c \nabla_a C_c = 0, \tag{5.106a}$$
$$C^a(\nabla_a C_b - \nabla_b C_a) = 0. \tag{5.106b}$$

Conformal properties of the alternate Euler equations

An important property of the Euler equations in this form is that they are invariant under a conformal rescaling, which means that it is possible to analyse radiation fluids and to derive evolution equations from a conformal point of view. First, we define a rescaled metric

$$\bar{g} = f^2 g, \tag{5.107}$$

using (1.3) and then analyse the behaviour of the second Euler equation under such a transformation. Applying (2.13) to (5.106b) yields

$$\bar{\nabla}_a C_b - \nabla_a C_b = -S_{ab}{}^{cd} \Upsilon_c C_d, \tag{5.108}$$

where $\Upsilon_c = f^{-1}\nabla_c f$. Then substituting in (2.13) into (5.108) and multiplying out gives

$$\bar{\nabla}_a C_b - \nabla_a C_b = -f^{-1}\nabla_a f C_b + f^{-1} C_a \nabla_b f + f^{-1} g_{ab} \nabla_c f C^c \tag{5.109}$$

We then use (5.96) to remove the derivative of the fluid index in (5.109); upon doing so and multiplying out one obtains the relation

$$\nabla_a C_b = \bar{\nabla}_a C_b + f^{-2} C^c \nabla_a C_c C_b + f^{-2} C_a C^c \nabla_b C_c - f^{-2} g_{ab} C^c C^d \nabla_c C_d. \tag{5.110}$$

By directly substituting (5.110) into (5.106b) and simplifying it can be verified that

$$C^a(\nabla_a C_b - \nabla_b C_a) = C^a(\bar{\nabla}_a C_b - \bar{\nabla}_b C_a) = 0. \tag{5.111}$$

So the equation is invariant under a conformal transformation and it is perfectly reasonable to utilize both of the Euler equations written in terms of the dynamic 4-velocity when making use of conformal methods. These particular transformation laws also help to give an actual physical understanding of the nature of the dynamic 4-velocity; if we multiply out (5.111), then it can be verified that

$$C^a \bar{\nabla}_a C_b = 0, \tag{5.112}$$

which is the same form as the geodesic equation. This means that C_a is a vector tangent to geodesics of the metric (5.107); crucially this is true only for this specific choice of metric.

5.4.3 Wave equations for the fluids

Now that we have obtained a more satisfactory set of equations for fluids from a conformal point of view, we will proceed to derive a set of wave equations that describe the evolution of fluids. However, before we proceed to perform such a task, we must mention some very important considerations for this particular case. Firstly, the wave equations will not be of the same form as any of the previous results; whereas the principal part of the previous wave equations was the d'Alembertian operator of the metric g_{ab}, this will not be the case for evolution equations for fluids. The reason for this is due to the fact that the d'Alembertian operator describes objects that propagate at the speed of light (evident by the $1/c^2$ term that appears in the d'Alembertian). However, in general, fluids do not propagate at light speeds, they propagate at the speed of sound. Although the individual particles of a radiation fluid are indeed massless, when one analyses this fluid using statistical mechanics it can be shown that in fact that speed of said fluid is less than light speed. As a result, the principal part of the wave equations will be a modified operator, whose precise form shall be obtained during the actual derivations.

The next point to take into consideration is that we will only be considering a very specific class of perfect fluid, an irrotational fluid. An irrotational fluid, as the name implies, is one where the fluid lines have no tendency to rotate or twist. In this case this limits the form of the dynamic 4-velocity

$$C_a = \nabla_a \Phi, \tag{5.113}$$

where Φ is a scalar field. For an irrotational fluid then (5.113) is a solution to (5.106b). Physically, this scalar field may be thought of as an imaginary flux surface that provides a way of quantifying the number of fluid lines that pass through a certain region.

Wave equation for Φ

To derive some wave equations for the fluid variables we start by rearranging (5.106a), then applying (5.113) to the contracted derivative of the dynamic 4-velocity and then we apply (5.105), which gives

$$\Box\Phi = P(1-\rho')C^a C^c \nabla_c C_a. \tag{5.114}$$

Now, using the equation of state (5.81), we can see that $\rho' = 3$; therefore the wave equation for the scalar field is

$$\Box\Phi = -2PC^aC^c\nabla_c\nabla_a\Phi. \tag{5.115}$$

We have not quite finished with this equation as we have yet to define the modified wave operator that was mentioned at the start of this section. The form of the modified wave operator will become much more apparent when we derive the equation for the vortex velocity.

Wave equation for C_a

To obtain a wave equation for the vortex velocity C_a we differentiate (5.106b) and apply (4.23), (3.29), (5.106a) and (5.106b), doing so gives

$$\begin{aligned}\Box C_b = 2L_b{}^aC_a + \tfrac{1}{6}RC_b - 2PC^a\nabla_aC_c\nabla_bC^c - 2C^aC^c\nabla_bP\nabla_cC_a \\ -2PC^a\nabla_bC^c\nabla_cC_a - 2PC^aC^c\nabla_c\nabla_aC_b.\end{aligned} \tag{5.116}$$

Examining both this equation and (5.115), we can clearly see the form of the modified wave operator, and can define the so-called *modified d'Alembertian*

$$\check{\Box} \equiv \Box + 2PC^aC^c\nabla_c\nabla_a, \tag{5.117}$$

this operator is still valid as a hyperbolic operator, because even though it contains second order derivatives other than the d'Alembertian, the dynamic 4-velocity is a small quantity compared to the speed of light and hence (5.117) is very close to the original d'Alembertian and retains the same properties as the d'Alembertian. So applying (5.117) to both (5.115) and (5.116) and the equations have the form

$$\check{\Box}\Phi = 0, \tag{5.118a}$$

$$\begin{aligned}&\check{\Box}C_b = 2L_b{}^aC_a + \tfrac{1}{6}RC_b - 2PC^a\nabla_aC_c\nabla_bC^c \\ &-2C^aC^c\nabla_bP\nabla_cC_a - 4PC^a\nabla_bC^c\nabla_cC_a.\end{aligned} \tag{5.118b}$$

This completes the derivation of the wave equations for the fluids, since the equations are written only in terms of the dynamic 4-velocity, which helps to simplify things considerably.

5.4.4 Subsidiary equations for the fluids

Now that we have derived some wave equations for irrotational fluids that are satisfactory from a conformal point of view, we now need to check that a solution to the wave equations implies a solution to the Euler equations, which means constructing a system of subsidiary equations for the Euler equations

$$H \equiv \nabla_c C^a + 2PC^a C^c \nabla_c C_a, \tag{5.119a}$$
$$\varepsilon_{ab} \equiv \nabla_a C_b - \nabla_b C_a, \tag{5.119b}$$
$$N_a \equiv C_a - \nabla_a \Phi. \tag{5.119c}$$

Wave Equation for H

As has been the case before, we will proceed to construct a system of homogeneous wave equations for the subsidiary variables. We start with the wave equation for (5.119a), differentiating said equation gives

$$\begin{aligned}\Box H = \Box \nabla_a C^a + 4C^a \nabla_a C^c \nabla_b C_c \nabla^b P + 2C^a C^b \nabla_b C_a \Box P + 2PC^a \nabla^b C_a \Box C_b \\ +2PC^a C^b \Box \nabla_b C_a + 4C^a C^b \nabla_c \nabla_b C_a \nabla^c P + 4C^a \nabla_b C_c \nabla^b P \nabla^c C_a \\ +4P\nabla^b C^a \nabla_c C_b \nabla^c C_a + 4PC^a \nabla_c \nabla_a C_b \nabla^c C^b + 4PC^a \nabla_c \nabla_b C_a \nabla^c C^b. \end{aligned} \tag{5.120}$$

Then, applying (4.23) to reorder the covariant derivatives, (2.12), (3.29) and (2.18) to remove the curvature components, plus (5.116) to get rid of the second order derivatives of the Vortex velocity and we get

$$\Box H = 0. \tag{5.121}$$

This is a trivially homogeneous solution, since we could easily state that a solution to (5.121) occurs when all the variables are equal to zero, hence the propagation of the constraints is satisfied.

Wave Equation for ε_{ab}

Next we examine the wave equation for the subsidiary variable of the second Euler equation. Differentiating (5.119b) and applying (4.23) gives

$$\begin{aligned}\Box \varepsilon_{ab} = \nabla_a \Box C_b - \nabla_b \Box C_a + C_d \nabla_c R^d{}_b{}^c{}_a + R^d{}_a{}^c{}_b \nabla_c C_d \\ -R^d{}_b{}^c{}_a \nabla_c C_d - R_{bc} \nabla^c C_a + R_{bc} \nabla^c C_b + R^d{}_{acb} \nabla^c C_d - R^d{}_{bca} \nabla^c C_d. \end{aligned} \tag{5.122}$$

The next step is to use the wave equation for the dynamic 4-velocity to eliminate the derivatives of C_a; doing so creates an equation with a very large number of terms. In spite of this, there is a great deal of symmetry among the various terms. We may take advantage of this symmetry by once again applying (4.23) in a very specific order, such that multiple second order derivative terms cancel out. Doing so and applying both (2.12) and (3.29) to eliminate the curvature terms and we get

$$\begin{aligned}\Box\varepsilon_{ab} &= \tfrac{1}{6}C_b\nabla_a R + C^c\nabla_a L_{bc} + \tfrac{1}{3}R\nabla_a C_b - \tfrac{1}{6}C_a\nabla_b R - C^c\nabla_b L_{ac} - \tfrac{1}{3}R\nabla_b C_a \\ &-C_b\nabla_c L_b{}^c + \Omega C^c\nabla_d d_{acb}{}^d - \Omega C^c\nabla_d d_a{}^d{}_{bc} + d_{acbd}C^c\nabla^d\Omega - d_{adbc}C^c\nabla^d\Omega + 2\Omega d_{acbd}\nabla^d C^c \\ &-2\Omega d_{adbc}\nabla^d C^c + 2\Omega P d_{abdf}C^cC^d\nabla^f C_c - 2\Omega P d_{adbf}C^cC^d\nabla^f C_c + 2\Omega d_{afbd}\nabla^f C_c. \quad (5.123)\end{aligned}$$

We then use both (4.47) and (2.18) to eliminate the derivatives of the Schouten tensor as well as (4.48) to remove the contracted derivatives of the Weyl tensor, which gives

$$\begin{aligned}\Box\varepsilon_{ab} &= \Omega C^c(T_{abc} - T_{acb} + T_{bca}) + \Omega C^c\Lambda_{abc} - \Omega C^c\Lambda_{bac} + C^c\Delta_{abc} \\ &+\tfrac{1}{3}R\nabla_a C_b - \tfrac{1}{3}R\nabla_b C_a + (d_{acbd} - d_{abcd} - d_{adbc})C^c\nabla^d\Omega + 2\Omega d_{acbd}\nabla^d C^c \\ &-2\Omega d_{adbc}\nabla^d C^c + 2\Omega P C^cC^d\nabla^f C_c(d_{abdf} - d_{adbf} + d_{afbd}), \quad (5.124)\end{aligned}$$

we can see that there are multiple terms that can be cancelled out and have been factorized as a result. Applying (4.11) and (2.30) to cancel out said terms and then using (5.119b) to eliminate the antisymmetric pair of derivatives of the dynamic 4-velocity and we are left with

$$\Box\varepsilon_{ab} = \tfrac{1}{3}R\varepsilon_{ab} + \Omega C^c\Lambda_{abc} - \Omega C^c\Lambda_{bac} + C^c\Delta_{abc} + 2\Omega d_{acbd}\nabla^d C^c - 2\Omega d_{adbc}\nabla^d C^c. \quad (5.125)$$

The final step is to apply the Bianchi identity to the remaining two terms that are not homogeneous in the subsidiary variables, upon doing so the result is

$$\Box\varepsilon_{ab} = \tfrac{1}{3}R\varepsilon_{ab} + \Omega C^c\Lambda_{abc} - \Omega C^c\Lambda_{bac} + C^c\Delta_{abc} + 2\Omega d_{abcd}\nabla^d C^c. \quad (5.126)$$

We can see that (5.126) is nearly homogeneous, except for the last term. To get rid of this term we will make use of the definition of the zero quantity N_a, differentiating (5.119c) and rearranging gives rise to the equation

$$\nabla^d C^c = \nabla^d\nabla^c\Phi - \nabla^d N^c. \quad (5.127)$$

Applying (5.127) to (5.126) and the term containing the second order derivative of the fluid scalar vanishes since the term $\nabla_a\nabla_b\Phi$ is a symmetric quantity, which is contracted with an antisymmetric pair of indices on the Weyl tensor and the contraction of a symmetric object with an antisymmetric object is zero. Hence (5.126) becomes

$$\Box\varepsilon_{ab} = \tfrac{1}{3}R\varepsilon_{ab} + \Omega C^c\Lambda_{abc} - \Omega C^c\Lambda_{bac} + C^c\Delta_{abc} - 2\Omega d_{abcd}\nabla^d N^c, \tag{5.128}$$

which is homogeneous in the subsidiary variables.

Wave equation for N_a

Finally, we proceed to construct a homogeneous wave equation for the quantity related to the fluid scalar N_a. Differentiating (5.119c) and applying (4.23), then using (5.115) and the result

$$\Box N_a = 2PC^bC^c\nabla_a\nabla_c\nabla_b\Phi + 2C^bC^c\nabla_a P\nabla_c\nabla_b\Phi + \Box C_a - R_{ac}\nabla^c\Phi. \tag{5.129}$$

Now, for reasons that will become apparent in a very short while, it is necessary to apply (4.23) to the third order derivative of the scalar field such that all contracted indices do not act directly on the scalar field. Doing so and applying (5.119c), (5.115) and (5.118b) gives

$$\begin{aligned}\Box N_a = \tfrac{1}{6}RN_a + 2PL_{bc}C^bC^cN^d - 2PL_{ac}C^bC^cN_b - 2PL_{bc}C_aC^bN^c\\ +2PL_{ac}C_bC^bN^c - 2\Omega Pd_{abcd}C^bC^cN^d - 2PC^b\varepsilon_{bc}\nabla_aC^c - 2PC^bC^c\nabla_c\varepsilon_{ab}\\ +2L_{ab}N^b - 2C^bC^c\nabla_aP\nabla_cN_b - 4PC^b\nabla_aC^c\nabla_cN_b - 2PC^bC^c\nabla_c\nabla_bN_a.\end{aligned} \tag{5.130}$$

It is here that we see the reason for commuting the derivatives of the third order derivative. In (5.130), there is a second order derivative of N_a, however, when we combine this with the d'Alembertian on the right hand side of the equation we notice that they form a modified d'Alembertian. Hence, (5.130) upon the application of (5.117) takes the form

$$\begin{aligned}\check{\Box} N_a = \tfrac{1}{6}RN_a + 2PL_{bc}C^bC^cN^d - 2PL_{ac}C^bC^cN_b - 2PL_{bc}C_aC^bN^c\\ +2PL_{ac}C_bC^bN^c - 2\Omega Pd_{abcd}C^bC^cN^d - 2PC^b\varepsilon_{bc}\nabla_aC^c - 2PC^bC^c\nabla_c\varepsilon_{ab}\\ +2L_{ab}N^b - 2C^bC^c\nabla_aP\nabla_cN_b - 4PC^b\nabla_aC^c\nabla_cN_b.\end{aligned} \tag{5.131}$$

Although this is the only one of the subsidiary equations to feature the modified d'Alembertian, this is not a problem since this modified d'Alembertian, as mentioned previously, possesses all the same properties as the standard d'Alembertian. Plus, this is not unusual since we are working with objects that do not propagate at light speed and hence one would expect some of the equations describing their behaviour to be different.

Summary

We have derived a set of valid wave equations for the evolution of an irrotational fluid. Furthermore, we have seen from the propagation of the constraints that any solution to the wave equations implies a solution to the Euler equations.

6 The Conformal Scalar Field System

Thus far in this thesis we have been analysing spacetimes coupled to trace-free matter purely from point of view of the CWEs, however, we will now attempt to derive results for the CFEs. More specifically, we will consider *spatially homogeneous* spacetimes coupled to a specific trace-free matter model. A spatially homogeneous spacetime is one that has the property that when a spacetime is described using the 3+1 decomposition, it is possible to choose a certain coordinate system such that all spatial derivatives of the metric are zero at a certain instant of time. It is not unreasonable to study spatially homogeneous spacetimes since one of the cornerstones of modern Cosmology states that the Universe is indeed spatially homogeneous on a large enough scale of distance.

6.1 Warped product metrics

A particularly useful way of analysing a spatially homogeneous spacetime is to express the metric of said spacetime as a *warped product*. Essentially, this technique decomposes the metric into two different block matrices; each block is described only by some of the coordinates [14]. These coordinates are written as $x^\mu = (x^A, x^i)$ where $A = 0, ..., m$, $i = m+1, ..., n$ and n is the dimension of the spacetime. With this set of coordinates, the general form of the warped product metric is given by

$$g = h_{AB} dx^A dx^B - f^2 k_{ij} dx^i dx^j, \tag{6.1}$$

where h_{AB} is a matrix whose components depend only on the x^A coordinates, k_{ij} is a matrix whose components depend on the x^i coordinates and f is a scalar that is a function of the x^A coordinates. In matrix form (6.2) is given as

$$g_{\mu\nu} = \left[\begin{array}{c|c} h_{AB} & 0 \\ \hline 0 & f k_{ij} \end{array}\right]. \tag{6.2}$$

Now, the precise form of (6.2) depends on the choice of coordinates that we use. For example, if the coordinates were spherical angular coordinates, then the two different matrices h_{AB} and $f k_{ij}$ in (6.2) would be two 2×2 matrices, one containing functions of the (t, r) coordinates and one containing functions of the

(θ, ϕ) coordinates. For our specific case, we will choose for the metric to be separated into time indices and space indices. So, for our specific case the Latin indices a, b, c, d refer to spacetime indices, A, B, C refer to time indices and i, j, k, l, m refer to spatial indices. For the case of a spatially homogeneous and isotropic spacetime the metric has the form

$$g = -d\tau^2 + l^2 k_{ij} dx^i dx^j, \tag{6.3}$$

where l is a scalar that takes the form of f and is simply denoted as such to distinguish the fact that we are working with cosmological models. It should be noted k_{ij} is a metric of constant curvature as this is a spatially homogeneous spacetime, meaning that the spatial derivatives of the curvature on an individual hypersurface must be zero. Written out in the form of (6.2) and (6.3) is equal to

$$g_{\mu\nu} = \left[\begin{array}{c|ccc} g_{00} & 0 & 0 & 0 \\ \hline 0 & k_{11} & k_{12} & k_{13} \\ 0 & k_{21} & k_{22} & k_{23} \\ 0 & k_{31} & k_{32} & k_{33} \end{array}\right]. \tag{6.4}$$

6.2 Computation of the geometric quantities for warped product metrics

6.2.1 Christoffel Symbols

Given the metric (6.4), one would like to compute the curvature; to do so requires one to compute the Christoffel symbols. When one computes the Christoffel symbols associated with this metric, it is important to keep in mind the two different types of indices that feature in (6.2). We will now compute the various Christoffel symbols for (6.4), starting with the symbol containing only pure timelike indices; substituting said indices into the metric gives

$$\Gamma^A{}_{BC} = \frac{1}{2} h^{AD} (\partial_C h_{BD} + \partial_B h_{CD} - \partial_D h_{BC}). \tag{6.5}$$

Since the only time index is $A = B = C = 0$ this means that the only pure timelike Christoffel symbol is $\Gamma^0{}_{00}$, which is equal to

$$\Gamma^0{}_{00} = \frac{1}{2} g^{0\rho} (\partial_0 g_{\rho 0} + \partial_0 g_{0\rho} - \partial_\rho g_{00}), \tag{6.6}$$

we can see from (6.4) that $g^{0\rho} = 0$ when $\rho \neq 0$, which gives

$$\Gamma^0{}_{00} = \frac{1}{2g_{00}} \partial_0 g_{00}. \tag{6.7}$$

Now, we can read from the metric that the only component is $g_{00} = -1$; substituting this value into (6.7) and we find that

$$\Gamma^0{}_{00} = 0. \tag{6.8}$$

Likewise, it is possible to compute an equation for Christoffel symbols with purely spatial indices via substitution into (6.4), which yields

$$\Gamma^j{}_{il} = \frac{1}{2}k^{jm}(\partial_l k_{im} + \partial_i k_{lm} - \partial_m k_{il}). \tag{6.9}$$

Finally, there are also Christoffel symbols with mixed indices to consider. Thankfully, due to the form of (6.4) and also the fact that $l = 1$, there are only a very small number of Christoffel symbols with mixed indices that are non-zero. We will show a direct computation of one of the non-vanishing Christoffel symbols as an example, namely $\Gamma^A{}_{ij}$. Substituting $a = 0$, $b = i$, $c = j$ into (6.4) gives

$$\Gamma^0{}_{ij} = \tfrac{1}{2}g^{A\mu}(\partial_i g_{j\mu} + \partial_j g_{\mu i} - \partial_\mu g_{ij}), \tag{6.10}$$

however, $g^{0\mu}$ is only non-zero if $0 = \mu$. Using this fact and the fact that $g_{AB} = h_{AB} = h_{00}$ and we can show that (6.10) has the form

$$\Gamma^i_{0j} = \Gamma^i_{j0} = \frac{1}{l}l'k^i{}_j, \tag{6.11}$$

where l' is a derivative of the function l with respect to time. Through an identical method, it can be shown that the other mixed Christoffel symbols vanish

$$\Gamma^0{}_{0i} = 0, \qquad \Gamma^i{}_{00} = 0. \tag{6.12}$$

6.2.2 Riemann tensor components

In addition to computing Christoffel symbols of pure and mixed indices, one can also do the same thing for the 4-dimensional curvature; by substituting (6.5), (6.9) and (6.11) into (2.7) then it can be verified that the components of the Riemann curvature tensor for a warped product metric are equal to

$$R^0{}_{000}[g] = R^0{}_{000}[k] = 0, \tag{6.13a}$$

$$R^i{}_{0j0} = -k^i{}_j\frac{1}{l}D_0D_0l, \tag{6.13b}$$

$$R^i{}_{jml}[g] = R^i{}_{jkl}[k] + 2(l')^2k^i_{[j}k_{k]l}, \tag{6.13c}$$

where

$$R_{ijml}[k] = 2\varepsilon k_{i[m}k_{l]j}. \tag{6.14}$$

In fact (6.14) is a direct result k_{ij} being of constant curvature; one can readily verify that $D_k R_{ijml}[k] = 0$.

6.2.3 Ricci tensor components

Once we have computed the Riemann curvature terms, the next logical step is compute the Ricci and Schouten tensors along with the Ricci scalar:

$$R_{00}[\tilde{g}] = R^a{}_{0a0}[\tilde{g}] = R^0{}_{000}[\tilde{g}] + R^i{}_{0i0}[\tilde{g}],$$

from which we can deduce that the purely time like Ricci tensor component is

$$R_{00}[\tilde{g}] = -k_0{}^0 \frac{1}{l} D_0 D_0 l = -\frac{3}{l} l''. \tag{6.15}$$

We now proceed to compute the purely spatial Ricci tensor component, which is a somewhat lengthy calculation

$$R_{ij}[\tilde{g}] = R^0{}_{i0j}[\tilde{g}] + R^k{}_{ikj}[\tilde{g}]. \tag{6.16}$$

While not immediately obvious, it can be shown that $R^0{}_{i0j}[\tilde{g}] = -R_{i0j0}[\tilde{g}]$. Begin with the equation

$$R_{i0j0} = g_{i0} R^0{}_{00j} + g_{in} R^n{}_{0j0}, \tag{6.17}$$

however, $R_{i0j0} = R_{0i0j}$ due to symmetry, so (6.17) becomes

$$R^0{}_{i0j} = g^{00} R_{0i0j}, \tag{6.18}$$

but $g^{00} = -1$, due to the metric signature, hence (6.18) can be written as

$$R^0{}_{i0j} = -R_{0i0j}. \tag{6.19}$$

Therefore, substituting in (6.13b) into (6.19) and we conclude that

$$R^0{}_{i0j} = k_{ij} l l''. \tag{6.20}$$

That is half of the terms in (6.16) calculated; we also need to calculate $R^k{}_{ikj}[\tilde{g}]$

$$R^k{}_{ikj}[\tilde{g}] = 2\varepsilon k^m_{[m} k_{j]i} + 2(l')^2 k^m_{[m} k_{j]i}. \tag{6.21}$$

Substituting in both (6.20) and (6.21) into (6.16) gives

$$R_{ij}[\tilde{g}] = k_{ij} l l'' + 2\varepsilon k^m_{[m} k_{j]i} + 2(l')^2 k^m_{[m} k_{j]i}. \tag{6.22}$$

Expanding the anti-commutator brackets and we find that the spatial Ricci tensor is given as

$$R_{ij} = (l''l + 2\varepsilon + 2(l')^2)k_{ij}. \tag{6.23}$$

From the Ricci tensor, we can compute the Ricci scalar

$$R = g^{ab}R_{ab}, \tag{6.24}$$

however, (6.24) is written in terms of spacetime indices, thus we need to decompose (6.24) into both time and space indices,

$$R = g^{00}R_{00} + g^{ij}R_{ij}. \tag{6.25}$$

Taking into account that $g^{00} = -1$ and $g^{ij} = l^{-2}k^{ij}$ and substituting in (6.15) and (6.23) into (6.25) then one obtains

$$R[\tilde{g}] = \frac{6}{l^2}(ll'' + \varepsilon + (l')^2). \tag{6.26}$$

Through a similar method, the Ricci curvature components with mixed indices can be verified to vanish

$$R_{0i} = R_{i0} = 0, \tag{6.27}$$

which is due to the fact that the Riemann curvature tensor components that one contracts are zero.

6.2.4 Schouten tensor components

So we have derived the Riemann and Ricci tensors, as well as the Ricci scalar for a spatially homogeneous spacetime. Given this it becomes natural to derive the Schouten tensor components for such a spacetime. Begin by computing the pure time components of the Schouten tensor by substituting time indices into (2.9)

$$L_{00}[\tilde{g}] = \frac{1}{2}R_{00} - \frac{1}{12}Rg_{00}. \tag{6.28}$$

Now, taking into account the metric signature and substituting in (6.15) we can see that, upon simplification, (6.28) can be written as

$$L_{00}[\tilde{g}] = \frac{1}{2l^2}(2ll'' + \varepsilon + (l')^2). \tag{6.29}$$

Next, we proceed to compute L_{ij} i.e. the spatial components of the Schouten tensor. By substituting in (6.23) and (6.26) into (2.9) and simplifying gives,

$$L_{ij}[\tilde{g}] = \frac{1}{2}k_{ij}(\varepsilon + (l')^2). \tag{6.30}$$

As for the other Schouten tensor components, again through simple substitution, one can easily verify that

$$L_{i0}[\tilde{g}] = L_{0i}[\tilde{g}] = 0. \tag{6.31}$$

So, we have derived all the curvature components of the spacetime, however, there is still more that we can do with the curvature terms by considering the Ricci scalar. The remarkable thing about the Ricci scalar is that it is not a variable that appears in the CFEs. As a result of this, one is free to choose the value of the Ricci scalar; naturally one always chooses R in such a way that it simplifies the equation as much as possible. In our case we will choose the value of the Ricci scalar to be $R[\tilde{g}] = 6\varepsilon$; upon substituting this value back into (6.26) one obtains

$$6\varepsilon = \frac{6l''}{l} + \frac{6\varepsilon}{l^2} + \frac{(l')^2}{l^2}.$$

We can clearly see one possible solution to the equation is when, $l = 1$ and $l' = l'' = 0$. Furthermore, a known result from the general theory of ODEs tells us that since this is a second order non-linear ODE this is the only possible solution. Hence, through a specific choice of gauge it is possible to deduce a specific value for the scalar function l and its derivatives. Substituting $l = 1$ and $l' = l'' = 0$ into (6.15), (6.23), (6.30) and (6.29) enables us to rewrite the curvature terms as

$$R_{ij} = 2\varepsilon k_{ij}, \quad R_{00} = 0, \quad L_{00} = \frac{\varepsilon}{2}, \quad L_{ij} = \frac{\varepsilon}{2}k_{ij}. \tag{6.32}$$

6.3 Evolution equations for the scalar field system

Recall in section 5.1, we first introduced the conformally invariant scalar matter model, which obeys a wave equation first defined as (5.2), and whose energy-momentum tensor is given by equation (5.8). Our goal will be to derive a system of evolution equations for a spatially homogeneous solution with matter content described by (5.8).

6.3.1 Components of the energy-momentum tensor

First, we discuss a few properties of this energy-momentum tensor. The first two most noteworthy things are that this tensor is both trace and divergence free, i.e. $g^{ab}T_{ab} = 0$ and $\nabla^a T_{ab} = 0$. Another important point that we should make is that because the spacetime is spatially homogeneous, the scalar field ϕ is a function of time only, hence $\nabla_\mu \phi = 0$ for $\mu \neq 0$. Armed with this piece of information and taking into account the warped-product nature of the spacetime, we can calculate the different components of the energy-momentum tensor. We will start by computing the time like components of the tensor, substituting time indices into (5.8) gives

$$T_{00} = \nabla_0\phi\nabla_0\phi - \frac{1}{4}g_{00}\nabla_c\phi\nabla^c\phi - \frac{1}{2}\phi\nabla_0\nabla_0\phi + \frac{1}{2}\phi^2 L_{00}, \tag{6.33}$$

letting $\nabla_0\phi = \phi'$ and substituting in (6.32) and it can be verified that (6.33) can be written as

$$T_{00} = (\phi')^2 + \frac{1}{4}\nabla_c\phi\nabla^c\phi - \frac{1}{2}\phi\phi'' + \frac{1}{4}\phi^2\varepsilon. \tag{6.34}$$

It is possible to simplify the above equation further, however, when analysing the $\nabla_c\phi\nabla^c\phi$ term one has to be careful due the metric signature

$$\frac{1}{4}\nabla_c\phi\nabla^c\phi = \frac{1}{4}g_{ab}\nabla^a\nabla^b\phi = g_{00}\nabla^0\nabla^0\phi, \tag{6.35}$$

and since $g_{00} = -1$, using (6.35) and (6.34) can be written as

$$T_{00} = \frac{1}{4}(3(\phi')^2 - 2\phi\phi'' + \phi^2\varepsilon). \tag{6.36}$$

It is possible to simplify this equation further using (5.2), once again taking into count that the scalar field is a function of time, meaning $\Box\phi = \phi''$ and remembering our choice of gauge i.e. $R = 6\varepsilon$, means that we may write out the wave equation as

$$\phi'' = -\varepsilon\phi. \tag{6.37}$$

We notice that this equation is of the same form as the harmonic oscillator equation, $\ddot{x} = -kx$ where k is a constant. This is quite a remarkable result, it means that we may find some physical interpretation of this system and indeed of this scalar field. Substituting (6.37) into (6.36) and we obtain

$$T_{00} = \frac{3}{4}((\phi')^2 + \varepsilon\phi^2). \tag{6.38}$$

We can write this in a nice compact form by using the following substitution

$$\rho \equiv ((\phi')^2 + \varepsilon\phi^2), \tag{6.39}$$

then (6.38) is equal to

$$T_{00} = \frac{3}{4}\rho. \tag{6.40}$$

We can also derive an equation for T_{ij}, i.e. the spatial components of the energy-momentum tensor, in a similar way. Substituting the necessary spatial components from (6.32), simplifying and writing in a compact form, again using (6.39), and the end result is

$$T_{ij} = \frac{1}{4}k_{ij}\rho. \tag{6.41}$$

We notice that our quantity ρ has a form that is quite similar to the energy of the harmonic oscillator. Hence we may interpret these components as being the energy density of the scalar field.

6.3.2 Derivation of the evolution equations

We next need to write out the CFEs in our goal to derive a set of evolution equations for the spacetime. Consider the general form of the CFEs (3.44a)-(3.44e). Another important fact about this spatially homogeneous spacetime is that it is conformally flat, which we first mentioned in section 2.2.1, meaning that the Weyl tensor vanishes. As a result of this (3.44c) and (3.44d) are trivially satisfied.

Another important point, which further helps to simplify the calculations, is to realize that this particular spacetime when conformally extended to a global region becomes equivalent to the de Sitter spacetime. This is due to a known result, which says that any spacetime when globally extended has a spacelike conformal boundary. A spacelike conformal boundary always has a positive value for the cosmological constant, exactly like de Sitter [23, 36]. In this way we are able to choose the value of the cosmological constant to be the same as deSitter i.e. $\lambda = 3$, which means that (3.44e) is equal to

$$2\Omega s + (\Omega')^2 = 1. \tag{6.42}$$

We may derive some more evolution equations for this particular choice of gauge. First, by substituting (6.32) and (6.38) into (3.44a), we get

$$\Omega'' = -\frac{\Omega\varepsilon}{2} - s + \frac{3}{8}\Omega^3\rho. \tag{6.43}$$

In the case of spatial indices (3.44a) becomes

$$-\frac{\Omega\varepsilon}{2} + s + \frac{1}{8}\Omega^3\rho = 0. \tag{6.44}$$

And finally (3.44b), which has only non-vanishing time components, reduces to

$$s' = -\frac{(\Omega')\varepsilon}{2} + \frac{3}{8}\Omega^2\Omega'\rho. \tag{6.45}$$

Our intention is to derive a set of evolution equations that uniquely determine the spacetime. To do so we must derive a set of evolution equations for the two variables that are most important to the structure of the spacetime, the conformal factor Ω and the scalar field ϕ. In this sense, only the equations for Ω'' and ϕ'' are important; the other equations may simply be interpreted as constraints. The next part is to write out the equations so that they do not depend on s, we may do this by rearranging (6.44), which gives

$$s = \frac{\Omega\varepsilon}{2} - \frac{1}{8}\Omega^3\rho. \tag{6.46}$$

We then rewrite (6.42), by using (6.46) to eliminate s

$$\Omega^2\varepsilon - \frac{1}{4}\Omega^4\rho + (\Omega')^2 = 1. \tag{6.47}$$

Next, we want to eliminate s from the evolution equation for Ω, which we may do by adding (6.43) and (6.44) together; this gives the very nice result

$$\Omega'' = \frac{1}{2}\Omega^3\rho - \Omega\varepsilon. \tag{6.48}$$

Remark: If one takes ∂_t of (6.38) then one can verify that $\partial_t T_{00} = 0$, so energy density is conserved.

6.3.3 Propagation of the constraints

So, we have derived a pair of elegant evolution equations; the next step would be to analyse its relation to the equations that have been dropped from the system (in other words the constraint equations), to which we use the propagation of the constraints. We write out (6.48), rearranging all terms on the LHS, however, rather than assuming that this equation is zero, we instead let this equation be equal to some other variable, which we will call Q

$$Q \equiv 1 - \Omega^2\varepsilon + \frac{1}{4}\Omega^4\rho - (\Omega')^2. \tag{6.49}$$

The next step is to calculate the value of the differential of Q, if we can show that this is a homogeneous equation in Q, then we will have proven that the solution of $Q = 0$ is valid at all times and hence our equation is correct. Differentiating (6.49) with respect to time gives

$$Q' = \frac{1}{2}\varepsilon\phi\Omega^4\phi' - 2\varepsilon\Omega\Omega' + \varepsilon\phi^2\Omega^3\Omega' + \Omega^3(\phi')^2\Omega' + \frac{1}{2}\Omega^4\phi'\phi'' - 2\Omega'\Omega''. \tag{6.50}$$

Substituting in both (6.48) and (6.37) into (6.50) and one finds that the end result is trivial i.e.

$$Q' = 0. \tag{6.51}$$

This means that the wave equation for Q is also trivial since differentiating (6.51) will give zero. As a result of this it means that the wave equation for Q is trivially homogeneous and the propagation of the constraints is satisfied.

6.3.4 Analysis of the evolution equations

We have successfully derived the evolution equations as well as checked their consistency. The next step is to try to analyse and solve the equations. In the case of the former we will see what we can learn about this particular spacetime model. There are three particular cases to analyse, when $\varepsilon = 0, 1, -1$, where as mentioned, ε denotes the curvature of the spatial sections of the Universe. We shall analyse in detail how each equation evolves in time. Before this, however, it is a good idea to discuss a model of the universe first proposed by Penrose called *Conformal cyclic cosmology* [25, 26]. This model postulates that the Universe that we live in is merely one of many known as aeons. Each one of these aeons can be conformally rescaled and each of these aeons is connected to each other at the conformal boundary. Whilst the physical mechanisms of this model are shrouded in mystery, it does provide us with a method of analysing the equations. In what follows we will proceed to solve the equations numerically and display the results. What we find in all cases is that the solutions to the equations and hence the behaviour of the conformal spacetimes are highly dependent on the initial conditions.

We can deduce the form of the initial data by using the constraint equation if we make the assumption that we start solving the equations at the conformal boundary; this is one of the advantages of using the CFEs since we are offered some freedom in assigning precisely where we start solving the equations. At the conformal boundary we know that $\Omega = 0$, so substituting into (6.47) we get the result that $(\Omega')^2 = 1$, so we know that the initial data is

$$\Omega = 0, \qquad \Omega' = 1. \tag{6.52}$$

Note that although we have chosen $\Omega' = 1$, it is possible to choose the initial value of the derivative of the conformal boundary to be -1. The reason why we have chosen $\Omega' = 1$ is because we are choosing to evolve the equations towards the future. Depending on the type analysis that one wishes to perform it may be more convenient to choose Ω' to be -1 and then evolve towards the past.

Analysis of the $\varepsilon = 0$ case

Begin with the case where $\varepsilon = 0$, i.e. for a flat universe. Whilst the values of the conformal factor and its derivative are fixed by the constraints, the values of the scalar field and its derivative are free for us to specify. To that end we will analyse the equations after assigning values for the initial data that should describe the behaviour of the matter and the conformal spacetime in all situations. We begin by choosing the initial conditions $\phi = 0$ and $\phi' = 0$, in the case where $\varepsilon = 0$ we can see that the evolution equation for the scalar field is trivial i.e.

$$\phi'' = 0. \tag{6.53}$$

Integrating (6.53) with respect to the proper time gives an equation for the derivative of the scalar field

$$\phi' = a_0, \tag{6.54}$$

where a_0 is a constant to be determined by the initial conditions. Integrating (6.54) with respect to proper time again gives a general expression for the scalar field of a flat universe

$$\phi = a_0\tau + a_1. \tag{6.55}$$

Substituting (6.52) into (6.48) gives an evolution equation for the conformal factor of a flat universe

$$\Omega'' = 0, \tag{6.56}$$

we will now proceed to analyse the behaviour of ϕ and Ω as given by the evolution equations (6.56) and (6.53) given some choice of initial conditions. We begin with a choice of data for ϕ and its derivative, the first choice of data will be $\phi = \phi' = 0$. Meanwhile, the choice for the conformal factor is always fixed and given by (6.52), courtesy of the constraint equation. Plotting the data for this choice of initial conditions gives the plot shown in Figure 6.1.

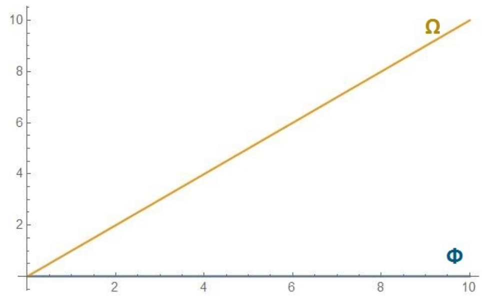

Figure 6.1: $\varepsilon = 0$ *spacetime with* $\phi = \phi' = 0$.

In all the graphs that will be displayed from this point on, the vertical axis represents the magnitude of the respective fields and the horizontal axis represents very large units of time. In the most recent figure we notice that the conformal factor continues to grow indefinitely. As a result of this it is impossible to rescale the spacetime such that it is compatible with conformal cyclic cosmology, since in order for there to be multiple aeons, there must be multiple points where the conformal factor vanishes. One has an almost identical scenario when $\phi = 1$ and $\phi' = 0$, as shown in Figure 6.2.

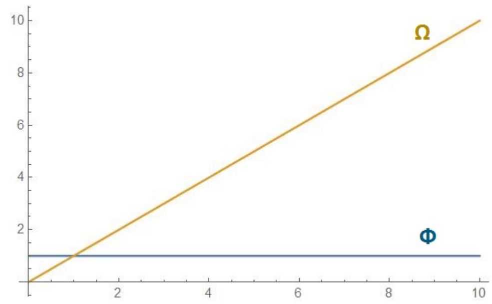

Figure 6.2: $\varepsilon = 0$ *spacetime with* $\phi = 1$ *and* $\phi' = 0$.

We see once again the conformal factor grows indefinitely, whereas the scalar field converges to a constant value over time. The situation is vastly different when the derivative of the scalar field is non-zero. Consider the case where $\phi = 0$ and $\phi' = 1$, then the result is shown in Figure 6.3.

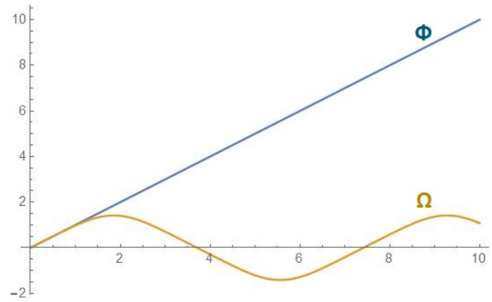

Figure 6.3: $\varepsilon = 0$ *spacetime with* $\phi = 0$ *and* $\phi' = 1$.

Here, the conformal factor does not grow indefinitely, but instead oscillates over time. This means that it is possible for this particular spacetime, with this choice of initial data, to be compatible with Penrose's idea, since conformal cyclic cosmology states that there are infinitely many aeons that are connected by the conformal boundary. The region between two points along the time axis where the conformal factor crosses the boundary represents one aeon. Since there are infinitely many points where the conformal factor vanishes, this means there are infinitely many points that can represent the end of one aeon and the beginning of another. Additionally, we see that the matter fields grow indefinitely over time. We see a similar result if $\phi = 0$ and $\phi' = -1$, as shown in Figure 6.4.

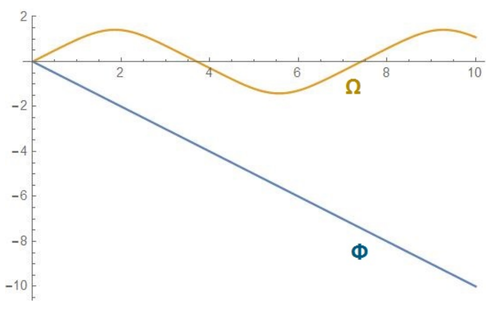

Figure 6.4: $\varepsilon = 0$ *spacetime with* $\phi = 0$ *and* $\phi' = -1$.

Here, we once again have a universe compatible with Penrose's idea of infinitely many aeons, since there are infinitely many points where the conformal factor vanishes. Unlike last time the fields decrease linearly over time. All the other remaining cases are essentially variations of these previous cases. Varying the value of ϕ simply affects the starting point of the matter fields and a positive or negative value of ϕ' determines whether the scalar field grows or decreases with time, respectively.

Hence, we can deduce that it is only possible for a conformal spacetime coupled with a conformally invariant trace-free scalar field with zero curvature to be compatible with conformal cyclic cosmology if the derivative of the scalar field is non-zero. Futhermore, the scalar field either grows or decreases linearly with time depending on whether the derivative of the scalar field is positive or negative, respectively.

Analysis of the $\varepsilon = 1$ case

Next, we analyse the for a positively curved universe i.e. where $\varepsilon = 1$. In this case (6.37) has the general solution

$$\phi = a_0 \cos\tau + a_1 \sin\tau, \tag{6.57}$$

and therefore the derivative of the scalar field with respect to time is

$$\phi' = -a_0 \sin\tau + a_1 \cos\tau. \tag{6.58}$$

Substituting (6.57) and (6.58) values into (6.48) and we obtain the following evolution equation for the conformal factor

$$\Omega'' = -\Omega + \frac{1}{2}\Omega^3(a_0^2 + a_1^2). \tag{6.59}$$

We now proceed to analyse (6.59) and (6.57) for different choices of initial data for the scalar field and the same choice of initial data for the conformal factor. Let us begin choosing $\phi = \phi' = 0$, then the variables ϕ and Ω evolve as shown in Figure 6.5

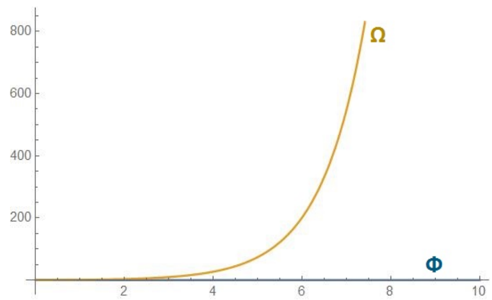

Figure 6.5: $\varepsilon = 1$ *spacetime with* $\phi = \phi' = 0$.

We can see that not only does the conformal factor tend to a constant value over time, but also the conformal factor grows exponentially with time. Hence, it is not possible for a Universe with positive curvature coupled to a conformally invariant scalar field with this choice of data. However, we notice something peculiar happens if we change the values of the data. When $\phi = 1$ and $\phi' = 0$ the variables evolve as shown in Figure 6.6

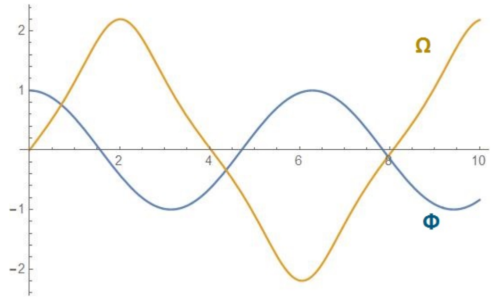

Figure 6.6: $\varepsilon = 1$ *spacetime with* $\phi = 1$ *and* $\phi' = 0$.

We see now that both the conformal factor and the scalar field have periodic behaviour. An almost identical result occurs if $\phi = 0$ and $\phi' = 1$, as displayed in Figure 6.7.

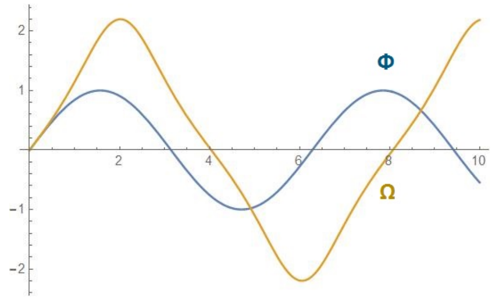

Figure 6.7: $\varepsilon = 1$ *spacetime with* $\phi = 0$ *and* $\phi' = 1$.

The only difference is that amplitude and initial value of the scalar field wave is different. In fact this is precisely the behaviour that is exhibited for all other values of the scalar field and its derivative.

So we can conclude that it is a conformal spacetime with positive curvature coupled to a conformally invariant scalar field is always compatible with conformal cyclic cosmology provided either the scalar field or its derivative are non-zero.

Analysis of the $\varepsilon = -1$ case

Finally, we analyse the $\varepsilon = -1$ case. When $\varepsilon = -1$ then (6.48) becomes

$$\Omega'' = \frac{1}{2}\Omega^3((\phi')^2 - \phi^2) + \Omega, \tag{6.60}$$

likewise (6.37) becomes

$$\phi'' = \phi,$$

which has a general solution

$$\phi = a_0 \cosh \tau + a_1 \sinh \tau. \tag{6.61}$$

Differentiating (6.61) with respect to the proper time τ gives

$$\phi' = a_0 \sinh \tau + a_1 \cosh \tau. \tag{6.62}$$

By substituting in (6.61) and (6.62) into (6.60) we conclude that

$$\Omega'' = \frac{1}{2}\Omega^3(a_1^2 - a_0^2) + \Omega. \tag{6.63}$$

Once again we make use of the initial data for the conformal factor and its derivative given by the constraints and choose the values for the scalar field.

What we find when plotting the evolution of (6.61) and (6.63) is that the situation is the same for the case where $\varepsilon = 1$, i.e it is always possible to construct a spacetime that compatible with conformal cyclic cosmology, provided that neither ϕ or ϕ' are non-zero.

6.4 Conformal geodesics

Before we construct the conformal geodesic equations for the spacetime that we are analysing, it is worth reviewing some of the key ideas behind the conformal geodesics. The starting point for this area is the standard geodesic equation in index free notation

$$\tilde{\nabla}_{\dot{x}}\dot{x} = 0. \tag{6.64}$$

The first step to constructing equations for a geodesic that has been reparametrized as a conformal geodesic is to do a conformal transformation of the derivative. We start with a curve $x(\lambda)$, which possesses a tangent vector $x' = dx/d\lambda$. This curve is a geodesic provided that it satisfies the standard geodesic equation $\tilde{\nabla}_{x'}x' = 0$. We can derive a conformal geodesic equation by performing a conformal transformation on (6.64)

$$(\nabla_a - \tilde{\nabla}_a)x' = S_{ac}{}^{bd}\upsilon_d x'^c, \tag{6.65}$$

then using (2.13) and (6.65) becomes

$$\nabla_{x'}x' = 2 < x', \Upsilon > x' - g(x', x')\Upsilon^\sharp = 0. \tag{6.66}$$

Now, it becomes convenient to introduce a new parameter $\tau = \tau(t)$ and let $\dot{x} = dx/d\tau$; using the chain rule enables us to write

$$\frac{dx}{d\tau} = \frac{d\lambda}{d\tau}\frac{dx}{d\lambda},$$

or equivalently $\dot{x} = d\lambda/d\tau x'$, which can be rearranged to $x' = \dot{x}\frac{d\tau}{d\lambda}$. This last term can be written more compactly as

$$x' = \dot{x}\tau'. \tag{6.67}$$

Substituting (6.67) into (6.66) and performing some manipulations gives

$$(\tau')^2\nabla_{\dot{x}}\dot{x} = (2 < x', \Upsilon > \tau' - \tau'')\dot{x} - g(x', x')\Upsilon^\sharp. \tag{6.68}$$

From (6.68), we can make a number of deductions regarding the conformal geodesics. We know that the general form of the geodesic equation is $\nabla_{\dot{x}}\dot{x} = 0$, hence in order for equation (6.68) to be conformally invariant we must choose our parameter τ in such a way that $\tau'' = 2 < x', \Upsilon > \tau'$; if this is the case then (6.68) reduces to

$$(\tau')^2\nabla_{\dot{x}}\dot{x} = -g(x', x')\Upsilon^\sharp. \tag{6.69}$$

For null like curves i.e. where $g(x', x') = 0$, (6.69) has the same form as (6.70). This means that although it is immediately possible to conformally rescale null-like geodesics, the same is not true for space and time-like geodesics. Of course, following this first step of the analysis, it becomes natural to ask if it is possible to construct a set of conformally invariant equations for time-like and space-like geodesics. It turns out that in order to construct a curve with conformal properties that describes either a space-like or time-like particle, one needs an additional variable along with the coordinates of the curve; this additional factor is a covector $\beta(\tau)$. What is the motivation for introducing this additional covector? Consider the standard geodesic equation in the unphysical spacetime

$$\nabla_{\dot{x}}\dot{x} = 0. \tag{6.70}$$

Physically, this tells us that geodesics do not have any acceleration; this is due to the fact that, when expanded, the principal part of the differential equation is a second order differential with respect to τ, i.e. an acceleration. Since this is zero, it is just a way of saying that geodesics do not have acceleration. However, for the conformally rescaled geodesic equation, the connection will be non-zero (with the exception of null geodesics) and in this way we say that conformal geodesics in general do have an acceleration. To quantify this particular acceleration we introduce the covector β; from this covector we may define a conformal geodesic. A conformal geodesic is a pair $(x(\tau), \beta(\tau))$ on a spacetime $(\tilde{\mathcal{M}}, \tilde{g})$, where τ is the proper time parameter of the curve $x(\tau)$. The curve in question possesses a tangent

$\dot{x}(\tau)$ and a covector $\beta(\tau)$, both of which satisfy the following two equations

$$\tilde{\nabla}_{\dot{x}}\dot{x} = -2 < \beta, \dot{x} > \dot{x} + \tilde{g}(\dot{x}, \dot{x})\beta^{\sharp}, \tag{6.71a}$$
$$\tilde{\nabla}_{\dot{x}}\beta =< \beta, \dot{x} > \beta - \frac{1}{2}g^{\sharp}(\beta, \beta)\dot{x}^{\flat} + L(\dot{x}, \cdot), \tag{6.71b}$$

where L is the Schouten tensor in index free notation. Hence, our goal in this section is to obtain such a pair for a spatially homogeneous and isotropic spacetime containing a conformally invariant scalar field [14, 16, 31]. How do we obtain such a pair? Previously, we obtained an equation for the unphysical spacetime given a particular choice of gauge $g = -d\tau^2 + k$. We may construct the conformal geodesics for such a spacetime in the following way, we start from the unphysical spacetime and work our way backwards into the physical picture. The first part of this task is to construct regular geodesics in the unphysical case. Any geodesic can be reparametrized as a conformal geodesic, but the opposite is not necessarily true. So we check if the coordinates of the unphysical spacetime satisfy the standard geodesic equation (6.70) which, when expanded into index notation, is equal to

$$\frac{d^2x^{\mu}}{dT} + \Gamma^{\mu}{}_{\alpha\beta}\frac{dx^{\alpha}}{dT}\frac{dx^{\beta}}{dT} = 0. \tag{6.72}$$

In the case where $\mu = 0$ (6.72) becomes

$$\frac{d^2x^{0}}{dT} + \Gamma^{0}{}_{\alpha\beta}\frac{dx^{\alpha}}{dT}\frac{dx^{\beta}}{dT} = 0. \tag{6.73}$$

Recalling in our convention that greek indices refer to spacetime indices in a particular coordinate system and that, for a warped product, these can expanded into timelike and spacelike indices, then from (6.73) we obtain

$$\frac{d^2}{dT^2}T + \Gamma^{0}{}_{00} + 2\Gamma^{0}{}_{0i}\frac{dT}{dT}\frac{dx^i}{dT} + \Gamma^{0}_{ij}\frac{dx^i}{dT}\frac{dx^j}{dT} = 0. \tag{6.74}$$

At this point we make use of the properties of our coordinates, namely that we are working with coordinates for a spatially homogeneous spacetime. For such a spacetime the coordiantes may be expressed as $(x^{\mu}(\tau)) = (T, x^1_*, x^2_*, x^3_*)$ where x^1_*, x^2_* and x^3_* are constants. This is simply a consequence of the spatial metric k_{ij} being of constant curvature, hence the spatial coordinates are constant. As a result of this all of the derivatives of the coordinates x^i are zero. This, combined with the fact that $\Gamma^0{}_{00} = \Gamma^0{}_{0i} = 0$, means that all the left hand terms in (6.74) vanish and the geodesic equation is satisfied. Similarly, when $\mu = i$ then we obtain

$$\Gamma^{i}{}_{00}\Big(\frac{dx^0}{dT}\Big)^2 = 0, \tag{6.75}$$

and since $\Gamma^i{}_{00} = 0$, (6.75) is satisfied. Now, we need to see if these geodesics in the spacetime can be recast as conformal geodesics, to do this, we must find a new parameter $\lambda = \lambda(\tau)$ and a covector β, such that (6.71a) and (6.71b) are satisfied. Due to the homogeneity of the spacetime, it is possible to deduce the form that the covector should have. As the curve points in one direction only (the timelike direction), this implies that the covector must also point in the same direction as the tangent vector. This suggests an anzatz of the form

$$\beta = \alpha \dot{x}, \tag{6.76}$$

where α is a scalar. Now, it can be shown that $\nabla_{x'} x' = \dot{x}\tau''$ if (6.70) is satisfied. To show this, we substitute (6.67) into $\nabla_{x'} x' = \dot{x}\tau''$, then expanding with the Leibnitz rule

$$\nabla_{x'} x' = \tau' \dot{x}^a (\dot{x}^b \nabla_a \tau' + \tau' \nabla_a \dot{x}^b), \tag{6.77}$$

then if (6.70) is satisfied then the second term vanishes and we obtain

$$\nabla_{x'} x' = \dot{x}\tau''. \tag{6.78}$$

With this result we can work out some useful relations. Substituting (6.78) into equation (6.71a) and we end up with

$$\dot{x}\tau'' = -2\tau < \alpha \dot{x}^\flat, x' > \dot{x} + g(x', x') \alpha \dot{x}, \tag{6.79}$$

which, upon substituting in (6.67), gives the ODE

$$\tau'' + \alpha \tau'^2 = 0. \tag{6.80}$$

Next, we need to derive another relation, this time by differentiating (6.76), which gives

$$\nabla_{x'} \beta = \nabla_{\tau' \dot{x}} (\alpha \dot{x}^\flat). \tag{6.81}$$

Expanding the RHS of (6.81) gives

$$\tau' \nabla_{\dot{x}} (\alpha \dot{x}^\flat) = \tau' (\nabla_{\dot{x}} \alpha) \dot{x}^\flat + \tau' \alpha \nabla_{\dot{x}} \dot{x}^\flat, \tag{6.82}$$

but due to the geodesic equation the second term on the RHS vanishes, dividing through by τ' and (6.82) becomes

$$\nabla_{x'} \beta = \alpha' \dot{x}^\flat. \tag{6.83}$$

Substituting (6.83) into (6.71b) and we get

$$\alpha' \dot{x}^\flat =< \beta, x' > \beta - \frac{1}{2} g^\sharp(\beta, \beta) x'^\flat + L(x', \cdot). \tag{6.84}$$

Then, substituting in both (6.76) and (6.67)

$$\alpha' \dot{x}^{\flat} = \tfrac{1}{2}\alpha^2 \tau' < \dot{x}, \dot{x} > \dot{x}^{\flat} + \tau' \tilde{L}(\dot{x}, \cdot),$$

which simplifies to

$$d\alpha = d\tau\Big(\frac{1}{2}\alpha^2 + \frac{\varepsilon}{2}\Big). \tag{6.85}$$

Now, there are three different cases to analyse which, just like in the analysis of the evolution equations depends upon whether or not the value of the curvature is $1, 0$ or -1. The general equation for the proper time can be found to be

$$\tau = -\frac{2}{\alpha} + 2\alpha\varepsilon + C \tag{6.86}$$

This completes the derivation of the conformal geodesics for conformal spacetimes whose matter content is described by a conformally invariant scalar field.

7 Initial Data for the Conformal Wave Equations

We have derived a set of wave equations that describe both the evolution of the geometry of conformally rescaled spacetimes perturbed by trace-free matter and how the trace-free matter itself evolves over time. Furthermore, we have shown through the propagation of the constraints that any solution to the CWEs also implies a solution to the CFEs, provided the initial data is suitably chosen. Given this, the next step is to try to solve the equations, which involves suitably choosing some initial data. This is due to the fact that the CWEs are differential equations, which are impossible to solve exactly without initial conditions. Unfortunately, even the process of choosing initial data is extremely complex; it not sufficient to simply pick any choice of data since only a select few initial conditions will give rise to a solution of the EFEs. In order for data to give an actual solution, it must satisfy a set of equations known as the constraint equations. The primary goal of this chapter is to highlight a particular property of initial data for the CWEs, namely, that given a specific set of basic quantities all other quantities can be derived. In order to do this one first needs to go through the necessary background material, relating to the constraint equations.

7.1 The 3+1 Decomposition in General Relativity

7.1.1 Spatial hypersurfaces

Recall in section 1.1.2 we described a process known as the 3+1 decomposition, which involves considering only the spatial components of the spacetime at an instant in time. To do so, we make use of a hypersurface, which physically describes all events that occur simultaneously. The visual representation of the hypersurface is like a leaf, where all points on the leaf correspond to events that occur at precisely the same instance in time. At an infinitesimal amount of time later simultaneous events may be described by a different hypersurface, so in this sense the spacetime ends up looking like a stack of leaves or a *foliation* of hypersurfaces. The 3+1 decomposition process involves choosing one such hypersurface at a specific instance in time and prescribing initial data on said surface, then solving the necessary equations to find a unique solution; this is akin to choosing some initial conditions for an ordinary differential equation and solving to find the constant of integration.

Let us examine some of the tools we need in order to work with the 3+1 decomposition. Since we are working with 3-dimensional hypersurfaces, we need tools that consider the spatial and time components of the spacetime individually. First, we need to be able to describe the direction of time with respect to the hypersurfaces. To this end we define a what is called a *global time function* t. The physical explanation of t is as follows: consider a certain spacetime and suppose that we are able to define time in a consistent manner for this spacetime for all events. In that case we can define a unit of time t to describe the progression of time for the entire spacetime. Leaves of the spacetime foliation associated to t are naturally the surfaces where t is constant. Another important point is that the leaves of the spacetime foliation never intersect, otherwise this would be a poor choice for the progression of time as it would be impossible to describe simultaneous events. From this t one defines the following covector

$$\varpi_a \equiv \nabla_a t, \tag{7.1}$$

this covector denotes the normal to the hypersurface and as such defines the direction that time flows. From this covector we can define the so-called *lapse function*

$$\alpha^2 \equiv \frac{1}{\varpi^a \varpi_a}. \tag{7.2}$$

The lapse function measures how much proper time elapses along neighbouring time slices along the direction specified by the normal ϖ_a. From both (7.1) and (7.2) we define the *unit normal*

$$n_a \equiv -\alpha \varpi_a. \tag{7.3}$$

Physically, this unit normal can be thought of as the 4-velocity of a normal observer, which is an observer that moves through the spacetime in a direction that is orthogonal to the hypersurface. The minus sign indicates that n^a points in the direction of increasing t. Using (7.2) it can be verified that

$$n^a n_a = -1. \tag{7.4}$$

This covers the separation of time components from the 4-dimensional spacetime; next we need to look at the curvature. The hypersurface is equipped with a purely spatial metric h_{ij}; the ij indices are used to indicate that this metric is a 3-dimensional object. Although h_{ij} does depend on time, as the shape of the hypersuraces will change as t changes, for all practical purposes we mostly consider a single hypersurace, where t is constant. In this scenario h_{ij} can be thought of as being independent of time. The purely spatial metric is related to the original spacetime metric via the formula

$$h_{ab} \equiv g_{ab} + n_a n_b. \tag{7.5}$$

Although h_{ij} is inherently a spatial object, in the previous equation the spatial metric has spacetime indices ab; this is due to the presence of the spacetime metric g_{ab} on the RHS of (7.5). In this sense h_{ab} is part of the decomposition of the spacetime metric, with h_{ab} representing the spatial parts of the metric and the unit normals $n_a n_b$ as representing the timelike parts of the metric. Just as g_{ab} is used to measure distances in spacetime, h_{ab} can be used to measure distances along a hypersurface. We can verify that h_{ab} is a purely spatial object since it has no component along the unit normal n_a which always points in the direction of time. This can be verified by contracting h_{ab} with n^a and making use of (7.4)

$$n^a h_{ab} = n^a g_{ab} + n_a n^a n_b = n_b - n_b = 0.$$

Effectively, h_{ab} separates any vector on the hypersurface into its components parallel to the unit normal and components parallel to the hypersurface. An alternative way of looking at h_{ab} is to think of it as an object analogous to the dot product; just as the dot product defines the projection of a certain vector to a surface, h_{ab} defines how much a certain vector is projected along a hypersurface. It is for this reason that h_{ab} is often referred to as the *projector tensor* or projector for short; we encountered a similar tensor in section 5.4.1. The inverse of the projector is obtained simply by raising the indices in (7.5)

$$h^{ab} \equiv g^{ab} + n^a n^b. \tag{7.6}$$

In most calculations, however, it is more convenient to consider the mixed version of the projector

$$h_a{}^b \equiv \delta_a{}^b + n_a n^b. \tag{7.7}$$

Using the projector we can accordingly define the spatial part of a vector as being

$$V_a^{\perp} \equiv h_a{}^c V_c, \tag{7.8}$$

the spatial part of a rank 2 tensor is likewise

$$T_{ab}^{\perp} \equiv h_a{}^c h_b{}^d T_{cd}, \tag{7.9}$$

with obvious extensions to higher dimensions.

7.1.2 3 Connections and spatial curvature

Up until now we have been analysing quantities living in 4-dimensional spacetime and as a result any derivatives of these quantities have also been 4-dimensional. However, as we are now working with quantities living in a 3-dimensional hypersurface, we need to adapt the notion of a derivative. The connection in question D_i is the 3-dimensional version of the Levi-Civita connection, which is to say, it describes how objects move on the hypersurface and satisfies the metric compatibility and torsion free conditions first defined in section 2.2.1 with the spatial metric h_{ij} in place of the spacetime metric g_{ab}. As before, it is useful to think of this 3-connection as being a part of the 4-dimensional spacetime connection ∇_a and therefore we will denote it using a spacetime index. The 3-connection D_a is defined when acting on a scalar ϕ as

$$D_a\phi \equiv {h_a}^b \nabla_b \phi. \tag{7.10}$$

Once again, the key ingredient for defining the 3-dimensional spatial analogues of a 4-dimensional spacetime object is the projector tensor. So essentially the 3-connection is obtained by taking the corresponding 4-dimensional connection and applying the projector to kill off the time components of the derivative. In accordance with (7.10) the 3-derivative of a covector is

$$D_a V_b \equiv {h_a}^c {h_b}^d \nabla_c V_d. \tag{7.11}$$

The 3-derivative of a mixed tensor of rank 2 is then given as

$$D_a {T^b}_c \equiv {h_a}^d {h_e}^b {h_c}^f \nabla_d {T^e}_f, \tag{7.12}$$

with obvious extensions to tensors of higher ranks. Associated to the metric h_{ab} and the 3-derivative D_a are the *spatial Christoffel symbols* which, in direct analogy to the standard Christoffel symbols, describe how vectors change when they are parallel transported across the hypersurface. The spatial Christoffel symbols are intuitively defined as

$$\gamma^i{}_{jk} = \tfrac{1}{2} h^{im}(\partial_j h_{km} + \partial_k h_{mj} - \partial_m h_{jk}). \tag{7.13}$$

Accordingly, one can also derive a purely spatial curvature tensor $r^d{}_{bac}$, by applying a commutator of 3-derivatives to an arbitrary vector v^a

$$D_a D_b v^c - D_b D_a v^c = r^c{}_{dab} v^d. \tag{7.14}$$

One can readily verify that $r^c{}_{dab} n^d = 0$. Similarly, one can also define the spatial Ricci tensor and scalar as

$$r_{db} \equiv r^c{}_{dcb}, \qquad r \equiv g^{ab} r_{ab}. \tag{7.15}$$

Likewise, one can also define the spatial Schouten tensor using (7.15)

$$l_{ab} \equiv r_{ab} - \tfrac{1}{4} h_{ab} r. \tag{7.16}$$

Just as the 4-dimensional spacetime curvature can be decomposed in terms of the Weyl and Schouten tensor, so too can the 3-dimensional spatial curvature, the main difference is that the Weyl tensor for a hypersurface does not exist. So then written in terms of the 3-dimensional Schouten tensor given in (7.16) $r^a{}_{bcd}$ is equal to

$$r_{abcd} = h_{bd} l_{ac} - h_{bc} l_{ad} - h_{ad} l_{bc} + h_{ac} l_{bd}. \tag{7.17}$$

The curvature tensor given by (7.14) is also known as the *intrinsic curvature tensor* because it only contains information about the hypersurface itself and not about how the hypersurface fits into the spacetime. This missing information is encoded in the so called *extrinsic curvature*, which is given as

$$K_{ab} \equiv -h_a{}^c h_b{}^d \nabla_c n_d. \tag{7.18}$$

Physically, this tensor measures how the hypersurface deforms as it is carried along the normal; this is evident by the derivative of the normal, since the normal changes as one moves from one hypersurface to another. A related quantity is the *acceleration* of the spacetime foliation, which is defined as

$$a_a \equiv n^b \nabla_b n_a. \tag{7.19}$$

Now, (7.18) may be re-expressed in terms of the acceleration; making use of (7.19), K_{ab} is equal to

$$K_{ab} = -\nabla_a n_b - n_a a_b. \tag{7.20}$$

This completes the background that one needs to start working in the 3 + 1 decomposition.

7.2 Decomposition of the Conformal Field Equations

Now that we have defined all the necessary quantities we next need to apply them to the CWEs. Regardless of whether or not we work in the physical spacetime, where the primary tool of analysis are the EFEs, or from the unphysical spacetime where the primary tool of analysis are the CFEs, we always start by prescribing initial data on some hypersurface. This means that for the process of solving the equations to find a unique solution to the CFEs, one still needs to decompose the variables of the CFEs into their respective spatial and timelike parts. In a manner analogous to the previous section, we will use the projector tensor and the normal in order to derive a decomposed version of all the CFEs.

7.2.1 The Decomposition of the CFE variables

Before proceeding to derive the constraint equations of the CFEs, let us first obtain the decomposed version of the variables that make up the CFEs, as these will naturally be used in the derivation of the constraints.

Decomposition of the derivative of the conformal factor

As the first CFE contains a derivative of the conformal factor we should obtain two expressions for the decomposition of the derivative of the conformal factor into two different components, one purely timelike term and a purely spatial term. To begin with, we define a shorthand notation for the derivative of the conformal factor

$$\nabla_a \Omega \equiv \Omega_a. \tag{7.21}$$

From here, we can obtain the necessary components of this vector by contracting with either the normal or the projector; the former is used to obtain the time components of the vector tangential to the normal and the latter is used to obtain the components that lie in the hypersurface. With that in mind, the equations that give the components parallel to the normal and tangential to the surface are, respectively

$$\omega \equiv \Omega_a n^a, \tag{7.22a}$$

$$\omega_a \equiv \Omega_b h_a{}^b. \tag{7.22b}$$

The derivative of the conformal factor (7.21), written in terms of its time and its spatial components is simply the sum of (7.22a) and (7.22b)

$$\Omega_a = n_a \omega + \omega_a. \tag{7.23}$$

7.2.2 Decomposition of the energy-momentum tensor

The next variable that we will decompose is the energy-momentum tensor. This is slightly more complex than in the previous section as the energy-momentum tensor is a rank 2 tensor as opposed to a vector, this means that it is possible to contract on more than one index with either the normal or the projector. The individual components are as follows:

$$\mu \equiv n^a n^b T_{ab}, \tag{7.24a}$$
$$\mu_a \equiv h_a{}^b n^d T_{bd}, \tag{7.24b}$$
$$\mu_{ab} \equiv h_a{}^c h_b{}^d T_{cd}. \tag{7.24c}$$

Physically, these variables may be interpreted as follows: μ is the energy density, which measures the amount of energy at specific points, μ_b is the flux vector, which measures the rate of the flow of energy along the surface and μ_{ab} is the stress tensor, which measures how the volume of a surface changes in response to the presence of matter. The energy-momentum tensor is then the sum of all possible contractions of the tensor with either the projector or the normal i.e.

$$T_{ab} = n_a n_b n^c n^d T_{cd} + h_a{}^c h_b{}^d T_{cd} + h_b{}^d n_a n^c T_{cd} + h_a{}^d n_b n^c T_{dc}. \tag{7.25}$$

Applying (7.24a)-(7.24c) and (7.25) becomes

$$T_{ab} = n_a n_b \mu + 2n_{(a}\mu_{b)} + \mu_{ab}. \tag{7.26}$$

By making note of the fact that T_{ab} is a trace-free quantity in all our calculations, it is possible to derive an important relation between the energy density and the stress tensor. Begin by contracting (7.26) with the metric

$$g^{ab}(\mu_{ab} - \mu_a n_b - \mu_b n_a + \mu n_a n_b) = 0. \tag{7.27}$$

However, as we stated beforehand, h_{ab} is a purely spatial object meaning $n^a h_{ab} = 0$, hence (7.27) becomes

$$\mu = \mu_a{}^a. \tag{7.28}$$

This means that a necessary condition for matter to be trace-free is that the energy density must be equal to the trace of the stress tensor.

Decomposition of the Schouten tensor

The next variable that we will decompose is the Schouten tensor; this process is identical to the previous decomposition for the energy-momentum tensor, since the Schouten tensor is also a symmetric rank 2 tensor. Once again the individual components are obtained from contractions with the normal and the projector

$$\theta = n^a n^b L_{ab}, \tag{7.29a}$$

$$\theta_a = h_a{}^b n^d L_{bd}, \tag{7.29b}$$

$$\theta_{ab} = h_a{}^c h_b{}^d L_{cd}. \tag{7.29c}$$

The Schouten tensor is then given as

$$L_{ab} = n_a n_b n^c n^d L_{cd} + h_a{}^c h_b{}^d L_{cd} + h_b{}^d n_a n^c L_{cd} + h_a{}^d n_b n^c L_{dc}, \tag{7.30}$$

then applying (7.29a), (7.29b) and (7.29c) causes (7.30) to become

$$L_{ab} = n_a n_b \theta + 2n_{(a}\theta_{b)} + \theta_{ab}. \tag{7.31}$$

Decomposition of the Weyl tensor

We now discuss the decomposition of the Weyl tensor, which is quite a lengthy task due to the fact that the Weyl tensor has four indices. Analogous to the electric and magnetic part of the Faraday tensor in electromagnetism, there are electric and magnetic parts of the Weyl tensor describing the electric and magnetic parts of the gravitational field, respectively. It should be noted that for a long period of time the magnetic part of the gravitational field was a theoretical concept [35], however the magnetic part of the gravitational field was eventually proven experimentally in 2002 to be a real physical object [30]. The electric part of the Weyl tensor is what we perceive as the force of gravity on a day to day basis, whereas the magnetic part is more subtle; it is responsible for such effects as the precession of a gyroscope. The electric part of the Weyl tensor is given as

$$d_{ef} = n^a n^b d_{eafb}. \tag{7.32}$$

This particular tensor can be verified to be trace-free i.e.

$$d_e{}^e = 0. \tag{7.33}$$

As is the case with the Faraday tensor, the magnetic part of the Weyl tensor is defined using the Hodge dual of the electric part

$$d^*{}_{ae} = \frac{1}{2}\epsilon_{ef}{}^{cd} n^b n^f d_{abcd}. \tag{7.34}$$

Alternatively, the magnetic part may be defined without the dual

$$d_{acd} = h_a{}^e h_c{}^f h_d{}^g n^b d_{ebfg}, \tag{7.35}$$

which is also trace-free, in other words

$$d^a{}_{ba} = 0. \tag{7.36}$$

The two definitions of the magnetic part of the Weyl tensor given by (7.34) and (7.35) encode the same information and are related to each other via the relationship

$$d_{agc} = \epsilon_{gceh} d^*{}_a{}^h n^e. \tag{7.37}$$

Finally, the Weyl tensor written in terms of the electric and magnetic parts is given as

$$d_{abcd} = d_{bd}h_{ac} - d_{bc}h_{ad} - d_{ad}h_{bc} + d_{ac}h_{bd} + d_{bd}n_a n_c - d_{ad}n_b n_c \\ -d_{bc}n_a n_d + d_{ac}n_b n_d - \epsilon_{cdef}d^*{}_b{}^f n_a n^e + \epsilon_{cdef}d^*{}_a{}^f n_b n^e - \epsilon_{abef}d^*{}_d{}^f n_c n^e + \epsilon_{abef}d^*{}_c{}^f n_d n^e. \tag{7.38}$$

Decomposition of the rescaled Cotton tensor

Finally we take a look at the decomposition of the rescaled Cotton tensor, which is a more lengthy task due to the rescaled Cotton tensor being a rank 3 tensor. Let us define all the individual components of the rescaled Cotton tensor as

$$\tau_{abc} = h_a{}^d h_b{}^e h_c{}^f T_{def}, \tag{7.39a}$$

$$\tau_{ab} = h_a{}^d h_b{}^e n^f T_{def}, \tag{7.39b}$$

$$\tau_a = h_a{}^d n^b n^{cd} T_{dbc}. \tag{7.39c}$$

These are the only possible non-vanishing components of the rescaled Cotton tensor, the others all vanish due to the antisymmetry of the rescaled Cotton tensor; for example $\tau = n^a n^b n^c T_{abc} = 0$ due to the fact that the rescaled Cotton tensor is antisymmetric on the a and b indices, but $n^a n^b$ is a symmetric quantity, so any contraction between the two is zero.

It is possible to re-express (7.39a)-(7.39c) in terms of the components of the energy-momentum tensor (7.24a)-(7.24c); the motivation for doing so will be explained shortly. We start by substituting (4.2) into (7.39a), and applying both (7.24b) and (7.24c) which gives

$$\tau_{abc} = \tfrac{3}{2}\mu_{bc}\omega_a - \tfrac{3}{2}\mu_{ac}\omega_b + \tfrac{1}{2}\mu_b h_{ac}\omega - \tfrac{1}{2}\mu_a h_{bc}\omega + \tfrac{1}{2}h_a{}^e h_{bc}\omega^d T_{de} \\ -\tfrac{1}{2}h_{ac}h_b{}^e\sigma^d T_{de} + \tfrac{1}{2}\Omega h_a{}^d h_b{}^e h_c{}^f \nabla_d T_{ef} - \tfrac{1}{2}\Omega h_a{}^d h_b{}^e h_c{}^f \nabla_e T_{df}. \tag{7.40}$$

Next, we proceed to remove the energy-momentum tensor terms; substituting in (7.26) and using (7.20) to remove the derivatives of the normal and one ends up with

$$\tau_{abc} = \tfrac{1}{2}\Omega K_{bc}\mu_a - \tfrac{1}{2}\Omega K_{ac}\mu_b + \tfrac{1}{2}\mu_b h_{ac}\omega - \tfrac{1}{2}\mu_a h_{bc}\omega + \tfrac{3}{2}\mu_{bc}\omega_a - \tfrac{3}{2}\mu_{ac}\omega_b$$
$$-\tfrac{1}{2}\mu_{bd}h_{ac}\omega^d + \tfrac{1}{2}\mu_{ad}h_{bc}\omega^d + \tfrac{1}{2}\Omega h_a{}^d h_b{}^e h_c{}^f \nabla_d \mu_{ef} - \tfrac{1}{2}\Omega h_a{}^d h_b{}^e h_c{}^f \nabla_e \mu_{df}. \qquad (7.41)$$

Likewise, one can express (7.39b) in terms of the components of the energy-momentum tensor. Substituting in (4.2) into (7.39b) and re-expressing the derivatives of the conformal factor using (7.21) followed by (7.23) gives

$$\tau_{ab} = \tfrac{3}{2}h_b{}^e n^d \omega_a T_{de} - \tfrac{3}{2}h_a{}^e n^d \sigma_b T_{de} + \tfrac{1}{2}\Omega h_a{}^e h_b{}^f n^d \nabla_e T_{df} - \tfrac{1}{2}\Omega h_a{}^e h_b{}^f n^d \nabla_f T_{de}. \qquad (7.42)$$

Next, we remove the energy-momentum tensor terms using the decomposition of the energy-momentum we defined a short while ago. Substituting (7.26) into (7.42), then using (7.20) to remove the derivatives of the normal and the end result is

$$\tau_{ab} = \tfrac{1}{2}\Omega K_b{}^d \mu_{ad} - \tfrac{1}{2}\Omega K_a{}^d \mu_{bd} + \tfrac{3}{2}\mu_b\omega_a - \tfrac{3}{2}\mu_a\omega_b + \tfrac{1}{2}\Omega h_{ae}h_{bd}\nabla^e\mu^d - \tfrac{1}{2}\Omega h_{ad}h_{be}\nabla^e\mu^d. \qquad (7.43)$$

Finally, we proceed to express (7.39c) in terms of the components of the energy-momentum tensor. Substituting in (4.2), then making use of (7.21), (7.23) and then applying the definitions of the energy-momentum tensor components (7.24a)-(7.24c) gives

$$\tau_a = -\mu_a\omega + \tfrac{3}{2}\mu\omega_a - \tfrac{1}{2}\mu_{ac}\omega^c. \qquad (7.44)$$

The quantities (7.41), (7.44) and (7.43) can be used to deduce a subtle, but important property of the rescaled Cotton tensor from the point of view of constructing initial data. Provided that one knows the extrinsic curvature and the energy-momentum tensor components then it is always possible to deduce the necessary information about the rescaled Cotton tensor. We will make use of this property when proving the main result of this chapter.

7.2.3 Decomposition of the metric CFEs

Now that we defined the necessary components we will now proceed to decompose the CFEs into components that lie along the hypersurface and that act as constraints. An important point is that any equation that has a contraction between the normal and a covariant derivative gives rise to a time derivative and is therefore an evolution equation. Hence, we will not derive any equations where a contraction occurs between the normal and the covariant derivative as our main interest is the constraints.

The First CFE

We start with the first CFE which, making use of (7.21) is equal to

$$\nabla_a \Omega_b = -\Omega L_{ab} + s g_{ab} + \tfrac{1}{2}\Omega^3 T_{ab}. \tag{7.45}$$

To find the first constraint, we contract (7.45) using the normal and the projector, which gives

$$h_b{}^c n^a \nabla_c \omega_a = -\Omega h_b{}^c n^a L_{ac} + s h_b{}^c n^a g_{ac} + \tfrac{1}{2}\Omega^3 h_b{}^c n^a T_{ab}. \tag{7.46}$$

We then decompose the various components of (7.46) using (7.31) and (7.23), which yields

$$h_{ba}\nabla^a \omega + h_{bc} n^a \omega \nabla^c n_a + h_{bc} n^a \nabla^c \omega_a = -\Omega \theta_b. \tag{7.47}$$

We then eliminate the derivative of the normal using (7.20). Then, transforming the spacetime derivatives into spatial derivatives using (7.9) and (7.47) becomes

$$D_b \omega = -\Omega \theta_b + K^a{}_b \omega_a + \tfrac{1}{2}\Omega^3 \mu_b. \tag{7.48}$$

This is one of the two constraint equations for the first CFE, any initial data set must satisfy this equation in order to be a solution to the EFEs.

A second constraint equation may be derived simply by repeating the same process as above, except this time contracting with two projectors as opposed to one normal and one projector. Doing so and following many of the same steps and it becomes possible to derive

$$D_b \omega_a = s h_{ab} - \Omega \theta_{ab} - K_{ab}\omega + \tfrac{1}{2}\Omega^3 \mu_{ab}. \tag{7.49}$$

The Second CFE

For the second CFE, there is only one free index and hence we must only derive one constraint equation. We start by applying (7.22b) to (3.44b) and contracting with the projector

$$h_b{}^a \nabla_a s = -\Omega^a h_b{}^c L_{ca} + \tfrac{1}{2}\Omega^a \Omega^2 h_b{}^c T_{ca}. \tag{7.50}$$

At this point one needs to apply (7.23), (7.26) and (7.31) to (7.50); doing so along with multiplying out and transforming the covariant derivatives into 3-derivatives gives

$$D_b s = -\tfrac{1}{2}\Omega^2 \mu_b \omega + \theta_b \omega + \tfrac{1}{2}\Omega^2 \mu_{ba}\omega^a - \theta_{ba}\omega^a. \tag{7.51}$$

The Third CFE

The constraint equations of the third CFE are somewhat more difficult to derive; we start by contracting (3.44c) with three projector tensors

$$h_a{}^d h_b{}^e h_c{}^f \nabla_d L_{ef} - h_a{}^d h_b{}^e h_c{}^f \nabla_e L_{df} = \Omega h_a{}^d h_b{}^e h_c{}^f T_{def} - h_a{}^d h_b{}^e h_c{}^g d_{degf} \Omega^f. \quad (7.52)$$

Applying (7.23), (7.31), (7.26) and (7.38) to decompose Ω^f, L_{ef}, d_{degf}, transforming the covariant derivatives into 3-derivatives and (7.52) becomes

$$\begin{aligned} D_a\theta_{bc} - D_b\theta_{ac} &= \Omega h_a{}^d h_b{}^e h_c{}^f T_{def} - K_{bc}\theta_a + K_{ac}\theta_b \\ -\epsilon_{defg} d^*{}_c{}^e h_a{}^f h_b{}^g n^d \omega &+ d_{bc}\omega_a - d_{ac}\omega_b - d_{bd}h_{ac}\omega^d + d_{ad}h_{bc}\omega^d. \end{aligned} \quad (7.53)$$

Finally, transforming the magnetic part of the Weyl tensor using (7.37) and applying (7.39a) causes (7.53) to become

$$D_a\theta_{bc} - D_b\theta_{ac} = \Omega\tau_{abc} - K_{bc}\theta_a + K_{ac}\theta_b + d_{cab}\omega + d_{bc}\omega_a - d_{ac}\omega_b - d_{bd}h_{ac}\omega^d + d_{ad}h_{bc}\omega^d. \quad (7.54)$$

This is the first constraint equation of the third CFE; the second is obtained by applying two projectors and one normal to (3.44c)

$$h_a{}^d h_b{}^e n^c \nabla_d L_{ec} - h_a{}^d h_b{}^e n^c \nabla_e L_{dc} = \Omega h_a{}^d h_b{}^e n^c T_{dec} - \Omega^f h_a{}^d h_b{}^e n^c d_{decf}. \quad (7.55)$$

Repeating the same steps as in the derivation of the first constraint and it can be verified that (7.55) reduces to

$$D_a\theta_b - D_b\theta_a = \Omega\tau_{ab} - K_b{}^c\theta_{ac} + K_a{}^c\theta_{bc} + d_{cab}\omega^c. \quad (7.56)$$

There are no other constraint equations for the third CFE due to symmetry.

The Fourth CFE

The constraint equations of the fourth CFE are derived in a similar manner, once again symmetries of certain tensors reduce the amount of effort that is required in the derivations. The first constraint of the fourth CFE is obtained by contracting the fourth CFE using a normal and two projectors

$$h_a{}^d h_b{}^e n^c \nabla_f d_{dec}{}^f = -h_a{}^d h_b{}^e n^c T_{dec}, \quad (7.57)$$

then applying (7.38) to decompose the Weyl tensor as well as using (7.20) to remove the derivatives of the normal in terms of the extrinsic curvature and (7.37) to remove the Hodge dual terms causes (7.57) to become

$$D_c d^c{}_{ab} = -d_b{}^c K_{ac} + d_a{}^c K_{bc} - a^c d_{cab} + h_a{}^f h_b{}^d n^c T_{fdc}. \tag{7.58}$$

Note, that in our case we are only considering an initial hypersurface, which means that the acceleration is zero. With this in mind and using (7.39b) to rewrite the last term in (7.58) and we end up with

$$D_c d^c{}_{ab} = d_a{}^c K_{bc} - d_b{}^c K_{ac} + \tau_{ab}. \tag{7.59}$$

Alternatively, one can express (7.59) in terms the Hodge dual by repeating the same steps as above and not making use of (7.37) to rewrite the various expressions; additionally, one multiplies the equation by a Levi-Civita alternating tensor. Performing this course of action gives

$$D_a d^{*}_k{}^a = d^{bc} \epsilon_{kacf} K_b{}^f n^a + \tfrac{1}{2} \epsilon_{kbfd} n^b \tau^{fd}. \tag{7.60}$$

The second constraint of the fourth CFE is obtained by contracting (3.44d) with two normals and one projector

$$h_b{}^d n^a n^c \nabla_f d_{adc}{}^f = -h_b{}^d n^a n^c T_{adc}. \tag{7.61}$$

Applying the decomposition of the Weyl tensor and switching to the three index representation of the magnetic part of the Weyl tensor, transforming the covariant derivatives into 3-derivatives, then using (7.37) and making use of both (7.20) and (7.39c) causes (7.61) to become

$$a^a d_{ba} - K^{ac} d_{abc} + D_c d_b{}^c = -h_b{}^f n^a n^c T_{afc}. \tag{7.62}$$

Finally, setting the acceleration to zero and making use of (7.39c) transforms (7.62) into

$$D_c d_b{}^c = K^{ac} d_{abc} + \tau_b. \tag{7.63}$$

The Fifth CFE

Finally, we will take a look at the constraint equation for the fifth CFE. In this case one only needs to express the equation in terms of the decomposed quantities, since λ is a scalar and we cannot contract with the metric projector or the normal as a result. Consequently, the constraint equation for the fifth CFE is obtained via simple substitution. First we consider the fifth CFE written in terms of Ω_a

$$\lambda = -3\Omega_a\Omega^a + 6\Omega s, \tag{7.64}$$

then substituting (7.22a) and (7.22b) into gives (7.64) gives the desired constraint equation

$$\lambda = 6\Omega s + 3\omega^2 - 3\omega_a\omega^a, \tag{7.65}$$

this completes the derivation of the constraint equations for the CFEs. Although we have derived every single constraint equation, for the result that we wish to achieve, in fact only few of them will be actually used to obtain the result that we desire, however one still needs to know the constraint equations in order to see why this is the case.

7.2.4 The conformal constraint equations

The conformal Gauss-Codazzi equation

Essential to both the 3+1 decomposition and the construction of initial data are the Gauss-Codazzi and the Codazzi-Mainardi equations. These two equations describe the relationship between the curvature of spacetime and the curvature of a hypersurface. To obtain these equations we need to make use of the definitions of the curvature. Let us begin by deriving the Gauss-Codazzi equation; the starting point are the definitions of curvature for a spacetime and a hypersurface given by (2.8) and (7.14), respectively. Applying the definition of a 3-derivative of a vector (7.11) to a second order covariant derivative of a vector gives

$$D_a D_b v^c = h_a{}^p h_b^q h_r{}^c \nabla_p \nabla_q v^r - K_{ab} h_r{}^c n^p \nabla_p v^r - K_a{}^c K_{bp} v^p. \tag{7.66}$$

Substituting this equation into (7.14) and multiplying out gives

$$r_{abcd} + K_{ac}K_{bd} - K_{ad}K_{cd} = h_a{}^p h_b{}^q h_c{}^r h_d{}^s R_{pqrs}. \tag{7.67}$$

This is the Gauss-Codazzi equation; in order to make this equation compatible with conformal methods we need to express it in terms of the conformal variables. Substituting in (2.12), (7.17) and (7.29c) then (7.67) becomes

$$l_{ac} = \Omega d_{ac} + K_a{}^d K_{cd} - K_{ac}K^d{}_d - \tfrac{1}{4}K_{db}K^{db}h_{ac} + \tfrac{1}{4}K^b{}_b K^d{}_d h_{ac} + \theta_{ac}, \tag{7.68}$$

which is the conformal Gauss-Codazzi equation.

The conformal Codazzi-Mainardi equation

Next we derive the Codazzi-Mainardi equation which, instead of relating the intrinsic curvature of the surface to the spacetime curvature, relates the extrinsic curvature to the spacetime curvature. The starting point of this derivation is to apply (7.12) to a covariant derivative of the extrinsic curvature

$$D_a K_{bc} = h_a{}^p h_b{}^q h_c{}^r \nabla_p K_{qr}. \tag{7.69}$$

Next we consider the commutator of the extrinsic curvature; substituting (7.69) into the commutator gives

$$D_b K_{ac} - D_a K_{bc} = h_a{}^p h_b{}^q h_c{}^r n^s R_{pqrs}, \tag{7.70}$$

this is the Codazzi-Mainardi equation. We can also obtain a version of this equation that is compatible with conformal methods by substituting in (2.12) into (7.70), doing so and expanding gives

$$D_c K_{bd} - D_d K_{bc} = \Omega d_{bcd} - h_{bd}\theta_c + h_{bc}\theta_d, \tag{7.71}$$

which is the conformal Codazzi-Mainardi equation.

7.3 Construction of initial data for the conformal evolution equations

We now proceed to the primary goal of this chapter, which concerns the nature of the initial data itself. Here is the claim: let us suppose that we have an initial hypersurface, if we are given h_{ij}, Ω, K_{ij}, and all the matter variables μ, μ_i, μ_{ij} then it is possible to obtain all the other variables for the evolution equations and the constraints.

Let us first consider the equation for the spatial Schouten tensor (7.16), we can see that provided we know the spatial metric h_{ij} it is possible to calculate l_{ij} since the spatial Ricci curvaure and scalar depend on the spatial Christoffel symbols which, as we can see from (7.13), depend on h_{ij}. The next variable that we consider is the Friedrich scalar; it is possible to obtain this variable from the constraint equation of the fifth CFE. Rearranging (7.65) gives

$$s = \frac{\lambda - 3\omega^2 + 3\omega_a\omega^a}{6\Omega}, \tag{7.72}$$

since the cosmological constant is a known quantity a priori and Ω_a is derived from Ω via (7.22b) this means that we know the value of the Friedrich scalar, which is consistent with our claim.

Next we examine the constraint equations for first CFE, starting with (7.48). We notice that the component of the Schouten tensor θ_a can be obtained again via simple rearrangement of said equation

$$\theta_i = \tfrac{1}{2}\Omega^2\mu_i - \frac{D_i\omega}{\Omega} + \frac{K_{ij}\omega^j}{\Omega}, \tag{7.73}$$

which is dependent on the required quantities and is consistent with the claim. The second constraint equation for the first CFE (7.49) is also used to prove our claim as we can obtain the component of the Schouten tensor θ_{ab}, rearranging gives

$$\theta_{ij} = \frac{\omega k_{ij} - D_i\omega_j + s h_{ij} - K_{ij}\omega + \frac{1}{2}\Omega^3\mu_{ij}}{\Omega}. \tag{7.74}$$

Again, we can see that (7.74) depends on all of the quantities that we claimed would be needed to completely determine all variables in the system, since we established from (7.72) that it is possible to determine the Friedrich scalar from all the required quantities.

In the final steps, we must make use of the conformal Gauss-Codazzi and the conformal Codazzi-Mainardi equations. Rearranging (7.68) gives

$$d_{ac} = \frac{l_{ac} - K_a{}^d K_{cd} + K_{ac}K^d{}_d + \frac{1}{4}K_{db}K^{db}h_{ac} - \frac{1}{4}K^b{}_b K^d{}_d h_{ac} - \theta_{ac}}{\Omega} \tag{7.75}$$

and doing the same to (7.71) yields

$$d_{ijk} = \frac{D_j K_{ki} - D_k K_{ji} + h_{ik}L_j - h_{ij}L_k}{\Omega}. \tag{7.76}$$

Looking at both (7.75) and (7.76), we can see that once again the RHS of the equations are comprised of quantities that are known. This means that once again we are able to deduce more variables of the system based on the ones that we claim determines the entire system. Recall in the previous section that we stated that in fact a large number of the constraint equations that were derived are in fact redundant with regards to proving this claim; by examining the equations we can now see why this is the case, it is because all of these terms can be written in terms of either the quantities that we have stated are needed to construct all the initial data variables or terms that depend on the same variables upon close inspection. For example, the constraint equation for the second CFE is formed from: the conformal factor and its derivatives, which are all determined if one knows the value of Ω, the matter terms μ_i and μ_{ij}, which we stated are necessary for constructing all the rest of the data and the Schouten tensor components, which are determined from (7.73) and (7.74). For the constraints for the third CFE, (7.54) and (7.56), they depend on the quantities that are fundamental to constructing data, quantities that are known from (7.73), (7.74), (7.75) and (7.76) and that can therefore be determined by the terms in our claim or components of the Cotton tensor. However, recall that the Cotton tensor can be expressed in terms of the energy momentum tensor, which is one of the fundamental quantities in our claim, therefore any term containing components of the Cotton tensor can be determined.

We have therefore shown that given the quantities $h_{ij}, \Omega, k_{ij}, \mu, \mu_i, \mu_{ij}$ then it is possible to determine all other pieces of information about the initial data. Whilst this is a known result it is by far the easiest way of showing this result to be true. Additionally, one needs to have knowledge about the properties of the initial data when it comes down to solving problems; one such area where information about the initial data is necessary is in the field of stability, which will be discussed in the next chapter.

7.3.1 The evolution equations

As mentioned beforehand, obtaining a suitable data set via the act of solving the constraint equations is only part of the process. The next step is to substitute said data set into the necessary evolution equations, which enables one to see how a certain spacetime with specific initial conditions will evolve in time. As solving the evolution equations is not relevant to any of the results obtained in this thesis, we will not give a derivation or a list of the equations. All that we will do is mention the method by which one obtains the evolution equations; essentially all one does is repeat a number of the steps performed in section 7.2, except this time contracting all the covariant derivatives with the normal as opposed to the projector. As we mentioned beforehand contracting the covariant derivative with the normal gives the components of the derivative that point in the time direction. Consequently, one is able to construct time derivatives and therefore evolution equations when contracting the covariant derivatives of the CFEs with the normal.

8 Stability of the de Sitter spacetime under trace-free matter perturbations

Throughout this thesis we have derived a number of general results relating to conformal methods in GR, however we have not yet used these results for anything in particular. This chapter shall be devoted to one such way in which the results can be applied, namely in showing that the de Sitter spacetime is both globally and nonlinearly stable.

8.1 Stability - basic ideas

One of the key ideas, as well as one of the primary subjects of interest in research of GR, concerns the analysis of the stability of spacetimes. Before looking at how the results that have been derived can be applied to stability it worth giving an overview of the subject. This subject is quite mathematically demanding and the results draw heavily on PDE theory, as such we will not go into full technical details of stability. Instead we will give a non-mathematical explanation of the main results of the theory and how this relates to GR.

To begin, we first need to answer the question: what is stability? In essence, stability asks the question of whether or not it is possible to gain information about how a perturbed system evolves in time based upon information about how the corresponding non-perturbed system evolves in time. To make this idea clearer, let us consider an example. Suppose one has an idealised background solution, the de Sitter spacetime, for example. The behaviour of this ideal, non-perturbed background solution is governed by equations that can be solved exactly; consequently it is possible using the evolution equations of this system to predict how the background solution evolves in time. Now, let us perturb the de Sitter background with the presence of matter. The question that we ask ourselves now is: how does this situation evolve in time? Also, we would like to know which properties of de Sitter are preserved, since in physics a large variety of problems in the real world are modelled using slight perturbations of idealized models. From the point of view of conformal methods, it is also important to ask if the conformal boundary is spacelike, since this is an indicator that any observer has existed for all eternity, meaning that the spacetime has also existed for the same amount of time. Now, due to a know result from PDE theory, known as "local existence", it is always guaranteed that for some small interval of time t_s that the perturbed version of this spacetime looks like the original version without the matter terms. This is a form of stability known as Cauchy stability, which we mentioned very briefly in section 1.1.2; it can only be applied if the equations have the correct form. The other form of stability is *Global stability* which, as the name implies, is the situation where a system is stable in the same sense as in Cauchy stability, but for infinite amounts of time. Our goal will be showing that de Sitter is stable in the global sense when perturbed specifically by trace-free matter, since all the equations that we have derived thus far describe trace-free matter. This idea of stability is of great interest in research in GR since it enables us to gain a great deal of information about physical systems, purely through knowledge of a well understood idealised model. Additionally, it enables the theory to make contact with the real world in situations where measurements with infinite precision are not possible.

8.1.1 Conditions for Cauchy stability

As mentioned in the previous section, our goal will be determining whether de Sitter perturbed by trace-free matter possesses global stability. This particular property of global stability follows from a conformal rescaling if a certain solution has Cauchy stability. If a system has Cauchy stability then the act of performing a conformal rescaling on said system essentially rescales and extends the finite stable region into an infinite region. Hence, global stability follows directly from Cauchy stability; it is therefore a good idea to review Cauchy stability and to see if the equations that have been derived thus far possess the necessary conditions for Cauchy stability. The key tool of determining whether a system possesses Cauchy stability is a result first proved by Hughes, Kato and Marsden (HKM) in 1977 [18]. This result is completely general and is therefore extremely useful. The HKM result says that any system is stable up to a finite time period if it satisfies the following conditions:

1. The perturbation of the background solution is described by wave equations;

2. The non-principal part of the wave equations depends in a *smooth* manner on the unknowns and their first order derivatives;

3. The background solution is also smooth and exists for a known amount of time;

4. The initial conditions that one starts with are *small* and smooth.

Provided all these conditions are met then the perturbed system will be close to the background system for finite time intervals. Within this checklist there are two words that deserve special attention: smooth and close. What do we mean by this? In everyday life, close and smooth have meanings that change according to the situation. You and your friend might be considered close to each other if you are sitting at the same table; similarly a building might be considered close if there is a high speed train that can take us there is a few minutes. For our purposes, the words close and smooth have very precise meanings that we will describe in detail.

A function is smooth if it is possible to take an infinite number of derivatives of this function. Even though one can take an infinite number of derivatives of a function, we only need to compute a total of four derivatives for the variables of the CWEs. The reason for this is that the primary function that we will be considering is the metric, since every single variable of the CWEs depends fundamentally on the metric. The highest order derivatives of our system are second order derivatives of the Weyl tensor. The Weyl tensor contains first order derivatives of the Christoffel symbols, which in turn contain first order derivatives of the metric; hence the highest order derivative that acts on the metric is four. From this we conclude that, in general, this particular number of derivatives needs to exist, otherwise quantities that you would compute using the wave equations could not be evaluated; for example it is impossible to solve a second order differential equation if the second order derivative with respect to the required function does not exist.

By close and small, we really mean that the value computed using the *Sobolev norm* is below a certain threshold. To make sense of this, let us give a definition of the Sobolev norm. A Sobolev norm is an object that generalizes the notion of distance to an infinite number of dimensions. Recall that the distance between two points, $\vec{p}(x, y, z)$ and $\vec{q}(x', y', z')$, in 3-dimensional space is given as

$$d(\vec{p}, \vec{q}) = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2},$$

we say that these two points are close if $d(\vec{p}, \vec{q}) < \varepsilon$, where ε is a small number. This small number is a value that is chosen by us; its value depends on the level of accuracy that one wishes to obtain. We can extend this idea to include continuous functions as well as discrete points. We first consider the simplest case of a function of a single parameter. Two arbitrary functions $f(x)$ and $g(x)$ on the real line are said to be close if

$$\int_{-\infty}^{\infty} \big(f(x) - g(x)\big)^2 dx < \varepsilon, \tag{8.1}$$

where the integral on the LHS of (8.1) is a generalisation of the distance of two points applied to functions. If the difference between these two functions is smaller than ε, then is it sufficient to say that $f(x)$ and $g(x)$ are close enough such that we obtain very similar results for both functions if we use them in the same calculation? The answer to this question is no, to understand why consider the following example: let us suppose that the function $f(x)$ is a constant value at all points and that the other function $g(x)$ is a constant that is infinitesimally close to $f(x)$ at all points except for a single point where the value of $g(x)$ is much bigger. In this particular situation, if one were to compute the difference between $f(x)$ and $g(x)$ using (8.1) then the result would still turn out to be less than the parameter ε, since we are integrating over such a large range and this particular information is lost. In stability it is absolutely critical that the functions remain close for all times, hence we need some way of checking for cases where such discrete discontinuities occur. A method of checking for such occurrences is to compare the differences between the derivatives of the functions, as in our example the differences between the derivatives will be very large. It is for this exact reason why it is not enough in HKM simply saying that the variables of the evolutions equations are close, we also need to determine if the derivatives are close; this leads to the concept of the Sobolev norm.

Let us suppose we are solving some equations on an initial hypersurface S, then the norm of the two functions under consideration is given by

$$\|f-g\|^2 \equiv \int_S \big(f(x)-g(x)\big)^2 d^3x,$$

from here we may define some operations that determine how close the derivatives of functions are

$$\begin{aligned}
\|\partial f-\partial g\|^2 &\equiv \sum \int_S (\partial f-\partial g) d^3x,\\
\|\partial^2 f-\partial^2 g\|^2 &\equiv \sum \int_S (\partial^2 f-\partial^2 g) d^3x,\\
&\vdots\\
\|\partial^\alpha f-\partial^\alpha g\|^2 &\equiv \sum \int_S (\partial^n f-\partial^n g) d^3x,
\end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha_1 + \alpha_2 + \alpha_3 = n$. The Sobolev norm is then defined as

$$\|f-g\|^2_{H^n} \equiv \|f-g\|^2 + \|\partial f-\partial g\|^2 + \ldots + \|\partial^n f-\partial^n g\|^2. \tag{8.2}$$

The functions f and g are stated to be close if $\|f-g\|^2 < \varepsilon$ for some chosen value of ε.

8.2 The de Sitter background solution

Now that we have taken a look at the basic ideas of stability, let us look at the background solution of our choosing, namely de Sitter. Recall that de Sitter is an exact solution to the EFEs, which models a universe with a positive cosmological constant λ. The metric for this spacetime is given as

$$\tilde{\mathring{g}} = dt^2 - \cosh^2 t \, \hbar, \tag{8.3}$$

where $\hbar$ is the metric of the 3-sphere which, in general can be written any reasonable system of coordinates, and we have introduced the notation that anything with a small circle above it is a variable of the de Sitter spacetime; $\mathring{g}$ is the metric of de Sitter, $\mathring{R}_{\mu\nu}$ is the Ricci curvature of de Sitter and so on. In order to perform a conformal rescaling of this spacetime, one needs to choose a value for the conformal factor. In order to do so, let us first look at the exact form of the conformal transformation from which we deduce the equation for the physical metric $\tilde{\mathring{g}} = \Omega^{-2}\mathring{g}$. With this in mind, we make an educated guess about the form of the conformal factor, which occurs by factoring out the $\cosh^2 t$ term from (8.3)

$$\tilde{\mathring{g}} = \cosh^2 t \Big(\frac{dt^2}{\cosh^2 t} - \hbar \Big). \tag{8.4}$$

From this we guess that the conformal factor should be

$$\Omega = \frac{1}{\cosh \ t}. \tag{8.5}$$

It is also convenient to define an alternate parameter for the time component of the metric; we can deduce from (8.4) an equation for the time parameter of the conformal de Sitter metric

$$dT = \frac{dt}{\cosh \ t}, \tag{8.6}$$

integrating both sides of (8.6) gives

$$T = 2 \arctan \ e^t. \tag{8.7}$$

From this we can conclude that the physical metric of the de Sitter spacetime can be written as

$$\tilde{\mathring{g}} = \cosh^2(-dT^2 + d\phi^2 + \sin^2 d\theta^2 + \sin^2\phi \sin^2\theta d\phi^2), \tag{8.8}$$

and therefore the conformal de Sitter metric is

$$\mathring{g} = -dT^2 + d\phi^2 + \sin^2 d\theta^2 + \sin^2\phi \sin^2\theta d\phi^2. \tag{8.9}$$

An important fact to mention about this metric is that it is conformally flat, which means that

$$\mathring{d}^{\rho}{}_{\sigma\delta\lambda} = 0.$$

The concept of conformal flatness was first mentioned in section 2.2.1; we stated that it is an important characteristic of certain spacetime metrics. The reason for this is because any quantity that we choose to calculate is dependent on Ω only, which helps to simplify things considerably.

Now that we have defined all the necessary concepts let us take an in depth look at the stability of de Sitter. To begin with the shape of the de Sitter metric is that of the Einstein cylinder. The Einstein cylinder is a conformal representation of Minkowski, de Sitter and anti-de Sitter spacetimes. More precisely, different sections of the cylinder correspond to one of these three spacetimes. All three spacetimes are conformally flat and so is the Einstein cylinder. Because of this, all of the spacetimes can be related to each other, however, as the Einstein cylinder is the largest it is possible for all three metrics to be represented by a section of the cylinder. The de Sitter metric is smooth because we can take the required number of derivatives of the metric components $\mathring{g}_{\mu\nu}$. Additionally, the quantities that describe the conformal de Sitter spacetime, $\mathring{\Gamma}^{\mu}{}_{\nu\lambda}$, $\mathring{d}^{\mu}{}_{\nu\lambda\rho}$, $\mathring{\Omega}$, $\mathring{L}_{\mu\nu}$ and $\mathring{s}$, are smooth since they depend on the metric $\mathring{g}$, which is itself a smooth quantity. Furthermore, all these quantities are smooth for all time on the Einstein cylinder. Moreover, these quantities are also a solution to the CWEs, since any solution to the EFEs implies a solution to the CFEs and any solution to the CFEs implies a solution to the CWEs, at least for the vacuum case. Naturally, we would like to extend this case to describe matter; to perform this task we consider an Anzatz of the form

$$u = \mathring{u} + \breve{u}, \tag{8.10}$$

where u is a shorthand notation for all the variables that describe the geometry of the conformal spacetime i.e. $u = (\Omega, s, g_{\mu\nu}, L_{\mu\nu}, d^{\mu}{}_{\nu\lambda\rho})$. So $\mathring{u}$ is a shorthand for all the components that describe the evolution of the background solution and $\breve{u}$ is a shorthand for all the components that describe the perturbation of the de Sitter spacetime that occur due to the presence of trace-free matter. Note that when we say our perturbation is small we really mean with respect to the Sobolev norm. With this shorthand notation our wave equations may be written in the form

$$\Box u = H(u, \partial u, \Theta, \partial\Theta), \tag{8.11a}$$
$$\Box \Theta = F(u, \partial u, \Theta, \partial\Theta), \tag{8.11b}$$

where Θ is shorthand for all the matter fields. For example, if we were doing an analysis of de Sitter coupled to the conformally invariant scalar field then $\Theta = (\phi, \phi_a, \phi_{ab})$. Also, both H and F are smooth functions with respect to their arguments. A noticeable fact about this system is that if you choose to express any of the variables in terms of coordinates, then the end result is still a smooth function.

So the wave equations have a smooth dependence on the unknowns, which is a requirement of the result of HKM. If we now substitute (8.10) into (8.11a) and

(8.11b) then the wave equations take the form

$$(\mathring{g}^{\mu\nu} + \breve{g}^{\mu\nu})\partial_\mu\partial_\nu\breve{u} + H'(\mathrm{u}, \partial\mathring{u}, \breve{u}, \partial\breve{u}, \Theta, \partial\Theta), \tag{8.12a}$$

$$(\mathring{g}^{\mu\nu} + \breve{g}^{\mu\nu})\partial_\mu\partial_\nu\breve{\Theta} + F'(\mathrm{u}, \partial\mathring{u}, \partial^2\mathring{u}, \breve{u}, \partial\breve{u}, \Theta, \partial\Theta). \tag{8.12b}$$

Note, the conformal de Sitter metric $\mathring{g}_{\mu\nu}$ is Lorentzian, since de Sitter is a Lorentzian spacetime. If $\mathring{g}_{\mu\nu}$ is Lorentzian then $g_{\mu\nu}$ will also be Lorentzian, provided the perturbation $\breve{g}_{\mu\nu}$ is small enough with respect to the Sobolev norm.

We want $g_{\mu\nu}$ to be a Lorentzian metric for all time as this is an important property of GR. The spacetime is 4-dimensional and has a structure where causality is preserved, which is described by the Lorentzian metric. We can say that $g_{\mu\nu}$ is Lorentzian on the initial hypersurface, if $\breve{g}_{\mu\nu}$ is small enough. If this is the case then HKM says that the perturbed metric is Lorentzian for all times and your solution is stable. Put in another way, the HKM result tells you that if the initial data $(\breve{u}, \breve{\Theta})$ is small enough then $\breve{u}$ and $\breve{\Theta}$, which solve (8.12a) and (8.12b), exist up to a finite amount of time. If this is the case, then the conformal extension of the spacetime causes the finite region to be rescaled into an infinite region, which guarantees that solution possesses global stability. As the CWEs with trace-free matter are smooth with respect to the unknowns and the de Sitter background exists for a known amount of time, this means that de Sitter perturbed with any of the forms of trace-free matter that have been analysed previously - the conformally invariant scalar field, Einstein-Maxwell fields, classical Yang-Mills and irrotational fluids - is stable for a finite amount of time. Furthermore, the fact that de Sitter can be conformally extended in such a way that it remains smooth means that the de Sitter spacetime perturbed with any of the aforementioned types of trace-free matter is stable for all times. Additionally as the conformal factor of the perturbed solution can be shown to vanish then the resulting conformal boundary is spacelike, just as de Sitter is. This observation makes the result global from the point of view of the physical metric.

Summary

We have shown that by the application of several known results that the de Sitter spacetime perturbed by the trace-free matter models analysed in Chapter 4 is stable in the local sense. Furthermore, the conformal nature of the evolution equations means that the local stability can be extended into global stability. Since there have been no linearisation carried out in any of the calculations, this means that such a spacetime is both globally and non linearly stable.

9 Conclusions

We have obtained a multitude of results for conformal spacetimes and trace-free matter. We have shown that it is possible for conformal spacetime perturbed by trace-free matter to be described by a system of wave equations. We have also shown that the trace-free matter itself can be described by a system of wave equations. Furthermore, we have shown that any solution to the wave equations implies a solution to the corresponding field equations. As a result of both the spacetime and the matter being described by wave equations, this means it is possible to formulate an initial value problem for conformal spacetimes perturbed by trace-free matter. We have also shown that these results can be used to prove that de Sitter is stable in the global sense when perturbed by trace-free matter.

We have also taken the time to analyse in depth the conformal field equations coupled with a conformally invariant scalar field. We have derived equations that describe a spacetime containing this particular spacetime. Additionally, we have analysed this spacetime and concluded that virtually all choices of initial data lead to a system that is compatible with Penrose's cyclic cosmology theory, with only a few exceptions.

There are a multitude of potential directions that this work could go in. For starters, whilst this whole thesis has been devoted entirely to trace-free matter, the question of non trace-free matter is still very much an open problem; if it were indeed possible to formulate a system of equations for matter that is not trace-free it would be an incredibly powerful tool. Even within the context of the equations already obtained there are many paths that this work could take because whilst this entire thesis has been almost exclusively devoted to the derivation and analysis of the equations, virtually no work has been done in either solving or analysing the equations to see what new information can be gathered.

From the point of view of the fifth chapter one could still do a similar analysis for other matter models apart from the relatively simple scalar field model. The fact that the Bach tensor has proven to couple to the field equations and the wave equations nicely means that it could also prove to be an object of interest in the future.

Either way, it is the hope of the author of this thesis that the usefulness and elegance of conformal methods in relativity has been displayed. Hopefully the reader should be convinced that conformal methods should no doubt prove to be an exciting and interesting field of research in the future.

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