ON THE TOPOLOGICAL COMPLEXITY OF ASPHERICAL SPACES

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ABSTRACT. The well-known theorem of Eilenberg and Ganea [12] expresses the Lusternik-Schnirelmann category of an aspherical space \( K(\pi, 1) \) as the cohomological dimension of the group \( \pi \). In this paper we study a similar problem of determining algebraically the topological complexity of the Eilenberg-MacLane spaces \( K(\pi, 1) \). One of our main results states that in the case when the group \( \pi \) is hyperbolic in the sense of Gromov the topological complexity \( TC(K(\pi, 1)) \) either equals or is by one larger than the cohomological dimension of \( \pi \times \pi \). We approach the problem by studying essential cohomology classes, i.e. classes which can be obtained from the powers of the canonical class (defined in [7]) via coefficient homomorphisms. We describe a spectral sequence which allows to specify a full set of obstructions for a cohomology class to be essential. In the case of a hyperbolic group we establish a vanishing property of this spectral sequence which leads to the main result.

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1. Introduction

In this paper we study a numerical topological invariant \( TC(X) \) of a topological space \( X \), originally introduced in [14], see also [15], [17]. The concept of \( TC(X) \) is related to the motion planning problem of robotics where a system (robot) has to be programmed to be able to move autonomously from any initial state to any final state. In this situation a motion of the system is represented by a continuous path in the configuration space \( X \) and a motion planning algorithm is a section of the path fibration

\[
p : PX \to X \times X, \quad p(\gamma) = (\gamma(0), \gamma(1)).
\]

Here \( PX \) denotes the space of all continuous paths \( \gamma : [0,1] \to X \) equipped with the compact-open topology. The topological complexity \( TC(X) \) is an integer reflecting the complexity of this fibration, it has several different characterisations, see [15]. Intuitively, \( TC(X) \) is a measure of the navigational complexity of \( X \) viewed as the configuration space of a system. \( TC(X) \) is similar in spirit to the classical Lusternik - Schnirelmann category \( \text{cat}(X) \). The invariants \( TC(X) \) and \( \text{cat}(X) \) are special cases of a more general notion of the genus of a fibration introduced by A. Schwarz [30]. A recent survey of the concept \( TC(X) \) and robot motion planning algorithms in interesting configuration spaces can be found in [18].
Definition 1.1. Given a path-connected topological space $X$, the topological complexity of $X$ is defined as the minimal number $TC(X) = k$ such that the Cartesian product $X \times X$ can be covered by $k$ open subsets $X \times X = U_1 \cup U_2 \cup \ldots U_k$ with the property that for any $i = 1, 2, \ldots, k$ there exists a continuous section $s_i : U_i \to PX, \pi \circ s_i = id$, over $U_i$. If no such $k$ exists we will set $TC(X) = \infty$.

Note that in the mathematical literature there is also a reduced version of the topological complexity which is one less compared to the one we are dealing with in this paper.

One of the main properties of $TC(X)$ is its homotopy invariance [14], i.e. $TC(X)$ depends only on the homotopy type of $X$. This property is helpful for the task of computing $TC(X)$ in various examples since cohomological tools can be employed. In the case when the configuration space $X$ is aspherical, i.e. $\pi_i(X) = 0$ for all $i > 1$, the number $TC(X)$ depends only on the fundamental group $\pi = \pi_1(X)$ and it was observed in [15] that one has to be able to express $TC(X)$ in terms of algebraic properties of the group $\pi$ alone.

A similar question for the Lusternik - Schnirelmann category $cat(X)$ was solved by S. Eilenberg and T. Ganea in 1957 in the seminal paper [12]. Their theorem relates $cat(X)$ and the cohomological dimension of the fundamental group $\pi$ of $X$.

The problem of computing $TC(K(\pi, 1))$ as an algebraic invariant of the group $\pi$ attracted attention of many mathematicians. Although no general answer is presently known, many interesting results were obtained.

The initial papers [14], [15] contained computations of $TC(X)$ for graphs, closed orientable surfaces and tori. In [19] the number $TC(X)$ was computed for the case when $X$ is the configuration space of many particles moving on the plane without collisions. D. Cohen and G. Pruidze [5] calculated the topological complexity of complements of general position arrangements and Eilenberg – MacLane spaces associated to certain right-angled Artin groups.

As a recent breakthrough, the topological complexity of closed non-orientable surfaces of genus $g \geq 2$ has only recently been computed by A. Dranishnikov for $g \geq 4$ in [11] and by D. Cohen and L. Vandembroucq for $g = 2, 3$ in [6]. In both these articles it is shown that $TC(K(\pi, 1))$ attains its maximum, i.e. coincides with $cd(\pi \times \pi) + 1$.

The estimates of M. Grant [20] give good upper bounds for $TC(K(\pi, 1))$ for nilpotent fundamental groups $\pi$. In [21], M. Grant, G. Lupton and J. Oprea proved that $TC(K(\pi, 1))$ is bounded below by the cohomological dimension of $A \times B$ where $A$ and $B$ are subgroups of $\pi$ whose conjugates intersect trivially. Using these estimates, M. Grant and D. Recio-Mitter [22] have computed $TC(K(\pi, 1))$ for certain subgroups of Artin’s braid groups.

Yuli Rudyak [29] showed that for any pair of positive integers $k, \ell$ satisfying $k \leq \ell \leq 2k$ there exists a finitely presented group $\pi$ such that $cd(\pi) = k$ and $TC(K(\pi, 1)) = \ell + 1$.

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2. Statements of the main results

In this section we state the main results obtained in the present paper.

**Theorem 1.** Let $X$ be a connected aspherical finite cell complex with hyperbolic fundamental group $\pi = \pi_1(X)$. Then the topological complexity $TC(X)$ equals either $\text{cd}(\pi \times \pi)$ or $\text{cd}(\pi \times \pi) + 1$.

The symbol $\text{cd}(\pi \times \pi)$ stands for the cohomological dimension of $\pi \times \pi$ and similarly for $\text{cd}(\pi)$. The Eilenberg-Ganea theorem [4] states that $\text{cd}(\pi \times \pi) = \text{gd}(\pi \times \pi)$ (with one possible exception) and hence the general dimensional upper bound for $TC(X)$ (see [14], [17]) gives

$$TC(X) \leq \text{gd}(\pi \times \pi) + 1 = \text{cd}(\pi \times \pi) + 1.$$  

Here $\text{gd}(\pi \times \pi)$ denotes *the geometric dimension of* $\pi \times \pi$, i.e. the minimal dimension of a cell complex with fundamental group $\pi \times \pi$. Thus Theorem 1 essentially states that the topological complexity $TC(K(\pi, 1))$, where $\pi$ is hyperbolic, is either maximal (as allowed by the dimensional upper bound) or is by one smaller than the maximum.

The notion of a hyperbolic group was introduced by M. Gromov in [23]; we also refer the reader to the monograph [3]. Hyperbolic groups are “typical”, i.e. they appear with probability tending to 1, in many models of random groups including Gromov’s well-known density model [24], [28].

As an example, consider the case of the fundamental group $\pi = \pi_1(\Sigma_g)$ of a closed orientable surface of genus $g \geq 2$. It is torsion-free hyperbolic and $TC(\Sigma_g) = \text{cd}(\pi \times \pi) + 1 = 5$, see [17], in accordance with the maximal option of Theorem 1. Similarly, if $\pi = F_\mu$ is a free group on $\mu$ generators then, according to [17], Proposition 4.42, $TC(K(F_\mu, 1)) = 3$ (for $\mu > 1$) and $\text{cd}(F_\mu \times F_\mu) = 2$; here again Theorem 1 is satisfied in the maximal version.

The only known to us example of an aspherical space $X = K(\pi, 1)$ with $\pi$ hyperbolic where $TC(X) = \text{cd}(\pi \times \pi)$ is the case of the circle $X = S^1$. It would be interesting to learn if some other examples of this type exist.

Theorem 1 follows from the following statement:

**Theorem 2.** Let $X$ be a connected aspherical finite cell complex with fundamental group $\pi = \pi_1(X)$. Suppose that (1) the centraliser of any nontrivial element $g \in \pi$ is cyclic and (2) $\text{cd}(\pi \times \pi) > \text{cd}(\pi)$. Then the topological complexity $TC(X)$ equals either $\text{cd}(\pi \times \pi)$ or $\text{cd}(\pi \times \pi) + 1$.

We do not know examples of finitely presented groups $\pi$ such that $\text{cd}(\pi \times \pi) = \text{cd}(\pi)$, i.e. such that the assumption (2) of Theorem 2 is violated. A. Dranishnikov [9] constructed examples with $\text{cd}(\pi_1 \times \pi_2) < \text{cd}(\pi_1) + \text{cd}(\pi_2)$; see also [8], page 157. In [9] he also proved that $\text{cd}(\pi \times \pi) = 2\text{cd}(\pi)$ for any Coxeter group $\pi$.

To state another main result of this paper we need to recall the notion of *TC-weight* of cohomology classes as introduced in [16]; this notion is similar but not identical to the concept of TC-weight introduced in [17], §4.5; both these notions were inspired by the notion of category weight of cohomology classes initiated by E. Fadell and S. Husseini [13].
Definition 2.1. Let $\alpha \in H^*(X \times X, A)$ be a cohomology class, where $A$ is a local coefficient system on $X \times X$. We say that $\alpha$ has weight $k \geq 0$ (notation $\text{wgt}(\alpha) = k$) if $k$ is the largest integer with the property that for any continuous map $f : Y \to X \times X$ (where $Y$ is a topological space) one has $f^*(\alpha) = 0 \in H^*(Y, f^*(A))$ provided the space $Y$ admits an open cover $U_1 \cup U_2 \cup \cdots \cup U_k = Y$ such that each restriction map $f|_{U_j} : U_j \to X \times X$ admits a continuous lift $U_j \to PX$ into the path-space fibration (1).

A cohomology class $\alpha \in H^*(X \times X, A)$ has a positive weight $\text{wgt}(\alpha) \geq 1$ if and only if $\alpha$ is a zero-divisor, i.e. if its restriction to the diagonal $\Delta_X \subset X \times X$ vanishes,

$$0 = \alpha|_{\Delta_X} \in H^*(X, \tilde{A}),$$

see [16], page 3341. Here $\tilde{A}$ denotes the restriction local system $A|_{\Delta_X}$. Note that in [16] the authors considered untwisted coefficients but all the arguments automatically extend to general local coefficient systems. In particular, by Proposition 2 from [16] we have

$$\text{wgt}(\alpha_1 \cup \alpha_2) \geq \text{wgt}(\alpha_1) + \text{wgt}(\alpha_2) \quad (3)$$

for cohomology classes $\alpha_i \in H^{d_i}(X \times X, A_i)$, $i = 1, 2$, where the cup-product $\alpha_1 \cup \alpha_2$ lies in $H^{d_1+d_2}(X \times X, A_1 \otimes_{\mathbb{Z}} A_2)$.

Theorem 3. Let $X$ be a connected aspherical finite cell complex. Suppose that the fundamental group $\pi = \pi_1(X)$ is such that the centraliser of any nontrivial element $g \in \pi$ is infinite cyclic. Then any degree $n$ zero-divisor $\alpha \in H^n(X \times X, A)$, where $n \geq 1$, has weight $\text{wgt}(\alpha) \geq n - 1$.

For obvious reasons this theorem is automatically true for $n = 1, 2$; it becomes meaningful only for $n > 2$.

Here is a useful corollary of Theorem 3:

Theorem 4. Under the assumptions of Theorem 2, one has

$$v^{n-1} \neq 0 \in H^{n-1}(\pi \times \pi, I^{n-1})$$

where $n = \text{cd}(\pi \times \pi)$. Here $v \in H^1(\pi \times \pi, I)$ denotes the canonical class, see §3 below.

The statement of Theorem 4 becomes false if we remove the assumptions on the fundamental group. For example in the case of an abelian group $\pi = \mathbb{Z}^k$ (see §6) we have $n = \text{cd}(\pi \times \pi) = 2k$ and $v^k$ is the highest nontrivial power of the canonical class.

Question: Let $\pi$ be a noncommutative hyperbolic group and let $n$ denote $\text{cd}(\pi \times \pi)$. Is it true that the $n$-th power of the canonical class $v^n \in H^n(\pi \times \pi, I^n)$ is nonzero, $v^n \neq 0$?

A positive answer to this question would imply that for any noncommutative hyperbolic group $\pi$ one has $\text{TC}(K(\pi, 1)) = \text{cd}(\pi \times \pi) + 1$.

The proofs of Theorems 1, 2, 3 and 4 are given in §9.

In §10, we present an application of Theorem 3 to the topological complexity of symplectically aspherical manifolds.
3. The canonical class

First we fix notations which will be used in this paper. We shall consider a discrete torsion-free group $\pi$ with unit element $e \in \pi$ and left modules $M$ over the group ring $\mathbb{Z}[\pi \times \pi]$. Any such module $M$ can be equivalently viewed as a $\pi - \pi$-bimodule using the convention

$$(g,h) \cdot m = gmh^{-1}$$

for $g,h \in \pi$ and $m \in M$. Recall that for two left $\mathbb{Z}[\pi \times \pi]$-modules $A$ and $B$ the module $\text{Hom}_\mathbb{Z}(A,B)$ has a canonical $\mathbb{Z}[\pi \times \pi]$-module structure given by $((g,h) \cdot f)(a) = gf(g^{-1}ah)h^{-1}$ where $g,h \in \pi$, $a \in A$ and $f : A \to B$ is a group homomorphism.

Besides, the tensor product $A \otimes \mathbb{Z} B$ has a left $\mathbb{Z}[\pi \times \pi]$-module structure given by $(g,h) \cdot (a \otimes b) = (gah^{-1}) \otimes (gbh^{-1})$ where $g,h \in \pi$ and $a \in A$, $b \in B$; we shall refer to this action as the diagonal action.

For a left $\mathbb{Z}[\pi \times \pi]$-module $A$ we shall denote by $\tilde{A}$ the same abelian group viewed as a $\mathbb{Z}[\pi]$-module via the conjugation action, i.e. $g \cdot a = gag^{-1}$ for $g \in \pi$ and $a \in A$.

The group ring $\mathbb{Z}[\pi]$ is a $\mathbb{Z}[\pi \times \pi]$-module with respect to the action

$$(g,h) \cdot a = gah^{-1}, \quad \text{where} \quad g,h,a \in \pi.$$  

The augmentation homomorphism $\epsilon : \mathbb{Z}[\pi] \to \mathbb{Z}$ is a $\mathbb{Z}[\pi \times \pi]$-homomorphism where we consider the trivial $\mathbb{Z}[\pi \times \pi]$-module structure on $\mathbb{Z}$. The augmentation ideal $I = \ker \epsilon$ is hence a $\mathbb{Z}[\pi \times \pi]$-module and we have a short exact sequence of $\mathbb{Z}[\pi \times \pi]$-modules

$$(4) \quad 0 \to I \to \mathbb{Z}[\pi] \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$  

In this paper we shall use the the formalism (described in [25], Chapter IV, §9) which associates a well defined class

$$\theta \in \text{Ext}^n_R(M,N)$$

with any exact sequence

$$0 \to N \to L_n \to L_{n-1} \to \cdots \to L_1 \to M \to 0$$

of left $R$-modules and $R$-homomorphisms, where $R$ is a ring. This construction can be briefly summarised as follows. If

$$\cdots \to C_2 \to C_1 \to C_0 \to M \to 0$$

is a projective resolution of $M$ over $R$, one obtains a commutative diagram

$$\begin{array}{cccccccc}
C_{n+1} & \to & C_n & \to & C_{n-1} & \to & \cdots & \to & C_1 & \to & C_0 & \to & M & \to & 0 \\
\downarrow & & \downarrow f & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \downarrow \epsilon & \downarrow & = & \\
0 & \to & N & \to & L_n & \to & \cdots & \to & L_2 & \to & L_1 & \to & M & \to & 0.
\end{array}$$

The homomorphism $f : C_n \to N$ is a cocycle of the complex $\text{Hom}_R(C_*,N)$, which is defined uniquely up to chain homotopy. The class $\theta$ is the cohomology class of this cocycle

$$\theta = \{ f \} \in H^n(\text{Hom}_R(C_*,N)) = \text{Ext}^n_R(M,N).$$

Note that the definition of Bourbaki (see [2] §7, n. 3) is slightly different since [2] introduces additionally a sign factor $(-1)^{n(n+1)/2}$. 

An important role plays the class 
\[ v \in \text{Ext}^1_{\mathbb{Z}[[\pi \times \pi]]}(\mathbb{Z}, I) = H^1(\pi \times \pi, I) \]
associated with the exact sequence (4). It was introduced in [7] under the name of the canonical class.

To describe the cocycle representing the canonical class \( v \) consider the bar resolution \( C_* \) of \( \mathbb{Z} \) over \( \mathbb{Z}[[\pi \times \pi]] \), see [4], page 19. Here \( \cdots \rightarrow C_1 \xrightarrow{d} C_0 \rightarrow \mathbb{Z} \rightarrow 0 \) where \( C_0 \) is a free \( \mathbb{Z}[[\pi \times \pi]] \)-module generated by the symbol \( [ ] \) and \( C_1 \) is the free \( \mathbb{Z}[[\pi \times \pi]] \)-module generated by the symbols \( [(g, h)] \) for all \( (g, h) \in \pi \times \pi \). The boundary operator \( d \) acts by
\[ d([(g, h)]) = ((g, h) - 1) [ ]. \]

We obtain the chain map
\[ C_2 \rightarrow C_1 \xrightarrow{d} C_0 \rightarrow \mathbb{Z} \rightarrow 0 \]
\[ \downarrow \quad \downarrow f \quad \downarrow \mu \quad \downarrow = \]
\[ 0 \rightarrow I \rightarrow \mathbb{Z}[[\pi]] \xrightarrow{\xi} \mathbb{Z} \rightarrow 0 \]
where \( \mu([ ]) = 1 \) and
\[ f([(g, h)]) = gh^{-1} - 1 \in I. \]

Thus, the cocycle \( f : C_1 \rightarrow I \) is given by the crossed homomorphism (5). Comparing with [7], we see that the definition of the canonical class given above coincides with the definition given in [7], page 110.

We shall also describe the cocycle representing the canonical class in the homogeneous standard resolution of \( \pi \times \pi \), see [4], page 18:

\[ \cdots \rightarrow C'_2 \xrightarrow{d} C'_1 \xrightarrow{d} C'_0 \rightarrow \mathbb{Z} \rightarrow 0. \]

Here \( C'_i \) is a free \( \mathbb{Z} \)-module generated by the \((i+1)\)-tuples \( ((g_0, h_0), \ldots, (g_i, h_i)) \) with \( g_j, h_j \in \pi \) for any \( j = 0, 1, \ldots, i \). Using (5) we obtain that the cocycle \( f' : C'_1 \rightarrow I \) representing the canonical class \( v \) is given by the formula
\[ f'((g_0, h_0), (g_1, h_1)) = g_1 h_1^{-1} - g_0 h_0^{-1}, \quad g_j, h_j \in \pi. \]

The canonical class \( v \) is closely related to the Berstein-Schwarz class (see [1], [10]); the latter is crucial for the study of the Lusternik-Schnirelmann category \( \text{cat} \). The Berstein-Schwarz class can be defined as the class
\[ b \in \text{Ext}^1_{\mathbb{Z}[\pi]}(\mathbb{Z}, I) = H^1(\pi, I) \]
which corresponds to the exact sequence (4) viewed as a sequence of left \( \mathbb{Z}[\pi] \)-modules via the left action of \( \pi = \pi \times 1 \subset \pi \times \pi \). For future reference we state
\[ v | \pi \times 1 = b; \]
here \( \pi \times 1 \subset \pi \times \pi \) denotes the left factor viewed as a subgroup.

The main properties of the canonical class \( v \) are as follows.
Let $X$ be a finite connected cell complex with fundamental group $\pi_1(X) = \pi$. We may view $I$ as a local coefficient system over $X \times X$ and form the cup-product $v \cup v \cup \cdots \cup v = v^k$ ($k$ times) which lies in the cohomology group

$$v^k \in H^k(X \times X, I^k),$$

where $I^k$ denotes the tensor product $I \otimes \cdots \otimes I$ of $k$ copies of $I$ viewed as a left $\mathbb{Z}[\pi \times \pi]$-module via the diagonal action as explained above. Let $n$ denote the dimension of $X$. It is known that in general the topological complexity satisfies $\text{TC}(X) \leq 2n + 1$ and the equality

$$\text{TC}(X) = 2 \dim(X) + 1 = 2n + 1$$

happens if and only if $v^{2n} \neq 0$; see [7], Theorem 7.

Another important property of $v$ is that it is a zero-divisor, i.e.

$$v \mid \Delta_v = 0 \in H^1(\pi, I)$$

where $\Delta_v \subset \pi \times \pi$ is the diagonal subgroup, $\Delta_v = \{ (g, g) ; g \in \pi \}$. This immediately follows from the observation that the cocycle $f$ representing $v$ (see (5)) vanishes on the diagonal $\Delta_v$.

Our next goal is to describe an exact sequence representing the power $v^n$ of the canonical class. The exact sequence (4) splits over $\mathbb{Z}$ and hence for any left $\mathbb{Z}[\pi \times \pi]$-module $M$ tensoring over $\mathbb{Z}$ we obtain an exact sequence

$$0 \to I \otimes M \to \mathbb{Z}[\pi] \otimes M \xrightarrow{i} M \to 0. \tag{10}$$

In (10) we consider the diagonal action of $\pi \times \pi$ on the tensor products. Taking here $M = I^s$ where $I^s = I \otimes \cdots \otimes I$ we obtain a short exact sequence

$$0 \to I^{s+1} \xrightarrow{\epsilon \otimes 1} \mathbb{Z}[\pi] \otimes I^s \xrightarrow{i \otimes 1} I^s \to 0. \tag{11}$$

Here $i : I \to \mathbb{Z}[\pi]$ is the inclusion and $\epsilon : \mathbb{Z}[\pi] \to \mathbb{Z}$ is the augmentation. Splicing exact sequences (11) for $s = 0, 1, \ldots, n - 1$ we obtain an exact sequence

$$0 \to I^n \to \mathbb{Z}[\pi] \otimes I^{n-1} \to \mathbb{Z}[\pi] \otimes I^{n-2} \to \cdots \mathbb{Z}[\pi] \otimes I \to \mathbb{Z}[\pi] \to \mathbb{Z} \to 0. \tag{12}$$

**Lemma 3.1.** The cohomology class

$$v^n \in H^n(\pi \times \pi, I^n) = \text{Ext}_{\mathbb{Z}[\pi \times \pi]}^n(\mathbb{Z}, I^n)$$

is represented by the exact sequence (12).

**Proof.** Consider again the homogeneous standard resolution (6) of $\pi \times \pi$. Define $\mathbb{Z}[\pi \times \pi]$-homomorphisms

$$\kappa_j : C'_j \to \mathbb{Z}[\pi] \otimes I^j, \quad \text{where} \quad j = 0, 1, \ldots, n - 1, \tag{13}$$

by the formula

$$\kappa_j((g_0, h_0), (g_1, h_1), \ldots, (g_j, h_j)) = x_0 \otimes (x_1 - x_0) \otimes \cdots \otimes (x_j - x_{j-1}),$$

where the symbol $x_i$ denotes $g_i h_i^{-1} \in \pi$ for $i = 0, \ldots, j$. 


We claim that the homomorphisms $\kappa_j$, for $j = 0, \ldots, n$, determine a chain map from the homogeneous standard resolution (6) into the exact sequence (12). In other words, we want to show that

$$\kappa_{j-1}(d((g_0, h_0), (g_1, h_1), \ldots, (g_j, h_j))) = (x_1 - x_0) \otimes \cdots \otimes (x_j - x_{j-1}).$$

This statement is obvious for $j = 1$. To prove it for $j > 1$ we apply induction on $j$. Denoting

$$\Pi_j(x_0, x_1, \ldots, x_{j-1}) = x_0 \otimes (x_1 - x_0) \otimes \cdots \otimes (x_{j-1} - x_{j-2})$$

we may write

$$\kappa_{j-1}(d((g_0, h_0), (g_1, h_1), \ldots, (g_j, h_j))) = \sum_{i=0}^{j-1} (-1)^i \Pi_j(x_0, \ldots, \hat{x}_i, \ldots, x_j).$$

The last two terms in this sum (for $i = j - 1$ and $i = j$) sum up to

$$(-1)^{i-1}x_0 \otimes (x_1 - x_0) \otimes \cdots (x_{j-2} - x_{j-3}) \otimes (x_j - x_{j-1}).$$

Thus we see that the LHS of (15) can be written as

$$\left[ \sum_{i=0}^{j-1} (-1)^i \Pi_{j-1}(x_0, \ldots, \hat{x}_i, \ldots, x_j) \right] \otimes (x_j - x_{j-1})$$

and our statement follows by induction.

The homomorphism $f_n : C'_n \to I^n$ which appears in the commutative diagram

$$C'_{n+1} \to C'_n \xrightarrow{d} C'_{n-1} \xrightarrow{d} C'_{n-2} \to \cdots$$

$$0 \to I^n \to \mathbb{Z}[[\pi]] \otimes I^{n-1} \to \mathbb{Z}[[\pi]] \otimes I^{n-2} \to \cdots$$

is given by the formula

$$f_n((g_0, h_0), (g_1, h_1), \ldots, (g_n, h_n)) = (x_1 - x_0) \otimes (x_2 - x_1) \otimes \cdots (x_n - x_{n-1})$$

where $x_i = g_i h_i^{-1}$. Since the cocycle representing $\mathfrak{v}$ is given by $x_1 - x_0$ (see (7)), using the diagonal approximation in the standard complex (see [4], page 108) we find that $f_n$ represents $\mathfrak{v}^n$.

**Remark 3.2.** Lemma 3.1 also follows by applying Theorems 4.2 and 9.2 from [27], Chapter VIII.

The canonical class $\mathfrak{v}$ allows to describe the connecting homomorphisms in cohomology as we shall exploit several times in this paper. Let $M$ be a left $\mathbb{Z}[[\pi \times \pi]]$-module. The Bockstein homomorphism

$$\beta : H^i(\pi \times \pi, M) \to H^{i+1}(\pi \times \pi, I \otimes M)$$

of the exact sequence (10) acts as follows

$$\beta(u) = \mathfrak{v} \cup u, \quad \text{for} \quad u \in H^i(\pi \times \pi, M).$$

This follows from Lemma 5 from [7] and from [4], chapter V, (3.3).
4. Universality of the Bernstein - Schwarz class

In the theory of Lusternik - Schnirelmann category an important role plays the following result which was originally stated (without proof) by A.S. Schwarz [30], Proposition 34. A recent proof can be found in [10].

**Theorem 5.** For any left $\mathbb{Z}[\pi]$-module $A$ and for any cohomology class $\alpha \in H^n(\pi, A)$ one may find a $\mathbb{Z}[\pi]$-homomorphism $\mu : I^n \to A$ such that $\alpha = \mu_*(b^n)$.

Recall that we view the tensor power $I^n = I \otimes_{\mathbb{Z}} I \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} I$ as a left $\mathbb{Z}[\pi]$-module using the diagonal action of $\pi$ from the left, i.e. $g \cdot (\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n) = g\alpha_1 \otimes g\alpha_2 \otimes \cdots \otimes g\alpha_n$ where $g \in \pi$ and $\alpha_i \in I$ for $i = 1, \ldots, n$.

In other words, Theorem 5 states that the powers of the Bernstein - Schwarz class $b^n$ are universal in the sense that any other degree $n$ cohomology class can be obtained from $b^n$ by a coefficient homomorphism. This result implies that the Lusternik - Schnirelmann category of an aspherical space is at least $\text{cd}(\pi) + 1$.

We include below a short proof of Theorem 5 (following essentially [10]) for completeness.

**Proof.** First one observes that $I^s$ is a free abelian group and hence $\mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I^s$ is free as a left $\mathbb{Z}[\pi]$-module; here we apply Corollary 5.7 from chapter III of [4]. Hence the exact sequence

$$
\cdots \to \mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I^n \to \mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I^{n-1} \to \cdots \to \mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I \to \mathbb{Z}[\pi] \to \mathbb{Z} \to 0
$$

is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\pi]$. The differential of this complex is given by

$$
\mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I^n = \mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I \otimes_{\mathbb{Z}} I^{n-1} \overset{\epsilon \otimes 1}{\longrightarrow} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I^{n-1} = \mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I^{n-1};
$$

here $\epsilon$ is the augmentation and $i : I \to \mathbb{Z}[\pi]$ is the inclusion.

Using resolution (18), any degree $n$ cohomology class $\alpha \in H^n(\pi, A)$ can be represented by an $n$-cocycle $f : \mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I^n \to A$ which is a $\mathbb{Z}[\pi]$-homomorphism vanishing on the image $I^{n+1}$ of the boundary homomorphism $\mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I^{n+1} \to \mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I^n$. In view of the short exact sequence

$$
0 \to I^{n+1} \to \mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I^n \overset{\epsilon \otimes 1}{\longrightarrow} I^n \to 0
$$

we see that there is a 1-1 correspondence between cocycles $f : \mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I^n \to A$ and homomorphisms $\mu : I^n \to A$.

Let $\mu : I^n \to A$ be the $\mathbb{Z}[\pi]$-homomorphism corresponding to a cocycle representing the class $\alpha$.

Using the definition of a class associated to an exact sequence (see beginning of §3), Lemma 3.1 and formula (8) we see that the identity map $I^n \to I^n$ corresponds to the $n$-th power of the Bernstein - Schwarz class $b^n$. Combining all these mentioned results we obtain $\mu_*(b^n) = \alpha$. □

5. Essential cohomology classes

It is easy to see that the analogue of Theorem 5 fails when we consider cohomology classes $\alpha \in H^n(\pi \times \pi, A)$ and ask whether such classes can be obtained from powers of
the canonical class \( v \in H^1(\pi \times \pi, I) \) by coeffient homomorphisms. The arguments of the proof of Theorem 5 are not applicable since the \( \mathbb{Z}[\pi \times \pi] \)-modules \( \mathbb{Z}[\pi] \otimes \mathbb{Z} I^s \) are neither free nor projective over the ring \( \mathbb{Z}[\pi \times \pi] \).

**Definition 5.1.** We shall say that a cohomology class \( \alpha \in H^n(\pi \times \pi, A) \) is essential if there exists a homomorphism of \( \mathbb{Z}[\pi \times \pi] \)-modules \( \mu : I^n \to A \) such that \( \mu_*(v^n) = \alpha \).

One wants to have verifiable criteria which guarantee that a given cohomology class \( \alpha \in H^n(\pi \times \pi, A) \) is essential. Since \( v \) and all its powers are zero-divisors, it is obvious that any essential class must also be a zero-divisor, i.e. satisfy

\[
\alpha | \Delta_\pi = 0 \in H^n(\pi, \tilde{A}),
\]

see above. For degree one cohomology classes this condition is sufficient, see Lemma 5.2. However, as we shall see, a degree \( n \geq 2 \) zero-divisor does not need to be essential.

Clearly, the set of all essential classes in \( H^n(\pi \times \pi, A) \) forms a subgroup.

Moreover, the cup-product of two essential classes \( \alpha_i \in H^n(\pi \times \pi, A_i) \), where \( i = 1, 2 \), is an essential class

\[
\alpha_1 \cup \alpha_2 \in H^{n_1+n_2}(\pi \times \pi, A_1 \otimes A_2).
\]

Indeed, suppose \( \mu_i : I^{n_i} \to A_i \) are \( \mathbb{Z}[\pi \times \pi] \)-homomorphisms such that \( \mu_i_*(v^{n_i}) = \alpha_i \), where \( i = 1, 2 \). Then \( \mu = \mu_1 \otimes \mu_2 : I^{n_1} \otimes I^{n_2} \to A_1 \otimes A_2 \) satisfies

\[
\mu_*(v^{n_1+n_2}) = \mu_*(v^{n_1} \cup v^{n_2}) = \mu_1(v^{n_1}) \cup \mu_2(v^{n_2}) = \alpha_1 \cup \alpha_2.
\]

**Lemma 5.2.** A degree one cohomology class \( \alpha \in H^1(\pi \times \pi, A) \) is essential if and only if it is a zero-divisor.

The proof will be postponed until we have prepared the necessary algebraic techniques.

**Lemma 5.3.** Consider two left \( \mathbb{Z}[\pi \times \pi] \)-modules \( M \) and \( N \). Let \( \tilde{M} \) and \( \tilde{N} \) denote the left \( \mathbb{Z}[\pi] \)-module structures on \( M \) and \( N \) correspondingly via conjugation, i.e. \( g \cdot m = gmg^{-1} \) and \( g \cdot n = gng^{-1} \) for \( g \in \pi \) and \( m \in M, n \in N \). Let

\[
\Phi : \text{Hom}_{\mathbb{Z}[\pi \times \pi]}(\mathbb{Z}[\pi] \otimes \mathbb{Z} M, N) \to \text{Hom}_{\mathbb{Z}[\pi]}(\tilde{M}, \tilde{N})
\]

be the map which associates with any \( \mathbb{Z}[\pi \times \pi] \)-homomorphism \( f : \mathbb{Z}[\pi] \otimes \mathbb{Z} M \to N \) its restriction \( f \mid e \otimes M \) onto

\[
M = e \otimes \mathbb{Z} M \subset \mathbb{Z}[\pi] \otimes \mathbb{Z} M
\]

where \( e \in \pi \) is the unit element. Then \( \Phi \) is an isomorphism.

**Proof.** The inverse map

\[
\Psi : \text{Hom}_{\mathbb{Z}[\pi]}(\tilde{M}, \tilde{N}) \to \text{Hom}_{\mathbb{Z}[\pi \times \pi]}(\mathbb{Z}[\pi] \otimes \mathbb{Z} M, N)
\]

can be defined as follows. Given \( \phi \in \text{Hom}_{\mathbb{Z}[\pi]}(\tilde{M}, \tilde{N}) \) let \( \hat{\phi} : \mathbb{Z}[\pi] \otimes \mathbb{Z} M \to N \) be defined by

\[
\hat{\phi}(g \otimes m) = g\phi(g^{-1}m) = \phi(mg^{-1})g
\]
for \( g \in \pi \) and \( m \in M \). For \( a, b \in \pi \) we have
\[
\hat{\phi}(agb^{-1} \otimes amb^{-1}) = a^g \hat{\phi}(bg^{-1}a^{-1} \cdot amb^{-1}) = a^g \phi(g \otimes m)b^{-1}
\]
which shows that \( \hat{\phi} \) is a \( \mathbb{Z}[\pi \times \pi] \)-homomorphism. We set \( \Psi(\phi) = \hat{\phi} \). One checks directly that \( \Psi \) and \( \Phi \) are mutually inverse. \( \square \)

As the next step we prove the following generalisation of the previous lemma.

**Lemma 5.4.** For two left \( \mathbb{Z}[\pi \times \pi] \)-modules \( M \) and \( N \) and any \( i \geq 0 \) the map
\[
\Phi : \text{Ext}^i_{\mathbb{Z}[\pi \times \pi]}(\mathbb{Z}[\pi] \otimes \mathbb{Z} M, N) \to \text{Ext}^i_{\mathbb{Z}[\pi]}(\tilde{M}, \tilde{N})
\]
is an isomorphism. The map \( \Phi \) acts by first restricting the \( \mathbb{Z}[\pi \times \pi] \)-module structure to the conjugate action of \( \mathbb{Z}[\pi] \) (where \( \pi = \Delta \pi \subset \pi \times \pi \) is the diagonal subgroup) and secondly by taking the restriction on the \( \mathbb{Z}[\pi] \)-submodule \( \tilde{M} = e \otimes_{\mathbb{Z}} M \subset \mathbb{Z}[\pi] \otimes_{\mathbb{Z}} M \).

**Proof.** Consider an injective resolution \( 0 \to N \to J_0 \to J_1 \to \ldots \) of \( N \) over \( \mathbb{Z}[\pi \times \pi] \); we may use it to compute \( \text{Ext}^i_{\mathbb{Z}[\pi \times \pi]}(\mathbb{Z}[\pi] \otimes \mathbb{Z} M, N) \). By Lemma 5.3, we have an isomorphism
\[
\Phi : \text{Hom}_{\mathbb{Z}[\pi \times \pi]}(\mathbb{Z}[\pi] \otimes \mathbb{Z} M, J_i) \to \text{Hom}_{\mathbb{Z}[\pi]}(\tilde{M}, \tilde{J}_i), \quad i = 0, 1, \ldots
\]
The statement of Lemma 5.4 follows once we show that each module \( \tilde{J}_i \) is injective with respect to the conjugate action of \( \mathbb{Z}[\pi] \).

Consider an injective \( \mathbb{Z}[\pi \times \pi] \)-module \( J \), two \( \mathbb{Z}[\pi] \)-modules \( X \subset Y \) and a \( \mathbb{Z}[\pi] \)-homomorphism \( f : X \to \tilde{J} \) which needs to be extended onto \( Y \). Note that \( \mathbb{Z}[\pi \times \pi] \) is free when viewed as a right \( \mathbb{Z}[\pi] \)-module where the right action is given by \((g, h) \cdot k = (gk, hk)\) for \((g, h) \in \pi \times \pi \) and \( k \in \pi \). Hence we obtain the \( \mathbb{Z}[\pi \times \pi] \)-modules
\[
\mathbb{Z}[\pi \times \pi] \otimes_{\mathbb{Z}[\pi]} X \xrightarrow{\cdot f} \mathbb{Z}[\pi \times \pi] \otimes_{\mathbb{Z}[\pi]} Y
\]
and the homomorphism \( f : X \to \tilde{J} \) determines
\[
f' : \mathbb{Z}[\pi \times \pi] \otimes_{\mathbb{Z}[\pi]} X \to J
\]
by the formula: \( f'(x) = f(x)h^{-1} \), where \( g, h \in \pi \) and \( x \in X \). It is obvious that \( f' \) is well-defined and is a \( \mathbb{Z}[\pi \times \pi] \)-homomorphism. Since \( J \) is \( \mathbb{Z}[\pi \times \pi] \)-injective, there is a \( \mathbb{Z}[\pi \times \pi] \)-extension
\[
f'' : \mathbb{Z}[\pi \times \pi] \otimes_{\mathbb{Z}[\pi]} Y \to J.
\]
The restriction of \( f'' \) onto \( Y = (e, e) \otimes_{\mathbb{Z}} Y \) is a \( \mathbb{Z}[\pi] \)-homomorphism \( Y \to \tilde{J} \) extending \( f \). Hence \( \tilde{J} \) is injective. This completes the proof. \( \square \)

**Proof of Lemma 5.2.** Consider the short exact sequence of \( \pi \times \pi \)-modules
\[
0 \to I \to \mathbb{Z}[\pi] \xrightarrow{\delta} \mathbb{Z} \to 0
\]
and the associated long exact sequence
\[
\cdots \to \text{Hom}_{\mathbb{Z}[\pi \times \pi]}(I, A) \xrightarrow{\delta} \text{Ext}^1_{\mathbb{Z}[\pi \times \pi]}(\mathbb{Z}, A) \xrightarrow{\epsilon} \text{Ext}^1_{\mathbb{Z}[\pi \times \pi]}(\mathbb{Z}[\pi], A) \to \cdots
\]
The condition that $\alpha \in H^1(\pi \times \pi, A) = \text{Ext}^1_{\mathbb{Z}[\pi \times \pi]}(\mathbb{Z}, A)$ is essential is equivalent to the requirement that $\alpha$ lies in the image of $\delta$. By exactness, it is equivalent to $\epsilon^*(\alpha) = 0 \in \text{Ext}^1_{\mathbb{Z}[\pi \times \pi]}(\mathbb{Z}[\pi], A)$. Consider the commutative diagram
\[
\begin{array}{ccc}
\text{Ext}^1_{\mathbb{Z}[\pi \times \pi]}(\mathbb{Z}, A) & \xrightarrow{\epsilon^*} & \text{Ext}^1_{\mathbb{Z}[\pi \times \pi]}(\mathbb{Z}[\pi], A) \\
\downarrow & & \downarrow \\
H^1(\pi \times \pi, A) & \xrightarrow{\Delta^*} & \text{Ext}^1_{\mathbb{Z}[\pi]}(\mathbb{Z}, \hat{A}) \cong H^1(\pi, \hat{A}).
\end{array}
\]
Here $\Delta : \pi \to \pi \times \pi$ is the diagonal. The isomorphism $\Phi$ is given by Lemma 5.4. The commutativity of the diagram follows from the explicit description of $\Phi$. Thus we see that a cohomology class $\alpha \in H^1(\pi \times \pi, A)$ is essential if and only if $\Delta^*(\alpha) = 0 \in H^1(\pi, A)$, i.e. if $\alpha$ is a zero-divisor.

**Corollary 5.5.** For any $\pi \times \pi$ module $A$ one has an isomorphism
\[
\Gamma : H^i(\pi \times \pi, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[\pi], A)) \to H^i(\pi, \hat{A}).
\]
This isomorphism acts as follows:
\[
v \mapsto \omega_{\pi}(v | \pi), \quad v \in H^i(\pi \times \pi, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[\pi], A))
\]
where $\pi \subset \pi \times \pi$ is the diagonal subgroup and $\omega : \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[\pi], A) \to A$ is the homomorphism $\omega(f) = f(e) \in A$. The symbol $e$ denotes the unit element $e \in \pi$.

**Proof.** Consider a free resolution $P_*$ of $\mathbb{Z}$ over $\mathbb{Z}[\pi \times \pi]$. Then
\[
\text{Hom}_{\mathbb{Z}[\pi \times \pi]}(P_*, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[\pi], A)) \cong \text{Hom}_{\mathbb{Z}[\pi \times \pi]}(P_* \otimes_{\mathbb{Z}} \mathbb{Z}[\pi], A) \cong \text{Hom}_{\mathbb{Z}[\pi]}(\hat{P}_*, \hat{A})
\]
according to Lemma 5.3. Our statement now follows since $\hat{P}_*$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\pi]$.  

6. **The case of an abelian group**

Throughout this section we shall assume that the group $\pi$ is abelian. We shall fully describe the essential cohomology classes in $H^n(\pi \times \pi, A)$.

First, we note that it makes sense to impose an additional condition on the $\mathbb{Z}[\pi \times \pi]$-module $A$.

For a $\mathbb{Z}[\pi \times \pi]$-module $B$ let $B' \subset B$ denote the submodule $B' = \{ b \in B; gb = bg \text{ for any } g \in \pi \}$. Any $\mathbb{Z}[\pi \times \pi]$-homomorphism $\mu : A \to B$ restricts to a homomorphism $\mu : A' \to B'$.

Clearly, $I' = I$ and similarly $(I^n)' = I^n$. Hence any $\mathbb{Z}[\pi \times \pi]$-homomorphism $\mu : I^n \to A$ takes values in the submodule $A' \subset A$. Hence discussing essential cohomology classes $\alpha \in H^n(\pi \times \pi, A)$ we may assume that $A' = A$.

Consider the map
\[
\phi : \pi \times \pi \to \pi, \quad \text{where} \quad \phi(x, y) = xy^{-1}.
\]
It is a group homomorphism (since \( \pi \) is abelian). Besides, let \( A \) be a \( \mathbb{Z}[\pi \times \pi] \)-module with \( A' = A \). Then there exists a unique \( \mathbb{Z}[^{\pi}] \)-module \( B \) such that \( A = \phi^*(B) \).

**Theorem 6.** Assume that the group \( \pi \) is abelian. Let \( B \) be a \( \mathbb{Z}[\pi] \)-module and let \( \alpha \in H^n(\pi \times \pi, \phi^*(B)) \) be a cohomology class. Then \( \alpha \) is essential if and only if \( \alpha = \phi^*(\beta) \) for some \( \beta \in H^n(\pi, B) \).

It follows from Theorem 6 that there are no nonzero essential cohomology classes \( \alpha \in H^n(\pi \times \pi, A) \) with \( n > \text{cd}(\pi) \). Moreover, we see that if \( \text{cd}(\pi) < n \leq \text{cd}(\pi \times \pi) \) then any cohomology class \( \alpha \in H^n(\pi \times \pi, A) \) is a zero-divisor which is not essential.

**Proof.** Assume that \( \alpha \in H^n(\pi \times \pi, \phi^*(B)) \) is such that \( \alpha = \phi^*(\beta) \) where \( \beta \in H^n(\pi, B) \). We want to show that \( \alpha \) is essential. By Theorem 5 there exists a \( \mathbb{Z}[\pi] \)-homomorphism \( \mu : I^n \to B \) such that \( \mu_*(b^n) = \beta \) where \( b \in H^1(\pi, I) \) is the Berstein - Schwarz class. Note that \( \phi^*(I) = I \) and

\[
\phi^*(b) = v
\]

where \( v \in H^1(\pi \times \pi, I) \) is the canonical class. To prove (24) we consider two subgroups \( G_1 = \pi \times 1 \subset \pi \times \pi \) and \( G_2 = \Delta_{\pi} \subset \pi \times \pi \) and since \( \pi \times \pi \simeq G_1 \times G_2 \) we can view the Eilenberg-MacLane space \( K(\pi \times \pi, 1) \) as the product \( K(G_1, 1) \times K(G_2, 1) \). The restriction of the classes \( \phi^*(b) \) and \( v \) onto \( G_1 \) coincide (as follows from (8) and from the definition of \( \phi \)). On the other hand, the restriction of the classes \( \phi^*(b) \) and \( v \) onto \( G_2 \) are trivial (as follows from (9) and from the definition of \( \phi \)). Now the equality (24) follows from the fact that the inclusion \( K(G_1, 1) \vee K(G_2, 1) \to K(G_1, 1) \times K(G_2, 1) \) induces a monomorphism on 1-dimensional cohomology with any coefficients.

Consider the commutative diagram

\[
\begin{array}{ccc}
H^n(\pi, I^n) & \xrightarrow{\mu_*} & H^n(\pi, B) \\
\phi^* \downarrow & & \downarrow \phi^* \\
H^n(\pi \times \pi, I^n) & \xrightarrow{\mu_*} & H^n(\pi \times \pi, \phi^*(B)).
\end{array}
\]

The upper left group contains the power \( b^n \) of the Berstein - Schwarz class which is mapped onto \( \beta = \mu_*(b^n) \) and \( \phi^*(\beta) = \alpha \). Moving in the other direction we find \( \alpha = \mu_*(\phi^*(b^n)) = \mu_*(v^n) \), i.e. \( \alpha \) is essential.

To prove the inverse statement, assume that a cohomology class

\[
\alpha \in H^n(\pi \times \pi, \phi^*(B))
\]

is essential, i.e. \( \alpha = \mu_*(v^n) \) for a \( \mathbb{Z}[\pi \times \pi] \)-homomorphism \( \mu : I^n \to \phi^*(B) \). We may also view \( \mu \) as a \( \mathbb{Z}[\pi] \)-homomorphism \( I^n \to B \) which leads to the commutative diagram (25). Using (24) we find that \( \phi^*(\mu_*(b^n)) = \mu_*(\phi^*(b^n)) = \mu_*(v^n) = \alpha \). Hence we see that \( \alpha = \phi^*(\beta) \) where \( \beta = \mu_*(b^n) \).

\[\square\]
If we wish to be specific, let $\pi = \mathbb{Z}^N$ and consider the trivial coefficient system $A = \mathbb{Z}$. Then $N$ is the highest dimension in which essential cohomology classes $\alpha \in H^N(\mathbb{Z}^N \times \mathbb{Z}^N; \mathbb{Z})$ exist. Due to Theorem 6, all $N$-dimensional essential cohomology classes are integral multiples of a single class which we are going to describe.

Denote by $x_1, \ldots, x_N \in H^1(\mathbb{Z}^N, \mathbb{Z})$ a set of generators. Then each class $\alpha_i = x_i \otimes 1 - 1 \otimes x_i \in H^1(\mathbb{Z}^N \times \mathbb{Z}^N; \mathbb{Z}), \; i = 1, \ldots, N,$ is a zero-divisor and hence is essential by Lemma 5.2. Their product $\alpha = \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_N \in H^N(\mathbb{Z}^N \times \mathbb{Z}^N; \mathbb{Z})$ is essential as a product of essential classes. We may write $\alpha$ as the sum of $2^N$ terms

$$\alpha = (-1)^N \cdot \sum_K (-1)^{|K|} x_K \otimes x_{K^c}$$

where $K \subset \{1, 2, \ldots, N\}$ runs over all subsets of the index set and $K^c$ denotes the complement of $K$. For $K = \{i_1, i_2, \ldots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$ the symbol $x_K$ stands for the product $x_{i_1}x_{i_2}\ldots x_{i_k}$.

7. The spectral sequence

In this and in the subsequent sections we abandon the assumption that $\pi$ is abelian and return to the general case, i.e. we consider an arbitrary discrete group $\pi$.

Let $A$ be a left $\mathbb{Z}[\pi \times \pi]$-module. We shall describe an exact couple and a spectral sequence which will allow us to find a sequence of obstructions for a cohomology class $\alpha \in H^*(\pi \times \pi, A)$ to be essential.

We introduce the following notations:

$$E_r^{s,0} = \text{Ext}^r_{\mathbb{Z}[\pi \times \pi]}(\mathbb{Z}[\pi] \otimes \mathbb{Z} I^s, A) \quad \text{and} \quad D_r^{s,0} = \text{Ext}^r_{\mathbb{Z}[\pi \times \pi]}(I^s, A).$$

The long exact sequence associated to the short exact sequence (11) can be written in the form

$$\cdots \to E_r^{s,0} \xrightarrow{k_0} D_0^{r,s+1} \xrightarrow{i_0} D_0^{r+1,s} \xrightarrow{j_0} E_0^{r+1,s} \to \cdots$$

Here $i_0 : D_r^{s,0} \to D_0^{r+1,s-1}$ is the connecting homomorphism

$$\text{Ext}^r_{\mathbb{Z}[\pi \times \pi]}(I^s, A) \to \text{Ext}^r_{\mathbb{Z}[\pi \times \pi]}(I^{s-1}, A)$$

corresponding to the exact sequence (11). Note that

$$D_0^{n,0} = H^n(\pi \times \pi, A) \quad \text{and} \quad D_0^{0,n} = \text{Hom}_{\mathbb{Z}[\pi \times \pi]}(I^n, A).$$

**Lemma 7.1.** The set of essential cohomology classes in $H^n(\pi \times \pi, A)$ coincides with the image of the composition of $n$ maps $i_0$:

$$\text{Hom}_{\mathbb{Z}[\pi \times \pi]}(I^n, A) = D_0^{n,0} \xrightarrow{i_0} D_0^{1,n-1} \xrightarrow{i_0} \cdots \xrightarrow{i_0} D_0^{n,0} = H^n(\pi \times \pi, A).$$
Proof. Applying the technique described in [25], chapter IV, §9, we obtain that the image of a homomorphism $f \in \text{Hom}_{\mathbb{Z}[\pi \times \pi]}(I^n, A)$ under the composition $i_0^n$ is an element of $\text{Ext}^2_{\mathbb{Z}[\pi \times \pi]}(\mathbb{Z}, A)$ represented by the exact sequence

$$0 \to A \to X_f \to \mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I^{n-2} \to \mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I^{n-3} \to \cdots \to \mathbb{Z}[\pi] \to \mathbb{Z} \to 0$$

where $X_f$ appears in the push-out diagram

$\begin{array}{ccc}
I^n & \to & \mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I^{n-1} \\
\downarrow f & & \downarrow \\
A & \to & X_f
\end{array}$

Using Lemma 3.1 we see that the same exact sequence represents the element $f_*(v^n)$. □

A different proof of Lemma 7.1 will be given later in this section.

The exact sequences (26) can be organised into a bigraded exact couple as follows. Denote

$$E_0 = \bigoplus_{r,s \geq 0} E_0^{rs} = \bigoplus_{r,s \geq 0} \text{Ext}^r_{\pi \times \pi}(\mathbb{Z}[\pi] \otimes_{\mathbb{Z}} I^s, A),$$

and

$$D_0 = \bigoplus_{r,s \geq 0} D_0^{rs} = \bigoplus_{r,s \geq 0} \text{Ext}^r_{\pi \times \pi}(I^s, A).$$

The exact sequence (26) becomes an exact couple

$$\begin{array}{ccc}
D_0 & \xrightarrow{i_0} & D_0 \\
\downarrow k_0 & & \downarrow j_0 \\
E_0 & & \\
\end{array}$$

Here the homomorphism $i_0$ has bidegree $(1, -1)$, the homomorphism $k_0$ has bidegree $(0,1)$, and the homomorphism $j_0$ has bidegree $(0,0)$. Applying the general formalism of exact couples, we may construct the $p$-th derived couple

$$\begin{array}{ccc}
D_p & \xrightarrow{i_p} & D_p \\
\downarrow k_p & & \downarrow j_p \\
E_p & & \\
\end{array}$$

where $p = 0, 1, \ldots$. The module $D_p^{rs}$ is defined as

$$D_p^{rs} = \text{Im}[i_{p-1} : D_{p-1}^{r-1,s+1} \to D_{p-1}^{rs}] = \text{Im}[i_0 \circ \cdots \circ i_0 : D_0^{r-p,s+p} \to D_0^{rs}].$$

and

$$E_p^{*,*} = H(E_p^{*,*}, d_{p-1}).$$
is the homology of the previous term with respect to the differential \(d_{p-1} = j_{p-1} \circ k_{p-1}\).

The degrees are as follows:

\[
\begin{align*}
\deg j_p &= (-p, p), \\
\deg i_p &= (1, -1), \\
\deg k_p &= (0, 1), \\
\deg d_p &= (-p, p + 1).
\end{align*}
\]

Using this spectral sequence we can express the set of essential classes as follows:

**Corollary 7.2.** The group \(D_{n,0} \subset H^n(\pi \times \pi, A) = D_{0,0}^{n,0}\) coincides with the set of all essential cohomology classes in \(H^n(\pi \times \pi, A)\).

**Proof.** This is equivalent to Lemma 7.1. \(\square\)

We want to express the homomorphism \(i_0 : D_0^{r,s+1} \to D_0^{r+1,s}\) with \(s \geq 0\) through the canonical class \(v \in H^1(\pi \times \pi, I)\). This will be used to give a different proof of Lemma 26 and will have some other interesting applications. According to the definition, \(i_0\) is the connecting homomorphism

\[
i_0 : \operatorname{Ext}^{r}_{Z[\pi \times \pi]}(I^{s+1}, A) \to \operatorname{Ext}^{r+1}_{Z[\pi \times \pi]}(I^s, A)
\]

corresponding to the short exact sequence (11). Note that

\[
D_0^{r,s} = \operatorname{Ext}^{r}_{Z[\pi \times \pi]}(I^s, A) = H^r(\pi \times \pi, \operatorname{Hom}Z(I^s, A)),
\]

see [4], chapter III, Proposition 2.2. Under this identification \(i_0\) turns into the Bockstein homomorphism

\[
\beta : H^r(\pi \times \pi; \operatorname{Hom}Z(I^{s+1}, A)) \to H^{r+1}(\pi \times \pi; \operatorname{Hom}Z(I^s, A))
\]

(32)

corresponding to the short exact sequence of \(Z[\pi \times \pi]\)-modules

\[
0 \to \operatorname{Hom}Z(I^s, A) \to \operatorname{Hom}Z(Z[\pi] \otimes Z I^s, A) \to \operatorname{Hom}Z(I^{s+1}, A) \to 0.
\]

(33)

The sequence (33) is obtained by applying the functor \(\operatorname{Hom}Z(\cdot, A)\) to the exact sequence (11) (note that (11) splits over \(Z\)).

Let \(\text{ev} : I \otimes Z \operatorname{Hom}Z(I^{s+1}, A) \to \operatorname{Hom}Z(I^s, A)\)

denote the homomorphism given by

\[
x_0 \otimes f \mapsto (x_1 \otimes \cdots \otimes x_s \mapsto f(x_0 \otimes x_1 \otimes \cdots \otimes x_s))
\]

for \(f \in \operatorname{Hom}Z(I^s, A)\) and \(x_i \in I\) for \(i = 0, 1, \ldots, s\).

**Proposition 7.3.** For any cohomology class \(u \in H^r(\pi \times \pi, \operatorname{Hom}Z(I^{s+1}, A))\) one has

\[
\beta(u) = -\text{ev}_*(v \cup u),
\]

(34)

where \(v \in H^1(\pi \times \pi, I)\) denotes the canonical class.
Proof. Using [4], chapter V, property (3.3) and Lemma 5 from [7], we obtain \( \delta(u) = \mathfrak{u} \cup u \), where

\[
\delta : H^r(\pi \times \pi; \text{Hom}_\mathbb{Z}(I^{s+1}, A)) \to H^{r+1}(\pi \times \pi; I \otimes \text{Hom}_\mathbb{Z}(I^s, A))
\]

is the Bockstein homomorphism associated with the short exact coefficient sequence

\[
0 \to I \otimes \mathbb{Z} \text{Hom}_\mathbb{Z}(I^{s+1}, A) \to \mathbb{Z}[\pi] \otimes \mathbb{Z} \text{Hom}_\mathbb{Z}(I^{s+1}, A) \xrightarrow{\epsilon \otimes \text{id}} \text{Hom}_\mathbb{Z}(I^{s+1}, A) \to 0.
\]

The latter sequence is obtained by tensoring (4) with \( \text{Hom}_\mathbb{Z}(I^{s+1}, A) \) over \( \mathbb{Z} \). To prove Proposition 7.3 it is enough to show that \( \beta = -ev \circ \delta \). Having this goal in mind, we denote by

\[
F : \mathbb{Z}[\pi] \otimes \mathbb{Z} \text{Hom}_\mathbb{Z}(I^{s+1}, A) \to \text{Hom}_\mathbb{Z}(\mathbb{Z}[\pi] \otimes I^s, A)
\]

the homomorphism which extends \( \mathbb{Z} \)-linearly the following map

\[
F(x \otimes f)(z \otimes y) = f((z - x) \otimes y)
\]

for \( x, z \in \pi, y \in I^s \) and \( f \in \text{Hom}_\mathbb{Z}(I^{s+1}, A) \). We compute:

\[
F((g, h) \cdot (x \otimes f))(z \otimes y) = F(gxh^{-1} \otimes (g, h)f)(z \otimes y)
\]

\[
= ((g, h)f)((z - gxh^{-1}) \otimes y) = gf((g^{-1}zh - x) \otimes g^{-1}y)h^{-1}
\]

\[
= gF(x \otimes f)(g^{-1}zh \otimes g^{-1}y)h^{-1} = ((g, h) \cdot F(x \otimes f))(z \otimes y).
\]

Hence \( F \) is a \( \mathbb{Z}[\pi \times \pi] \)-homomorphism. Next we claim that the following diagram with exact rows commutes:

\[
\begin{array}{ccc}
0 & \to & I \otimes \mathbb{Z} \text{Hom}_\mathbb{Z}(I^{s+1}, A) \xrightarrow{i \otimes \text{id}} \mathbb{Z}[\pi] \otimes \mathbb{Z} \text{Hom}_\mathbb{Z}(I^{s+1}, A) \xrightarrow{\epsilon \otimes \text{id}} \text{Hom}_\mathbb{Z}(I^{s+1}, A) \to 0 \\
& \downarrow \text{ev} & \downarrow F & \downarrow \text{id} \\
0 & \to & \text{Hom}_\mathbb{Z}(I^s, A) \xrightarrow{i^*} \text{Hom}_\mathbb{Z}(\mathbb{Z}[\pi] \otimes I^s, A) \xrightarrow{i^*} \text{Hom}_\mathbb{Z}(I^{s+1}, A) \to 0
\end{array}
\]

Indeed, we compute for \( g, h \in \pi, y \in I^s \) and \( f \in \text{Hom}_\mathbb{Z}(I^{s+1}, A) \),

\[
(-ev \circ ev)((g - 1) \otimes f)(h \otimes y) = -ev((g - 1) \otimes y) = -e((g - 1) \otimes y)
\]

and

\[
(F \circ (i \otimes \text{id}))((g - 1) \otimes f)(h \otimes y)
\]

\[
= (F(g \otimes f))(h \otimes y) - (F(1 \otimes f))(h \otimes y)
\]

\[
= f((h - g) \otimes y) - f((h - 1) \otimes y)
\]

\[
= -f((g - 1) \otimes y).
\]

Hence, we see that the left square in the above diagram commutes. We further observe that for all \( g, h \in \pi, y \in I^s \) and \( f \in \text{Hom}_\mathbb{Z}(I^{s+1}, A) \) one has

\[
((i^* \otimes F)(g \otimes f))(h - 1) \otimes y = F(g \otimes f)(h \otimes y) - F(g \otimes f)(1 \otimes y)
\]

\[
= f((h - g) \otimes y) - f((1 - g) \otimes y) = f((h - 1) \otimes y)
\]

\[
= \epsilon(g)f((h - 1) \otimes y) = ((\epsilon \otimes \text{id})(g \otimes f))((h - 1) \otimes y),
\]

and hence the right square of the diagram commutes as well.
The commutativity of the above diagram implies that the Bockstein homomorphisms satisfy
\[ \ev_* \circ \delta = \beta' \circ \text{id} = \beta' , \]
where \( \beta' \) denotes the Bockstein homomorphism of the bottom row exact sequence. Since this sequence coincides with the sequence associated with \( \beta \) up to a sign change in the first map, one derives from the snake lemma that \( \beta' = -\beta \). This completes the proof.

**Corollary 7.4.** Let \( \alpha \in H^n(\pi \times \pi, A) \) be a cohomology class and let \( k = 1, 2, \ldots, n - 1 \) be an integer. Then the following conditions are equivalent:

1. \( \alpha \) lies in \( D_{k}^{n,0} \).
2. \( \alpha = \psi_*(v^k \cup u) \) for a cohomology class \( u \in H^{n-k}(\pi \times \pi, \text{Hom}_Z(I^k, A)) \) where
   \[ \psi : I^k \otimes \text{Hom}_Z(I^k, A) \to A \]
   is the coefficient pairing
   \[ \psi(x_1 \otimes \cdots \otimes x_k \otimes f) = f(x_k \otimes x_{k-1} \otimes \cdots \otimes x_1) . \]

**Proof.** The condition \( \alpha \in D_{k}^{n,0} \) means that \( \alpha = \psi_{0}^{k}(u) \) for some \( u \in D_{0}^{n-k,k} \). We know that \( D_{0}^{n-k,k} = \text{Ext}_{Z[\pi \times \pi]}^{n-k}(I^k, A) = H^{n-k}(\pi \times \pi, \text{Hom}_Z(I^k, A)) \). Our statement follows by applying iteratively Proposition 7.3. \( \square \)

We may use Proposition 7.3 to give another proof of Lemma 7.1. By Corollary 7.4, classes \( \alpha \in D_{n}^{n,0} \) are characterised by the property \( \alpha = v^u \cup u \) where the cup product is given with respect to the pairing \( I^n \otimes \text{Hom}_Z(I^n, A) \to A \) given by the formula (35) for some \( u \in H^0(\pi \times \pi, \text{Hom}_Z(I^n, A)) = \text{Hom}_Z[\pi \times \pi](I^n, A) \). Thus \( u \) is a \( Z[\pi \times \pi] \)-homomorphism \( I^n \to A \) and applying the definition of the cup product (see [4], chapter V, §3) we see that \( \alpha \in D_{n}^{n,0} \) if and only if \( \alpha = \phi_{*}(v^n) \) for a \( Z[\pi \times \pi] \)-homomorphism \( \phi : I^n \to A \).

**Corollary 7.5.** If for some \( Z[\pi \times \pi] \)-module \( A \) and for an integer \( k \) the module \( D_{k}^{n,0} \) is nonzero then \( \text{TC}(K(\pi, 1)) \geq k + 1 \).

**Proof.** Using Corollary 7.4 we see that \( D_{k}^{n,0} \neq 0 \) then for \( \alpha \in D_{k}^{n,0} \), \( \alpha \neq 0 \), we have \( \alpha = \psi_*(v^k \cup u) \) and hence \( v^k \neq 0 \). Since \( v \) is a zero-divisor, our statement follows from [17], Corollary 4.40. \( \square \)

Using the spectral sequence we may describe a complete set of \( n \) obstructions for a cohomology class \( \alpha \in H^n(\pi \times \pi, A) = D_{0}^{n,0} \) to be essential. We shall apply Lemma 7.1 and act inductively. The class \( \alpha \) is essential if it lies in the image of the composition of \( n \) maps \( i_0 \). For this to happen we first need to guarantee that \( \alpha \) lies in the image of the last map \( i_0 : D_{0}^{n-1,1} \to D_{0}^{n,0} \). Because of the exact sequence
\[ \cdots \to D_{0}^{n-1,1} \xrightarrow{i_0} D_{0}^{n,0} \xrightarrow{j_0} E_{0}^{n,0} \xrightarrow{k_0} \cdots \]
we see that \( \alpha \) lies in the image of \( i_0 \) if and only if
\[ j_0(\alpha) = 0 \in E_{0}^{n,0} . \]
We have the commutative diagram
\[
\begin{array}{ccc}
\text{Ext}_{\mathbb{Z}[\pi \times \pi]}^n(\mathbb{Z}, A) & \xrightarrow{j_0} & \text{Ext}_{\mathbb{Z}[\pi \times \pi]}^n(\mathbb{Z}[\pi], A) \\
\downarrow & & \downarrow \Phi \simeq \\
H^n(\pi \times \pi, A) & \xrightarrow{r^*} & H^n(\pi, \tilde{A}).
\end{array}
\]

Here \( \Phi \) is the isomorphism of Lemma 5.4 and \( j_0 = \epsilon^* \) is the homomorphism induced by the augmentation; the homomorphism \( r^* \) is induced by the inclusion \( r : \pi \to \pi \times \pi \) of the diagonal subgroup. We see that the class \( \alpha \) lies in the image of \( i_0 \) if and only if it is a zero divisor, i.e. \( r^*(\alpha) = 0 \).

To describe the second obstruction let us assume that \( \alpha \in H^n(\pi \times \pi, A) \) is a zero-divisor, i.e. (36) is satisfied. Then \( \alpha \in D_{1}^{n,0} \). One has \( \alpha \in D_{2}^{n,0} \) if and only if
\[
(37) \quad j_1(\alpha) = 0 \in E_1^{n-1,1}.
\]
This follows from the exact sequence
\[
\cdots \to D_{1}^{n-1,1} \xrightarrow{i_1} D_{1}^{n,0} \xrightarrow{j_1} E_1^{n-1,1} \xrightarrow{k_1} D_{1}^{n-1,2} \to \cdots
\]
where \( i_1 \) is the restriction of \( i_0 \) onto \( D_1 \subset D_0 \).

Continuing these arguments and using the exact sequences
\[
\cdots \to D_{p}^{n-p-1,1} \xrightarrow{i_{p-1}} D_{p}^{n-p,0} \xrightarrow{j_{p-1}} E_{p}^{n-p-1,1} \xrightarrow{k_{p-1}} D_{p}^{n-p-1,2} \to \cdots
\]
we arrive at the following conclusion:

**Corollary 7.6.** Let \( k \) and \( n \) be integers with \( 0 < k \leq n \).

1. A cohomology class \( \alpha \in H^n(\pi \times \pi, A) \) lies in the group \( D_k^{n,0} = \text{Im}[i_k^*: D_{0}^{n-k,k} \to D_{0}^{n,0}] \) if and only if the following \( k \) obstructions
\[
(38) \quad j_s(\alpha) \in E_s^{n-s,s}, \text{ where } s = 0, 1, \ldots, k - 1,
\]
vanish.

2. The condition \( j_0(\alpha) = 0 \) is equivalent for \( \alpha \) to be a zero-divisor.

3. Each obstruction \( j_s(\alpha) \) is defined once the previous obstruction \( j_{s-1}(\alpha) \) vanishes.

4. The triviality of all obstructions \( j_0(\alpha), j_1(\alpha), \ldots, j_{n-1}(\alpha) \) is necessary and sufficient for the cohomology class \( \alpha \) to be essential.

Figure 1 shows the locations of the obstructions \( j_k(\alpha) \in E_k^{n-k,k} \).

8. Computing the term \( E_{r,s}^{r,s} \) for \( s \geq 1 \)

In this section we compute the initial term \( E_{0}^{r,s} \) of the spectral sequence. Using Lemma 21 we find
\[
E_{0}^{r,s} = \text{Ext}_{\mathbb{Z}[\pi \times \pi]}^{r}(\mathbb{Z}[\pi] \otimes \mathbb{Z} I^s, A) \simeq \text{Ext}_{\mathbb{Z}[\pi]}^{r}(I^s, \tilde{A})
\]
(39)
where in the second line the tilde \( \sim \) above the corresponding modules means that we consider these modules with respect to the conjugation action, i.e. \( g \cdot a = gag^{-1} \) for \( a \in A \) and \( g \in \pi \). We exploit below the simple structure of the module \( \tilde{I}^s \) where \( s \geq 1 \) and compute explicitly the \( E_0 \)-term.

For \( s \geq 1 \) consider the action of \( \pi \) on the Cartesian power \( \pi^s = \pi \times \pi \times \cdots \times \pi \) (\( s \) times) via conjugation, i.e. \( g \cdot (g_1, \ldots, g_s) = (gg_1g^{-1}, \ldots, gg_sg^{-1}) \) for \( g, g_1, \ldots, g_s \in \pi \). The orbits of this action are joint conjugacy classes of \( s \)-tuples of elements of \( \pi \). We denote by \( C_{\pi^s} \) the set of orbits and let \( C'_{\pi^s} \subset C_{\pi^s} \) denote the set of orbits of nontrivial elements, i.e. such that \( g_i \neq 1 \) for all \( i = 1, \ldots, s \).

Let \( C \in C'_{\pi^s} \) be an orbit. The isotropy subgroup \( N_C \subset \pi \) of an \( s \)-tuple \((g_1, \ldots, g_s) \in C \subset \pi^s\) is the intersection of the centralisers of the elements \( g_1, \ldots, g_s \). The subgroup \( N_C \), viewed up to conjugation, depends only on the orbit \( C \).

**Theorem 7.** For any left \( \mathbb{Z}[[\pi \times \pi]] \)-module \( A \) and for integers \( r \geq 0 \) and \( s \geq 1 \) one has

\[
E_0^{r,s} \simeq \prod_{C \in C'_{\pi^s}} H^r(N_C, A|N_C)
\]

Here \( A|N_C \) denotes \( A \) viewed as \( \mathbb{Z}[N_C] \)-module with \( N_C \subset \pi = \Delta_\pi \subset \pi \times \pi \).

**Proof.** For any \( C \in C'_{\pi^s} \) consider the set \( J_C \subset \tilde{I}^s \) generated over \( \mathbb{Z} \) by the tensors of the form

\[
(g_1 - 1) \otimes \cdots \otimes (g_s - 1)
\]

for all \((g_1, \ldots, g_s) \in C\). It is clear that \( J_C \) is a \( \mathbb{Z}[[\pi]] \)-submodule of \( \tilde{I}^s \) (since we consider the conjugation action). Moreover, we observe that

\[
\tilde{I}^s = \bigoplus_{C \in C'_{\pi^s}} J_C.
\]
Indeed, the elements $g - 1$ with various $g \in \pi^*$ (where we denote $\pi^* = \pi - \{1\}$) form a free $\mathbb{Z}$-basis of $I$; therefore elements of the form $(g_1 - 1) \otimes \cdots \otimes (g_s - 1)$ with all possible $g_1, \ldots, g_s \in \pi^*$ form a free $\mathbb{Z}$-basis of $I^s$. The formula (42) is now obvious.

For $C \in \mathcal{C}_r^N$, let $\mathbb{Z}[C]$ denote the free abelian group generated by $C$. Since $\pi$ acts on $C$, the group $\mathbb{Z}[C]$ is naturally a left $\mathbb{Z}[\pi]$-module which is isomorphic to $J_C$ via the isomorphism

$$(g_1, \ldots, g_s) \mapsto (g_1 - 1) \otimes \cdots \otimes (g_s - 1),$$

where $(g_1, \ldots, g_s) \in C$. For a left $\mathbb{Z}[\pi]$-module $B$ we have

$$\text{Hom}_{\mathbb{Z}[\pi]}(J_C, B) = \text{Hom}_{\mathbb{Z}[\pi]}(\mathbb{Z}[C], B|_{\mathcal{N}_C}) = \text{Hom}_{\mathcal{N}_C}(\mathbb{Z}, B) = H^0(\mathcal{N}_C, B|_{\mathcal{N}_C}).$$

Here we used the fact that the action of $\pi$ on $C$ is transitive and hence a $\mathbb{Z}[\pi]$-homomorphism $f : \mathbb{Z}[C] \to B$ is uniquely determined by one of its values $f(c)$ where $c \in C$.

Consider a free resolution

$$P_* : \cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

of $\mathbb{Z}$ over $\mathbb{Z}[\pi]$. Since $\mathbb{Z}[C]$ is free as an abelian group we have the exact sequence

$$(43) \quad \cdots \to \mathbb{Z}[C] \otimes_{\mathbb{Z}} P_n \to \mathbb{Z}[C] \otimes_{\mathbb{Z}} P_{n-1} \to \cdots \to \mathbb{Z}[C] \otimes_{\mathbb{Z}} P_0 \to \mathbb{Z}[C] \to 0$$

of $\mathbb{Z}[\pi]$-modules. It is easy to see that each module $\mathbb{Z}[C] \otimes_{\mathbb{Z}} P_n$ (equipped with the diagonal action) is free as a $\mathbb{Z}[\pi]$-module (see [4], chapter III, Corollary 5.7). Thus we see that (43) is a free $\mathbb{Z}[\pi]$-resolution of $\mathbb{Z}[C]$ and we may use it to compute $\text{Ext}^r_{\mathbb{Z}[\pi]}(\mathbb{Z}[C], B)$. We have

$$\text{Hom}_{\mathbb{Z}[\pi]}(\mathbb{Z}[C] \otimes_{\mathbb{Z}} P_n, B) = \text{Hom}_{\mathbb{Z}[\pi]}(\mathbb{Z}[C], \text{Hom}_{\mathbb{Z}}(P_n, B)) = \text{Hom}_{\mathbb{Z}[\mathcal{N}_C]}(P_n, B|_{\mathcal{N}_C}).$$

Thus we see that the complex $\text{Hom}_{\mathbb{Z}[\pi]}(\mathbb{Z}[C] \otimes P_*, B)$ which computes $\text{Ext}^r_{\mathbb{Z}[\pi]}(J_C, B)$, coincides with the complex

$$(44) \quad \text{Hom}_{\mathbb{Z}[\mathcal{N}_C]}(P_*, B|_{\mathcal{N}_C}).$$

Since $P_n$ is free as a $\mathbb{Z}[\mathcal{N}_C]$-module, we see that the cohomology of the complex (44) equals $H^r(\mathcal{N}_C, B|_{\mathcal{N}_C})$. Thus we obtain isomorphisms

$$(45) \quad \text{Ext}^r_{\mathbb{Z}[\pi]}(J_C, B) \simeq \text{Ext}^r_{\mathbb{Z}[\pi]}(\mathbb{Z}[C], B) \simeq H^r(\mathcal{N}_C; B|_{\mathcal{N}_C}).$$

Combining the isomorphisms (39), (42) and (45) we obtain the isomorphism (40). \hfill \Box

**Corollary 8.1.** Let $\pi$ be a discrete torsion-free group such that the centraliser of any nontrivial element $g \in \pi$, $g \neq 1$ is infinite cyclic. Then

$$B^r_{\pi,s} = 0$$

for all $r > 1$ and $s \geq 1$.

**Proof.** Applying Theorem 7 we see that each group $\mathcal{N}_C$, where $C \in \mathcal{C}_r^N$, is a subgroup of $\mathbb{Z}$ and hence it is either $\mathbb{Z}$ or trivial. The result now follows from (40) since we assume that $r > 1$. \hfill \Box
Figure 2. The nontrivial groups in the $E_0$-term of the spectral sequence.

Figure 2 shows potentially nontrivial groups in the $E_0$-term in the case when all centralisers of nontrivial elements are cyclic.

9. Proofs of Theorems 1, 2, 3 and 4

Proof of Theorem 3. Let $X = K(\pi, 1)$ where the group $\pi$ satisfies our assumption that the centraliser of any nonzero element is infinite cyclic. Let $\alpha \in H^n(X \times X, A)$ be a zero-divisor. Here $A$ is a local coefficient system over $X \times X$. By Corollary 7.6, statement (2), we have $j_0(\alpha) = 0$. Besides, applying Corollary 8.1 we see that the obstructions $j_s(\alpha) \in E_s^{n-s,s}$ vanish for $s = 1, 2, \ldots, n - 2$ since they lie in the trivial groups. Thus we obtain

$$\alpha \in D^{n,0}_{n-1}.$$  

Next we apply Corollary 7.4 which gives

$$\alpha = \psi_s(v^{n-1} \cup u)$$  

for some $u \in H^1(\pi \times \pi, \text{Hom}_\mathbb{Z}(I^{n-1}, A))$ where $\psi$ is given by (35).

To prove that $\text{wgt}(\alpha) \geq n - 1$ we observe that the canonical class $v$ has positive weight, $\text{wgt}(v) \geq 1$, since it is a zero-divisor, see (9). Hence using (3) we obtain $\text{wgt}(v^{n-1}) \geq n - 1$. Let $f : Y \to X \times X$ be a continuous map as in Definition 2.1. Then

$$f^*(\alpha) = \psi_s(f^*(v^{n-1}) \cup f^*(u)) = 0$$  

since $f^*(v^{n-1}) = 0$. This completes the proof. □

Proof of Theorem 2. Suppose that we are in the situation of Theorem 2, i.e. let $X$ be an aspherical finite cell complex such whose fundamental group $\pi = \pi_1(X)$ has the properties (1) and (2). Denote $n = \text{cd}(\pi \times \pi)$. We may find a local coefficient system $A$ over $X \times X$ and a nonzero cohomology class $\alpha \in H^n(X \times X, A)$. Since $n > \text{cd}(\pi)$ we obtain that $\alpha$
is a zero-divisor. Next we apply Theorem 3 which implies that the weight of $\alpha$ satisfies $\text{wgt}(\alpha) \geq n - 1$. Thus we obtain that $\text{TC}(X) \geq n$.

The inequality $\text{TC}(X) \leq n + 1$ follows from the Eilenberg - Ganea theorem and general dimensional upper bound for the topological complexity $\text{TC}(X) \leq \dim(X \times X) + 1$.

Hence, $\text{TC}(X)$ is either $n$ or $n + 1$.

**Proof of Theorem 4.** As in the proof of Theorem 2 we may find a nonzero cohomology class $\alpha \in H^n(X \times X, A)$, where $n = \text{cd}(\pi \times \pi)$, which is automatically a zero-divisor, since $n > \text{cd}(\pi)$. Applying the arguments used in the proof of Theorem 3 we find that $\alpha = \psi_*(w^{n-1} \cup u)$, see (46), implying that $w^{n-1} \neq 0$.

**Proof of Theorem 1.** Let $X$ be an aspherical finite cell complex with $\pi = \pi_1(X)$ hyperbolic. Then $\pi$ is torsion-free and we may assume that $\pi \neq 1$ since in the simply connected case our statement is obvious. The centraliser

$$Z(g) = \{h \in \pi; hgh^{-1} = g\}$$

of any nontrivial element $g \in \pi$ is virtually cyclic, see [3], Corollary 3.10 in chapter III. It is well known that any torsion-free virtually cyclic group is cyclic. Thus, we see that the assumption (1) of Theorem 2 is satisfied.

Next we show that the assumption (2) of Theorem 2 is satisfied as well, i.e. $\text{cd}(\pi \times \pi) > \text{cd}(\pi)$.

We know that $\pi$ has a finite $K(\pi, 1)$ and hence there exists a finite free resolution $P_\ast$ of $\mathbb{Z}$ over $\mathbb{Z}[\pi]$. Here each $\mathbb{Z}[\pi]$-module $P_i$ is finitely generated and free and $P_i$ is nonzero only for finitely many $i$. Note that $P_\ast \otimes_{\mathbb{Z}} P_\ast$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\pi \times \pi]$ (see [4], Proposition 1.1 in chapter V). For any two left $\mathbb{Z}[\pi]$-modules $A_1, A_2$ we have the natural isomorphism of chain complexes

$$\text{Hom}_{\mathbb{Z}[\pi]}(P_\ast, A_1) \otimes_{\mathbb{Z}} \text{Hom}_{\mathbb{Z}[\pi]}(P_\ast, A_2) \to \text{Hom}_{\mathbb{Z}[\pi \times \pi]}(P_\ast \otimes_{\mathbb{Z}} P_\ast, A_1 \otimes_{\mathbb{Z}} A_2).$$

If at least one of these chain complexes is flat over $\mathbb{Z}$, the Künneth theorem is applicable and we obtain the monomorphism

$$H^j(\pi, A_1) \otimes_{\mathbb{Z}} H^j(\pi, A_2) \to H^{j+j'}(\pi \times \pi, A \otimes_{\mathbb{Z}} A_2)$$

given by the cross-product $\alpha \otimes \alpha' \mapsto \alpha \times \alpha'$. We see that a certain cross-product $\alpha \times \alpha'$ is nonzero provided that the tensor product of the abelian groups $H^j(\pi, A_1) \otimes_{\mathbb{Z}} H^j(\pi, A_2)$ is nonzero.

Let $A_1, A_2$ be left $\mathbb{Z}[\pi]$-modules chosen such that $H^n(\pi, A_1) \neq 0$, where $n = \text{cd}(\pi)$, and $H^1(\pi, A_2) \simeq \mathbb{Z}$. We may take $A_1 = \mathbb{Z}[\pi]$, see [4], chapter VIII, Proposition 6.7. Hence $\text{Hom}_{\mathbb{Z}[\pi]}(P_\ast, A_1)$ is a chain complex of free abelian groups. For some nonzero elements $\alpha \in H^n(\pi, A_1)$ and $\alpha' \in H^1(\pi, A_2)$ we shall have $\alpha \times \alpha' \neq 0$ since $H^n(\pi, A_1) \otimes_{\mathbb{Z}} H^1(\pi, A_2) \simeq H^n(\pi, A_1) \neq 0$. Thus, we obtain $\text{cd}(\pi \times \pi) \geq n + 1 = \text{cd}(\pi) + 1$ as required.

We explain below how to construct the module $A_2$. Fix a non-unit element $g \in \pi$ and let $C = C_g \subset \pi$ denote the conjugacy class of $g$. Let $A_2 = \mathbb{Z}^C = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[C], \mathbb{Z})$ denote the set of all functions $f : C \to \mathbb{Z}$; here $\mathbb{Z}[C]$ denotes the free abelian group generated by

$$\mathbb{Z}.$$
C. Since \( \pi \) acts on \( C \) (via conjugation) we obtain the induced action of \( \pi \) on \( \mathbb{Z}^C \), where \( (g \cdot f)(h) = f(g^{-1}hg) \).

Let \( P_* \) denote a free resolution of \( \mathbb{Z} \) over \( \mathbb{Z}[\pi] \). We have the isomorphisms of chain complexes (which are similar to those used in the proof of Theorem 7)

\[
\text{Hom}_{\mathbb{Z}[\pi]}(P_*, A_2) \cong \text{Hom}_{\mathbb{Z}[\pi]}(P_* \otimes_{\mathbb{Z}} \mathbb{Z}[C], \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}[\pi]}(\mathbb{Z}[C], \text{Hom}_{\mathbb{Z}}(P_*, \mathbb{Z})) \cong \text{Hom}_{\mathbb{Z}[\pi]}(P_*, \mathbb{Z}).
\]

The complex \( P_* \) is a free resolution of \( \mathbb{Z} \) over \( \mathbb{Z}[Z(\gamma)] \) by restriction. Thus, \( H^i(\pi, A_2) \cong H^i(Z(\gamma), \mathbb{Z}) \) for any \( i \). Since we know that the centraliser \( Z(\gamma) \) is isomorphic to \( \mathbb{Z} \) we have \( H^1(\pi, A_2) \cong \mathbb{Z} \).

This completes the proof. \( \square \)

10. An application

A symplectic manifold \((M, \omega)\) is said to be symplectically aspherical [26] if

\[ \int_{S^2} f^* \omega = 0 \]

for any continuous map \( f : S^2 \to M \). It follows from the Stokes’ theorem that a symplectic manifold which is aspherical in the usual sense is also symplectically aspherical.

**Theorem 8.** Let \((M, \omega)\) be a closed \(2n\)-dimensional symplectically aspherical manifold. Suppose that the fundamental group \( \pi = \pi_1(M) \) is of type \( FL \) and the centraliser of any nontrivial element of \( \pi \) is cyclic. Then the topological complexity \( TC(M) \) is either \( 4n \) or \( 4n + 1 \).

**Proof.** Our assumption on \( \pi \) being of type \( FL \) implies that there exists a finite cell complex \( K = K(\pi, 1) \) (see [4], chapter VIII, Theorem 7.1). Let \( g : M \to K \) be a map inducing an isomorphism of the fundamental groups. Let \( u \in H^2(M, \mathbb{R}) \) denote the class of the symplectic form. Then \( u^n \neq 0 \) since \( \omega^n \) is a volume form on \( M \).

Next we show that there exists a unique class \( v \in H^2(K, \mathbb{R}) \) with \( g^*(v) = u \). With this goal in mind we consider the Hopf exact sequence (see [4], chapter II, Proposition 5.2)

\[ \pi_2(M) \xrightarrow{h} H_2(M) \xrightarrow{g_*} H_2(K) \to 0 \]

where \( h \) denotes the Hurewicz homomorphism and all homology groups are with integer coefficients. We may view \( u \) as the homomorphism \( u_* : H_2(M) \to \mathbb{R} \); it vanishes on the image of \( h \) due to (47). Hence there exists a unique \( v_* : H_2(K) \to \mathbb{R} \) with \( u_* = v_* \circ g_* \) and this \( v_* \) is the desired cohomology class \( v \in H^2(K, \mathbb{R}) \).

Denote \( \bar{u} = u \times 1 - 1 \times u \in H^2(M \times M, \mathbb{R}) \) and also \( \bar{v} = v \times 1 - 1 \times v \in H^2(K \times K, \mathbb{R}) \). These classes are zero-divisors and \((g \times g)^*(\bar{v}) = \bar{u} \). Note that

\[ \bar{u}^{2n} = \pm \binom{2n}{n} u^n \times u^n \neq 0. \]

Thus we see that \( \bar{v}^{2n} \neq 0 \) since \( \bar{u}^{2n} = (g \times g)^*(\bar{v}^{2n}) \).
Applying Theorem 3 to the class $\alpha = \bar{u}^{2n} \in H^{4n}(K, \mathbb{R})$ we obtain $\text{wgt}(\alpha) \geq 4n - 1$. We claim that $\text{wgt}(\bar{u}^{2n}) = \text{wgt}((g \times g)^*(\alpha)) \geq 4n - 1$. Indeed, let $f : Y \to M \times M$ be a map satisfying the properties of the Definition 2.1 with $k = 4n - 1$. Then

$$f^*(\bar{u}^{2n}) = f^*((g \times g)^*(\alpha)) = [(g \times g) \circ f]^*(\alpha) = 0.$$ 

Since $\bar{u}^{2n} \neq 0$, the inequality $\text{wgt}(\bar{u}^{2n}) \geq 4n - 1$ implies that $\text{TC}(M) \geq 4n$. The upper bound $\text{TC}(M) \leq 4n + 1$ is standard (see [14], Theorem 4). Thus, $\text{TC}(M) \in \{4n, 4n + 1\}$ as claimed. This completes the proof.

\[\square\]

References


