

**LINEAR MAPS ON REAL C*-ALGEBRAS
AND RELATED STRUCTURES**

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Abstract

In this thesis we obtain new results on the structures of real C^* -algebras and non-surjective isometries between them. Some of the results have been published in [1].

We prove a spectral inequality for real Banach*-algebras and give characterizations of real C^* -algebras among Banach*-algebras.

We study the ideal and facial structures in real C^* -algebras and show that there is a bijection from the class of norm-closed left ideals I of a real C^* -algebra A to the class of weak*-closed faces F of the state space $S(A)$. The bijection is given by $I \mapsto F = \{\rho \in S(A) : \rho(a^*a) = 0 \text{ for all } a \in I\}$, with inverse $F \mapsto I = \{a \in A : \rho(a^*a) = 0 \text{ for all } \rho \in F\}$.

As an application, we use the structures of faces to show an algebraic property of linear maps on real C^* -algebras. We prove that if $T : A \rightarrow B$ is a linear contraction between real C^* -algebras A and B , then there is a projection p in the second dual B'' of B such that

$$T(aa^*a)p = T(a)T(a)^*T(a)p \quad (a \in A).$$

If T is an isometry, not necessarily surjective, we obtain a stronger result which also extends a celebrated result of Kadison on surjective isometries between complex C^* -algebras. We prove the following theorem.

Let T be a linear isometry between two real C^* -algebras A and B , which can be non-surjective. Then for each $a \in A$ there exists a partial isometry $u \in B''$ and a projection $p \in B''$ such that

- (i) $\{u, T(\{f, g, h\}), u\} = \{u, \{T(f), T(g), T(h)\}, u\}$;
- (ii) $T(\{f, g, h\})p = \{T(f), T(g), T(h)\}p$,

for all f, g, h in the real JB^* -triple $A(a)$ generated by $a \in A$, where $\{f, g, h\}$ is the triple product defined by $2\{f, g, h\} = fg^*h + hg^*f$. Moreover, $\{u, T(\cdot).u\} : A(a) \rightarrow B''$ and $T(\cdot)p : A(a) \rightarrow B''$ are isometries.

This theorem cannot be proved by simple complexification. We give an example of a real linear isometry which cannot be complexified to a complex isometry.

We conclude by proving a theorem which states that a Jordan*-homomorphism $T : A \rightarrow B$ between real C^* -algebras A and B is a sum of a C^* -homomorphism and a C^* -antihomomorphism, extending a well-known result for complex C^* -algebras.

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Chapter 1

Introduction

This thesis investigates real C^* -algebras and in particular, the structures of linear continuous maps between real C^* -algebras. We are interested in answering questions that arise naturally in the study of real C^* -algebras when considering well-known results in complex C^* -algebras, for example, the Banach-Stone theorem [2, 48], Kadison's results on surjective isometries [28] and the result of Jacobson, Rickart and Kadison on Jordan*-homomorphisms [27, 28].

The theory of complex C^* -algebras, first introduced as *rings of operators* by Von Neumann and Murray [33, 34, 35] in 1930's and developed by Gelfand, Naimark [19, 20, 21] and Segal [46, 47] in 1940's, has advanced significantly and found numerous applications in quantum physics, K-theory and operator theory. On the other hand, real C^* -algebras have not been developed extensively, although they have been studied by several authors [4, 22, 31, 37]. Nevertheless, there are several recent applications of real C^* -algebras in KK-theory [5, 29], operator theory [8, 12] and even in fractals and computer science [32]. Further, real C^* -algebras play a significant role in the theory of JB*-triples and infinite dimensional geometry. We mention as examples the papers [8, 9, 13, 26, 39] which appeared recently.

Motivated by the pioneering works in these papers and as a follow-up, we develop the structure theory of real C^* -algebras further and systematically, giving a comprehensive theory of ideal structures, facial structures and linear geometric structures of real C^* -algebras, thereby extending a significant part of the theory of complex C^* -algebras to real C^* -algebras.

We begin in Chapter 2 with some preliminaries needed for later development. In Section 2.3, we present a new result on Pták's inequality for real Banach*-algebras, give characterisations for a real Banach*-algebra to be a real C^* -algebra and prove a real analogue of the Vidav-Palmer theorem. These results have been published in [1]. In Section 2.4, we discuss the ordering of projections in real W^* -algebras and prove several lemmas that are used later in the thesis. Finally, in Section 2.5, we consider real C^* -algebras as real JB^* -triples and present the lemmas that will be used in Section 4.3 for the proof of the main results concerning linear isometries between real C^* -algebras and their connection to Jordan structures.

In Chapter 3, we first describe the structures of weak*-closed left ideals and norm-closed faces of the normal state space of a real W^* -algebra. We then use these results in the setting of real C^* -algebras to show that there is a one to one correspondence between norm-closed left ideals and weak*-closed faces of the state space of a real C^* -algebra. More precisely, we prove the following main result of this chapter.

Theorem. *Let A be a real C^* -algebra. There is a one to one inclusion preserving correspondence between the norm-closed left ideals of A and the weak*-closed faces of the state space $S(A)$. If the face F corresponds to the ideal J , then*

- (i) $F = \{\rho \in S(A) \text{ and } \rho(a^*a) = 0 \text{ for all } a \in J\}$;
- (ii) $J = \{a \in A : \rho(a^*a) = 0 \text{ for all } \rho \in F\}$.

In the last chapter of the thesis, we study continuous linear maps between real C^* -algebras. A seminal result on isometries between C^* -algebras is the Banach-Stone theorem which describes completely the structure of a surjective isometry between two complex abelian C^* -algebras. We begin in Section 4.1 by proving a Banach-Stone theorem for real abelian C^* -algebras. Namely, we show that a linear surjective isometry between real abelian C^* -algebras is a composition operator. Our proof reveals the fact that a linear surjective isometry between two real C^* -algebras can be complexified to an isometry between the complexified C^* -algebras. However, this property fails in the case of non-surjective isometries on real C^* -algebras. We show in Section 4.4 that a non-surjective isometry $T : A \rightarrow B$ between real C^* -algebras A and B cannot always be complexified to an isometry. In Section 4.2 we use the results of Chapter 3, specifically the correspondence between ideals and faces in real C^* -algebras, to prove the following result which extends the result in [11, Proposition 2.2].

Proposition. *Let $T : A \rightarrow B$ be a linear contraction between real C^* -algebras A and B . Then there is a largest projection p in B'' such that*

- (i) $T(aa^*a)p = T(a)T(a)^*T(a)p$;
- (ii) $pT(a)^*T(a) = T(a)^*T(a)p \quad (a \in A)$,

where B'' denotes the second dual of B .

We then proceed to establish the main result of the thesis on non-surjective isometries between real C^* -algebras, extending the seminal result of Kadison on surjective isometries between complex C^* -algebras as well as those of [10] and [11]. We prove the following theorem.

Theorem. *Let T be a linear isometry between two real C^* -algebras A and B , which can be non-surjective. Then for each $a \in A$ there exists a partial isometry $u \in B''$ and a projection $p \in B''$ such that*

- (i) $\{u, T(\{f, g, h\}), u\} = \{u, \{T(f), T(g), T(h)\}, u\}$;

$$(ii) T(\{f, g, h\})p = \{T(f), T(g), T(h)\}p,$$

for all f, g, h in the real JB^* -triple $A(a)$ generated by $a \in A$, where $\{f, g, h\}$ is the triple product defined by $2\{f, g, h\} = fg^*h + hg^*f$. Moreover, $\{u, T(\cdot).u\} : A(a) \rightarrow B''$ and $T(\cdot)p : A(a) \rightarrow B''$ are isometries.

This result cannot be proved using complexification since T need not complexify to a complex isometry, as noted above. We develop some new techniques, based on the Jordan approach in [10], to accomplish the proof.

Finally, in Section 4.5, motivated by the fact that linear isometries between real C^* -algebras preserve the Jordan structures and are in essence Jordan homomorphisms, we study Jordan*-homomorphisms between real C^* -algebras and prove a real analogue of the well-known result that a complex Jordan*-homomorphism is a sum of a complex C^* -homomorphism and C^* -antihomomorphism [27, 28].

Theorem. *Let $T : A \rightarrow B$ be a Jordan*-homomorphism between real C^* -algebras A and B . Then there exists a projection $z \in B''$ such that*

$$T(\cdot) = T(\cdot)z + T(\cdot)(1 - z)$$

where $T(\cdot)z : A \rightarrow B''$ is a C^* -homomorphism and $T(\cdot)(1 - z) : A \rightarrow B''$ is a C^* -antihomomorphism.

Chapter 2

Real Operator algebras

We will present the basic theory of real operator algebras and mainly real C^* -algebras. Unlike complex C^* -algebras, real C^* -algebras are less developed and we choose to discuss their structures in a self-contained manner. We also highlight several important differences between real and complex operator algebras.

2.1 Complexifications of real normed algebras

A useful technique when studying real linear spaces and real algebras is complexification. For completeness, we recall the construction below (for details see [43]).

Definition 2.1.1. Let X be a real linear normed space. We define X_c to be the cartesian product $X_c = X \times X$ with algebraic operations defined by

- (i) $(x, y) + (u, v) = (x + u, y + v)$ for $x, y, u, v \in X$;
- (ii) $(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \alpha y + \beta x)$ for $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$;
- (iii) $(x, y)(u, v) = (xu - yv, xv + yu)$ for $x, y, u, v \in X$, if X is an algebra.

Then X_c is a complex linear space and the mapping $x \in X \mapsto (x, 0) \in X_c$ is a real isomorphism from X into X_c . It is an algebra isomorphism if X is an algebra. We can define a norm on X_c , for instance,

$$\|(x, y)\| = \sup_{\theta \in \mathbb{R}} \left\{ \frac{1}{\sqrt{2}} (\|x \cos \theta - y \sin \theta\| + \|y \cos \theta + x \sin \theta\|) \right\}$$

so that X_c becomes a normed linear space and the isomorphism $x \in X \mapsto (x, 0) \in X_c$ is an isometry. If X is a normed algebra, then X_c is also a normed algebra. We call $(X_c, \|\cdot\|)$ a *complexification* of X and denote $X_c = X + iX$. We also write $x + iy$ for $(x, y) \in X_c$. In the sequel, we always assume that a suitable norm has been chosen for X_c .

We note that X_c is a Banach space if and only if X is a Banach space.

Given a real or complex Banach space X , we denote by $B(X)$ the Banach algebra of bounded linear operators on X .

Let X be a real normed linear space. For $T \in B(X)$, define a linear operator $T_c : X_c \rightarrow X_c$ by

$$T_c(x + iy) = T(x) + iT(y) \quad (x + iy \in X_c).$$

Then $T_c \in B(X_c)$ and the mapping $T \rightarrow T_c$ is an isometric real isomorphism of the algebra $B(X)$ into $B(X_c)$.

We denote by X^* the dual of a complex Banach space X and by X' the dual of a real Banach space X . The real restriction X_r of a complex Banach space X is defined to be the space X with scalar multiplication restricted to the real numbers. Henceforth, we will identify a real normed space X as a closed subspace of its complexification X_c via the embedding $x \in X \mapsto (x, 0) \in X_c$. For a real or complex normed linear space X , we denote by X_1 the closed unit ball

$$X_1 = \{x \in X : \|x\| \leq 1\}.$$

Let X be a real Banach space and X_c its complexification. We define a map

$\tau : X_c \rightarrow X_c$ by

$$\tau(x + iy) = x - iy \quad (x, y \in X)$$

which is a conjugate linear isometric isomorphism of period 2. Moreover, we have

$$X = \{x \in X_c : \tau(x) = x\}.$$

We define $\tau^* : X_c^* \rightarrow X_c^*$ by

$$\tau^*(f)(x + iy) = \overline{f(\tau(x + iy))} \quad (x, y \in X, f \in X_c^*).$$

Then τ^* is a period 2 conjugate linear isometry.

Lemma 2.1.2. *Let X be a real Banach space and X_c be its complexification. The following assertions hold.*

- (i) *Let $f \in X_c^*$ with $\tau^*(f) = f$. Then $f|_X \in X'$ and $\|f|_X\| = \|f\|$.*
- (ii) *For $g \in X'$, define $g_c : X_c \rightarrow \mathbb{C}$ by*

$$g_c(x + iy) = g(x) + ig(y)$$

for all $x, y \in X$. Then $g_c \in X_c^$, $\tau^*(g_c) = g_c$ and $\|g\| = \|g_c\|$. If $f \in X_c^*$ and $\tau^*(f) = f$, then $(f|_X)_c = f$.*

Proof. See, for example, [31, Proposition 1.1.4]. □

It follows by Lemma 2.1.2(ii) that X' can be isometrically embedded into X_c^* via $g \in X' \mapsto g_c \in X_c^*$. Let $f \in X_c^*$. We observe that $f = \frac{1}{2}(f + \tau^*(f)) + i\frac{1}{2i}(f - \tau^*(f))$ and thus we obtain

$$X' = \{f \in X_c^* : \tau^*(f) = f\} \text{ and } X_c^* = X' + iX'.$$

Moreover, $\tau^*(f + ig) = f - ig$ for all $f, g \in X'$.

2.2 Real Banach algebras

This section contains a brief introduction to real Banach algebras. We prove a spectral inequality which has been published in [1].

Definition 2.2.1. A real (resp. complex) algebra A that is a real (resp. complex) Banach space with a norm satisfying $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$, is called a *real* (resp. *complex*) *Banach algebra*.

Given a real Banach algebra A , its complexification A_c is a complex Banach algebra. Further, all norms on A_c making an isometric embedding $x \in A \mapsto (x, 0) \in A_c$ are equivalent (cf. [43, Theorem 1.3.2]).

As in the complex case, if a real Banach algebra A does not have an identity it can be isometrically embedded into a real Banach algebra \tilde{A} with identity, where the *unit extension* \tilde{A} is defined by $\tilde{A} = A \oplus \mathbb{R}$ with norm

$$\|a \oplus \lambda\| = \|a\| + |\lambda| \quad (a \in A, \lambda \in \mathbb{R})$$

and product

$$(a \oplus \lambda)(b \oplus \mu) = (ab + \lambda b + \mu a) \oplus \lambda\mu \quad (a, b \in A, \lambda, \mu \in \mathbb{R}).$$

For a real Banach algebra A with identity, we denote its identity by 1 and we write, if no confusion is likely, λ for $\lambda \cdot 1$ with $\lambda \in \mathbb{R}$. Likewise, for a complex Banach algebra B with identity, we also denote its identity by 1 and we write λ for $\lambda \cdot 1$ with $\lambda \in \mathbb{C}$.

A key concept frequently used in the study of operator algebras is that of a spectrum. The spectrum of an element b in a complex unital Banach algebra B is the set of all $\lambda \in \mathbb{C}$ such that $b - \lambda$ is not invertible in B . If A is a unital real Banach algebra, we may wish to define the spectrum of an element a in A as the

set

$$\{\lambda \in \mathbb{R} : a - \lambda \text{ not invertible in } A\}.$$

Unfortunately, in this definition, the spectrum of a may be empty. For example, if A is the algebra of 2×2 real matrices, then $a = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ has empty spectrum according to the above definition. For this reason, we define the spectrum of an element in a real Banach algebra as follows.

Definition 2.2.2. Let A be a unital real Banach algebra and $a \in A$. The *spectrum* of a is defined by

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda \text{ not invertible in } A_c\}$$

where A_c is the complexification of A .

By the complex result, $\sigma(a)$ is a non-empty compact subset of \mathbb{C} .

We define the *spectral radius* $r(a)$ of $a \in A$ by

$$r(a) = \max\{|\lambda| : \lambda \in \sigma(a)\}.$$

Definition 2.2.3. Let A be a real Banach algebra with identity. A is said to be *divisible* if every non-zero element is invertible.

One fundamental difference between real and complex algebras is that the only complex divisible Banach algebra is \mathbb{C} whereas the real divisible Banach algebras are \mathbb{R} , \mathbb{C} and the quaternion algebra \mathbb{H} [22, Theorem 9.7].

We now consider involutions of real Banach algebras.

Definition 2.2.4. A real Banach algebra A is called a *real Banach*-algebra* if there is a real linear map $*$: $A \rightarrow A$ with period 2, called *involution*, such that

$$(ab)^* = b^*a^* \quad (a, b \in A).$$

Although we do not assume continuity of the involution $*$ in the above definition, we will see in the proof of Proposition 2.3.6 that in real C^* -algebras continuity of the involution is automatic.

Remark 2.2.5. We can always extend the involution to the complexification $*$: $A_c \rightarrow A_c$ as follows:

$$(a + ib)^* = a^* - ib^* \quad (a, b \in A).$$

Definition 2.2.6. An element a in a real Banach $*$ -algebra A is called:

- (i) *hermitian* if $a^* = a$;
- (ii) *skew-hermitian* if $a^* = -a$;
- (iii) *unitary* if $a^*a = aa^* = 1$.

We denote by A_h , A_{sh} and $U(A)$ the set of hermitian, skew hermitian and unitary elements in A respectively.

We note another difference between real and complex algebras. Let A be a real Banach $*$ -algebra and let $[U(A)]_{\mathbb{R}}$ be the real linear span of the set $U(A)$ of unitary elements. Then we have $A_{sh} \subset [U(A)]_{\mathbb{R}}$ (cf. [31, Lemma 3.1.3]) and $A = A_h + A_{sh}$ but $A \neq [A_h]_{\mathbb{R}}$. On the other hand, for a complex Banach $*$ -algebra B , we have $B = [U(B)]_{\mathbb{C}} = [B_h]_{\mathbb{C}}$ where $[\cdot]_{\mathbb{C}}$ is the complex linear span.

In what follows, all Banach $*$ -algebras are assumed to have an identity. Let A be a real Banach $*$ -algebra and A_c be the complexification of A . A real Banach $*$ -algebra A is called *hermitian* if every hermitian element has real spectrum and *skew hermitian* if every skew hermitian element has imaginary spectrum. A real Banach $*$ -algebra A is called *symmetric* if for every $a \in A$, the element $1 + a^*a$ is invertible. We note that symmetry is equivalent to being hermitian and skew hermitian (see [31, Theorem 3.6.5] and [37, Lemma 1]) and that for complex

Banach*-algebras, the conditions of being hermitian, skew hermitian and symmetric are all equivalent.

Given a complex hermitian Banach*-algebra A , the following spectral inequality

$$r(a) \leq r(a^*a)^{\frac{1}{2}} \quad (a \in A)$$

has been proved by Pták in [42], where $r(a)$ denotes the spectral radius of a . This inequality is referred to as Pták's inequality.

Pták's inequality holds for all elements in an abelian real symmetric Banach*-algebra. For arbitrary non-abelian real symmetric Banach*-algebras however, the inequality has been proved only for elements with real spectrum [31, Theorem 6.3.2(1)] although it has been claimed in [52] that the proof for the complex algebras can be extended to all Banach*-algebras. There seems to be a gap in the claim of [52]. Nevertheless, we will show that Pták's inequality also holds in real C*-algebras.

We will make use of the following square-root result in [17].

Lemma 2.2.7. *Let A be a real Banach*-algebra and let $h \in A$ be hermitian. If $r(1-h) < 1$, then there is a unique hermitian element u in A such that $r(1-u) < 1$ and $u^2 = h$.*

We now prove Pták's inequality for an element with purely imaginary spectrum.

Proposition 2.2.8. *Let A be a real Banach*-algebra which is hermitian and skew hermitian and let $a \in A$ be such that $\sigma(a) \subseteq i\mathbb{R}$. Then $r(a) \leq r(a^*a)^{\frac{1}{2}}$.*

Proof. Let $a \neq 0$ and suppose $r(a)^2 > r(a^*a)$ for contradiction.

Then there exists $\varepsilon > 0$ such that $r(a) - \varepsilon > r(a^*a)^{\frac{1}{2}}$. Therefore, we can pick $\lambda \in \sigma(a) \subseteq i\mathbb{R}$ such that $|\lambda| > r(a) - \varepsilon$ and $a - \lambda$ is not invertible in A_c .

We note that $|\lambda| > r(a^*a)^{\frac{1}{2}} > 0$. Let $b = \frac{a}{\lambda} \in A_c$. We have $b^* = \frac{a^*}{\lambda} \in A_c$.

Since $r(b^*b) = \frac{r(a^*a)}{\lambda\lambda} < 1$, the element $1 - b^*b$ is invertible and $r(1 - (1 - b^*b)) = r(b^*b) < 1$, where $(1 - b^*b)^* = 1 - b^*b$.

Moreover, $1 - b^*b = 1 - \frac{a^*a}{\lambda\lambda} = 1 - \frac{a^*a}{|\lambda|^2} \in A$ and by Lemma 2.2.7 there is a unique element $u = u^* \in A$ such that $u^2 = 1 - b^*b$.

By invertibility of $1 - b^*b$, the hermitian element u is invertible with hermitian inverse.

Observe that,

$$\begin{aligned} (1 + b^*)(1 - b) &= 1 - b^*b + b^* - b \\ &= u^2 + b^* - b \\ &= u(1 + u^{-1}(b^* - b)u^{-1})u. \end{aligned}$$

Letting $\lambda = i\lambda_1$ with $\lambda_1 \in \mathbb{R}$, we have

$$\begin{aligned} u^{-1}(b^* - b)u^{-1} &= u^{-1}\left(\frac{a^*}{\lambda} - \frac{a}{\lambda}\right)u^{-1} \\ &= u^{-1}\left(\frac{a^*}{-i\lambda_1} - \frac{a}{i\lambda_1}\right)u^{-1} \\ &= -\frac{u^{-1}(a^* + a)u^{-1}}{i\lambda_1}. \end{aligned}$$

Since $(u^{-1}(a^* + a)u^{-1})^* = u^{-1}(a^* + a)u^{-1}$ and A is hermitian, we have $\sigma(u^{-1}(a^* + a)u^{-1}) \subseteq \mathbb{R}$.

By the spectral theorem,

$$\sigma\left(-\frac{u^{-1}(a^* + a)u^{-1}}{i\lambda_1}\right) = -\frac{\sigma(u^{-1}(a^* + a)u^{-1})}{i\lambda_1} \subseteq i\mathbb{R}$$

and therefore $-1 \notin \sigma(u^{-1}(b^* - b)u^{-1})$ which implies that $1 + u^{-1}(b^* - b)u^{-1}$ is invertible.

Hence $(1 + b^*)(1 - b)$ is the product of invertible elements and $1 - b$ has a left inverse.

Similar arguments for the element bb^* show that $1 - b$ has a right inverse.

Therefore $1 - \frac{a}{\lambda} = 1 - b$ is invertible which is a contradiction. We conclude that $r(a) \leq r(a^*a)^{\frac{1}{2}}$. \square

We note that the skew hermitian condition cannot be dropped in Proposition 2.2.8 as shown by the next example.

Example 2.2.9. Consider the Banach*-algebra \mathbb{R}^2 with pointwise multiplication, addition and involution $(a, b)^* = (b, a)$. Then $(\mathbb{R}^2, \|\cdot\|_\infty)$ is a hermitian Banach *-algebra but it is not skew hermitian, Pták's inequality does not hold and its complexification \mathbb{C}^2 is not hermitian. Indeed, if $(a, b)^* = (a, b)$, then $(a, b) = (b, a)$ and $\sigma((a, b)) = \sigma((a, a)) = \{a\} \subset \mathbb{R}$. Thus A is hermitian.

If $(a, b)^* = -(a, b)$, then $(a, b) = (-b, -a)$ and $\sigma((a, b)) = \sigma((a, -a)) = \{\lambda \in \mathbb{R} : \lambda = \pm a\} \subset \mathbb{R}$. Thus A is not skew hermitian.

Let $x = (3, 2) \in \mathbb{R}^2$. Then $r(x) = 3$ and $r(x^*x) = r((6, 6)) = 6$. Clearly, $r(x) > r(x^*x)^{\frac{1}{2}}$.

2.3 Real C*-algebras

In this section, we introduce real C*-algebras and present some basic results. An important feature in the study of real C*-algebras is their connection to the complex ones. Indeed, complexification will be often used throughout the thesis. However, complexification cannot be applied to all cases since it does not always preserve the algebraic or geometric structures [14, 24]. We will highlight the fundamental differences between complex and real C*-algebras and justify the methods that will be used for some crucial results when motivating the study of the real structures. The basic theory of real C*-algebras can be found in [22], [31] and [37].

Definition 2.3.1. Let A be a real Banach*-algebra. Then A is called a *real C*-algebra* if $\|a^*a\| = \|a\|^2$ and $1 + a^*a$ is invertible for all $a \in A$ if A has a unit. If A is non-unital, then we require that $1 + a^*a$ is invertible in the unit extension $A \oplus \mathbb{R}$.

It has been proven in [43, Lemma 4.1.13] that if A is a non-unital real C*-algebra, then the unit extension $\tilde{A} = A \oplus \mathbb{R}$ is a real C*-algebra with norm given by

$$\|(a, \lambda)\| = \sup\{\|au + \lambda u\| : u \in A, \|u\| = 1\} \quad (a \in A, \lambda \in \mathbb{R}).$$

Remark 2.3.2. Every complex C*-algebra is automatically hermitian and thus symmetric (see for example [42]). In contrast, the condition "1 + a^*a invertible" is necessary in the definition of a real C*-algebra. In fact, if we consider the complex plane \mathbb{C} with modulus norm and the identity involution $z^* = z$, then we have $|z^*z| = |z|^2$ for all $z \in \mathbb{C}$ but $1 + i^*i$ is not invertible. We also note that every complex C*-algebra B is a real C*-algebra if we consider its real restriction B_r .

By [37, Proposition 1 and Theorem 1], we have the following equivalent definitions for a real C*-algebra.

Theorem 2.3.3. *Let A be a real Banach*-algebra. The following conditions are equivalent.*

- (i) A is a real C*-algebra.
- (ii) A is hermitian and $\|a^*a\| = \|a\|^2$ for all $a \in A$.
- (iii) The complexification $A_c = A + iA$ can be normed to become a complex C*-algebra and keeps the original norm on A .
- (iv) A is isometrically *-isomorphic to a norm-closed real*-algebra of bounded operators on a real Hilbert space.

Therefore every real C*-algebra A comes from a complex C*-algebra. More precisely, we have the following from Section 2.1 (see also [31, p.78]).

Proposition 2.3.4. *Every real C*-algebra A is the real form of a complex C*-algebra (B, τ) , that is, $A = \{b \in B : \tau(b) = b\}$, where $B = A + iA$ is the complexification of A and $\tau : B \rightarrow B$ is a conjugate linear *-isomorphism with period 2. Moreover, $B^* = A' + iA'$ is the complexification of A' .*

Corollary 2.3.5. *Every closed real *-subalgebra of a real C*-algebra is also a real C*-algebra.*

We now give necessary and sufficient conditions for a real Banach*-algebra A to have an equivalent C* norm. Pták's inequality holds for these algebras. In fact, $p(x) = r(x^*x)^{1/2}$ is a C*-seminorm in a real Banach*-algebra (cf. [31, Theorem 6.3.2(7)]) and the issue is when $p(\cdot)$ is a C* norm. The following result has been published in [1].

Proposition 2.3.6. *Let $(A, \|\cdot\|)$ be a real Banach*-algebra which is hermitian and skew hermitian. The following are equivalent:*

- (i) A has an equivalent C*-norm;
- (ii) p is a complete norm;

(iii) p is equivalent to the original norm of A ;

(iv) $r(h) \geq \alpha \|h\|$ for all $h = \pm h^*$ and some $\alpha > 0$.

In the above, Pták's inequality $r(x) \leq p(x)$ holds for all $x \in A$.

Proof. (i) \Rightarrow (ii) Let $(A, \|\cdot\|)$ be *-isomorphic to a real C*-algebra $(A, \|\cdot\|_1)$ via $\phi : (A, \|\cdot\|) \rightarrow (A, \|\cdot\|_1)$. Then we have $p(a) = p(\phi(a)) = r(\phi(a)^*\phi(a))^{1/2} = \|\phi(a)^*\phi(a)\|_1^{1/2} = \|\phi(a)\|_1$ for all $a \in A$ and in particular $p(\cdot)$ is a complete norm.

(ii) \Rightarrow (iii) We first show that the involution is bounded.

Let (x_n) be a sequence in A such that $x_n \rightarrow 0$ and $x_n^* \rightarrow y$ as $n \rightarrow \infty$. Then we have

$$\begin{aligned} p(y) &= p(y - x_n^* + x_n^*) \\ &\leq p(y - x_n^*) + p(x_n^*) \\ &\leq \|y - x_n^*\|^{1/2} \|y^* - x_n\|^{1/2} + \|x_n^*\|^{1/2} \|x_n\|^{1/2} \rightarrow 0 \end{aligned}$$

which gives $p(y) = 0$. Since p is a norm, we have $y = 0$. Hence by the closed graph theorem, there exists $\alpha > 0$ such that $\|x^*\| \leq \alpha \|x\|$ for all $x \in A$. It follows that the identity map $I : (A, p(\cdot)) \rightarrow (A, \|\cdot\|)$ is continuous, since $p(x) = r(x^*x)^{1/2} \leq \|x^*x\|^{1/2} \leq \|x^*\|^{1/2} \|x\|^{1/2} \leq \alpha^{1/2} \|x\|$. Hence $p(\cdot)$ is equivalent to $\|\cdot\|$ by the open mapping theorem.

(iii) \Rightarrow (iv) Let $h = \pm h^*$. We note that every hermitian or skew hermitian element is normal. The equivalence of $p(\cdot)$ and the norm $\|\cdot\|$ of A implies that $\alpha \|\cdot\| \leq p(\cdot)$ for some $\alpha > 0$. In particular we have $\alpha \|h\| \leq p(h) = r(h)$, since h is normal.

(iv) \Rightarrow (i) Let $x \in A$. Then $x = h + k$ with $h^* = h$ and $k^* = -k$. By the symmetry of A we have $r(h) \leq p(x)$ and $r(k) \leq p(x)$ (see [31, Theorem

3.6.3(2)]. Hence by (iv), there is some $\alpha > 0$ such that

$$\alpha\|x\| \leq \alpha\|h\| + \alpha\|k\| \leq r(h) + r(k) \leq 2p(x).$$

By [31, Lemma 7.1.3], the involution is continuous in norm and thus, bounded by some $\beta > 0$. We then have

$$p(x) = r(x^*x)^{1/2} \leq \|x^*x\|^{1/2} \leq \|x^*\|^{1/2}\|x\|^{1/2} \leq \beta\|x\|.$$

Hence $p(\cdot)$ is equivalent to $\|\cdot\|$ and (i) holds.

Let ϕ be the isomorphism in the proof of (i) \Rightarrow (ii). Then, for every $a \in A$, we have $\sigma(\phi(a)) = \sigma(a)$ and

$$\begin{aligned} r(a)^2 &= r(\phi(a))^2 \\ &\leq \|\phi(a)\|_1^2 \\ &= \|\phi(a)^*\phi(a)\|_1 \\ &= r(\phi(a)^*\phi(a)) \\ &= r(\phi(a^*a)) \\ &= r(a^*a) = p(a)^2 \end{aligned}$$

□

Remark 2.3.7. The condition (iv) above cannot be replaced by $r(x) \geq \alpha\|x\|$ for all $x \in A$. For example, if we consider the real C*-algebra $M_2(\mathbb{R})$ of 2×2 real matrices and the elements $x_n = \begin{pmatrix} 1/n & 0 \\ n & 0 \end{pmatrix}$. Then, $r(x_n) = 1/n$ and $\|x_n\| = r(x_n^*x_n)^{1/2} = (n^2 + (1/n)^2)^{1/2}$. Clearly, it is not possible to find $\alpha > 0$ such that $r(x_n) \geq \alpha\|x_n\|$ for all n .

A hermitian and skew hermitian real Banach*-algebra need not have an equivalent C* norm. For example, the abelian group algebra $L_1(\mathbb{R})$ of real integrable functions on \mathbb{R} , with convolution product $(f * g)(s) = \int_{\mathbb{R}} f(t)g(s-t)dt$ and in-

volution $f^*(s) = f(s^{-1})$, does not have an equivalent C* norm. However, Pták's inequality holds for every element in $L_1(\mathbb{R})$, indeed $p(s) = r(s)^2$ for $s \in L_1(\mathbb{R})$.

Using the numerical range in a real Banach*-algebra A , we introduce the concept of a numerical range hermitian and skew hermitian algebra and show that it characterises real C*-algebras among Banach*-algebras.

We recall that the numerical range of an element a in A is defined to be

$$V(a, A) = \{f(a) : f \in A', f(1) = 1 = \|f\|\}$$

where A' is the dual of A . If B is a complex algebra, the numerical range $V(b, B)$ of an element $b \in B$ is defined by

$$V(b, B) = \{f(b) : f \in B^*, f(1) = 1 = \|f\|\}$$

where B^* denotes the complex dual of B (cf. [6]).

Let

$$H(B) = \{b \in B : V(b, B) \subseteq \mathbb{R}\}.$$

A characterisation of complex C*-algebras by numerical range is the well-known Vidav-Palmer theorem (cf. [36] and [51]), which states that a complex Banach*-algebra B is isometrically *-isomorphic to a complex C*-algebra if and only if $B = H(B) + iH(B)$. The Vidav-Palmer theorem is a complex result and we cannot expect a real analogue since, by definition, the numerical range of each element in a real algebra is already real. Nevertheless, we can introduce the concept of *complexified numerical range* for a real Banach*-algebra, defined via complexification, just as the way we have defined the spectrum in a real algebra. This enables us to characterise real C*-algebras using complexified numerical ranges.

Definition 2.3.8. Let A be a real Banach*-algebra and A_c its complexification. Let $a \in A$. The set

$$V(a, A_c) = \{f(a) : f \in A_c^*, f(1) = 1 = \|f\|\}$$

is called the *complexified numerical range* of a .

Definition 2.3.9. Let A be a real Banach*-algebra.

- (i) A is called *numerical range hermitian* if for every $h = h^* \in A$, we have $V(h, A_c) \subseteq \mathbb{R}$.
- (ii) A is called *numerical range skew hermitian* if for every $k = -k^* \in A$, we have $V(k, A_c) \subseteq i\mathbb{R}$.

We define the sets

$$H_c(A) = \{a \in A : V(a, A_c) \subseteq \mathbb{R}\} \text{ and } SH_c(A) = \{a \in A : V(a, A_c) \subseteq i\mathbb{R}\}.$$

Theorem 2.3.10. *Let A be a real Banach*-algebra. Then A is a real C*-algebra if and only if A is numerical range hermitian and skew hermitian.*

Proof. Let A be a real C*-algebra and A_c its complexification. Then for every element $h^* = h$ in A , we have $h^* = h$ in A_c and hence $V(h, A_c) \subset \mathbb{R}$ by [6, Example 5] and $h \in H_c(A)$. Similarly, let $k^* = -k$ in A . Then $(ik)^* = ik \in A_c$ and $V(ik, A_c) \subset \mathbb{R}$. Hence $V(k, A_c) \subset i\mathbb{R}$ and $k \in SH_c(A)$. This shows that A is numerical range hermitian and skew hermitian.

Conversely, let A be a numerical range hermitian and skew hermitian Banach*-algebra, with complexification A_c . Let $h^* = h \in A_c$ be such that $h = a + ib$, $a, b \in A$. Then, $a + ib = h = h^* = a^* - ib^*$ which implies that $a^* = a$, $b^* = -b$ and by the assumption, we have $V(a, A_c) \subseteq \mathbb{R}$ and $V(b, A_c) \subseteq i\mathbb{R}$. Therefore, $V(h, A_c) \subseteq V(a, A_c) + iV(b, A_c) \subseteq \mathbb{R} + i(i\mathbb{R}) = \mathbb{R}$ and $h \in H(A_c)$. Take an arbitrary $x \in A_c$ and write $x = \frac{x+x^*}{2} + i\frac{x-x^*}{2i}$. Let $u = \frac{x+x^*}{2}$ and $v = \frac{x-x^*}{2i}$. Then, $u^* = u \in A_c$ and $v^* = v \in A_c$. By the previous argument, we have $u, v \in H(A_c)$ and it follows that $A_c = H(A_c) + iH(A_c)$. By the Vidav-Palmer theorem, A_c is a complex C*-algebra and therefore A is a real C*-algebra. \square

Under the conditions of Theorem 2.3.10, we have $A = H_c(A) + SH_c(A)$.

We will now discuss real abelian C*-algebras. By $C_0(X)$ we denote the Banach space of complex continuous functions on a locally compact Hausdorff space X , vanishing at infinity. If X is compact, then $C_0(X)$ is the space $C(X)$ of all continuous complex functions. With pointwise product and complex conjugation as involution, $C_0(X)$ is an abelian complex C*-algebra. Conversely, it is well-known that every complex abelian C*-algebra is of the form $C_0(X)$ for some locally compact Hausdorff space X .

Let X be a locally compact Hausdorff space and $\sigma : X \rightarrow X$ a period 2 homeomorphism. In the sequel we let

$$C_0(X, \sigma) = \{f \in C_0(X) : f \circ \sigma(x) = \overline{f(x)}, x \in X\}$$

which is a real abelian C*-algebra with the supremum norm, the pointwise product and involution $f^*(x) = \overline{f(x)}$ for all $x \in X$.

It is immediate that if σ is the identity map $id : X \rightarrow X$, then $C_0(X, id)$ is just the space $C_0(X, \mathbb{R})$ of real continuous functions on X vanishing at infinity.

Definition 2.3.11. We define the *spectrum* X of a real abelian C*-algebra A to be the set of all non-zero multiplicative real linear functionals $: A \rightarrow \mathbb{C}$.

We note that X is a locally compact Hausdorff space in the pointwise topology by [22, Proposition 10.5].

Let A be an abelian real C*-algebra and let X be its spectrum. We define the *Gelfand transform* of an element $a \in A$ to be the mapping $\hat{a} : X \rightarrow \mathbb{C}$ given by

$$\hat{a}(x) = x(a) \quad (x \in X).$$

The mapping $a \in A \mapsto \hat{a} \in C_0(X)$ is a homomorphism from A into $C_0(X)$ and is called the *Gelfand transform* of A (cf. [22, Theorem 10.7]). We have the following well known characterisation of real abelian C*-algebras and we refer to [22, Theorem 12.5] for a proof.

Proposition 2.3.12. *Let A be a real abelian C*-algebra and X its spectrum. Then A is isometrically *-isomorphic to $C_0(X, \sigma)$ for some period 2 homeomorphism $\sigma : X \rightarrow X$.*

Definition 2.3.13. Let A be a real C*-algebra. An element $a \in A$ is called *positive* if $a^* = a$ and $\sigma(a) \subset [0, \infty)$ where $\sigma(a)$ is the spectrum of a .

We denote the set of positive elements by A_+ .

The following results can be obtained via complexification.

Lemma 2.3.14. *Let A be a real C*-algebra with complexification A_c . The following assertions hold.*

- (i) $A_+ = (A_c)_+ \cap A$ is a closed cone.
- (ii) Let $a \in A_+$. Then there exists a unique $a^{\frac{1}{2}} \in A_+$ such that $(a^{\frac{1}{2}})^2 = a$ and $a^{\frac{1}{2}}$ is the limit of a sequence of polynomials in A without constant term.
- (iii) Let $a \in A$. Then $a \in A_+$ if and only if there exists $b \in A$ such that $a = b^*b$.
- (iv) For any $a^* = a$ there are unique a_+ and a_- in A_+ such that

$$a = a_+ - a_- \quad \text{and} \quad a_+ a_- = 0.$$

We make note of another difference between real and complex C*-algebras. In a real C*-algebra A only the set of hermitian elements A_h is the real linear span of the positive elements, i.e. $A_h = [A_+]_{\mathbb{R}}$. For a complex C*-algebra, however, the whole algebra is the complex linear span of the positive elements.

We now consider the dual space A' of a real C*-algebra A . For a real linear functional $f : A \rightarrow \mathbb{R}$ we define $f^* : A \rightarrow \mathbb{R}$ by

$$f^*(a) = f(a^*) \quad (a \in A).$$

The weak*-topology on the dual A' of a real C*-algebra A is defined to be the smallest topology for which all the mappings $f \mapsto f(x)$ are continuous for all $f \in A'$ and $x \in A$.

Definition 2.3.15. Let A be a real C*-algebra.

(i) A real linear functional $f : A \rightarrow \mathbb{R}$ is said to be *hermitian* if

$$f(a^*) = f(a)$$

for all $a \in A$. In this case, f vanishes on skew hermitian elements.

(ii) A real linear functional $f : A \rightarrow \mathbb{R}$ is said to be *positive*, in symbols $f \geq 0$, if f is hermitian and $f(a^*a) \geq 0$ for all $a \in A$.

(iii) A positive functional f is called a *real state* if $\|f\| = 1$. If A is unital, this is equivalent to saying $f(1) = 1$. The real state space of A , denoted by $S(A)$, is the set of all real states of A , equipped with the weak*-topology.

If f is a functional on a real C*-algebra A , then, by Lemma 2.1.2, we can extend it to a functional $f_c : A_c \rightarrow \mathbb{C}$ on the complexification A_c such that $f_c(a + ib) = f(a) + if(b)$ with $a, b \in A$. Unlike the real case, a complex functional f on a complex C*-algebra B satisfying $f(a^*a) \geq 0$ for all $a \in B$ is automatically hermitian.

Lemma 2.3.16. Let A be a real C*-algebra and $f : A \rightarrow \mathbb{R}$ a positive functional on A . Then the complexification $f_c : A_c \rightarrow \mathbb{C}$ is positive.

Proof. We prove that $f(x^*x) \geq 0$ for every $x \in A_c$. Let $x \in A_c$ with $x = a + ib$ for $a, b \in A$. Then $x^*x = a^*a + b^*b + i(a^*b - b^*a)$. Hence

$$f_c(x^*x) = f(a^*a + b^*b) + if(a^*b - b^*a) = f(a^*a) + f(b^*b) \geq 0$$

where we have used the fact that $f \geq 0$ implies that f vanishes on skew hermitian elements. \square

The following result is the well known Schwarz inequality which will be used in many places.

Lemma 2.3.17. *Let f be a positive functional on a real C^* -algebra A . Define $f_b(\cdot) = f(b^* \cdot b)$ for $b \in A$. Then f_b is a positive functional and the following inequalities hold.*

- (i) $f(b^*a)^2 \leq f(a^*a)f(b^*b)$ (Schwarz inequality).
- (ii) $|f_b(a)| \leq f(b^*b)r(a^*a)^{\frac{1}{2}}$.

Proof. We apply the complex results (see for example [50, Proposition I.9.5 and Lemma I.9.10]) to the complexification $(f_b)_c : A_c \rightarrow \mathbb{C}$ where

$$(f_b)_c(x + iy) = f_b(x) + if_b(y) = f(b^*xb) + if(b^*yb) = f_c(b^*(x + iy)b).$$

□

Lemma 2.3.18. *Let A be a real C^* -algebra. Let $f \in A'$ and $f_c = f + if \in A^*$. The following conditions hold.*

- (i) *Suppose there exists $a \in A_+$ with $\|a\| \leq 1$ and $\|f\| = f(a)$. Then $f \geq 0$.*
- (ii) *Let f be hermitian. Then there exist unique positive functionals $f_+, f_- \in A'$ such that*

$$f = f_+ - f_- \text{ and } \|f\| = \|f_+\| + \|f_-\|.$$

Proof. (i) We have, by Lemma 2.1.2, that $\|f_c\| = \|f\| = f(a) = f_c(a)$ for $a \in A_+ \subset (A_c)_+$ and $\|a\| \leq 1$. Hence by [50, Lemma III.3.2], $f_c \geq 0$ and $f \geq 0$.

- (ii) Since f_c is also hermitian it follows by [50, Theorem III.4.2(ii)] that there exist unique positive functionals $\phi_+, \phi_- \geq 0$ in A_c^* such that $f_c = \phi_+ - \phi_-$ and $\|f\| = \|f_c\| = \|\phi_+\| + \|\phi_-\|$. Let $f_{\pm} = \text{Re}(\phi_{\pm}|A)$. Then $f_{\pm} \geq 0$ and

$\|f_{\pm}\| \leq \|\phi_{\pm}\|$. Therefore we obtain

$$\begin{aligned} f &= f_c|A = (\phi_+ - \phi_-)|A = \mathbf{Re}((\phi_+ - \phi_-)|A) = f_+ - f_- \\ \|f\| &\leq \|f_+\| + \|f_-\|. \end{aligned}$$

Since also

$$\|f\| = \|f_c\| = \|\phi_+\| + \|\phi_-\| \geq \|f_+\| + \|f_-\|,$$

we obtain

$$\|f\| = \|f_+\| + \|f_-\|.$$

If $f = \rho_+ - \rho_-$ is another decomposition, then $f_c = (\rho_+)_c - (\rho_-)_c$ and $\|f_c\| = \|f\| = \|\rho_+\| + \|\rho_-\| = \|(\rho_+)_c\| + \|(\rho_-)_c\|$. By the uniqueness in the complex case, we have $\phi_{\pm} = (\rho_{\pm})_c$ and hence $f_{\pm} = \rho_{\pm}$.

□

The decomposition of a hermitian functional $f \in A'$ as $f = f_+ - f_-$ is called the *Jordan decomposition* of f . We observe that in a real C*-algebra A , the set A'_h of hermitian functionals is the real linear span of the set A'_+ of positive functionals. This is different from a complex C*-algebra B for which the whole B^* is the complex linear span of B^*_+ .

Definition 2.3.19. Let A be a real C*-algebra. We define the *real pure state space* of A to be the set $P(A)$ of extreme points of the real state space $S(A)$: $P(A) = \text{ex}S(A)$, equipped with the weak*-topology. Each $\rho \in P(A)$ is called a *real pure state* of A .

We refer to [31, Proposition 5.3.2] for a proof of the following result.

Proposition 2.3.20. *Let A be a unital abelian real C*-algebra. We identify A with $C(X, \sigma)$, where X is a compact Hausdorff space and $\sigma : X \rightarrow X$ is a homeomorphism of period 2. Then the real state space of A is given by*

$$S(A) = \{\mu \in C(X, \sigma)' : \|\mu\| = 1 = \mu(1), \mu(f) = \mu(f \circ \sigma) \forall f \in C(X, \sigma)\}$$

and its pure real state space is

$$P(A) = \left\{ \frac{1}{2}(\delta_x + \delta_{\sigma(x)}) : x \in X \right\}$$

where δ_a denotes the point mass at $a \in X$.

In contrast to complex C*-algebras, a real pure state of a real abelian C*-algebra is not always multiplicative. Indeed, for $\rho \in P(C(X, \sigma))$ and $f \in C(X, \sigma)$ we have $\rho(f) = \text{Re}(f(x))$ for some $x \in X$ which implies that ρ need not be multiplicative. We also note that when σ is the identity map $id : X \rightarrow X$, then $C(X, id) = C(X, \mathbb{R})$ and $P(C(X, id)) = P(C(X, \mathbb{R})) = \{\delta_x : x \in X\}$. The real pure states are multiplicative in this case.

As in the case of complex C*-algebras, a positive functional of a real C*-algebra induces a representation of the algebra via the Gelfand-Naimark-Segal construction.

Definition 2.3.21. A real linear subspace J of a real C*-algebra A is called a *left ideal* if $AJ \subset J$. It is called a *right ideal* if $JA \subset J$. A left (or right) ideal is called *maximal* if it is not properly contained in any other left (or right) ideal.

Let A be a real C*-algebra. For a positive functional $f : A \rightarrow \mathbb{R}$ we define the *left kernel* L_f by

$$L_f = \{a \in A : f(a^*a) = 0\}$$

which is a left ideal of A by the Schwarz inequality. We define an inner product on the quotient space A/L_f by

$$\langle \tilde{a}, \tilde{b} \rangle = f(b^*a)$$

where $\tilde{a} = a + L_f$ and $\tilde{b} = b + L_f$ belong to A/L_f . The completion of A/L_f with respect to this inner product is denoted by H_f which is a real Hilbert space. For

$a \in A$, we define a continuous linear map $\pi_f(a) : A/L_f \rightarrow A/L_f$ by

$$\pi_f(a)\tilde{b} = \tilde{ab} \quad (\tilde{b} \in A/L_f).$$

We can extend $\pi_f(a)$ uniquely to a bounded linear operator on H_f , still denoted by $\pi_f(a)$.

Definition 2.3.22. Let A be a real C*-algebra. The pair $\{\pi, H\}$ is called a **-representation* of A if H is a real Hilbert space and $\pi : A \rightarrow B(H)$ is a **-homomorphism*.

The following well-known representation of a real C*-algebra can be proved as in the complex case, see, for example, [31, p.42-43].

Theorem 2.3.23. (*GNS representation*) Let A be a real C*-algebra and let $f : A \rightarrow \mathbb{R}$ be a positive functional. Then $\{\pi_f, H_f\}$ is a **-representation* of A such that

$$f_b(a) = \langle \pi_f(a)\tilde{b}, \tilde{b} \rangle \quad (a, b \in A).$$

Let A have identity. Then $\xi_f = \tilde{1}$ is a cyclic vector in H_f and

$$f(a) = \langle \pi_f(a)\xi_f, \xi_f \rangle \quad (a \in A).$$

Definition 2.3.24. Let A be a real C*-algebra. A representation $\{\pi, H, \xi\}$ is called *faithful* if for all $x \in A$, $\pi(x^*x) = 0$ implies $x = 0$. A positive functional $f \in A'$ is called *faithful* if for all $x \in A$ with $x \neq 0$ we have $f(x^*x) > 0$.

If f is faithful, then the GNS-representation π_f is faithful.

2.4 Real W^* -algebras

We now introduce real W^* -algebras. We present new results that will be used in Chapter 3. For a detailed analysis of real W^* -algebras, we refer to [25] and [31].

We recall that the weak*-topology on a real or complex Banach space X with predual X_* is defined to be the smallest topology for which all the mappings $x \mapsto x(f)$ are continuous for all $f \in X_*$ and $x \in X$. We denote the weak*-topology by $\sigma(X, X_*)$ or simply w^* if X_* is understood.

Definition 2.4.1. Let A be a real C^* -algebra. We call A a *real W^* -algebra* if A is linearly isometric to the dual space E' of a real Banach space E , in which case, E is unique up to linear isometric isomorphism.

In the initial definition of real W^* -algebras the condition "multiplication in A is separately $\sigma(A, E)$ -continuous" was included in literature but was proved superfluous in [25]. We denote the (unique) predual of a (real or complex) C^* -algebra A by A_* and there is no ambiguity of speaking of *the* weak*-topology on a (real or complex) W^* -algebra. By [9, Section 1 and Section 2], we have the following results.

Theorem 2.4.2. *Let A be a real C^* -algebra. Then its second dual A'' is a real W^* -algebra.*

Theorem 2.4.3. *Let A be a real W^* -algebra. Then its complexification A_c is a W^* -algebra. Moreover A is $\sigma(A_c, A_{c*})$ -closed in A_c and for $a, a_\alpha \in A$*

$$\sigma(A, E) - \lim_{\alpha} a_\alpha = a \Leftrightarrow \sigma(A_c, A_{c*}) - \lim_{\alpha} a_\alpha = a.$$

We note that we have $A_c^{**} = A'' + iA''$.

Corollary 2.4.4. *Every real W^* -algebra has an identity.*

We will always denote the identity of a real W^* -algebra by 1.

Corollary 2.4.5. *Let A be a real C^* -algebra. Then A is a real W^* -algebra if and only if it can be faithfully represented as a weak*-closed real *-subalgebra of $B(H)$ for some real Hilbert space H , where the predual of $B(H)$ is the Banach space $T(H)$ of trace class operators on H .*

Definition 2.4.6. For a subset S of a real W^* -algebra A we define the *commutant* of S to be the algebra

$$S^c = \{b \in A : ab = ba \text{ for all } a \in S\}.$$

The set $Z(S) = S \cap S^c$ is called the *centre* of S . We have $Z(A) = A^c$. If $Z(A) = \{\alpha 1 : \alpha \in \mathbb{R}\}$, then we call A a *factor*.

Like the complex W^* -algebras, real W^* -algebras contain plenty of projections.

Definition 2.4.7. Let A be a real W^* -algebra. An element $p \in A$ is called a *projection* if $p = p^* = p^2$.

We denote by

$$P_A = \{p \in A : p = p^* = p^2\}$$

the set of projections in A . A projection $p \in A$ is called *abelian* if the W^* -algebra pAp is abelian. A projection $p \in A$ is called *central* if it belongs in $Z(A)$. Two projections p and q in A are called *orthogonal* if $pq = 0$. A projection p is *majorised* by q , i.e. $p \leq q$ in the partial order of the cone A_+ , if and only if $p(1 - q) = 0$.

Definition 2.4.8. Let A be a real W^* -algebra. A is called *discrete* or of *Type I* if for any non-zero central projection $p \in A$, there exists a non-zero abelian projection $q \in A$ such that $q \leq p$.

Definition 2.4.9. Let A be a real W^* -algebra. An element $u \in A$ is called a *partial isometry* if u^*u is a projection. We call u^*u the *initial projection* of u .

It follows that uu^* is also a projection in A and we call uu^* the *final projection* of u .

Given an element $a \in A$, the smallest projection $p \in A$ such that $pa = a$ is called the *left support* of a and is denoted by $s_l(a)$. Likewise, we define the *right support*, in symbols $s_r(a)$, of a to be the smallest projection $q \in A$ such that $aq = a$. It is immediate that if a is hermitian, then the left support $s_l(a)$ and right support $s_r(a)$ are the same and we denote them by $s(a)$. Using the complex result [38, Proposition 2.2.9] and Theorem 2.4.3, we deduce the following lemma.

Lemma 2.4.10. *Let A be a real W^* -algebra and let $a \in A$. Then there is a unique partial isometry $u \in A$ with the property $a = u|a|$ where $|a| = (a^*a)^{\frac{1}{2}}$ and uu^* is the left support of a .*

We call $a = u|a|$ the *polar decomposition* of a .

Lemma 2.4.11. *Let A be a real W^* -algebra and let A_* be its predual. Let $a \in A$ be hermitian. Then there exists a family of spectral projections $\{e(\lambda) \in A : \lambda \in \mathbb{R}\}$ such that*

$$a = \int_{-\infty}^{+\infty} \lambda de(\lambda) = \int_{-\|a\|}^{+\|a\|} \lambda de(\lambda)$$

where the spectral integral converges in the $\sigma(A, A_*)$ -topology.

Proof. See [9, Proposition 3.9] or [31, Proposition 4.3.4(3)]. \square

We note from the above lemma that in a real W^* -algebra A , the set A_h of hermitian elements is the norm closure $\overline{[P_A]}_{\mathbb{R}}$ of the real linear span of projections. This is different from a complex W^* -algebra B , where the whole algebra is the norm closure of the complex linear span of the set of projections.

Taking into consideration that $P_A \subset [U(A)]_{\mathbb{R}}$, where $U(A) = \{u \in A : u^*u = uu^* = 1\}$ is the set of unitary elements (since $P_A \ni p = \frac{1}{2}(2p - 1) + \frac{1}{2}1 \in [U(A)]_{\mathbb{R}}$), and the fact that $A_{sh} \subset [U(A)]_{\mathbb{R}}$ (cf. Section 2.2), we see that $A = \overline{[U(A)]}_{\mathbb{R}}$.

We now discuss the equivalence relation of projections. Let A be a real W^* -algebra. Two projections p and q in A are said to be *equivalent* if there exists a partial isometry $u \in A$ such that $u^*u = p$ and $uu^* = q$. We denote this by $p \sim q$.

If p is not equivalent to q , we write $p \not\sim q$.

In the real W^* -algebra $B(H)$ of bounded operators on a real Hilbert space H , two projections $p, q \in B(H)$ are equivalent if and only if $\dim p(H) = \dim q(H)$.

If a projection p is equivalent to a projection p_1 ($p \sim p_1$) with $p_1 \leq q$, then we write $p \preceq q$. If, in addition, $p \not\sim q$, then we write $p \prec q$.

Definition 2.4.12. Let A be a real W^* -algebra. The *central support* $c(p)$ of a projection $p \in A$ is the smallest central projection in A majorising p .

Letting $A \subset B(H)$ for some real Hilbert space H , we note that $c(p)$ is the projection from H onto $\overline{[ApH]}_{\mathbb{R}}$.

We have the following relations regarding projections in a real W^* -algebra (see [31, Section 4.3 and Section 4.4]).

Lemma 2.4.13. Let A be a real W^* -algebra on a real Hilbert space H . Then the supremum $\bigvee_{i \in I} p_i$ of a family $\{p_i\}_{i \in I}$ of projections in A exists with respect to the ordering induced by the cone A_+ . Further, we have

$$\bigvee_{i \in I} p_i : H \rightarrow \overline{[\bigcup_{i \in I} p_i H]}.$$

is the projection onto the closed linear span $\overline{[\bigcup_{i \in I} p_i H]}$ of $\bigcup_{i \in I} p_i H$.

Lemma 2.4.14. Given that $\{p_i\}_{i \in I}$ and $\{q_i\}_{i \in I}$ are two orthogonal families of projections in A with $p_i \sim q_i$ for all $i \in I$, we have $\sum_{i \in I} p_i \sim \sum_{i \in I} q_i$ and $\bigvee_{i \in I} p_i \sim \bigvee_{i \in I} q_i$

Lemma 2.4.15. Let A be a real W^* -algebra. Let p, q be projections in A . Then the following statements hold.

- (i) $p \preceq q$ and $q \preceq p$ imply $p \sim q$.
- (ii) $p \preceq q$ implies $c(p) \leq c(q)$ and $p \sim q$ implies $c(p) = c(q)$.
- (iii) Let $q \leq p$. Then $c(q)p$ is the central support of q in pAp .

For the remainder of this section, A denotes a real W*-algebra with predual A_* . We note that the predual A_* identifies with a subspace of the dual A' . A functional $\phi \in A'$ is called *normal* if it falls into A_* . In fact, $\phi \in A_*$ if and only if ϕ is $\sigma(A, A_*)$ -continuous. We write

$$\langle a, \phi \rangle = \phi(a) \quad (a \in A, \phi \in A_*).$$

Similar to complex W*-algebras, a positive normal functional $\phi : A \rightarrow \mathbb{R}$ is *completely additive* (cf.[31, Theorem 4.5.3]), that is, for any orthogonal family $\{p_i\}_{i \in I}$ of projections in A we have

$$\phi\left(\sum_{i \in I} p_i\right) = \sum_{i \in I} \phi(p_i).$$

Given a real C*-algebra A , we denote

$$\begin{aligned} A'_h &= \{\phi \in A' : \phi^* = \phi\}; \\ A'_+ &= \{\phi \in A' : \phi \geq 0\}. \end{aligned}$$

Let A be a real W*-algebra. The left translation of a functional $\phi \in A_*$ by $b \in A$ is the mapping $a \in A \rightarrow \phi(ba)$ and is denoted by $b\phi$. We note that $\|b\phi\| \leq \|b\|\|\phi\|$.

Similarly we define the right translation of a functional $\phi \in A_*$ by $b \in A$ to be the mapping $a \in A \rightarrow \phi(ab)$ and is denoted by ϕb .

We denote by ϕ_b the functional $b^*\phi b$ and note that the mapping $\phi \in A' \rightarrow \phi_b \in A'$ is norm continuous.

Definition 2.4.16. We define the *support* of a functional $\phi \in A'_+ \cap A_*$ to be the least projection $e \in A$ such that $\phi = \phi_e$. We write $s(\phi)$ for the support of ϕ .

Remark 2.4.17. Clearly since $1 \in A$ every functional $\phi \in A'_+ \cap A_*$ has a support. We observe that $\phi(e) = \phi(1) = \|\phi\|$ and $\phi(1 - e) = 0$. We note that by [9, Proposition 3.4], two functionals are *orthogonal* if and only if they have

orthogonal supports. For two orthogonal functionals $\phi, \theta \in A'_+ \cap A_*$, we have $\|\phi + \theta\| = \|\phi\| + \|\theta\|$ and vice versa.

If $\phi, \theta \in A'_+ \cap A_*$ satisfies $\theta \leq \phi$, then $s(\theta) \leq s(\phi)$.

Moreover, for $\phi = \sum_{i=1}^{\infty} \phi_i$ where the sum converges in norm and $\phi_i \in A'_+ \cap A_*$, we have

$$s(\phi) = s\left(\sum_{i=1}^{\infty} \phi_i\right) = \bigvee_{i=1}^{\infty} s(\phi_i).$$

For the Jordan decomposition $\phi = \phi^+ - \phi^-$ of a hermitian normal functional ϕ on A , we have $\phi^+, \phi^- \in A'_+ \cap A_*$ with orthogonal supports and $\|\phi\| = \|\phi^+\| + \|\phi^-\|$. If ϕ is a positive functional on A , then we have $\|\phi\| = \phi(1)$. In fact, we have the following result.

Lemma 2.4.18. *Let ϕ be a normal functional on a real W^* -algebra A . Then ϕ is positive if and only if $\phi(a) = \|\phi\|$ for some $a \in A_+$ with $\|a\| \leq 1$.*

Proof. This follows from Lemma 2.3.18. □

The proof of the following lemma is the same as in the complex case (see for example [50, Lemma III.4.1]) and we present it below for completeness.

Lemma 2.4.19. *Let ϕ be a normal functional on a real W^* -algebra A and $e \in A$ be a projection. Then $\|e\phi\| = \|\phi\|$ implies $e\phi = \phi$.*

Proof. Assume not. Without loss of generality, we can take $\|\phi\| = 1$. Let $f = 1 - e$. We can choose $a, b \in A$ with $\|a\|, \|b\| \leq 1$ such that $e\phi(a) = 1$ and $f\phi(b) = \delta$ with $\delta > 0$. Then

$$\begin{aligned} \|ea + \delta fb\|^2 &= \|(ea + \delta fb)^*(ea + \delta fb)\| \\ &= \|(ea)^*(ea) + \delta^2(fb)^*(fb)\| \\ &\leq 1 + \delta^2. \end{aligned}$$

We have $\phi(ea + \delta fb)^2 = (e\phi)(a)^2 + 2\delta(e\phi)(a)(f\phi)(b) + \delta^2(f\phi)(b)^2 = 1 + 2\delta^2 + \delta^4 \geq 1 + \delta^2$ which contradicts the assumption that $\|\phi\| = 1$. □

Lemma 2.4.20. *Let $\omega, \eta \in A'_+ \cap A_*$ with $\omega(1) = \eta(1)$ and suppose $|\omega(a)|^2 \leq \eta(a^*a)$ for all $a \in A$. Then $\omega = \eta$.*

Proof. For any $x^* = x$ and $\varepsilon > 0$ we have $\omega(1 + \varepsilon x)^2 \leq \eta((1 + \varepsilon x)^2)$. By $\omega(1) = \eta(1)$, we have $2\omega(x) + \varepsilon\omega(x)^2 \leq 2\eta(x) + \varepsilon\eta(x^2)$. Since this is true for all ε we must have $\omega(x) \leq \eta(x)$. It follows that $(\eta - \omega)(a^*a) \geq 0$ for all $a \in A$. We then have that the functional $\eta - \omega$ is positive and by Lemma 2.4.18 $\|\eta - \omega\| = (\eta - \omega)(1) = \eta(1) - \omega(1) = 1 - 1 = 0$. It follows that $\omega = \eta$. \square

As in the complex case, we also have a polar decomposition of normal functionals in real W^* -algebras. This fact is often stated in literature without proof (see for example [31, Theorem 4.5.6] and [9, Proposition 3.7]). We give a proof for completeness, and in fact, our uniqueness arguments are different from those in complex C^* -algebras (cf. [50, Theorem III.4.2]).

Proposition 2.4.21. *Let A be a real W^* -algebra. Let $\phi \in A_*$ be a normal functional on A . Then there exists a unique positive normal functional $\eta \in A'_+ \cap A_*$ and partial isometry $v \in A$ such that $v^*v = s(\eta)$ and $\phi = v\eta$. Moreover, $\|\eta\| = \|\phi\|$ and $|\phi(a)|^2 \leq \|\eta\|\eta(a^*a)$ for all $a \in A$. In addition $\eta = v^*\phi$.*

Proof. We may assume that $\|\phi\| = 1$.

By the $\sigma(A, A_*)$ -continuity of ϕ and the weak*-compactness of the closed unit ball in A' , there exists an element $a \in A$ with $\|a\| \leq 1$ such that $\langle a, \phi \rangle = \|\phi\|$. We let $a^* = u|a^*|$ be the polar decomposition of a^* . Then we have

$$\|\phi\| = \langle a, \phi \rangle = \langle |a^*|u^*, \phi \rangle = \langle |a^*|, u^*\phi \rangle.$$

We let $\eta = u^*\phi$. Then

$$\|\eta\| \leq \|\phi\| = \langle |a^*|, u^*\phi \rangle = \langle |a^*|, \eta \rangle \leq \|\eta\|.$$

Hence $\|\eta\| = \|\phi\|$. Since $|a^*| \leq 1$ and $|a^*| \in A_+$, it follows by Lemma 2.4.18 that η is positive.

Let $e = uu^*$. Then $u\eta = uu^*\phi = e\phi$ and

$$\|\phi\| = \langle a, \phi \rangle = \langle ae, \phi \rangle = \langle a, e\phi \rangle \leq \|e\phi\| \leq \|\phi\|.$$

Hence $\|\phi\| = \|e\phi\|$ and by Lemma 2.4.19 we obtain $e\phi = \phi$ and $\phi = u\eta$.

Since $u^*u\eta = u^*\phi = \eta$ we have $s(\eta) \leq u^*u$. Therefore $v = us(\eta)$ is a partial isometry in A with $\phi = v\eta$ and $v^*v = s(\eta)$.

By the Schwarz inequality,

$$|\phi(x)|^2 = |\eta(vx)|^2 \leq \eta(vv^*)\eta(x^*x) \leq \|\eta\|\eta(x^*x) = \eta(x^*x).$$

We now show that this decomposition is unique. Let $\psi \in A'_+ \cap A_*$, with norm $\|\psi\| = 1$, be another functional that satisfies all conditions of the proposition and $\phi = w\psi$ with $w^*w = s(\psi)$. Then

$$|\eta(x)|^2 = |\phi(v^*x)|^2 \leq \|\psi\|\psi(x^*vv^*x) \leq \psi(x^*x)$$

and by Lemma 2.4.20, we obtain that $\psi = \eta$.

Moreover, $\eta = v^*\phi = v^*w\psi = v^*w\eta$ and then $\eta(1) = \eta(v^*w)$. Hence $v^*w = s(\eta) = v^*v$. Likewise, $w^*v = s(\psi) = w^*w$. It follows that $v^*v = w^*w$. We have

$$\begin{aligned} w &= ww^*w = ww^*v \\ ww^* &= ww^*vv^*ww^* \\ 0 &= ww^*(1 - vv^*)ww^* \\ 0 &= (1 - vv^*)ww^* \\ ww^* &= vv^*ww^* = vv^* \end{aligned}$$

which implies that $w = ww^*w = vv^*w = vv^*v = v$. □

We denote η by $|\phi|$ and the expression $\phi = v|\phi|$ is called the *polar decomposition* of ϕ .

2.5 Jordan structures in real C*-algebras

One of our goals is to show that the isometries of real C*-algebras preserve a specific product, known as the Jordan triple product. For this purpose we present in this section the basic theory of Jordan triple structures related to C*-algebras as well as the results that will be used to obtain our main theorem in Chapter 4.

A *complex JB*-triple* is a complex Banach space A equipped with a triple product $\{\cdot, \cdot, \cdot\} : A \times A \times A \rightarrow A$ which is linear and symmetric in the outer variables, conjugate linear in the middle one and satisfies the following conditions:

(i) (Jordan identity) for $a, b, x, y, z \in A$,

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}; \quad (2.5.1)$$

(ii) $D(a, a) : A \rightarrow A$ is a Hermitian linear operator with non-negative spectrum, where $D(a, b)(x) = \{a, b, x\}$ with $a, b, x \in A$;

(iii) $\|\{x, x, x\}\| = \|x\|^3$ for all $x \in A$.

A complex subspace B of A is called a *subtriple* if it is closed with respect to the triple product, that is, $x, y, z \in B$ implies $\{x, y, z\} \in B$. A real subtriple of A is a real subspace of A which is closed with respect to the triple product. We define a *real JB*-triple* to be a closed real subtriple of a complex JB*-triple. It has been shown in [26] that a real JB*-triple can be complexified to a complex JB*-triple. A complex (real) C*-algebra A is a complex (real) JB*-triple in the triple product:

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a) \quad (a, b, c \in A).$$

We will always assume the above triple product in a real or complex C*-algebra. We note that one basic difference between complex and real JB*-triples is the polarization of the triple product. For a complex JB*-triple B , we have the following

polarization formulas:

$$4\{a, x, a\} = \sum_{k=0}^3 (-1)^k \{i^k a + x, i^k a + x, i^k a + x\} \quad (2.5.2)$$

$$2\{a, x, b\} = \{a + b, x, a + b\} - \{a, x, a\} - \{b, x, b\} \quad (a, b, x, y \in B).$$

Thus the triple product in a complex JB*-triple is uniquely determined by the cube product $x^{(3)} = \{x, x, x\}$.

This is not the case for a real JB*-triple A in which the triple product is determined by the product $\{a, x, a\}$ but not the cube product

$$2\{a, x, b\} = \{a + b, x, a + b\} - \{a, x, a\} - \{b, x, b\} \quad (a, b, x \in A). \quad (2.5.3)$$

In contrast to (2.5.2), we have

$$2\{a, x, a\} + 4\{a, a, x\} = (a + x)^3 + (x - a)^3 - 2x^3 \quad (a, x \in A). \quad (2.5.4)$$

We define below the tripotents in complex (real) JB*-triples which are exactly the partial isometries in complex (real) C*-algebras and play a key role in the characterisation of linear isometries on C*-algebras. For details of the results below see [13] and [40].

Definition 2.5.1. An element e in a complex or real JB*-triple A is called a *tripotent* if $\{e, e, e\} = e$.

Two tripotents $e, f \in A$ are called *orthogonal* if $\{e, f, x\} = 0$ for all $x \in A$, in which case we also have $\{f, e, x\} = 0$ for all $x \in A$.

An element e in A is called a *minimal tripotent* if it cannot be written as the sum of two orthogonal non-zero tripotents.

We note that for a complex JB*-triple A , a tripotent e is minimal if and only if for all $x \in A$, we have $\{e, x, e\} = \lambda e$ for some $\lambda \in \mathbb{C}$. If A is a real JB*-triple then e is minimal if and only if for all $x \in A$, we have $\{e, x, e\} = \lambda e$ for some $\lambda \in \mathbb{R}$.

Let A be a real or complex JB*-triple. For a tripotent $e \in A$, we define the operators $D(e) : A \rightarrow A$ and $Q(e) : A \rightarrow A$ by

$$D(e)(x) = D(e, e)(x) \text{ and } Q(e)(x) = \{e, x, e\} \quad (x \in A).$$

The Peirce projections associated with the tripotent e are defined by

$$\begin{aligned} P_2(e) &= Q(e)^2 \\ P_1(e) &= 2(D(e) - Q(e)^2) \\ P_0(e) &= I - 2D(e) + Q(e)^2 \end{aligned}$$

where $P_0(e) + P_1(e) + P_2(e) = I$. Moreover, if we denote by $A_j(e)$ the space $P_j(e)(A)$, then $A_j(e)$ is the kernel of $D(e) - \frac{j}{2}I$ for $j = 0, 1, 2$. We also have the following Peirce multiplication rules:

- (i) $\{P_i(e), P_j(e), P_k(e)\} \subset P_{i-j+k}(e)$ if $i - j + k \in \{0, 1, 2\}$;
- (ii) $\{P_i(e), P_j(e), P_k(e)\} = \{0\}$ if $i - j + k \notin \{0, 1, 2\}$;
- (iii) $\{P_0(e), P_2(e), P_i(e)\} = \{P_2(e), P_0(e), P_i(e)\} = 0$ for all $i = 0, 1, 2$.

A real or complex JB*-triple is called a *JBW*-triple* if it has a (unique) predual. Given a tripotent in a JBW*-triple A , the space $P_2(e)A$ is also a JBW*-triple. We now prove that minimal tripotents in real C*-algebras give rise to minimal projections. A projection p in a real C*-algebra A is called *minimal* if for any projection $e \in A$ that satisfies $e = ep = pe$, we have $e = 0$ or $e = p$.

Lemma 2.5.2. *Let u be a minimal tripotent in a real C*-algebra B regarded as a real JB*-triple. Then $p = u^*u$ and $q = uu^*$ are minimal projections in B .*

Proof. It is immediate that if u is a tripotent, then $p = u^*u$ and $q = uu^*$ are projections. Now let e be a projection such that $e = ep = pe$. We show that $e = p$ or $e = 0$. It follows from minimality of u that $\{u, ue, u\} = \lambda u$ for some $\lambda \in \mathbb{R}$. This gives $\lambda u = u(ue)^*u = ueu^*u = u(u^*ueu^*u) = u(pep) = ue$

and $\lambda p = \lambda u^*u = u^*ue = pe = e$. Multiplying by e on the left side yields $e = \lambda ep = \lambda e$. Hence $e = 0$ or $\lambda = 1$. The latter implies $e = p$.

Similarly, q is also a minimal projection. \square

The following lemmas will be used in Section 4.3 on isometries between real C^* -algebras.

Lemma 2.5.3. *Let $\{p_\phi\}_{\phi \in Q}$ be a family of projections in a real W^* -algebra W . Let $p = \bigvee_{\phi \in Q} p_\phi$. If $ap_\phi = 0$ for some $a \in W$ and all $\phi \in Q$, then $ap = 0$.*

Proof. Let $W \subseteq B(H)$ for some real Hilbert space H . Then by the definition of p and Lemma 2.4.13, we have $p(H) = \overline{\bigcup_{\phi \in Q} p_\phi(H)}$, which is the closed real linear span of $\bigcup_{\phi \in Q} p_\phi(H)$ in H . Let $\xi \in H$. Then

$$p\xi = \lim_{n \rightarrow \infty} \xi_n$$

with $\xi_n = \sum_{m=1}^{k_n} \lambda_m^n p_{\phi_m}(h_m)$ for some $\lambda_m^n \in \mathbb{R}$, $\phi_m \in Q$ and $h_m \in H$. We have $a\xi_n = \sum_{m=1}^{k_n} \lambda_m^n ap_{\phi_m}(h_m) = 0$ for every n . Hence

$$ap\xi = \lim_{n \rightarrow \infty} a\xi_n = 0$$

and it follows that $ap = 0$. \square

Remark 2.5.4. For any two minimal projections p_ϕ, q_ϕ in a real W^* -algebra W , their central supports $c(p_\phi)$ and $c(q_\phi)$ are either equal or orthogonal. Indeed, if $p_\phi c(q_\phi) = 0$, then $p_\phi(1 - c(q_\phi)) = p_\phi$ and $c(p_\phi) \leq 1 - c(q_\phi)$ which implies $c(p_\phi)c(q_\phi) = 0$. Likewise if $q_\phi c(p_\phi) = 0$, then $c(q_\phi)c(p_\phi) = 0$. Now if $p_\phi c(q_\phi) \neq 0 \neq q_\phi c(p_\phi)$, then $p_\phi c(q_\phi)$ is a projection and $p_\phi c(q_\phi) \leq p_\phi$ which gives $p_\phi = p_\phi c(q_\phi)$ and $c(p_\phi) \leq c(q_\phi)$. Similarly we obtain $c(q_\phi) \leq c(p_\phi)$ and therefore $c(p_\phi) = c(q_\phi)$. Now if $p_\phi = u_\phi^*u_\phi$ and $q_\phi = u_\phi u_\phi^*$ for a minimal tripotent u_ϕ , then p_ϕ and q_ϕ are equivalent projections and their central supports are equal since if their central supports were orthogonal, then we would have $u_\phi^* \in p_\phi W q_\phi = c(p_\phi)p_\phi W q_\phi c(q_\phi) = 0$ which is impossible. We note that $Wc(p_\phi) = Wc(q_\phi)$ is a

type I factor and can therefore be represented as the full operator algebra $B(K)$ on some real Hilbert space K (cf. [31, p.188]). To simplify notation, we will denote the central supports $c(p_\phi) = c(q_\phi)$ by $c(\phi)$.

Lemma 2.5.5. *Let $\{p_\phi\}_{\phi \in Q}$ and $\{q_\phi\}_{\phi \in Q}$ be two families of distinct minimal projections in a W*-algebra W with $p_\phi \sim q_\phi$ for all $\phi \in Q$. For each $\phi \in Q$, we let*

$$p_{c(\phi)} = \bigvee_{\substack{\psi \in Q \\ c(p_\psi) = c(\phi)}} p_\psi \quad \text{and} \quad q_{c(\phi)} = \bigvee_{\substack{\psi \in Q \\ c(q_\psi) = c(\phi)}} q_\psi.$$

Then $p_{c(\phi)} \sim q_{c(\phi)}$ in $Wc(\phi) \subset W$.

Proof. Let $\psi \in Q$. Then by Remark 2.5.4, $c(p_\psi) = c(q_\psi)$ since p_ψ is equivalent to q_ψ . Hence $c(p_\psi) = c(\phi)$ iff $c(q_\psi) = c(\phi)$. We have $p_\psi, q_\psi \in Wc(\phi)$ for $c(p_\psi) = c(\phi)$. Identifying $Wc(\phi)$ with $B(K)$ for some real Hilbert space K , we have $\dim p_\psi(K) = \dim q_\psi(K) = 1$ since p_ψ and q_ψ are minimal projections. We note that for $\psi, \psi' \in Q$, we have $p_\psi(K) \neq p_{\psi'}(K)$ since p_ψ and $p_{\psi'}$ are distinct minimal projections. The same applies to q_ψ and $q_{\psi'}$. It follows that

$$\dim p_{c(\phi)}(K) = \dim \left[\overline{\bigcup_{\substack{\psi \in Q \\ c(p_\psi) = c(\phi)}} p_\psi(K)} \right] = \dim \left[\overline{\bigcup_{\substack{\psi \in Q \\ c(q_\psi) = c(\phi)}} q_\psi(K)} \right] = \dim q_{c(\phi)}(K).$$

Hence $p_{c(\phi)} \sim q_{c(\phi)}$. □

A fruitful technique of studying isometries between C*-algebras is to study the behavior of the isometries on the extreme points of the dual ball. We will use this technique in Section 4.3 to obtain our main result on isometries between real C*-algebras. To facilitate this, we present below some essential results concerning the structure of the dual of a real JB*-triple.

If B is a real JB*-triple, then its second dual B'' is also a real JB*-triple. Moreover, B is the real form of a complex JB*-triple B_c , that is, there is a conjugate linear isometry $\tau : B_c \rightarrow B_c$ of period 2 such that $B = \{b \in B_c : \tau(b) = b\}$. Further, the dual map $\sigma = \tau^{**} : B_c^{**} \rightarrow B_c^{**}$ is a conjugate linear isometry of period 2 and

$B'' = \{b \in B_c^{**} : \sigma(b) = b\}$, where the dual map $\tau^* : B_c^* \rightarrow B_c^*$ is defined by

$$\tau^*(\phi)(b) = \overline{\phi(\tau(b))} \quad (\phi \in B_c^*, b \in B_c)$$

and τ^* is a conjugate linear isometry of period 2. We denote $\sigma_* = \tau^*$. We observe that if u is a tripotent in B_c^{**} , then $\sigma(u)$ is also a tripotent in B_c^{**} . In fact, letting $U(B_c^{**})$ be the set of all tripotents in B_c^{**} , the set $U(B'')$ of tripotents in B'' is $U(B_c^{**}) \cap B'' = \{u \in U(B_c^{**}) : \sigma(u) = u\}$.

For a functional $\phi \in B_c^*$, there exists a unique tripotent $u_\phi \in B_c^{**}$, called the *support tripotent* of ϕ , such that $\phi = \phi \circ P_2(u_\phi)$ and $\phi|_{P_2(u_\phi)(B_c^{**})}$ is a faithful normal positive functional on the JBW*-triple $P_2(u_\phi)(B_c^{**})$. We note that if $u_\phi \in B_c^{**}$ is the support tripotent of $\phi \in B_c^*$, then $\sigma(u_\phi)$ is the support tripotent of $\sigma_*(\phi)$. On the other hand, if ϕ is in B' , its complex extension $\phi_c \in B_c^*$ has support tripotent $u_{\phi_c} \in B_c^{**}$ such that $\sigma(u_{\phi_c}) = u_{\phi_c}$ since $\sigma_*(\phi_c) = \phi_c$. Hence $u_{\phi_c} \in U(B'')$ and we call it the *support tripotent* of ϕ in B'' , denoted by u_ϕ . Finally, we note that in the case where ϕ is an extreme point of B'_1 , the corresponding support tripotent u_ϕ is minimal [40, Corollary 2.1]. It also follows that, as $\phi(u_\phi) = 1$,

$$\{u_\phi, x, u_\phi\} = \phi(x)u_\phi \quad (2.5.5)$$

and $\phi(x) = 0$ if and only if $\{u_\phi, x, u_\phi\} = 0$ since $\phi(x) = \phi(\{u_\phi, x, u_\phi\})$ and $\{u_\phi, x, u_\phi\} = \lambda u_\phi$ for some $\lambda \in \mathbb{R}$. Based on these observations and the complex results in [3, Proposition 1.2] we have the following lemma (see also [39]).

Lemma 2.5.6. *Let B'' be the second dual of a real C^* -algebra B and let $\phi \in B'$ with $\|\phi\| = 1$. If there exists $u \in B''$ such that $\phi(u) = \|u\| = 1$ then*

(i) $\phi\{x, y, u\} = \phi\{y, x, u\}$;

(ii) $\phi\{x, x, u\} \geq 0$;

(iii) $|\phi\{x, y, u\}|^2 \leq \phi\{x, x, u\}\phi\{y, y, u\}$ (*Cauchy-Schwarz inequality*),

for all $x, y \in B''$.

Lemma 2.5.7. *Let B'' be the second dual of a real C^* -algebra B and let $\phi \in B'$ be an extreme point in the closed unit ball B'_1 . Let $N_\phi = \{b \in B'' : \phi\{b, b, u\} = 0\}$ where u is the support tripotent of ϕ . Then $N_\phi = P_0(u)(B'')$.*

Proof. We note that $\|\phi\| = 1$, and if $b \in P_0(u)(B'')$, then $\{u, u, b\} = 0$. Moreover, $P_2(u)(b) = 0$ implies $\{u, \{u, b, u\}, u\} = 0$ and then $\{u, b, u\} = 0$. Therefore,

$$\begin{aligned} 0 &= \{u, b, \{b, u, u\}\} \\ &= \{\{u, b, b\}, u, u\} - \{b, \{b, u, u\}, u\} + \{b, u, \{u, b, u\}\} \\ &= \{\{u, b, b\}, u, u\}. \end{aligned}$$

Hence we get by Lemma 2.5.6(i) that

$$\phi\{b, b, u\} = \phi\{u, \{u, b, b\}, u\} = \phi\{\{u, b, b\}, u, u\} = 0.$$

We conclude that $b \in N_\phi$.

Conversely let $\phi\{b, b, u\} = 0$. We show $b \in P_0(u)B''$. By the Cauchy-Schwarz inequality, we have $\phi\{u, b, u\} = \phi\{b, u, u\} = 0$ and by (2.5.5),

$$\{u, \{u, u, b\}, u\} = \{u, \{u, b, u\}, u\} = 0$$

which implies that $b \in P_0(u)(B'') \oplus P_1(u)(B'')$. Write $b = b_0 + b_1$ with $b_0 \in P_0(u)(B'')$ and $b_1 \in P_1(u)(B'')$. Since $\phi\{b_0, b_0, u\} = \phi\{b_1, b_0, u\} = \phi\{b_0, b_1, u\} = 0$ by Peirce multiplication rules, we obtain $\phi\{b_1, b_1, u\} = 0$. We aim to show $b_1 = 0$.

Since u supports ϕ , from $\phi\{b_1, b_1, u\} = 0$ we obtain $\{u, \{b_1, b_1, u\}, u\} = 0$. Again by Peirce multiplication rules, $\{b_1, b_1, u\} \in P_2(u)$ which gives $0 = \{b_1, b_1, u\}$ and $b_1 b_1^* u + u b_1^* b_1 = 0$.

Therefore,

$$\begin{aligned}
b_1 b_1^* u + u b_1^* b_1 &= 0 \\
u^* b_1 b_1^* u + u^* u b_1^* b_1 &= 0 \\
u^* b_1 b_1^* u u^* u + u^* u b_1^* b_1 u^* u &= 0 \\
u^* b_1 b_1^* u + u^* u b_1^* b_1 u^* u &= 0 \\
(u^* b_1)(u^* b_1)^* + (b_1 u^* u)^*(b_1 u^* u) &= 0
\end{aligned}$$

which implies $u^* b_1 = 0$ and $b_1 u^* u = 0$.

From $b_1 = P_1(u)b_1 = uu^*b_1 + b_1u^*u$ we conclude $b_1 = 0$ and $b = b_0 \in P_0(u)(B'')$. \square

Chapter 3

Structures of real C^* -algebras

In this chapter, we study the ideal and facial structures in real C^* -algebras and show that these two structures are closely related. Specifically, we show that there exists a one to one correspondence between norm-closed left ideals and weak*-closed faces in real C^* -algebras. This extends the same correspondence in complex C^* -algebras. We do not use complexification to prove these results, however, most arguments are similar to the complex case in [41], apart from the polar decomposition proved in Proposition 2.4.21. Some results concerning faces and ideals in complex C^* -algebras are in fact not true for real C^* -algebras as pointed out in Remark 3.2.13 and Remark 3.3.9. We wish to mention that a paper regarding the structures of faces of real JBW^* -triples has appeared recently in literature [13].

3.1 Ideals in real W^* -algebras

Throughout we let A be a real W^* -algebra with identity 1. Recall that A_h denotes the hermitian elements in A and A_+ the positive elements in A . As usual, A_1 denotes the closed unit ball of A .

Lemma 3.1.1. *Let e be projection in A and a in $A_+ \cap A_1$. Then $a \leq e$ if and only if $ae = ea = a$.*

Proof. Let $a \leq e$. Let $f = 1 - e$ and $b = a^{\frac{1}{2}}$. Then $0 \leq faf \leq fef = 0$ and $0 = faf = fb^2f = (bf)^*(bf)$ which implies $bf = 0$ and $fb = (bf)^* = 0$. It follows that $af = fa = 0$ and $a = ae = ea$. Conversely, assume $a = ae = ea$. Then $a \leq 1$ gives $a = eae \leq e1e = e$. \square

The proof of the next lemma is not included in [41] but we can apply the arguments of [50, Lemma I.10.1(ii)] and we present it below for completeness.

Lemma 3.1.2. *Let $e \in A_+ \cap A_1$. Then e is an extreme point of $A_+ \cap A_1$ if and only if it is a projection.*

Proof. Let e be an extreme point of $A_+ \cap A_1$ and assume for contradiction that e is not a projection. Since $1 \in A$, using functional calculus we can represent e by a continuous function f on a compact Hausdorff space X such that $0 \leq f \leq 1$. By our assumption, f is extreme in $C(X)_+ \cap C(X)_1$ but not a projection. We have $0 < f(x_0) < 1$ for some $x_0 \in X$ and then there exists a compact neighborhood U of x_0 such that $0 < f(x) < 1$ for all $x \in U$. We can select a continuous function g on X such that $g(x_0) = 1$, $g(x) = 0$ for all $x \notin U$ and $0 \leq g(x) \leq 1$ for all $x \in X$. For a small $\varepsilon > 0$ we have $0 \leq f(1 \pm \varepsilon g) \leq 1$ and $fg \neq 0$. Hence $f = \frac{1}{2}(f(1 + \varepsilon g) + f(1 - \varepsilon g))$ which contradicts the extremeness of f .

Conversely, suppose that e is a projection such that $e = \frac{1}{2}(a + b)$ for some $a, b \in A_+ \cap A_1$. We may assume $e \neq 0$. Then $\frac{1}{2}a \leq e$ and by Lemma 3.1.1, $ae = ea = a$. Thus the subalgebra generated by a, b, e is abelian. We can represent e, a, b by continuous functions f, g, h respectively, on a compact Hausdorff space X such that $0 \leq f, g, h \leq 1$ and $f = \frac{1}{2}(g + h)$. Since e is a projection, f is a projection and $f = \chi_U$ where χ is the characteristic function and U is a subset of X . It is immediate that $f = g = h$. Hence $a = b = e$ and e is extreme. \square

Lemma 3.1.3. *Let a be a non-zero positive element of A . Then there exists a non-zero projection e in A and an element b in A_+ such that $ab = ba = e$.*

Proof. We note that the spectrum $\sigma(a)$ of a cannot be $\{0\}$ since a is non-zero and A has no non-zero positive nilpotent elements. Consider the spectral decomposition $a = \int_{\sigma(a)} \lambda de_\lambda$ with $\lambda \geq 0$ (cf. Lemma 2.4.11). Let $b = \int_{\sigma(a)} f(\lambda) de_\lambda$, where $f(\lambda) = \frac{1}{\lambda}$ for $\lambda \neq 0$ and $f(0) = 0$ if $0 \in \sigma(a)$. It follows that $b \in A_+$ and $ab = ba = \int_{\sigma(a)} de_\lambda = e$ is a projection in A . \square

Definition 3.1.4. Let A be a real W^* -algebra and $J \subset A$. A projection $e \in A$ is called a *right support* of J , if $ae = a$ for all $a \in J$.

The proof of the next lemma is similar to [44, Proposition 1.10.1].

Lemma 3.1.5. *Let J be a left ideal of A . Then J is weak*-closed if and only if it contains a right support e . In this case $J = Ae$ and $J \cap J^* = eAe$ and e is unique. The ideal J is two-sided if and only if e is a central projection.*

Proof. Assume that J contains a right support e . Then $J = Ae$ and e is a projection in the W^* -algebra A . Since the multiplication is weak*-continuous in A (cf. Section 2.4) it follows that J is weak*-closed.

Conversely, assume J is a weak*-closed left ideal of A . We let $J^* = \{x^* : x \in J\}$. Then $N = J \cap J^*$ is a real C^* -algebra and since the involution is weak*-continuous, N is a real W^* -subalgebra of A . We let e be the identity in N which is a projection in A . Since J is a left ideal, we have $Ae \subseteq J$. Now, for any $a \in J$ we note that $a^*a \in N$ and $ea^*ae = ea^*a = a^*ae = a^*a$ which implies $(1 - e)a^*a(1 - e) = 0$ and $a(1 - e) = 0$. Thus, $a = ae$ and $J \subseteq Ae$ leading to $J = Ae$.

It is immediate that $J^* = eA$. For $a \in J \cap J^*$ we have $a = ae = ea$ and $J \cap J^* \subset eAe$. Conversely, if $a \in eAe$, then $a = ae = ea$ and hence $a \in J$ and $a \in J^*$.

To prove uniqueness of e let us assume that $J = Ap$ for some projection $p \in A$. Then $p \in J$ and there exists $a \in A$ such that $p = ae$ and $p = p^*p = ea^*ae$ which leads to $ep = pe = p$ and $p \leq e$. Similarly, $e \leq p$ and we conclude that $p = e$.

Now, if J is a two-sided ideal, we have $J = Ae$ and $J^* = Ae'$ for some projections $e, e' \in A$. Then e, e' are both identities of $J \cap J^*$ which gives $e = e'$ and

hence $J = J^*$. Thus J is a real W^* -algebra with identity e . We observe that for $x \in A$, $xe = e(xe) = exe = (ex)e = ex$ and e is central. Conversely, if e is a central projection, then it is immediate that $J = Ae$ is a weak*-closed two-sided ideal in A . \square

We now describe the correspondence between weak*-closed left ideals in a real W^* -algebra A and weak*-closed faces of the cone A_+ .

Definition 3.1.6. A non-empty convex set $F \subseteq A_+$ is called a *face* of A_+ if $\lambda a + (1 - \lambda)b \in F$ for any $a, b \in A_+$ and $0 < \lambda < 1$ implies $a, b \in F$.

Given a linear subspace J of A , the intersection $J \cap A_+$ is a face of A_+ if $0 \leq b \leq a$ implies $b \in J \cap A_+$.

We note that, if F is a face of A_+ , then $0 \in F$ and $a \in F$ implies $\lambda a \in F$ for all $\lambda \geq 0$. This follows from the fact that for $\lambda \geq 1$, we have $a = (1 - \frac{1}{\lambda})0 + \frac{1}{\lambda}(\lambda a)$ which implies $\lambda a \in F$. For $0 \leq \lambda \leq 1$ we have $\lambda a = \lambda a + (1 - \lambda)0 \in F$ since F is convex. We will use the same definition for faces of the positive cone in real C^* -algebras in Section 3.3.

Lemma 3.1.7. *Let $a \in A$. The smallest norm-closed face of A_+ containing a is $\{b \in A_+ : b \leq \lambda a \text{ for some } \lambda > 0\}$.*

Proof. Let $F = \{b \in A_+ : b \leq \lambda a \text{ for some } \lambda > 0\}$. Let $b_1, b_2 \in F$. It follows that $b_1 \leq \lambda_1 a$ and $b_2 \leq \lambda_2 a$ for some $\lambda_1, \lambda_2 > 0$. Then for all $0 < \mu < 1$, we have $x = \mu b_1 + (1 - \mu)b_2 \leq (\mu\lambda_1 + (1 - \mu)\lambda_2)a \in F$ and F is convex.

Let $b \in F$ with $b \leq \lambda a$ for some $\lambda > 0$. Assume $b = \mu b_1 + (1 - \mu)b_2$ with $b_1, b_2 \in A_+$ and $0 < \mu < 1$. Then $b_1 \leq \frac{\lambda}{\mu}a$ and $b_2 \leq \frac{\lambda}{1-\mu}a$. Hence $b_1, b_2 \in F$ and F is a face.

We now show that F is norm-closed. Let $b_n \in F$ with $b_n \rightarrow b$ in norm. We may assume $b \neq 0$. We observe that $b_n \leq \lambda_n a$ for $\lambda_n > 0$. Then for a sufficiently large n , $0 < \frac{\|b\|}{2} \leq \|b_n\| \leq \lambda_n \|a\|$ which implies $\lambda_n \geq \frac{\|b\|}{2\|a\|} > 0$. Hence we can choose a subsequence $\{\lambda_{n_k}\}$ such that $\lambda_{n_k} \rightarrow \lambda > 0$. Therefore $b = \lim b_{n_k} \leq \lim \lambda_{n_k} a = \lambda a$ and $b \in F$.

It is clear that $a \in F$ and thus the smallest norm-closed face S of A_+ containing a is a subset of F . Conversely, let $b \in F$. We may assume that $b \leq \lambda a$ for some $\lambda > 1$. Then $a = (1 - \frac{1}{\lambda})\frac{\lambda a - b}{\lambda - 1} + \frac{1}{\lambda}b \in S$ implies $b \in S$ and $F \subset S$. This completes the proof. \square

We note that Lemma 3.1.7 also holds for real C^* -algebras.

Lemma 3.1.8. *Let J be a weak*-closed left ideal of A and let $F = J \cap A_+$. Then F is a weak*-closed face of A_+ and $J = AF$. If $J = Ae$, then $F = eA_+e$.*

Proof. Since J and A_+ are weak*-closed, $F = J \cap A_+$ is weak*-closed and convex. We let $J = Ae$ with e the right support of J . Then $eA_+e \subset F$. If $a \in F$, then $a = ae = (ae)^* = ea = eae$ and therefore $a \in eA_+e$. Hence $F = eA_+e$.

For $a \in F$ and $b \in A_+$ with $0 \leq b \leq a$ we note by the above that $a = ae = ea$. By Lemma 3.1.1, $a \leq e$ which implies $b \leq e$ and $b = eb = be = ebe$. Hence $b \in F$ and F is a face of A_+ .

Finally, $AF \subset J$ and $J = Ae \subset AF$ gives $J = AF$. \square

Lemma 3.1.9. *Let a be a positive element of A with right support e . Then the smallest weak*-closed face of A_+ containing a is eA_+e .*

Proof. Let F be the smallest weak*-closed face containing a . We note that a right support for a is the least projection $e \in A$ such that $ae = a$. Since a is positive, a is hermitian and then $ae = a = a^* = ea = eae$ and $a \leq e$ (cf. Lemma 3.1.1). By Lemma 3.1.8, eA_+e is a weak*-closed face of A_+ . Since eA_+e contains a , we have $F \subset eA_+e$. We note that the W^* -algebra $W^*(a)$ generated by a is the weak*-closure of real polynomials $p(a)$ of a without constant term. Letting 1_a be the identity of $W^*(a)$, we have $(1_a - e)p(a) = 0$ for all polynomials $p(a)$; hence $(1_a - e)1_a = 0$. Clearly, $e \leq 1_a$ and therefore $e = 1_a \in W^*(a)$. Now $e \in W^*(a)$ implies that e is the weak*-limit of a net of polynomials $p_\beta(a) \geq 0$ without constant term. Since $0 \leq a \leq \|a\|$, we have $0 \leq a^2 \leq \|a\|a$ and $0 \leq a^n \leq \|a\|^{n-1}a$. It follows that $p_\beta(a) = \sum \alpha_n a^n \leq \alpha a$ for some $\alpha > 0$. Hence $p_\beta(a) \in F$ and as F is weak*-closed, we have $e \in F$. Since for any

$b \in A_+$, we have $ebe \leq \|b\|e$, it is immediate that $eA_+e \subset F$. We conclude that the smallest weak*-closed face of A_+ containing a is eA_+e . \square

Lemma 3.1.10. *Let F be a weak*-closed face of A_+ . Then F contains a projection e which is the largest projection in F and $F = eA_+e$.*

Proof. By Lemma 3.1.9, we observe that if $a \in F$, then the right support of a is also in F . If $a, b \in F$ with right supports f, g respectively, then $a + b$ has right support $f \vee g$. It follows that the set of right supports in F forms an upward directed set bounded above by 1 and hence converges weak* to its least upper bound which we call e . Since F is weak*-closed, we have $e \in F$ and as F is a face it follows that $eA_+e \subset F$. Now, for $a \in F$ we note that $a = ae = (ae)^* = ea = eae$ and $a \in eA_+e$. Hence $F = eA_+e$. Now, let f be a projection in F . Then $f = fe = (fe)^* = ef$ and by Lemma 3.1.1 $f \leq e$. Hence e is the largest projection in F . \square

Lemma 3.1.11. *Let F be a weak*-closed face of A_+ and $J = AF$. Then J is a weak*-closed left ideal of A and $F = J \cap A_+$. If $F = eA_+e$, then $J = Ae$.*

Proof. Let e be the largest projection in F by Lemma 3.1.10. Then Ae is a weak*-closed left ideal. Moreover, $Ae \subset AF = J$. On the other hand, $J = AF = A(eA_+e) \subset Ae$ and hence $J = Ae$. Finally, we observe that $F = eA_+e = Ae \cap A_+ = J \cap A_+$. \square

Lemma 3.1.12. *Let J be a weak*-closed left ideal of A and $a \in A$. Then $a \in J$ if and only if $a^*a \in J \cap A_+$.*

Proof. Clearly if $a \in J$, then $a^*a \in J \cap A_+$. If $a^*a \in J \cap A_+$, then, $(a^*a)^{\frac{1}{2}} \in J \cap A_+$ by Lemma 2.3.14. Considering the polar decomposition of a , we have $a = w(a^*a)^{\frac{1}{2}}$ for some partial isometry $w \in A$ and we conclude that $a \in J$. \square

We can now summarize our results in the following theorem.

Theorem 3.1.13. *Let A be a real W^* -algebra. There is a one to one inclusion preserving correspondence between the weak*-closed left ideals J of A and the*

weak*-closed faces F of the positive cone A_+ . Namely, if the face F corresponds to the ideal J , then

$$(i) \quad F = J \cap A_+ = eA_+e;$$

$$(ii) \quad J = AF = Ae = \{a \in A : a^*a \in F\},$$

where e is the right support of J .

Remark 3.1.14. By the above theorem, we have the right ideal $J^* = FA = eA = \{a \in A : aa^* \in F\}$.

3.2 Faces in preduals of real W^* -algebras

In this section we investigate the structure of faces of the predual of a real W^* -algebra. We show that there exists a one to one correspondence between norm-closed faces of the normal state space of a real W^* -algebra A and the weak*-closed ideals of A via a projection $e \in A$. We will make use of several results proved earlier in Section 2.4.

We let A be a real W^* -algebra with dual A' and predual A_* . As usual we identify the predual A_* with the normal functionals in A' . We recall the following notations:

$$\begin{aligned} A'_h &= \{\phi \in A' : \phi^* = \phi\}; \\ A'_+ &= \{\phi \in A' : \phi \geq 0\}; \\ S(A) &= \{\phi \in A'_+ : \|\phi\| = 1\}. \end{aligned}$$

We recall that the left translation $b\phi$ of a functional $\phi \in A'$ by $b \in A$ is the mapping $a \in A \rightarrow \phi(ba)$ and the right translation ϕb of a functional $\phi \in A'$ by $b \in A$ is the mapping $a \in A \rightarrow \phi(ab)$. We let $\phi_b = b^*\phi b$. The support of a functional $\phi \in A'_+ \cap A_*$ is the least projection $e \in A$ such that $\phi = \phi_e$. It is immediate that if $\phi(e) = \phi(p)$ with $p \in A$ projection, then $\phi(p - e) = 0$ and $\phi(pap) = \phi(a)$ for all $a \in A$. Hence $e \leq p$.

Definition 3.2.1. A subset Y of A' is called *left invariant* if $\phi \in Y$ implies $b\phi \in Y$ for all $b \in A$. A subset Y of A' is called *right invariant* if $\phi \in Y$ implies $\phi b \in Y$ for all $b \in A$. We call Y *invariant* if it left and right invariant.

We note that A'_h , A'_+ and A_* are invariant under the mapping $\phi \mapsto \phi_b$ with $b \in A$.

We define the *annihilator* L^\perp of a subspace L of A_* by

$$L^\perp = \{a \in A : \phi(a) = 0 \text{ for all } \phi \in L\}.$$

Given a subset $X \subset A$, we define its *annihilator* X_\perp in A_* by

$$X_\perp = \{\phi \in A_* : \phi(x) = 0 \text{ for all } x \in X\}.$$

By the bipolar theorem (cf. [45, Theorem IV.1.5]), we have $L = (L^\perp)_\perp$ if L is a norm-closed subspace of A_* and $X = (X_\perp)^\perp$ if X is a weak*-closed subspace of A .

Lemma 3.2.2. *Let L be a left invariant norm-closed subspace of A_* . Then there exists a projection $e \in A$ such that $L = \{\phi \in A_* : \phi = \phi e\}$.*

Proof. Let L^\perp be the annihilator of L in A . Then L^\perp is a weak*-closed left ideal in A by the left invariance of L . By Lemma 3.1.5, $L^\perp = Af$ where f is the right support of L^\perp . Let $e = 1 - f$. Then $\phi \in A_*$ vanishes on L^\perp if and only if $\phi = \phi e$. Since $L = (L^\perp)_\perp$, we conclude that $\phi \in L$ if and only if $\phi = \phi e$. \square

Remark 3.2.3. By the arguments in the above proof, we note that there is a one to one correspondence between the left invariant norm-closed subspaces L of A_* and the weak*-closed left ideals J of A . Namely, $L^\perp = J$ and $J_\perp = L$. If $L = \{\phi \in A_* : \phi = \phi e\}$, then $J = A(1 - e)$. Applying the same arguments as in the above proof, there is a one to one correspondence between the norm-closed right invariant subspaces of A_* and the weak*-closed right ideals of A . Namely, the weak*-closed right ideal $I = fA$ with left support f corresponds to the norm-closed right invariant subspace $I_\perp = \{\phi \in A_* : \phi = f\phi\}$ of A_* . By the proof of Lemma 3.1.5, if J is a two-sided ideal with support e , then $J = eAe$ and $J_\perp = \{\phi \in A_* : \phi = e\phi e\}$ is the corresponding norm-closed invariant subspace of A_* . Let $\phi \in A_*$ positive with support $s(\phi)$. Then the left kernel $L_\phi = \{a \in A : \phi(a^*a) = 0\}$ is a weak*-closed left ideal in A by the Schwarz inequality. If $a \in A(1 - s(\phi))$, then $as(\phi) = 0$ and it follows that $a \in L_\phi$. Hence

$A(1 - s(\phi)) \subset L_\phi$. Conversely, since L_ϕ is a weak*-closed left ideal, it follows by Lemma 3.1.5 that $L_\phi = Ae$ for some projection $e \in A$. Therefore $\phi(e) = 0$ and $\phi(1 - e) = \phi((1 - e)s(\phi)) = \phi(s(\phi))$. Hence $1 - e \geq s(\phi)$ and $e \leq 1 - s(\phi)$, i.e. $L_\phi = Ae \subset A(1 - s(\phi))$. We conclude $L_\phi = A(1 - s(\phi))$.

We define the face of the positive normal functionals $A'_+ \cap A_*$ of a real W^* -algebra A as in Definition 3.1.6 replacing A_+ with A'_+ and note that applying similar arguments as in Lemma 3.1.7, the smallest norm-closed face of $A'_+ \cap A_*$ containing $\phi \in A'_+ \cap A_*$ is $\{\phi' \in A'_+ \cap A_* : \lambda\phi' \leq \phi, \lambda > 0\}$.

Lemma 3.2.4. *Let $\phi \in A_*$ be positive with support $e \in A$. Let T_ϕ be the smallest norm-closed face of $A'_+ \cap A_*$ containing ϕ . Then $T_\phi = \{\theta_e : \theta \in A'_+ \cap A_*\}$.*

Proof. We observe that $T_\phi = \{\phi' \in A'_+ \cap A_* : \lambda\phi' \leq \phi, \lambda > 0\}$.

The set $F = \{\theta_e : \theta \in A'_+ \cap A_*\}$ is a face since for $\psi \in A'_+ \cap A_*$ with $\psi \leq \theta_e$ we have $s(\psi) \leq s(\theta_e) \leq e$ and thus $\psi = \psi_e$ and $\psi \in F$. Since $s(\phi) = e$, we have $\phi = \phi_e$ and $\phi \in F$. As T_ϕ is the smallest face containing ϕ , we get $T_\phi \subseteq F$. It remains to show the reverse inclusion. For this it suffices to show that if $\theta \in A'_+ \cap A_*$, then $\theta_e \in T_\phi$. We consider θ_e as a normal functional on the reduced algebra eAe .

Let $\pi : eAe \rightarrow B(H)$ be the GNS representation induced by ϕ on eAe , with cyclic vector $z \in H$. We show that z is a separating vector in H . We know $\phi(a) = \langle \pi(a)z, z \rangle$ for all $a \in eAe$. Let $\pi(a)z = 0$. Then $\phi(a^*a) = 0$ and $a \in Ae_\phi$, where by Remark 3.2.3, $L_\phi = \{b \in A : \phi(b^*b) = 0\} = Ae_\phi$ with $e_\phi = 1 - s(\phi)$. Hence if $\psi \in A'_+ \cap A_*$ with $s(\psi) \leq e = s(\phi)$, then $e_\psi = 1 - s(\psi) \geq e_\phi$ and $L_\psi = \{b \in A : \psi(b^*b) = 0\} = Ae_\psi \supset Ae_\phi$. It follows that all normal functionals ψ on eAe must satisfy $\psi(a^*a) = 0$ and hence $a = 0$. This proves that z is a separating vector for $\pi(eAe)$. Hence z is a generating vector for the commutant $\pi(eAe)^c$ in $B(H)$ and $\pi(eAe)^c z$ is dense in H (cf. [31, Proposition 4.6.2(3)]).

Let $s \in \pi(eAe)^c$ with $\|s\| \leq 1$ and $\psi(a) = \langle \pi(a)sz, sz \rangle$ for all $a \in eAe$. We have $\|s^*s\| \leq 1$ and $0 \leq s^*s \leq 1$.

Letting $t = (1 - s^*s)^{\frac{1}{2}}$ we obtain

$$\begin{aligned}
\phi(a) - \psi(a) &= \langle \pi(a)z, z \rangle - \langle \pi(a)sz, sz \rangle \\
&= \langle \pi(a)(1 - s^*s)z, z \rangle \\
&= \langle \pi(a)t^2z, z \rangle \\
&= \langle \pi(a)tz, tz \rangle \geq 0 \quad (a \in (eAe)_+)
\end{aligned}$$

and we conclude that $\phi - \psi \geq 0$ on eAe . Define $\psi' \in A_+$ by $\psi'(\cdot) = \psi(e \cdot e)$. Since e supports ϕ , we have $\phi(\cdot) \geq \psi'(\cdot)$. The fact that T_ϕ is a face implies that $\psi' \in T_\phi$. This shows that T_ϕ contains all the functionals of the form $\langle \pi(e \cdot e)x, x \rangle$ with $x \in \pi(eAe)^c z$.

Now, since $\pi(eAe)^c z$ is dense in H and the mapping $x \in H \rightarrow \langle \pi(e \cdot e)x, x \rangle \in \mathbb{R}$ is strongly continuous, T_ϕ contains all the functionals of the form $\langle \pi(e \cdot e)x, x \rangle$ with $x \in H$. Noting that every positive normal functional on eAe is of the form $\langle \pi(e \cdot e)x, x \rangle$ with $x \in H$ (using similar arguments as in [44, Theorem 2.7.9]), we conclude that $\theta_e \in T_\phi$. Therefore $T_\phi = \{\theta_e : \theta \in A'_+ \cap A_*\}$. \square

Definition 3.2.5. Let T be a norm-closed face of $A'_+ \cap A_*$. We define the *support* of T , in symbols $s(T)$, as the least projection $e \in A$ such that $\phi_e = \phi$ for all $\phi \in T$.

By Lemma 3.2.4, we observe that if T is the smallest norm-closed face containing ϕ , then $s(T) = s(\phi)$.

Lemma 3.2.6. Let $e \in A$ be a projection. Then the set $T = \{\theta_e : \theta \in A'_+ \cap A_*\}$ is a norm-closed face of $A'_+ \cap A_*$, consisting of all $\psi \in A'_+ \cap A_*$ with $s(\psi) \leq e$. Moreover, $s(T) = e$.

Proof. This is clear by the arguments in the proof of the Lemma 3.2.4 and the definition of the support of a face. \square

Lemma 3.2.7. Let T be a norm-closed face of $A'_+ \cap A_*$ and $\{e_\alpha\}$ be any maximal orthogonal family of projections in A such that e_α is the support of some $\phi_\alpha \in T$. Then $\sum_\alpha e_\alpha = s(T)$.

Proof. Let $e = \sum_{\alpha} e_{\alpha}$ and $f = s(T) - e$ and assume $f \neq 0$. If $\phi(f) = 0$ for all $\phi \in T$, then by the definition of the support of a face, $\phi(s(\phi)) = \phi(s(T)) = \phi(e)$. This implies $s(\phi) \leq e$ for all $\phi \in T$ and it follows that $s(T) = e$ which contradicts our assumption. Hence we must have $\phi'(f) \neq 0$ for some $\phi' \in T$.

Let $g = s(\phi')$ and $h = (e \vee g) - e$. If $h = 0$, then $g \leq e$ and $\phi'(s(T)) = \phi'(g) = \phi'(e)$ which implies $\phi'(f) = 0$, a contradiction. Therefore $h \neq 0$ and $h \leq s(T)$. Since $s(\phi'_h) \leq h$ and $h \leq s(T)$, we have $s(\phi'_h) \leq s(T)$ and $\phi'_h \in T$ by Lemma 3.2.6. Now, $s(\phi'_h)e = s(\phi'_h)he = 0$ implies that $\{e_{\alpha}\}$ is not maximal which contradicts the hypothesis. We conclude, $f = 0$ and $s(T) = e$. \square

Lemma 3.2.8. *Let T be a norm-closed face of $A'_+ \cap A_*$ with $s(T) = e$. Then $T = \{\theta_e : \theta \in A'_+ \cap A_*\}$.*

Proof. We first observe that $T \subseteq \{\theta_e : \theta \in A'_+ \cap A_*\}$. For the reverse inclusion let $\theta \in A'_+ \cap A_*$ with $s(\theta) \leq e$. We choose a maximal orthogonal family $\{e_{\alpha}\}$ of projections in A such that e_{α} supports ϕ_{α} in T . By Lemma 3.2.7, $e = \sum_{\alpha} e_{\alpha}$ and since θ is normal, $\theta(e) = \theta(\sum_{\alpha} e_{\alpha}) = \sum_{\alpha} \theta(e_{\alpha})$. Hence $\theta(e_{\alpha})$ vanishes except for a countable subset of $\{e_{\alpha}\}$ which we denote by $\{e_i\}$. We have $\theta(e - \sum_i e_i) = 0$ and $s(\theta) \leq \sum_i e_i$. We can choose $\phi_i \in T$ with $\|\phi_i\| = 1$ and $s(\phi_i) = e_i$ for all i . Let $\phi = \sum_i 2^{-i} \phi_i$ which belongs to T since T is norm-closed and convex. By $s(\phi) = \sum_i e_i \geq s(\theta)$, we obtain $\theta_{s(\phi)} = \theta$ and by Lemma 3.2.4, $\theta \in T_{\phi}$ where T_{ϕ} is the smallest norm-closed face of $A'_+ \cap A_*$ containing ϕ . Since $T_{\phi} \subset T$, we conclude that $\theta \in T$. Therefore $T = \{\theta_e : \theta \in A'_+ \cap A_*\}$. \square

Lemma 3.2.9. *Let L be a left invariant norm-closed subspace of A_* . Then $L \cap A'_+$ is a norm-closed face of $A'_+ \cap A_*$ and there is a projection $e \in A$ such that*

$$L \cap A'_+ = e(A'_+ \cap A_*)e = \{|\phi| : \phi \in L\}.$$

Moreover, we have $L = A(L \cap A'_+)$.

Proof. By Lemma 3.2.2, we have $L = \{\phi \in A_* : \phi = \phi e\}$ for some projection e . If $\psi \in L \cap A'_+$, then $\psi(a) = \psi(ea) = \psi((ea)^*) = \psi(a^*e) = \psi(ea^*e) =$

$\psi((ea^*e)^*) = \psi(eae) = \psi_e(a)$. Conversely, if $\psi \in \{\phi_e : \phi \in A'_+ \cap A_*\}$, then $\psi(a) = \psi(eae) = \psi(e(ea)e) = \psi(ea)$ and $\psi \in L \cap A'_+$. Hence $L \cap A'_+ = \{\phi_e : \phi \in A'_+ \cap A_*\} = e(A'_+ \cap A_*)e$, which is a norm-closed face of $A'_+ \cap A_*$ by Lemma 3.2.6.

It is evident that if $\psi \in L \cap A'_+$, then the positivity of ψ implies $|\psi| = \psi$ and then $\psi \in \{|\phi| : \phi \in L\}$. For $\psi \in L$, we consider the polar decomposition $\psi = v|\psi|$ for some partial isometry $v \in A$. It is immediate that $|\psi| = v^*\psi \in L \cap A'_+$ and then $\{|\phi| : \phi \in L\} \subset L \cap A'_+$ and $L \subset A(L \cap A'_+)$. Clearly, if $\psi \in A(L \cap A'_+)$, then by the left invariance of L , $\psi \in L$ and $L = A(L \cap A'_+)$. \square

Lemma 3.2.10. *Let T be a norm-closed face of $A'_+ \cap A_*$ with support e . Then $AT = \{a\theta : \theta \in T, a \in A\}$ is a left invariant norm-closed subspace of A_* and $T = (AT) \cap A'_+$. Moreover,*

$$AT = \{\phi \in A_* : \phi e = \phi\} = \{\phi \in A_* : |\phi| \in T\}.$$

Proof. If e is the support of T , then by Lemma 3.2.8, $T = \{\theta_e : \theta \in A'_+ \cap A_*\}$. Therefore $AT = \{a\theta_e : \theta \in A'_+ \cap A_*, a \in A\}$ which is a left invariant norm-closed subspace of A_* . We will show that

$$AT = \{\phi \in A_* : \phi e = \phi\} = \{\phi \in A_* : |\phi| \in T\}.$$

For any $\phi \in A_*$ with $\phi = \phi e$, we consider the polar decomposition of $\phi = v|\phi|$ with partial isometry $v \in A$. Then $|\phi|(a) = |\phi|(ae)$. Since $|\phi|$ is hermitian, $|\phi|(a^*) = |\phi|(a) = |\phi|(ae) = |\phi|((ae)^*) = |\phi|(ea^*) = |\phi|(ea^*e)$. Hence $|\phi| \in T$ and $\phi = v|\phi|_e$ which gives $\phi \in AT$. Therefore we have shown

$$\begin{aligned} \{\phi \in A_* : \phi e = \phi\} &\subset AT \quad \text{and} \\ \{\phi \in A_* : \phi e = \phi\} &\subset \{\phi \in A_* : |\phi| \in T\}. \end{aligned}$$

Conversely, if $\phi \in AT$, then $\phi = a\psi_e$ for some $a \in A$ and $\psi \in A'_+ \cap A_*$. It is immediate that $\phi e = \phi$. Lastly, if $\phi \in A_*$ such that $|\phi| \in T$, then $|\phi|_e = |\phi|$ and

since $\phi = v|\phi|$, we have $\phi e = v|\phi|e = v|\phi|_e e = v|\phi|_e = v|\phi| = \phi$. Therefore we have shown

$$\begin{aligned} \{\phi \in A_* : \phi e = \phi\} &\supset AT \quad \text{and} \\ \{\phi \in A_* : \phi e = \phi\} &\supset \{\phi \in A_* : |\phi| \in T\}. \end{aligned}$$

We now prove $T = AT \cap A'_+$. We first observe that if $\phi \in T \subset A'_+$, then $\phi_e = \phi$ and $\phi e = \phi$, i.e. $\phi \in AT$. Conversely, if $\phi \in AT \cap A'_+$, then $\phi e = \phi$ and since ϕ is hermitian, it follows that $\phi_e = \phi$, i.e. $\phi \in T$ by Lemma 3.2.8. We conclude that $T = AT \cap A'_+$. \square

We can now summarize our results in the following theorem.

Theorem 3.2.11. *Let A be a real W^* -algebra. There is a one to one inclusion preserving correspondence between left invariant norm-closed subspaces L of A_* and norm-closed faces T of $A'_+ \cap A_*$. If the subspace L corresponds to the face T , then*

- (i) $T = L \cap A'_+ = e(A'_+ \cap A_*)e = \{|\phi| : \phi \in L\}$;
- (ii) $L = AT = A_*e = \{\phi : |\phi| \in T\}$,

where e is the support of T .

Let $N(A) = \{\phi \in A_* : \|\phi\| = 1 = \phi(1)\} = A_* \cap S(A)$ be the *normal state space* of a real W^* -algebra A . The following theorem gives a correspondence between the norm-closed faces of the normal state space $N(A)$ and the weak*-closed left ideals of A . We first note that there is a natural one to one inclusion preserving correspondence between the faces of $A'_+ \cap A_*$ and the faces of $N(A)$. If F is a face of $N(A)$, then $T = \mathbb{R}^+F$ is a face of $A'_+ \cap A_*$. Conversely, given a face T of $A'_+ \cap A_*$, the intersection $F = T \cap N(A)$ is a face of $N(A)$. Since $N(A)$ is norm-closed, F is norm-closed if and only if T is norm-closed.

Theorem 3.2.12. *Let A be a real W^* -algebra. There is a one to one inclusion preserving correspondence between the weak*-closed left ideals of A and the norm-closed faces of $N(A)$. If the face F corresponds to the ideal J , then*

$$(i) \quad F = \{\rho \in N(A) : \rho(a^*a) = 0 \text{ for all } a \in J\};$$

$$(ii) \quad J = \{a \in A : \rho(a^*a) = 0 \text{ for all } \rho \in F\}.$$

If e is the right support of the ideal J , then $1 - e$ is the support of the face F .

Proof. If F is a norm-closed face of $N(A)$, then $T = \mathbb{R}^+F$ is a face of $A'_+ \cap A_*$ with support $s(T)$ and the corresponding left invariant subspace L is of the form $L = A_*(1 - e)$, where $e = 1 - s(T)$ and vice versa by Theorem 3.2.11. By Remark 3.2.3, L is a norm-closed left invariant subspace of A_* if and only if $J = L^\perp$ is a weak*-closed left ideal of A . Then $J = L^\perp = Ae$.

We show (ii). Let $a \in J$ and $\rho \in F$. Then $a^*a \in J = L^\perp$ and $\rho \in \mathbb{R}^+F = L \cap A'_+ \subset L$. Hence $\rho(a^*a) = 0$. Conversely, let $a \in A$ and $\rho(a^*a) = 0$ for all $\rho \in F$. Each $\psi \in L = AT = A(\mathbb{R}^+F)$ is of the form $\psi = b\phi$ with $b \in A$ and $\phi \in \mathbb{R}^+F$. Therefore $|\psi(a)|^2 = |\phi(ba)|^2 \leq \phi(b^*b)\phi(a^*a) = 0$, giving $a \in L^\perp = J$. Similarly, we can prove (i). \square

Remark 3.2.13. In [41, Corollary 3.19] for a complex W^* -algebra B with a left ideal $J = Bf$ where f is projection in B and corresponding face $F = \{\rho \in N(B) : \rho(a^*a) = 0 \text{ for all } a \in J\}$ of $N(B)$, we have $J + J^* = Bf + fB = \bigcap_{\rho \in F} \rho^{-1}(0)$. However, this result does not hold for real W^* -algebras. The proof for the complex case relies on the fact that all elements ω of F vanish at $a \in B$ if and only if $(1 - f)a(1 - f) = 0$, where $s(F) = 1 - f$. In contrast, for a real W^* -algebra A , every functional in $N(A)$ vanishes on skew hermitian elements and hence $\bigcap_{\rho \in F} \rho^{-1}(0)$ includes all the skew hermitian elements and need not equal to $J + J^*$. A counterexample can be found in [31, p.96-97].

3.3 Ideals in real C*-algebras

We now extend the results obtained in Section 3.1 for real W*-algebras to real C*-algebras. For the rest of this section, A is a real C*-algebra and A_h denotes the subspace of hermitian elements, A_+ the set of positive elements and A_1 the closed unit ball of A . From the discussion in Section 2.3 we may assume without loss of generality that the algebra contains an identity 1.

For any set $X \subset A$, we let \overline{X} denote the norm closure of X in A .

Lemma 3.3.1. *Let S be a norm-closed face of A_+ . Then the following are equivalent for an element $a \in A_h$:*

- (i) $a \in S - S$;
- (ii) $a^2 \in S$;
- (iii) $|a| \in S$;
- (iv) $a^+, a^- \in S$.

Proof. (i) \Rightarrow (ii). Let $a \in S - S$. Then $a = b - c$ with $b, c \in S$. If $x \in S$, then $x^2 \leq \|x\|x$ and it follows that $x^2 \in S$. Hence $b^2, c^2 \in S$ and $b^2 + c^2 \in S$. Since $(b + c)^2 + (b - c)^2 = 2(b^2 + c^2)$, we have $(b - c)^2 \leq 2(b^2 + c^2)$. Thus $a = (b - c)^2 \in S$.

(ii) \Rightarrow (iii). For $a^2 \in S \subset A_+$, $|a| = (a^2)^{\frac{1}{2}}$ is the limit of a sequence of polynomials $p(a^2)$ in A without constant term. Since $p(a^2) \leq \lambda a^2$ for some $\lambda \geq 0$ and S is norm-closed, it follows that $|a| \in S$.

(iii) \Rightarrow (iv). This is immediate from $a^+, a^- \leq (a^2)^{\frac{1}{2}}$ (cf. Lemma 2.3.14).

(iv) \Rightarrow (i). By Lemma 2.3.14, $a = a^+ - a^- \in S - S$.

□

The following two technical lemmas can be proved via functional calculus as in [41, Lemma 4.3] and [41, Lemma 4.4].

Lemma 3.3.2. *Every $a \in A$ may be uniformly approximated by elements of the form $b_\varepsilon(a^*a)^{\frac{1}{2}}$ where $b_\varepsilon = (\varepsilon 1 + aa^*)^{-1}a(a^*a)^{\frac{1}{2}}$ and $\varepsilon > 0$.*

Lemma 3.3.3. *Let $a, b \in A_+$ with $b \leq a \leq 1$. Then b may be uniformly approximated by elements of the form $a^{\frac{1}{2}}c_\varepsilon a^{\frac{1}{2}}$ where $0 \leq c_\varepsilon \leq 1$.*

Corollary 3.3.4. *Let $a \in A$ be positive. Then the smallest norm-closed face S of A_+ containing a is $\overline{aA_+a}$.*

Proof. By Lemma 3.3.1, the smallest norm-closed face of A_+ containing a is the smallest norm-closed face of A_+ containing a^2 . By Lemma 3.1.7, if $b \in S$, then there exists a $\lambda > 0$ such that $\lambda b \leq a^2$. Since $0 \leq \frac{\lambda b}{\|a\|^2} \leq \frac{a^2}{\|a\|^2} \leq 1$, by Lemma 3.3.3, $\frac{\lambda b}{\|a\|^2}$ is uniformly approximated by elements of the form $\frac{a}{\|a\|}c_\varepsilon \frac{a}{\|a\|}$ with $0 \leq c_\varepsilon \leq 1$. Hence $b \in \overline{aA_+a}$, i.e. $S \subset \overline{aA_+a}$.

Conversely, if $b \in \overline{aA_+a}$, then b is approximated by elements of the form $ac_\varepsilon a$ with $c_\varepsilon \in A_+$. Therefore, $ac_\varepsilon a \leq \|c_\varepsilon\|a^2$ and $ac_\varepsilon a \in S$. Since S is norm-closed, we have $b \in S$, i.e. $\overline{aA_+a} \subset S$. \square

Lemma 3.3.5. *Let J be a norm-closed left ideal in A and let $S = J \cap A_+$. Then S is a norm-closed face of A_+ and $\overline{aS} = J$.*

Proof. Evidently, $J \cap A_+$ is a norm-closed convex subset of A_+ . Now, if $a \in J \cap A_+$, then $a^{\frac{1}{2}} \in J \cap A_+$ and $a^{\frac{1}{2}}ca^{\frac{1}{2}} \in J \cap A_+$ for all $c \in A_+$. By Lemma 3.3.3, if $b \in A_+$ with $b \leq a$, then b can be uniformly approximated by elements of the form $a^{\frac{1}{2}}c_\varepsilon a^{\frac{1}{2}}$ with $c_\varepsilon \geq 0$. Since $J \cap A_+$ is norm-closed, it follows that $b \in J \cap A_+$. Hence $S = J \cap A_+$ is a norm-closed face of A_+ .

We observe that $\overline{aS} \subset J$. If $a \in J$, then $a^*a \in J \cap A_+ = S$ and $(a^*a)^{\frac{1}{2}} \in S$ by Lemma 3.3.1. Thus, letting $b_\varepsilon = (\varepsilon 1 + aa^*)^{-1}a(a^*a)^{\frac{1}{2}}$, as in Lemma 3.3.2, we obtain $b_\varepsilon(a^*a)^{\frac{1}{2}} \in aS$. By the same lemma, a is uniformly approximated by $b_\varepsilon(a^*a)^{\frac{1}{2}}$ and hence $a \in \overline{aS}$. Thus $J = \overline{aS}$. \square

Lemma 3.3.6. *Let S be a norm-closed face of A_+ and $J = \overline{AS}$. Then J is a norm-closed left ideal of A and $J \cap A_+ = S$.*

Proof. Let $a, b \in A$, Then we have

$$(i) \quad (a+b)^*(a+b) \leq 2(a^*a + b^*b), \text{ since } (a+b)^*(a+b) + (a-b)^*(a-b) = 2(a^*a + b^*b);$$

$$(ii) \quad (\lambda a)^*(\lambda a) = \lambda^2 a^*a \text{ for } \lambda \in \mathbb{R};$$

$$(iii) \quad b^*a^*ab \leq \|a\|^2 b^*b.$$

Let $I = \{a \in A : a^*a \in S\}$. By (i), (ii) and (iii), I is a left ideal. Since S is norm-closed, it follows that I is norm-closed. By Lemma 3.3.1, if $a \in S \subset A_+$, then $a^*a = a^2 \in S$ and $a \in I$, i.e. $S \subset I$. Now we have, $J = \overline{AS} \subset I$ since I is norm-closed.

Conversely, if $a \in I$, then $a^*a \in S$ and by Lemma 3.3.1, $(a^*a)^{\frac{1}{2}} \in S$. Letting $b_\varepsilon = (\varepsilon 1 + aa^*)^{-1} a (a^*a)^{\frac{1}{2}}$ as in Lemma 3.3.2, we obtain $b_\varepsilon (a^*a)^{\frac{1}{2}} \in AS$. Since a is approximated by $b_\varepsilon (a^*a)^{\frac{1}{2}}$, we have $a \in \overline{AS} = J$ which gives $I \subset J$ and $I = J$.

Finally, if $a \in J \cap A_+ = I \cap A_+$, then $a^2 \in S$ by the definition of I and then by Lemma 3.3.1, $a \in S$. Conversely, if $a \in S$, then $a^*a = a^2 \in S$ and hence $a \in I \cap A_+$. Therefore $S = I \cap A_+ = J \cap A_+$. \square

Lemma 3.3.7. *Let J be a norm-closed left ideal of A and $a \in A$. Then $a \in J$ if and only if $a^*a \in J \cap A_+$.*

Proof. If $a \in J$, then it is immediate that $a^*a \in J \cap A_+$. Conversely, if $a^*a \in J \cap A_+$, then by the function representation of a^*a , we have $(a^*a)^{\frac{1}{2}} \in J \cap A_+$ and thus $b_\varepsilon (a^*a)^{\frac{1}{2}} \in J$, where $b_\varepsilon = (\varepsilon 1 + aa^*)^{-1} a (a^*a)^{\frac{1}{2}}$ as in Lemma 3.3.2. By the same lemma, a is approximated by $b_\varepsilon (a^*a)^{\frac{1}{2}} \in J$ and since J is norm-closed it follows that $a \in J$. \square

We conclude with the following extension of Theorem 3.1.13 for real C*-algebras.

Theorem 3.3.8. *Let A be a real C*-algebra. There is a one to one inclusion preserving correspondence between the norm-closed left ideals of A and the norm-closed faces of A_+ . If the face S of A_+ corresponds to the ideal J of A , then $S = J \cap A_+$ and $J = \overline{AS}$, where $a \in J$ if and only if $a^*a \in S$.*

Remark 3.3.9. In [41, Theorem 4.9], it was shown that if J is a norm-closed right ideal in a complex C*-algebra B , then $J \cap J^*$ is the complex linear span of the face $J \cap B_+$. However, in real C*-algebras, the positive elements do not span the whole algebra and the statement does not hold. For example, consider the real abelian C*-algebra $C(D, -)$ of complex continuous functions f on the unit disk D satisfying $f(\bar{x}) = \overline{f(x)}$. We observe that every left ideal $J \in A$ is also a two-sided ideal and if we take $J = A$, then $S = A_+$ and we immediately have $J \neq [A_+]_{\mathbb{R}}$.

In [41, Lemma 4.10], it was also shown that if J is a norm-closed left ideal of a complex C*-algebra A , then J is two-sided if and only if $a \in J$ implies $u^*au \in J$ for all u in the unitary set $U(A)$. The proof depends on the fact that in a complex C*-algebra the unitary elements span the whole algebra. This is not true for a real C*-algebra.

3.4 Faces in duals of real C*-algebras

Analogous to the case of real W*-algebras, our objective in this section is to show the correspondence between the norm-closed left ideals of a real C*-algebra A and the weak*-closed faces of its state space $S(A)$.

Let A be a real C*-algebra with dual A' . Given a subset Y of A' , we say that Y is *left invariant* if $\phi \in Y$ implies $a\phi \in Y$ for all $a \in A$ where $a\phi$ is the left translation of ϕ by a . We say that Y is *invariant* if $\phi \in Y$ implies $a\phi b \in Y$ for all $a, b \in A$, where $a\phi b$ is defined by $(a\phi b)(x) = \phi(axb)$ for all $x \in A$.

Given a set $X \subset A'$, we denote by \overline{X}^{w*} its weak*-closure in A' .

Lemma 3.4.1. *Let L be a left invariant weak*-closed subspace of A' . Then $\phi \in L$ implies $a\phi \in L$ for all $a \in A''$.*

Proof. Let $\phi \in L$ and $a \in A''$. We must show that $a\phi \in L$. Choose a net $\{a_\alpha\} \in A$ with a_α converging weak* to a . Then $a_\alpha\phi \in L$ and $a_\alpha\phi$ converges weak* to $a\phi$. Since L is weak*-closed it follows that $a\phi \in L$. \square

Let A'' be the second dual of a real C*-algebra A . We always identify A as a weak*-dense C*-subalgebra of A'' . In this case, we have $A' = (A'')_*$ and $A'_+ = (A'')'_+ \cap (A'')_*$. This enables us to apply the results on faces and ideals in real W*-algebras to real C*-algebras.

Lemma 3.4.2. *Let T be a weak*-closed face of A'_+ and let $L = \overline{AT}^{w*}$. Then L is a left invariant weak*-closed subspace of A' and $T = L \cap A'_+$.*

Proof. Consider T as a norm-closed face of the cone A'_+ of the positive normal functionals of the real W*-algebra A'' . By Lemma 3.2.10, $A''T$ is a left invariant norm-closed subspace of A' and $T = A''T \cap A'_+$.

If $a \in A''$ and $\{a_\alpha\} \in A$ is a net converging weak* to a , then $a_\alpha\phi$ converges weak* to $a\phi$ for all $\phi \in T$. Hence $AT \subset A''T \subset \overline{AT}^{w*}$. It remains to demonstrate that $A''T = \overline{AT}^{w*}$, i.e. that $A''T$ is weak*-closed. For this it suffices to show that

$A''T \cap A'_1$ is weak* compact, by the Krein-Smulian theorem.

We show $A''T \cap A'_1$ is weak*-closed in A'_1 . Let $\{\phi_\alpha\}$ be a net in $A''T \cap A'_1$ weak*-converging to some $\phi \in \overline{AT}^{w*} \cap A'_1$. Using the polar decomposition of ϕ_α and left invariance of $A''T$, we see that $|\phi_\alpha| \in T \cap A'_1$ which is a weak*-compact set. By taking a subnet, we may assume that $|\phi_\alpha|$ weak*-converges to some $\omega \in T \cap A'_1$. Then for any $a \in A$, we have $\phi_\alpha(a) \rightarrow \phi(a)$ and $|\phi_\alpha|(a^*a) \rightarrow \omega(a^*a)$. It follows that $|\phi|(a)^2 \leq \omega(a^*a)$ for all $a \in A$ and thus for all $a \in A''$. By Lemma 2.4.20, we obtain $\omega = |\phi|$. Therefore $|\phi| \in T \cap A'_1$ and hence $\phi \in A''T \cap A'_1$. Therefore, $A''T$ is weak*-closed and $A''T = \overline{AT}^{w*}$. \square

Lemma 3.4.3. *Let L be a left invariant weak*-closed subspace of A' and let $T = L \cap A'_+$. Then T is a weak*-closed face of A'_+ and $L = \overline{AT}^{w*}$.*

Proof. By Lemma 3.2.9 and Lemma 3.4.1, $T = L \cap A'_+$ is a norm-closed face of A'_+ and is also weak*-closed since L and A'_+ are weak*-closed. Using the aforementioned lemmas and the proof in Lemma 3.4.2, $L = A''T = \overline{AT}^{w*}$. \square

Lemma 3.4.4. *Let L be a left invariant weak*-closed subspace of A' . Then for all $\phi \in A'$, we have $\phi \in L$ if and only if $|\phi| \in L \cap A'_+$.*

Proof. Let $T = L \cap A'_+$. By Lemma 3.4.2 and Lemma 3.4.3, $L = \overline{AT}^{w*} = A''T$. Considering the polar decomposition $\phi = v|\phi|$ and $|\phi| = v^*\phi$ for some partial isometry $v \in A''$ the conclusion is immediate. \square

We can now generalize Theorem 3.2.11 to real C*-algebras.

Theorem 3.4.5. *Let A be a real C*-algebra. There is a one to one inclusion preserving correspondence between the left invariant weak*-closed subspaces of A' and the weak*-closed faces of A'_+ . If the subspace L corresponds to the face T , then*

$$(i) \quad T = L \cap A'_+ = \{|\phi| : \phi \in L\};$$

$$(ii) \quad L = \overline{AT}^{w*} = \{\phi : |\phi| \in T\}.$$

Given a norm-closed left ideal J in $A \subset A''$, its weak*-closure \overline{J}^{w*} is a weak*-closed left ideal in A'' and therefore by Theorem 3.2.12, corresponds to a norm-closed face F in the normal state space $N(A'')$, where $N(A'')$ is just the state space $S(A)$ of A . On the other hand, if F is a weak*-closed face of $S(A)$, then it is also norm-closed and by Theorem 3.2.12, it corresponds to a weak*-closed left ideal J_F in A'' and $J = J_F \cap A$ is a norm-closed left ideal in A .

We observe that there is a natural one to one correspondence between the weak*-closed faces T of A'_+ and the weak*-closed faces F of the state space $S(A)$. Indeed, $T = \mathbb{R}^+ F$ and $F = T \cap S(A)$. Therefore we obtain the following main theorem relating the weak*-closed faces $S(A)$ of A and the norm-closed left ideals in A .

Theorem 3.4.6. *Let A be a real C^* -algebra. There is a one to one inclusion preserving correspondence between the norm-closed left ideals of A and the weak*-closed faces of $S(A)$. If the face F corresponds to the ideal J , then*

- (i) $F = \{\rho \in S(A) \text{ and } \rho(a^*a) = 0 \text{ for all } a \in J\}$;
- (ii) $J = \{a \in A : \rho(a^*a) = 0 \text{ for all } \rho \in F\}$.

Definition 3.4.7. Let F be a face of $S(A)$. We call F *invariant* if $\rho \in F$ implies $\rho_a \in F$ for all $a \in A$ with $\rho(a^*a) \neq 0$, where $\rho_a(\cdot) = \frac{\rho(a^*\cdot a)}{\rho(a^*a)}$.

Corollary 3.4.8. *A weak*-closed face F of $S(A)$ is invariant if and only if the corresponding norm-closed left ideal $J = \{a \in A : \rho(a^*a) = 0 \text{ for all } \rho \in F\}$ is two-sided.*

Proof. It is immediate that if J is a norm-closed two-sided ideal, then $x \in J$ implies $xa \in J$ for all $a \in A$. Hence for $\rho \in F$ and $\rho(a^*a) \neq 0$ we have $\rho_a(x^*x) = \frac{\rho(a^*x^*xa)}{\rho(a^*a)} = \frac{\rho((xa)^*(xa))}{\rho(a^*a)} = 0$; that is, $\rho_a \in F$ and F is invariant. Conversely, if F is invariant, then $\rho_a \in F$ for all $\rho \in F$ with $\rho(a^*a) \neq 0$ and therefore $\rho((xa)^*(xa)) = \rho(a^*x^*xa) = \rho(a^*a)\rho_a(x^*x) = 0$ for all $a \in A$ and $x \in J$. Hence $xa \in J$ and J is a also right ideal. \square

Chapter 4

Linear maps of real C^* -algebras

In this last chapter, we study continuous linear maps of real C^* -algebras and in particular, we extend the work of [10] and [11] on linear isometries between complex C^* -algebras. The study of isometries between complex C^* -algebras dates back to the seminal results of Banach [2] and Stone [48] in 1930's, now known as the Banach-Stone theorem which states that the linear isometries between two spaces of continuous functions on compact Hausdorff spaces are exactly the weighted composition operators. This is a result about complex abelian C^* -algebras. The full generalisation of the Banach-Stone theorem to non-abelian complex C^* -algebras is the celebrated result of Kadison [28] which states that a surjective linear isometry T between two complex C^* -algebras is a "composition operator" in the sense that T is of the form $uJ(\cdot)$ where u is a unitary in (the unit extension of) the codomain and J is a Jordan*-isomorphism. This result shows, in essence, that the surjective linear isometries between complex C^* -algebras are exactly the Jordan triple isomorphisms, that is, linear maps preserving the Jordan triple product. If T is non-surjective, this result is false. Nevertheless, it has been shown in [11] that T still preserves the Jordan triple product locally. Our objective is to extend all these results to real C^* -algebras. Due to the fact that the complexification of a non-surjective real isometry need not be a complex isometry, we have to abandon the complexification method and instead, we develop some new

techniques to accomplish our task. We refer the reader to [15] and [16] for an extensive study of isometries between Banach spaces.

We begin by studying contractive linear maps between real C^* -algebras and generalise the complex result in [11, Proposition 2.2] to the real case. In Section 4.2, we consider surjective linear isometries between real abelian C^* -algebras and obtain an analogue of the Banach-Stone theorem, that is, we show that these isometries are composition operators. In Section 4.3, we study non-surjective isometries between real C^* -algebras and show that they preserve the Jordan triple product locally, thus extending the complex result in [10, Theorem 2] and [11, Corollary 3.12]. We give an example in Section 4.4 of a real isometry whose complexification is not an isometry hence showing that the complexification method cannot be used. In the final section, we show that a Jordan*-homomorphism between two real C^* -algebras is the sum of a C^* -homomorphism and a C^* -antihomomorphism.

4.1 Contractions of real C^* -algebras

We will make use of the results on faces obtained in Chapter 3 and specifically in Section 3.4 to prove that every contraction between real C^* -algebras preserves the cube product (cf. Section 2.5) cut down by a projection. The arguments in [11, Proposition 2.2] can be extended to our setting of real C^* -algebras.

As before, we denote the dual of a real C^* -algebra A by A' , the closed unit ball of A by A_1 and the closed unit ball of A' by A'_1 . We let

$$Q(A) = \{\phi \in A'_1 : \phi \geq 0\}$$

be the quasi-state space of A which is weak* compact and convex (cf. [31, Proposition 5.2.7(3)]). Every weak*-closed face of $Q(A)$ containing zero is of the form $F(p) = \{\phi \in Q(A) : \phi(1 - p) = 0\}$ for some projection $p \in A''$, called the support projection of the face by Theorem 3.2.12. We note by Theorem 3.4.6 that

if the face $F(p)$ corresponds to the norm-closed ideal J of A , then for $\phi \in F(p)$ we have $\phi(x^*x) = 0$ for all $x \in J$. We can naturally consider the weak*-closed ideal $\overline{J^{w*}}$ in the real W^* -algebra A'' . Since p is the support projection of the face $F(p)$, the support of the ideal $\overline{J^{w*}}$ is $1 - p$ which implies that $x \in J \subset \overline{J^{w*}}$ if and only if $xp = 0$.

Each $\phi \in F(p)$ induces a Gelfand-Naimark-Segal representation of A which we denote by $(\pi_\phi, H_\phi, \omega_\phi)$ and without loss of generality we use the same notation for the extended representation of A'' (cf. [31, Proposition 5.5.4]). For $x \in A''$ we write $x\omega_\phi$ for $\pi_\phi(x)\omega_\phi$. It follows that $x\omega_\phi = 0$ if and only if $\phi(x^*x) = 0$ and by the arguments in the previous paragraph, if and only if, $xp = 0$.

We write $a^{(3)} = \{a, a, a\} = aa^*a$.

Proposition 4.1.1. *Let $T : A \rightarrow B$ be a linear contraction between real C^* -algebras A and B . Then there is a largest projection p in B'' such that*

- (i) $T(a^{(3)})p = T(a)^{(3)}p$;
- (ii) $pT(a)^*T(a) = T(a)^*T(a)p$ for all a in A .

Proof. Let

$$\begin{aligned} F_1 &= \bigcap_{a \in A_1} \{\phi \in Q(B) : T(a^{(3)})\omega_\phi = T(a)^{(3)}\omega_\phi\} \\ &= \bigcap_{a \in A_1} \{\phi \in Q(B) : \phi((T(a^{(3)}) - T(a)^{(3)})^*(T(a^{(3)}) - T(a)^{(3)})) = 0\}. \end{aligned}$$

Then F_1 is a weak*-closed face of $Q(B)$ containing zero.

Let $a \in A_1$. We define a weak*-continuous affine map $\Phi_a : Q(B) \rightarrow Q(B)$ by

$$\Phi_a(\phi)(\cdot) = \phi(T(a)^*T(a) \cdot T(a)^*T(a)).$$

For $n = 1, 2, \dots$, we define

$$F_{n+1} = \{\phi \in F_n : \Phi_a(\phi) \in F_n, \forall a \in A_1\} = \bigcap_{a \in A_1} F_n \cap \Phi_a^{-1}(F_n).$$

The sets $\{F_n\}$ form a decreasing sequence of weak*-closed faces of $Q(B)$. It follows that the intersection $F = \bigcap_{n=1}^{\infty} F_n$ is a weak*-closed face of $Q(B)$ containing 0. By the discussion before the proposition, there exists a projection p in B'' supporting F . We then have

$$F = F(p) = \{\phi \in Q(B) : \phi(1 - p) = 0\}.$$

Thus for each $a \in A_1$ and $\phi \in F$ we get

$$\Phi_a(\phi)(\cdot) = \phi(T(a)^*T(a) \cdot T(a)^*T(a)) \in F.$$

It follows that

$$\begin{aligned} 0 &= \Phi_a(\phi)(1 - p) = \phi(T(a)^*T(a)(1 - p)T(a)^*T(a)) \\ &= \phi(((1 - p)T(a)^*T(a))^*(1 - p)T(a)^*T(a)). \end{aligned}$$

Hence by the arguments before the proposition, $(1 - p)T(a)^*T(a)p = 0$ which implies $T(a)^*T(a)p = pT(a)^*T(a)p$.

Taking the involution we obtain $pT(a)^*T(a)p = pT(a)^*T(a)$ and therefore

$$pT(a)^*T(a) = T(a)^*T(a)p \quad (a \in A).$$

We note that $T(a^{(3)})\omega_\phi = T(a)^{(3)}\omega_\phi$ for all $\phi \in F$ which implies that

$$T(a^{(3)})p = T(a)^{(3)}p \quad (a \in A).$$

Now if $q \in B''$ is another projection satisfying the conditions of the proposition, then we have $\Phi_a(F(q)) \subseteq F(q)$ and $F(q) \subseteq F_1$. Therefore

$$F(q) = \{\phi \in Q(B) : \phi(1 - q) = 0\} \subseteq F_n, \quad n = 1, 2, \dots$$

It follows that $F(q) \subseteq F(p)$ and $q \leq p$. □

Remark 4.1.2. We note that for complex C^* -algebras A and B and a linear contraction $T : A \rightarrow B$, complex polarisation implies

$$T(\{a, b, c\})p = \{T(a), T(b), T(c)\}p \quad (a, b, c \in A)$$

whenever $T(a^{(3)})p = T(a)^{(3)}p$ with $p \in B^{**}$. This may not be achievable in the real case as demonstrated by the example in Section 4.4.

4.2 Composition operators on abelian C*-algebras

We will consider later the special case of Section 4.1 when T is an isometry. In this section we show that the surjective linear isometries on real abelian C*-algebras are exactly the composition operators, thus extending the Banach-Stone theorem to real abelian C*-algebras.

We note that a triple isomorphism between C*-algebras is always an isometry by spectral theory. Conversely, it was shown in [28] that a surjective isometry $T : A \rightarrow B$ between complex C* algebras A and B is a triple homomorphism

$$T(ab^*c + cb^*a) = T(a)T(b)^*T(c) + T(c)T(b)^*T(a) \quad (a, b, c \in A).$$

This result has been extended to real C* algebras in [9, Theorem 6.4].

It follows from the above results that a surjective linear isometry between real C*-algebras can always be complexified to an isometry, unlike the non-surjective isometries which do not share this property as shown in Section 4.4.

Proposition 4.2.1. *Let $T : A \rightarrow B$ be a real linear bijective map between real C*-algebras A and B . Then T is an isometry if and only if the complexification $T_c : A_c \rightarrow B_c$ is an isometry.*

Proof. If T_c is an isometry, then the restriction T is clearly an isometry. If T is an isometry, then it is also a triple homomorphism by [9, Theorem 6.4]. Consequently, T_c is a triple homomorphism and hence an isometry. \square

We now recall that a real abelian C*-algebra A is real C* isomorphic to the algebra $C_0(X, \sigma)$ for some locally compact Hausdorff space X , where $\sigma : X \rightarrow X$ is a homeomorphism of period 2 and

$$C_0(X, \sigma) = \{f \in C_0(X) : f(\sigma(x)) = \overline{f(x)} \text{ for all } x \in X\}, \quad (4.2.1)$$

$C_0(X)$ being the C*-algebra of complex continuous functions on X vanishing at infinity. In particular, taking σ to be the identity map, $C_0(X, \sigma)$ is the algebra $C_0(X, \mathbb{R})$ of real continuous functions on X vanishing at infinity. The investigation of linear isometries between C*-algebras was motivated by the seminal result of Banach and Stone, proved in 1932-37, which determines completely the structure of linear isometries on $C(X, \mathbb{R})$ for compact X , namely, they are exactly the weighted composition operators. This is stated more precisely as follows.

Theorem 4.2.2. *Let K and L be compact Hausdorff spaces. Then $C(K, \mathbb{R})$ is isometric to $C(L, \mathbb{R})$ if and only if K and L are homeomorphic. Moreover, every surjective linear isometry $T : C(L, \mathbb{R}) \rightarrow C(K, \mathbb{R})$ is of the form*

$$T(f)(k) = \alpha(k) \cdot (f \circ h)(k) \quad (k \in K)$$

where $h : K \rightarrow L$ is a homeomorphism and $\alpha : K \rightarrow \mathbb{R}$ is a continuous function satisfying $|\alpha(k)| = 1$.

In fact, the above Banach-Stone theorem is true for all complex abelian C*-algebras $C_0(X)$ (see for example [18]). We complete the picture by showing that it is actually true for all real abelian C*-algebras.

Definition 4.2.3. A mapping $T : C_0(X, \sigma) \rightarrow C_0(Y, \tau)$ is called a *weighted composition operator* if it is of the form $T(f) = u \cdot (f \circ \delta)$ where $u : Y \rightarrow \mathbb{C}$ is a continuous function satisfying $|u(x)| = 1$ for all $x \in Y$ and $\delta : Y \rightarrow X$ a homeomorphism.

Theorem 4.2.4. *A linear bijection between real abelian C* algebras is an isometry if and only if it is a weighted composition operator.*

Proof. Consider $T : C_0(X, \sigma) \rightarrow C_0(Y, \tau)$ for some locally compact Hausdorff spaces X and Y , with period-2 homeomorphisms $\sigma : X \rightarrow X$ and $\tau : Y \rightarrow Y$. If T is a weighted composition operator then there exists a homeomorphism $\delta : Y \rightarrow X$ such that $T(f) = u \cdot f \circ \delta$ for all $f \in C_0(X, \sigma)$. It is immediate

that T is an isometry. Conversely, if T is an isometry, then its complexification $T_c : C_0(X) \rightarrow C_0(Y)$ is an isometry by Proposition 4.3.2. Hence T_c is a weighted composition operator

$$T_c(f) = u \cdot f \circ \delta \quad (f \in C_0(X))$$

for some homeomorphism $\delta : Y \rightarrow X$ and $u \in C_0(Y)$ with $|u(x)| = 1$, $x \in Y$. It follows that $T(f) = u \cdot f \circ \delta$ for all $f \in C_0(X, \sigma)$. \square

4.3 Isometries between real C*-algebras

Our goal in this section is to show that any linear isometry $T : A \rightarrow B$, surjective or not, between real C*-algebras A and B reduces locally to a Jordan triple homomorphism by a projection $p = u^*u \in B''$ for some partial isometry u .

In [9, Theorem 6.4] it is shown that if T is surjective, then it is a global triple homomorphism. For complex C*-algebras A and B , it has been proved in [11, Corollary 3.12] that if $T : A \rightarrow B$ is a (possibly non-surjective) complex linear isometry, then there exists a largest projection $p_a \in B^{**}$ such that $T(\cdot)p_a : A(a) \rightarrow B^{**}$ is an isometry and a triple homomorphism, that is

$$T(\{f, g, h\})p_a = \{T(f), T(g), T(h)\}p_a$$

for all $f, g, h \in A(a)$, where $A(a)$ is the complex JB*-triple generated by a .

The above result has been generalized in [10, Theorem 2] with a different technique to complex JB*-triples. In particular for the complex C*-algebra isometry $T : A \rightarrow B$ above, there exists a tripotent $u \in B^{**}$ such that $\{u, T(\cdot), u\} : A(a) \rightarrow B^{**}$ is an isometry and

$$\{u, T(\{f, g, h\}), u\} = \{u, \{T(f), T(g), T(h)\}, u\}$$

for all $f, g, h \in A(a)$.

It is our aim not only to extend these results to real C*-algebras but also to clarify the connection between the projection p_a and the tripotent u in the aforementioned complex results. In fact, we will show that the tripotent u is the partial isometry that gives rise to the projection p_a , that is, $p_a = u^*u$. Our proof cannot be achieved by complexification and is substantially different from that of [10, 11].

A key element in our arguments is that given an element a in a real C*-algebra

A , the real JB^* -subtriple $A(a)$ generated by a is linearly isometric to the JB^* -triple $C_0(X, \mathbb{R})$ of real continuous functions on X vanishing at infinity (see [30]). For the basic theory of JB^* -triples the reader is referred to Section 2.5. The closed unit ball of a Banach space E will be denoted by E_1 and the set of extreme points of a convex set S will be denoted by ϑS .

For $f \in C_0(X, \mathbb{R})$, the triple product $f^{(3)} = \{f, f, f\}$ is just the pointwise cube f^3 on X . In the following proof, we use repeatedly the Jordan identity in (2.5.1) for the triple product $\{a, b, c\}$ in a C^* -algebra.

Theorem 4.3.1. *Let B be a real C^* -algebra and $T : C_0(X, \mathbb{R}) \rightarrow B$ be a linear isometry which need not be surjective. Then there exists a projection $p \in B''$ such that*

$$(i) \quad T(\{f, g, h\})p = \{T(f), T(g), T(h)\}p;$$

$$(ii) \quad \|T(f)p\| = \|f\| \quad (f, g, h \in C_0(X, \mathbb{R})).$$

Proof. Let $A = C_0(X, \mathbb{R})$ and $E = T(A)$. Let $T' : E' \rightarrow A'$ be the dual map of the surjective isometry $T : A \rightarrow E$. Then T' is a real linear surjective isometry.

Let ψ be an extreme point of the unit ball E'_1 of E' . Then we can naturally extend ψ to an extreme point ϕ of B'_1 . We also denote by $T' : B' \rightarrow A'$ the dual map of $T : A \rightarrow B$ since confusion is unlikely. One sees readily that $T'(\phi) = T'(\psi)$. Also, the set

$$Q = \{\phi \in \vartheta B'_1 : \phi|_E \in \vartheta E'_1\}$$

is non-empty.

Let $\phi \in Q$. Then $\psi = \phi|_E$ is an extreme point in E'_1 and hence $T'(\psi)$ is an extreme point in A'_1 since $T' : E' \rightarrow A'$ is a surjective isometry. We note that the extreme points of A'_1 are of the form $\alpha\delta_x$ with $x \in X$ and $\alpha = \pm 1$ (cf. Section 2.3.5). Hence

$$T'(\phi)(f) = T'(\psi)(f) = \alpha f(x) \quad (f \in A)$$

for some $x \in X$ and $\alpha \in \{1, -1\}$.

We note from Section 2.5.2 that the minimal tripotents of B'' are the support tripotents of the extreme points of B'_1 . Let $u_\phi \in B''$ be the support tripotent of $\phi \in Q$. Then $\phi(\{u_\phi, \cdot, u_\phi\}) = \phi(\cdot)$ and $\{u_\phi, \cdot, u_\phi\} = \phi(\cdot)u_\phi$ by (2.5.5) since u_ϕ is minimal. Therefore $\phi \circ T(f) = T'(\phi)(f) = \alpha f(x)$ implies

$$\{u_\phi, T(f), u_\phi\} = \alpha f(x)u_\phi. \quad (4.3.1)$$

We also note that $\{u_\phi, T(f), u_\phi\}^{(3)} = \alpha f(x)^{(3)}u_\phi = \{u_\phi, T(f^{(3)}), u_\phi\}$. Moreover,

$$\begin{aligned} \{u_\phi, T(\{f, g, h\}), u_\phi\} &= \alpha f(x)g(x)h(x)u_\phi \\ &= \{\alpha f(x)u_\phi, \alpha g(x)u_\phi, \alpha h(x)u_\phi\} \\ &= \{\{u_\phi, T(f), u_\phi\}, \{u_\phi, T(g), u_\phi\}, \{u_\phi, T(h), u_\phi\}\} \end{aligned}$$

which shows that $\{u_\phi, T(\cdot), u_\phi\}$ is a triple homomorphism.

Now, using the Jordan identity (2.5.1) and (4.3.1), we have

$$\begin{aligned} \{u_\phi, \{T(f), u_\phi, T(f)\}, u_\phi\} &= \{\{u_\phi, T(f), u_\phi\}, T(f), u_\phi\} \\ &\quad - \{u_\phi, T(f), \{u_\phi, T(f), u_\phi\}\} \\ &\quad + \{u_\phi, T(f), \{u_\phi, T(f), u_\phi\}\} \\ &= f(x)^2 u_\phi \end{aligned}$$

which gives $\phi(\{u_\phi, \{T(f), u_\phi, T(f)\}, u_\phi\}) = f(x)^2$ and since u_ϕ supports ϕ , we have

$$\phi(\{T(f), u_\phi, T(f)\}) = f(x)^2. \quad (4.3.2)$$

We show $\phi(T(f)^{(3)}) = \alpha f^{(3)}(x) = \phi(T(f^{(3)}))$. It will then follow immediately that $\{u_\phi, T(f)^{(3)}, u_\phi\} = \{u_\phi, T(f^{(3)}), u_\phi\}$.

We will show first that $\{u_\phi, u_\phi, T(h)\} = u_\phi$ for $h \in A$ satisfying $\|h\| = 1$ and $h(x) = \alpha$.

By the Cauchy-Schwarz inequality (cf. Lemma 2.5.6), we have

$$\begin{aligned} 1 &= |\phi \circ T(h)|^2 = |\phi(\{u_\phi, T(h), u_\phi\})|^2 \\ &\leq \phi(\{u_\phi, u_\phi, u_\phi\})\phi(\{T(h), T(h), u_\phi\}) \\ &\leq \|T(h)\|^2 = \|h\|^2 = 1 \end{aligned}$$

which implies

$$\phi(\{T(h), T(h), u_\phi\}) = 1.$$

Let $N_\phi = \{b \in B'' : \phi(\{b, b, u_\phi\}) = 0\}$.

By Lemma 2.5.7,

$$N_\phi = P_0(u_\phi)(B''). \quad (4.3.3)$$

We will now show that $T(h) - u_\phi \in N_\phi$. Indeed we have

$$\begin{aligned} &\phi(\{T(h) - u_\phi, T(h) - u_\phi, u_\phi\}) \\ &= \phi(\{T(h), T(h), u_\phi\}) - \phi(\{u_\phi, T(h), u_\phi\}) - \phi(\{T(h), u_\phi, u_\phi\}) + \phi(u_\phi) \\ &= 1 - \alpha h(x) - \alpha h(x) + 1 = 0 \end{aligned}$$

taking into account that $\phi(\{T(h), u_\phi, u_\phi\}) = \phi(\{u_\phi, T(h), u_\phi\})$ by Lemma 2.5.6. Hence by (4.3.3), $T(h) - u_\phi \in P_0(u_\phi)(B'')$ and then $\{u_\phi, u_\phi, T(h) - u_\phi\} = 0$ and

$$\{u_\phi, u_\phi, T(h)\} = u_\phi.$$

We next show that $\phi(\{T(g), T(g), u_\phi\}) = 0$ whenever $g \in A$ satisfies $g(x) = 0$. Without loss of generality we can take $\|g\| = 1$. We may assume by Urysohn's lemma that g vanishes in a neighborhood of x in which case we can choose $k \in A$ such that $\|k\| = 1$, $k(x) = \alpha$ and $kg = 0$. Then $\|k + g\| = 1$ and $(k + g)(x) = \alpha$. By the above arguments we have $T(k + g) + N_\phi = u_\phi + N_\phi = T(k) + N_\phi$ which implies $T(g) \in N_\phi$ and $\phi(\{T(g), T(g), u_\phi\}) = 0$.

Let $f \in A$ with $\|f\| = 1$. Pick $h \in A$ such that $\|h\| = 1$ and $h(x) = \alpha$. Then $(f - \alpha f(x)h)(x) = 0$ and by the arguments of the previous paragraph, we have

$T(f - \alpha f(x)h) \in N_\phi$. Hence

$$\begin{aligned} \{u_\phi, u_\phi, T(f - \alpha f(x)h)\} &= 0 \quad \text{and} \\ \{u_\phi, u_\phi, T(f)\} &= \{u_\phi, u_\phi, T(\alpha f(x)h)\} = \alpha f(x)u_\phi. \end{aligned} \quad (4.3.4)$$

Using the above result and the Jordan identity (2.5.1), we obtain

$$\begin{aligned} \alpha f(x)\{u_\phi, T(f), u_\phi\} &= \{u_\phi, T(f), \{u_\phi, u_\phi, T(f)\}\} \\ &= \{\{u_\phi, T(f), u_\phi\}, u_\phi, T(f)\} \\ &\quad - \{u_\phi, \{T(f), u_\phi, u_\phi\}, T(f)\} \\ &\quad + \{u_\phi, u_\phi, \{u_\phi, T(f), T(f)\}\} \\ &= f(x)^2 u_\phi - f(x)^2 u_\phi + \{u_\phi, u_\phi, \{u_\phi, T(f), T(f)\}\} \\ &= \{u_\phi, u_\phi, \{u_\phi, T(f), T(f)\}\}. \end{aligned} \quad (4.3.5)$$

Therefore, using Lemma 2.5.6,

$$\begin{aligned} \phi(\{u_\phi, T(f), T(f)\}) &= \phi(\{u_\phi, \{u_\phi, T(f), T(f)\}, u_\phi\}) \\ &= \phi(\{u_\phi, u_\phi, \{u_\phi, T(f), T(f)\}\}) \\ &= \alpha f(x)\phi(\{u_\phi, T(f), u_\phi\}) \\ &= \alpha f(x)\alpha f(x) \\ &= f(x)^2. \end{aligned} \quad (4.3.6)$$

Combining (4.3.2), (4.3.4) and (4.3.6) we obtain

$$\begin{aligned}
\phi(\{u_\phi, T(f)^{(3)}, u_\phi\}) &= \phi(\{u_\phi, u_\phi, \{T(f), T(f), T(f)\}\}) \\
&= \phi(\{\{u_\phi, u_\phi, T(f)\}, T(f), T(f)\}) \\
&\quad - \phi(\{T(f), \{u_\phi, u_\phi, T(f)\}, T(f)\}) \\
&\quad + \phi(\{T(f), T(f), \{u_\phi, u_\phi, T(f)\}\}) \\
&= \alpha f(x)(2\phi(\{u_\phi, T(f), T(f)\}) - \phi(\{T(f), u_\phi, T(f)\})) \\
&= 2\alpha f(x)f(x)^2 - \alpha f(x)f(x)^2 \\
&= \alpha f^3(x)
\end{aligned}$$

which yields

$$\{u_\phi, T(f)^{(3)}, u_\phi\} = \{u_\phi, T(f^3), u_\phi\}. \quad (4.3.7)$$

Taking into consideration (4.3.1), (4.3.4) and (4.3.5) we have

$$\begin{aligned}
\{u_\phi, T(f), T(f)\} &= \{\{u_\phi, u_\phi, u_\phi\}, T(f), T(f)\} \\
&= \{u_\phi, u_\phi, \{u_\phi, T(f), T(f)\}\} \\
&\quad - \{u_\phi, T(f), \{u_\phi, u_\phi, T(f)\}\} \\
&\quad + \{u_\phi, \{u_\phi, u_\phi, T(f)\}, T(f)\} \\
&= f(x)^2 u_\phi
\end{aligned} \quad (4.3.8)$$

and then

$$\begin{aligned}
(\alpha f(x))^3 u_\phi &= \alpha f(x) \{u_\phi, T(f), T(f)\} \\
&= \{\{u_\phi, T(f), u_\phi\}, T(f), T(f)\} \\
&= \{T(f), T(f), \{u_\phi, T(f), u_\phi\}\} \\
&= \{T(f), T(f), \{T(f), u_\phi, u_\phi\}\} \\
&= \{\{T(f), T(f), T(f)\}, u_\phi, u_\phi\} \\
&\quad - \{T(f), \{T(f), T(f), u_\phi\}, u_\phi\} \\
&\quad + \{T(f), u_\phi, \{T(f), T(f), u_\phi\}\} \\
&= \{T(f)^{(3)}, u_\phi, u_\phi\}.
\end{aligned}$$

Since

$$\{u_\phi, u_\phi, T(f^{(3)})\} = \alpha f^3(x) u_\phi,$$

we get

$$\{T(f)^{(3)}, u_\phi, u_\phi\} = \{T(f^{(3)}), u_\phi, u_\phi\} \quad (4.3.9)$$

that is, $\{T(f)^{(3)} - T(f^{(3)}), u_\phi, u_\phi\} = 0$ in B'' . We conclude that

$$(T(f)^{(3)} - T(f^{(3)}))u_\phi^* u_\phi + u_\phi u_\phi^* (T(f)^{(3)} - T(f^{(3)})) = 0. \quad (4.3.10)$$

We now wish to prove that (4.3.7) and (4.3.9) hold for the Jordan triple product $\{T(f), T(g), T(h)\}$ for all $f, g, h \in A$, despite the unavailability of the polarisation in the real case.

By (4.3.1), (4.3.8) and the Jordan identity we observe that

$$\begin{aligned}
& \{u_\phi, \{T(f), T(f), T(g)\}, u_\phi\} \\
&= 2\{\{T(f), T(f), u_\phi\}, T(g), u_\phi\} - \{T(f), T(f), \{u_\phi, T(g), u_\phi\}\} \\
&= 2\{(\alpha f(x))^2 u_\phi, T(g), u_\phi\} - \{T(f), T(f), \alpha g(x) u_\phi\} \\
&= 2(\alpha f(x))^2 \alpha g(x) u_\phi - \alpha g(x) (\alpha f(x))^2 u_\phi \\
&= \alpha f(x)^2 g(x) u_\phi \\
&= \{u_\phi, T(\{f, f, g\}), u_\phi\}.
\end{aligned} \tag{4.3.11}$$

It follows from the identity (2.5.4) that

$$\begin{aligned}
& \{u_\phi, 2T(\{f, g, f\}) + 4T(\{f, f, g\}), u_\phi\} \\
&= \{u_\phi, T((f+g)^{(3)}) + T((g-f)^{(3)}) - 2T(g^{(3)}), u_\phi\} \\
&= \{u_\phi, T(f+g)^{(3)} + T(g-f)^{(3)} - 2T(g)^{(3)}, u_\phi\} \\
&= \{u_\phi, 2\{T(f), T(g), T(f)\} + 4\{T(f), T(f), T(g)\}, u_\phi\}
\end{aligned}$$

which implies $\{u_\phi, T(\{f, g, f\}), u_\phi\} = \{u_\phi, \{T(f), T(g), T(f)\}, u_\phi\}$ by (4.3.11), and then by (2.5.3),

$$\{u_\phi, \{T(f), T(g), T(h)\}, u_\phi\} = \{u_\phi, T(\{f, g, h\}), u_\phi\}. \tag{4.3.12}$$

The identity

$$\begin{aligned}
& \{\{u_\phi, u_\phi, \{u_\phi, T(g), T(f)\}\} = \{\{u_\phi, u_\phi, u_\phi\}, T(g), T(f)\} \\
& \quad - \{u_\phi, \{u_\phi, u_\phi, T(g)\}, T(f)\} \\
& \quad + \{u_\phi, T(g), \{u_\phi, u_\phi, T(f)\}\} \\
& = \{u_\phi, T(g), T(f)\}
\end{aligned}$$

implies that $\{u_\phi, T(g), T(f)\} \in P_2(u_\phi)$.

Hence by the Jordan identity (2.5.1), (4.3.1) and (4.3.4), we get

$$\begin{aligned}
\{u_\phi, T(g), T(f)\} &= \{u_\phi, \{u_\phi, \{u_\phi, T(g), T(f)\}, u_\phi\}, u_\phi\} \\
&= \{u_\phi, 2\{\{T(g), u_\phi, u_\phi\}, T(f), u_\phi\} - \{T(g), u_\phi, \{u_\phi, T(f), u_\phi\}\}, u_\phi\} \\
&= \{u_\phi, 2g(x)f(x)u_\phi - g(x)f(x)u_\phi, u_\phi\} = g(x)f(x)u_\phi.
\end{aligned}$$

Moreover,

$$\begin{aligned}
f(x)^2u_\phi &= \{T(f), \{u_\phi, T(f), u_\phi\}, u_\phi\} \\
&= \{\{T(f), u_\phi, T(f)\}, u_\phi, u_\phi\} \\
&\quad - \{T(f), u_\phi, \{T(f), u_\phi, u_\phi\}\} \\
&\quad + \{T(f), u_\phi, \{T(f), u_\phi, u_\phi\}\} \\
&= \{\{T(f), u_\phi, T(f)\}, u_\phi, u_\phi\}
\end{aligned}$$

which gives

$$\begin{aligned}
\{T(f), u_\phi, T(f)\} &= \{T(f), \{u_\phi, u_\phi, u_\phi\}, T(f)\} \\
&= \{\{u_\phi, u_\phi, T(f)\}, u_\phi, T(f)\} \\
&\quad - \{u_\phi, u_\phi, \{T(f), u_\phi, T(f)\}\} \\
&\quad + \{T(f), u_\phi, \{u_\phi, u_\phi, T(f)\}\} \\
&= f(x)^2u_\phi.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\{u_\phi, u_\phi, \{T(f), T(g), T(f)\}\} \\
&= 2\{\{u_\phi, u_\phi, T(f)\}, T(g), T(f)\} - \{T(f), \{u_\phi, u_\phi, T(g)\}, T(f)\} \\
&= \alpha f(x)\{u_\phi, T(g), T(f)\} - \alpha g(x)\{T(f), u_\phi, T(f)\} \\
&= 2\alpha f(x)g(x)f(x)u_\phi - \alpha g(x)(\alpha f(x))^2u_\phi \\
&= \alpha f(x)^2g(x)u_\phi = \{u_\phi, u_\phi, T(\{f, g, f\})\}.
\end{aligned}$$

Again, using the identity (2.5.3), we obtain

$$\{u_\phi, u_\phi, \{T(f), T(g), T(h)\}\} = \{u_\phi, u_\phi, T(\{f, g, h\})\} \quad (4.3.13)$$

and hence $\{\{T(f), T(g), T(h)\} - T(\{f, g, h\}), u_\phi, u_\phi\} = 0$ in B'' , in other words

$$\begin{aligned} 0 &= (\{T(f), T(g), T(h)\} - T(\{f, g, h\}))u_\phi^*u_\phi \\ &\quad + u_\phi u_\phi^*(\{T(f), T(g), T(h)\} - T(\{f, g, h\})). \end{aligned} \quad (4.3.14)$$

Let $p_\phi = u_\phi^*u_\phi$ and $q_\phi = u_\phi u_\phi^*$ be the initial and final minimal projections of the minimal tripotent u_ϕ (see Lemma 2.5.2). By (4.3.10), we get

$$\begin{aligned} (T(f)^{(3)} - T(f^{(3)}))p_\phi + q_\phi(T(f)^{(3)} - T(f^{(3)})) &= 0 \\ (T(f)^{(3)} - T(f^{(3)}))p_\phi + q_\phi(T(f)^{(3)} - T(f^{(3)}))p_\phi &= 0. \end{aligned}$$

Since $\{u_\phi, T(f)^{(3)}, u_\phi\} = \{u_\phi, T(f^{(3)}), u_\phi\}$ in (4.3.7) implies

$$\{u_\phi, \{u_\phi, T(f)^{(3)} - T(f^{(3)}), u_\phi\}, u_\phi\} = 0,$$

or equivalently

$$q_\phi(T(f)^{(3)} - T(f^{(3)}))p_\phi = 0,$$

we have

$$(T(f)^{(3)} - T(f^{(3)}))p_\phi = 0. \quad (4.3.15)$$

Let $p = \bigvee_{\phi \in Q} p_\phi$ be the lattice supremum in B'' .

By Lemma 2.5.3 and (4.3.15), we have $(T(f)^{(3)} - T(f^{(3)}))p = 0$, in other words

$$T(f)^{(3)}p = T(f^{(3)})p \quad (f \in A). \quad (4.3.16)$$

Applying the same arguments to (4.3.14), we obtain

$$\{T(f), T(g), T(h)\}p = T(\{f, g, h\})p \quad (f, g, h \in A). \quad (4.3.17)$$

Hence we have proved statement (i). To prove statement (ii) we observe that for any $f \in A$, we can pick $x_f \in X$ with $\|f\| = |f(x_f)|$. Let $\psi \in \vartheta E'_1$ be such that $T'(\psi) = \delta_{x_f}$. Let $\phi \in \vartheta B'_1$ be an extension of ψ . Then $\phi \in Q$ and $T'(\phi) = \delta_{x_f}$. Hence

$$\begin{aligned} \|T(f)\| &\geq \|T(f)p\| \geq \|T(f)pp_\phi\| = \|T(f)p_\phi\| = \|T(f)u_\phi^*u_\phi\| \\ &\geq \|u_\phi u_\phi^* T(f) u_\phi^* u_\phi\| = \|\{u_\phi, \{u_\phi, T(f), u_\phi\}, u_\phi\}\| = \|f(x_f)u_\phi\| = \|f\| \end{aligned}$$

which implies $\|T(f)p\| = \|f\|$ for all $f \in A$. □

Remark 4.3.2. In the above proof, we observe that (4.3.8) implies

$$u_\phi T(f)^* T(f) + T(f) T(f)^* u_\phi = f(x)^2 u_\phi$$

and then

$$\begin{aligned} u_\phi^* u_\phi T(f)^* T(f) &= f(x)^2 u_\phi^* u_\phi - u_\phi^* T(f) T(f)^* u_\phi \\ &= (f(x)^2 u_\phi^* - u_\phi^* T(f) T(f)^*) u_\phi \\ &= T(f)^* T(f) u_\phi^* u_\phi, \end{aligned}$$

that is, $p_\phi T(f)^* T(f) = T(f)^* T(f) p_\phi$.

Multiplying by p on the left, $p T(f)^* T(f) p_\phi = p_\phi T(f)^* T(f) = T(f)^* T(f) p_\phi$.

By Lemma 2.5.3, we conclude $p T(f)^* T(f) p = T(f)^* T(f) p$ and thus

$$p T(f)^* T(f) = T(f)^* T(f) p.$$

In particular, if the domain A is the algebra $C_0(X, \mathbb{R})$, then the largest projection in Proposition 4.1.1 is non-zero. By the previous arguments, the projection p of Theorem 4.3.1 satisfies the conditions of Proposition 4.1.1. Let p_T be the largest projection for $A = C_0(X, \mathbb{R})$ in Proposition 4.1.1, then we have $p \leq p_T$ and $\|T(f)\| = \|T(f)p\| = \|T(f)pp_T\| \leq \|T(f)p_T\| \leq \|T(f)\|$. It follows that

$\|T(f)p_T\| = \|T(f)\|$. This extends the complex result in [11, Theorem 3.10].

The next result reveals the connection between Theorem 1 in [10] and Theorem 3.10 in [11]. In fact, it will be shown that p in Theorem 4.3.1 is actually the initial projection of the tripotent u in Theorem 4.3.3 below.

Theorem 4.3.3. *Let B be a real C^* -algebra and $T : C_0(X, \mathbb{R}) \rightarrow B$ be a linear isometry, not necessarily surjective. Then there exists a partial isometry $u \in B''$ such that*

- (i) $\{u, T(\{f, g, h\}), u\} = \{u, \{T(f), T(g), T(h)\}, u\}$ with $f, g, h \in C_0(X, \mathbb{R})$;
- (ii) $\{u, T(\cdot), u\} : C(X, \mathbb{R}) \rightarrow B''$ is an isometry.

Proof. We let $A = C_0(X, \mathbb{R})$, $E = T(A)$ and $Q = \{\phi \in \vartheta B'_1 : \phi|_E \in \vartheta E'_1\}$. As in the proof of Theorem 4.3.1, for each $\phi \in Q$, there corresponds a minimal tripotent $u_\phi \in B''$ such that

$$\{u_\phi, T(\{f, g, h\}), u_\phi\} = \{u_\phi, \{T(f), T(g), T(h)\}, u_\phi\}.$$

Let $p_\phi = u_\phi^* u_\phi$ and $q_\phi = u_\phi u_\phi^*$ be the initial and final minimal projections of the minimal tripotent u_ϕ . We note that for $\phi, \phi' \in Q$, $\phi \neq \phi'$ implies $p_\phi \neq p_{\phi'}$. We can choose from Q a maximal subfamily Q' such that for $\phi \neq \phi' \in Q'$, the corresponding central projections satisfy $c(\phi) \perp c(\phi')$, where $c(\phi)$ denotes $c(p_\phi) = c(q_\phi)$ and $c(\phi')$ denotes $c(p_{\phi'}) = c(q_{\phi'})$. Define

$$p_{c(\phi)} = \bigvee_{\substack{\psi \in Q \\ c(p_\psi) = c(\phi)}} p_\psi \quad \text{and} \quad q_{c(\phi)} = \bigvee_{\substack{\psi \in Q \\ c(q_\psi) = c(\phi)}} q_\psi \quad (\phi \in Q').$$

Then we have $p_{c(\phi)} \sim q_{c(\phi)}$ by Lemma 2.5.5. By Remark 2.5.4, we get

$$\begin{aligned} p &= \bigvee_{\phi \in Q} p_\phi = \bigvee_{\phi \in Q'} p_{c(\phi)} \\ q &= \bigvee_{\phi \in Q} q_\phi = \bigvee_{\phi \in Q'} q_{c(\phi)}. \end{aligned}$$

We note that the projection p as defined above is the same as the one in Theorem 4.3.1. Since for $\phi \neq \phi' \in Q'$ we have $p_{c(\phi)} \perp p_{c(\phi')}$ and $q_{c(\phi)} \perp q_{c(\phi')}$, it follows that

$$p = \bigvee_{\phi \in Q'} p_{c(\phi)} \sim q = \bigvee_{\phi \in Q'} q_{c(\phi)}.$$

Hence there exists a tripotent $u \in B''$ such that $p = u^*u$ and $q = uu^*$. By (4.3.17) we have

$$\begin{aligned} \{u, \{u, T(\{f, g, h\}), u\}, u\} &= qT(\{f, g, h\})p \\ &= q\{T(f), T(g), T(h)\}p \\ &= \{u, \{u, \{T(f), T(g), T(h)\}, u\}, u\} \end{aligned}$$

and taking the triple product with the tripotent u in both sides we get

$$\{u, T(\{f, g, h\}), u\} = \{u, \{T(f), T(g), T(h)\}, u\} \quad (f, g, h \in A). \quad (4.3.18)$$

Finally, for any f we pick $x_f \in X$ as in Theorem 4.3.1 with $\|f\| = |f(x_f)|$. Let $\psi \in \vartheta E'_1$ such that $T'(\psi) = \delta_{x_f}$. Let $\phi \in \vartheta B'_1$ be an extension of ψ . Then $\phi \in Q$ and $T'(\phi) = \delta_{x_f}$. Therefore,

$$\begin{aligned} \|T(f)\| &\geq \|\{u, T(f), u\}\| \geq \|\{u, \{u, T(f), u\}, u\}\| \geq \|u_\phi u_\phi^* u u^* T(f) u^* u u_\phi^* u_\phi\| \\ &\geq \|u_\phi u_\phi^* T(f) u_\phi^* u_\phi\| = \|\{u_\phi, \{u_\phi, T(f), u_\phi\}, u_\phi\}\| = \|\{u_\phi, T(f), u_\phi\}\| \\ &= \|f(x_f)u_\phi\| = \|f\|. \end{aligned}$$

We conclude that

$$\|\{u, T(f), u\}\| = \|f\| \quad (f \in A). \quad (4.3.19)$$

□

Remark 4.3.4. Using the previous results, we can now show that if T is an isometry between two real C^* -algebras A and B , then T is a local triple homomorphism via a tripotent in B'' . That does not however imply that T is a triple homomorphism on the whole algebra A as demonstrated in the example of Section 4.4. The following theorem is an extension of the results in [10, 11].

Theorem 4.3.5. *Let $T : A \rightarrow B$ be a linear isometry between two real C^* -algebras A and B . We do not assume that T is surjective. Then for each $a \in A$, there exist a partial isometry $u \in B''$ and a projection $p \in B''$ such that*

$$(i) \quad \{u, T(\{f, g, h\}), u\} = \{u, \{T(f), T(g), T(h)\}, u\};$$

$$(ii) \quad T(\{f, g, h\})p = \{T(f), T(g), T(h)\}p,$$

for all f, g, h in the real JB^* -triple $A(a) \subseteq A$ generated by a .

Moreover,

$$\{u, T(\cdot), u\} : A(a) \rightarrow B'' \quad \text{and} \quad T(\cdot)p : A(a) \rightarrow B''$$

are isometries. In addition, p is the initial projection of the partial isometry u .

Proof. Since $A(a)$ is linearly isometric to the JB^* -triple $C_0(X, \mathbb{R})$ of continuous functions on X vanishing at infinity (cf. [30]), the result follows immediately from Theorem 4.3.1 and Theorem 4.3.3. □

4.4 Examples

We show that Theorem 4.3.1 and Theorem 4.3.3 cannot be extended to *all* real abelian C*-algebras which are of the form $C_0(X, \sigma)$ (cf. (4.2.1) or Section 2.3.3). We will give an example of a non-surjective real linear isometry $T : C(D, \sigma) \rightarrow B$ for a compact Hausdorff space D such that there is no projection $p \in B''$ nor a tripotent $u \in B''$ satisfying all the conditions of Theorem 4.3.1 and Theorem 4.3.3 respectively. In addition we show that the complexification T_c of the isometry T is not an isometry.

Example 4.4.1. Let D be the closed unit disc in \mathbb{C} . Let $C(D, \sigma)$ be the real abelian C*-algebra of complex continuous functions on D that satisfy $f(\sigma(x)) = \overline{f(x)}$ where $\sigma : D \rightarrow D$ is defined by conjugation $\sigma(x) = \bar{x}$. Let $C(D, M_2(\mathbb{R}))$ be the real C*-algebra of continuous $M_2(\mathbb{R})$ -valued functions on D , where $M_2(\mathbb{R})$ denotes the real C*-algebra of 2×2 real matrices.

Define $T : C(D, \sigma) \rightarrow C(D, M_2(\mathbb{R}))$ by

$$T(f) = \begin{pmatrix} \operatorname{Re}(f) & \operatorname{Im}(f) \\ 0 & 0 \end{pmatrix} \quad (f \in C(D, \sigma)).$$

Then T is a real linear isometry that cannot be complexified to an isometry on the complexification $C(D, \sigma) \oplus iC(D, \sigma)$. Moreover there exists neither a projection $p \in C(D, M_2(\mathbb{R}))''$ nor a tripotent $u \in C(D, M_2(\mathbb{R}))''$ satisfying all the conditions of Theorem 4.3.1 and Theorem 4.3.3 respectively.

It is evident that the map T is real linear. The norm of $f \in C(D, \sigma)$ is the supremum norm $\|f\| = \sup\{|f(x)| : x \in D\}$. We have

$$\|f\|^2 = \sup\{\operatorname{Re}(f(x))^2 + \operatorname{Im}(f(x))^2 : x \in D\}.$$

The C*-norm of $T(f)$ in $M_2(\mathbb{R})$ is given by

$$\begin{aligned}
\|T(f)\|^2 &= \left\| \begin{pmatrix} \operatorname{Re}(f) & \operatorname{Im}(f) \\ 0 & 0 \end{pmatrix} \right\|^2 \\
&= \left\| \begin{pmatrix} \operatorname{Re}(f) & \operatorname{Im}(f) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \operatorname{Re}(f) & 0 \\ \operatorname{Im}(f) & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} \operatorname{Re}(f)^2 + \operatorname{Im}(f)^2 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \sup\{\operatorname{Re}(f(x))^2 + \operatorname{Im}(f(x))^2 : x \in D\}.
\end{aligned}$$

This shows that T is an isometry.

We have

$$\begin{aligned}
T(f)^3 &= \begin{pmatrix} \operatorname{Re}(f) & \operatorname{Im}(f) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \operatorname{Re}(f) & 0 \\ \operatorname{Im}(f) & 0 \end{pmatrix} \begin{pmatrix} \operatorname{Re}(f) & \operatorname{Im}(f) \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} (\operatorname{Re}(f)^2 + \operatorname{Im}(f)^2)\operatorname{Re}(f) & (\operatorname{Re}(f)^2 + \operatorname{Im}(f)^2)\operatorname{Im}(f) \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \operatorname{Re}(|f|^2 f) & \operatorname{Im}(|f|^2 f) \\ 0 & 0 \end{pmatrix} \\
&= T(|f|^2 f) \\
&= T(f^3)
\end{aligned}$$

and the largest projection p in Proposition 4.1.1 for T is just the identity. We observe, however, that $T(\{f, g, h\}) \neq \{T(f), T(g), T(h)\}$.

We will now show that there is no projection $p \in C(D, M_2(\mathbb{R}))''$ such that

$$\{T(f), T(g), T(h)\}p = T(\{f, g, h\})p \quad \text{and} \quad \|T(f)p\| = \|f\|.$$

Let p be such a projection for contradiction. We observe that for a projection $p \in C(D, M_2(\mathbb{R}))''$ to satisfy $\{T(f), T(g), T(h)\}p = T(\{f, g, h\})p$, it is equivalent

to satisfying

$$\{T(f), T(g), T(f)\}p = T(\{f, g, f\})p. \quad (4.4.1)$$

Now,

$$\begin{aligned} \{T(f), T(g), T(f)\} &= \begin{pmatrix} \operatorname{Re}(f) & \operatorname{Im}(f) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \operatorname{Re}(g) & 0 \\ \operatorname{Im}(g) & 0 \end{pmatrix} \begin{pmatrix} \operatorname{Re}(f) & \operatorname{Im}(f) \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \operatorname{Re}^2(f)\operatorname{Re}(g) & \operatorname{Im}^2(f)\operatorname{Im}(g) \\ +\operatorname{Re}(f)\operatorname{Im}(f)\operatorname{Im}(g) & +\operatorname{Re}(f)\operatorname{Im}(f)\operatorname{Im}(g) \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

On the other hand we have,

$$\begin{aligned} \{f, g, f\} &= (\operatorname{Re}(f) + i\operatorname{Im}(f))(\operatorname{Re}(g) - i\operatorname{Im}(g))(\operatorname{Re}(f) + i\operatorname{Im}(f)) \\ &= (\operatorname{Re}^2(f)\operatorname{Re}(g) - \operatorname{Im}^2(f)\operatorname{Re}(g) + 2\operatorname{Re}(f)\operatorname{Im}(f)\operatorname{Im}(g) \\ &\quad + i(\operatorname{Im}^2(f)\operatorname{Im}(g) - \operatorname{Re}^2(f)\operatorname{Im}(g) + 2\operatorname{Re}(f)\operatorname{Im}(f)\operatorname{Re}(g))) \end{aligned}$$

and hence

$$T(\{f, g, f\}) = \begin{pmatrix} \operatorname{Re}^2(f)\operatorname{Re}(g) & \operatorname{Im}^2(f)\operatorname{Im}(g) \\ -\operatorname{Im}^2(f)\operatorname{Re}(g) & -\operatorname{Re}^2(f)\operatorname{Im}(g) \\ +2\operatorname{Re}(f)\operatorname{Im}(f)\operatorname{Im}(g) & +2\operatorname{Re}(f)\operatorname{Im}(f)\operatorname{Re}(g) \\ 0 & 0 \end{pmatrix}.$$

Therefore (4.4.1) requires that

$$\begin{aligned} & \begin{pmatrix} \operatorname{Re}^2(f)\operatorname{Re}(g) & \operatorname{Im}^2(f)\operatorname{Im}(g) \\ +\operatorname{Re}(f)\operatorname{Im}(f)\operatorname{Im}(g) & +\operatorname{Re}(f)\operatorname{Im}(f)\operatorname{Im}(g) \\ 0 & 0 \end{pmatrix} p \\ &= \begin{pmatrix} \operatorname{Re}^2(f)\operatorname{Re}(g) & \operatorname{Im}^2(f)\operatorname{Im}(g) \\ -\operatorname{Im}^2(f)\operatorname{Re}(g) & -\operatorname{Re}^2(f)\operatorname{Im}(g) \\ +2\operatorname{Re}(f)\operatorname{Im}(f)\operatorname{Im}(g) & +2\operatorname{Re}(f)\operatorname{Im}(f)\operatorname{Re}(g) \\ 0 & 0 \end{pmatrix} p \end{aligned}$$

which simplifies to

$$\begin{pmatrix} \operatorname{Re}(f)\operatorname{Im}(f)\operatorname{Im}(g) & \operatorname{Re}(f)\operatorname{Im}(f)\operatorname{Re}(g) \\ -\operatorname{Im}^2(f)\operatorname{Re}(g) & -\operatorname{Re}^2(f)\operatorname{Im}(g) \\ 0 & 0 \end{pmatrix} p = 0. \quad (4.4.2)$$

Let $g \in C(D, \sigma)$ be the constant function $g(x) = 1$ and define $f \in C(D, \sigma)$ to be the function $f(x) = i\operatorname{Im}(x)$. Substituting f, g into (4.4.2) we obtain

$$\begin{pmatrix} -\operatorname{Im}^2(x) & 0 \\ 0 & 0 \end{pmatrix} p = 0 \quad (x \in D). \quad (4.4.3)$$

Now let $h \in C(D, \sigma)$ be the function $h(x) = \operatorname{Im}^2(x)$. Then (4.4.3) implies

$$\|T(h)p\| = \left\| \begin{pmatrix} \operatorname{Im}^2(x) & 0 \\ 0 & 0 \end{pmatrix} p \right\| = 0 \neq \|h\|.$$

Hence there exists no projection p satisfying all conditions in Theorem 4.3.1. We now consider the requirements of Theorem 4.3.3. For contradiction, let u be a tripotent in $C(D, M_2(\mathbb{R}))''$ satisfying all conditions of Theorem 4.3.3 for T de-

fined above. By similar computation as for the projection p before, we get

$$\left\{ u, \begin{pmatrix} \operatorname{Re}(f)\operatorname{Im}(f)\operatorname{Im}(g) & \operatorname{Re}(f)\operatorname{Im}(f)\operatorname{Re}(g) \\ -\operatorname{Im}^2(f)\operatorname{Re}(g) & -\operatorname{Re}^2(f)\operatorname{Im}(g) \\ 0 & 0 \end{pmatrix}, u \right\} = 0.$$

Substituting f and g defined above, we also have

$$\left\{ u, \begin{pmatrix} -\operatorname{Im}^2(x) & 0 \\ 0 & 0 \end{pmatrix}, u \right\} = 0 \quad (x \in D).$$

Hence the same function $h \in C(D, \sigma)$ with $h(x) = \operatorname{Im}^2(x)$ gives the following contradiction:

$$\|\{u, T(h), u\}\| = \left\| \left\{ u, \begin{pmatrix} \operatorname{Im}^2(x) & 0 \\ 0 & 0 \end{pmatrix}, u \right\} \right\| = 0 \neq \|h\|.$$

Therefore there is no u satisfying all the conditions of Theorem 4.3.3 for the isometry T .

Finally, consider the complexification T_c :

$$\begin{aligned} T_c(g + if) &= T(g) + iT(f) \\ &= \begin{pmatrix} \operatorname{Re}(g) + i\operatorname{Re}(f) & \operatorname{Im}(g) + i\operatorname{Im}(f) \\ 0 & 0 \end{pmatrix} \in C(D, M_2(\mathbb{C})). \end{aligned}$$

Let $g(x) = 1$ be the constant function and $f(x) = i\operatorname{Im}(x)$ as before. Then

$$\begin{aligned} \|g + if\| &= \|1 - \operatorname{Im}(x)\| = \sup\{|1 - \operatorname{Im}(x)| : x \in D\} = 2 \quad \text{while} \\ \|T_c(g + if)\| &= \left\| \begin{pmatrix} 1 & i\operatorname{Im}(x) \\ 0 & 0 \end{pmatrix} \right\| = \sup\{\sqrt{1^2 + \operatorname{Im}(x)^2} : x \in D\} = \sqrt{2}. \end{aligned}$$

Therefore $T_c : C(D) \rightarrow C(D, M_2(\mathbb{C}))$ is not an isometry.

4.5 Jordan*-homomorphisms

We have so far studied linear maps between real C^* -algebras that in some form preserve the Jordan ternary structure. We are now going to study maps that preserve the Jordan binary product and the involution, namely, the Jordan*-homomorphisms. We conclude the thesis by showing that if $T : A \rightarrow B$ is a Jordan*-homomorphism from a real C^* -algebra A to a real C^* -algebra B , then

$$T(\cdot) = T(\cdot)z + T(\cdot)(1 - z)$$

for some projection $z \in B''$, where $T(\cdot)z$ is a C^* -homomorphism and $T(\cdot)(1 - z)$ is a C^* -antihomomorphism. The proof follows similar arguments as in [27] and [50, p.187].

Definition 4.5.1. We call a linear map $T : A \rightarrow B$ between real C^* -algebras A and B a *Jordan*-homomorphism* if $T(ab + ba) = T(a)T(b) + T(b)T(a)$ and $T(a^*) = T(a)^*$ for all $a, b \in A$.

A linear map $T : A \rightarrow B$ is called a *C^* -homomorphism* if $T(ab) = T(a)T(b)$ and $T(a^*) = T(a)^*$. A *C^* -antihomomorphism* is such that $T(ab) = T(b)T(a)$ and $T(a^*) = T(a)^*$.

We note that the condition $T(ab+ba) = T(a)T(b)+T(b)T(a)$ for all $a, b \in A$, is equivalent to $T(a^2) = T(a)^2$ for all $a \in A$ since $ab + ba = (a + b)^2 - a^2 - b^2$. Moreover, by similar arguments as in [49, p.439] a Jordan*-homomorphism is automatically continuous.

Lemma 4.5.2. *Let $T : A \rightarrow B$ be a Jordan*-homomorphism between real C^* -algebras A and B . Then the following properties hold for all a, b, c in A ,*

$$(i) \quad T(abc + cba) = T(a)T(b)T(c) + T(c)T(b)T(a);$$

$$(ii) \quad (T(ab) - T(a)T(b))(T(ab) - T(b)T(a)) = 0 \text{ and} \\ (T(ab) - T(b)T(a))(T(ab) - T(a)T(b)) = 0;$$

(iii) If $a = a^2$ and $ab = ba$, then $T(a)T(b) = T(b)T(a)$.

(iv) If A is a real abelian C^* -algebra, then T is a C^* -homomorphism.

Proof. (i) If we denote by $a \circ b = \frac{1}{2}(ab + ba)$ the Jordan binary product in A , then the ternary product is given by $abc + cba = 2(a \circ b) \circ c + 2a \circ (b \circ c) - 2b \circ (a \circ c)$. Hence the result follows.

(ii) By (i), we have

$$\begin{aligned} & (T(ab) - T(a)T(b))(T(ab) - T(b)T(a)) \\ &= T(ab)^2 + T(a)T(b)^2T(a) - T(ab)T(b)T(a) - T(a)T(b)T(ab) \\ &= T(abab) + T(ab^2a) - T(abba + abab) \\ &= T(abab + abba - abba - abab) = 0. \end{aligned}$$

Likewise we prove $(T(ab) - T(b)T(a))(T(ab) - T(a)T(b)) = 0$.

(iii) Since $ab = ba = aba$, we have $ab + ba = 2aba$ and by (i), $T(a)T(b) + T(b)T(a) = 2T(a)T(b)T(a)$. Moreover, as $T(a)^2 = T(a^2) = T(a)$, multiplication on the left by $T(a)$ gives $T(a)T(b) = T(a)T(b)T(a)$. Hence $T(b)T(a) = T(a)T(b)$.

(iv) Let $a = a^*$ and $b = b^*$ in A . Then by (ii),

$$\begin{aligned} & (T(ab) - T(a)T(b))(T(ab) - T(a)T(b))^* \\ &= (T(ab) - T(a)T(b))(T(b^*a^*) - T(b^*)T(a^*)) \\ &= (T(ab) - T(a)T(b))(T(ba) - T(b)T(a)) \\ &= (T(ab) - T(a)T(b))(T(ab) - T(b)T(a)) = 0 \end{aligned}$$

which implies $T(ab) = T(a)T(b) = T(b)T(a)$.

Now let $a = a^*$ and $b = -b^*$, or $a = -a^*$ and $b = b^*$ in A . By (ii),

$$\begin{aligned}
& (T(ab) - T(a)T(b))(T(ab) - T(a)T(b))^* \\
&= (T(ab) - T(a)T(b))(T(b^*a^*) - T(b^*)T(a^*)) \\
&= (T(ab) - T(a)T(b))(-T(ba) + T(b)T(a)) \\
&= -(T(ab) - T(a)T(b))(T(ab) - T(b)T(a)) = 0
\end{aligned}$$

which again implies $T(ab) = T(a)T(b) = T(b)T(a)$.

Finally let $a = -a^*$ and $b = -b^*$ in A . We have

$$\begin{aligned}
& (T(ab) - T(a)T(b))(T(ab) - T(a)T(b))^* \\
&= (T(ab) - T(a)T(b))(T(b^*a^*) - T(b^*)T(a^*)) \\
&= (T(ab) - T(a)T(b))(T(ba) - T(b)T(a)) \\
&= (T(ab) - T(a)T(b))(T(ab) - T(b)T(a)) = 0.
\end{aligned}$$

which gives $T(ab) = T(a)T(b) = T(b)T(a)$.

Now take any $a, b \in A$. We consider the decomposition $a = \frac{a+a^*}{2} + \frac{a-a^*}{2}$ and $b = \frac{b+b^*}{2} + \frac{b-b^*}{2}$. Then using the previous results as well as the fact that $a + a^*, b + b^*$ are hermitian elements and $a - a^*, b - b^*$ are skew hermitian elements which all commute with each other, we obtain

$$\begin{aligned}
T(ab) &= T\left(\left(\frac{a+a^*}{2} + \frac{a-a^*}{2}\right)\left(\frac{b+b^*}{2} + \frac{b-b^*}{2}\right)\right) \\
&= T\left(\frac{a+a^*}{2} + \frac{a-a^*}{2}\right)T\left(\frac{b+b^*}{2} + \frac{b-b^*}{2}\right) \\
&= T(a)T(b)
\end{aligned}$$

□

Remark 4.5.3. It is immediate that if T is a Jordan*-homomorphism, then it preserves the triple product $T(\{a, b, c\}) = \frac{1}{2}T(ab^*c + cb^*a) = \frac{1}{2}(T(a)T(b)^*T(c) +$

$$T(c)T(b)^*T(a) = \{T(a), T(b), T(c)\}.$$

Definition 4.5.4. Let $n \geq 2$. A family $\{u_{ij} : 1 \leq i, j \leq n\}$ of elements in a real W^* -algebra A is called a *matrix unit* if

$$(i) \quad u_{ij}^* = u_{ji};$$

$$(ii) \quad u_{ij}u_{kl} = \delta_{jk}u_{il};$$

$$(iii) \quad \sum_{i=1}^n u_{ii} = 1,$$

for all $1 \leq i, j, k, l \leq n$, where δ_{ij} is the Kronecker delta and $\{u_{ii}\}$ are mutually orthogonal projections.

Let $T : A \rightarrow B$ be a Jordan*-homomorphism and let A have a matrix unit in the following lemmas.

Lemma 4.5.5. Let $L = \{u_{ij} : 1 \leq i, j \leq n\}^c$ where $\{u_{ij} : 1 \leq i, j \leq n\}$ is a matrix unit in A and $\{u_{ij} : 1 \leq i, j \leq n\}^c$ denotes the commutant in A . Then for every $x \in A$, we have $x = \sum_{i,j=1}^n x_{ij}u_{ij}$ with $x_{ij} \in L$ and $T(x) = \sum_{i,j=1}^n T(x_{ij})T(u_{ij})$.

Proof. Let $x_{ij} = \sum_{k=1}^n u_{ki}x u_{jk}$. Then $u_{mn}x_{ij} = u_{mi}x u_{jn} = x_{ij}u_{mn}$, hence $x_{ij} \in L$. Moreover, $x = \sum_{i,j=1}^n u_{ii}x u_{jj} = \sum_{i,j=1}^n (\sum_{k=1}^n u_{ki}x u_{jk})u_{ij} = \sum_{i,j=1}^n x_{ij}u_{ij}$. For any $1 \leq i, j, k, l \leq n$, let $u = u_{ij} + u_{ii}$ where $i \neq j$. Then $u^2 = u$ and $u x_{kl} = x_{kl}u$. By Lemma 4.5.2(iii), $T(x_{kl}u_{ij}) + T(x_{kl}u_{ii}) = T(x_{kl}u) = T(x_{kl})T(u) = T(x_{kl})T(u_{ij}) + T(x_{kl})T(u_{ii})$. Moreover by the same lemma, since u_{ii} is an idempotent, we have $T(x_{kl}u_{ii}) = T(x_{kl})T(u_{ii})$. Therefore $T(x_{kl}u_{ij}) = T(x_{kl})T(u_{ij})$ which gives the last identity. \square

Lemma 4.5.6. Let $v_{ij} = T(u_{ii})T(u_{ij})T(u_{jj})$ and $w_{ij} = T(u_{ii})T(u_{ji})T(u_{jj})$ for $i \neq j$. Then $T(u_{ij}) = v_{ij} + w_{ji}$.

Proof. From $u_{ij} = u_{ii}u_{ij}u_{jj} + u_{jj}u_{ij}u_{ii}$ and Lemma 4.5.2, we obtain

$$\begin{aligned} T(u_{ij}) &= T(u_{ii}u_{ij}u_{jj} + u_{jj}u_{ij}u_{ii}) \\ &= T(u_{ii})T(u_{ij})T(u_{jj}) + T(u_{jj})T(u_{ij})T(u_{ii}) \\ &= v_{ij} + w_{ji} \end{aligned}$$

□

Lemma 4.5.7. $\{T(u_{ii}) : 1 \leq i \leq n\}$ consists of orthogonal projections.

Proof. For $i \neq j$, we have $u_{ii}u_{jj} = u_{jj}u_{ii} = 0$ and by Lemma 4.5.2(iii), we have $T(u_{ii})T(u_{jj}) = T(u_{jj})T(u_{ii}) = 0$. Since T is a Jordan*-homomorphism and u_{ii} is a projection, we have $T(u_{ii}) = T(u_{ii}^*) = T(u_{ii})^*$ and $T(u_{ii}) = T(u_{ii}^2) = T(u_{ii})^2$, hence $\{T(u_{ii}) : 1 \leq i \leq n\}$ consists of orthogonal projections. □

Lemma 4.5.8. For v_{ij} and w_{ij} defined in Lemma 4.5.6, we have

- (i) $v_{ij} = T(u_{ii})T(u_{ij}) = T(u_{ij})T(u_{jj})$;
- (ii) $w_{ij} = T(u_{ii})T(u_{ji}) = T(u_{ji})T(u_{jj})$.

Proof. By Lemma 4.5.6 and Lemma 4.5.7, we have

$$\begin{aligned} T(u_{ij})T(u_{jj}) &= T(u_{ii})T(u_{ij})T(u_{jj})T(u_{jj}) + T(u_{jj})T(u_{ij})T(u_{ii})T(u_{jj}) \\ &= T(u_{ii})T(u_{ij})T(u_{jj}) = v_{ij} \end{aligned}$$

and

$$\begin{aligned} T(u_{ii})T(u_{ij}) &= T(u_{ii})T(u_{ii})T(u_{ij})T(u_{jj}) + T(u_{ii})T(u_{jj})T(u_{ij})T(u_{ii}) \\ &= T(u_{ii})T(u_{ij})T(u_{jj}) = v_{ij}. \end{aligned}$$

Similarly we obtain $w_{ij} = T(u_{ii})T(u_{ji}) = T(u_{ji})T(u_{jj})$. □

Lemma 4.5.9. For all $1 \leq i, j, k, l \leq n$, we have

(i) $v_{ij}v_{kl} = \delta_{jk}v_{il}$ and $w_{ij}w_{kl} = \delta_{jk}w_{il}$;

(ii) $v_{ij}^* = v_{ji}$ and $w_{ij}^* = w_{ji}$.

Proof. (i) For $j \neq k$ we have by Lemma 4.5.7 and Lemma 4.5.8

$$\begin{aligned} v_{ij}v_{kl} &= T(u_{ij})T(u_{jj})T(u_{kk})T(u_{kl}) = 0 & \text{and} \\ w_{ij}w_{kl} &= T(u_{ji})T(u_{jj})T(u_{kk})T(u_{lk}) = 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} v_{ij}v_{jl} &= T(u_{ii})T(u_{ij})T(u_{jj})T(u_{jl}) \\ &= T(u_{ii})T(u_{ij})T(u_{jj})T(u_{jl}) + T(u_{ii})T(u_{jl})T(u_{jj})T(u_{ij}) \\ &= T(u_{ii})(T(u_{ij})T(u_{jj})T(u_{jl}) + T(u_{jl})T(u_{jj})T(u_{ij})) \\ &= T(u_{ii})T(u_{ij}u_{jj}u_{jl} + u_{jl}u_{jj}u_{ij}) \\ &= T(u_{ii})T(u_{il}) \\ &= v_{il} \end{aligned}$$

where we have used Lemma 4.5.2(i) and the fact that $u_{ii}u_{jl} = u_{jl}u_{ii} = 0$ implies that $T(u_{ii})T(u_{jl}) = 0$ by Lemma 4.5.2(iii). Similarly we prove $w_{ij}w_{jl} = w_{il}$.

(ii) $v_{ij}^* = (T(u_{ii})T(u_{ij}))^* = T(u_{ij})^*T(u_{ii})^* = T(u_{ji})T(u_{ii}) = v_{ji}$ and likewise $w_{ij}^* = w_{ji}$. □

Lemma 4.5.10. *Let $i \neq j$ and let $v_{ii} = v_{ij}v_{ji}$ and $w_{ii} = w_{ij}w_{ji}$. Then v_{ii} and w_{ii} are projections.*

Proof. By Lemma 4.5.9, $v_{ii}^* = (v_{ij}v_{ji})^* = v_{ji}^*v_{ij}^* = v_{ij}v_{ji} = v_{ii}$. In addition, $v_{ii}^2 = v_{ij}v_{ji}v_{ij}v_{ji} = v_{ij}v_{jk}v_{ki}v_{ij}v_{ji} = v_{ik}v_{ki} = v_{ii}$. Likewise w_{ii} is a projection. □

Remark 4.5.11. It is immediate by Lemma 4.5.9 and Lemma 4.5.10 that the sets $\{v_{ii} : 1 \leq i \leq n\}$ and $\{w_{ii} : 1 \leq i \leq n\}$ consist of orthogonal projections. Moreover by Lemma 4.5.8, $v_{ii} = v_{ij}v_{ji} = T(u_{ij})T(u_{jj})T(u_{jj})T(u_{ji}) = T(u_{ij})T(u_{jj})T(u_{ji}) = T(u_{ij})T(u_{ji})T(u_{ii})$ and $w_{ii} = T(u_{ji})T(u_{jj})T(u_{ij}) = T(u_{ji})T(u_{ij})T(u_{ii})$. Hence

$$\begin{aligned} v_{ii} + w_{ii} &= (T(u_{ij})T(u_{ji}) + T(u_{ji})T(u_{ij}))T(u_{ii}) \\ &= T(u_{ij}u_{ji} + u_{ji}u_{ij})T(u_{ii}) \\ &= T(u_{ii}). \end{aligned}$$

Lemma 4.5.12. *For all $1 \leq i, j, k, l \leq n$, we have $v_{ij}w_{kl} = w_{kl}v_{ij} = 0$.*

Proof. Since we may write $v_{ii} = v_{ij}v_{ji}$ and $w_{kk} = w_{kl}w_{lk}$ for any j and l , we need only consider the case when $i \neq j$ and $k \neq l$.

For $j \neq k$, we have $v_{ij}w_{kl} = T(u_{ij})T(u_{jj})T(u_{kk})T(u_{kl}) = 0$.

For $j = k$, we have

$$\begin{aligned} v_{ij}w_{jl} &= v_{ij}w_{ji}w_{il} \\ &= T(u_{ij})T(u_{jj})T(u_{jj})T(u_{ij})w_{il} \\ &= T(u_{ij})T(u_{jj})T(u_{ij})w_{il} \\ &= T(u_{ij}u_{jj}u_{ij})w_{il} = 0. \end{aligned}$$

Using similar arguments we obtain $w_{kl}v_{ij} = 0$. □

Lemma 4.5.13. *Let $e = \sum_{i=1}^n v_{ii}$ and $f = \sum_{i=1}^n w_{ii}$. Then $\{v_{ij} : 1 \leq i, j \leq n\}$ is a matrix unit in the reduced real W^* -algebra eBe and $\{w_{ij} : 1 \leq i, j \leq n\}$ is a matrix unit in the reduced real W^* -algebra fBf . Moreover, $T(1) = e + f$.*

Proof. The first statement follows directly by Lemma 4.5.9, Lemma 4.5.10, and the definition of e, f . By Remark 4.5.11, $T(1) = T(\sum_{i=1}^n u_{ii}) = \sum_{i=1}^n T(u_{ii}) = \sum_{i=1}^n (v_{ii} + w_{ii}) = e + f$. □

Lemma 4.5.14. *The elements e, f are orthogonal projections in B commuting with $T(A)$.*

Proof. From the definition of e and f , Lemma 4.5.10 and Lemma 4.5.12, it is clear that e and f are orthogonal projections. It remains to prove that they commute with $T(A)$.

We recall that $T(x) = \sum_{i,j=1}^n T(x_{ij})T(u_{ij})$. For all i, j, k, l with $i \neq j$ we observe that $v_{ij} = T(u_{ii})T(u_{ij})$ commutes with $T(x_{kl})$ by the proof of Lemma 4.5.5, since x_{kl} commutes with all u_{ij} . For $i = j$ we note that $v_{ii} = v_{ij}v_{ji}$ and we can apply the previous argument to deduce that v_{ii} commutes with $T(x_{kl})$. Now using the previous results

$$eT(u_{ij}) = \sum_{k=1}^n v_{kk}T(u_{ij}) = \sum_{k=1}^n v_{kk}(v_{ij} + w_{ji}) = v_{ij}$$

and

$$T(u_{ij})e = T(u_{ij}) \sum_{k=1}^n v_{kk} = (v_{ij} + w_{ji}) \sum_{k=1}^n v_{kk} = v_{ij}.$$

Therefore,

$$\begin{aligned} T(x)e &= \sum_{i,j=1}^n T(x_{ij})T(u_{ij})e \\ &= \sum_{i,j=1}^n T(x_{ij})v_{ij} \\ &= \sum_{i,j=1}^n v_{ij}T(x_{ij}) \\ &= \sum_{i,j=1}^n eT(u_{ij})T(x_{ij}) \\ &= eT(x). \end{aligned}$$

Hence e commutes with $T(A)$. Likewise $fT(u_{ij}) = T(u_{ij})f = w_{ji}$ and the same

arguments show that f commutes with $T(A)$. \square

We now arrive at the following result concerning Jordan*-homomorphisms on real W^* -algebras with a matrix unit.

Proposition 4.5.15. *Let A be a real W^* -algebra with a matrix unit and B a real W^* -algebra. Let $T : A \rightarrow B$ be a Jordan*-homomorphism. Then there are projections $e, f \in B$ such that e and f commute with $T(A)$ and $T = T(\cdot)e + T(\cdot)f$ where $T(\cdot)e : A \rightarrow eBe$ is a C^* -homomorphism and $T(\cdot)f : A \rightarrow fBf$ is a C^* -antihomomorphism.*

Proof. Let a, b be arbitrary elements in $L = \{u_{ij} : 1 \leq i, j \leq n\}^c$. Then $abu_{ii} + bau_{jj} = (au_{ij} + bu_{ji})^2$ and by the proof of Lemma 4.5.5, we have

$$\begin{aligned} T(ab)T(u_{ii}) + T(ba)T(u_{jj}) &= (T(a)T(u_{ij}) + T(b)T(u_{ji}))^2 \\ &= T(a)T(b)T(u_{ij})T(u_{ji}) + T(b)T(a)T(u_{ji})T(u_{ij}). \end{aligned}$$

Multiplying on the right by v_{ij} , we obtain $T(ab)v_{ij} = T(a)T(b)v_{ij}$ and multiplying on the right by w_{ji} , we obtain $T(ab)w_{ji} = T(b)T(a)w_{ji}$.

Now let $x = \sum_{i,j=1}^n x_{ij}u_{ij}$ and $y = \sum_{k,l=1}^n y_{kl}u_{kl}$ with $x_{ij}, y_{kl} \in L$. Then $xy = \sum_{i,j,l=1}^n x_{ij}y_{jl}u_{il}$ and $T(xy) = \sum_{i,j,l=1}^n T(x_{ij}y_{jl})T(u_{il})$. Let $e = \sum_{i=1}^n v_{ii} \in B$. We have

$$\begin{aligned} T(xy)e &= \sum_{i,j,l=1}^n T(x_{ij}y_{jl})T(u_{il})e = \sum_{i,j,l=1}^n T(x_{ij}y_{jl})v_{il} \\ &= \sum_{i,j,l=1}^n T(x_{ij})T(y_{jl})v_{il} = \sum_{i,j,l=1}^n T(x_{ij})T(y_{jl})v_{ij}v_{jl} \\ &= \sum_{i,j,l=1}^n T(x_{ij})v_{ij}T(y_{jl})v_{jl} = \sum_{i,j=1}^n T(x_{ij})v_{ij} \sum_{k,l=1}^n T(y_{kl})v_{kl} \\ &= T(x)eT(y)e. \end{aligned}$$

We also note that $T(x^*)e = T(x)^*e = (eT(x))^* = (T(x)e)^*$ since T preserves

the involution and e commutes with $T(A)$. Hence $T(x)e$ is a \mathbf{C}^* -homomorphism. Similarly, for $f = \sum_{i=1}^n w_{ii}$ we have $T(xy)f = T(y)fT(x)f$ and $T(\cdot)f$ is a \mathbf{C}^* -antihomomorphism. For the last statement we observe that $T(1) = e + f$ and $T(x) = T(x)e + T(x)f$. \square

To extend the above theorem to all real \mathbf{W}^* -algebras we first need to introduce the classification of real \mathbf{W}^* -algebras similar to the complex case.

Definition 4.5.16. Let A be a real \mathbf{W}^* -algebra. Let $p \in A$ be a projection. Then p is *finite* if for any projection q with $q \leq p$ and $q \sim p$ we have $q = p$. A projection p is *infinite* if it is not finite.

A is called *finite (infinite)* if the identity is finite (infinite). If A does not contain any non-zero finite central projection, then it is called *properly infinite*.

Definition 4.5.17. Let A be a real \mathbf{W}^* -algebra. A projection $p \in A$ is called *abelian* if the algebra pAp is abelian. A projection $p \in A$ is called *semi-abelian* if the algebra $pA_h p$ is abelian.

We note that in a complex \mathbf{C}^* -algebra B , a semi-abelian projection p is abelian.

Definition 4.5.18. Let A be a real \mathbf{W}^* -algebra. A is called *semi-discrete* if for any non-zero central projection $p \in A$, there exists a non-zero semi-abelian projection $q \in A$ such that $q \leq p$.

A is called *continuous* if it does not contain any non-zero semi-abelian projection.

A real \mathbf{W}^* -algebra A has a decomposition:

$$A = A_1 \oplus A_2$$

where A_1 is semi-discrete and A_2 is continuous (cf. [31, Theorem 8.1.9]).

We can further decompose a semi-discrete real \mathbf{W}^* -algebra A_1 as $A_1 = A_{11} + A_{12}$ where A_{11} is finite and A_{12} is properly infinite (cf. [31, Theorem 8.1.4]).

Now A_{11} can be decomposed as the sum of matrix algebras (see [31, Proposition 8.6.2]):

$$A_{11} = A_0 \oplus \sum_{\alpha \in I} \oplus M_\alpha$$

where A_0 is either 0 or a real abelian W^* -algebra and, M_α is an $n_\alpha \times n_\alpha$ matrix algebra, with $n_\alpha \in \mathbb{N} \setminus \{1\}$, which contains orthogonal mutually equivalent projections $p_1, p_2, \dots, p_{n_\alpha}$ such that $p_1 + p_2 + \dots + p_{n_\alpha} = 1_\alpha$ where 1_α is the identity of M_α . Let $\{u_i : 1 \leq i \leq n_\alpha\} \in M_\alpha$ be partial isometries such that $u_i^* u_i = p_1$ and $u_i u_i^* = p_i$. Define $u_{ij} = u_i u_j^*$. Then $\{u_{ij} : 1 \leq i, j \leq n_\alpha\}$ is a matrix unit in M_α . Since A_{12} is properly infinite, the halving lemma [31, Theorem 8.3.1] implies that there is a matrix unit $\{u_{ij} : 1 \leq i, j \leq 2\}$ in A_{12} . Hence there is a family $\{z_0, z_2\} \cup \{z_\alpha : \alpha \in I\}$ of mutually orthogonal central projections in A_1 such that

$$1_1 = z_0 + \sum_{\alpha \in I} z_\alpha + z_2$$

and $A_0 = Az_0$, $M_\alpha = Az_\alpha$ and $A_{12} = Az_2$.

Corollary 4.5.19. *Let A_1 be a semi-discrete real W^* -algebra and B a real W^* -algebra. Let $T : A_1 \rightarrow B$ be a weak*-continuous Jordan*-homomorphism. Then there are projections $e, f \in B$ such that*

$$T(\cdot) = T(\cdot)e + T(\cdot)f$$

where $T(\cdot)e : A_1 \rightarrow B$ is a C^* -homomorphism and $T(\cdot)f : A_1 \rightarrow B$ is a C^* -antihomomorphism.

Proof. Let $A_1 = A_0 \oplus \sum_{\alpha \in I} \oplus M_\alpha \oplus A_{12}$ be the above decomposition with corresponding central projections z_β , $\beta = 0, 2, \alpha$. By Lemma 4.5.2(iii), we can write $T = \oplus_\beta T_\beta$ where $T_\beta : Az_\beta \rightarrow B$ is the restriction of T to Az_β :

$$T_\beta(az_\beta) = T(az_\beta) = T(a)T(z_\beta) \quad (a \in A)$$

and $\{T(z_\beta)\}_\beta$ are mutually orthogonal projections in B .

Each T_β can be decomposed as the sum

$$T_\beta(\cdot) = T_\beta(\cdot)e_\beta + T_\beta(\cdot)f_\beta$$

of a C^* -homomorphism $T_\beta(\cdot)e_\beta$ and a C^* -antihomomorphism $T_\beta(\cdot)f_\beta$ by Proposition 4.5.15, where e_β and f_β are mutually orthogonal projections in B and $T_\beta(z_\beta) = e_\beta + f_\beta$.

Let $e = \sum_\beta e_\beta$ and $f = \sum_\beta f_\beta$. Then e and f are orthogonal projections in B and $T(1) = \sum_\beta T(z_\beta) = e + f$. We have

$$T(\cdot) = T(\cdot)e + T(\cdot)f$$

where $T(\cdot)e : A_1 \rightarrow B$ is a C^* -homomorphism and $T(\cdot)f : A_1 \rightarrow B$ is a C^* -antihomomorphism. \square

Corollary 4.5.20. *Let A_2 be a continuous real W^* -algebra and B a real W^* -algebra. Let $T : A_2 \rightarrow B$ be a Jordan*-homomorphism. Then there are projections $e', f' \in B$ such that*

$$T(\cdot) = T(\cdot)e' + T(\cdot)f'$$

where $T(\cdot)e' : A_2 \rightarrow B$ is a C^* -homomorphism and $T(\cdot)f' : A_2 \rightarrow B$ is a C^* -antihomomorphism.

Proof. Since A_2 is continuous, the identity is the sum of two equivalent orthogonal projections p and q [31, Proposition 8.6.4]. Let $p = u^*u$ and $q = uu^*$ for some partial isometry $u \in A_2$. Then $\{u_{11} = p, u_{12} = u^*, u_{21} = u, u_{22} = q\}$ is a matrix unit in A_2 . Therefore we can apply Proposition 4.5.15 to deduce that there exist orthogonal projections e' and f' in B commuting with $T(A_2)$ such that $T(\cdot)e' : A_2 \rightarrow B$ is a C^* -homomorphism and $T(\cdot)f' : A_2 \rightarrow B$ is a C^* -antihomomorphism, where $T(z_2) = e' + f'$. \square

Remark 4.5.21. Let $T : A \rightarrow B$ be a weak*-continuous Jordan*-homomorphism between real W*-algebras A and B where $A = A_1 \oplus A_2$ is the direct sum of a semi-discrete A_1 and a continuous A_2 real W*-algebra. Following similar arguments as in the proof of Corollary 4.5.19, we have

$$T(\cdot) = T(\cdot)z + T(\cdot)z'$$

where $T(\cdot)z : A \rightarrow B$ is a C*-homomorphism and $T(\cdot)z' : A \rightarrow B$ is a C*-antihomomorphism. Since $T(A)(1 - z - z') = \{0\}$, we can replace z' by $1 - z$ above. Given a Jordan*-homomorphism $T : A \rightarrow B$ between real C*-algebras A and B , then it is continuous as mentioned in the beginning of the section and the second dual map $T'' : A'' \rightarrow B''$ is a weak*-continuous Jordan*-homomorphism between real W*-algebras A'' and B'' . We conclude in the following theorem which determines the structures of real Jordan*-homomorphisms and extends the result of [27, 28, 50] to real C*-algebras.

Theorem 4.5.22. *Let $T : A \rightarrow B$ be a Jordan*-homomorphism between real C*-algebras A and B . Then there exists a projection $z \in B''$ such that*

$$T(\cdot) = T(\cdot)z + T(\cdot)(1 - z)$$

where $T(\cdot)z : A \rightarrow B''$ is a C*-homomorphism and $T(\cdot)(1 - z) : A \rightarrow B''$ is a C*-antihomomorphism.

Remark 4.5.23. The decomposition of Jordan homomorphisms on complex W*-algebras have also been proved with a different technique in [23, p.163]. Recently, Jordan homomorphisms between associative algebras have been studied in [7] which extends the work of the aforementioned authors.

Glossary

X_c : the complexification of a real Banach space X

X' : the dual of a real Banach space X

X^* : the dual of a complex Banach space X

X_* : the predual of a real or complex Banach space X

X_1 : the closed unit ball of a real or complex Banach space X

X_r : the real restriction of a complex Banach space X

$B(X)$: the algebra of bounded operators from a real or complex Banach space X to itself

$C_0(X)$: the Banach space of complex continuous functions on a locally compact Hausdorff space X , vanishing at infinity.

$C_0(X, \mathbb{R})$: the Banach space of real continuous functions on a locally compact Hausdorff space X , vanishing at infinity.

For a real or complex Banach*-algebra A :

\tilde{A} : the unit extension of A

A_h : the set of hermitian elements of A

A_{sh} : the set of skew-hermitian elements of A

$U(A)$: the set of unitary elements of A

P_A : the set of projections of A

A_+ : the set of positive elements of A

$\sigma(a)$: the spectrum of an element $a \in A$

$r(a)$: the spectral radius of an element $a \in A$.

For a real C^* -algebra A :

\hat{a} : the Gelfand transform of an element $a \in A$

$s_l(a)$: the left support of an element $a \in A$

$s_r(a)$: the right support of an element $a \in A$

$s(a)$: the support of an element $a \in A_h$

$c(p)$: the central support of a projection $p \in A$

A^c : the commutant of a real C^* -algebra A

A'_h : the set of hermitian (continuous) linear functionals on A

A'_+ : the set of positive (continuous) linear functionals on A

$S(A)$: the real state space of A

$P(A)$: the real pure state space of A .

For a real W^* -algebra A :

$N(A)$: the real normal state space of A

$b\phi$: the left translation of a functional $\phi \in A_*$ by $b \in A$ where $b\phi(\cdot) = \phi(b\cdot)$

ϕb : the right translation of a functional $\phi \in A_*$ by $b \in A$ where $\phi b(\cdot) = \phi(\cdot b)$

ϕ_b : the left and right translation $b^*\phi b$

$s(\phi)$: the support of a functional $\phi \in A'_+ \cap A_*$.

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