CLASSIFICATION OF DIGITAL AFFINE NONCOMMUTATIVE GEOMETRIES

SHAHN MAJID & ANNA PACHOŁ

ABSTRACT. It is known that connected translation invariant n-dimensional noncommutative differentials $\mathrm{d}x^i$ on the algebra $k[x^1,\cdots,x^n]$ of polynomials in n-variables over a field k are classified by commutative algebras V on the vector space spanned by the coordinates. This data also applies to construct differentials on the Heisenberg algebra 'spacetime' with relations $[x^\mu,x^\nu]=\lambda\Theta^{\mu\nu}$ where Θ is an antisymmetric matrix as well as to Lie algebras with pre-Lie algebra structures. We specialise the general theory to the field $k=\mathbb{F}_2$ of two elements, in which case translation invariant metrics (i.e. with constant coefficients) are equivalent to making V a Frobenius algebras. We classify all of these and their quantum Levi-Civita bimodule connections for $n=2,3$, with partial results for $n=4$. For $n=2$ we find 3 inequivalent differential structures admitting 1,2 and 3 invariant metrics respectively. For $n=3$ we find 6 differential structures admitting $0,1,2,3,4,7$ invariant metrics respectively. We give some examples for $n=4$ and general n. Surprisingly, not all our geometries for $n\geq 2$ have zero quantum Riemann curvature. Quantum gravity is normally seen as a weighted 'sum' over all possible metrics but our results are a step towards a deeper approach in which we must also 'sum' over differential structures. Over $\mathbb{F}_2$ we construct some of our algebras and associated structures by digital gates, opening up the possibility of 'digital geometry'.

1. Introduction

A standard technique in physics and engineering is to replace geometric backgrounds by discrete approximations such as a lattice or graph, thereby rendering systems more calculable. In recent years it has become clear that this can be handled by noncommutative geometry not because the 'coordinate algebras' A are noncommutative (they remain commutative) but because differentials and functions do not commute, see [10] and references therein. The formalism of noncommutative differential geometry does not require functions and differentials to commute, so is more general even when the algebra is classical. In the present work we use such noncommutative differential geometry to explore a different and complementary kind of 'discretisation scheme' in which the field $\mathbb{C}$ or $\mathbb{R}$ that we work over is replaced by the field $\mathbb{F}_2$ of two elements $0,1$ and which we call *digital geometry.*

We use a 'bottom up' constructive approach to noncommutative differential geometry that grew in the 1990s out of (but not limited to) the differential geometry of quantum groups, rather than one of the powerful operator algebra approach to

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noncommtutative geometry as in [5]. This is more explicit (albeit mathematically less deep) and has the merit that one can work over any field k. Often characteristic 2 (which includes $\mathbb{F}_2$) is excluded for simplicity so one must be a little careful (notably tensors cannot be decomposed into symmetric and antisymmetric parts) but most of the theory including differential forms (as differential graded algebras $\Omega(A)$), vector bundles, principal bundles, connections and Riemannian metrics work over any field. We refer to our LTCC lectures [7] for a recent introduction. A small part of the formalism is recapped in Section 2 along with a recent classification theorem [13] for translation invariant differentials on Hopf algebras with linear (additive) coproduct, which will be our starting point. To keep a lid on the classification problem we insist that our metrics are invertible, which is known [4] to require that the metric is central (commutes with functions) and we assume that our connections $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ are *bimodule connections* [15, 6]. This means that their right handed derivation rule is expressed in a 'generalised braiding' $\sigma : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ and we require this to be invertible. The 'quantum groups' approach to noncommutative differential geometry was particularly developed using bimodule connections in recent works such as [3, 4, 10, 2].

The present paper follows on from [2] where we studied the de Rham cohomology of $\mathbb{F}_2[x]$ (polynomials in one variable) with noncommutative differential structures, which turned out to be surprisingly rich. This led to nice family of 'finite' geometries over $\mathbb{F}_2$ as finite dimensional commutative Hopf algebras A_d for every $d \in \mathbb{N}$ (and over $\mathbb{F}_p$ for any prime p). By contrast, we will now be interested in affine or 'flat space' $A = \mathbb{F}_2[x^1, \cdots, x^n]$ but it turns out that the classification of its differential structures of dimension n already amounts to the classification of finite geometries in the form of n-dimensional commutative algebras V over $\mathbb{F}_2$, so our results now include the classification of all of these up to dimension $n = 4$ (we find that there are 16 of these up to isomorphism if we ask for them to be unital, see Section 5), along with more complete results for $n = 2, 3$ for metrics and connections on A for each of the respectively 3 and 6 possible choices of V in these dimensions, see Sections 3, 4. These bimodule noncommutative geometries are explored under the restriction that the metric and Christoffel symbol coefficients are constants in keeping with our view of $A = \mathbb{F}_2[x^1, \cdots, x^n]$ as 'flat space', i.e. we are looking at translation invariant geometries over $\mathbb{F}_2$. It is interesting that for $n \geq 3$ some of the possible geometries nevertheless have quantum curvature $R_\nabla \neq 0$, which we regard as a purely quantum phenomenon.

We envisage many applications throughout mathematical physics and engineering wherever classical differential geometry plays a role. It is not our goal to develop these here but we conclude with an extended discussion in Section 6 of some that we have in mind. Our own motivation for noncommutative geometry has come from quantum gravity in which the proposal of *quantum spacetime* and concrete models [8, 12, 1] emerged out of quantum groups (and was the origin of one of the two main classes of quantum groups, namely the bicrossproduct ones). In this context one could in principle 'sum over all geometries' so our classification is a peek into a restricted part of this. More generally our classification is a tool for model building and one can explore each of our geometries much further, for example solving wave equations. Clearly we would like to go further and explore all geometries not just the translation invariant ones in the present work. Also embedded in our above explanation and surprisingly forced on us by translation invariance of the differentials $\mathrm{d}x^i$ is the set up of classical and quantum field theory in which we work with the space of functions on a linear space V which is itself the space of functions on an underlying geometry. This suggests a different envisaged application in which spacetime would be the coordinate algebra V and $A = k[x_1, \cdots, x_n]$ or more abstractly $k[V]$ would be the algebra of functionals *on* V as the vector space of functions. Differentials also automatically extend to A the Heisenberg algebra, so the first steps of quantum field theory also arise out of the natural possibilities for the noncommutative geometry of affine spaces. In this case our spacetime geometry is built on differentials and Riemannian structures on $\Omega(V)$ as in [2] and as will be classified in a sequel [11] in preparation. The discussion ends with a translation of algebra over $\mathbb{F}_2$ into digital electronics, thereby justifying our terminology and opening up a new front of applications in which geometric ideas can be translated into electronics.

We made extensive use of the numerical package R to enumerate all possible values of our structure constants, preceded and followed by symbolic calculations on Mathematica.

2. Calculi on $k[x^1, \cdots, x^n]$ and Heisenberg algebras

If A is a possibly noncommutative 'coordinate' algebra, by differential calculus on A we mean an A-bimodule Ω^1 and a map $\mathrm{d}: A \to \Omega^1$ obeying the Leibniz rule $\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b$ with the map $A \otimes A \to \Omega^1$ given by $a \otimes b \mapsto a\mathrm{d}b$ surjective. Here a bimodule means we can associatively multiply such 1-forms by elements of A from the left and the right. The calculus is called connected if $\ker \mathrm{d} = k.1$ where we work over the field k. If A is a Hopf algebra or 'quantum group' the coproduct expresses 'group translation' and there is a standard notion of the differential calculus being left and right covariant under this. We refer to [7] for an introduction.

We build on the Majid-Tao theorem [13] which states that connected translation invariant differential structures of classical dimension on 'quantum spaces' consisting of enveloping algebras $U(\mathfrak{g})$ where $\mathfrak{g}$ is a Lie algebra, are classified by pre-Lie structures $\circ$ on $\mathfrak{g}$. A pre-Lie algebra structure is a 'product' $\circ$ on $\mathfrak{g}$ such that

$$v \circ w - w \circ v = [v, w] \tag{2.1}$$

(i.e. we recover the given Lie bracket) and

$$(v \circ w) \circ z = (v \circ z) \circ w + v \circ (w \circ z - z \circ w), \quad \forall v, w, z \in g. \tag{2.2}$$

The differential calculus has generators $\mathrm{d}x^\mu$ where $\{x^\mu\}$ is a basis of $\mathfrak{g}$ and bimodule relations

$$[\mathrm{d}x^\mu, x^\nu] = \lambda \mathrm{d}(x^\mu \circ x^\nu)$$

where $\lambda \neq 0$ is the deformation parameter. Clearly the Jacobi identity

$$[[\mathrm{d}x^\mu, x^\nu], x^\rho] = [[\mathrm{d}x^\mu, x^\rho], x^\nu] + [\mathrm{d}x^\mu, [x^\nu, x^\rho]]$$

for the bimodule relations translates immediately in this context to (2.2). The Leibniz rule $\mathrm{d}[x^\mu, x^\nu] = [\mathrm{d}x^\mu, x^\nu] + [x^\mu, \mathrm{d}x^\nu]$ is the other part (2.1). If the pre-Lie algebra is unital with the left unit e then clearly the calculus is inner in the sense of existence of element $\theta \in \Omega^1$ such that $\mathrm{d} = [\theta,]$ on A (and on forms if we use graded commutator), with $\theta = \lambda^{-1}\mathrm{d}e$. Note that the calculus could be inner in some other way with θ not the differential of an element of the pre-Lie algebra. Isomorphisms of the pre-Lie algebra are induced by linear coordinate transformations that do not change the differential structure.

In the commutative case of $A = k[x^1, \cdots, x^n]$ regarded as the enveloping algebra of an Abelian Lie algebra, we need $\circ$ commutative and in this case (2.2) says that $\circ$ is associative, so the data is that of an n-dimensional commutative algebra. Since $\mathrm{d}1 = 0$ and 1 is central, a quick look at the proof above tells us that this works just as well for the Heisenberg algebra regarded as noncommutative space,

$$[x^\mu, x^\nu] = \lambda\Theta^{\mu\nu}$$

which includes the commutative case with $\Theta = 0$ (this can also be seen as a Lie algebra with a central generator on the right hand side to which we apply the pre-Lie theory and then set the central generator to 1).

There is in fact no need for $\mathfrak{g}$ to be finite dimensional. It can be an infinite dimensional vector space V with an antisymmetric bilinear form $\Theta : V \times V \to k$ and the data for a calculus of the above form on the associated algebra with relations $[v, w] = \lambda\Theta(v, w)$ is precisely products $\circ : V \times V \to V$ making $(V, \circ)$ an associative commutative algebra. In the unital case this will be inner as before. An example is $V = C^\infty(M)$ on a manifold M in which case the above is a canonical noncommutative differential calculus or 'noncommutative variational calculus' on the space of functionals on V, or more precisely on the symmetric algebra $S(V)$ or its Heisenberg 'quantum field theory' version.

Next we consider quantum metrics. In the constructive approach to noncommutative geometry this means a nondegenerate element $g \in \Omega^1 \otimes_A \Omega^1$ which commutes with elements of A. The latter is known [4] to be necessary for the existence of a bimodule inner product $(\ ,\ ) : \Omega^1 \otimes_A \Omega^1 \to A$ inverse to g in the sense $(\omega, g_1)g_2 = \omega = g_1(g_2, \omega)$ for all $\omega \in \Omega^1$, where $g = g_1 \otimes g_2$ with sums of such terms understood. We can also construct the latter directly as a bimodule inner product, which in our case has the form $(\mathrm{d}v, \mathrm{d}w) = B(v, w)$ for some bilinear map $V \times V \to A$ obeying

$$B(v \circ w, z) + B(v, w \circ z) = \lambda^{-1}[B(v, z), w] \tag{2.3}$$

for all $v, w, z \in V$. Here we require

$$((\mathrm{d}v)w, \mathrm{d}z) = wB(v, z) + \lambda B(v \circ w, z) = B(v, z)w - \lambda B(v, z \circ w) = (\mathrm{d}v, w\mathrm{d}z)$$

using the commutation relations of differentials and functions, which in the middle is the condition stated. For the inner product to be 'real' we need $\overline{B(v, w)} = B(w^*, v^*)$ which in a self-adjoint basis with B symmetric requires its coefficients to be real.

However, if $\circ$ has an identity e (so that the calculus is inner by a coordinate differential) and the field has characteristic not 2 then there is no such map other than $B = 0$ at the algebra level. So see this we set $w = e$ so that $B(v, z) = \frac{1}{2\lambda}[B(v, z), e]$ from our condition, which has no solution at an algebraic level since the second expression has strictly lower degree when $B(v, z)$ is written in a standard normal-ordered form. There could still be non-algebraic examples and there could still be non-unital inner and non-inner examples or we could be in characteristic 2.

In this paper we will focus on this latter possibility for invariant metrics on unital inner calculi by taking $k = \mathbb{F}_2$ the field of two elements $0, 1$. We take $\lambda = 1$ and trivial $*$-structure as the only choices that make sense when there are only two elements. In the commutative polynomial algebra case condition (2.3) becomes

$$B(v\circ w,z)=B(v,w\circ z)$$

and we can keep this also in the Heisenberg case if B has its values in the constants, which means the coefficients of the metric (given by B^{-1}) are constants (an 'invariant metric'). In this case the condition on B means that the data is precisely that of a commutative Frobenius algebra over $\mathbb{F}_2$. So for each of these we obtain a differential calculus and metric on the symmetric algebra on the vector space of the algebra.

We close our generalities with a few general classes of examples:

(i) Let X be a finite set of order n and $V=k(X)$ the algebra of functions on X with pointwise product $\circ$. We let $x^\mu=\delta_\mu$ the delta function at point $\mu\in X$ and since $x^\mu\circ x^\nu=\delta_{\mu\nu}x^\mu$ we have

$$[\mathrm{d}x^\mu,x^\nu]=\delta_{\mu\nu}\mathrm{d}x^\mu$$

with inner element $\theta=\sum_\mu \mathrm{d}x^\mu$. For a quantum metric $g=\sum g_{\mu\nu}\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu$ we require

$$[g,x^\rho]=\sum_\mu g_{\mu\rho}\mathrm{d}x^\mu\otimes\mathrm{d}x^\rho+\sum_\mu g_{\rho\mu}\mathrm{d}x^\rho\otimes\mathrm{d}x^\mu=0$$

for all ρ which implies $g_{\rho\rho}=0$ unless we are in characteristic 2, and $g_{\mu\rho}=0$ for all $\mu\neq\rho$. So there is no metric unless we work in characteristic 2 but in this case

$$g=\sum_\mu \mathrm{d}x^\mu\otimes\mathrm{d}x^\mu$$

is the unique quantum metric (the Euclidean metric), for example over $\mathbb{F}_2$.

(ii) $V=k\mathbb{Z}_n=k[x]/\langle x^n-e\rangle$ with basis x^μ the different powers of x with respect to $\circ$ and $e=x^0$. We have commutation relations

$$[\mathrm{d}x^\mu,x^\nu]=\mathrm{d}x^{\mu+\nu}$$

with indices treated mod n. At least n quantum metrics exist over $\mathbb{F}_2$, namely the n metrics

$$g=\sum_\mu \mathrm{d}x^\mu\otimes\mathrm{d}x^{m-\mu};\quad m=0,1,\cdots n-1.$$

We check that $[g,x^\rho]:=\sum\mathrm{d}x^{\mu+\rho}\otimes\mathrm{d}x^{m-\mu}+\mathrm{d}x^\mu\otimes\mathrm{d}x^{m-\mu+\rho}=0$ after a relabelling $\mu+\rho\to\mu$ in the first sum to get 2 copies. In addition the elements $c=\sum_\mu\mathrm{d}x^\mu$ and hence $c\otimes c$ are central and adding the latter gives a complementary metric where all the coefficients are reversed $0\leftrightarrow 1$. We will see that for $n=2$ this gives no new metrics and indeed find just the above two, and for $n=3$ complementary metrics are degenerate so again give no more nondegenerate metrics and we find just the above 3 (this will not be the case for $n=4$ where we obtain 8 metrics).

(iii) With $n=p^d$ and working over $\mathbb{F}_p$ where p is prime, there is a natural algebra $V=A_d=k[x]/\langle x^{p^d}-x\rangle$ which plays an important role in the theory of field extensions. We have x^μ the powers under $\circ$ with $e=x^0$ and $\mu=0,\cdots,n-1$. We focus on $p=2$.

A_1 is 2-dimensional with e a unit and $x\circ x=x$. The calculus is $[\mathrm{d}e,e]=\mathrm{d}e$, $[\mathrm{d}e,x]=[\mathrm{d}x,e]=[\mathrm{d}x,x]=\mathrm{d}x$. This is case B among the algebras for $n=2$ in the next section and we find there that there is exactly one quantum metric $g=\mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}e$. In fact this is isomorphic to (i) for 2 points. A_2

is 4-dimensional with e a unit, $x^\mu \circ x^\nu = x^{\mu+\nu}$ if $\mu+\nu < 4$ and reduced by $x^4 = x$ otherwise. Its own NCG was studied in [2] but now we are not studying its NCG but rather that of $k[x^0, x^1, x^2, x^3]$ as a 4-dimensional noncommutative spacetime. We will find 3 metrics for this calculus in Section 5.

Returning to the general theory over a field k, after we have found a calculus and metric the next step is to try to find a quantum torsion free metric compatible or 'quantum Levi-Civita' bimodule connecton (QLC for short). By 'bimodule connection' on Ω^1 we mean a left connection, i.e. $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ such that $\nabla(a\omega) = a(\nabla\omega) + \mathrm{d}a \otimes \omega$ for all $a \in A, \omega \in \Omega^1$ and in addition there exists a bimodule map σ so that

$$\nabla(\omega a) = (\nabla\omega)a + \sigma(\omega \otimes \mathrm{d}a), \quad \sigma : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1.$$

Here σ if it exists is uniquely determined. In [10] it is shown that in the inner case (with θ) the construction of a bimodule connection is equivalent to the construction of bimodule maps σ and $\alpha : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$. Then

$$\nabla\omega = \theta \otimes \omega - \sigma(\omega \otimes \theta) + \alpha\omega. \tag{2.4}$$

Such a bimodule connection is metric compatible if

$$\theta \otimes g + (\alpha \otimes \mathrm{id})\, g + \sigma_{12}\, (\mathrm{id} \otimes (\alpha - \sigma_\theta))\, g = 0 \tag{2.5}$$

where $\sigma_\theta = \sigma((\ )\otimes\theta)$. This condition results in quadratic relations for the coefficients of σ. Finally, the curvature and torsion of a connection are

$$R_\nabla = (\mathrm{d} \otimes \mathrm{id} - (\wedge \otimes \mathrm{id})\,(\mathrm{id} \otimes \nabla))\,\nabla \quad : \Omega^1 \to \Omega^2 \otimes_A \Omega^1 \tag{2.6}$$

$$T_\nabla = \wedge\nabla - \mathrm{d} \quad : \Omega^1 \to \Omega^2$$

where $\wedge : \Omega^1 \otimes_A \Omega^1 \to \Omega^2$ is the exterior product. One generally requires that Ω is generated by A and Ω^1, and this will always be our case also. The construction of a torsion free bimodule connection in the inner case is then equivalent [10] to the bimodule maps σ and α satisfying

$$\wedge\sigma = -\wedge, \quad \wedge\alpha = 0. \tag{2.7}$$

In order to solve these equations, we write out all our conditions in terms of structure tensors starting with the pre-Lie algebra in the form

$$x^\mu \circ x^\nu = V^{\mu\nu}{}_\rho x^\rho, \quad V^{\mu\nu}{}_\rho \in k. \tag{2.8}$$

For our polynomial or Heisenberg cases we need symmetry of the product so

$$V^{\mu\nu}{}_\rho = V^{\nu\mu}{}_\rho \tag{2.9}$$

and from (2.2) we need

$$V^{\rho\nu}{}_\lambda V^{\lambda\mu}{}_\gamma = V^{\rho\mu}{}_\lambda V^{\lambda\nu}{}_\gamma \tag{2.10}$$

which given commutativity is associativity of the product $\circ$ in this case. For an inner calculus we have additionally

$$\theta \cdot V = \mathrm{id}, \quad \theta_\mu V^{\mu\nu}{}_\rho = \delta^\nu_\rho$$

for some 1-form $\theta = \theta_\mu \mathrm{d}x^\mu$, which expresses that θ corresponds to the identity for $\circ$. If this exists, it is unique. The differential calculus induced by the pre-Lie algebra structure has the commutation relations:

$$[\mathrm{d}x^\rho, x^\nu] = V^{\rho\nu}{}_\mu \mathrm{d}x^\mu. \tag{2.11}$$

These are the results already discussed (and work the same way over any field) to associate an inner calculus to a unital commutative associative algebra V).

The conditions for quantum metric $g = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \in \Omega^1 \otimes_A \Omega^1$ come down to the metric central in the sense

$$g_{\lambda\nu}V^{\lambda\rho}{}_{\mu} + g_{\mu\gamma}V^{\gamma\rho}{}_{\nu} = 0, \tag{2.12}$$

$g_{\mu\nu} = g_{\nu\mu}$ for quantum symmetry and nonzero determinant for invertibility. The centrality here has no nonzero solution in the unital case of θ unless we are in characteristic 2.

For a bimodule connection (for inner calculi) we require bimodule maps σ and $\alpha : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ as above. For the alpha map on the basis 1-forms, taking $\alpha(\mathrm{d}x^\mu) = \alpha^\mu{}_{\nu\rho}\mathrm{d}x^\nu \otimes \mathrm{d}x^\rho$ and $\alpha^\mu{}_{\nu\rho} = \alpha^\mu{}_{\rho\nu}$, we require

$$\alpha^\rho{}_{\gamma\sigma}V^{\gamma\nu}{}_{\lambda} + \alpha^\rho{}_{\lambda\gamma}V^{\gamma\nu}{}_{\sigma} = V^{\rho\nu}{}_{\mu}\alpha^\mu{}_{\lambda\sigma}. \tag{2.13}$$

These conditions come from the compatibility of α with the differential calculus, i.e. from equality $\alpha([\mathrm{d}x^\rho, x^\nu]) = V^{\rho\nu}{}_{\mu}\alpha(\mathrm{d}x^\mu)$ calculating the left hand side $\alpha([\mathrm{d}x^\rho, x^\nu]) = [\alpha(\mathrm{d}x^\rho), x^\nu] = \alpha^\rho{}_{\gamma\sigma}[\mathrm{d}x^\gamma \otimes \mathrm{d}x^\sigma, x^\nu] = \alpha^\rho{}_{\gamma\sigma}V^{\gamma\nu}{}_{\lambda}\mathrm{d}x^\lambda \otimes \mathrm{d}x^\sigma +$ $+\alpha^\rho{}_{\lambda\gamma}V^{\gamma\nu}{}_{\sigma}\mathrm{d}x^\lambda \otimes \mathrm{d}x^\sigma$ and from the right hand side $V^{\rho\nu}{}_{\mu}\alpha(\mathrm{d}x^\mu) = V^{\rho\nu}{}_{\mu}\alpha^\mu{}_{\lambda\sigma}\mathrm{d}x^\lambda \otimes \mathrm{d}x^\sigma$ gives the above relation (2.13).

Similarly for the sigma map, we assume $\sigma(\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu) = \sigma^{\mu\nu}{}_{\rho\lambda}\mathrm{d}x^\rho \otimes \mathrm{d}x^\lambda$ and require compatibility: $\sigma([\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu, x^\gamma]) = [\sigma(\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu), x^\gamma]$. From the left hand side we obtain $\sigma([\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu, x^\gamma]) = V^{\mu\gamma}{}_{\alpha}\sigma(\mathrm{d}x^\alpha \otimes \mathrm{d}x^\nu) + V^{\nu\gamma}{}_{\beta}\sigma\left(\mathrm{d}x^\mu \otimes \mathrm{d}x^\beta\right) = \left(V^{\mu\gamma}{}_{\alpha}\sigma^{\alpha\nu}{}_{\omega\sigma} + V^{\nu\gamma}{}_{\beta}\sigma^{\mu\beta}{}_{\omega\sigma}\right)\mathrm{d}x^\omega \otimes \mathrm{d}x^\sigma$ and from the right hand side we obtain $[\sigma(\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu), x^\gamma] = \left[\sigma^{\mu\nu}{}_{\lambda\rho}\mathrm{d}x^\lambda \otimes \mathrm{d}x^\rho, x^\gamma\right] = \left(\sigma^{\mu\nu}{}_{\lambda\sigma}V^{\lambda\gamma}{}_{\omega}\mathrm{d}x^\omega \otimes \mathrm{d}x^\sigma + \sigma^{\mu\nu}{}_{\omega\rho}V^{\rho\gamma}{}_{\sigma}\mathrm{d}x^\omega \otimes \mathrm{d}x^\sigma\right)$ which results in the condition

$$V^{\mu\gamma}{}_{\alpha}\sigma^{\alpha\nu}{}_{\omega\sigma} + V^{\nu\gamma}{}_{\beta}\sigma^{\mu\beta}{}_{\omega\sigma} = \sigma^{\mu\nu}{}_{\lambda\sigma}V^{\lambda\gamma}{}_{\omega} + \sigma^{\mu\nu}{}_{\omega\rho}V^{\rho\gamma}{}_{\sigma}. \tag{2.14}$$

Additionally, we are interested in σ invertible as an $n^2 \times n^2$ k-valued matrix

$$\sigma = \begin{pmatrix} \sigma^{11}{}_{11} \, \sigma^{11}{}_{12} \cdots \sigma^{11}{}_{nn} \\ \sigma^{12}{}_{11} & \\ \cdots & \cdots \\ \sigma^{nn}{}_{11} & \sigma^{nn}{}_{nn} \end{pmatrix} \quad \text{with} \quad \det(\sigma) \neq 0.$$

Metric compatibility (2.5) in the unital inner case (for $\alpha = 0$, which turns out to be the case in our considerations) becomes the non-linear conditions for the sigma coefficients,

$$\theta_\rho g_{\mu\nu} = \sigma^{\lambda\gamma}{}_{\beta\nu}g_{\alpha\lambda}\theta_\gamma\sigma^{\alpha\beta}{}_{\rho\mu}. \tag{2.15}$$

3. Classification and their quantum geometries for $n = 2$

For $n = 1$ there are up to isomorphism only two algebras of dimension 1 over any field k, namely $x \circ x = 0$ which is nonunital and gives the classical calculus $[\mathrm{d}x, x] = 0$ on $k[x]$ and $e \circ e = e$ which gives the finite difference calculus $[\mathrm{d}e, e] = \lambda \mathrm{d}e$ on $k[e]$

for deformation parameter λ in agreement with [9] for 1-dimensional calculi. The only candidate for a quantum metric is up to normalisation $g = \mathrm{d}x \otimes \mathrm{d}x$ which is central as the calculus is commutative and similarly $g = \mathrm{d}e \otimes \mathrm{d}e$ which is only central over $\mathbb{F}_2$. We are interested in this case, with $\lambda = 1$.

Our goal in this section is to give a full classification of the $n = 2$ unital (hence inner) case. Clearly, 2-dimensional commutative unital algebras have the form e, x as basis and $e \circ e = e, e \circ x = x = x \circ e$ and $x \circ x = \alpha e + \beta x$ for constants α, β for free parameters α, β. Over $\mathbb{F}_2$ this means four possibilities

$$x \circ x = 0, \quad x \circ x = e, \quad x \circ x = x, \quad x \circ x = e + x$$

with the first two isomorphic by $x \mapsto x + e$. Thus there are three inequivalent unital algebras hence three calculi. We used a computer package 'R' to check this explicitly (as a warm up to the next section which is best done by computer), then also find the metrics in each case and the metric compatible quantum Levi-Civita connections for each metric. We check every possible $0, 1$ value for the various structure constants. We summarise our results in Table 1.

<table>
<tr><th></th><th>Relations
$[\mathrm{d}e, e] = \mathrm{d}e$
$[\mathrm{d}e, x] = \mathrm{d}x = [\mathrm{d}x, e]$</th><th>Orbit order</th><th>Isotropy group</th><th>Quantum metrics and nonzero QLCs</th></tr>
<tr><td rowspan="2">A</td><td rowspan="2">$x \circ x = 0$
$[\mathrm{d}x, x] = 0$</td><td rowspan="2">2×3</td><td rowspan="2">$\{1\}$</td><td>$g_{A.I} = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e$
$\nabla \mathrm{d}e = \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = 0$
$\nabla \mathrm{d}e = \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}x$
$\nabla \mathrm{d}e = \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}x$
$\nabla \mathrm{d}e = 0, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}e$
$\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x$</td></tr>
<tr><td>$g_{A.II} = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x$
$\nabla \mathrm{d}e = \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = 0$
$\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e, \quad \nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}x$
$\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}x$</td></tr>
<tr><td>B</td><td>$x \circ x = x$
$[\mathrm{d}x, x] = \mathrm{d}x$</td><td>$1 \times 3$</td><td>$\{1, u\} = \mathbb{Z}_2$</td><td>$g_B = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e$
$\nabla \mathrm{d}e = 0, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}e$</td></tr>
<tr><td rowspan="3">C</td><td rowspan="3">$x \circ x = e + x$
$[\mathrm{d}x, x] = \mathrm{d}e + \mathrm{d}x$</td><td rowspan="3">$1 \times 3$</td><td rowspan="3">$\{1, v\} = \mathbb{Z}_2$</td><td>$g_{C.I} = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x$
$\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}x$</td></tr>
<tr><td>$g_{C.II} = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e$
$\nabla \mathrm{d}e = 0, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}e$</td></tr>
<tr><td>$g_{C.III} = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x$
$\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e$</td></tr>
</table>

Table 1. All possible unital inner noncommutative geometries on $\mathbb{F}_2[e, x]$. Note that $g_{A.II} = g_{C.I}$ and $g_B = g_{C.II}$.

We now explain how these results were obtained. *A priori* the noncommutative geometry of interest is that of $\mathbb{F}_2[x^1, x^2]$, defined as the universal enveloping algebra of an Abelian Lie algebra generated by basis elements x^1, x^2, the commutative algebra product (2.8) induces the differential calculus (2.11). Notice that in 2 dimensions with variables x^1 and x^2 we have three possibilities of inner calculi with θ as the differential of an element of the pre-Lie algebra and they are $\theta = \mathrm{d}x^1$ or $\mathrm{d}x^2$ or $\mathrm{d}x^1 + \mathrm{d}x^2$. (Equivalently, the algebra has a unit $e = x^1, x^2$ or $x^1 + x^2$.)

We find all the possible solutions of the commutative pre-Lie algebra structure in 2 dimensions which induces inner differential calculus, i.e. satisfying (2.1) and (2.2). Let $S = \{s_1, ..., s_{12}\}$ be the set of all solutions and they can be grouped as follows:

• 4 cases of inner calculus with $\theta = \mathrm{d}x^1$, all have the following commutation relations $\left[\mathrm{d}x^1, x^1\right] = \mathrm{d}x^1, \left[\mathrm{d}x^1, x^2\right] = \mathrm{d}x^2 = \left[\mathrm{d}x^2, x^1\right]$ and the remaining commutators we order as follows: $s_1 : \left[\mathrm{d}x^2, x^2\right] = 0$, $s_2 : \left[\mathrm{d}x^2, x^2\right] = \mathrm{d}x^2$, $s_3 : \left[\mathrm{d}x^2, x^2\right] = \mathrm{d}x^1 + \mathrm{d}x^2$, $s_4 : \left[\mathrm{d}x^2, x^2\right] = \mathrm{d}x^1$;

• 4 cases of inner calculus with $\theta = \mathrm{d}x^2$ with $\left[\mathrm{d}x^2, x^i\right] = \mathrm{d}x^i = \left[\mathrm{d}x^i, x^2\right]$ and $s_5 : \left[\mathrm{d}x^1, x^1\right] = 0$, $s_6 : \left[\mathrm{d}x^1, x^1\right] = \mathrm{d}x^2$, $s_7 : \left[\mathrm{d}x^1, x^1\right] = \mathrm{d}x^1$, $s_8 : \left[\mathrm{d}x^1, x^1\right] = \mathrm{d}x^1 + \mathrm{d}x^2$;

• 4 cases of inner calculus with $\theta = \mathrm{d}x^1 + \mathrm{d}x^2$, such that $\left[\mathrm{d}x^1 + \mathrm{d}x^2, x^i\right] = \mathrm{d}x^i$:

$s_9 : \left[\mathrm{d}x^1, x^1\right] = 0 \quad , \quad \left[\mathrm{d}x^1, x^2\right] = \mathrm{d}x^1 = \left[\mathrm{d}x^2, x^1\right] \quad , \quad \left[\mathrm{d}x^2, x^2\right] = \mathrm{d}x^1 + \mathrm{d}x^2,$

$s_{10} : \left[\mathrm{d}x^1, x^1\right] = \mathrm{d}x^2 \quad , \quad \left[\mathrm{d}x^1, x^2\right] = \mathrm{d}x^1 + \mathrm{d}x^2 = \left[\mathrm{d}x^2, x^1\right] \quad , \quad \left[\mathrm{d}x^2, x^2\right] = \mathrm{d}x^1,$

$s_{11} : \left[\mathrm{d}x^1, x^1\right] = \mathrm{d}x^1 \quad , \quad \left[\mathrm{d}x^1, x^2\right] = 0 = \left[\mathrm{d}x^2, x^1\right] \quad , \quad \left[\mathrm{d}x^2, x^2\right] = \mathrm{d}x^2 ,$

$s_{12} : \left[\mathrm{d}x^1, x^1\right] = \mathrm{d}x^1 + \mathrm{d}x^2 \quad , \quad \left[\mathrm{d}x^1, x^2\right] = \mathrm{d}x^2 = \left[\mathrm{d}x^2, x^1\right] \quad , \quad \left[\mathrm{d}x^2, x^2\right] = 0.$

One can show that due to the action of the group of isomorphisms G on the set S of these solutions we get only three inequivalent families, corresponding to the orbits of the action of the group.

The group of isomorphisms in 2 dimensions over $\mathbb{F}_2$ is $G = SL(2,2) = PSL(2,2) = S_3$ (of order 6) with the elements

$$\mathtt{1} = \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix}, u = \begin{pmatrix} 1\,0 \\ 1\,1 \end{pmatrix}, w = \begin{pmatrix} 0\,1 \\ 1\,0 \end{pmatrix}, vu = \begin{pmatrix} 0\,1 \\ 1\,1 \end{pmatrix}, uv = \begin{pmatrix} 1\,1 \\ 1\,0 \end{pmatrix}, v = \begin{pmatrix} 1\,1 \\ 0\,1 \end{pmatrix}.$$

Its action on the set of solutions S results in the change of variables, e.g. the action of the element u corresponds to the change of variables

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} 1\,0 \\ 1\,1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}. \tag{3.1}$$

Note that already the set of the first 4 solutions (for inner calculi with $\mathrm{d}x^1$,i.e. $S_1 = \{s_1, s_2, s_3, s_4\}$ splits into the three orbits under the action of the group G. It is enough to consider the element $u \in G$ and the change of variables (3.1) to obtain that $s_1 \simeq s_4$.

Recall that if G acts on a set S the orbits of this action are the sets

$$O_s = \{s' \in S \quad | \quad g \cdot s = s' \quad \text{for} \quad g \in G\}.$$

We obtain the following:

For the calculus A the orbit consist of the elements: $O_{s_1} = \{s_1, s_{4,} s_5, s_6, s_9, s_{12}\}, |O_{s_1}| = 6$ and the isotropy group of element $Hs_1 = \{\mathtt{1}\}$.

For the calculus B: $O_{s_2} = \{s_2, s_{7,} s_{11}\}, |O_{s_2}| = 2$ and the isotropy group of the element $Hs_2 = \{\mathtt{1}, u\}$.

For the calculus C: $O_{s_3} = \{s_3, s_{8,} s_{10}\}, |O_{s_3}| = 2$ and the isotropy group of the element $Hs_3 = \{\mathtt{1}, v\}$.

As an example we present the explicit calculation for the orbit containing element s_2: $e \cdot s_2 = s_2; u \cdot s_2 = s_2;$ $w \cdot s_2 = s_7; uv \cdot s_2 = s_7;$ $vu \cdot s_2 = s_{11};$ $v \cdot s_2 = s_{11}$. Therefore the corresponding orbit is

$$O_{s_2} = \{s_2, s_7, s_{11}\}$$

and the isotropy groups of its elements are

$$H_{s_2} = \{e, u\}, \quad H_{s_7} = \{w, uv\}, \quad H_{s_{11}} = \{vu, v\}.$$

Other orbits are calculated analogously. These three orbits exhaust the elements of the whole set S implying there are only three non-isomorphic families of differential calculi as collected in the first column of Table 1, choosing s_1 as case A, s_2 as case B and s_3 as case C, with $x^1 = e$ the identity element for the $\circ$ product and $\theta = \mathrm{d}x^1 = \mathrm{d}e$, and $x^2 = x$.

For each of these differential calculi A, B and C we next look for the quantum metrics $g \in \Omega^1 \otimes_A \Omega^1$ with $\wedge(g) = 0$ in the form

$$g = g_{11}\mathrm{d}e \otimes \mathrm{d}e + g_{12}\left(\mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e\right) + g_{22}\mathrm{d}x \otimes \mathrm{d}x$$

with constant coefficients, i.e. $g_{11}, g_{12}, g_{22} \in \mathbb{F}_2$. Then we look for bimodule connections, which take the form (2.4) including bimodule maps α and σ. Here $\nabla \mathrm{d}e = \nabla \mathrm{d}x = 0$ and σ = flip on the generators are always torsion free metric compatible bimodule connections but each of the calculi has an additional QLCs which are collected along with the possible metrics in the last column of Table 1.

We show some of the calculations behind the A case explicitly, with similar arguments for the other cases. Thus, working with the A calculus, we first calculate: $[g, e] = 0$ and $[g, x] = g_{11}\left(\mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x\right) = 0 \Rightarrow g_{11} = 0$. The possible (non degenerate) solutions for metric coefficients are

i) $g_{12} = 1, g_{22} = 0$ resulting in $g_{A.I} = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e$

ii) $g_{12} = 1, g_{22} = 1$ resulting in $g_{A.II} = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x$.

Next, to find the bimodule connections, we look for the bimodule maps α and σ, taking the former in the form $\alpha\left(\mathrm{d}x\right) = a\mathrm{d}e \otimes \mathrm{d}e + b\left(\mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e\right) + c\mathrm{d}x \otimes \mathrm{d}x$ and we calculate that $\alpha\left([\mathrm{d}x, e]\right) = [\alpha\left(\mathrm{d}x\right), e] = 0$. On the other hand, for this calculus, $\alpha\left([\mathrm{d}x, e]\right) = \alpha\left(\mathrm{d}x\right) = a\mathrm{d}e \otimes \mathrm{d}e + b\left(\mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e\right) + c\mathrm{d}x \otimes \mathrm{d}x$. Therefore we have $a, b, c = 0$.

Similarly for $\alpha\left(\mathrm{d}e\right) = a'\mathrm{d}e \otimes \mathrm{d}e + b'\left(\mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e\right) + c'\mathrm{d}x \otimes \mathrm{d}x$ we calculate that $\alpha\left([\mathrm{d}e, e]\right) = [\alpha\left(\mathrm{d}e\right), e] = 0$, while on the other hand $\alpha\left([\mathrm{d}e, e]\right) = \alpha\left(\mathrm{d}e\right) = a'\mathrm{d}e \otimes \mathrm{d}e + b'\left(\mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e\right) + c'\mathrm{d}x \otimes \mathrm{d}x$ implying $a', b', c' = 0$. Hence there are no non-zero module maps α.

For the sigma map we assume (2.14) and the metric compatibility (with $\alpha = 0$) (2.5). We solve the relations (2.14) and (2.15) over the field $\mathbb{F}_2$ by computer, which gives rise to the following torsion free metric compatible 'quantum Levi-Civita' bimodule connections. This gives us

i) For $g_{A.I} = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e$ we have five solutions

(A.I.1) $\nabla \mathrm{d}e = \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = 0;$

(A.I.2) $\nabla \mathrm{d}e = \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}x;$

(A.I.3) $\nabla \mathrm{d}e = \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}x;$

(A.I.4) $\nabla \mathrm{d}e = 0, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}e;$

(A.I.5) $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x.$

ii) For $g_{A.II} = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x$ we have three solutions

(A.II.1) $\nabla \mathrm{d}e = \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = 0;$

(A.II.2) $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e, \quad \nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}x;$

(A.II.3) $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}x.$

We solve the B, C cases similarly, all results being collected in Table 1 above. We have not listed the associated σ as these are uniquely determined by ∇ and the commutation relations.

As a check we see that calculus case A corresponds to $V \simeq \mathbb{F}_2\mathbb{Z}_2$ as described in the general analysis, see example (ii), in Section 2. The metrics $g_{A.II}, g_{A.I}$ recover the two metrics there for $m = 0, 1$ after the change of variables to $x^1 = e + x$, $x^0 = e$ (where the superscript on the left is a label not an exponent). The complementary metrics in the general analysis duplicate these. The calculus case B corresponds to $V \simeq \mathbb{F}_2(2 \text{ points})$ in the general analysis, example (i) in Section 2, with the metric g_B agreeing with the Euclidean metric there on change of variables $x^0 = e + x$, $x^1 = x$.

Proposition 3.1. *For $n = 2$ all quantum Levi-Civita connections as listed in Table 1 are flat.*

Proof. Explicitly using (2.6) we demonstrate the calculation for the first two bimodule connections compatible with the first metric $g_{A.I}$ in the family A. For (A.I.1) this is

$R_\nabla \mathrm{d}e = (\mathrm{d} \otimes \mathrm{id} - (\wedge \otimes \mathrm{id})(\mathrm{id} \otimes \nabla))\nabla \mathrm{d}e = -(\wedge \otimes \mathrm{id})(\mathrm{id} \otimes \nabla)(\mathrm{d}x \otimes \mathrm{d}x) = 0,$

$R_\nabla \mathrm{d}x = (\mathrm{d} \otimes \mathrm{id} - (\wedge \otimes \mathrm{id})(\mathrm{id} \otimes \nabla))\nabla \mathrm{d}x = 0,$

while for (A.I.2) the calculation is

$R_\nabla \mathrm{d}e = -(\wedge \otimes \mathrm{id})(\mathrm{id} \otimes \nabla)(\mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x) = \mathrm{d}x \wedge \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \wedge \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}e \wedge \mathrm{d}x \otimes \mathrm{d}x = 0,$

$R_\nabla \mathrm{d}x = -(\wedge \otimes \mathrm{id})(\mathrm{id} \otimes \nabla)(\mathrm{d}x \otimes \mathrm{d}x) = -\mathrm{d}x \wedge \mathrm{d}x \otimes \mathrm{d}x = 0.$

Similarly for all the other bimodule connections in Table 1 above. We refer only to bimodule quantum Levi-Civita connections with constant coefficients and invertible σ as listed in the table. □

4. Classification for $n = 3$

For the $n = 3$ inner case over $\mathbb{F}_2$ we will find six inequivalent unital algebras. In each case we take e, x, y as basis and have $e \circ e = e, e \circ x = x = x \circ e, e \circ y = y = y \circ e$, with the remaining relations as:

A: $\quad x \circ y = 0 = y \circ x, \quad x \circ x = 0 = y \circ y,$

B: $\quad x \circ y = 0 = y \circ x, \quad x \circ x = x, \quad y \circ y = y,$

C: $x \circ y = 0 = y \circ x, \quad x \circ x = x, \quad y \circ y = 0,$

D: $x \circ y = x + y = y \circ x, \quad x \circ x = y, \quad y \circ y = x,$

E: $x \circ y = 0 = y \circ x, \quad x \circ x = y, \quad y \circ y = 0,$

F: $x \circ y = x + y = y \circ x, \quad x \circ x = e + x + y, \quad y \circ y = x.$

These six inequivalent commutative unital algebras imply six noncommutative differential calculi as shown in Table 2. For each of them we show the number of quantum metrics and for each metric the number of torsion free cotorsion free ('quantum Levi-Civita') bimodule connections. The metrics are listed in detail in Table 4.

	Relations $[\mathrm{d}e, e] = \mathrm{d}e$ $[\mathrm{d}e, x] = \mathrm{d}x = [\mathrm{d}x, e]$ $[\mathrm{d}e, y] = \mathrm{d}y = [\mathrm{d}y, e]$	Orbit order	Isotropy Group	Quantum metrics	Nonzero QLC	$R_\nabla \neq 0$
A	$[\mathrm{d}x, y] = 0 = [\mathrm{d}y, x]$ $[\mathrm{d}x, x] = 0 = [\mathrm{d}y, y]$	$\lvert O_{s_1}\rvert = 4 \times 7$	$\{\imath, \tilde{w}, \tilde{u}\tilde{v}, \tilde{v}, \tilde{u}, \tilde{v}\tilde{u}\} = S_3$	0	-	-
B	$[\mathrm{d}x, y] = 0 = [\mathrm{d}y, x]$ $[\mathrm{d}x, x] = \mathrm{d}x, [\mathrm{d}y, y] = \mathrm{d}y$	$\lvert O_{s_{34}}\rvert = 4 \times 7$	$\{\imath, \tilde{w}, \tilde{\tilde{v}}\tilde{\tilde{u}}, \tilde{\tilde{u}}, \tilde{\tilde{v}}, \tilde{\tilde{u}}\tilde{\tilde{v}}\} = S_3$	1	3	0
C	$[\mathrm{d}x, y] = 0 = [\mathrm{d}y, x]$ $[\mathrm{d}x, x] = \mathrm{d}x, [\mathrm{d}y, y] = 0$	$\lvert O_{s_2}\rvert = 24 \times 7$	$\{\imath\}$	2	13 each g_C	2 for $g_{C.I}$ 3 for $g_{C.II}$
D	$[\mathrm{d}x, y] = \mathrm{d}x + \mathrm{d}y = [\mathrm{d}y, x]$ $[\mathrm{d}x, x] = \mathrm{d}y, [\mathrm{d}y, y] = \mathrm{d}x$	$\lvert O_{s_{23}}\rvert = 12 \times 7$	$\{\imath, \tilde{w}\} = \mathbb{Z}_2$	3	3 each g_D	1 for $g_{D.I}$ 0 for $g_{D.II}$ 1 for $g_{D.III}$
E	$[\mathrm{d}x, y] = 0 = [\mathrm{d}y, x]$ $[\mathrm{d}x, x] = \mathrm{d}y, [\mathrm{d}y, y] = 0$	$\lvert O_{s_3}\rvert = 12 \times 7$	$\{\imath, \tilde{v}\} = \mathbb{Z}_2$	4	13 each g_E	2 for $g_{E.I}$ 3 for $g_{E.II}$ 5 for $g_{E.III}$ 4 for $g_{E.IV}$
F	$[\mathrm{d}x, y] = \mathrm{d}e + \mathrm{d}x = [\mathrm{d}y, x]$ $[\mathrm{d}x, x] = \mathrm{d}e + \mathrm{d}x + \mathrm{d}y$ $[\mathrm{d}y, y] = \mathrm{d}x$	$\lvert O_{s_{20}}\rvert = 8 \times 7$	$\{\imath, \tilde{\tilde{v}}\tilde{\tilde{u}}, \tilde{\tilde{u}}\tilde{\tilde{v}}\} = \mathbb{Z}_3$	7	3 each g_F	2 for each g_F except 0 for $g_{F.II}$

Table 2. All possible unital inner noncommutative geometries on $\mathbb{F}_2[e, x, y]$.

We now outline how these results were obtained. For $\mathbb{F}_2[x^1, x^2, x^3]$ defined as universal enveloping algebra of Abelian Lie algebra generated by the basis elements x^1, x^2, x^3, the commutative product (2.9) induces the differential calculus, as before. In 3 dimensions there is seven possibilities for element θ (as the differential of an element of the pre-Lie algebra) for inner calculi, namely

$$\theta = \mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3, \mathrm{d}x^1 + \mathrm{d}x^2, \mathrm{d}x^1 + \mathrm{d}x^3,\ \mathrm{d}x^2 + \mathrm{d}x^3 \text{ and } \mathrm{d}x^1 + \mathrm{d}x^2 + \mathrm{d}x^3.$$

Finding explicitly the solutions to (2.1), (2.2) gives 7×64 cases, i.e. 64 solutions for each of the seven possible θ. The 64 cases with inner calculus with $\theta = \mathrm{d}x^1$ are listed in Table 3, there are similarly 64 for each of the other 6 cases.

	$V^{11}{}_1$	$V^{11}{}_2$	$V^{11}{}_3$	$V^{12}{}_1$	$V^{12}{}_2$	$V^{12}{}_3$	$V^{13}{}_1$	$V^{13}{}_2$	$V^{13}{}_3$	$V^{22}{}_1$	$V^{22}{}_2$	$V^{22}{}_3$	$V^{23}{}_1$	$V^{23}{}_2$	$V^{23}{}_3$	$V^{33}{}_1$	$V^{33}{}_2$	$V^{33}{}_3$
[1,]	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0
[2,]	1	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0
[3,]	1	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0
[4,]	1	0	0	0	1	0	0	0	1	0	1	1	0	0	0	0	0	0
[5,]	1	0	0	0	1	0	0	0	1	1	0	0	0	0	1	0	0	0
[6,]	1	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0
[7,]	1	0	0	0	1	0	0	0	1	1	0	1	0	0	1	0	0	0
[8,]	1	0	0	0	1	0	0	0	1	0	1	1	0	0	1	0	0	0
[9,]	1	0	0	0	1	0	0	0	1	0	0	0	0	1	0	1	0	0
[10,]	1	0	0	0	1	0	0	0	1	0	1	0	0	1	0	1	0	0
[11,]	1	0	0	0	1	0	0	0	1	1	0	1	0	1	0	1	0	0
[12,]	1	0	0	0	1	0	0	0	1	1	1	1	0	1	0	1	0	0
[13,]	1	0	0	0	1	0	0	0	1	1	0	0	1	1	1	1	0	0
[14,]	1	0	0	0	1	0	0	0	1	0	1	0	1	1	1	1	0	0
[15,]	1	0	0	0	1	0	0	0	1	0	0	1	1	1	1	1	0	0
[16,]	1	0	0	0	1	0	0	0	1	1	1	1	1	1	1	1	0	0
[17,]	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0
[18,]	1	0	0	0	1	0	0	0	1	0	0	1	1	0	0	0	1	0
[19,]	1	0	0	0	1	0	0	0	1	0	1	0	0	1	0	0	1	0
[20,]	1	0	0	0	1	0	0	0	1	1	1	1	1	1	0	0	1	0
[21,]	1	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0
[22,]	1	0	0	0	1	0	0	0	1	0	1	1	1	0	1	0	1	0
[23,]	1	0	0	0	1	0	0	0	1	0	0	1	0	1	1	0	1	0
[24,]	1	0	0	0	1	0	0	0	1	1	0	0	1	1	1	0	1	0
[25,]	1	0	0	0	1	0	0	0	1	0	1	0	0	0	0	1	1	0
[26,]	1	0	0	0	1	0	0	0	1	0	1	1	1	0	0	1	1	0
[27,]	1	0	0	0	1	0	0	0	1	0	0	0	0	1	0	1	1	0
[28,]	1	0	0	0	1	0	0	0	1	1	0	1	1	1	0	1	1	0
[29,]	1	0	0	0	1	0	0	0	1	1	0	0	0	0	1	1	1	0
[30,]	1	0	0	0	1	0	0	0	1	1	0	1	1	0	1	1	1	0
[31,]	1	0	0	0	1	0	0	0	1	1	1	1	0	1	1	1	1	0
[32,]	1	0	0	0	1	0	0	0	1	0	1	0	1	1	1	1	1	0
[33,]	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1
[34,]	1	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	1
[35,]	1	0	0	0	1	0	0	0	1	1	0	1	0	0	0	0	0	1
[36,]	1	0	0	0	1	0	0	0	1	1	1	1	0	0	0	0	0	1
[37,]	1	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	1
[38,]	1	0	0	0	1	0	0	0	1	0	1	0	0	1	0	0	0	1
[39,]	1	0	0	0	1	0	0	0	1	0	0	1	0	1	0	0	0	1
[40,]	1	0	0	0	1	0	0	0	1	0	1	1	0	1	0	0	0	1
[41,]	1	0	0	0	1	0	0	0	1	1	0	0	0	0	1	0	0	1
[42,]	1	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	1
[43,]	1	0	0	0	1	0	0	0	1	0	0	1	0	0	1	0	0	1
[44,]	1	0	0	0	1	0	0	0	1	1	1	1	0	0	1	0	0	1
[45,]	1	0	0	0	1	0	0	0	1	1	0	0	1	1	1	0	0	1
[46,]	1	0	0	0	1	0	0	0	1	0	1	0	1	1	1	0	0	1
[47,]	1	0	0	0	1	0	0	0	1	1	0	1	1	1	1	0	0	1
[48,]	1	0	0	0	1	0	0	0	1	0	1	1	1	1	1	0	0	1
[49,]	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	1
[50,]	1	0	0	0	1	0	0	0	1	1	0	1	1	0	0	0	1	1
[51,]	1	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	1	1
[52,]	1	0	0	0	1	0	0	0	1	0	0	1	1	1	0	0	1	1
[53,]	1	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	1	1
[54,]	1	0	0	0	1	0	0	0	1	1	1	1	1	0	1	0	1	1
[55,]	1	0	0	0	1	0	0	0	1	0	1	1	0	1	1	0	1	1
[56,]	1	0	0	0	1	0	0	0	1	0	1	0	1	1	1	0	1	1
[57,]	1	0	0	0	1	0	0	0	1	0	1	0	0	0	0	1	1	1
[58,]	1	0	0	0	1	0	0	0	1	1	1	1	1	0	0	1	1	1
[59,]	1	0	0	0	1	0	0	0	1	0	1	0	0	1	0	1	1	1
[60,]	1	0	0	0	1	0	0	0	1	0	1	1	1	0	1	1	1	1
[61,]	1	0	0	0	1	0	0	0	1	1	0	0	0	0	1	1	1	1
[62,]	1	0	0	0	1	0	0	0	1	0	0	1	1	0	1	1	1	1
[63,]	1	0	0	0	1	0	0	0	1	1	0	1	0	1	1	1	1	1
[64,]	1	0	0	0	1	0	0	0	1	1	0	0	1	1	1	1	1	1

Table 3. Structure coefficients for unital inner differential calculi with $\theta = dx^1$

The set of the 7×64 solutions, denoted by $S = \{s_1, ..., s_{64}, ...s_{7\times 64}\}$, splits into the six orbits under the action of a group of isomorphisms over $\mathbb{F}_2$: $G = GL(3,\mathbb{F}_2) = PSL(2,7)$. The order of the group of isomorphisms is $|G| = 168$ and we write only some of its elements (only the ones needed to list the isotropy groups explicitly in Table 2) :

$$\mathbb{1}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ \tilde{u}=\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix},\ \tilde{w}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\ \tilde{u}\tilde{v}=\begin{pmatrix}1&0&0\\0&1&1\\0&1&0\end{pmatrix},\ \tilde{v}=\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix},$$
$$\tilde{v}\tilde{u}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&1\end{pmatrix},\ \tilde{\tilde{v}}\tilde{\tilde{u}}=\begin{pmatrix}1&0&0\\1&1&1\\0&1&0\end{pmatrix},\ \tilde{\tilde{u}}=\begin{pmatrix}1&0&0\\1&1&1\\0&0&1\end{pmatrix},\ \tilde{\tilde{v}}=\begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix},\ \tilde{\tilde{u}}\tilde{\tilde{v}}=\begin{pmatrix}1&0&0\\0&0&1\\1&1&1\end{pmatrix}.$$

One can present the sketch of a proof, by considering only the solutions for one of the possible θ, for which we take $\theta = \mathrm{d}x^1$. These are the first 64 solutions we denote by $S_1 = \{s_1, ..., s_{64}\}$ listed in each row of Table 3, where coefficients $V^{\mu\nu}{}_\rho$ (as solutions of (2.1), (2.2) are collected. The other choices of θ have similar structure by a $GL(3,\mathbb{F}_2)$ transformation mapping $\mathrm{d}x^1$ to any other θ. The space S_1 of such restricted solutions again splits into the six orbits under the action of $GL(3,\mathbb{F}_2)$.

Thus, s_i corresponds to the case with coefficients $V^{\mu\nu}{}_\rho$ listed in the row $[i]$ in Table 3 and falls to the orbit denoted as $O_{s_i} \cap O_{r_i}$ (which is equivalent to the one of the families A - F given in the Table 2 above). We write $O_{s_i} \cap O_{p_i}$ to underline that we list only part of the orbit (the one coming from the first 64 solutions with inner calculus $\mathrm{d}x^1$ as O_{p_i}, when in fact the full size of O_{s_i} is 7 ×bigger, as indicated in Table 2).

The first 64 solutions split into the following orbits:

$O_{s_1} \cap O_{p_1} = \{s_1, s_5, s_9, s_{13}\}$;

$O_{s_{34}} \cap O_{p_{34}} = \{s_{34}, s_{38}, s_{42}, s_{46}\}$;

$O_{s_2} \cap O_{p_2} = \{s_2, s_4, s_6, s_8, s_{10}, s_{12}, s_{14}, s_{16}, s_{19}, s_{21}, s_{25}, s_{32}, s_{33}, s_{35}, s_{37}, s_{39}, s_{41}, s_{43}, s_{45}, s_{47}, s_{49}, s_{51}, s_{61}, s_{64}\}$;

$O_{s_{23}} \cap O_{p_{23}} = \{s_{23}, s_{18}, s_{28}, s_{30}, s_{36}, s_{40}, s_{44}, s_{48}, s_{53}, s_{56}, s_{57}, s_{59}\}$;

$O_{s_3} \cap O_{p_3} = \{s_3, s_7, s_{11}, s_{15}, s_{17}, s_{24}, s_{27}, s_{29}, s_{54}, s_{55}, s_{58}, s_{60}\}$;

$O_{s_{20}} \cap O_{p_{20}} = \{s_{20}, s_{22}, s_{26}, s_{31}, s_{50}, s_{52}, s_{62}, s_{63}\}$.

Already for the first 64 solutions we get the six orbits giving six (A - F) inequivalent noncommutative differential calculi.

For the isotropy groups one can calculate that, e.g. for H_{s_1}: $\mathbb{1}\cdot s_1 = s_1; \tilde{w}\cdot s_1 = s_1; \tilde{u}\tilde{v}\cdot s_1 = s_1; \tilde{v}\cdot s_1 = s_1; \tilde{u}\cdot s_1 = s_1; \tilde{v}\tilde{u}\cdot s_1 = s_1$ or for H_{s_3}: $\mathbb{1}\cdot s_3, \tilde{v}\cdot s_3$ or for $H_{s_{23}}$: $\mathbb{1}\cdot s_{23} = s_{23}$ and $\tilde{w}\cdot s_{23} = s_{23}$. Similarly one can calculate the isotropy groups for the remaining elements.

Below we also show some examples of isomorphisms of certain solutions in S_1 to the six presented above families.

One element of G namely $\begin{pmatrix}1&0&0\\0&0&1\\1&1&0\end{pmatrix}$ gives the change of variables

$$y^1 = x^1;\quad y^2 = x^3;\quad y^3 = x^1 + x^2$$
$$\mathrm{d}y^1 = \mathrm{d}x^1;\quad \mathrm{d}y^2 = \mathrm{d}x^3;\quad \mathrm{d}y^3 = \mathrm{d}x^1 + \mathrm{d}x^2$$

under which we immediately see that

$$[\mathrm{d}y^1, y^1] = \mathrm{d}y^1; \quad [\mathrm{d}y^1, y^2] = \mathrm{d}y^2 = [\mathrm{d}y^2, y^1]; \quad [\mathrm{d}y^1, y^3] = \mathrm{d}y^3 = [\mathrm{d}y^3, y^1]$$

i.e. the result is still an inner differential calculus with $\theta = \mathrm{d}y^1 = \mathrm{d}x^1$.

One can check that the remaining commutators will always fall into one of the six families above A - F:

- In case A, s_1 is isomorphic to s_9, with $[\mathrm{d}y^2, y^2] = 0; [\mathrm{d}y^2, y^3] = \mathrm{d}y^2 = [\mathrm{d}y^3, y^2]; [\mathrm{d}y^3, y^3] = \mathrm{d}y^1$ i.e. with the non-zero coefficients, ${V^{23}}_2 = {V^{32}}_2 = 1$ and ${V^{33}}_1 = 1$, cf. row [9] in Table 3 above.
- In case B, s_{34} is isomorphic to s_{38}, with $[\mathrm{d}y^2, y^2] = \mathrm{d}y^2; [\mathrm{d}y^2, y^3] = \mathrm{d}y^2 = [\mathrm{d}y^3, y^2]; [\mathrm{d}y^3, y^3] = \mathrm{d}y^3$.
- In case C, s_2 is isomorphic to s_{37}, with $[\mathrm{d}y^2, y^2] = 0; [\mathrm{d}y^2, y^3] = \mathrm{d}y^2 = [\mathrm{d}y^3, y^2]; [\mathrm{d}y^3, y^3] = \mathrm{d}y^3$.
- In case D, s_{23} is isomorphic to s_{30}, with $[\mathrm{d}y^2, y^2] = \mathrm{d}y^1 + \mathrm{d}y^3; [\mathrm{d}y^2, y^3] = \mathrm{d}y^1 + \mathrm{d}y^3 = [\mathrm{d}y^3, y^2]; [\mathrm{d}y^3, y^3] = \mathrm{d}y^1 + \mathrm{d}y^2$.
- In case E, s_3 is isomorphic to s_{27}, with $[\mathrm{d}y^2, y^2] = 0; [\mathrm{d}y^2, y^3] = \mathrm{d}y^2 = [\mathrm{d}y^3, y^2]; [\mathrm{d}y^3, y^3] = \mathrm{d}y^1 + \mathrm{d}y^2$.
- In case F, s_{20} is isomorphic to s_{63}, with $[\mathrm{d}y^2, y^2] = \mathrm{d}y^1 + \mathrm{d}y^3; [\mathrm{d}y^2, y^3] = \mathrm{d}y^2 + \mathrm{d}y^3 = [\mathrm{d}y^3, y^2]; [\mathrm{d}y^3, y^3] = \mathrm{d}y^1 + \mathrm{d}y^2 + \mathrm{d}y^3$.

4.1. **Quantum metrics.** For each of the differential calculi A - F in the first column of Table 2 we next look for the quantum metrics $g \in \Omega^1 \otimes_A \Omega^1$ with $\wedge(g) = 0$ in the form

$$\begin{aligned} g = {} & g_{11}\mathrm{d}e \otimes \mathrm{d}e + g_{12}(\mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e) + g_{13}(\mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e) + g_{22}\mathrm{d}x \otimes \mathrm{d}x \\ & + g_{23}(\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x) + g_{33}\mathrm{d}y \otimes \mathrm{d}y \end{aligned}$$

with constant coefficients, i.e. $g_{\mu\nu} \in \mathbb{F}_2$, similarly as outlined in the $n = 2$ case. The results for the possible quantum metrics corresponding to the above differential calculi are shown in Table 4.

As a check we see that the calculus case B corresponds to $V \simeq \mathbb{F}_2(3 \text{ points})$ with the metric g_B agreeing with the Euclidean metric in the general analysis (example (i) from Section 2 after the change of variables: $e = x^0 + x^1 + x^2$, $x^1 = x$, $x^2 = y$. The case D corresponds to $V \simeq \mathbb{F}_2\mathbb{Z}_3$ with the metrics $g_{D.II}, g_{D.III}, g_{D.I}$ recovering the $m = 0, 1, 2$ metrics for example (ii) in Section 2 after the change of variables $x^0 = e, x^1 = e + x, x^2 = e + y$ (where the superscript of x is a label not a square; this element being $(x^1)^2$ in $\mathbb{F}_2\mathbb{Z}_3$). The complementary metrics are not listed in our table as they are degenerate.

Next we look for bimodule connections, which take the form (2.4) including bimodule maps α and σ. As for $n = 2$, careful analysis shows that there are no nonzero module maps α. For the σ map we assume metric compatibility (2.5) and impose the torsion free condition (2.7). The curvature is calculated from (2.6). $\nabla \mathrm{d}e = \nabla \mathrm{d}x = \nabla \mathrm{d}y = 0$ and σ = flip on the generators are always torsion free metric compatible bimodule connections but we also find additional non-zero QLCs, some

	Quantum metrics
A	
B	$g_B = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x$
C	$g_{C.I} = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x$ $g_{C.II} = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y$
D	$g_{D.I} = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x$ $g_{D.II} = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x$ $g_{D.III} = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}y\otimes\mathrm{d}y$
E	$g_{E.I} = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x$ $g_{E.II} = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x$ $g_{E.III} = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y$ $g_{E.IV} = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y$
F	$g_{F.I} = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x$ $g_{F.II} = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x$ $g_{F.III} = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x$ $g_{F.IV} = \mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}y\otimes\mathrm{d}y$ $g_{F.V} = \mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y$ $g_{F.VI} = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y$ $g_{F.VII} = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y$

Table 4. All possible quantum metrics for each of the possible calculi in Table 2. Note that $g_{E.II} = g_{C.I}$, $g_{E.IV} = g_{C.II}$, $g_{F.I} = g_{E.I}$ and $g_{F.II} = g_{D.II} = g_B$.

of which have non-zero curvature R_∇ (their numbers are given in the Table 2). This methodology is the same as for $n = 2$, therefore we omit the details and list the resulting QLCs and their curvatures. The maps σ although computed in the analysis are not listed as they are uniquely determined by ∇ and the commutation relations. There is no case A as this did not have any quantum metrics.

Case B (for metric g_B):
(B.1) $\nabla\mathrm{d}e = 0,\quad \nabla\mathrm{d}x = 0,\quad \nabla\mathrm{d}y = \mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x,\quad R_\nabla = 0;$
(B.2) $\nabla\mathrm{d}e = 0,\quad \nabla\mathrm{d}x = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}y\otimes\mathrm{d}y,\quad \nabla\mathrm{d}y = 0,\quad R_\nabla = 0;$
(B.3) $\nabla\mathrm{d}e = 0,\quad \nabla\mathrm{d}x = \mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y = \nabla\mathrm{d}y,\quad R_\nabla = 0.$

Case C
- for metric $g_{C.I}$:
(C.I.1) $\nabla\mathrm{d}e = 0,\quad \nabla\mathrm{d}x = 0,\quad \nabla\mathrm{d}y = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x\ ;\quad R_\nabla = 0;$

(C.I.2) $\nabla\mathrm{d}e = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x,\quad \nabla\mathrm{d}x = 0,$
$\nabla\mathrm{d}y = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y,\quad R_\nabla = 0;$

(C.I.3) $\nabla\mathrm{d}e = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e,\quad \nabla\mathrm{d}x = \mathrm{d}y\otimes\mathrm{d}y = \nabla\mathrm{d}y,\quad R_\nabla = 0;$

(C.I.4) $\nabla\mathrm{d}e = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x,\quad \nabla\mathrm{d}x = 0,$
$\nabla\mathrm{d}y = \mathrm{d}y\otimes\mathrm{d}y,\quad R_\nabla = 0;$

(C.I.5) $\nabla\mathrm{d}e = \mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x,\quad \nabla\mathrm{d}x = \mathrm{d}y\otimes\mathrm{d}y,\quad \nabla\mathrm{d}y = 0,\quad R_\nabla = 0;$

(C.I.6) $\nabla\mathrm{d}e = \mathrm{d}y\otimes\mathrm{d}y,\quad \nabla\mathrm{d}x = 0,\quad \nabla\mathrm{d}y = 0,\quad R_\nabla = 0;$

(C.I.7) $\nabla\mathrm{d}e = \mathrm{d}y\otimes\mathrm{d}y+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e,\quad \nabla\mathrm{d}x = \mathrm{d}y\otimes\mathrm{d}y = \nabla\mathrm{d}y,\quad R_\nabla = 0;$

(C.I.8) $\nabla\mathrm{d}e = \mathrm{d}y\otimes\mathrm{d}y+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x,\quad \nabla\mathrm{d}x = 0,$
$\nabla\mathrm{d}y = \mathrm{d}y\otimes\mathrm{d}y,\quad R_\nabla = 0;$

(C.I.9) $\nabla \mathrm{d}e = \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x$, $\nabla\ \mathrm{d}x = \mathrm{d}y \otimes \mathrm{d}y$, $\nabla \mathrm{d}y = 0$, $R_\nabla = 0$;

(C.I.10) $\nabla \mathrm{d}e = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e$, $\nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x$, $\nabla \mathrm{d}y = \mathrm{d}y \otimes \mathrm{d}y$, $R_\nabla = 0$;

(C.I.11) $\nabla \mathrm{d}e = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x$, $\nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x$, $\nabla \mathrm{d}y = 0$,

$$R_\nabla \mathrm{d}e = \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y, \quad R_\nabla \mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y, \quad R_\nabla \mathrm{d}y = 0;$$

(C.I.12) $\nabla \mathrm{d}e = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e$, $\nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x$, $\nabla \mathrm{d}y = \mathrm{d}y \otimes \mathrm{d}y$, $R_\nabla = 0$;

(C.I.13) $\nabla \mathrm{d}e = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x$, $\nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x$, $\nabla \mathrm{d}y = 0$,

$$R_\nabla \mathrm{d}e = \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x, \quad R_\nabla \mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y, \quad R_\nabla \mathrm{d}y = 0.$$

- for metric $g_{C.II}$:
(C.II.1) $\nabla \mathrm{d}e = 0$, $\nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e$,
$\nabla \mathrm{d}y = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}e$, $R_\nabla = 0$;

(C.II.2) $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x$, $\nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e$, $\nabla \mathrm{d}y = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}y$,

$$R_\nabla \mathrm{d}e = \mathrm{d}y \wedge \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}e \wedge \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}e \wedge \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}e \wedge \mathrm{d}y \otimes \mathrm{d}x,$$

$$R_\nabla \mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}x \wedge \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y,$$

$$R_\nabla \mathrm{d}y = \mathrm{d}e \wedge \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x.$$

The remaining solutions (C.II.3) - (C.II.13) are the same as for the metric $g_{C.I}$, i.e. are equal to cases (C.I.3) - (C.I.13) respectively.

Case D
- for metric $g_{D.I}$:
(D.I.1) $\nabla \mathrm{d}e = 0$, $\nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}e$, $\nabla \mathrm{d}y = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e$, $R_\nabla = 0$;

(D.I.2) $\nabla \mathrm{d}e = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y$, $\nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x$, $\nabla \mathrm{d}y = \mathrm{d}y \otimes \mathrm{d}y$, $R_\nabla = 0$;

(D.I.3) $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}y$,
$\nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}y$,
$\nabla \mathrm{d}y = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y$,

$$R_\nabla \mathrm{d}e = \mathrm{d}x \wedge \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \wedge \mathrm{d}e \otimes \mathrm{d}e,$$

$$R_\nabla \mathrm{d}x = \mathrm{d}e \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}x \wedge \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}x \wedge \mathrm{d}e \otimes \mathrm{d}y,$$

$$R_\nabla \mathrm{d}y = \mathrm{d}e \wedge \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}y \wedge \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}e.$$

- metric $g_{D.II}$:
(D.II.1) $\nabla \mathrm{d}e = 0$, $\nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x$,
$\nabla \mathrm{d}y = \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}y$, $R_\nabla = 0$;

(D.II.2) $\nabla \mathrm{d}e = 0$, $\nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x$,
$\nabla \mathrm{d}y = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x$, $R_\nabla = 0$;

(D.II.3) $\nabla \mathrm{d}e = 0$, $\nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y$,
$\nabla \mathrm{d}y = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y$, $R_\nabla = 0$.

- for metric $g_{D.III}$

(D.III.1) $\nabla \mathrm{d}e = 0, \quad \nabla \mathrm{d}x = \mathrm{d}e\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}e, \quad \nabla \mathrm{d}y = \mathrm{d}e\otimes \mathrm{d}e, \quad R_\nabla = 0;$

(D.III.2) $\nabla \mathrm{d}e = \mathrm{d}x\otimes \mathrm{d}x + \mathrm{d}x\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{d}x\otimes \mathrm{d}x,$
$\nabla \mathrm{d}y = \mathrm{d}x\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}x, \quad R_\nabla = 0;$

(D.III.3) $\nabla \mathrm{d}e = \mathrm{d}e\otimes \mathrm{d}e + \mathrm{d}e\otimes \mathrm{d}x + \mathrm{d}e\otimes \mathrm{d}y + \mathrm{d}x\otimes \mathrm{d}e + \mathrm{d}x\otimes \mathrm{d}x + \mathrm{d}y\otimes \mathrm{d}e,$
$\nabla \mathrm{d}x = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x, \quad \nabla \mathrm{d}y = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}e,$

$$R_\nabla \mathrm{d}e = (\mathrm{d}x + \mathrm{d}y)\wedge \mathrm{d}e\otimes \mathrm{d}e + \mathrm{d}e\wedge \mathrm{d}y\otimes \mathrm{d}y,$$
$$R_\nabla \mathrm{d}x = \mathrm{d}e\wedge \mathrm{d}x\otimes \mathrm{d}x + \mathrm{d}x\wedge \mathrm{d}e\otimes \mathrm{d}e + \mathrm{d}x\wedge \mathrm{d}y\otimes \mathrm{d}y + \mathrm{d}x\wedge \mathrm{d}e\otimes \mathrm{d}y + \mathrm{d}x\wedge \mathrm{d}y\otimes \mathrm{d}e,$$
$$R_\nabla \mathrm{d}y = \mathrm{d}e\wedge \mathrm{d}y\otimes \mathrm{d}y + \mathrm{d}y\wedge \mathrm{d}e\otimes \mathrm{d}e + \mathrm{d}y\wedge \mathrm{d}x\otimes \mathrm{d}e + \mathrm{d}y\wedge \mathrm{d}e\otimes \mathrm{d}x.$$

Case E

- for metric $g_{E.I}$:

(E.I.1) $\nabla \mathrm{d}e = 0, \quad \nabla \mathrm{d}x = \mathrm{d}e\otimes \mathrm{d}x + \mathrm{d}x\otimes \mathrm{d}e + \mathrm{d}e\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}e,$
$\nabla \mathrm{d}y = \mathrm{d}e\otimes \mathrm{d}x + \mathrm{d}x\otimes \mathrm{d}e + \mathrm{d}e\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}e + \mathrm{d}e\otimes \mathrm{d}e, \quad R_\nabla = 0;$

(E.I.2) $\nabla \mathrm{d}e = \mathrm{d}e\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}e, \quad \nabla \mathrm{d}x = \mathrm{d}y\otimes \mathrm{d}y = \nabla \mathrm{d}y, \quad R_\nabla = 0;$

(E.I.3) $\nabla \mathrm{d}e = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x, \quad \nabla \mathrm{d}x = 0, \quad \nabla \mathrm{d}y = \mathrm{d}y\otimes\mathrm{d}y, \quad R_\nabla = 0;$

(E.I.4) $\nabla \mathrm{d}e = \mathrm{d}e\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}e + \mathrm{d}x\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{d}e\otimes \mathrm{d}x + \mathrm{d}x\otimes \mathrm{d}e + \mathrm{d}e\otimes$ $\mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}e, \quad \nabla \mathrm{d}y = \mathrm{d}e\otimes \mathrm{d}x + \mathrm{d}x\otimes \mathrm{d}e + \mathrm{d}e\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}e + \mathrm{d}e\otimes \mathrm{d}e + \mathrm{d}y\otimes \mathrm{d}y,$

$$R_\nabla \mathrm{d}e = \mathrm{d}y\wedge \mathrm{d}e\otimes \mathrm{d}e + \mathrm{d}e\wedge \mathrm{d}x\otimes \mathrm{d}x + \mathrm{d}e\wedge \mathrm{d}x\otimes \mathrm{d}y + \mathrm{d}e\wedge \mathrm{d}y\otimes \mathrm{d}x + \mathrm{d}e\wedge \mathrm{d}y\otimes \mathrm{d}y,$$
$$R_\nabla \mathrm{d}x = \mathrm{d}x\wedge \mathrm{d}e\otimes \mathrm{d}y + \mathrm{d}x\wedge \mathrm{d}y\otimes \mathrm{d}e + \mathrm{d}x\wedge \mathrm{d}y\otimes \mathrm{d}y + \mathrm{d}y\wedge \mathrm{d}x\otimes \mathrm{d}x,$$
$$R_\nabla \mathrm{d}y = \mathrm{d}e\wedge \mathrm{d}y\otimes \mathrm{d}y + \mathrm{d}y\wedge \mathrm{d}x\otimes \mathrm{d}x + \mathrm{d}x\wedge \mathrm{d}y\otimes \mathrm{d}y;$$

(E.I.5) $\nabla \mathrm{d}e = \mathrm{d}x\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}x\ , \quad \nabla \mathrm{d}x = \mathrm{d}y\otimes \mathrm{d}y, \quad \nabla \mathrm{d}y = 0, \quad R_\nabla = 0;$

(E.I.6) $\nabla \mathrm{d}e = \mathrm{d}y\otimes \mathrm{d}y, \quad \nabla \mathrm{d}x = 0\ , \quad \nabla \mathrm{d}y = 0, \quad R_\nabla = 0;$

(E.I.7) $\nabla \mathrm{d}e = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}y\otimes\mathrm{d}y, \quad \nabla \mathrm{d}x = \mathrm{d}y\otimes\mathrm{d}y\ , \quad \nabla \mathrm{d}y = \mathrm{d}y\otimes\mathrm{d}y, \quad R_\nabla = 0;$

(E.I.8) $\nabla \mathrm{d}e = \mathrm{d}e\otimes \mathrm{d}y + \mathrm{d}x\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}e + \mathrm{d}y\otimes \mathrm{d}x + \mathrm{d}y\otimes \mathrm{d}y\ , \quad \nabla \mathrm{d}x = 0\ , \quad \nabla \mathrm{d}y =$ $\mathrm{d}y\otimes \mathrm{d}y, \quad R_\nabla = 0;$

(E.I.9) $\nabla \mathrm{d}e = \mathrm{d}x\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}x + \mathrm{d}y\otimes \mathrm{d}y, \quad \nabla \mathrm{d}x = \mathrm{d}y\otimes \mathrm{d}y, \quad \nabla \mathrm{d}y = 0, \quad R_\nabla = 0;$

(E.I.10) $\nabla \mathrm{d}e = \mathrm{d}e\otimes \mathrm{d}y + \mathrm{d}x\otimes \mathrm{d}x + \mathrm{d}y\otimes \mathrm{d}e\ , \quad \nabla \mathrm{d}x = \mathrm{d}x\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}x\ ,$
$\nabla \mathrm{d}y = \mathrm{d}y\otimes \mathrm{d}y, \quad R_\nabla = 0;$

(E.I.11) $\nabla \mathrm{d}e = \mathrm{d}x\otimes \mathrm{d}x + \mathrm{d}x\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}x\ , \quad \nabla \mathrm{d}x = \mathrm{d}x\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}x\ , \quad \nabla \mathrm{d}y =$ $0, \quad R_\nabla = 0;$

(E.I.12) $\nabla \mathrm{d}e = \mathrm{d}e\otimes \mathrm{d}y + \mathrm{d}x\otimes \mathrm{d}x + \mathrm{d}y\otimes \mathrm{d}e + \mathrm{d}y\otimes \mathrm{d}y\ , \quad \nabla \mathrm{d}x = \mathrm{d}x\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}x,$
$\nabla \mathrm{d}y = \mathrm{d}y\otimes \mathrm{d}y, \quad R_\nabla = 0;$

(E.I.13) $\nabla \mathrm{d}e = \mathrm{d}x\otimes \mathrm{d}x + \mathrm{d}x\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}x + \mathrm{d}y\otimes \mathrm{d}y\ , \quad \nabla \mathrm{d}x = \mathrm{d}x\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}x,$
$\nabla \mathrm{d}y = 0,$

$$R_\nabla \mathrm{d}e = \mathrm{d}x\wedge \mathrm{d}y\otimes \mathrm{d}y + \mathrm{d}y\wedge \mathrm{d}x\otimes \mathrm{d}x, \quad R_\nabla \mathrm{d}x = \mathrm{d}x\wedge \mathrm{d}y\otimes \mathrm{d}y, \quad R_\nabla \mathrm{d}y = 0.$$

- for metric $g_{E.II}$:

(E.II.1) $\nabla \mathrm{d}e = 0\ , \quad \nabla \mathrm{d}x = \mathrm{d}e\otimes \mathrm{d}e + \mathrm{d}e\otimes \mathrm{d}x + \mathrm{d}x\otimes \mathrm{d}e + \mathrm{d}x\otimes \mathrm{d}x,$
$\nabla \mathrm{d}y = \mathrm{d}e\otimes \mathrm{d}x + \mathrm{d}e\otimes \mathrm{d}y + \mathrm{d}x\otimes \mathrm{d}e + \mathrm{d}x\otimes \mathrm{d}y + \mathrm{d}y\otimes \mathrm{d}e + \mathrm{d}y\otimes \mathrm{d}x, \quad R_\nabla = 0;$

(E.II.2) $\nabla \mathrm{de} = \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y, \quad \nabla \mathrm{d}y = 0,$

$$R_\nabla \mathrm{de} = \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y, \quad R_\nabla \mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y, \quad R_\nabla \mathrm{d}y = 0;$$

(E.II.3) $\nabla \mathrm{de} = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{de} \otimes \mathrm{de} + \mathrm{de} \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{de} + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y,$
$\nabla \mathrm{d}y = \mathrm{de} \otimes \mathrm{d}x + \mathrm{de} \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{de} + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{de} + \mathrm{d}y \otimes \mathrm{d}x,$

$$R_\nabla \mathrm{de} = \mathrm{d}y \wedge \mathrm{de} \otimes \mathrm{de} + \mathrm{de} \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{de} \wedge \mathrm{d}y \otimes \mathrm{d}x + \mathrm{de} \wedge \mathrm{d}x \otimes \mathrm{d}y,$$

$$R_\nabla \mathrm{d}x = \mathrm{d}x \wedge \mathrm{de} \otimes \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{de},$$

$$R_\nabla \mathrm{d}y = \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y + \mathrm{de} \wedge \mathrm{d}y \otimes \mathrm{d}y;$$

(E.II.4) $\nabla \mathrm{de} = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y, \quad \nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y, \quad \nabla \mathrm{d}y = 0,$

$$R_\nabla \mathrm{de} = \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y, \quad R_\nabla \mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y, \quad R_\nabla \mathrm{d}y = 0.$$

And the remaining ones are: (E.II.5.)=(E.I.11.), (E.II.6.)=(E.I.5.), (E.II.7.)=(E.I.2.), (E.II.8.)=(E.I.3.), (E.II.9.)=(E.I.6.), (E.II.10.)=(E.I.13.), (E.II.11.)=(E.I.9.), (E.II.12.)=(E.I.7.), (E.II.13.)=(E.I.8.).

- for metric $g_{E.III}$

(E.III.1) $\nabla \mathrm{de} = \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x\ , \quad \nabla \mathrm{d}y = 0,$

$$R_\nabla \mathrm{de} = \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x, \quad R_\nabla \mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y, \quad R_\nabla \mathrm{d}y = 0;$$

(E.III.2) $\nabla \mathrm{de} = 0, \quad \nabla \mathrm{d}x = \mathrm{de} \otimes \mathrm{de} + \mathrm{de} \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{de} + \mathrm{d}y \otimes \mathrm{d}y,$
$\nabla \mathrm{d}y = \mathrm{de} \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{de} + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x, \quad R_\nabla = 0;$

(E.III.3) $\nabla \mathrm{de} = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{de} \otimes \mathrm{de} + \mathrm{de} \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{de},$
$\nabla \mathrm{d}y = \mathrm{de} \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{de} + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x,$

$R_\nabla \mathrm{de} = \mathrm{d}y \wedge \mathrm{de} \otimes \mathrm{de} + \mathrm{de} \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{de} \wedge \mathrm{d}y \otimes \mathrm{d}y, \quad R_\nabla \mathrm{d}x = \mathrm{d}x \wedge \mathrm{de} \otimes \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{de} + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y,$

$$R_\nabla \mathrm{d}y = \mathrm{de} \wedge \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}x;$$

(E.III.4) $\nabla \mathrm{de} = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y,$
$\nabla \mathrm{d}y = 0,$

$$R_\nabla \mathrm{de} = \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x, \quad R_\nabla \mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y, \quad R_\nabla \mathrm{d}y = 0;$$

(E.III.5) $\nabla \mathrm{de} = \mathrm{de} \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{de}, \quad \nabla \mathrm{d}x = 0, \quad \nabla \mathrm{d}y = \mathrm{d}y \otimes \mathrm{d}y, \quad R_\nabla = 0;$

(E.III.6) $\nabla \mathrm{de} = \mathrm{de} \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{de} + \mathrm{d}y \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{d}y \otimes \mathrm{d}y = \nabla \mathrm{d}y, \quad R_\nabla = 0;$

(E.III.7) $\nabla \mathrm{de} = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y, \quad \nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x, \quad \nabla \mathrm{d}y = 0,$

$$R_\nabla \mathrm{de} = \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x, \quad R_\nabla \mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y, \quad R_\nabla \mathrm{d}y = 0;$$

(E.III.8) $\nabla \mathrm{de} = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y,$
$\nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y, \quad \nabla \mathrm{d}y = 0,$

$$R_\nabla \mathrm{de} = \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x, \quad R_\nabla \mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y, \quad R_\nabla \mathrm{d}y = 0;$$

(E.III.9) $\nabla \mathrm{de} = \mathrm{de} \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{de} + \mathrm{d}y \otimes \mathrm{d}y, \quad \nabla \mathrm{d}x = 0\ , \quad \nabla \mathrm{d}y = \mathrm{d}y \otimes \mathrm{d}y, \quad R_\nabla = 0;$

(E.III.10) $\nabla \mathrm{de} = \mathrm{de} \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{de} + \mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y, \quad \nabla \mathrm{d}x = \mathrm{d}y \otimes \mathrm{d}y,$
$\nabla \mathrm{d}y = \mathrm{d}y \otimes \mathrm{d}y, \quad R_\nabla = 0.$

The remaining ones are: (E.III.11) = (E.I.5), (E.III.12)=(E.I.6), (E.III.13)=(E.I.9).

- for metric $g_{E.IV}$:

(E.IV.1) $\nabla \mathrm{d}e = 0, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}e,$
$\nabla \mathrm{d}y = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}e, \quad R_\nabla = 0;$

(E.IV.2) $\nabla \mathrm{d}e = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}y, \quad \nabla \mathrm{d}y = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}e,$

$$R_\nabla \mathrm{d}e = \mathrm{d}y \wedge \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}e \wedge \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}e \wedge \mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}e \wedge \mathrm{d}y \otimes \mathrm{d}y,$$

$$R_\nabla \mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}e,$$

$$R_\nabla \mathrm{d}y = \mathrm{d}e \wedge \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y;$$

And (E.IV.3)=(E.I.11), (E.IV.4)=(E.I.5),(E.IV.5)=(E.II.2), (E.IV.6)=(E.I.2), (E.IV.7)=(E.I.3), (E.IV.8)=(E.I.6), (E.IV.9)=(E.II.4), (E.IV.10)=(E.I.13), (E.IV.11)=(E.I.9), (E.IV.12)=(E.I.7), (E.IV.13)=(E.I.8).

Case F
- for metric $g_{F.I}$

(F.I.1) $\nabla \mathrm{d}e = 0, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}e, \quad \nabla \mathrm{d}y = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e, \quad R_\nabla = 0;$

(F.I.2) $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}y, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e,$
$\nabla \mathrm{d}y = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y,$

$$R_\nabla \mathrm{d}e = \mathrm{d}y \wedge \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \wedge \mathrm{d}x \otimes \mathrm{d}x, \quad R_\nabla \mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}e,$$

$$R_\nabla \mathrm{d}y = \mathrm{d}e \wedge \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x;$$

(F.I.3) $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}y,$
$\nabla \mathrm{d}y = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}e,$

$$R_\nabla \mathrm{d}e = \mathrm{d}y \wedge \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \wedge \mathrm{d}x \otimes \mathrm{d}x, \quad R_\nabla \mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}e,$$

$$R_\nabla \mathrm{d}y = \mathrm{d}e \wedge \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x.$$

- for metric $g_{F.II}$

(F.II.1) $\nabla \mathrm{d}e = 0, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}x,$
$\nabla \mathrm{d}y = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}y, \quad R_\nabla = 0;$

(F.II.2) $\nabla \mathrm{d}e = 0, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}y,$
$\nabla \mathrm{d}y = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}x, \quad R_\nabla = 0;$

(F.II.3) $\nabla \mathrm{d}e = 0, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x,$
$\nabla \mathrm{d}y = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}x, \quad R_\nabla = 0.$

- for metric $g_{F.III}$

(F.III.1) $\nabla \mathrm{d}e = 0, \quad \nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}x,$
$\nabla \mathrm{d}y = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x, \quad R_\nabla = 0;$

(F.III.2) $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y, \quad \nabla \mathrm{d}y = \mathrm{d}e \otimes \mathrm{d}y + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}e + \mathrm{d}y \otimes \mathrm{d}y,$

$$R_\nabla \mathrm{d}e = \mathrm{d}x \wedge \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}e \wedge \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}e \wedge \mathrm{d}y \otimes \mathrm{d}x,$$

$$R_\nabla \mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x,$$

$$R_\nabla \mathrm{d}y = \mathrm{d}y \wedge \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}y \wedge \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}y \wedge \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}x \wedge \mathrm{d}y \otimes \mathrm{d}y;$$

(F.III.3) $\nabla \mathrm{d}e = \mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}y\otimes\mathrm{d}y$, $\nabla\mathrm{d}x = \mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}y\otimes\mathrm{d}x$, $\nabla\mathrm{d}y = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}y\otimes\mathrm{d}x$,

$$R_\nabla\mathrm{d}e = \mathrm{d}x\wedge\mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\wedge\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}e\wedge\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}x,$$

$$R_\nabla\mathrm{d}x = \mathrm{d}x\wedge\mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\wedge\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\wedge\mathrm{d}x\otimes\mathrm{d}x,$$

$$R_\nabla\mathrm{d}y = \mathrm{d}y\wedge\mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}y\wedge\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}y\wedge\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}y\wedge\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}y.$$

- for metric $g_{F.IV}$

(F.IV.1) $\nabla\mathrm{d}e = 0$, $\nabla\mathrm{d}x = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e$, $\nabla\mathrm{d}y = \mathrm{d}e\otimes\mathrm{d}e$, $R_\nabla = 0$;

(F.IV.2) $\nabla\mathrm{d}e = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}y\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y$, $\nabla\mathrm{d}x = \mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x$, $\nabla\mathrm{d}y = \mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}y\otimes\mathrm{d}e$,

$$R_\nabla\mathrm{d}e = \mathrm{d}x\wedge\mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}y\wedge\mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}y,$$

$$R_\nabla\mathrm{d}x = \mathrm{d}e\wedge\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}x\wedge\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}y,$$

$$R_\nabla\mathrm{d}y = \mathrm{d}y\wedge\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}y\wedge\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}y;$$

(F.IV.3) $\nabla\mathrm{d}e = \mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y$,
$\nabla\mathrm{d}x = \mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}y\otimes\mathrm{d}x$,
$\nabla\mathrm{d}y = \mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}y\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y$,

$$R_\nabla\mathrm{d}e = \mathrm{d}x\wedge\mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}y\wedge\mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}y,$$

$$R_\nabla\mathrm{d}x = \mathrm{d}e\wedge\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}x\wedge\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}y,$$

$$R_\nabla\mathrm{d}y = \mathrm{d}y\wedge\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}y\wedge\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}y.$$

- for metric $g_{F.V}$

(F.V.1) $\nabla\mathrm{d}e = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}e$, $\nabla\mathrm{d}x = \mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x$,
$\nabla\mathrm{d}y = \mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x$,

$$R_\nabla\mathrm{d}e = \mathrm{d}x\wedge\mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\wedge\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}y,$$

$$R_\nabla\mathrm{d}x = \mathrm{d}e\wedge\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}y,$$

$$R_\nabla\mathrm{d}y = \mathrm{d}y\wedge\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}y\wedge\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}y\wedge\mathrm{d}x\otimes\mathrm{d}x;$$

(F.V.2) $\nabla\mathrm{d}e = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y$, $\nabla\mathrm{d}x = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}y\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y$, $\nabla\mathrm{d}y = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}y\otimes\mathrm{d}y$,

$$R_\nabla\mathrm{d}e = \mathrm{d}x\wedge\mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\wedge\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}y,$$

$$R_\nabla\mathrm{d}x = \mathrm{d}e\wedge\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}y,$$

$$R_\nabla\mathrm{d}y = \mathrm{d}y\wedge\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}y\wedge\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}y\wedge\mathrm{d}x\otimes\mathrm{d}x;$$

(F.V.3) $\nabla\mathrm{d}e = 0$, $\nabla\mathrm{d}x = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y$,
$\nabla\mathrm{d}y = \mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}e+\mathrm{d}y\otimes\mathrm{d}x$, $R_\nabla = 0$.

- for metric $g_{F.VI}$

(F.VI.1) $\nabla\mathrm{d}e = 0$, $\nabla\mathrm{d}x = \mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x$, $\nabla\mathrm{d}y = \mathrm{d}y\otimes\mathrm{d}y$, $R_\nabla = 0$;

(F.VI.2) $\nabla\mathrm{d}e = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x$, $\nabla\mathrm{d}x = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}e$, $\nabla\mathrm{d}y = \mathrm{d}e\otimes\mathrm{d}e+\mathrm{d}e\otimes\mathrm{d}x+\mathrm{d}e\otimes\mathrm{d}y+\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}e$,

$$R_\nabla\mathrm{d}e = \mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}x+\mathrm{d}e\wedge\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}y,$$

$$R_\nabla\mathrm{d}x = \mathrm{d}e\wedge\mathrm{d}x\otimes\mathrm{d}e+\mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}x+\mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}y,$$

$$R_\nabla \mathrm{d}y = \mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}e + \mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}y;$$

(F.VI.3) $\nabla\mathrm{d}e = \mathrm{d}e\otimes\mathrm{d}e + \mathrm{d}e\otimes\mathrm{d}y + \mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}x\otimes\mathrm{d}y + \mathrm{d}y\otimes\mathrm{d}e + \mathrm{d}y\otimes\mathrm{d}x + \mathrm{d}y\otimes\mathrm{d}y$,
$\nabla\mathrm{d}x = \mathrm{d}e\otimes\mathrm{d}e + \mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}x\otimes\mathrm{d}y + \mathrm{d}y\otimes\mathrm{d}x + \mathrm{d}y\otimes\mathrm{d}y$,
$\nabla\mathrm{d}y = \mathrm{d}e\otimes\mathrm{d}x + \mathrm{d}e\otimes\mathrm{d}y + \mathrm{d}x\otimes\mathrm{d}e + \mathrm{d}x\otimes\mathrm{d}y + \mathrm{d}y\otimes\mathrm{d}e + \mathrm{d}y\otimes\mathrm{d}x$,

$$R_\nabla \mathrm{d}e = \mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}x + \mathrm{d}e\wedge\mathrm{d}x\otimes\mathrm{d}y + \mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}y,$$

$$R_\nabla \mathrm{d}x = \mathrm{d}x\wedge\mathrm{d}e\otimes\mathrm{d}e + \mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}y + \mathrm{d}y\wedge\mathrm{d}x\otimes\mathrm{d}x,$$

$$R_\nabla \mathrm{d}y = \mathrm{d}y\wedge\mathrm{d}e\otimes\mathrm{d}e + \mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}y.$$

- for metric $g_{F.VII}$

(F.VII.1) $\nabla\mathrm{d}e = 0$, $\nabla\mathrm{d}x = \mathrm{d}e\otimes\mathrm{d}e + \mathrm{d}e\otimes\mathrm{d}x + \mathrm{d}e\otimes\mathrm{d}y + \mathrm{d}x\otimes\mathrm{d}e + \mathrm{d}y\otimes\mathrm{d}e$,
$\nabla\mathrm{d}y = \mathrm{d}e\otimes\mathrm{d}x + \mathrm{d}e\otimes\mathrm{d}y + \mathrm{d}x\otimes\mathrm{d}e + \mathrm{d}y\otimes\mathrm{d}e$, $R_\nabla = 0$;

(F.VII.2) $\nabla\mathrm{d}e = \mathrm{d}e\otimes\mathrm{d}y + \mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}y\otimes\mathrm{d}e$, $\nabla\mathrm{d}x = \mathrm{d}e\otimes\mathrm{d}x + \mathrm{d}x\otimes\mathrm{d}e + \mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}x\otimes\mathrm{d}y + \mathrm{d}y\otimes\mathrm{d}x$, $\nabla\mathrm{d}y = \mathrm{d}e\otimes\mathrm{d}y + \mathrm{d}x\otimes\mathrm{d}y + \mathrm{d}y\otimes\mathrm{d}e + \mathrm{d}y\otimes\mathrm{d}x$,

$$R_\nabla \mathrm{d}e = \mathrm{d}y\wedge\mathrm{d}e\otimes\mathrm{d}e + \mathrm{d}e\wedge\mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}e\wedge\mathrm{d}x\otimes\mathrm{d}y + \mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}x + \mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}y,$$

$$R_\nabla \mathrm{d}x = \mathrm{d}x\wedge\mathrm{d}e\otimes\mathrm{d}e + \mathrm{d}x\wedge\mathrm{d}e\otimes\mathrm{d}y + \mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}e + \mathrm{d}y\wedge\mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}y,$$

$$R_\nabla \mathrm{d}y = \mathrm{d}y\wedge\mathrm{d}e\otimes\mathrm{d}e + \mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}y + \mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}x + \mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}y;$$

(F.VII.3) $\nabla\mathrm{d}e = \mathrm{d}e\otimes\mathrm{d}e + \mathrm{d}e\otimes\mathrm{d}x + \mathrm{d}x\otimes\mathrm{d}e + \mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}x\otimes\mathrm{d}y + \mathrm{d}y\otimes\mathrm{d}x$, $\nabla\mathrm{d}x = \mathrm{d}e\otimes\mathrm{d}y + \mathrm{d}y\otimes\mathrm{d}e$, $\nabla\mathrm{d}y = \mathrm{d}e\otimes\mathrm{d}x + \mathrm{d}e\otimes\mathrm{d}y + \mathrm{d}x\otimes\mathrm{d}e + \mathrm{d}x\otimes\mathrm{d}y + \mathrm{d}y\otimes\mathrm{d}e + \mathrm{d}y\otimes\mathrm{d}x$,

$$R_\nabla \mathrm{d}e = \mathrm{d}y\wedge\mathrm{d}e\otimes\mathrm{d}e + \mathrm{d}e\wedge\mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}e\wedge\mathrm{d}x\otimes\mathrm{d}y + \mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}x + \mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}y,$$

$$R_\nabla \mathrm{d}x = \mathrm{d}x\wedge\mathrm{d}e\otimes\mathrm{d}e + \mathrm{d}x\wedge\mathrm{d}e\otimes\mathrm{d}y + \mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}e + \mathrm{d}y\wedge\mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}y,$$

$$R_\nabla \mathrm{d}y = \mathrm{d}y\wedge\mathrm{d}e\otimes\mathrm{d}e + \mathrm{d}y\wedge\mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}e\wedge\mathrm{d}y\otimes\mathrm{d}y + \mathrm{d}x\wedge\mathrm{d}y\otimes\mathrm{d}y.$$

5. Partial results for $n = 4$

For $n = 4$ the analysis is rather more complicated but the outlined methods work for the classification of 4-dimensional unital algebras and we find 16 up to isomorphism, and hence this many inner differential structures for polynomials in 4 variables over $\mathbb{F}_2$. These are summarised in Table 5.

The methods are the same we have seen for $n = 2, 3$ so we will be brief. For the inner case with $\theta = \mathrm{d}x^1$ by computer we get 5216 solutions to eqs. (2.1)-(2.2) for $\mathbb{F}_2[x^1, x^2, x^3, x^4]$. The isomorphisms group G has the order $|G| = 20160$. Checking the isomorphisms between all of the solutions of the inner case with $\theta = dx^1$ there are only 16 inequivalent differential calculi. The remaining possibilities for θ are isomorphic to the one with $\theta = \mathrm{d}x^1$. We renamed $\theta = \mathrm{d}e$ and the remaining variables as $x^2 = x$, $x^3 = y$, $x^4 = z$ and listed the calculi in the Table 5 along with the corresponding number of quantum metrics.

Most of these calculi have metrics leading to too many geometries to study so explicitly as we did for $n = 2, 3$, so we focus on some that fit with the general examples (i)-(iii) in Section 2.

	$[\mathrm{d}e,e]=\mathrm{d}e$; $[\mathrm{d}e,x]=\mathrm{d}x=[\mathrm{d}x,e]$; $[\mathrm{d}e,y]=\mathrm{d}y=[\mathrm{d}y,e]$; $[\mathrm{d}e,z]=\mathrm{d}z=[\mathrm{d}z,e]$	Quantum metrics
A	$[\mathrm{d}x,x]=0$; $[\mathrm{d}x,y]=0=[\mathrm{d}y,x]$; $[\mathrm{d}x,z]=0=[\mathrm{d}z,x]$ $[\mathrm{d}y,z]=0=[\mathrm{d}z,y]$; $[\mathrm{d}y,y]=0=[\mathrm{d}z,z]$	0
B	$[\mathrm{d}x,x]=\mathrm{d}z$; $[\mathrm{d}x,y]=0=[\mathrm{d}y,x]$; $[\mathrm{d}x,z]=0=[\mathrm{d}z,x]$ $[\mathrm{d}y,z]=0=[\mathrm{d}z,y]$; $[\mathrm{d}y,y]=0=[\mathrm{d}z,z]$	0
C	$[\mathrm{d}x,x]=\mathrm{d}x$; $[\mathrm{d}x,y]=0=[\mathrm{d}y,x]$; $[\mathrm{d}x,z]=0=[\mathrm{d}z,x]$ $[\mathrm{d}y,z]=0=[\mathrm{d}z,y]$; $[\mathrm{d}y,y]=0=[\mathrm{d}z,z]$	0
D	$[\mathrm{d}x,x]=\mathrm{d}x$; $[\mathrm{d}x,y]=\mathrm{d}y=[\mathrm{d}y,x]$; $[\mathrm{d}x,z]=0=[\mathrm{d}z,x]$ $[\mathrm{d}y,z]=0=[\mathrm{d}z,y]$; $[\mathrm{d}y,y]=0=[\mathrm{d}z,z]$	4
E	$[\mathrm{d}x,x]=0$; $[\mathrm{d}x,y]=\mathrm{d}z=[\mathrm{d}y,x]$; $[\mathrm{d}x,z]=0=[\mathrm{d}z,x]$ $[\mathrm{d}y,z]=0=[\mathrm{d}z,y]$; $[\mathrm{d}y,y]=0=[\mathrm{d}z,z]$	8
F	$[\mathrm{d}x,x]=\mathrm{d}z$; $[\mathrm{d}x,y]=\mathrm{d}z=[\mathrm{d}y,x]$; $[\mathrm{d}x,z]=0=[\mathrm{d}z,x]$ $[\mathrm{d}y,z]=0=[\mathrm{d}z,y]$; $[\mathrm{d}y,y]=0=[\mathrm{d}z,z]$	8
G	$[\mathrm{d}x,x]=\mathrm{d}y$; $[\mathrm{d}x,y]=\mathrm{d}z=[\mathrm{d}y,x]$; $[\mathrm{d}x,z]=0=[\mathrm{d}z,x]$ $[\mathrm{d}y,z]=0=[\mathrm{d}z,y]$; $[\mathrm{d}y,y]=0=[\mathrm{d}z,z]$	8
H	$[\mathrm{d}x,x]=\mathrm{d}e+\mathrm{d}x$; $[\mathrm{d}x,y]=\mathrm{d}y+\mathrm{d}z=[\mathrm{d}y,x]$; $[\mathrm{d}x,z]=\mathrm{d}y=[\mathrm{d}z,x]$ $[\mathrm{d}y,z]=0=[\mathrm{d}z,y]$; $[\mathrm{d}y,y]=0=[\mathrm{d}z,z]$	12
I	$[\mathrm{d}x,x]=\mathrm{d}y$; $[\mathrm{d}x,y]=\mathrm{d}x+\mathrm{d}y=[\mathrm{d}y,x]$; $[\mathrm{d}x,z]=0=[\mathrm{d}z,x]$; $[\mathrm{d}y,z]=0=[\mathrm{d}z,y]$; $[\mathrm{d}y,y]=\mathrm{d}x$; $[\mathrm{d}z,z]=0$	6
J	$[\mathrm{d}x,x]=\mathrm{d}x+\mathrm{d}z$; $[\mathrm{d}x,y]=\mathrm{d}x+\mathrm{d}z=[\mathrm{d}y,x]$; $[\mathrm{d}x,z]=0=[\mathrm{d}z,x]$; $[\mathrm{d}y,z]=0=[\mathrm{d}z,y]$; $[\mathrm{d}y,y]=\mathrm{d}x$; $[\mathrm{d}z,z]=0$	4
K	$[\mathrm{d}x,x]=\mathrm{d}x$; $[\mathrm{d}x,y]=0=[\mathrm{d}y,x]$; $[\mathrm{d}x,z]=0=[\mathrm{d}z,x]$ $[\mathrm{d}y,z]=0=[\mathrm{d}z,y]$; $[\mathrm{d}y,y]=\mathrm{d}y$; $[\mathrm{d}z,z]=0$	2
L	$[\mathrm{d}x,x]=\mathrm{d}z$; $[\mathrm{d}x,y]=0=[\mathrm{d}y,x]$; $[\mathrm{d}x,z]=\mathrm{d}e+\mathrm{d}y=[\mathrm{d}z,x]$ $[\mathrm{d}y,y]=\mathrm{d}y$; $[\mathrm{d}y,z]=0=[\mathrm{d}z,y]$; $[\mathrm{d}z,z]=\mathrm{d}x$	3
M	$[\mathrm{d}x,x]=\mathrm{d}e+\mathrm{d}x+\mathrm{d}y+\mathrm{d}z$; $[\mathrm{d}x,y]=0=[\mathrm{d}y,x]$; $[\mathrm{d}x,z]=\mathrm{d}e+\mathrm{d}x+\mathrm{d}y=[\mathrm{d}z,x]$ $[\mathrm{d}y,z]=0=[\mathrm{d}z,y]$; $[\mathrm{d}y,y]=\mathrm{d}y$; $[\mathrm{d}z,z]=\mathrm{d}x$	7
N	$[\mathrm{d}x,x]=\mathrm{d}z$; $[\mathrm{d}x,y]=0=[\mathrm{d}y,x]$; $[\mathrm{d}y,z]=0=[\mathrm{d}z,y]$ $[\mathrm{d}x,z]=\mathrm{d}x+\mathrm{d}z=[\mathrm{d}z,x]$; $[\mathrm{d}y,y]=\mathrm{d}e+\mathrm{d}x+\mathrm{d}y+\mathrm{d}z$; $[\mathrm{d}z,z]=\mathrm{d}x$	9
O	$[\mathrm{d}x,x]=\mathrm{d}e+\mathrm{d}z$; $[\mathrm{d}x,y]=\mathrm{d}z=[\mathrm{d}y,x]$; $[\mathrm{d}x,z]=\mathrm{d}e+\mathrm{d}y=[\mathrm{d}z,x]$ $[\mathrm{d}y,z]=\mathrm{d}e=[\mathrm{d}z,y]$; $[\mathrm{d}y,y]=\mathrm{d}x+\mathrm{d}y$; $[\mathrm{d}z,z]=\mathrm{d}e$	15
P	$[\mathrm{d}x,x]=\mathrm{d}x$; $[\mathrm{d}x,y]=0=[\mathrm{d}y,x]$; $[\mathrm{d}x,z]=0=[\mathrm{d}z,x]$ $[\mathrm{d}y,z]=0=[\mathrm{d}z,y]$; $[\mathrm{d}y,y]=\mathrm{d}y$; $[\mathrm{d}z,z]=\mathrm{d}z$	1

Table 5. All possible inner differential structures for on $\mathbb{F}_2[e,x,y,z]$.

(i) $V=\mathbb{F}_2(X)$. We carefully make the same change of variable noted for $n=2$ and $n=3$ above (case B in those tables but case P in the $n=4$ table) to determine the quantum Levi-Civita connection in the general x^μ coordinate system where our basis of delta-functions on X is labelled by $\mu=1,2,\cdots,n=|X|$ (or abstractly by $\mu\in X$ as indexing set). The change of coordinate only concerns $e=\sum_\mu x^\mu$ as the algebra identity element. For $n=2$ we let $t=e+x$ an $x^\mu=t,x$ as natural 'space' or 'spacetime' coordinates. Note that over $\mathbb{F}_2$ there is no difference between Euclidean or Minkowski signature and there is no particular significance to our choice of symbol t here. The metric and quantum Levi-Civita connection found in Section 3 are

$$g_B=\mathrm{d}t\otimes\mathrm{d}t+\mathrm{d}x\otimes\mathrm{d}x,\quad \nabla\mathrm{d}t=\nabla\mathrm{d}x=(\mathrm{d}t+\mathrm{d}x)\otimes(\mathrm{d}t+\mathrm{d}x).$$

Similarly for $n = 3$ we let $t = e + x + y$ and $x^\mu = t, x, y$ as more natural 'coordinates'. The metric and three quantum Levi-Civita connection found in Section 4 are $g_B = \mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y$ as expected and:

(B.1) $\nabla \mathrm{d}x = 0,\ \nabla \mathrm{d}t = \nabla \mathrm{d}x = (\mathrm{d}t + \mathrm{d}x) \otimes (\mathrm{d}t + \mathrm{d}x)$;

(B.2) $\nabla \mathrm{d}y = 0,\ \nabla \mathrm{d}t = \nabla \mathrm{d}y = (\mathrm{d}t + \mathrm{d}y) \otimes (\mathrm{d}t + \mathrm{d}y)$;

(B.3) $\nabla \mathrm{d}t = 0,\ \nabla \mathrm{d}x = \nabla \mathrm{d}y = (\mathrm{d}x + \mathrm{d}y) \otimes (\mathrm{d}x + \mathrm{d}y)$.

From these formulae we can now extrapolate from $n = 2, 3$ to a general construction for quantum Levi-Civita connections for this case.

Proposition 5.1. *For calculus defined by $V = \mathbb{F}_2(X)$, we can partition X into a subset $T \subseteq X$ with $|X| - |T|$ even and the remainder into unordered pairs,*

$$X = T \sqcup (\sqcup_\alpha X_\alpha)$$

where the $(|X|-|T|)/2$ subsets X_α each have two elements and are omitted if $T = X$. Each such partition gives a quantum Levi-Civita connection for the Euclidean metric in the form

$$g = \sum_{t\in T} \mathrm{d}x^t \otimes \mathrm{d}x^t + \sum_\alpha \sum_{s\in X_\alpha} \mathrm{d}x^s \otimes x^s,$$

namely

$$\nabla \mathrm{d}x^t = 0, \quad \forall t \in T, \quad \nabla \mathrm{d}x^s = (\mathrm{d}x^s + \mathrm{d}x^{\bar s}) \otimes (\mathrm{d}x^s + \mathrm{d}x^{\bar s}),$$
$$\sigma(\mathrm{d}x^s \otimes \mathrm{d}x^s) = \mathrm{d}x^{\bar s} \otimes \mathrm{d}x^{\bar s}, \quad \sigma(\mathrm{d}x^s \otimes \mathrm{d}x^{\bar s}) = \mathrm{d}x^s \otimes \mathrm{d}x^{\bar s}$$

where $s \in X_\alpha$ for some α and $\bar s$ denotes the other element of X_α, otherwise the flip map on basis elements. These connections have zero curvature.

Proof. Once we obtained the formula based on our computer results for $n = 2, 3$ it is not hard to verify directly that this is quantum metric compatible and quantum torsion free from the general form of the commutation relations $[\mathrm{d}x^\mu, \mathrm{d}x^\nu] = \delta_{\mu\nu}\mathrm{d}x^\mu$. The σ is then uniquely determined from ∇ and comes out as stated. In general we set $m = |T|, n = |X|$ with $\binom{n}{m}$ choices for T and then $(n - m - 1)!! = (n - m)!/(2^{\frac{n-m}{2}}(\frac{n-m}{2})!)$ choices for the number of partitions of the remaining elements into pairs. The latter is a well-known observation easily proven as follows. Pick an element of $X \setminus T$. There are $n - m - 1$ choices for which other element to pair it with. We then remove both elements from consideration and repeat the process. The quantum metric is always the same Euclidean one regardless of the partition of indices. Note that if we do not care at all about the labelling of indices (geometrically these are all equivalent) then we have just the integer part of $n/2$ choices for m for the number of basis elements with zero connection, but in our tables we have been distinguishing these. The zero curvature is immediate from the formulae for ∇. □

For example, if $n = 2$ then we can take $m = 2$ (the zero connection) or $m = 0$ with one choice for the connection in this case. If $n = 3$ then we can take $m = 3$ (the zero connection) or $m = 1$ with three choices for which element to take for T and then a unique connection for each choice. This agrees with the results from the previous tables as described above. For $n = 4$ of interest here (case P in Table 5) we can take $m = 4$ (the zero connection), $m = 2$ which has six choices for T and for each of these a unique connection, or $m = 0$ with T empty and three choices for how to pair off the four indices. Here if the indices are $X = \{0,1,2,3\}$, say, then we have three possible partitions into pairs, namely

$$X = \{0,1\} \sqcup \{2,3\}, \quad X = \{0,2\} \sqcup \{1,3\}, \quad X = \{0,3\} \sqcup \{1,2\},$$

giving three quantum Levi-Civita connections. If we write $x^\mu = t, x, y, z$ then

$$g = \mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z$$

and our three quantum Levi-Civita connections with all $\nabla \mathrm{d}x^\mu$ nonzero are

$$\nabla \mathrm{d}t = \nabla \mathrm{d}x = (\mathrm{d}t + \mathrm{d}x) \otimes (\mathrm{d}t + \mathrm{d}x), \quad \nabla \mathrm{d}y = \nabla \mathrm{d}z = (\mathrm{d}y + \mathrm{d}z) \otimes (\mathrm{d}y + \mathrm{d}z);$$

$$\nabla \mathrm{d}t = \nabla \mathrm{d}y = (\mathrm{d}t + \mathrm{d}y) \otimes (\mathrm{d}t + \mathrm{d}y), \quad \nabla \mathrm{d}x = \nabla \mathrm{d}z = (\mathrm{d}x + \mathrm{d}z) \otimes (\mathrm{d}x + \mathrm{d}z);$$

$$\nabla \mathrm{d}t = \nabla \mathrm{d}z = (\mathrm{d}t + \mathrm{d}z) \otimes (\mathrm{d}t + \mathrm{d}z), \quad \nabla \mathrm{d}x = \nabla \mathrm{d}y = (\mathrm{d}x + \mathrm{d}y) \otimes (\mathrm{d}x + \mathrm{d}y).$$

We also had the zero connection and six connections with $\nabla \mathrm{d}x^\mu = 0$ for two of the indices. As before, there is no special significance to the labelling of the generators.

(ii) $V = \mathbb{F}_2\mathbb{Z}_n$**.** For $n = 4$ this is case G in Table 5 after a change of variables to $x^0 = e, x^1 = e{+}x, x^2 = e{+}y, x^3 = e{+}x{+}y{+}x$ (on the left are labels not exponents, albeit exponents with the $\circ$ product). The same methods as above for $n = 2,3$ give us 8 quantum metrics for $n = 4$ with matrices, written in basis order $\mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3, \mathrm{d}x^0$,

$$\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}, \begin{bmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{bmatrix}, \begin{bmatrix}1&1&0&1\\1&0&1&1\\0&1&1&1\\1&1&1&0\end{bmatrix}, \begin{bmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{bmatrix}, \begin{bmatrix}0&1&1&1\\1&1&1&0\\1&1&0&1\\1&0&1&1\end{bmatrix}$$

of which the first four are in the general example (ii) in Section 2 (for $m = 1,0,3,2$ respectively). The other four are their complementary metrics with de-Morgan dual coefficients which for $n = 4$ are distinct and nondegenerate. Thus the general construction together with duals gives all metrics at least for $n \le 4$ (and we suppose for all n). Experience with $n = 2,3$ tells us to expect more than one nonzero quantum Levi-Civita connection for each metric when $n = 4$.

(iii) $V = A_2$**.** This appears as case L in Table 5 after a change of variables to $x^0 = e, x^1 = x, x^2 = z$ and $x^3 = e + y$ (on the left are labels not exponents, albeit exponents for the $\circ$ product) to match the basis in the general example (iii) in Section 2. In this basis the relations for the calculus on $\mathbb{F}_2[x^0, x^1, x^2, x^3]$ are

$$[\mathrm{d}x^0, x^0] = x^0, \quad [\mathrm{d}x^0, x^i] = [\mathrm{d}x^i, x^0] = \mathrm{d}x^i, \quad [\mathrm{d}x^1, x^1] = [\mathrm{d}x^2, x^3] = [\mathrm{d}x^3, x^2] = \mathrm{d}x^2$$

$$[\mathrm{d}x^2, x^1] = [\mathrm{d}x^1, x^2] = [\mathrm{d}x^3, x^3] = \mathrm{d}x^3, \quad [\mathrm{d}x^3, x^1] = [\mathrm{d}x^2, x^2] = [\mathrm{d}x^1, x^3] = \mathrm{d}x^1$$

and by computer one has three quantum metrics,

$$g_I = \mathrm{d}x^1 \otimes \mathrm{d}x^3 + x^3 \otimes x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2 + \mathrm{d}x^3 \otimes \mathrm{d}x^3 + \mathrm{d}x^3 \otimes \mathrm{d}x^0 + \mathrm{d}x^0 \otimes \mathrm{d}x^3 + \mathrm{d}x^0 \otimes \mathrm{d}x^0$$

$$g_{II} = \mathrm{d}x^1 \otimes \mathrm{d}x^2 + \mathrm{d}x^2 \otimes \mathrm{d}x^1 + \mathrm{d}x^3 \otimes \mathrm{d}x^0 + \mathrm{d}x^0 \otimes \mathrm{d}x^3 + \mathrm{d}e \otimes \mathrm{d}x^0$$

$$g_{III} = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^3 + \mathrm{d}x^3 \otimes \mathrm{d}x^2 + \mathrm{d}x^3 \otimes \mathrm{d}x^3 + \mathrm{d}x^3 \otimes \mathrm{d}x^0 + \mathrm{d}x^0 \otimes \mathrm{d}x^3 + \mathrm{d}x^0 \otimes \mathrm{d}x^0$$

In both cases (ii),(iii) one can clearly go ahead and look for quantum Levi-Cita connections but would need a more powerful computer.

	Relations	Quantum metrics and QLCs
D	$[\mathrm{d}e, e] = e$ $[\mathrm{d}x, e] = 0 = [\mathrm{d}e, x]$ $[\mathrm{d}x, x] = 0$	$g_D = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x$ $\nabla \mathrm{d}e = \alpha \mathrm{d}x \otimes \mathrm{d}e, \quad \nabla \mathrm{d}x = \beta \mathrm{d}x \otimes \mathrm{d}x$ $\nabla \mathrm{d}e = \alpha \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}e + \beta \mathrm{d}x \otimes \mathrm{d}x$ $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}x + \alpha \mathrm{d}x \otimes \mathrm{d}e, \quad \nabla \mathrm{d}x = \mathrm{d}e \otimes \mathrm{d}e + \beta \mathrm{d}x \otimes \mathrm{d}x$
E	$[\mathrm{d}e, e] = x,$ $[\mathrm{d}x, e] = 0 = [\mathrm{d}e, x]$ $[\mathrm{d}x, x] = 0$	$g_{E.I} = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e$ $\nabla \mathrm{d}e = \alpha \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \beta \mathrm{d}e \otimes \mathrm{d}e$ $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}x + \alpha \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \beta \mathrm{d}e \otimes \mathrm{d}e$ $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}e + \alpha \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \beta \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e$ $\nabla \mathrm{d}e = \mathrm{d}x \otimes \mathrm{d}e + \alpha \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \beta \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x$ $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \alpha \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \beta \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x$ $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}e + \alpha \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \beta \mathrm{d}e \otimes \mathrm{d}e + +\mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x$ $g_{E.II} = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x$ $\nabla \mathrm{d}e = \alpha \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \beta \mathrm{d}e \otimes \mathrm{d}e$ $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}x + \alpha \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \beta \mathrm{d}e \otimes \mathrm{d}e$ $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x + \alpha \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \beta \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x$ $\nabla \mathrm{d}e = \mathrm{d}x \otimes \mathrm{d}e + \alpha \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \beta \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x$ $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \alpha \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \beta \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}x \otimes \mathrm{d}x$ $\nabla \mathrm{d}e = \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e + \alpha \mathrm{d}x \otimes \mathrm{d}x, \quad \nabla \mathrm{d}x = \beta \mathrm{d}e \otimes \mathrm{d}e + \mathrm{d}e \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}e$

Table 6. All possible non-inner noncommutative geometries on $\mathbb{F}_2[e, x]$. Here $\alpha, \beta \in \mathbb{F}_2$ are parameters.

6. Discussion

In this paper our main focus was on inner differential calculi and as such we classified all noncommutative Riemannian geometries on $\mathbb{F}_2[x^1, \cdots, x^n]$ i.e. in n-dimensions and with constant coefficients, for $n \le 3$ and some results for $n = 4$ or higher. There are several remarks to be made.

First of all, the inner case was a useful restriction which is typical of strictly noncommutative geometries, taking the view that classical geometry is a somewhat special and unrepresentative limit. However, a similar analysis and classification can be done without this requirement, it just produces many more calculi. For example, for $n = 2$ we find two additional families (to the three given in Table 1) namely $D : [\mathrm{d}e, e] = e$ and $E : [\mathrm{d}e, e] = \mathrm{d}x$ for the non-zero commutors. For calculus D there exists one quantum metric and for calculus E there exist two quantum metrics. All quantum metrics and quantum Levi-Civita connections (parametrised by $\alpha, \beta \in \mathbb{F}_2$) are shown in Table 6. These are in addition to the classical, commutative, calculus which has the zero algebra (all products zero).

Next, we should note that while our 'coordinate algebra' $A = \mathbb{F}_2[x^1, \cdots, x^n]$ has been classical, the same formulae for differential geometries hold identically if we have commutation relations of Heisenberg/Clifford type (there being no difference over $\mathbb{F}_2$) defined by some $\Theta_{\mu\nu}$. For the example in $n = 4$ given after Proposition 5.1 with $V = \mathbb{F}_2(4 \text{ points})$, the structure of the connection for the calculus suggests pair-wise grouping with relations

$$xt + tx = 1, \quad yz + zy = 1$$

for the algebra A. The geometry is not affected by this change of relations as explained in Section 2, since $\mathrm{d}1 = 0$ and since our formulae have constant coefficients.

This brings us to a main limitation of the paper, namely to constant coefficients in the metric and connection in our $\mathrm{d}x^\mu$ basis. This means that our geometries are in some sense 'flat space' and indeed we checked that many of them have zero curvature. What is surprising is that even so there are so many rich possibilities for the quantum Levi-Civita connection other than $\nabla \mathrm{d}x^\mu = 0$ and $\sigma = $ flip which is the obvious 'flat' connection, and indeed some of them are even curved. This non-uniqueness of the torsion free metric compatible bimodule connection for a given metric is also seen in some other noncommutative models, such as [4]. It is also remarkable that we can't take any constant coefficients for the metric, which is a rigidity phenomenon for noncommutative geometry again seen in other models [4, 14]. In our case the number of metrics is far less than the potentially $2^{n(n+1)/2}$ possible coefficients values and gave our rich classification.

We now consider applications of such noncommutative geometries. Our motivation here is that they model a quantum space or spacetime, but one could also apply them in many other contexts such as 'digital' models of quantum mechanics phase spaces or other 'geometric' applications in engineering. Apart from enumerating the different geometries (which would be relevant to a sector of quantum gravity where we sum over geometries) we can generally explore particles and fields on each noncommutative-geometric background, for example solutions of wave equations and Maxwell equations. Here the natural scalar Laplacian in our approach to noncommutative geometry is defined by $\square = (\ ,\ )\nabla \mathrm{d}$ where $(\ ,\ ) : \Omega^1 \otimes_A \Omega^1 \to A$ is the inverse metric [4]. To make this concrete we take the differential calculus be given by $V = \mathbb{F}_2(X)$ for a finite indexing set X (so spacetime coordinates are x^μ where $\mu \in X$). We are then forced to the Euclidean metric and have connections as in Proposition 5.1. The non-commutation relations in each variable and the fact that they mutually commute gives us

$$\mathrm{d}f(x^1,\cdots,x^n) = \sum_\mu (\partial_\mu f)\mathrm{d}x^\mu, \quad \partial_\mu f(x^1,\cdots,x^n) = f(x^1,\cdots,x^\mu{+}1,\cdots,x^n) - f(x^1,\cdots,x^n)$$

i.e. the partial derivatives are finite difference operators. Then the Leibniz properties of a connection and evaluation against the inner product give

$$\square f(x^1,\cdots,x^n) = \sum_\mu \partial_\mu \partial_\mu f$$

independently of the connection (this is because $(\mathrm{d}x^s + \mathrm{d}x^{\bar{s}}, \mathrm{d}x^s + \mathrm{d}x^{\bar{s}}) = 0$ over $\mathbb{F}_2$). For example, we have

$$f = \sum_{i_1,\cdots,i_n} a_{i_1 i_2 \cdots i_n} (x^1)^{2^{i_1}} \cdots (x^n)^{2^{i_n}}, \quad \partial^\mu f = \sum_{i_1,\cdots,i_n} a_{i_1 i_2 \cdots i_n} (x^1)^{2^{i_1}} \cdots \widehat{(x^\mu)^{2^{i_\mu}}} \cdots (x^n)^{2^{i_n}}$$

where we leave out the x^μ. Hence such functions are automatically zero modes of $\square$. This is probably the simplest example; other V will lead to other commutation relations and other geometries. The properties and applications of such geometric wave operators would be an interesting topic for further work. Our idea is that such equations could be used to propagate information much as in a quantum computer but here modelled with 'digital geometry'.

A more speculative direction to be explored here is to use the above as a model of classical and quantum field theory. Thus we have been thinking of $A = \mathbb{F}_2(V)$ by which we mean polynomials in generators x^μ arising as a basis of a commutative algebra with vector space V. But what if V is actually the spacetime coordinate algebra? For example if $V = \mathbb{F}_2(X)$ and X is a discrete spacetime with basis $x^\mu = \delta_\mu$ of delta-functions at $\mu \in X$ then A would be functionals on X (i.e. functions on the vector space of functions on X). We can also allow these functionals to have Heisenberg-type commutation relations as explained in Section 2 above and if we are not interested in a metric then we do not need to work over $\mathbb{F}_2$ so we can get closer to conventional classical or quantum field theory. What we see, however, is that in this context the structure of V (the classical spacetime geometry in some sense) determines a noncommutative differential on the algebra of functionals, i.e a noncommutative variational calculus. It would be interesting to reformulate such things as noncommutative Euler-Lagrange equations from this point of view now much more tied to the classical spacetime geometry. This applies even if we keep A classical, i.e. are studying noncommutative variations or differentials of classical fields on X.

This brings us to a different classification problem. If we are interested in commutative algebras V as 'spacetime coordinate algebras' with the classical or possibly quantum field theory interpretation of A, then we should also be interested in the differential geometry of V as our spacetime differential geometry. This will then connect through to any variational field equations just as it does classically when $V = C^\infty(M)$ for a manifold M. We again can make things simpler by letting V be finite-dimensional (a finite geometry) and going 'digital' by working over $\mathbb{F}_2$. We have already done part of the classification since we classified unital algebras over $\mathbb{F}_2$ up to dimension 4 (the latter being too numerous to list explicitly in the present paper). Beyond this we should consider noncommutative differential and Riemannian structures over each V, which is a classification problem we will address by computer methods similar to the above, in [11]. Such a 'finite digital geometry' was obtained for the 4-dimensional algebra A_2 in [2, Prop. 5.7], where it shown that there is a natural 2-dimensional differential calculus with 3 possible metrics with constant coefficients, and for each of these the paper found one quantum Levi-Civita connection other than the zero one, with zero curvature.

For the three unital algebras V of dimension 2 identified in Section 3 we have only the zero calculus or the universal calculus of the maximal dimension $n-1$, i.e. 1-dimensional over the algebra. The relations of the latter for each algebra are obtained by applying d to the algebra relations, giving

$$\text{A:}\quad [\mathrm{d}x, x] = 0;\quad \text{B: } [\mathrm{d}x, x] = \mathrm{d}x;\quad \text{C:}\quad [\mathrm{d}x, x] = \mathrm{d}x$$

for the three $\Omega^1(V)$, along with $e = 1$ central and killed by d. In each case we have $g = \mathrm{d}x \otimes \mathrm{d}x$ as the only metric and $\nabla \mathrm{d}x = 0$ or $\nabla \mathrm{d}x = \mathrm{d}x \otimes \mathrm{d}x$ as quantum Levi-Civita connections. For $n > 2$ we have different differential structures form the zero up to the universal of dimension $n-1$ over the algebra with more nontrivial geometries arising.

We can also allow our algebras to be noncommutative and look for other algebraic structures including Hopf algebras and solutions of the Yang-Baxter or braid relations over $\mathbb{F}_2$. The nice thing about doing algebra over $\mathbb{F}_2$ is that it could in theory be realised both in software machine code or indeed in actual silicon by means of logic gates, as follows. Apart from the motivation given, geometric elements of quantum computing could then be implemented digitally while possibly keeping some of the benefits.

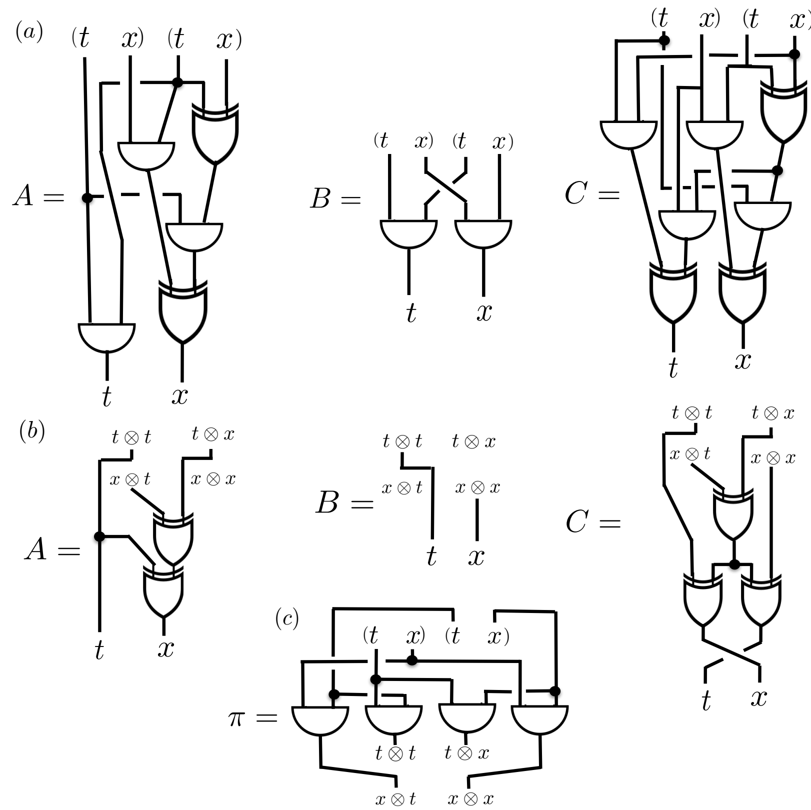

FIGURE 1. Electronic circuit diagrams for each of the 3 unital products A,B,C of dimension 2 over $\mathbb{F}_2$, (a) as $V\times V\to V$ and (b) as $V\otimes V\to V$. (c) shows the canonical map $V\times V\to V\otimes V$

First, we can represent a vector space V of dimension m and fixed basis $\{e_i\}_{i=1}^{i=m}$ by a ribbon cable of m wires. Then any $v\in V$ corresponds to a pattern of 0s and 1s in the wires according to $v=\sum v_i e_i$ where v_i is the digital signal (0 or 1) in the i'th wire. Thus v has e_i each time there is a 1 in the i-the wire. Equivalently, the binary number $v_{1_1}\cdots v_{i_m}$ represents the vector $v\in V$ (there are 2^m states of each). If W similarly has basis $\{f_j\}$ where $j=1,\cdots,p$ then we identify $V\otimes W$ with mp wires by basis $E_{i,j}=e_i\otimes f_j$, which we can organise as m bundles of p-wire cables (one could imagine them stacked below each other). Direct sum $V\oplus W$ corresponds to a $m+p$-wire cable given by placing the m-wire cable for V next to that of W, as does $V\times W$. Algebraic operations can then be written as digitial gates assigning to all input truth table values an output truth table, as expressed in the state of the wires.

To illustrate this we show the circuit diagrams for the three $n=2$ unital algebras over $\mathbb{F}_2$ (cases A,B,C in Section 3). Here V is 2-dimensional so has 2 wires. We chose basis $t=e+x,x$ for V (the labelling is arbitrary but recall that in the example of $\mathbb{F}_2$(2 points) we had t and x as the natural basis of δ-functions for the two points).

Thus the correspondence we use between vectors and digital states will be

$$0 = 00, \quad x = 01, \quad t = e + x = 10, \quad e = 11.$$

The easiest representation is to define the algebra product as a map $V \times V \to V$ with two 2-wire inputs one for each element of V and a 2-wire output which give the 3 different algebra products shown in part (a). The flat edged 'product' denotes AND which is 1 exactly when both inputs are. The other curve-edged 'product' denotes symmetric difference or XOR (exclusive OR) which is 1 exactly when the two inputs are different. The desired outcomes can be expressed as Boolean algebra or more precisely as a Boolean ring (using AND as product and XOR as addition) and then converted easily to the diagrams shown. These 'naive products' do define the product of any two vectors in V but one should note that they do not define it on nondecomposable ('entangled') vectors such as $t \otimes t + x \otimes x$ since these are not in the image of the map $\otimes : V \times V \to V \otimes V$ (the image has 10 elements including 0). We would hardly worry about this in linear algebra since the product is linear but since we have not encoded such a property it is better to define the products more fully as maps $V \otimes V \to V$ which we do in part (b). The products in part (a) factors through the maps in part (b) via the canonical map $\pi : V \times V \to V \otimes V$ which as a diagram consists of 4 AND gates connecting up as shown in part (c). One can check with a little Boolean algebra that following this by the maps (b) gives the maps (a) so we can pull back to them, but the maps in (b) carry a little more information as explained. This language is obviously more tricky and for example associativity of the two ways to form the iterated product $V \otimes V \otimes V \to V$ ideally would be drawn in 4D with the input requiring a cube of wire-ends, one dimension for each tensor factor. In [11] we will describe some small examples of geometric Laplacians associated to finite Riemannian geometries in this language.

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School of Mathematics, Mile End Rd, London E1 4NS, UK

E-mail address: s.majid@qmul.ac.uk, a.pachol@qmul.ac.uk