Rigidity of Frameworks

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Declaration

I, Hakan Guler, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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Details of collaboration and publications:
Chapters 1, 3 and 4 are collaborations with my PhD supervisor Bill Jackson.

Abstract

A d-dimensional *(bar-and-joint) framework* is a pair (G, p) where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$ is a function which is called the *realisation* of the framework (G, p). A *motion* of a framework (G, p) is a continuous function $P : [0, 1] \times V \to \mathbb{R}^d$ which preserves the edge lengths for all $t \in [0, 1]$. A motion is *rigid* if it also preserves the distances between non-adjacent pairs of vertices of G. A framework is *rigid* if all of its motions are rigid motions.

An *infinitesimal motion* of a d-dimensional framework (G, p) is a function $q : V \to \mathbb{R}^d$ such that $[p(u) - p(v)] \cdot [q(u) - q(v)] = 0$ for all $uv \in E$. An infinitesimal motion of the framework (G, p) is *rigid* if we have $[p(u) - p(v)] \cdot [q(u) - q(v)] = 0$ also for non-adjacent pairs of vertices. A framework (G, p) is *infinitesimally rigid* if all of its infinitesimal motions are rigid infinitesimal motions. A d-dimensional framework (G, p) is *generic* if the coordinates of the positions of vertices assigned by p are algebraically independent. For generic frameworks rigidity and infinitesimal rigidity are equivalent.

We construct a matrix of size $|E| \times d|V|$ for a given d-dimensional framework (G, p) as follows. The rows are indexed by the edges of G and the set of d consecutive columns corresponds to a vertex of G. The entries of a row indexed by $uv \in E$ contain the d coordinates of $p(u) - p(v)$ and $p(v) - p(u)$ in the d consecutive columns corresponding to u and v, respectively, and the remaining entries are all zeros. This matrix is the *rigidity matrix* of the framework (G, p) and denoted by $R(G, p)$. Translations and rotations of a given framework (G, p) give rise to a subspace of dimension $\binom{d+1}{2}$ of the null space of $R(G, p)$ when $p(v)$ affinely spans $\mathbb{R}^d$. Therefore we have $\operatorname{rank} R(G, p) \leq d|V| - \binom{d+1}{2}$ if $p(v)$ affinely spans $\mathbb{R}^d$, and the framework is infinitesimally rigid if equality holds.

We construct a matroid corresponding to the framework (G, p) from the rigidity matrix $R(G, p)$ in which $F \subseteq E$ is independent if and only if the rows of $R(G, p)$ indexed by F are linearly independent. This matroid is called the *rigidity matroid* of the framework (G, p). It is clear that any two generic realisations of G give rise to the same rigidity matroid.

In this thesis we will investigate rigidity properties of some families of frameworks.

We first investigate rigidity of linearly constrained frameworks i.e., 3-dimensional bar-and-joint frameworks for which each vertex has an assigned plane to move on. Next we characterise rigidity of 2-dimensional bar-and-joint frameworks (G, p) for which three distinct vertices $u, v, w \in V(G)$ are mapped to the same point, that is $p(u) = p(v) = p(w)$, and this is the only algebraic dependency of p. Then we characterise rigidity of a family of non-generic body-bar frameworks in 3-dimensions. Finally, we give an upper bound on the rank function of a d-dimensional bar-and-joint framework for $1 \leq d \leq 11$.

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I am very happy that I met Huseyin Ozcan and his family. I owe them a big "thank you" for regarding me like a family member during the time I spent in London.

This thesis is dedicated to my parents.

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Chapter 0

Introduction

0.1 Graphs, Frameworks and Rigidity

Definition 0.1.1. A *(simple) graph* G is an ordered pair (V, E) consisting of a finite set V of *vertices*, and a set E of *edges* consisting of unordered pairs of distinct vertices.

Let $G = (V, E)$ be a graph. To denote a member of E we use the form xy rather than $\{x, y\}$. For an edge $e = xy$, the vertices x, y are called *endpoints* of e and we say x and y are *adjacent*. We also say that the vertices x and y are *incident* with the edge e. The *neighbourhood*, $N_G(v)$ (or $N(v)$ when it is clear), of a vertex $v \in V$ is the set of all vertices that are adjacent to v in G. The *closed neighbourhood* of v is the set $N_G(v) \cup \{v\}$, and denoted by $N_G[v]$ (or N[v]). The *degree* of v is the size of $N_G(v)$ and denoted by $d_G(v)$ (or $d(v)$). We use $\delta(G)$, respectively $\Delta(G)$ to denote the *minimum*, respectively the *maximum* value of $d_G(v)$ over all vertices.

A graph $H = (S, F)$ is called a *subgraph* of $G = (V, E)$, if $S \subseteq V$ and $F \subseteq E$. A subgraph H of G is called an *induced subgraph* of G, if for every $x, y \in S$ with $xy \in E$, we have $xy \in F$. If H is an induced subgraph of G with vertex set S, we say the set S *induces* H in G, and denote H as $G[S]$. The set and the number of edges of $G[S]$ are denoted by $E_G(S)$ and $i_G(S)$, respectively.

Definition 0.1.2. A *d-dimensional bar-and-joint framework* is a pair (G, p) where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$ is a map. We say that p is a *realisation* (or *configuration*) of the framework (G, p).

We consider the framework to be a straight line realisation of G in $\mathbb{R}^d$ and the length of an edge $uv \in E$ given by Euclidean distance between $p(u)$ and $p(v)$.

Definition 0.1.3. Two given frameworks (G, p_0) and (G, p_1) are
• *equivalent* if $\|p_0(u) - p_0(v)\| = \|p_1(u) - p_1(v)\|$, $\forall uv \in E$, and
• *congruent* if $\|p_0(u) - p_0(v)\| = \|p_1(u) - p_1(v)\|$, $\forall u, v \in V$.

Definition 0.1.4. A *motion* of a d-dimensional framework (G, p) is a function $P : [0, 1] \times V \to \mathbb{R}^d$ such that $P(0, v) = p(v)$ for all $v \in V$ and
(M1): $\|P(t, u) - P(t, v)\| = \|P(0, u) - P(0, v)\|$, $\forall t \in [0, 1]$ and $\forall uv \in E$;
(M2): $P(t, v)$ is a continuous function of t, $\forall v \in V$.
We say that a motion P is from (G, p_0) to (G, p_1) if $P(0, v) = p_0(v)$ and $P(1, v) = p_1(v)$, $\forall v \in V$.

We can imagine a motion as a continuous path in the algebraic variety $W = \{q \in \mathbb{R}^{dn} | (G, q)$ is equivalent to $(G, p_0)\}$ which goes from p_0 to p_1, where q is regarded as a dn-tuple (d entries for each vertex) for each framework (G, q).

Definition 0.1.5. A motion is a *rigid motion* if $\forall u, v \in V$ and $\forall t \in [0, 1]$ we have $\|P(t, u) - P(t, v)\| = \|P(0, u) - P(0, v)\|$.

Example 0.1.1. Let (G, p) be the framework in Figure 1.

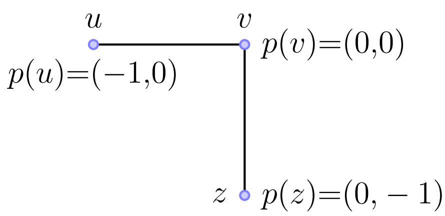

Figure 1: A framework in $\mathbb{R}^2$.

Now consider three different functions.
• $P(t, x) = \begin{cases} p(x), & x = u, v \\ p(x) + (0, -t), & , x = z \end{cases}$
This function is not a motion of the framework, since it increases the length of the edge vz.

- $P(t,x) = \begin{cases} p(x), & x = u, v \\ p(x) + (-t, 1 - \sqrt{1-t^2}), & x = z \end{cases}$

This function defines a motion but not a rigid one, since the distance between u and z is decreasing as illustrated in Figure 2.

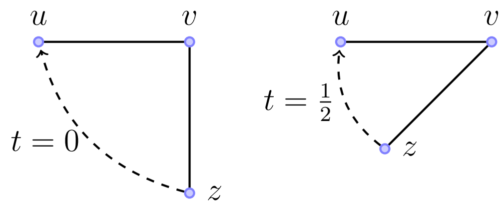

Figure 2: The function does not deform the edges but it changes the distance between u and z.

- $P(t,x) = p(x) + (t, 0), \ \forall x \in V.$

This function corresponds to a translation and hence a rigid motion, since translations preserve the distance between any two points.

Definition 0.1.6. A framework (G,p) is *rigid* if all of its motions are rigid motions. Equivalently, a framework (G,p) is rigid if and only if every motion of (G,p) results in a framework which is congruent to (G,p). If a framework is not rigid then we say it is *flexible*.

A framework (G,p) has combinatorial properties arising from the graph G as well as geometric properties arising from the realisation p. It is natural to ask whether considering the graph G is enough to determine the rigidity of the framework (G,p) or not. It is clear that any framework whose underlying graph is a complete graph is rigid, since a complete graph has no non-adjacent pairs of vertices. Let us consider some other examples in Figure 3.

The framework on the left is not rigid, since it can be continuously deformed to the framework in the middle without changing edge lengths. However, if we add a diagonal edge, then the resulting framework is rigid.

The two frameworks in Figure 4 have the same underlying graph with different realisations. The framework on the left is not rigid, we can move the vertices e and f without moving the others or changing the edge lengths. Such a motion changes

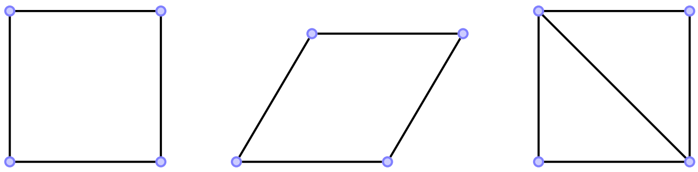

Figure 3: Rigid and flexible frameworks.

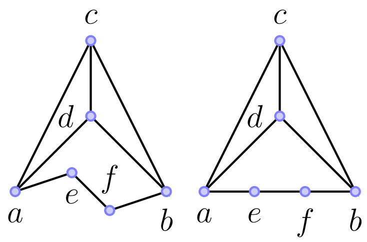

Figure 4: A graph with a non-rigid and rigid realization.

the distance between e or f and c or d. The framework on the right is rigid. Since a triangle is a complete graph hence rigid, we cannot deform the edges of triangles adc and bcd. Therefore, we just need to consider about the line on which the vertices a, e, f, b lie. If we try to move e and f as we did for the framework on the left, then we need to decrease the distance between a and b. If the distance between a and b decreases, then d must go up and this results in a deformation of the edge cd. Thus moving e and f which is the only candidate for a non-rigid motion is not a motion. Hence, the framework on the right is rigid. The key thing here is the fact that a, e, f and b are collinear, and this is not a generic realisation. We first need to define when a realisation is considered as generic.

Definition 0.1.7. A set $A = \{a_1, a_2, \ldots, a_m\}$ of distinct real numbers is said to be *algebraically dependent* if there exists a non-zero polynomial $f(x_1, x_2, \ldots, x_m)$ with rational coefficients satisfying $f(a_1, a_2, \ldots, a_m) = 0$. If A is not algebraically dependent, it is called *algebraically independent* or *generic*.

A realisation p of a d-dimensional framework (G, p) with $V(G) = \{v_1, v_2, \ldots, v_n\}$ is *generic* if the dn-tuple $(p(v_1), p(v_2), \ldots, p(v_n))$ is generic. We can now see that for the framework on the right in the previous example there are four vertices on the same line causing this realisation not to be generic.

Asimov and Roth [2] showed an equivalent statement for the definition of rigidity which is the following.

Theorem 0.1.1. *[2] A framework* (G, p_0) *is rigid if and only if there exist an* $\epsilon > 0$ *such that every framework* (G, p_1) *which is equivalent to* (G, p_0) *and satisfies* $\|p_0(v) - p_1(v)\| < \epsilon$, $\forall v \in V$, *is congruent to* (G, p_0).

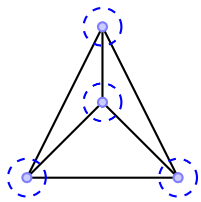

Figure 5: An ϵ neighbourhood of a framework. All the dashed circles have radius ϵ.

Since the existence of a continuous path between two points of an algebraic variety implies the existence of a differentiable path between those points, see [2] for details, the existence of a motion between two frameworks implies the existence of a differentiable motion between them. Therefore, we can consider motions as differentiable motions, and write

$$\|P(t,u) - P(t,v)\|^2 = \|p_0(u) - p_0(v)\|^2.$$

Then we get the following by differentiating with respect to t.

$$[P(t,u) - P(t,v)] \cdot [P'(t,u) - P'(t,v)] = 0.$$

Note that $P'(t,v)$ is the instantaneous velocity of the vertex v at time t. If we let $t = 0$ and $P'(0,v) = q(v)$ for all $v \in V$ we get

$$[p_0(u) - p_0(v)] \cdot [q(u) - q(v)] = 0, \ \forall uv \in E. \tag{1}$$

With these in mind we can define infinitesimal motions.

Definition 0.1.8. An *infinitesimal motion* of a d-dimensional framework (G, p) is a function $q : V \to \mathbb{R}^d$ such that $[p_0(u) - p_0(v)] \cdot [q(u) - q(v)] = 0$ for all $uv \in E$.

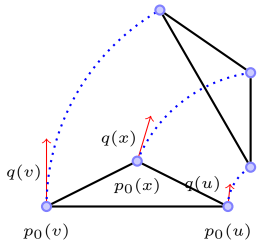

Figure 6: A motion (dashed blue arcs) of a framework and corresponding instantaneous velocity vectors (red vectors).

Definition 0.1.9. An infinitesimal motion of a framework (G, p) is an *infinitesimal rigid motion* if $[p_0(u) - p_0(v)] \cdot [q(u) - q(v)] = 0$ for all $u, v \in V$.

Definition 0.1.10. A framework (G, p) is *infinitesimally rigid* if all of its infinitesimal motions are infinitesimal rigid motions.

Definition 0.1.11. *The rigidity matrix* $R(G, p)$ of a d-dimensional framework (G, p) is the matrix of the system of equations (1). It is a $|E| \times d|V|$ matrix whose rows are indexed by the edges of G and the set of d consecutive columns corresponds to a vertex of G. The entries in the row corresponding to an edge $e \in E$ and columns corresponding to a vertex $u \in V$ are given by the vector $p(u) - p(v)$ if $e = uv$ and is the zero vector if e is not incident with u.

Definition 0.1.12. A d-dimensional framework (G, p) is called *independent* (or *dependent*) if the rows of $R(G, p)$ are linearly independent (or dependent).

Example 0.1.2. Let (G, p) be the framework shown below, and let $p(v_i) = (x_i, y_i)$.

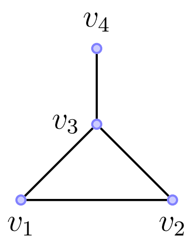

Figure 7: A framework in $\mathbb{R}^2$.

Then the rigidity matrix $R(G,p)$ is the following.

$$\begin{array}{c} v_1v_2 \\ v_1v_3 \\ v_2v_3 \\ v_3v_4 \end{array}\begin{pmatrix} x_1-x_2 & y_1-y_2 & x_2-x_1 & y_2-y_1 & 0 & 0 & 0 & 0 \\ x_1-x_3 & y_1-y_3 & 0 & 0 & x_3-x_1 & y_3-y_1 & 0 & 0 \\ 0 & 0 & x_2-x_3 & y_2-y_3 & x_3-x_2 & y_3-y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3-x_4 & y_3-y_4 & x_4-x_3 & y_4-y_3 \end{pmatrix}$$

The space of infinitesimal motions of a d-dimensional framework (G,p) is equal to the null space (kernel) of $R(G,p)$. We know that infinitesimal translations and rotations are trivially in the null space of $R(G,p)$. Therefore we get the dimension of the null space of $R(G,p)$ is at least

$$\binom{d}{1}+\binom{d}{d-2}=\binom{d}{1}+\binom{d}{2}=\binom{d+1}{2}$$

when $p(v)$ affinely spans $\mathbb{R}^d$, since infinitesimal translations along each vector in the standard basis ($\binom{d}{1}$ vectors) together with infinitesimal rotations about each $(d-2)$-dimensional subspace spanned by $(d-2)$ vectors in the standard basis ($\binom{d}{d-2}$ such subspaces) are linearly independent. Thus we get $\operatorname{rank} R(G,p) \leq d|V|-\binom{d+1}{2}$. Asimov and Roth [2] showed that the equality holds if and only if (G,p) is infinitesimally rigid when G has at least $d+2$ vertices. Note that as $\binom{|V|}{2} \leq d|V|-\binom{d+1}{2}$ when $|V| \leq d+1$, we have a different condition for this case.

Theorem 0.1.2. *[2] A d-dimensional framework (G,p) is infinitesimally rigid if and only if*

$$\operatorname{rank} R(G,p)=\begin{cases} d|V|-\binom{d+1}{2}, & |V| \geq d+2 \\ \binom{|V|}{2}, & |V| \leq d+1 \end{cases}$$

We see from Theorem 0.1.2 that a d-dimensional generic framework (G,p) with fewer than $d+2$ vertices is infinitesimally rigid if and only if G is a complete graph.

Theorem 0.1.3. *[2] Let (G,p) be an infinitesimally rigid framework. Then (G,p) is rigid.*

The converse of Theorem 0.1.3 does not hold. To see this let (G,p) the framework in Figure 8, $q(v_5)$ be the red vector and $q(v_i)$ be the zero vector, for $i \neq 5$. This framework is rigid since we cannot deform the triangles, and to deform the line on which v_1, v_2, v_5 lie we need to deform the triangles. For infinitesimal rigidity, first our assignment of velocity vectors corresponds to an infinitesimal motion i.e., $[p(u)-p(v)]\cdot[q(u)-q(v)] = 0$ for all $uv \in E$. However, this is not an infinitesimal rigid motion since $q(v_5) - q(v_3)$ is not perpendicular to the line through $p(v_3)$ and $p(v_5)$ which implies $[p(v_5) - p(v_3)] \cdot [q(v_5) - q(v_3)] \neq 0$. Thus (G, p) is not infinitesimally rigid. The following theorem shows that the converse of Theorem 0.1.3 holds for generic frameworks.

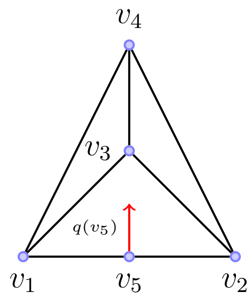

Figure 8: A non-infinitesimally rigid but rigid framework in $\mathbb{R}^2$.

Theorem 0.1.4. *[2] Let (G, p) be a d-dimensional generic framework. Then (G, p) is rigid if and only if (G, p) is infinitesimally rigid.*

Theorem 0.1.4 implies that the rigidity is a generic property.

Theorem 0.1.5. *[2] Let (G, p) and (G, p') be d-dimensional generic frameworks. Then (G, p) is rigid if and only if (G, p') is rigid.*

Theorem 0.1.5 implies that being rigid or infinitesimally rigid is a generic property. This allows us to talk about rigidity of a graph instead of a framework by restricting the realisations to generic ones. Therefore we can define the rigidity of a graph as follows.

Definition 0.1.13. A graph G is *rigid*, respectively *independent*, or *dependent* in $\mathbb{R}^d$ if there exists a generic realisation p of G in $\mathbb{R}^d$ such that (G, p) is rigid, respectively independent, or dependent.

Since rank $R(G,p)$ is maximised when (G,p) is generic let us denote rank $R(G,p)$ when p is a generic realisation of G in $\mathbb{R}^d$ by $r_d(G)$ or $r(G)$ if the dimension is obvious. Then a d-dimensional generic framework (G,p) with at least $d+2$ vertices is rigid if and only if $r(G) = d|V| - \binom{d+1}{2}$. It is clear that if we have a rigid framework and we add more edges then the resulting framework will still be rigid. However, if we delete an edge from a rigid framework it may not be rigid anymore. This motivates us to define minimal rigidity.

Definition 0.1.14. A framework (G,p) is *minimally rigid* if (G,p) is rigid and $(G-e,p)$ is not rigid for all $e \in E(G)$. Similarly, a graph G is *minimally rigid* if G is rigid and $G-e$ is not rigid for all $e \in E(G)$.

0.1.1 Rigidity Matroid

Definition 0.1.15. A *matroid* M is an ordered pair $(E,\mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E having the following properties:
- (I1) $\emptyset \in \mathcal{I}$
- (I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$
- (I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there is an element $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

Let $M = (E,\mathcal{I})$ be a matroid. The set E is called the *ground set* of M and each member of $\mathcal{I}$ is called an *independent set*. We say a set $E' \subseteq E$ with $E' \notin \mathcal{I}$ is a *dependent set*. We see that all maximal independent sets have the same size by (I3). The *rank* of a set $E' \subseteq E$ in M is the maximum size of a subset in $\mathcal{I}$ and denoted by $r_M(E')$ or $r(E')$ if the matroid M is clear from the context. A minimally dependent set of M is called a *circuit*. A maximal independent set of M is said to be a *base* of M.

There are several other equivalent definitions of a matroid and Definition 0.1.15 is the most common. Let us give another definition of a matroid that we will use in this thesis.

Definition 0.1.16. A *matroid* M is an ordered pair $(E,\mathcal{I})$ of a finite set E and a collection of independent subsets $\mathcal{I}$ of E satisfying the following three conditions:
- (I1) $\emptyset \in \mathcal{I}$

• (I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$
• (I3′) For every $E' \subseteq E$, all maximal independent subsets of E' have the same cardinality.

We can construct the *rigidity matroid*, $\mathcal{R}(G,p)$, of a d-dimensional framework (G,p) on $E(G)$, by using the rigidity matrix $R(G,p)$. A set of edges $F \subseteq E$ is *independent*, respectively *dependent* in $\mathcal{R}(G,p)$ if the corresponding rows of $R(G,p)$ are linearly independent, respectively dependent. A minimally dependent set of edges $F \subseteq E$ in $\mathcal{R}(G,p)$ is called a *circuit* in $\mathcal{R}(G,p)$; that is F is a circuit, if F is dependent and $F-e$ is independent in $\mathcal{R}(G,p)$ for all $e \in F$. We also say that a subgraph H of G is *independent*, respectively *dependent*, or a *circuit* in $\mathcal{R}(G,p)$, if $E(H)$ is independent, respectively dependent or a circuit in $\mathcal{R}(G,p)$. Since all generic realisations, in d-dimensions, of a graph G give rise to the same independence relations between the rows of the corresponding rigidity matrix, we obtain the same matroid from all d-dimensional generic realisations of G. This allows us to talk about the generic rigidity matroid of a graph instead of that of a framework. Let $\mathcal{R}_d(G)$ denote the d-dimensional generic rigidity matroid of G. We say G is *independent* in $\mathbb{R}^d$ if $|E| = r_d(G) = r(\mathcal{R}_d(G))$ (similarly, if $|E| = r(\mathcal{R}(G,p))$, we say that (G,p) is *independent* in $\mathbb{R}^d$). A graph G is called *dependent* in $\mathbb{R}^d$ if G is not independent in $\mathbb{R}^d$. A graph G is called a *circuit* $\in \mathbb{R}^d$, if G is dependent in $\mathbb{R}^d$ and $G-e$ is independent in $\mathbb{R}^d$ for all $e \in E(G)$. Using these we can have an equivalent definition of minimal rigidity as follows: a generic d-dimensional framework (G,p) with at least $d+2$ vertices is minimally rigid in $\mathbb{R}^d$ if $r_d(G) = |E| = d|V| - \binom{d+1}{2}$. Here the condition $r_d(G) = d|V| - \binom{d+1}{2}$ ensures the rigidity of G and the condition $r_d(G) = |E|$ prevents G from having unnecessary edges to guarantee the rigidity of G. The *rank*, $r_d(F)$, of a set of edges $F \subseteq E$ in $\mathcal{R}_d(G)$ is the maximum number of independent rows corresponding to the edges in F in any generic rigidity matrix.

Example 0.1.3. Let (G,p) be the framework below.

$G=(V,E)$ is not minimally rigid since G is K_4 and we have seen K_4-e is rigid for any generic realisation in $\mathbb{R}^2$. We can easily see that K_4-e is minimally rigid since $d|V| - \binom{d+1}{2} = 2\cdot 4 - \binom{3}{2} = 5$ is the minimum number of edges necessary to construct a 2-dimensional generically rigid framework on four vertices. Therefore $F \subseteq E$ is independent in the rigidity matroid, $\mathcal{R}_2(G)$, if $|F| \leq 5$. The only dependent set is

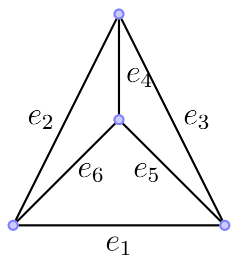

Figure 9: A generic framework in $\mathbb{R}^2$.

E.

For $X \subseteq V$ let $E_G(X)$ denote the set and $i_G(X)$ the number of edges in the subgraph of G induced by X. If it is clear from the context we will simply use $E(X)$ and $i(X)$ for $E_G(X)$ and $i_G(X)$, respectively. A graph $G = (V, E)$ is (k, l)-*sparse* if $i(X) \leq k|X| - l$ for all $X \subseteq V$ with $|X| \geq k$.

Lemma 0.1.6. *[18] Let (G, p) be a d-dimensional framework. Suppose (G, p) is independent in $\mathbb{R}^d$. Then G is $(d, \binom{d+1}{2})$-sparse.*

Proof. Suppose $i(X) > d|X| - \binom{d+1}{2}$ for some $X \subseteq V$ with $|X| \geq d$. Then $E_G(X) \subseteq E$ is dependent in $\mathcal{R}(G, p)$ since

$$|E_G(X)| = i(X) > d|X| - \binom{d+1}{2} \geq \text{rank } R(G[X], p|_X) = r_{(G,p)}(E_G(X))$$

where $r_{(G,p)}(E_G(X))$ is the rank of $E_G(X)$ in $\mathcal{R}(G, p)$. Then we must have E to be dependent in $\mathcal{R}(G, p)$ since it has a dependent subset, namely $E_G(X)$. $\square$

0.1.2 Graph Operations

In this section we shall discuss some graph operations which preserve the rigidity of a framework. These operations are also called Henneberg operations since they were introduced by L. Henneberg [9].

Definition 0.1.17. Let G and H be graphs. If $H = G - v$ for some vertex v of degree at most d we say that G is a (d-dimensional) 0-*extension* of H. If $H = G - v + u\omega$

for some vertex v of degree $d+1$ and non-adjacent neighbours u, ω of v, then we say that G is a (d-dimensional) 1-*extension* of H.

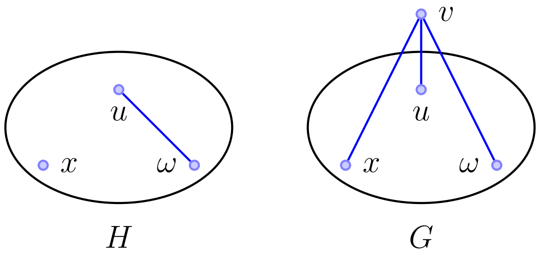

Figure 10: G is a 2-dimensional 1-extension of H.

Lemma 0.1.7. *Let (G,p) and $(H,p|_H)$ be two d-dimensional frameworks. Suppose G is a 0-extension of H, and the rows of $R(H,p|_H)$ are linearly independent. Suppose further that $H = G - v$, $d(v) = d$ and $u_1, u_2, \ldots, u_d$ are neighbours of v in G, and $p(v)$, $p(u_1), \ldots,$ $p(u_d)$ do not lie on a $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$. Then the rows of $R(G,p)$ are linearly independent.*

Proof. Let first d rows of $R(G,p)$ be the rows corresponding to the edges u_1v, $u_2v, \ldots,$ u_dv respectively; and first $d+1$ d-tuples of columns correspond to the vertices v, $u_1, \ldots,$ u_d respectively. Then $R(G,p)$ is

$$\left(\begin{array}{c|cccccc}
p(v)-p(u_1) & p(u_1)-p(v) & 0\cdots0 & \cdots & 0\cdots0 & \cdots & 0\cdots0 \\
p(v)-p(u_2) & 0\cdots0 & p(u_2)-p(v) & \cdots & 0\cdots0 & \cdots & 0\cdots0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p(v)-p(u_d) & 0\cdots0 & 0\cdots0 & \cdots & p(u_d)-p(v) & \cdots & 0\cdots0 \\
\hline
0\cdots0 & & & & & & \\
\vdots & & & R(H,p|_H) & & & \\
0\cdots0 & & & & & &
\end{array}\right)$$

Then rank $R(G,p)$=rank $R(H,p|_H)$+rank A, where A is the d-by-d matrix obtained from the first d rows and the first d columns of $R(G,p)$. Since $p(v)$, $p(u_1), \ldots,$ $p(u_d)$ do not lie on a $d-1$-dimensional affine subspace we have rank $A = d$ which means the rows of $R(G,p)$ are linearly independent.

$\square$

Lemma 0.1.8. *Let (H, q) be a d-dimensional framework and G be a 1-extension of H with $H = G - v + u_1u_2$ where u_1, u_2 are non-adjacent neighbours of v in G. Let (G, p) be a d-dimensional framework such that $p|_H = q$ and that every algebraic dependency of p arises from $p|_H = q$. Suppose the rows of $R(H, p|_H) = R(H, q)$ are linearly independent. Then the rows of $R(G, p)$ are linearly independent.*

Proof. First note that since G is a d-dimensional 1-extension of H we have $d_G(v) = d + 1$. Let $u_1, u_2, \ldots, u_{d+1}$ be neighbours of v. Let $(G + u_1u_2, \tilde{p})$ be a non-generic realisation of $G + u_1u_2$ obtained by putting $\tilde{p}(z) = p(z)$, $\forall z \neq v$ and $\tilde{p}(v)$ equal to a point on the line through $p(u_1)$ and $p(u_2)$ such that $p(u_1) \neq \tilde{p}(v) \neq p(u_2)$, see Figure 11. Note that this implies $\tilde{p}|_H = p|_H = q$.

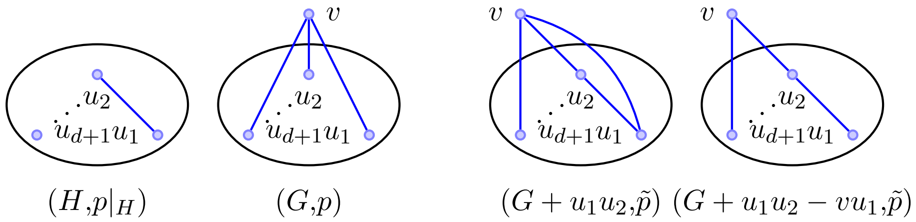

Figure 11: Generic (p) and non-generic $(\tilde{p})$ realizations of G and H.

Since the rows of $R(H, p|_H)$ are linearly independent the rows of $R(G + u_1u_2 - vu_1, \tilde{p})$ are linearly independent by Lemma 0.1.7 (since $\tilde{p}(u_2), \ldots, \tilde{p}(u_{d+1})$, $\tilde{p}(v)$ do not lie on a $d-1$-dimensional affine subspace). Now consider the submatrix of $R(G + u_1u_2, \tilde{p})$ with the rows corresponding to vu_1, vu_2, u_1u_2. Then this submatrix looks like

$$\begin{pmatrix} \tilde{p}(v) - \tilde{p}(u_1) & \tilde{p}(u_1) - \tilde{p}(v) & 0 \cdots 0 & 0 \cdots 0 & \cdots & 0 \cdots 0 \\ \tilde{p}(v) - \tilde{p}(u_2) & 0 \cdots 0 & \tilde{p}(u_2) - \tilde{p}(v) & 0 \cdots 0 & \cdots & 0 \ldots 0 \\ 0 \cdots 0 & \tilde{p}(u_1) - \tilde{p}(u_2) & \tilde{p}(u_2) - \tilde{p}(u_1) & 0 \cdots 0 & \cdots & 0 \cdots 0 \end{pmatrix}$$

Since $\tilde{p}(v)$, $\tilde{p}(u_1)$, $\tilde{p}(u_2)$ are collinear we have

$$\tilde{p}(v) - \tilde{p}(u_1) = a\big(\tilde{p}(v) - \tilde{p}(u_1)\big) = b\big(\tilde{p}(u_1) - \tilde{p}(u_2)\big)$$

for some non-negative scalars a and b. This gives us a dependence of the rows of this submatrix. That is, the rows of $R(G + u_1u_2\tilde{p})$ indexed by vu_1, vu_2, u_1u_2 are linearly dependent. Therefore if we delete one of these rows (one of corresponding edges from the graph) the rank of the matrix will remain the same. Note that if we delete vu_1 the resulting framework will be $(G + u_1u_2 - vu_1, \tilde{p})$ and if we delete u_1u_2 the resulting framework will be $(G, \tilde{p})$. Therefore we have

$$\text{rank } R(G, \tilde{p}) = \text{rank } R(G + u_1u_2 - vu_1, \tilde{p}) = |E(G + u_1u_2 - vu_1)| = |E(G)|$$

where the second equality follows from the fact that the rows of $R(G+u_1u_2-vu_1, \tilde{p})$ are linearly independent. Then rank $R(G, p) \geq$rank $R(G, \tilde{p}) = |E(G)|$ since every algebraic dependency of p arises from $q = p|_H = \tilde{p}|_H$. Hence, rank $R(G, p) = |E(G)|$ implying that the rows of $R(G, p)$ are linearly independent. $\square$

0.2 Some Known Results

A (k, l)-sparse graph $G = (V, E)$ is called (k, l)-*tight* if $|E| = k|V| - l$ holds. Similarly, we say a set $X \subseteq V$ is (k, l)-*tight* if $i_G(X) = k|X| - l$. The 0- and 1-extension moves can be used to characterise rigidity in $\mathbb{R}^1$ and $\mathbb{R}^2$.

Theorem 0.2.1. *A graph G is minimally rigid in $\mathbb{R}^1$ if and only if G is $(1, 1)$-tight.*

Laman [16] in 1970 gave a characterisation for the rigidity of graphs in $\mathbb{R}^2$.

Theorem 0.2.2. *[16] A graph G is minimally rigid in $\mathbb{R}^2$ if and only if G is $(2, 3)$-tight.*

Given a graph $G = (V, E)$, a *cover* of G is a family $\mathcal{X} = \{X_1, X_2, \ldots, X_t\}$ of subsets of V with $|X_i| \geq 2$ for all i such that $\bigcup_{i=1}^{t} E(X_i) = E$. The cover $\mathcal{X}$ is k-*thin* if $|X_i \cap X_j| \leq k$ for all $i \neq j$.

Lovász and Yemini [17] gave the following min-max identity for the rank function in $\mathcal{R}_2$.

Theorem 0.2.3. *[17] Let $G = (V, E)$ be a graph. Then*

$$r_2(G) = \min_{\mathcal{X}} \sum_{X \in \mathcal{X}} (2|X| - 3)$$

where the minimum is taken over all 1*-thin covers* $\mathcal{X}$ *of* G*.*

For the case when $d \geq 3$, the characterisation of rigidity in $\mathbb{R}^d$ remains open. By Theorems 0.2.1 and 0.2.2, when $d = 1, 2$, a graph G is minimally rigid in $\mathbb{R}^d$ if and only if G is $(d, \binom{d+1}{2})$-tight. However, being $(3, 6)$-tight does not guarantee the rigidity of a graph in $\mathbb{R}^3$. To see this consider the graph B_3 in Figure 12. The graph B_3 is $(3, 6)$-tight but it is not rigid. Let us take a generic realisation (B_3, p). We can rotate one of the two copies of $K_5 - uv$ about the line defined by $p(u)$ and $p(v)$ and keep other vertices fixed. Since this is a non-rigid motion, the graph B_3 is not rigid in $\mathbb{R}^3$.

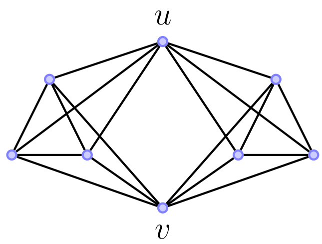

Figure 12: The 3-dimensional double banana graph (B_3).

By Theorems 0.2.1 and 0.2.2, for $d = 1, 2$, a graph $G = (V, E)$ is a circuit in $\mathbb{R}^d$ if and only if $|E| = d|V| - \binom{d+1}{2} + 1$ and $i(X) \leq d|X| - \binom{d+1}{2}$ for all $X \subsetneq V$ with $|X| \geq d$. Hence, for $d = 1, 2$, if G is a circuit in $\mathbb{R}^d$, then G is rigid in $\mathbb{R}^d$. However, this does not hold in higher dimensions. The graph B_3 is dependent in $\mathbb{R}^3$, as it has $|E(B_3)| = 3|V(B_3)| - 6$ edges and it is not rigid in $\mathbb{R}^3$. It can be shown that, for all $e \in E(B_3)$, we can obtain $B_3 - e$ from a single vertex by 0- and 1-extension operations. Therefore $B_3 - e$ is independent in $\mathbb{R}^3$ for all $e \in E(B_3)$ by Lemmas 0.1.7 and 0.1.8. Hence B_3 is a circuit. Since it is not rigid in $\mathbb{R}^3$, it is a non-rigid circuit in $\mathbb{R}^3$.

A non-rigid circuit in $\mathbb{R}^d$ is $(d, \binom{d+1}{2})$-sparse. To see this let $G = (V, E)$ be a non-rigid circuit in $\mathbb{R}^d$. Suppose a set $X \subseteq V$ with $|X| \geq d$ satisfies $i(X) > d|X| - \binom{d+1}{2}$. Since G is a non-rigid circuit, we have $|E| - 1 = r_d(G) < d|V| - \binom{d+1}{2}$. This implies that $X \neq V$. The fact that G is a circuit implies that $G - e$ is independent for all $e \in E$. Combining this with Lemma 0.1.6, we obtain a contradiction to the existence of X. Since a rigid circuit in $\mathbb{R}^d$ is not $(d, \binom{d+1}{2})$-sparse, we can say a circuit in $\mathbb{R}^d$ is non-rigid if and only if it is $(d, \binom{d+1}{2})$-sparse.

Chapter 1

Rigidity of Linearly Constrained Frameworks

1.1 Introduction

In this chapter we will give a combinatorial characterisation for the generic rigidity of frameworks in 3 dimensions each of whose vertices are allowed to move only on a specific plane. We call such a framework a linearly constrained framework. We say a linearly constrained framework is *rigid* if it has no motion. That is, the only continuous motion of the vertices which satisfies the plane constraints for the vertices and the length constraints for the edges is the trivial motion which keeps each vertex fixed. We also say that a linearly constrained framework is *infinitesimally rigid* if it has only the infinitesimal motion which assigns zero velocity to each vertex. We will give precise definitions for these terms later.

We can generalise this problem to d dimensions for all $d \geq 1$. That is, a linearly constrained framework in d-dimensions is a d-dimensional bar-and-joint framework such that each vertex is allowed to move on a specific hyperplane. We denote linearly constrained frameworks by a triple (G, p, q) where $G = (V, E)$ is a graph $p : V \to \mathbb{R}^d$ is the realisation map for the vertices and $q : V \to \mathbb{R}^d$ is the map that assigns unit vectors to the vertices that are normal to the associated hyperplanes. We say (G, p, q) is *generic* if (p, q) is algebraicly independent over the rationals.

In 1-dimension the characterisation of the rigidity of linearly constrained frameworks is straightforward as the only hyperplane contains only the zero vector. This means every linearly constrained framework in 1-dimension is infinitesimally rigid.

In 2 dimensions Streinu and Theran [25] characterised a more general version of the problem. The frameworks they consider may have vertices that are allowed to move along a specific line and vertices that are allowed to move freely in $\mathbb{R}^2$. If we specify that each vertex has exactly one assigned line to move along, then their result implies the following theorem.

Theorem 1.1.1. *[25] A generic linearly constrained framework* (G, p, q) *in* $\mathbb{R}^2$ *is rigid if and only if* G *contains a spanning* $(1, 0)$*-tight subgraph.*

In 3-dimensions some non-generic cases were studied by Nixon, Owen and Power [20, 21]. They worked on frameworks (G, p, q) whose vertices are realised on a surface and the associated plane for each vertex v is the tangent plane of this surface at the point $p(v)$. They classify the surfaces with respect to the number of continuous isometries they have. In this chapter, we will reserve the term ellipsoid for an ellipsoid whose principal axes have different lengths. Similarly, an elliptical cylinder will refer to an elliptical cylinder such that the principal axes of the corresponding ellipse have different lengths. A surface $\mathcal{M}$ is said to be of *type* k, if the dimension of the space of continuous isometries of $\mathcal{M}$ is k. For example, an ellipsoid is of type 0, an elliptical cylinder is of type 1, a circular cylinder is of type 2, a sphere is of type 3.

Definition 1.1.1. A framework on a surface $\mathcal{M}$ is *rigid* on $\mathcal{M}$ if the continuous isometries of $\mathcal{M}$ are the only motions of the framework.

Note that in Theorem 1.1.2 below, we use (G, p) instead of (G, p, q). By having the map p and the surface $\mathcal{M}$, we do not need to specify the map q that assigns the tangent planes to the vertices of the graphs. Also the term generic in this theorem means the every algebraic dependence of p can be obtained from the formula of the surface.

Theorem 1.1.2. *[20, 21] Let* $G = (V, E)$ *be a simple graph and* $\mathcal{M}$ *be an irreducible surface of type* $k = 1, 2$. *Then a generic framework* (G, p) *on* $\mathcal{M}$ *is rigid on* $\mathcal{M}$ *if and only if* G *has a spanning* $(2, k)$*-tight subgraph.*

Before stating the main result of this chapter we need to give a definition.

Definition 1.1.2. A graph G is $(2,0)^*$-*sparse*, respectively $(2,0)^*$-*tight*, if G is K_5-free, and $(2,0)$-sparse, respectively $(2,0)$-tight.

The theorem below is the main result of this chapter.

Theorem 1.1.3. *Let $G = (V, E)$ be a simple graph. Then a generic linearly constrained framework (G, p, q) in $\mathbb{R}^3$ is rigid if and only if G has a spanning subgraph which is $(2,0)^*$-tight.*

The proof of one direction of this theorem is straightforward, that is, if (G, p, q) in $\mathbb{R}^3$ is generic and rigid, then G has a spanning subgraph which is $(2,0)^*$-tight. To prove this we will use the rigidity matrix $R(G, p, q)$ (which will be defined later) of (G, p, q) and some basic properties of matrices.

The proof of the other direction has two parts, one is combinatorial and the other is geometric. For the combinatorial part, we will use some extension, respectively reduction moves that increase, respectively decrease the number of vertices and edges of the graph in consideration and preserve $(2,0)^*$-tightness. We will then give a recursive characterisation of the $(2,0)^*$-tight simple graphs by using these moves starting from a set of base graphs. For the geometric part, we use some results from Nixon, Owen and Power [21] that tell us most of our extension moves preserve independence and infinitesimal rigidity of the linearly constrained framework. We will show the extension moves that are not considered in [21] also preserve the infinitesimal rigidity and independence of the linearly constrained framework.

We will prove the sufficiency direction of Theorem 1.1.3 by induction. The base case of the induction will be a set of minimal $(2,0)^*$-tight simple graphs, for which we will give specific infinitesimally rigid realisations that are calculated by a computer program. We will show that when G is not 4-regular, we can obtain G from a disjoint union of the base graphs by a sequence of extension moves that preserve independence of a linearly constrained framework. When G is 4-regular, we do not know whether the move we use in the recursive construction preserves independence of a linearly constrained framework, so we will use an ad hoc argument based on Theorem 1.1.2.

1.2 Some Properties of $(2,0)^*$-Sparse Graphs

In this section we will derive some basic properties of $(2,0)^*$-tight graphs. Let $G=(V,E)$ be a $(2,0)^*$-sparse graph. We say a set $X \subseteq V$ is $(2,0)^*$-*sparse (-tight)*, if the graph $G[X]$ is $(2,0)^*$-sparse (-tight).

Lemma 1.2.1. *Let $G=(V,E)$ be a $(2,0)^*$-sparse graph. Suppose $X, Y \subseteq V$ are $(2,0)^*$-tight sets. Then $X \cap Y$ and $X \cup Y$ are also $(2,0)^*$-tight.*

Proof: As the graphs $G[X\cap Y]$ and $G[X\cup Y]$ are subgraphs of G and G is K_5-free, we obtain that $G[X\cap Y]$ and $G[X\cup Y]$ are K_5-free. Therefore we only need to show that the sets $X\cap Y$ and $X \cup Y$ are $(2,0)$-tight. Since X and Y are $(2,0)$-tight we have $i(X)=2|X|$ and $i(Y)=2|Y|$. Using the fact that $i: 2^V \to \mathbb{N}$ is supermodular we obtain

$$\begin{aligned}
2|X|+2|Y| =& i(X)+i(Y)\\
\leq& i(X\cap Y)+i(X\cup Y)\\
\leq& 2|X\cap Y|+2|X\cup Y|\\
=& 2|X\cap Y|+2(|X|+|Y|-|X\cap Y|)\\
=& 2|X|+2|Y|,
\end{aligned}$$

implying equality holds throughout. In particular, the second and the third lines are equal and this completes the proof. ■

Lemma 1.2.2. *Let $G=(V,E)$ be a $(2,0)^*$-sparse graph and choose $X, Y \subseteq V$ with $i(X)=2|X|-p$ and $i(Y)=2|Y|-q$. Suppose $i(X\cup Y) \leq 2|X\cup Y|-1$. Then $i(X\cap Y)\geq 2|X\cap Y|-p-q+1$.*

Proof: The supermodularity of $i: 2^V \to \mathbb{N}$ gives

$$\begin{aligned}
2|X|-p+2|Y|-q =& i(X)+i(Y)\\
\leq& i(X\cap Y)+i(X\cup Y)\\
\leq& i(X\cap Y)+2|X\cup Y|-1.
\end{aligned}$$

Hence $2|X| - p + 2|Y| - q \leq i(X \cap Y) + 2|X \cup Y| - 1$. This together with the fact that $|X \cup Y| = |X| + |Y| - |X \cap Y|$ gives the desired result. ■

Lemma 1.2.1 implies that in a $(2, 0)^*$-sparse graph, there is at most one maximal $(2, 0)^*$-tight set.

Lemma 1.2.3. *Let $G = (V, E)$ be the union of two $(2, 0)^*$-sparse graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \{u\}$. Suppose $|E_1| = 2|V_1| - 1$ and $|E_2| = 2|V_2| - 1$ and there are no $(2, 0)^*$-tight sets in G_1 and G_2 that contain u. Then G is $(2, 0)^*$-tight.*

Proof: First note that as G_1 and G_2 are K_5-free and $E_G(V_1 \setminus \{u\}, V_2 \setminus \{u\}) = \emptyset$, the graph G is K_5-free. Therefore we only need to show that G is $(2, 0)$-tight.

For a contradiction let us assume G is not $(2, 0)$-tight. Since $|E| = |E_1| + |E_2| = 2|V_1| + 2|V_2| - 2 = 2|V|$, there must be a set $X \subseteq V$ with $2|X| < i_G(X)$. Let $X_1 = V_1 \cap X$ and $X_2 = V_2 \cap X$. First suppose $u \in X$. Then

$$2|X_1| + 2|X_2| = 2|X| + 2 < i_G(X) + 2 = i_{G_1}(X_1) + i_{G_2}(X_2) + 2.$$

Hence either $2|X_1| < i_{G_1}(X_1) + 1$ or $2|X_2| < i_{G_2}(X_2) + 1$. Then either X_1 is $(2, 0)$-tight in G_1 or X_2 is $(2, 0)$-tight in G_2. Since $u \in X_1$ and $u \in X_2$, this gives a contradiction. Now suppose $u \notin X$. Then

$$2|X_1| + 2|X_2| = 2|X| < i_G(X) = i_{G_1}(X_1) + i_{G_1}(X_2).$$

Therefore either $2|X_1| < i_{G_1}(X_1)$ or $2|X_2| < i_{G_2}(X_2)$, contradicting the fact that G_1 and G_2 are $(2, 0)$-sparse. ■

Lemma 1.2.4. *Suppose $G = (V, E)$ is a $(2, 0)^*$-tight graph. Then $\delta(G) \geq 2$.*

Proof: Suppose the contrary. Let $v \in V$ be with $d(v) \leq 1$. Then we have $2|V \setminus \{v\}| = 2|V| - 2 = i(V) - 2 < i(V \setminus \{v\})$, a contradiction. ■

1.3 Graph Operations

In this section we will introduce some extension moves that preserve $(2,0)^*$-tightness. We will also describe some special cases for which the inverse moves preserve $(2,0)^*$-sparsity.

1.3.1 Henneberg Moves

Figure 1.1 illustrates (2-dimensional) 0- and 1-extensions which are defined in Chapter 0. Note that we normally use a d-dimensional 0- or 1-extension for bar-and-joint frameworks in $\mathbb{R}^d$. In this chapter we will use 2-dimensional version of these moves even though we are working in $\mathbb{R}^3$. This is because of the fact that each vertex has two degrees of freedom in a linearly constrained framework in $\mathbb{R}^3$ whereas a vertex in a bar-and-joint framework has three degrees of freedom in $\mathbb{R}^3$.

We refer to the inverse operation of the 0-extension as a 0-*reduction*. Namely, the 0-reduction operation removes a vertex of degree two and its incident edges from the original graph. We call the inverse operation of a 1-extension a 1-*reduction* operation. A 1-reduction deletes a vertex of degree three that has a pair of non-adjacent neighbours x, y and then adds the edge xy.

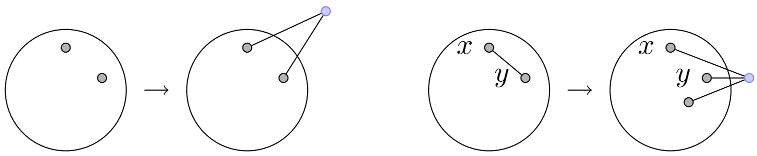

Figure 1.1: 0- and 1-extensions.

Lemma 1.3.1. *Let $G = (V, E)$ be a graph. Suppose $H = (V \cup \{v\}, F)$ can be obtained from G by a 0-extension. Then G is $(2,0)^*$-tight if and only if H is $(2,0)^*$-tight.*

Proof: Since G is a subgraph of H and $d_H(v) = 2$, it is straightforward that $(2,0)^*$-tightness of H implies $(2,0)^*$-tightness of G.

Now suppose G is $(2,0)^*$-tight but H is not. Since G is K_5-free and the only vertex $v \in V(H) \setminus V(G)$ has degree 2 in H, the graph H is K_5-free. Therefore we only need to show that H is $(2,0)$-tight. Moreover since $|E(H)| = 2|V(H)|$, it is enough to show that H is $(2,0)$-sparse.

Suppose the contrary and let $X \subset V(H)$ be a set with $v \in X$ and $2|X| < i_H(X)$. Then we have

$$2|X \setminus \{v\}| = 2|X| - 2 < i_H(X) - 2 = i_H(X \setminus \{v\}) = i_G(X \setminus \{v\}),$$

contradicting the fact that G is $(2,0)$-sparse. ■

Lemma 1.3.2. *Let $G = (V,E)$ be a $(2,0)^*$-tight graph. Suppose $H = (V \cup \{v\}, F)$ can be obtained from G by a 1-extension. Then H is $(2,0)^*$-tight.*

Proof: First note that as $d_H(v) = 3$ and G is K_5-free, the graph H is K_5-free. Therefore we only need to show that H is $(2,0)$-tight. Moreover, since $|E(H)| = 2|V(H)|$, it is enough to show that H is $(2,0)$-sparse.

Suppose the contrary. We may assume xy is the deleted edge under the 1-extension operation. Then there exists a set X with $v \in X$ and $2|X| < i_H(X)$, as $H - v$ is a subgraph of G. If $|X \cap N_H(v)| \leq 2$, then we would have $2|X \setminus \{v\}| < i_G(X \setminus \{v\})$, contradicting the fact that G is $(2,0)$-sparse. Hence we may assume $N(v) \subseteq X$. This implies $x, y \in X$ and so

$$2|X \setminus \{v\}| = 2|X| - 2 < i_H(X) - 2 = i_H(X \setminus \{v\}) + 1 = i_G(X \setminus \{v\}),$$

contradicting the fact that G is $(2,0)$-sparse. ■

Lemma 1.3.3. *Let $H = (V \cup \{v\}, F)$ be a $(2,0)$-sparse graph. Suppose $d_H(v) = 3$ and the closed neighbourhood of v, $N_H[v]$, does not induce a copy of K_4. Then there exists a $(2,0)$-sparse graph $G = (V,E)$ which can be obtained from H by a 1-reduction operation at v.*

Proof: Suppose none of the possible reductions at v gives a $(2,0)$-sparse graph. Let x, y, z be neighbours of v in H. Since $N_H[v]$ does not induce a copy of K_4, at least one of the edges xy, xz, yz is missing in H.

Suppose exactly one of these edges, say xy, is missing. Then there must be a $(2,0)$-tight set X in $H-v$ with $x, y \in X$, as otherwise, applying a 1-reduction at v which adds the edge xy would give a $(2,0)$-sparse graph. If $z \in X$, then adding v and its three incident edges to $H-v$ the set $X \cup \{v\}$ breaks the $(2,0)$-sparsity of H. Therefore we may assume $z \notin X$. Then we have $i_H(X \cup \{z, v\}) = i_H(X) + 2 + 3 = 2|X| + 2 + 3 = 2|X \cup \{z, v\}| + 1$. In particular, $2|X \cup \{z, v\}| < i_H(X \cup \{z, v\})$, contradicting the $(2,0)$-sparsity of H.

Suppose at least two of xy, xz, yz are missing. Then as above there must be $(2,0)$-tight sets X and Y in H for each missing edge. Using Lemma 1.2.1, we have $X \cup Y$ is $(2,0)$-tight. Clearly $x, y, z \in X \cup Y$. Then

$$i_H(X \cup Y \cup \{v\}) = i_H(X \cup Y) + 3 = 2|X \cup Y| + 3 = 2|X \cup Y \cup \{v\}| + 1.$$

In particular, $2|X \cup Y \cup \{v\}| < i_H(X \cup Y \cup \{v\})$, contradicting $(2,0)$-sparsity of H.

$\blacksquare$

We use K_5^- to denote a copy of the graph on five vertices with nine edges.

Lemma 1.3.4. *Let H be a $(2,0)^*$-tight graph and $v \in V$ with $d(v) = 3$. Suppose $i(N(v)) \leq 1$. Then there exists a 1-reduction at v resulting in a $(2,0)^*$-tight graph G.*

Proof: Suppose the contrary and and let $N(v) = \{a, b, c\}$. We may assume $ab, ac \notin E(H)$. Then we see that the pairs of vertices a, b and a, c are contained in either a copy of K_5^- or a $(2,0)^*$-tight set in $H-v$.

First assume both pairs a, b and a, c are contained in $(2,0)^*$-tight sets X and Y in H, respectively. Then by Lemma 1.2.1, $X \cup Y$ is $(2,0)^*$-tight. This implies that for the set $Z := X \cup Y \cup \{v\}$, $i_H(Z) = 2|Z| + 1$, a contraction as H is $(2,0)$-sparse.

Next assume both pairs a, b and a, c are contained in copies of K_5^- in H with vertex sets X and Y, respectively. As there is only one missing edge in a K_5^- and $ab, ac \notin E(H)$, $X \neq Y$ holds. If $X \cup Y$ is $(2,0)^*$-tight, we would get a contradiction to the $(2,0)$-sparsity of H as in the previous paragraph. Hence we may assume $X \cup Y$ is not $(2,0)^*$-tight, and so $i_H(X \cup Y) \leq 2|X \cup Y| - 1$ holds. Then by Lemma 1.2.2, $i_H(X \cap Y) \geq 2|X \cap Y| - 1$. As $|X \cap Y| \leq 4$ and the only graph K on at most four vertices that satisfies $i(K) \geq 2|V(K)| - 1$ is the empty graph, we have $|X \cap Y| = 0$, contradicting the fact that $a \in X \cap Y$.

Finally assume one of the pairs of vertices, say a, b, belongs to a K_5^- with vertex set X and the other pair a, c belongs to a $(2,0)^*$-tight set Y in $H - v$. As in the previous paragraphs, $X \cup Y$ cannot be $(2,0)$-tight due to $(2,0)$-sparsity of H and hence $i(X \cup Y) \leq 2|X \cup Y| - 1$. Then by Lemma 1.2.2, $i(X \cap Y) = 2|X \cap Y|$. However, this is a contradiction as $a \in X \cap Y$, X induces a K_5^- which has no $(2,0)$-tight subgraph other than the empty graph. ■

1.3.2 P_3-to-C_4 and K_2-to-K_3 Moves

Let $G = (V, E)$ be a graph and v be a vertex with incident edges $vu_0, vu_1, \ldots, vu_k$, $vw_0, vw_1, \ldots, vw_m$. The *P_3-to-C_4* move at v removes the edges $vw_1, \ldots, vw_k$ and adds a new vertex v' with incident edges $v'u_0, v'w_0, v'w_1, \ldots, v'w_k$.

The *K_2-to-K_3* move removes the edges $vw_1, \ldots, vw_k$ and adds a new vertex v' with incident edges $v'u_0, v'w_1, \ldots, v'w_k$.

Figure 1.2 illustrates these moves. Both moves are also referred to as *vertex split* moves in the literature, see [31].

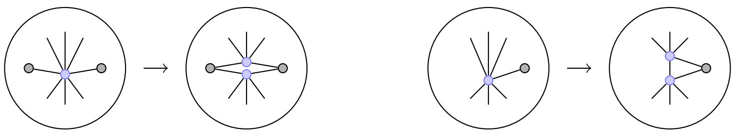

Figure 1.2: P_3-to-C_4 on the left and K_2-to-K_3 on the right. The edges whose second endpoint is undefined may or may not exist. Also the number of such edges is arbitrary.

Lemma 1.3.5. *Let $G = (V, E)$ be a $(2,0)^*$-tight graph. Suppose $H = (V \cup \{v'\}, F)$ is obtained from G by a P_3-to-C_4 move or a K_2-to-K_3 move. Then H is also $(2,0)^*$-tight.*

Proof: It is easy to see that H is K_5-free. We will show that H is $(2,0)$-tight and this will complete the proof. Let v be the vertex we split in G. Suppose H is not $(2,0)$-tight. Then as $|E(H)| = 2|V(H)|$, H is not $(2,0)$-sparse and there exists a set $X \subseteq V(H)$ with $2|X| < i_H(X)$. If $v, v' \in X$ then we have $2|X \setminus \{v'\}| = 2|X| - 2 < i_H(X) - 2 = i_G(X \setminus \{v'\})$, contradicting the $(2,0)$-sparsity of G. If $v, v' \notin X$, then $2|X| < i_H(X) = i_G(X)$, contradicting the $(2,0)$-sparsity of G. Hence we may assume exactly one of v and v' is contained in X. Let x denote this vertex. Then we have $2|X| < i_H(X) \le i_G(X \setminus \{x\} \cup \{v\})$, a contradiction. ■

We refer to the inverse operations of the P_3-to-C_4 and the K_2-to-K_3 moves as *C_4-to-P_3* and *K_3-to-K_2*, respectively. We can sometimes use the C_4-to-P_3 move as an alternative to 1-reduction when there are no possible 1-reductions at a degree three vertex.

Lemma 1.3.6. *Let H be a $(2,0)^*$-tight graph and $v \in V(H)$ with $d(v) = 3$. Suppose $N_H[v]$ induces a copy of K_4 that is not contained in a K_5^- in H and there exists a vertex x with $vx \notin E(H)$ and $|N(x) \cap N(v)| = 2$. Then applying a C_4-to-P_3 move which identifies x and v results in a $(2,0)^*$-tight graph G.*

Proof: Let $N(v) = \{a, b, c\}$ and see Figure 1.3 for an illustration.

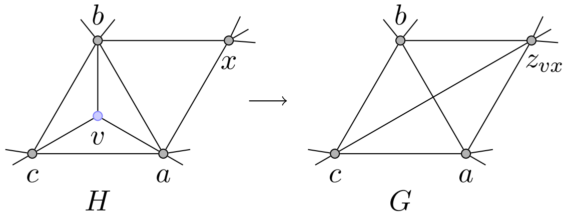

Figure 1.3: A C_4-to-P_3 move on the C_4 whose vertices are v, a, x, b.

If G is $(2,0)$-sparse, then by the edge count G is $(2,0)$-tight. Hence we may assume that G is either not $(2,0)$-sparse or G contains a copy of K_5.

Then there exists a set $X \subseteq V(G)$ such that either $2|X| < i_G(X)$ holds or X induces a copy of K_5. Let z_{vx} denote the vertex obtained from contracting x and v. Since $G - z_{vx}$ and $G - c$ are isomorphic to subgraphs of H and H is $(2,0)^*$-sparse, we have $z_{vx}, c \in X$. Let us set $X' := X \setminus \{z_{vx}\} \cup \{x\}$.

First suppose $a, b \in X$. If $2|X| < i_G(X)$ holds, then we have

$$2|X' \cup \{v\}| = 2|X| + 2 < i_G(X) + 2 = i_H(X' \cup \{v\}).$$

In particular, $2|X' \cup \{v\}| < i_H(X' \cup \{v\})$, a contradiction to the $(2,0)$-sparsity of H. If X induces a copy of K_5 in G, then the set X' induces a copy of K_5^- in H.

The facts that $a, b, c, x \in X'$ and cx is the missing edge of this K_5^- imply that $N[v]$ is contained in a K_5^-, a contradiction.

For the remaining cases we will consider the possibilities $2|X| < i_G(X)$ and X induces a copy of K_5 together. As $i(K_5) = 2|V(K_5)|$, we can combine these possibilities and obtain $2|X| \leq i_G(X)$.

Suppose one of a, b say $a \in X$. Then we have

$$2|X' \cup \{v, b\}| = 2|X| + 4 \leq i_G(X) + 4 = i_H(X' \cup \{v, b\}) - 1.$$

In particular $2|X' \cup \{v, b\}| < i_H(X' \cup \{v, b\})$, a contradiction.

Finally suppose $a, b \notin X$. Then we have

$$2|X' \cup \{v, a, b\}| = 2|X| + 6 \leq i_G(X) + 6 = i_H(X' \cup \{v, a, b\}) - 1.$$

In particular, $2|X' \cup \{v, a, b\}| < i_H(X' \cup \{v, a, b\})$, a contradiction. Hence we conclude that G is $(2, 0)^*$-tight. ■

Let $G = (V, E)$ be a graph and $F \subset E$. We say a subgraph H of G is *generated* by F, if the set of endpoints of the edges in F induce H in G.

Lemma 1.3.7. *Let H be a $(2, 0)^*$-tight graph obtained from the disjoint union of K_5^- and some arbitrary graph H' by adding two edges e, f between K_5^- and H' that generate a copy of K_3 or C_4 in H, see Figure 1.4. If $\{e, f\}$ generates a C_3, then there exists a K_3-to-K_2 move on this K_3 that results in a $(2, 0)^*$-tight graph G. If $\{e, f\}$ generates a C_4, then there exists a C_4-to-P_3 move on this C_4 that results in a $(2, 0)^*$-tight graph G.*

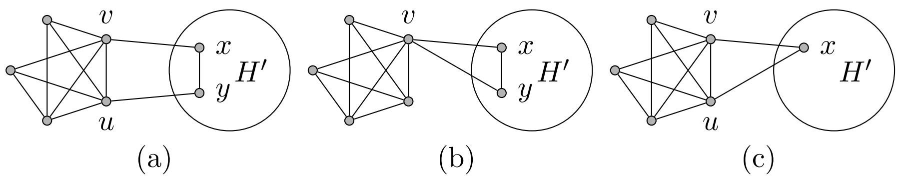

Figure 1.4: The edges joining K_5^- to H' are arbitrary as long as they induce a K_3 or a C_4 in H.

Proof: First note that since H is $(2,0)^*$-tight, the facts that $E(K_5^-) = 2|V(K_5^-)| - 1$ and there are two edges joining K_5^- to H' imply $E(H') = 2|V(H')| - 1$. First consider the case illustrated in Figure 1.4 (c). We will contract the edge vx (contracting ux works as well). Let z_{vx} denote the modified vertex. If there is a $(2,0)^*$-tight set X in H' with $x \in X$, then we have

$$2|X \cup V(K_5^-)| = 2|X| + 10 = i_{H'}(X) + 10 = i_H(X \cup V(K_5^-)) - 1.$$

In particular, we have $2|X \cup V(K_5^-)| < i_H(X \cup V(K_5^-))$, a contradiction. Hence we may assume that there are no $(2,0)^*$-tight sets in H' containing x. Now if we set $G_1 = G[V(K_5^-) \setminus \{v\} \cup \{z_{vx}\}]$ and $G_2 = G[V(H') \setminus \{x\} \cup \{z_{vx}\}]$, and apply Lemma 1.2.3, we deduce that G is $(2,0)^*$-tight.

Now consider the cases (a) and (b) illustrated in Figure 1.4. First note that for case (a), there are two possible C_4-to-P_3 moves that contract vertices u and x or v and y. Similarly, for case (b), there are two possible K_3-to-K_2 moves that contract the edges vx or vy.

Note also that, for both (a) and (b), there cannot be a $(2,0)^*$-tight set X in H' with both $x, y \in X$, as adding $V(K_5^-)$ to X would give

$$2|X \cup V(K_5^-)| = 2|X| + 10 < i_{H'}(X) + 10 = i_H(X \cup V(K_5^-)) - 1.$$

In particular, we would have $2|X \cup V(K_5^-)| < i_H(X \cup V(K_5^-))$, a contradiction. Combining this with the fact that the union of any two $(2,0)^*$-tight sets is $(2,0)^*$-tight in a $(2,0)^*$-sparse graph (Lemma 1.2.1), we deduce that either x or y, say y, is not contained in a $(2,0)^*$-tight set in H'. Let z_{vy} denote the modified vertex for the C_4-to-P_3 move which contracts v and y in case (a), and for the K_3-to-K_2 move which contracts vy in case (b). Then if we set $G_1 = G[V(K_5^-) \setminus \{v\} \cup \{z_{vy}\}]$ and $G_2 = G[V(H') \setminus \{y\} \cup \{z_{vy}\}]$, and apply Lemma 1.2.3, we deduce that G is $(2,0)^*$-tight. ■

Lemma 1.3.8. *Let H be a $(2,0)^*$-tight graph obtained from the disjoint union of two graphs K and H' by adding two edges $e = vx, f = vy$ between K and H' where $v \in V(K)$ and $x, y \in V(H')$. Suppose $xy \notin E(H')$ and x and y are contained in a* *subgraph K' of H and that K and K' are both isomorphic to K_5^-. Then there exists a C_4-to-P_3 move that identifies v and t that results in a $(2,0)^*$-tight graph G where $t \in V(K') \setminus \{x, y\}$.*

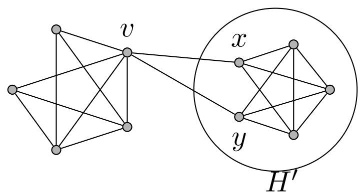

Figure 1.5: The vertex v in the K_5^- on the left is arbitrary as long as it is adjacent to both x and y in H'.

Proof: Let z_{vt} denote the modified vertex and H'_z denote the graph obtained from H' by relabelling t as z_{vt}. We also let S denote the vertex set of K.

First assume that z_{vt} is contained in a $(2,0)^*$-tight set X_z in H'_z. Then the set $X' := X_z \setminus \{z_{vt}\} \cup \{t\}$ is $(2,0)^*$-tight in H'. Since $t \in V(K')$ and $K' \cong K_5^-$, the set X' must contain every vertex of K', as otherwise, adding the remaining vertices of this K' to X' would break $(2,0)$-sparsity of H'. In particular, $x, y \in X'$. Then adding S to X', we obtain

$$2|S \cup X'| = 2|S| + 2|X'| = 2 \cdot 5 + i_{H'}(X') = 10 + i_H(X') = i_H(S \cup X') - 1.$$

In particular $2|S \cup X'| < i_H(S \cup X')$, contradicting the $(2,0)$-sparsity of H.

We next assume that z_{vt} is not contained in a $(2,0)^*$-tight set in H'_z. The graph H'_z is $(2,0)^*$-sparse as it is isomorphic to H'. Let $S_z := S \setminus \{v\} \cup \{z_{vt}\}$. We also know that $G[S_z]$ is $(2,0)^*$-sparse and satisfies $|E(G[S_z])| = 2|V(G[S_z])| - 1$ as it is isomorphic to K_5^-. We now set $G_1 := H'_z$ and $G_2 := G[S_z]$ and apply Lemma 1.2.3 to complete the proof. ■

Let $G_6^1, \ldots, G_6^4$ denote the $(2,0)^*$-tight graphs on six vertices: see the graphs on the top row in Figure 1.13 for an illustration. These are the $(2,0)^*$-tight graphs with the minimum number of vertices. We will use G_6^* to denote an arbitrary graph from this family.

Lemma 1.3.9. *Let H be a $(2,0)^*$-tight graph and $v \in V$ with $d(v) = 3$. Suppose v is not contained in a G_6^*, $i(N(v)) = 2$, and the missing edge in $H[N(v)]$ belongs to a K_5^-. Then there exists a C_4-to-P_3 move in H resulting in a $(2,0)^*$-tight graph G.*

Proof: We may assume $N(v) = \{a, b, c\}$ and some set $X \subset V(H)$ containing a, b induces a K_5^- with the property that ab is the missing edge. Then as $i(N(v)) = 2$, we have $ac, bc \in E(H)$. Since v is not contained in a G_6^*, we have $c \notin X$, see Figure 1.6 for an illustration.

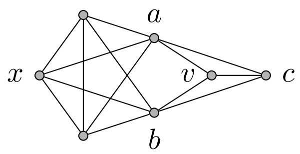

Figure 1.6: The subgraph $H[X \cup N[v]]$.

Let $x \in X \setminus \{a, b\}$ be a vertex. We will show that performing the C_4-to-P_3 move on the vertices x, a, b, v by identifying x and v will result in a $(2,0)^*$-tight graph G. First note that as $v, c \notin X$, X induces a K_5^-, v has two neighbours in X, and c and v are adjacent, the vertex c has at most two neighbours in X. Otherwise, the set $X \cup \{c, v\}$ would break $(2,0)$-sparsity. Combining this with the fact that $a, b \in X$ are two neighbours of c, we obtain $xc \notin E(H)$.

Suppose performing the C_4-to-P_3 move on the vertices x, a, b, v by identifying x and v does not result in a $(2,0)^*$-tight graph G. Let z_{xv} denote the vertex in G we obtain after identifying x and v. Since $G - z_{xv}c$ is a subgraph of H, x and c must be contained in either a copy of K_5^- or a $(2,0)$-tight set in $H - v$. First assume there exists a copy of K_5^- containing both x and c. Let Y be the vertex set of this K_5^-. We may assume that $X \cup Y$ is not $(2,0)$-tight, as otherwise, adding v to this set with three incident edges would break $(2,0)$-sparsity of H. Hence $i(X \cup Y) \leq 2|X \cup Y| - 1$. Then by Lemma 1.2.2 we have $i(X \cap Y) \geq 2|X \cap Y| - 1$. However, this is a contradiction as $X \cap Y$ is non-empty and the only graph K on at most four vertices with $i(K) \geq 2|V(K)| - 1$ is the empty graph.

Now assume there exists a $(2,0)$-tight set Y with $x, c \in Y$. Again due to $(2,0)$-sparsity of H, we must have $i(X \cup Y) \leq 2|X \cup Y| - 1$. Then by Lemma 1.2.2, we have $i(X \cap Y) = 2|X \cap Y|$. However, since $X \cap Y$ is non-empty, has size at most five and H is K_5-free, this is a contraction. ■

1.3.3 Vertex-to-K_4 Move

Let $G = (V, E)$ be a graph and $v \in V$. A *vertex-to-K_4* move at v replaces v by a copy of K_4 and replaces each edge vu by an edge xu where x is an arbitrary vertex of the K_4 we just created, see Figure 1.7.

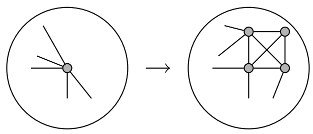

Figure 1.7: Vertex-to-K_4 move. Note that for each edge incident with the vertex we replace by a K_4, we are free to choose any vertex of the K_4 as its second endpoint after replacing the vertex by the K_4.

Lemma 1.3.10. *Let $G = (V, E)$ be a $(2,0)^*$-tight graph and $v \in V$. Suppose $H = (V \setminus \{v\} \cup \{v_1, v_2, v_3, v_4\}, F)$ is obtained from G by a vertex-to-K_4 move. Then H is $(2,0)^*$-tight.*

Proof: Suppose H is not $(2,0)$-tight. Then since $|E(H)| = 2|V(H)|$, there exists a set $X \subseteq V(H)$ with $2|X| < i_H(X)$. Let $k = |X \cap \{v_1, v_2, v_3, v_4\}|$. If $k = 0$, then $2|X| < i_H(X) = i_G(X)$, a contradiction. Hence $k \geq 1$. Set $S = X \setminus (X \cap \{v_1, v_2, v_3, v_4\}) \cup \{v\}$. Then $2|S| = 2|X| - 2k + 2 < i_H(X) - 2k + 2 \leq i_H(X) - \binom{k}{2} = i_G(S)$ holds, since $2k + 2 \geq \binom{k}{2}$ for $1 \leq k \leq 4$. In particular, $2|S| < i_G(S)$, a contradiction. Hence H is $(2,0)$-tight.

It remains to show that H is K_5-free. Suppose there is a copy of K_5 in H with vertex set X. Since the graph $H[V \setminus \{v\} \cup \{v_i\}]$ is isomorphic to a subgraph of G for all $1 \leq i \leq k$ and G is K_5-free, we obtain $|X \cap \{v_1, v_2, v_3, v_4\}| \geq 2$. However, as v_i, v_j, for all $1 \leq i < j \leq 4$, do not have a common neighbour in $V \setminus \{v\}$, the set X cannot induce a copy of K_5, a contradiction. Hence H is K_5-free and so $(2,0)^*$-tight. ■

We refer to the inverse operation of the vertex-to-K_4 move as a K_4-*contraction*. We will denote the graphs in Figure 1.8 by $K_4 \circ K_4$. These are the graphs obtained from a K_5 by a vertex-to-K_4 move that do not contain a K_5. One can see that for each graph in Figure 1.8, contracting the K_4 on the left results in a K_5.

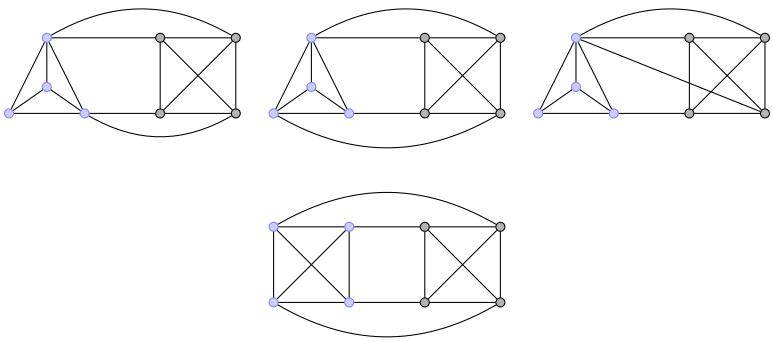

Figure 1.8: The graphs $K_4 \circ K_4$.

Lemma 1.3.11. *Let H be a $(2,0)^*$-tight graph and $T \subset V$ induce a K_4 in H. Suppose that $|N_H(x) \cap T| \leq 1$, for all $x \in V(H) \setminus T$ and that T is not contained in a $K_4 \circ K_4$ in H. Then contracting the K_4 induced by T gives a $(2,0)^*$-tight graph G.*

Proof: First note that the fact that T is not contained in a $K_4 \circ K_4$ in H imply that G is K_5-free. Therefore we only need to show that G is $(2,0)$-tight.

Suppose G is not $(2,0)$-tight. Since $|E(G)| = 2|V(G)|$ we may assume that G is not $(2,0)$-sparse. Thus there exists a set $X \subseteq V(G)$ with $2|X| < i_G(X)$. Then $z \in X$ where z represents the vertex arising from contracting the K_4 spanned by T, as otherwise $G[X]$ would be a subgraph of H. This gives

$$2|X \setminus \{z\} \cup T| = 2|X| + 6 < i_G(X) + 6 = i_H(X \setminus \{z\} \cup T),$$

since $|E(H[T])| = 6$. In particular, $2|X \setminus \{z\} \cup T| < i_H(X \setminus \{z\} \cup T)$, contradicting the fact that H is $(2,0)$-sparse.

Hence we conclude that G is $(2,0)$-tight, and so $(2,0)^*$-tight. ■

1.3.4 2-Extension Move

Let $G = (V, E)$ be a graph and $xy, zt \in E$ be two non-adjacent edges. A 2-*extension move* also known as X-*replacement* in the literature, see [31], is an operation that removes xy and zt and adds a new vertex v with incident edges vx, vy, vz, vt, see Figure 1.9. The inverse operation of a 2-extension move is called a 2-*reduction*. Namely, the 2-reduction operation in a graph G at a vertex v of degree four with neighbours x, y, z, t and missing edges xy, zt, removes v and adds xy and zt to the edge set.

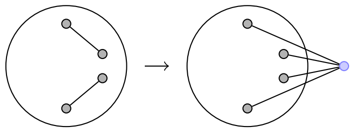

Figure 1.9: A 2-extension move.

Lemma 1.3.12. *Let $G = (V, E)$ be a $(2, 0)^*$-tight graph with non-adjacent edges $xy, zt \in E$. Suppose $H = (V \cup \{v\}, F)$ is obtained from G by a 2-extension move on xy and zt. Then H is also $(2, 0)^*$-tight.*

Proof: First note that as $H - v$ is a subgraph of G and there are at least two missing edges in $H[N[v]]$, the graph H is K_5-free. Thus we only need to show that H is $(2, 0)$-tight. Since $|E(H)| = 2|V(H)|$ holds it is enough to show that H is $(2, 0)$-sparse.

Suppose H is not $(2, 0)$-sparse. Therefore there exists a set $X \subseteq V(H)$ with $2|X| < i_H(X)$. If $v \notin X$, then $X \subseteq V(G)$ and we have $2|X| < i_H(X) \leq i_G(X)$, a contradiction.

Hence we may assume $v \in X$. If at most two neighbours of v are in X, then $2|X \setminus \{v\}| = 2|X| - 2 < i_H(X) - 2 \leq i_G(X \setminus \{v\})$ holds. In particular we have $2|X \setminus \{v\}| < i_G(X \setminus \{v\})$, contradicting the $(2, 0)$-sparsity of G. If three neighbours of v are in X, then $H[X]$ can be obtained from $G[X \setminus \{v\}]$ by a 1-extension move. By Lemma 1.3.2 this gives a contradiction to the fact that G is $(2, 0)$-sparse. If all four neighbours x, y, z, t of v are in X, then $2|X \setminus \{v\}| = 2|X| - 2 < i_H(X) - 2 = i_G(X \setminus \{v\})$ holds. In particular, $2|X \setminus \{v\}| < i_G(X \setminus \{v\})$, a contradiction.

Hence we conclude that H is $(2,0)$-tight and so $(2,0)^*$-tight. ■

Lemma 1.3.13. *Let $G = (V, E)$ be a 4-regular graph. Then G is $(2,0)$-tight.*

Proof: Since G is 4-regular, we have $|E| = 2|V|$. Hence showing that G is $(2,0)$-sparse is enough.

For a contradiction suppose G is not $(2,0)$-sparse and let $X \subset V$ be a set with $i(X) > 2|X|$. This implies that the graph $G[X]$ has average degree strictly bigger than four. Therefore there exists a vertex $x \in X$ with $d_{G[X]}(x) > 4$. Since G is 4-regular and $G[X]$ is a subgraph of G, this gives a contraction. ■

Lemma 1.3.14. *Let H be a connected $(2,0)^*$-tight 4-regular graph and $v \in V(H)$. Suppose H is K_4-free. Then either H is the unique 4-regular graph on six vertices or there exists a 2-reduction move at a vertex $x \in N_H[v]$ that results in a $(2,0)^*$-tight graph G.*

Proof: Let $N_H(v) = \{a, b, c, d\}$ be the neighbourhood of v. First note that the 2-reduction move preserves 4-regularity. Note also that since H is K_4-free, if a 2-reduction move creates a copy of K_5, then this K_5 must contain the two edges added after the 2-reduction. Combining these with 4-regularity and the connectivity of H, we deduce that H is the unique 4-regular graph on six vertices.

Therefore we may assume that if there is a 2-reduction move at $x \in N_H[v]$, then the resulting graph G is K_5-free. Combining this with Lemma 1.3.13 and the fact that 2-reduction move preserves 4-regularity, we obtain that if we can find a vertex at which we can apply a 2-reduction move, then this must result in a $(2,0)^*$-tight graph G. Hence we may assume that we cannot apply a 2-reduction move at v. This is possible only when, for every pair of distinct missing edges $e_1, e_2 \in E(N_H(v))$, the edges e_1 and e_2 have a common endpoint. This together with the fact that H is K_4-free implies that up to isomorphism there is only one possibility for $H[N[v]]$ which is shown in Figure 1.10.

Now consider vertex b in Figure 1.10. Let $N(b) = \{y, z, v, d\}$. Since H is 4-regular and v, d have four neighbours in $N_H[v]$, we see that $E_H(\{y, z\}, \{v, d\}) = \emptyset$. Thus we can apply a 2-reduction move at b which removes b and adds the edges yv

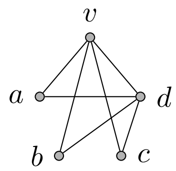

Figure 1.10: The only case when we cannot apply 2-reduction at v up to isomorphism.

and zd to obtain a $(2,0)^*$-tight graph G. ■

Lemma 1.3.15. *Let H be a connected $(2,0)^*$-tight 4-regular graph. Suppose that there is a K_4 with vertex set T in H that is not contained in a K_5^- and that there is a vertex x with $|N_H(x) \cap T| = 2$. Then there exists a 2-reduction move at x that results in a $(2,0)^*$-tight graph G.*

Proof: Let a, b denote the two neighbours of x in T and y, z denote the two neighbours of x in $V \setminus T$. As each of a, b has three incident edges in $H[T]$ and there is no K_5^- containing the vertices in T, we see that there is no K_5^- containing a or b. Then since $N_H(x) = \{a, b, y, z\}$ and there are no edges from $\{a, b\}$ to $\{y, z\}$, we can apply a 2-reduction move at x that removes x, and adds edges ay and bz to obtain a graph G. Since this operation preserves 4-regularity G is $(2,0)$-tight by Lemma 1.3.13. Note that since there is no K_5^- containing a or b, and there are no edges in H from $\{a, b\}$ to $\{y, z\}$, we cannot obtain a copy of K_5 in G after the 2-reduction at x. Therefore G is indeed $(2,0)^*$-tight. ■

Lemma 1.3.16. *Let H be a 4-regular $(2,0)^*$-tight graph. Suppose the set $X \subset V(H)$ induces a K_5^- in H with uv being the missing edge of this K_5^-. Suppose further that there exists a vertex $x \in V(H) \setminus X$ that is adjacent to both u and v. Then there exists a 2-reduction move at x that results in a $(2,0)^*$-tight graph G.*

Proof: Let $N_H(x) = \{u, v, z, t\}$. Since H is 4-regular, the copy of K_5^- induced by X is the only copy of K_5^- that contains u or v and $E_H(\{u, v\}, \{z, t\}) = \emptyset$. Then we can apply a 2-reduction move at x which removes the vertex x and adds the edges uz, vt to obtain a graph G. Since this operation preserves 4-regularity G is $(2,0)$-tight by Lemma 1.3.13. The facts that u and v are contained only in one K_5^-, and $E(\{u,v\},\{z,t\}) = \emptyset$ imply this operation does not create a copy of K_5. Thus G is indeed $(2,0)^*$-tight. ■

Lemma 1.3.17. *Let H be a 4-regular $(2,0)^*$-tight graph that contains the graph in Figure 1.11 as a subgraph. Suppose the set $\{x,y,z,t\}$ does not induce a K_4. Then there exists a 2-reduction move at x resulting in a $(2,0)^*$-tight graph G.*

Proof: First note that combining the fact that 2-reduction move preserves 4-regularity and Lemma 1.3.13 we only need to find a 2-reduction move at x that does not create a K_5. Note also that as H is 4-regular a vertex in H belongs to at most one K_5^-.

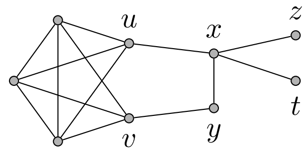

Figure 1.11: The vertices x, y, z, t do not induce a K_4.

We claim that y cannot be contained in a K_5^-. To see this suppose the contrary. Then this K_5^- must contain x, y and must not contain u, v due to 4-regularity. Since the edge xy is present in H, xy would not be the missing edge of this K_5^-. Hence x or y, say y, must have four neighbours in this K_5^-. Since the K_5^- that contains x, y does not contain v, this forces y to have degree at least five in H, a contraction as H is 4-regular.

Since $H[\{x,y,z,t\}] \ncong K_4$, at least one of the edges yz, yt, zt is missing. First assume yt or yz, say yt is missing. Then as y does not belong to a K_5^- we can now apply a 2-reduction move at x which removes the vertex x and adds the edges yt and uz and obtain a K_5-free graph G.

Now assume zt is missing. Due to 4-regularity we have $E_H(\{u\},\{y,z,t\}) = \emptyset$. If there is no K_5^- containing both z and t in H, then we can apply a 2-reduction at x which removes the vertex x, and adds the edges zt, uy to obtain a graph G. If there is a K_5^- containing z, t induced by a set $T \subset V(H)$, then we have $x, y \notin T$, since H is 4-regular and $zt \notin E(H)$. As otherwise, x, y must have four neighbours in T, since the missing edge of this K_5^- is zt. Then this implies x and y have degree at least five, since $ux, vy \in E(H)$ and $u, v \notin T$. This and 4-regularity of H imply that also the edges yz and yt are missing, since t and z have three neighbours in T and $xt, xz \in E(H)$. Then as u and z, t already belong to some copies of K_5^- that do not contain x, the 2-reduction move at x which removes the vertex x and adds the edges uz and yt gives a K_5-free graph G. ■

1.3.5 K_5^- Moves

Let $G = (V, E)$ be a graph. A $(K_5^-, 0)$-*extension* is an operation that adds a copy of K_5^- and connects this K_5^- and G with an edge, see (A) and (B) in Figure 1.12. Similarly, a $(K_5^-, 1)$-*extension* on $xy \in E$ is an operation that removes an edge xy from G, adds a copy of K_5^- and connects x and y to this K_5^- by two edges, see (C,D,E,F,G) in Figure 1.12.

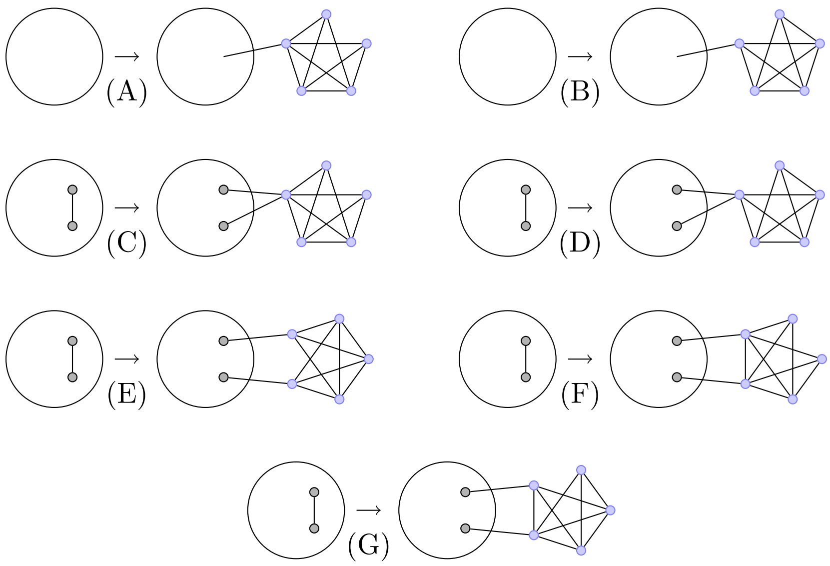

Figure 1.12: K_5^- Moves.

Lemma 1.3.18. *Let $G = (V, E)$ be a graph. Suppose H is a graph obtained from G by a $(K_5^-, 0)$-extension. Then G is $(2,0)^*$-tight if and only if H is $(2,0)^*$-tight.*

Proof: First note that if H is $(2,0)^*$-tight, then so is G, since G is a subgraph of H and $|E| = 2|V|$ holds.

For the other direction let $e = uv$ denote the edge that connects G and K_5^- where $v \in V$ and $u \in V(K_5^-)$. We now set $G_1 := G + u + uv$ and $G_2 := K_5^-$, and apply Lemma 1.2.3 to complete the proof. ■

Lemma 1.3.19. *Let $G = (V, E)$ be a graph. Suppose H is a graph obtained from G by a $(K_5^-, 1)$-extension on an edge $xy \in E$ that does not belong to a K_5 in G. Then G is $(2,0)^*$-tight if and only if H is $(2,0)^*$-tight.*

Proof: First note that the condition that xy does not belong to a copy of K_5 in G implies that G is K_5-free if and only if H is K_5-free. Therefore by using this and the edge counts we only need to show that G is $(2,0)$-sparse if and only if H is $(2,0)$-sparse.

Let $v_1, \ldots, v_5$ denote the vertices of the copy of K_5^-, e, f denote the edges that connect G and K_5^-, and x, y be the endpoints of these edges in G. It is easy to check that we have $i_H(Y) < 2|Y|$ for all $Y \subseteq \{x, y, v_1, \ldots, v_5\}$.

Suppose G is $(2,0)$-sparse but H is not. Then there exists a set $X \subseteq V(H)$ such that $2|X| < i_H(X)$. Since G is $(2,0)$-sparse and $i_H(Y) < 2|Y|$ for all $Y \subseteq \{v_1, \ldots, v_5\}$, $X \cap V$ and $T := X \cap \{v_1, \ldots, v_5\}$ are non-empty. If $|\{e, f\} \cap E(X)| \leq 1$, then we would have

$$2|X \setminus T| = 2|X| - 2|T| < i_H(X) - i_H(T) - 1 \leq i_G(X \setminus T).$$

In particular, $2|X \setminus T| < i_G(X \setminus T)$, a contradiction. Hence we may assume $e, f \in E(X)$. This implies $x, y \in X$. Then we have

$$2|X \setminus T| = 2|X| - 2|T| < i_H(X) - i_H(T) - 2 + 1 = i_G(X \setminus T),$$

where the -2 term corresponds to the edges e, f and the $+1$ term corresponds to the edge xy. In particular we have $2|X \setminus T| < i_G(X \setminus T)$, a contradiction.

Now suppose H is $(2,0)$-sparse but G is not. Then there exists a set $X \subseteq V$ with $2|X| < i_G(X)$. We see that $x, y \in X$, as otherwise, $G[X]$ would be a subgraph of H, and so would be $(2,0)$-sparse. Let $S = X \cup \{v_1, \ldots, v_5\}$. Then

$$2|S| = 2|X| + 10 < i_G(X) + 10 = i_H(X) + 11 = i_H(S).$$

In particular $2|S| < i_H(S)$, a contradiction. ■

1.4 Recursive Construction for $(2,0)^*$-Tight Graphs

In this section we will give a recursive construction for $(2,0)^*$-tight simple graphs. We will give an inductive construction for such graphs by using the moves in Section 1.3.

The graphs in Figure 1.13 are the base graphs in our recursive construction. We will show that every $(2,0)^*$-tight graph can be obtained from these graphs by the extension moves described in Section 1.3. We will sometimes need to consider a base graph with respect to the number of vertices it has. When this is the case we will use G_6^* for a base graph on six vertices, $K_5^- \cdot K_5^-$ for a base graph on nine vertices, and $K_5^-|K_5^-$ for a base graph on ten vertices. It is easy to see that a $K_5^- \cdot K_5^-$ can be obtained from two copies of K_5^- by letting them intersect at a single vertex. Similarly, a $K_5^-|K_5^-$ is obtained from the disjoint union of two copies of K_5^- by adding two edges e, f that connect these copies of K_5^- such that there is no C_4 or K_3 that contains both e and f.

Lemma 1.4.1. *Let $G = (V, E)$ be a $(2,0)^*$-sparse graph. Suppose $X_1, X_2, \ldots, X_k \subset V$ are sets that induce a K_5^-, a G_6^*, a $K_5^-|K_5^-$, or a $K_4 \circ K_4$ in G. Then the set $X = \bigcup_{i=1}^k X_k$, can be partitioned in such a way that each part induces K_5^-, $K_5^- \cdot K_5^-$, $K_5^-|K_5^-$, G_6^* or $K_4 \circ K_4$ in G.*

Proof: Consider distinct X_i and X_j with $X_i \cap X_j \neq \emptyset$ for some $1 \leq i < j \leq k$. Assume both X_i and X_j induce a copy of K_5^-. First note that as $1 \leq |X_i \cap X_j| \leq 4$, we have $i(X_i \cap X_j) \leq 2|X_i \cap X_j| - 2$. Then

$$2|X_i| + 2|X_j| = 20 = 18 + 2 = i(X_i) + i(X_j) + 2$$

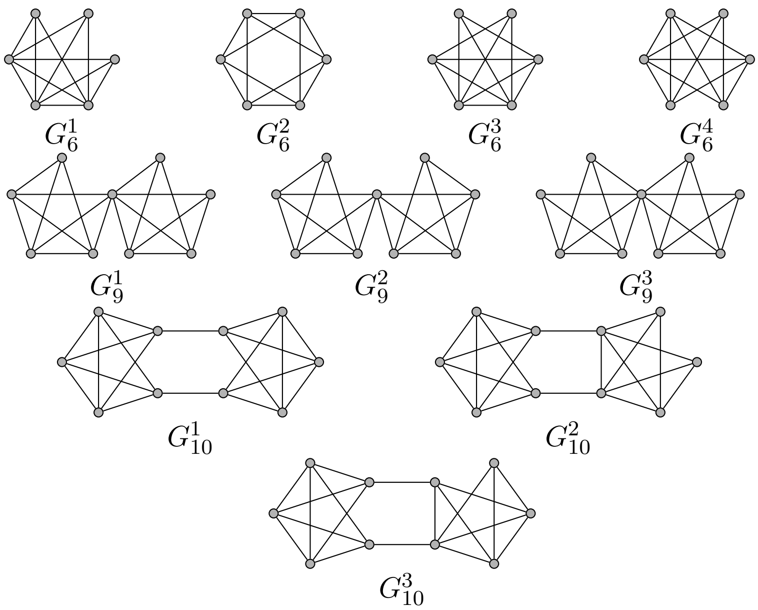

Figure 1.13: Base graphs for $(2,0)^*$-tightness.

$$
\begin{aligned}
&\leq i(X_i \cap X_j) + i(X_i \cup X_j) + 2 \\
&\leq 2|X_i \cap X_j| + 2|X_i \cup X_j| - 2 + 2 \\
&= 2|X_i| + 2|X_j|.
\end{aligned}
$$

Hence equality holds throughout. In particular, we have $i(X_i \cap X_j) = 2|X_i \cap X_j| - 2$. This is only possible when $|X_i \cap X_j| = 1$ or $|X_i \cap X_j| = 4$. The former case implies $X_i \cup X_j$ induces a $K_5^- \cdot K_5^-$ and the latter case implies $X_i \cup X_j$ induces a G_6^*. Therefore we conclude that in a $(2,0)^*$-sparse graph two distinct copies of K_5^-, are either disjoint, or form a $K_5^- \cdot K_5^-$ or G_6^*.

Now consider a $(2,0)^*$-tight set Y and a set X_i that induces a K_5^- with $X_i \cap Y \neq \emptyset$. Then $i(X_i \cap Y) \leq 2|X_i \cap Y| - 1$ holds as $|X \cap Y| \leq 5$ and G is $(2,0)^*$-sparse. This implies

$$
\begin{aligned}
2|X_i| + 2|Y| &= i(X_i) + i(Y) + 1 \\
&\leq i(X_i \cap Y) + i(X_i \cup Y) + 1 \\
&\leq 2|X_i \cap Y| - 1 + 2|X_i \cup Y| + 1
\end{aligned}
$$

$$= 2|X_i| + 2|Y|.$$

Hence equality holds throughout. In particular, $i(X_i \cap Y) = 2|X_i \cap Y| - 1$. Since X_i induces a K_5^- and $X_i \cap Y \neq \emptyset$, this is only possible when $X_i \subseteq Y$.

Finally consider two $(2,0)^*$-tight sets Y, Z that have no proper non-empty $(2,0)$-tight subsets. By Lemma 1.2.1, $Y \cap Z$ must be $(2,0)$-tight. Hence $Y \cap Z = \emptyset$.

We can combine these deductions with the fact that $K_5^- \cdot K_5^-$, $K_5^-|K_5^-$, G_6^* and $K_4 \circ K_4$ are $(2,0)^*$-tight and have no proper non-empty $(2,0)^*$-tight subgraph to complete the proof. ■

When we apply one of the reduction moves defined in Section 1.3 on a $(2,0)^*$-tight graph it may not result in a $(2,0)^*$-tight graph. We say a reduction move is *admissible* if it preserves $(2,0)^*$-tightness.

Lemma 1.4.2. *Let $G = (V, E)$ be a $(2,0)^*$-tight graph that is not a disjoint union of the base graphs drawn in Figure 1.13. Then*

(a) If G is not 4-regular, then there exists at least one admissible 0-, 1-reduction, K_3-to-K_2, C_4-to-P_3, K_4-contraction, $(K_5^-, 0)$- or $(K_5^-, 1)$-reduction move on G.

(b) If G is 4-regular, then there exists at least one admissible 2-reduction, C_4-to-P_3, K_4-contraction, $(K_5^-, 0)$- or $(K_5^-, 1)$-reduction move on G.

Proof: We may assume that G is connected and not isomorphic to any of the base graphs drawn in Figure 1.13, as otherwise, we can take a connected component of G that is not isomorphic to any of the base graphs and proceed in the same way. The fact that $|E| = 2|V|$ implies that the average degree of G is 4. Hence either G contains a degree 2 or 3 vertex, or G is 4-regular.

Proof of (a): We will split the proof into two cases.

Case 1. There exists a vertex v with $d(v) = 2$.

Then the 0-reduction move at v is admissible by Lemma 1.3.1.

Case 2. $\delta(G) = 3$.

Case 2.1. There exists a vertex v with $d(v) = 3$ that does not belong to a K_5^-, G_6^* or a $K_4 \circ K_4$.

We split this case into three sub-cases depending on $N(v)$.

Case 2.1.1. $i(N(v)) \leq 1$.

Then there exists an admissible 1-reduction at v by Lemma 1.3.4.

Case 2.1.2. $i(N(v)) = 2$.

By Lemma 1.3.3, we can assume that the only possible 1-reduction at v creates a copy of K_5. Then combining this with the main assumption of Case 2.1 and Lemma 1.3.9, we see that there exists an admissible C_4-to-P_3 move.

Case 2.1.3. $i(N(v)) = 3$.

Then $N[v]$ induces a copy of K_4. Let $N(v) = \{a, b, c\}$. Since v does not belong to a K_5^-, every vertex $x \in V \setminus N(v)$ can be adjacent to at most two vertices in $N(v)$.

First assume there exists a vertex $x \in V \setminus N(v)$ that has two neighbours in $N(v)$, say a and b. Then there exists an admissible C_4-to-P_3 move that identifies the vertices x and v on the C_4 whose vertices are v, a, x, b by Lemma 1.3.6.

Next assume every vertex $x \in V \setminus N(v)$ has at most one neighbour in $N(v)$. Then as the K_4 that is induced by $N[v]$ is not contained in a $K_4 \circ K_4$ (main assumption of Case 2.1), there exists an admissible K_4-contraction move by Lemma 1.3.11.

Case 2.2 Every vertex of degree three in G belongs to a K_5^-, G_6^*, or a $K_4 \circ K_4$.

First note that by Lemma 1.4.1, we can obtain a family $Q = \{Q_1, \ldots, Q_l\}$ of pairwise disjoint subsets of V such that Q_i induces a K_5^-, $K_5^- \cdot K_5^-$, $K_5^-|K_5^-$, G_6^* or a $K_4 \circ K_4$ and the vertex set of every copy of a K_5^-, G_6^* and $K_4 \circ K_4$ in G is contained in a Q_i, for some $1 \leq i \leq l$. Next consider $Q' \subset Q$, where $Q_i \in Q'$ if Q_i has a vertex of degree three in G, $1 \leq i \leq l$. Note that every degree three vertex is contained in some $Q_i \in Q'$, $1 \leq i \leq l$. Figure 1.14 shows all possibilities of how a set $X \in Q'$ may connect to other vertices of G classified by the number of edges from X to $V \setminus X$. We will consider each of the cases (a) to (k) illustrated in Figure 1.14 in turn. For case (a), by Lemma 1.3.18, there exists an admissible $(K_5^-, 0)$-reduction. For case (b), if there is a K_5^- that contains both vertices drawn in $G - K_5^-$, then by Lemma 1.3.8 there exists an admissible C_4-to-P_3 move. If there is no K_5^- that contains both vertices drawn in $G - K_5^-$ for case (b), then by Lemma 1.3.19, there exists an admissible $(K_5^-, 1)$-reduction. For case (c) we may assume there is no K_5^- that contains both vertices drawn in $G - K_5^-$, as the case when there is such a K_5^- corresponds to case (j). Therefore by Lemma 1.3.19, there exists an admissible $(K_5^-, 1)$-reduction for the case (c), since the $(K_5^-, 1)$-reduction move does not create a copy of K_5, as those two vertices drawn in $G - K_5^-$ do not belong to a K_5^-. For the cases (d,e), by Lemma 1.3.7, there exists an admissible K_3-to-K_2 move.

For the case (f), by Lemma 1.3.7, there exists an admissible C_4-to-P_3 move. When $k = 0$ for cases (h,i,j) we obtain a base graph since G is connected. When $k = 0$ for case (k), G is isomorphic to one of the graphs denoted by $K_4 \circ K_4$. Referring to Figure 1.15 one can check that the K_3-to-K_2 moves for the graphs on top and the C_4-to-P_3 moves for the graph on the bottom left that identify the blue vertices are admissible. Note that as G is not 4-regular G cannot be the graph drawn on the bottom right in Figure 1.15.

Hence we may assume that only case (g) and cases (h,i,j,k) with $k \geq 1$ can occur. It is straightforward to calculate that the average degree in G of the vertices in $X \in Q'$ is strictly bigger than four, for the cases (g) and (h,i,j,k) when $k \geq 1$. Combining this with the fact that every vertex of degree three belongs to an $X \in Q'$, and that Q' consists of pairwise disjoint sets, we may deduce that the average degree of G is strictly bigger than four. This contradicts the fact that $|E| = 2|V|$.

This completes the proof of (a)

Proof of (b). We split the proof into three cases.

Case 1. G is K_4-free.

Then there exists an admissible 2-reduction by Lemma 1.3.14.

Case 2. There exists a K_4 that is not contained in a K_5^- in G.

Take such a copy of K_4. Let T be the vertex set of this K_4. Consider $F := E(T, V \setminus T)$. Since this K_4 is not contained in a copy of K_5^-, for all $x \in V \setminus T$, we have $|N_G(x) \cap T| \leq 2$.

First suppose there exists $x \in V \setminus T$ with $|N_G(x) \cap T| = 2$. Then by Lemma 1.3.15, there exists an admissible 2-reduction.

Hence we may assume that $|N_G(x) \cap T| \leq 1$ for all $x \in V \setminus T$. Let S denote the set of vertices in $V \setminus T$ that has a neighbour in T. As G is 4-regular and every vertex in T has a distinct neighbour in $V \setminus T$, we have $|S| = 4$. If S does not induce a copy of K_4, then by Lemma 1.3.11, there exists an admissible K_4-contraction. If S induces a copy of K_4, then due to 4-regularity and connectivity of G, G must be the 4-regular graph drawn in Figure 1.15 on the bottom right. In this case, one can check that the C_4-to-P_3 move that identifies the blue vertices is admissible.

Case 3. Every K_4 is contained in a K_5^- and there exists a K_4 in G.

Then G contains a copy of K_5^-. Let X denote the vertex set of this K_5^- and uv be the missing edge. Since G is 4-regular, and u and v have three neighbours in this

K_5^-, each of u and v has another neighbour outside this K_5^-. Let x and y be these neighbours of u and v, respectively.

Case 3.1. $x = y$.

Then by Lemma 1.3.16, there exists an admissible 2-reduction.

Case 3.2. $x \neq y$.

Then there are two possibilities depending on whether the edge xy exists or not.

Case 3.2.1. $xy \notin E$.

If there exists a K_5^- containing both x and y in G, then due to 4-regularity, G must be one of the base graphs, namely G_{10}^1. If there is no K_5^- containing both x and y, then by Lemma 1.3.19, there exists an admissible $(K_5^-, 1)$-reduction.

Case 3.2.2. $xy \in E$.

Let $N[x] = \{x, u, y, z, t\}$. First suppose that there exists a K_4 in $G[N[x]]$. Since G is 4-regular and u is contained in a K_5^- that is induced by X, we have $G[\{x, y, z, t\}] = K_4$. Also the fact that every K_4 is contained in a K_5^- and G is 4-regular imply that $Y := \{x, y, z, t, s\}$ induces a K_5^- for some $s \in V \setminus X$. As $xy \in E$, at least one of x and y, say x, has four neighbours in Y. Combining this with the fact that x is also adjacent to u, we obtain $d(x) \geq 5$, contradicting the fact that G is 4-regular.

Therefore we may assume $N[x]$ is K_4-free, hence $G[\{x, y, z, t\}] \neq K_4$. Then by Lemma 1.3.17, there exists an admissible 2-reduction. ■

Theorem 1.4.3. *Let $G = (V, E)$ be a simple graph. Then G is $(2,0)^*$-tight if and only if G can be obtained from a disjoint union of the base graphs in Figure 1.13 by a sequence (possibly empty) of 0-, 1-, 2-extensions, K_2-to-K_3, P_3-to-C_4, vertex-to-K_4 moves and $(K_5^-, 0)$- and $(K_5^-, 1)$-extensions.*

Proof: The facts that the base graphs are $(2,0)^*$-tight and the moves listed in the statement preserve being $(2,0)^*$-tight imply that G is $(2,0)^*$-tight if it can be constructed as in the theorem.

For the other direction suppose G is $(2,0)^*$-tight and cannot be obtained from a disjoint union of the base graphs by a sequence of 0-, 1-, 2-extensions, K_2-to-K_3, P_3-to-C_4, vertex-to-K_4 moves and $(K_5^-, 0)$- and $(K_5^-, 1)$-extensions and that G has the minimum number of vertices over all such graphs.

By Lemma 1.4.2, there exists at least one admissible 0-, 1-, 2-reduction, K_3-to-K_2, C_4-to-P_3, K_4-contraction, $(K_5^-, 0)$- or $(K_5^-, 1)$-reduction for G. Let H denote the graph that is obtained by G by an admissible reduction. By the minimality of $|V|$, H satisfies the statement of the theorem. The fact that we can obtain G, from H by one of the 0-, 1-, 2-extensions, K_2-to-K_3, P_3-to-C_4, vertex-to-K_4 moves and $(K_5^-, 0)$- and $(K_5^-, 1)$-extensions now gives a contradiction. ■

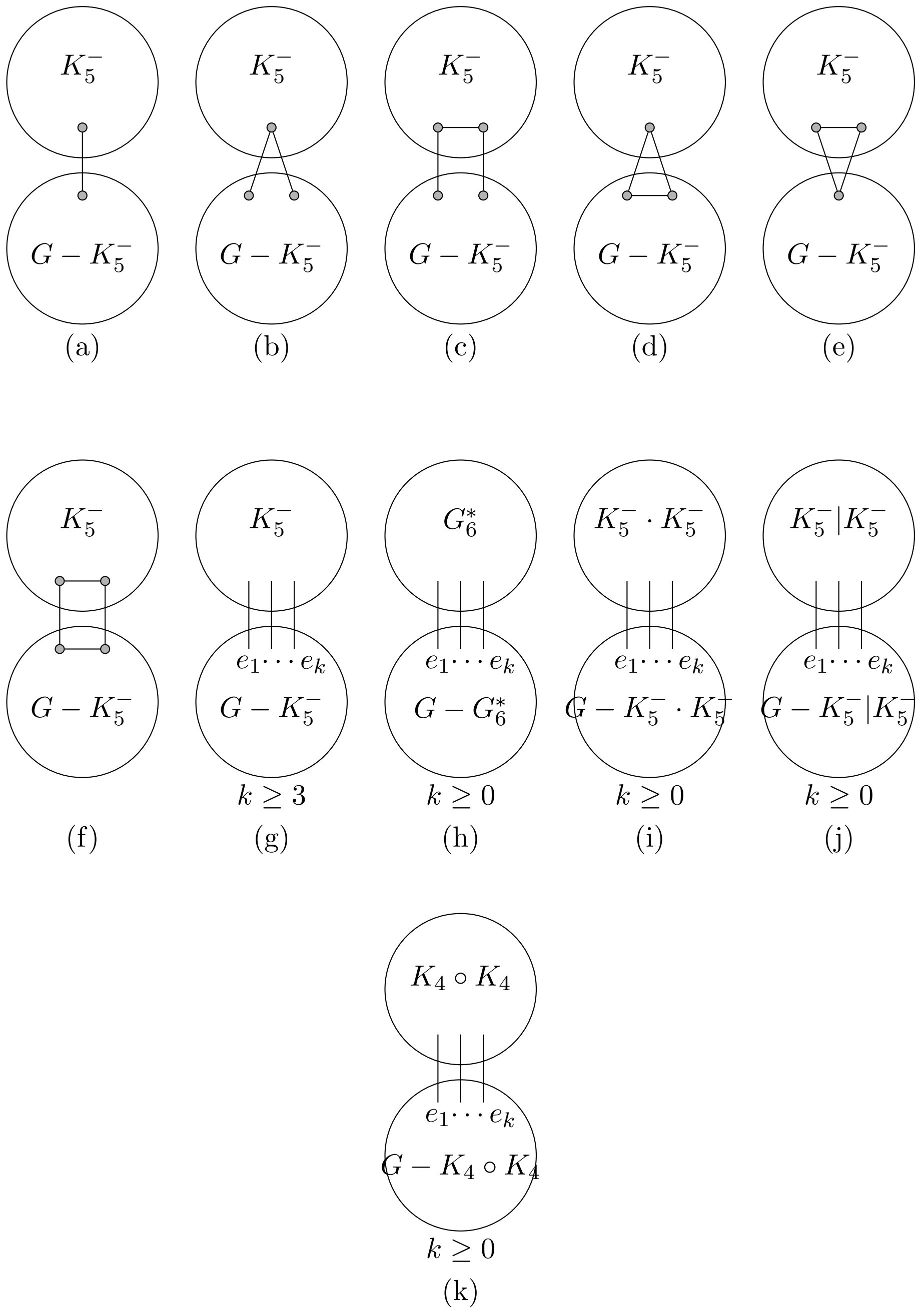

Figure 1.14: Possibilities of how a member of Q' connects to other vertices of G. The edges whose endpoints are undefined can be incident with any vertex as long as there is a vertex of degree three in the part drawn on top for each case.

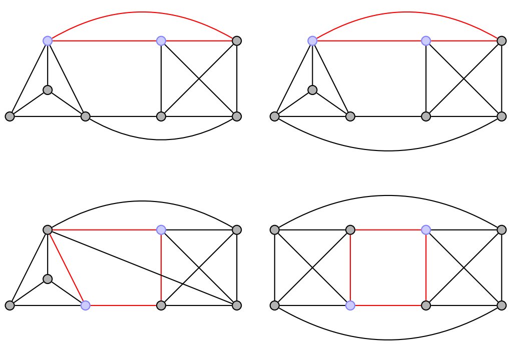

Figure 1.15: The K_3-to-K_2 moves for the graphs on top and the C_4-to-P_3 moves for the graphs on the bottom that identify the blue vertices are admissible.

1.5 Geometric Matroid

In this section we will give a characterisation for the rigidity of generic linearly constrained frameworks in $\mathbb{R}^3$. In order to do this we will need some extension moves that preserve independence and rigidity of generic linearly constrained frameworks.

Definition 1.5.1. A *linearly constrained framework in* $\mathbb{R}^3$ is a triple (G,p,q) where $G=(V,E)$ is a graph, $p:V\to\mathbb{R}^3$ and $q:V\to\mathbb{R}^3$. It is *generic* if (p,q) is algebraically independent over $\mathbb{Q}$.

In this definition the map p assigns positions in $\mathbb{R}^3$, and the map q assigns planes in $\mathbb{R}^3$ ($q(v)$ is the normal vector of a plane that contains the point $p(v)$) to the vertices.

Definition 1.5.2. A *motion* of (G,p,q) is a continuous map $P:V\times[0,1]\to\mathbb{R}^3$, such that

- $P(v,0)=p(v)$ for all $v\in V$,
- $|P(v,t)-P(u,t)|=|p(v)-p(u)|$ for all $uv\in E$, and
- $(P(v,t)-p(v))\cdot q(v)=0$ for all $t\in[0,1]$ and for all $v\in V$.

Definition 1.5.3. A linearly constrained framework (G,p,q) is *rigid* if its only motion is the zero motion, that is, $P(v,t)=p(v)$ for all $v\in V$ and for all $t\in[0,1]$.

Definition 1.5.4. An *infinitesimal motion* of (G,p,q) is a map $m:V\to\mathbb{R}^3$ satisfying the system of linear equations

$$(p(u)-p(v))\cdot(m(u)-m(v))=0 \text{ for all } uv\in E \tag{1.1}$$

$$q(v)\cdot m(v)=0 \text{ for all } v\in V. \tag{1.2}$$

Definition 1.5.5. The *rigidity matrix* $R(G,p,q)$ of the framework (G,p,q) is the matrix of coefficients of this system of equations in (1.1) and (1.2) for the unknowns $m(v)$ for all $v\in V$.

Definition 1.5.6. A framework (G,p,q) is *infinitesimally rigid* if its only infinitesimal motion is $m=0$, or equivalently if $\operatorname{rank} R(G,p,q)=3|V|$. We say that a graph G is *rigid* (as a linearly constrained framework) if $\operatorname{rank} R(G,p,q)=3|V|$ for some (p,q), or equivalently if $\operatorname{rank} R(G,p,q)=3|V|$ for all generic (p,q).

The rigidity matrix $R(G,p,q)$ of a linearly constrained framework (G,p,q) can be obtained from the rigidity matrix of the 3-dimensional bar-and-joint framework (G,p) by adding a new row for each $v \in V$ whose entries are $q(v)$ in the columns corresponding to the vertex v and zeros elsewhere. We say (G,p,q) is *independent*, respectively *dependent* if the rows of $R(G,p,q)$ are independent, respectively dependent. Therefore, if the 3-dimensional framework (G,p) is dependent, then the linearly constrained framework (G,p,q) will be dependent for every choice of q.

Let $G = (V,E)$ be a simple graph. The generic *linearly constrained rigidity matroid* $\mathcal{R}(G)$ of G is the matroid on E obtained from the rigidity matrix $R(G,p,q)$ of a generic linearly constrained framework (G,p,q). A set of edges $F \subseteq E$ is *independent*, respectively *dependent* in $\mathcal{R}(G)$ if the rows corresponding to the edges in F and the rows corresponding to $q(v)$ for all $v \in V$ are linearly independent, respectively dependent in $R(G,p,q)$. A set of edges $F \subseteq E$ is a *circuit* in $\mathcal{R}(G)$ if F is dependent and $F-e$ is independent for all $e \in F$. We also say a subgraph H of G is *independent (dependent, a circuit)*, if $E(H)$ is independent (dependent, a circuit) in $\mathcal{R}(G)$.

By using the same argument for bar-and-joint frameworks Asimow and Roth used in [2] (i.e., changing the entries of the rigidity matrix into generic values), we can deduce the following result.

Lemma 1.5.1. *Let (G,p,q) be an independent linearly constrained framework. Suppose (G,p',q') is a generic linearly constrained framework. Then (G,p',q') is independent.*

We say a framework (G,p) on a surface $\mathcal{M}$ is *independent* on $\mathcal{M}$, if the rigidity matrix $R(G,p,q)$ of the linearly constrained framework (G,p,q) has linearly independent rows, where $q(v)$ is the unit normal vector to $\mathcal{M}$ at the point $p(v)$ for all $v \in V$. We say a graph $G = (V,E)$ is *independent* on $\mathcal{M}$, if there exists a framework (G,p) which is independent on $\mathcal{M}$. Since a framework on an algebraic surface $\mathcal{M}$ can be regarded as a linearly constrained framework, we see that Lemma 1.5.1 implies our next result.

Lemma 1.5.2. *Let $\mathcal{M}$ be an irreducible surface of type k, $0 \leq k \leq 2$. Suppose (G,p) is an independent framework on $\mathcal{M}$ and (G,p',q) is a generic linearly constrained framework. Then (G,p',q) is independent.*

The lemma below implies an important step in the proof of our main result and is due to Nixon, Owen and Power [21].

Lemma 1.5.3. *[21] Let $\mathcal{M}$ be an irreducible surface of type k, $0 \leq k \leq 2$. Let H be an independent graph on $\mathcal{M}$. Suppose G is a graph obtained from H by a move one of the following types: 0-extension, 1-extension, K_2-to-K_3, P_3-to-C_4 or vertex-to-K_4. Then G is independent on $\mathcal{M}$.*

We can now combine Lemma 1.5.2 and Lemma 1.5.3 to obtain the following useful result. Note that we only need the case $k = 0$ in Lemmas 1.5.2 and 1.5.3 in order to deduce this result.

Lemma 1.5.4. *Let H be a rigid graph (as a linearly constrained framework). Suppose G is a graph obtained from H by one of 0-, 1-extensions, K_2-to-K_3, P_3-to-C_4 or vertex-to-K_4 moves. Then G is rigid (as a linearly constrained framework).*

We will now show that $(K_5^-, 0)$- and $(K_5^-, 1)$-extension moves preserve generic independence of linearly constrained frameworks.

Lemma 1.5.5. *Let (H, p, q) be a minimally infinitesimally rigid linearly constrained framework with $x \in V(H)$. Let $s \neq p(x)$ be a point in $\mathbb{R}^3$. Suppose G is a graph obtained from H by a $(K_5^-, 0)$-extension move for which ux is the edge that connects H and K_5^-. Then there exists a minimally infinitesimally rigid linearly constrained framework (G, p', q') such that $p'|_{V(H)} = p$, $q'|_{V(H)} = q$, and $p'(v), p'(u), p'(x)$ and s are collinear for some arbitrary vertex $v \in V(K_5^-)$ with $u \neq v$.*

Proof: We first fix $v \neq u \in V(K_5^-)$ and take a generic realisation of $(K_5^-, \hat{p})$ on an elliptical cylinder $\mathcal{Y}$. We can translate and rotate $\mathcal{Y}$ in $\mathbb{R}^3$ and obtain a framework $(K_5^-, \bar{p})$ on an elliptical cylinder $\mathcal{Y}'$ such that the points $p(x), s, \bar{p}(u), \bar{p}(v)$ are collinear. By genericity of $\hat{p}$, the axis of $\mathcal{Y}'$ is not orthogonal to the line through the points $p(x)$ and $\bar{p}(u)$. Therefore the tangent plane T_u at $\bar{p}(u)$ to $\mathcal{Y}'$ is not orthogonal to $\bar{p}(u) - p(x)$.

Now let (G, p', q') be the linearly constrained framework for which $p'|_{V(H)} = p$, $q'|_{V(H)} = q$, $p'|_{V(K_5^-)} = \bar{p}$ and $q'(w)$ is the normal to the tangent plane of $\mathcal{Y}'$ at $p'(w)$ for all $w \in V(K_5^-)$.

Now consider an infinitesimal motion m of the framework (G, p', q'). Since (H, p, q) is infinitesimally rigid, $m(t) = 0$ for all $t \in V(H)$. This and the fact that $[m(u) - m(x)] \cdot [p'(u) - p'(x)] = 0$ imply that $m(u)$ is orthogonal to $p'(u) - p'(x)$. However, since $m(u) \in T_u$ and T_u is not orthogonal to $p'(u) - p'(x)$ (as T_u is not orthogonal to $\bar{p}(u) - p(x)$), we see that $m(u) = 0$. Combining this with the fact that the only infinitesimal motion of $(K_5^-, p'|_{V(K_5^-)}, q'|_{V(K_5^-)})$ is the translation along the axis of $\mathcal{Y}'$ (by Theorem 1.1.2), we obtain that $m(t) = 0$ for all $t \in V(K_5^-)$. Hence $m(t) = 0$ for all $t \in V(G)$. Therefore (G, p', q') is infinitesimally rigid. The fact that G has $2|V(G)|$ edges tells us (G, p', q') is minimally infinitesimally rigid. ■

Lemma 1.5.6. *Let H be a minimally rigid graph (as a linearly constrained framework) and let G be obtained from H by a $(K_5^-, 1)$-extension move. Then G is minimally rigid (as a linearly constrained framework).*

Proof: Let (H, p, q) be a generic realisation of H. Let $e = xy \in E(H)$ be the edge on which $(K_5^-, 1)$-extension move is applied. Let e_1 and e_2 be the edges in G that connect $V(H)$ and $V(K_5^-)$. Let u and v be the endpoints of the edges $e_1 = xu$ and $e_2 = yv$ in $V(K_5^-)$, respectively. Note that we may have $u = v$.

We first perform a $(K_5^-, 0)$-extension on (H, p, q) by applying Lemma 1.5.5 with $s = p(y)$ to obtain a minimally infinitesimally rigid linearly constrained framework $(G - e_2 + xy, p', q')$ with $p'(u), p'(v), p'(x), p'(y)$ collinear. We will show that for the framework $(G - e_2 + xy, p', q')$, replacing the edge xy by the edge $e_2 = yv$ preserves independence. In order to do this let us consider the linearly constrained framework $(G + xy, p', q')$ and its rigidity matrix $R(G + xy, p', q')$. We have two cases.

Case 1. $u = v$.

Since $(G - e_2 + xy, p', q')$ is minimally infinitesimally rigid, $e_1 = xu$ and $xy \in E(G - e_2 + xy)$ and $p'(x), p'(y), p'(u)$ are collinear, the rows of $R(G + xy, p', q')$ have a unique linear dependence which is obtained from the rows corresponding to the edges e_1, e_2, xy. Therefore deleting any of these rows makes the matrix have linearly independent rows. Hence, we delete the row corresponding to the edge xy and deduce that the linearly constrained framework (G, p', q') is minimally infinitesimally rigid.

Case 2. $u \neq v$.

We split this case into two sub-cases depending on whether uv is an edge in $G - e_2 + xy$ or not.

Case 2.1. $uv \in E(G - e_2 + xy)$.

Since $(G - e_2 + xy, p', q')$ is minimally infinitesimally rigid, $e_1 = xu, xy, uv \in E(G-e_2+xy)$ and $p'(x)$, $p'(y)$, $p'(u)$, $p'(v)$ are collinear, the rows of $R(G+xy, p', q')$ have a unique linear dependence which is obtained from the rows corresponding to the edges e_1, e_2, xy, uv. Therefore deleting any of these rows makes the matrix have linearly independent rows. Hence, we delete the row corresponding to the edge xy and deduce that the linearly constrained framework (G, p', q') is minimally infinitesimally rigid.

Case 2.2. $uv \notin E(G - e_2 + xy)$.

Since the vertices u, v are contained in the $K_5^- \subset G - e_2 + xy$ and $uv \notin E(G - e_2 + xy)$, we see that uv is the missing edge of the K_5^-. Let X denote the vertex set of the K_5^-. We first add the edge uv to the K_5^- and remove another edge f instead so that the K_5^- remains as another K_5^-. Then for this $K_5^- + uv - f$, we proceed as in Case 2.1 to deduce that the framework $(G + uv - f, p', q')$ is minimally infinitesimally rigid. We next add the edge f back and consider the framework $(G+uv, p', q')$ and its rigidity matrix $R(G+uv, p', q')$. Since the set X induces a K_5 in $G+uv$ and $(G+uv-f, p', q')$ is independent, the rows corresponding to the edges in $E_{G+uv}(X)$ of $R(G + uv, p', q')$ form a minimally linearly dependent set. Now the fact that $uv \in E_{G+uv}(X)$ implies $R(G, p', q')$ has linearly independent rows. Hence the linearly constrained framework (G, p', q') is minimally infinitesimally rigid. ■

The following lemma is due to Jackson and Jordán [12] and will be a useful tool in the proof of our main result. Note that the lemma is stated for independence of a graph in the 3-dimensional bar-and-joint rigidity matroid $\mathcal{R}_3$. Since the rigidity matrix $R(G, p, q)$ of a linearly constrained framework (G, p, q) in $\mathbb{R}^3$ contains the rigidity matrix $R(G, p)$ of the bar-and-joint framework (G, p) in $\mathbb{R}^3$ as a $|E(G)| \times 3|V(G)|$ submatrix, this result will allow us to deduce that the rows corresponding to the edges in $R(G, p, q)$ are linearly independent for some cases.

Lemma 1.5.7. *[12] Let G be a connected graph with $\Delta(G) \leq 5$ and $\delta(G) \leq 4$. Then G is independent in $\mathbb{R}^3$ (as a bar-and-joint framework) if and only if G is*

$(3,6)$-*sparse.*

We will use the lemma below in the proof of the main theorem for the case when the graph G is 4-regular. (We do not know whether the 2-extension move preserves independence of linearly constrained frameworks, so we cannot use 2-extension to solve this case as in the proof of Theorem 1.4.3.)

Lemma 1.5.8. *Let $G = (V,E)$ be a simple graph and (G,p) be a generic (bar-and-joint) realisation of G in $\mathbb{R}^3$. Let (G,p,q) be the linearly constrained framework we get by choosing a family of concentric elliptical cylinders $\mathcal{Z}$, defined by the equations $x^2 + 2y^2 = r_i$ and choosing the r_i such that each vertex $v \in V$ lies on a unique elliptical cylinder in $\mathcal{Z}$ and $q(v)$ to be the unit normal vector at the point $p(v)$ to the cylinder that contains v. Suppose G is connected and $(2,1)$-tight. Then the rows of $R(G,p,q)$ are linearly independent and the only infinitesimal motions of (G,p,q) are translations in the direction of the z-axis.*

Proof: We first consider an elliptical cylinder $\mathcal{Y} \in \mathcal{Z}$ and a generic framework (G,p') on $\mathcal{Y}$. By Theorem 1.1.2 with $k = 1$, (G,p') is rigid on $\mathcal{Y}$ and so the only infinitesimal motions of (G,p') on $\mathcal{Y}$ are translations in the direction of the z-axis. Now consider the linearly constrained framework (G,p',q). Since every algebraic dependency of (p,q) is an algebraic dependency of (p',q) but not vice versa, we have $\dim\ker R(G,p,q) \leq \dim\ker R(G,p',q)$. This gives $1 \leq \dim\ker R(G,p,q) \leq \dim\ker R(G,p',q) = 1$. Therefore equality holds throughout. In particular, we have $\dim\ker R(G,p,q) = 1$. Hence the only infinitesimal motions of (G,p,q) are translations in the direction of the z-axis. As G is $(2,1)$-tight this also implies that the rows of $R(G,p,q)$ are linearly independent. ■

The following is our main result of this chapter.

Theorem 1.5.9. *Let $G = (V,E)$ be a simple graph. Then G can be realised as an infinitesimally rigid linearly constrained framework in $\mathbb{R}^3$ if and only if G has a spanning subgraph which is $(2,0)^*$-tight.*

Proof: We first prove necessity and suppose G can be realised as an infinitesimally rigid linearly constrained framework (G,p,q) in $\mathbb{R}^3$. We may assume that $|E| = 2|V|$ and (G,p,q) is a minimally infinitesimally rigid linearly constrained framework (by deleting some edges). Then $R(G, p, q)$ has linearly independent rows. For a contradiction, suppose G is not $(2, 0)^*$-tight. Then there exists a set $X \subseteq V$ such that either X induces a copy of K_5 in G or $i(X) > 2|X|$. As K_5 is dependent in $\mathbb{R}^3$ as a bar-and-joint framework, every linearly constrained framework whose underlying graph is K_5 is dependent. Therefore since $R(G, p, q)$ has linearly independent rows, G does not contain a copy of K_5. This implies that the only possibility that breaks $(2, 0)^*$-tightness of G is having a set $X \subseteq V$ with $i(X) > 2|X|$. Let $R(G, p, q)$ be the rigidity matrix of (G, p, q). Consider the submatrix $R(G[X], p|_X, q|_X)$ of $R(G, p, q)$ induced by the row corresponding to $E(X)$ and the columns corresponding to X. We can reorder the columns and rows of $R(G, p, q)$ such that the rows corresponding to edges in $E(X)$ come before the other rows and the columns corresponding to the vertices in X come before the other columns and obtain the matrix below.

$$R(G, p, q) = \begin{bmatrix} R(G[X], p|_X, q|_X) & 0 \\ * & * \end{bmatrix}$$

Since $R(G[X], p|_X, q|_X)$ has $3|X|$ columns and more than $3|X|$ rows (as $i(X) > 2|X|$), we see that the rows of $R(G[X], p|_X, q|_X)$ are linearly dependent. Therefore the rows of $R(G, p, q)$ are linearly dependent. Now the fact that $|E| = 2|V|$ implies $R(G, p, q)$ has rank strictly less than $3|V|$, contradicting the infinitesimal rigidity of (G, p, q).

We prove sufficiency by induction on $|V| + |E|$. We may assume that G is connected and $|E| = 2|V|$.

Case 1. G is one of the graphs drawn in Figure 1.13.

We give infinitesimally rigid realisations for these graphs in Figure 1.16. For each graph, the coordinates in the figure correspond to the positions of its vertices in $\mathbb{R}^3$. If we set $p(v) := (p_v^1, p_v^2, p_v^3)$ and $q(v) := (p_v^1, 2p_v^2, 3p_v^3)$, then the linearly constrained framework (G, p, q) is infinitesimally rigid for each of the graphs drawn in Figure 1.13.

Case 2. G is neither one of the graphs drawn in Figure 1.13 nor 4-regular.

Then we apply Lemma 1.4.2 (a) to G and obtain a $(2, 0)^*$-tight graph H with $|V(H)| + |E(H)| < |V| + |E|$. Hence the graph H satisfies the statement of the theorem by the induction hypothesis. That is H can be realised as an infinitesimally

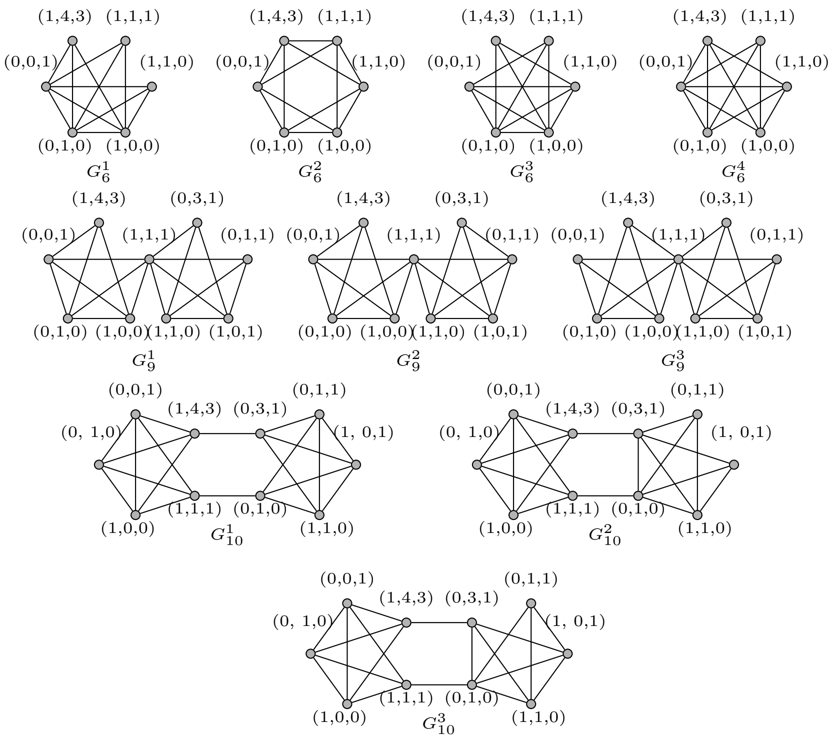

Figure 1.16: Infinitesimally rigid realisations for base graphs.

rigid linearly constrained framework in $\mathbb{R}^3$. The fact that H is obtained from G by one of 0-, 1-reduction, K_3-to-K_2, C_4-to-P_3, K_4-contraction moves and $(K_5^-, 0)$- and $(K_5^-, 1)$-reductions implies that G can be obtained from H by one of 0-, 1-extension, K_2-to-K_3, P_3-to-C_4, vertex-to-K_4 moves and $(K_5^-, 0)$- and $(K_5^-, 1)$-extensions. Lemmas 1.5.4, 1.5.5 and 1.5.6 now imply that G is rigid.

Case 3. G is 4-regular and not a base graph.

We first show that G is a circuit in the $(2, 1)$-sparsity matroid. Suppose the contrary. Then the fact that $|E| = 2|V|$ implies that there exists a set $X \subsetneq V$ with $i(X) = 2|X|$. Combining this with the connectivity and 4-regularity of G, we obtain a contradiction.

Therefore G is a circuit in the $(2, 1)$-sparsity matroid. Let (G, p) be a generic realisation of G in $\mathbb{R}^3$. Let (G, p, q) be the linearly constrained framework we get by choosing a family of concentric elliptical cylinders $\mathcal{Z}$, defined by the equations $x^2+2y^2 = r_i$ and choosing the r_i such that each $v \in V$ lies on a unique elliptical cylinder in $\mathcal{Z}$ and $q(v)$ to be the unit normal vector at the point $p(v)$ to the cylinder that contains v. Then by Lemma 1.5.8, the only infinitesimal motions of $(G-e,p,q)$ are translations in the direction of z-axis and the rows of $R(G-e,p,q)$ are linearly independent for all $e \in E$. Since adding e back to $G-e$ does not cancel out translations in the direction of the z-axis, this implies that $R(G,p,q)$ has a unique row dependence (ω,λ) up to scalar multiplication and $\omega_e \neq 0$ for all $e \in E$, where ω_e is the coefficient of the row corresponding to e and λ_v is the coefficient of the row corresponding to v for all $e \in E$ and $v \in V$. Since (G,p) is independent in the 3-dimensional rigidity matroid by Lemma 1.5.7 (as $(2,0)^*$-sparsity implies $(3,6)$-sparsity), we have $\lambda_v \neq 0$ for some $v \in V$. It follows that the matrix R_v obtained from $R(G,p,q)$ by deleting the row indexed by v has $\operatorname{rank} R_v = \operatorname{rank} R(G,p,q) = 3|V|-1$ and $\ker R_v = \ker R(G,p,q) = \langle (0,0,1,0,0,1,\dots,0,0,1) \rangle$. Let $(G,p,\tilde{q})$ be the constrained framework with $\tilde{q}(u) = q(u)$ for all $u \in V-v$ and $\tilde{q}(v) = (0,0,1)$. Then $\ker R(G,p,\tilde{q}) \subseteq \ker R_v$ and $(0,0,1,0,0,1,\dots,0,0,1) \notin \ker R(G,p,\tilde{q})$. Hence $\ker R(G,p,\tilde{q}) = \{0\}$ and $(G,p,\tilde{q})$ is an infinitesimally rigid linearly constrained framework. ■

Chapter 2

Rigidity of Frameworks with Three Coincident Points in $\mathbb{R}^2$

2.1 Introduction

In this chapter we will give a characterisation for the rigidity of a family of non-generic 2-dimensional bar-and-joint frameworks. The frameworks we will investigate have three vertices mapped to the same point, and this is the only algebraic dependency of the realisation. The problem where there are two vertices mapped to the same point was solved by Fekete, Jordán and Kaszanitzky in [6].

To set up the problem let $G = (V, E)$ be a graph and $u, v, w \in V$ be distinct vertices. As the characterisation highly depends on the three vertices mapped to the same point, we fix these vertices u, v, w.

Definition 2.1.1. Let (G, p) be a 2-dimensional framework and let $u, v, w \in V$ be distinct vertices. We say p is a *uvw-coincident* realisation if $p(u) = p(v) = p(w)$ holds. We also say that a uvw-coincident realisation p is *generic*, if the framework $(G - v - w, p|_{V\setminus\{v,w\}})$ is generic.

As all generic uvw-coincident realisations of a graph G give rise to the same matroid, the generic uvw-coincident rigidity matroid, on the edge set, we can say a graph G is *uvw-rigid*, if (G, p) is rigid for a generic uvw-coincident realisation p. Let $\mathcal{R}_{uvw}(G)$ denote the generic uvw-rigidity matroid of G for some fixed distinct vertices $u, v, w \in V$ and let r_{uvw} be the rank function of the matroid $\mathcal{R}_{uvw}(G)$. For

$T \subseteq V(G)$, we use G_T to denote the simple graph obtained from G by contracting the vertices in T and deleting multiple edges and loops. When T has a small size, say for example, $T = \{x, y\}$, we also use G_{xy} to denote the same graph G_T.

For the case when there are two coincident vertices, u and v with $p(u) = p(v)$, which is studied in [6], we can define uv-rigidity, $\mathcal{R}_{uv}(G)$ and r_{uv} in the same way. The theorems below are the main results in [6].

Theorem 2.1.1. *[6] Let $G = (V, E)$ be a graph and $u, v \in V$ be distinct vertices. Then G is uv-rigid in $\mathbb{R}^2$ if and only if $G - uv$ and G_{uv} are both rigid in $\mathbb{R}^2$.*

Theorem 2.1.2. *[6] Let $G = (V, E)$ be a graph and $u, v \in V$ be distinct vertices. Then $r_{uv}(G) = \min\{r_2(G - uv), r_2(G_{uv} + 2)\}$.*

The two theorems below will be the main results of this chapter.

Theorem 2.1.3. *Let $G = (V, E)$ be a graph and let $u, v, w \in V$ be distinct vertices and $G' = G - uv - uw - vw$. Then G is uvw-rigid in $\mathbb{R}^2$ if and only if G' is rigid in $\mathbb{R}^2$ and G'_S is rigid in $\mathbb{R}^2$ for all $S \subseteq \{u, v, w\}$ with $|S| \geq 2$.*

Theorem 2.1.4. *Let $G = (V, E)$ be a graph, $u, v, w \in V$ be distinct vertices and $G' = G - uv - uw - vw$. Then $r_{uvw}(G) = \min\{r_2(G'), r_2(G'_{uv}) + 2, r_2(G'_{uw}) + 2, r_2(G'_{vw}) + 2, r_2(G'_{uvw}) + 4\}$.*

We will proceed in a similar way to [6]. We will first define a count matroid $\mathcal{M}_{uvw}(G)$ on the edge set of G in Section 2.2. We will then show that this matroid is equal to $\mathcal{R}_{uvw}(G)$.

The independent sets of the matroid $\mathcal{M}_{uvw}(G)$ will be defined to satisfy the general sparsity condition, $(2, 3)$-sparsity, for all set of edges F, and some special sparsity conditions if $|V(F) \cap \{u, v, w\}| \geq 2$. The assumption that $p(u) = p(v) = p(w)$ implies that if there is an edge e whose both endpoints are in $\{u, v, w\}$, then e corresponds to a zero row in the rigidity matrix of a uvw-coincident realisation (G, p). Hence such an edge e is a circuit in $\mathcal{R}_{uvw}(G)$. This illustrates why we need a special sparsity condition when an edge set F satisfies $|V(F) \cap \{u, v, w\}| \geq 2$.

Some lemmas we use to characterise $\mathcal{M}_{uvw}(G)$ will have very similar proof techniques to the lemmas Fekete, Jordán and Kaszanitzky used to characterise $\mathcal{M}_{uv}(G)$ in [6]. For the sake of completeness we will give detailed proofs of all our lemmas.

After characterising $\mathcal{M}_{uvw}(G)$ we will focus on the matroid $\mathcal{R}_{uvw}(G)$. We will give Henneberg type moves that preserve independence in $\mathcal{R}_{uvw}(G)$.

We will later show that $\mathcal{M}_{uvw}(G) \cong \mathcal{R}_{uvw}(G)$. Showing that independence in $\mathcal{R}_{uvw}(G)$ implies independence in $\mathcal{M}_{uvw}(G)$ will be the easy direction of this statement. For the other direction, we will proceed by induction and use the Henneberg type moves starting from a set of base graphs. We will give some specific realisations for the base graphs and the independence of these realisations were verified via a computer program.

Finally, using the fact that $\mathcal{M}_{uvw}(G) \cong \mathcal{R}_{uvw}(G)$, we will prove Theorems 2.1.3 and 2.1.4.

2.2 The Count Matroid

Let $G = (V, E)$ be a graph. For some $X \subseteq V$ let $G[X]$ denote the subgraph induced by X and let $E_G(X)$ be the set and $i_G(X)$ be the number of edges of $G[X]$. For a family $\mathcal{S} = \{S_1, S_2, \dots, S_k\}$, where $S_i \subseteq V$ for all $i = 1, \dots, k$, we define $E_G(\mathcal{S}) = \bigcup_{i=1}^k E_G(S_i)$ and put $i_G(\mathcal{S}) = |E_G(\mathcal{S})|$. We also define $\mathrm{cov}(\mathcal{S}) = \{(x, y) : \{x, y\} \subseteq S_i, \text{ for some } 1 \leq i \leq k\}$. We say that $\mathcal{S}$ *covers* $F \subseteq E$ if $F \subseteq \mathrm{cov}(\mathcal{S})$. A collection $\mathcal{K} = \{\mathcal{S}_1, \dots, \mathcal{S}_k\}$ of families of subsets of V is a *cover* of F if $F \subseteq \bigcup_{i=1}^k \mathrm{cov}(\mathcal{S}_i)$. The *degree* of a vertex v is denoted by $d_G(v)$ and the *neighbourhood* of v is denoted by $N_G(v)$. We may omit the subscripts referring to G if the graph is clear from the context.

Let G be a graph and $u, v, w \in V$ be three distinct vertices of G. Let $\mathcal{H} = \{H_1, \dots, H_k\}$ be a family with $H_i \subseteq V$, $1 \leq i \leq k$ and let $S \subseteq \{u, v, w\}$ with $|S| \geq 1$. We say that $\mathcal{H}$ is *S-compatible* if $S \subseteq H_i$ and $|H_i| \geq |S| + 1$ holds for all $1 \leq i \leq k$. The *S-value* of subsets H of V of size at least two is $2|H| - 3$ if $H \not\subseteq S$, and is 0 if $H \subseteq S$. It is denoted by $\mathrm{val}_S(H)$. The *value* of an S-compatible family $\mathcal{H}$ is

$$\mathrm{val}_S(\mathcal{H}) := \sum_{i=1}^{k} (2|H_i \setminus S| - 1) + 2(|S| - 1).$$

Let us give some motivation for the definition of $\mathrm{val}_S(\mathcal{H})$. We will use $\mathrm{val}_S(\mathcal{H})$ to characterise the rank function of $\mathcal{M}_{uvw}$, in a similar way that 1-thin covers were used to characterise the rank function of $\mathcal{R}_2(G)$ in Theorem 0.2.3. If we rewrite the function $\sum_{X\in\mathcal{X}}(2|X|-3)$ in this theorem by replacing $\mathcal{X}$ by $\mathcal{H}$, we obtain $\sum_{i=1}^{k}(2|H_i|-3)$. The families $\mathcal{X}$ in Theorem 0.2.3 are 1-thin ($|X_i \cap X_j| \leq 1$) whereas $H_i \cap H_j = S$ for distinct $H_i, H_j \in \mathcal{H}$. In some sense we want $\mathcal{H}$ to behave like a 1-thin family $\mathcal{X}$. We do this by regarding the set S as a single vertex. Then applying this idea in the count $\sum_{i=1}^{k}(2|H_i|-3)$, we obtain that

$$\sum_{i=1}^{k}(2|H_i|-3) - 2(|S|-1)(k-1)$$
$$= \sum_{i=1}^{k}(2|H_i \setminus S|-1) + 2(|S|-1) = \mathrm{val}_S(\mathcal{H})$$

Let $G=(V,E)$ be a graph and $u,v,w \in V$ be distinct vertices and $S \subseteq \{u,v,w\}$ be non-empty.

Definition 2.2.1. A graph $G=(V,E)$ is S-*sparse*, if for all $H \subseteq V$ with $|H| \geq 2$, we have $i_G(H) \leq \mathrm{val}_S(H)$ and for all S-compatible families $\mathcal{H}$ we have $i_G(\mathcal{H}) \leq \mathrm{val}_S(\mathcal{H})$.

We see that if G is S-sparse, then there is no edge between any distinct pair of vertices in S. It is easy to see that S-sparsity is just (2,3)-sparsity when $|S|=1$. Therefore will focus on the case $|S| \geq 2$.

Example 2.2.1. If G is S-sparse for all $S \subseteq \{u,v,w\}$ with $|S|=2$, then this does not imply that G is $\{u,v,w\}$-sparse. Let G be the graph on the left in Figure 2.1. Then G is S-sparse for all $S \subseteq \{u,v,w\}$ with $|S|=2$. However, it is not $\{u,v,w\}$-sparse as for the $\{u,v,w\}$-compatible family $\mathcal{H}=\{\{u,v,w,x_i\} : 1 \leq i \leq 5\}$ we have $i_G(\mathcal{H}) = 10 > 9 = \mathrm{val}_{\{u,v,w\}}(\mathcal{H})$.

We also know that $\{u,v,w\}$-sparsity does not imply S-sparsity for all $S \subseteq \{u,v,w\}$ with $|S|=2$. Let G be the graph on the right in Figure 2.1. It is $\{u,v,w\}$-sparse but not $\{u,v\}$-sparse as for the $\{u,v\}$-compatible family $\mathcal{H}=\{\{u,v,x_i\} : 1 \leq i \leq 3\}$ we have $i_G(\mathcal{H}) = 6 > 5 = \mathrm{val}_{\{u,v\}}(\mathcal{H})$.

2.2.1 Preliminary Results on Compatible Families

In this subsection we will give some useful tools that will help us to define and characterise $\mathcal{M}_{uvw}(G)$ for a graph G.

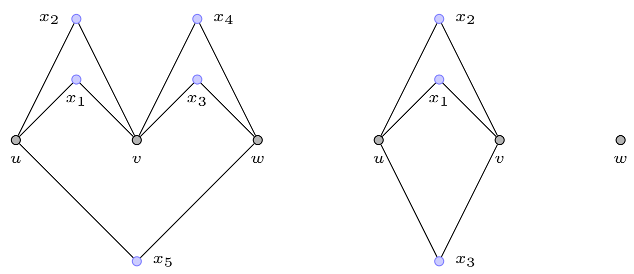

Figure 2.1: Comparison of S-sparsity for $S \subseteq \{u, v, w\}$ with $|S| \geq 2$.

Lemma 2.2.1. *Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be an S-compatible family for some $S \subseteq \{u, v, w\}$ with $|S| \geq 2$. Suppose $|H_i \cap H_j| \geq |S| + 1$ for some pair $1 \leq i < j \leq k$. Then there is an S-compatible family $\overline{\mathcal{H}}$ with $\operatorname{cov}(\mathcal{H}) \subseteq \operatorname{cov}(\overline{\mathcal{H}})$ for which $\operatorname{val}_S(\overline{\mathcal{H}}) \leq \operatorname{val}_S(\mathcal{H}) - 1$.*

Proof: We may assume that $i = k-1$ and $j = k$. Let $\overline{\mathcal{H}} = \{H_1, \ldots, H_{k-2}, (H_{k-1} \cup H_k)\}$. Then we have

$$\begin{aligned}
\operatorname{val}_S(\mathcal{H}) &= \sum_{l=1}^{k} (2|H_l \setminus S| - 1) + 2(|S| - 1) \\
&= \sum_{l=1}^{k-2} (2|H_l \setminus S| - 1) + 2(|S| - 1) + (2|H_{k-1} \setminus S| - 1) + (2|H_k \setminus S| - 1) \\
&= \sum_{l=1}^{k-2} (2|H_l \setminus S| - 1) + 2(|S| - 1) + (2|(H_{k-1} \cup H_k) \setminus S| - 1) \\
&\quad + (2|(H_{k-1} \cap H_k) \setminus S| - 1) \\
&\geq \operatorname{val}_S(\overline{\mathcal{H}}) + 1.
\end{aligned}$$

Clearly, $\operatorname{cov}(\mathcal{H}) \subseteq \operatorname{cov}(\overline{\mathcal{H}})$ holds. ■

We define a set $H \subseteq V(G)$ with $|H| \geq 2$ to be S-*tight*, if $i_G(H) = \operatorname{val}_S(H)$. Note that in an S-sparse graph G, if $H \not\subseteq S$, then H is S-tight if and only if H is $(2,3)$-tight. In this chapter we will use the terminology tight instead of $(2,3)$-tight for the sets $H \subseteq V(G)$. Similarly an S-compatible family $\mathcal{H}$ is S-*tight* or just *tight* when it is clear what S we refer to, if $i_G(\mathcal{H}) = \operatorname{val}_S(\mathcal{H})$.

Lemma 2.2.2. *Let $\mathcal{H} = \{H_1, \dots, H_k\}$ be an S-compatible family for some $S \subseteq \{u, v, w\}$ with $|S| \geq 2$ and $H_i \cap H_j = S$ for all $1 \leq i < j \leq k$, and $Y \subseteq V$ be a set of vertices with $|Y \cap S| \leq 1$ and $|Y \cap H_i| \geq 2$ for some $1 \leq i \leq k$. Then there is an S-compatible family $\overline{\mathcal{H}}$ with $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\overline{\mathcal{H}})$ for which $\mathrm{val}_S(\overline{\mathcal{H}}) \leq \mathrm{val}_S(\mathcal{H}) + \mathrm{val}_S(Y)$. Furthermore, if G is S-sparse and $\mathcal{H}$ and Y are both S-tight, then $\overline{\mathcal{H}}$ is also S-tight.*

Proof: By renumbering the sets of $\mathcal{H}$, if necessary, we may assume that $|Y \cap H_i| \geq 2$ if $i \geq j$ for some $j \leq k$, and $|Y \cap H_i| \leq 1$ for all $i \leq j-1$. Let $X = Y \cup \bigcup_{i=j}^{k} H_i$ and $\overline{\mathcal{H}} = \{H_1, \dots, H_{j-1}, X\}$. Then we have $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\overline{\mathcal{H}})$, and

$$
\begin{aligned}
&\mathrm{val}_S(\mathcal{H}) + \mathrm{val}_S(Y) = \sum_{i=1}^{k} (2|H_i \setminus S| - 1) + 2(|S| - 1) + (2|Y| - 3) \\
&= \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + 2(|S| - 1) + \sum_{i=j}^{k} (2|H_i \setminus S| - 1) + (2|Y| - 3) \\
&= \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + 2(|S| - 1) + (2|X \setminus S| - 1) \\
&\quad + 2|Y \cap S| - (k-j) + 2\sum_{i=j}^{k} |Y \cap (H_i \setminus S)| - 3 \\
&= \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + (2|X \setminus S| - 1) + 2(|S| - 1) \\
&\quad + 2|Y \cap S| - (k-j) + 2\sum_{i=j}^{k} |Y \cap H_i| - 2\sum_{i=j}^{k} |Y \cap S| - 3 \\
&= \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + (2|X \setminus S| - 1) + 2(|S| - 1) \\
&\quad - (k-j) + 2\sum_{i=j}^{k} |Y \cap (H_i)| - 2|Y \cap S|(k-j) - 3 \\
&\geq \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + (2|X \setminus S| - 1) + 2(|S| - 1) + 2\sum_{i=j}^{k} |Y \cap H_i| - 3(k-j+1) \\
&= \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + (2|X \setminus S| - 1) + 2(|S| - 1) + \sum_{i=j}^{k} (2|Y \cap H_i| - 3)
\end{aligned}
$$

$$= \mathrm{val}_S(\overline{\mathcal{H}}) + \sum_{i=j}^{k} \mathrm{val}_S(Y \cap H_i)$$

where for the inequality step we use $|Y \cap S| \leq 1$.

Now suppose that $\mathcal{H}$ and Y are S-tight. Then we have

$$i(\overline{\mathcal{H}}) + \sum_{i=j}^{k} i(Y \cap H_i) \geq i(\mathcal{H}) + i(Y) = \mathrm{val}_S(\mathcal{H}) + \mathrm{val}_S(Y)$$

$$\geq \mathrm{val}_S(\overline{\mathcal{H}}) + \sum_{i=j}^{k} \mathrm{val}_S(Y \cap H_i) \geq i(\overline{\mathcal{H}}) + \sum_{i=j}^{k} i(Y \cap H_i),$$

where the first inequality follows since the edges spanned by $\mathcal{H}$ or Y are spanned by $\overline{\mathcal{H}}$ and if some edge is spanned by both $\mathcal{H}$ and Y, then it is spanned by $Y \cap H_i$ for some i. The first equality holds because $\mathcal{H}$ and Y are S-tight, and the second inequality holds by our calculations above. The last inequality holds because G is S-sparse. Hence equality must hold everywhere, which implies that $\overline{\mathcal{H}}$ is also S-tight. ■

Lemma 2.2.3. *Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be an S-compatible family for some $S \subseteq \{u, v, w\}$ with $|S| \geq 2$ and $H_i \cap H_j = S$ for all $1 \leq i < j \leq k$, and let $Y \subseteq V$ be a set of vertices with $Y \cap S = \emptyset$ and $|Y \cap H_i| \leq 1$ for all $1 \leq i \leq k$, for which $|Y \cap H_i| = |Y \cap H_j| = 1$ for some pair $1 \leq i < j \leq k$. Then there is an S-compatible family $\overline{\mathcal{H}}$ with $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\overline{\mathcal{H}})$ for which $\mathrm{val}_S(\overline{\mathcal{H}}) = \mathrm{val}_S(\mathcal{H}) + \mathrm{val}_S(Y)$. Furthermore, if G is S-sparse and $\mathcal{H}$ and Y are both S-tight, then $\overline{\mathcal{H}}$ is also S-tight.*

Proof: We may assume that $i = k-1$ and $j = k$. Let $\overline{\mathcal{H}} = \{H_1, \ldots, H_{k-2}, (H_{k-1} \cup H_k \cup Y)\}$. Then we have

$$\mathrm{val}_S(\mathcal{H}) + \mathrm{val}_S(Y) = \sum_{i=1}^{k} (2|H_i \setminus S| - 1) + 2(|S| - 1) + (2|Y| - 3)$$

$$= \sum_{i=1}^{k-2} (2|H_i \setminus S| - 1) + 2(|S| - 1) + (2|H_{k-1} \setminus S| - 1) + (2|H_k \setminus S| - 1) + (2|Y| - 3)$$

$$
\begin{aligned}
&= \sum_{i=1}^{k-2} (2|H_i \setminus S| - 1) + 2(|S| - 1) + (2(|H_{k-1} \setminus S| + |H_k \setminus S| + |Y|) - 1) - 4 \\
&= \sum_{i=1}^{k-2} (2|H_i \setminus S| - 1) + (2|(H_{k-1} \cup H_k \cup Y) \setminus S| - 1) + 2(|S| - 1) \\
&\quad + 2|Y \cap (H_{k-1} \setminus S)| + 2|Y \cap (H_k \setminus S)| - 4 \\
&= \mathrm{val}_S(\overline{\mathcal{H}}).
\end{aligned}
$$

Clearly, we have $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\overline{\mathcal{H}})$. Now suppose that G is S-sparse and $\mathcal{H}$ and Y are S-tight. then we have

$$
i(\mathcal{H}) + i(Y) = \mathrm{val}_S(\mathcal{H}) + \mathrm{val}_S(Y) = \mathrm{val}_S(\overline{\mathcal{H}}) \geq i(\overline{\mathcal{H}}) \geq i(\mathcal{H}) + i(Y)
$$

where the last inequality follows since $|Y \cap H_i| \leq 1$ for all $1 \leq i \leq k$. Hence equality must hold everywhere which implies that $\overline{\mathcal{H}}$ is also S-tight. ■

Lemma 2.2.4. *Let $G = (V, E)$ be S-sparse for some $S \subseteq \{u, v, w\}$ with $|S| \geq 2$ and let $X, Y \subseteq V$ be S-tight sets in G with $|X \cap Y| \geq 2$ and $X, Y \not\subseteq S$. Then $X \cap Y \not\subseteq S$, and $X \cup Y$ and $X \cap Y$ are S-tight.*

Proof: First note that as G is S-sparse we have

$$
\begin{aligned}
2|X| - 3 + 2|Y| - 3 = \mathrm{val}_S(X) + \mathrm{val}_S(Y) &= i(X) + i(Y) \\
&\leq i(X \cap Y) + i(X \cup Y) \\
&\leq \mathrm{val}_S(X \cap Y) + \mathrm{val}_S(X \cup Y) \\
&= \mathrm{val}_S(X \cap Y) + 2|X \cup Y| - 3
\end{aligned}
$$

Suppose $X \cap Y$ is a subset of S. Then $\mathrm{val}_S(X \cap Y) = 0$ and putting this in the above equations gives $2|X| - 3 + 2|Y| - 3 \leq 2|X \cup Y| - 3 = 2|X| + 2|Y| - 2|X \cap Y| - 3 \leq 2|X| + 2|Y| - 7$, a contradiction.

Hence $X \cap Y$ is not a subset of S. Then we have $\mathrm{val}_S(X \cap Y) = 2|X \cap Y| - 3$ and hence equality holds throughout. In particular, $\mathrm{val}_S(X \cup Y) = i(X \cup Y)$ and $\mathrm{val}_S(X \cap Y) = i(X \cap Y)$, so $X \cup Y$ and $X \cap Y$ are S-tight. ■

Lemma 2.2.5. *Let $G = (V, E)$ be S-sparse for all $S \subseteq \{u, v, w\}$ with $|S| \geq 2$. Suppose that there is a tight U-compatible family in G for some $U \subseteq \{u, v, w\}$ with $|U| \geq 2$. Then there is a unique family $\mathcal{H}_{\max}$ with the properties that $\mathcal{H}_{\max}$ is a tight T-compatible family for some $U \subseteq T \subseteq \{u, v, w\}$, and $\operatorname{cov}(\mathcal{H}) \subseteq \operatorname{cov}(\mathcal{H}_{\max})$ for all tight S-compatible families $\mathcal{H}$ of G for all $S \subseteq \{u, v, w\}$ with $|S| \geq 2$.*

Proof: It follows from Lemma 2.2.1 that if $\mathcal{H} = \{H_1, \ldots, H_k\}$ is a tight S-compatible family in G then $H_i \cap H_j = S$ for all $1 \leq i \leq k$. Now consider a pair $\mathcal{H}_1 = \{H_1, \ldots, H_k\}$ and $\mathcal{H}_2 = \{\overline{H}_1, \ldots, \overline{H}_l\}$ of tight S_i-compatible families with $S_i \subseteq \{u, v, w\}$ and $|S_i| \geq 2$ for $i = 1, 2$.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the bipartite graph with bipartition $(\mathcal{H}_1, \mathcal{H}_2)$, and edge set

$$\mathcal{E} := \{H_i\overline{H}_j : |(H_i \setminus S_1) \cap (\overline{H}_j \setminus S_2)| \geq 1, 1 \leq i \leq k, 1 \leq j \leq l\}.$$

Let $(\mathcal{V}_i, \mathcal{F}_i)$, $1 \leq i \leq r$ be the connected components of $\mathcal{G}$. Define $V_i = \bigcup_{H \in \mathcal{V}_i} H$, $1 \leq i \leq r$ and put

$$\mathcal{H}_{\text{union}} := \{V_i \cup S_1 \cup S_2 : 1 \leq i \leq r\},$$

$$\mathcal{H}_{\text{int}} := \{H_i \cap \overline{H}_j : H_i\overline{H}_j \in \mathcal{E}\},$$

Note that $\mathcal{H}_{\text{union}}$ and $\mathcal{H}_{\text{int}}$ are $(S_1 \cup S_2)$- and $(S_1 \cap S_2)$-compatible, respectively. We see that every edge in E which is covered by either $\mathcal{H}_1$ or $\mathcal{H}_2$ is covered by $\mathcal{H}_{\text{union}}$ and every edge covered by both $\mathcal{H}_1$ and $\mathcal{H}_2$ is covered by $\mathcal{H}_{\text{int}}$. This implies that $i(\mathcal{H}_1) + i(\mathcal{H}_2) \leq i(\mathcal{H}_{\text{union}}) + i(\mathcal{H}_{\text{int}})$. Since $|\mathcal{V}| = k + l$ and r is the number of connected components of $\mathcal{G}$,

$$r + |\mathcal{E}| \geq k + l. \tag{2.1}$$

We also have

$$\begin{aligned}
&\sum_{i=1}^{r} (|V_i \cup S_1 \cup S_2| - |S_1 \cup S_2|) + \sum_{H_i\overline{H}_j \in \mathcal{E}} (|H_i \cap \overline{H}_j| - |S_1 \cap S_2|) \\
=&\sum_{i=1}^{k} (|H_i| - |S_1|) + \sum_{i=1}^{l} (|\overline{H}_i| - |S_2|)
\end{aligned} \tag{2.2}$$

as a vertex $x \notin S_1 \cup S_2$ contributes the same amount (one or two) to both sides of (2.2), and a vertex $s \in S_1 \cup S_2$ contributes zero to both sides of (2.2).

Then we have

$$\begin{aligned}
&\sum_{i=1}^{k}(2|H_i \setminus S_1| - 1) + 2(|S_1| - 1) + \sum_{i=1}^{l}(2|\overline{H}_i \setminus S_2| - 1) + 2(|S_2| - 1) \\
&= \mathrm{val}_{S_1}(\mathcal{H}_1) + \mathrm{val}_{S_2}(\mathcal{H}_2) \\
&= i(\mathcal{H}_1) + i(\mathcal{H}_2) \\
&\leq i(\mathcal{H}_{\text{union}}) + i(\mathcal{H}_{\text{int}}) \\
&\leq \mathrm{val}_{S_1 \cup S_2}(\mathcal{H}_{\text{union}}) + \mathrm{val}_{S_1 \cap S_2}(\mathcal{H}_{\text{int}}) \\
&= \sum_{i=1}^{r}(2|(V_i \cup S_1 \cup S_2) \setminus (S_1 \cup S_2)| - 1) + 2(|S_1 \cup S_2| - 1) \\
&\quad + \sum_{H_i\overline{H}_j \in \mathcal{E}}(2|(H_i \cap \overline{H}_j) \setminus (S_1 \cap S_2)| - 1) + 2(|S_1 \cap S_2| - 1) \\
&= \sum_{i=1}^{r} 2(|V_i \cup S_1 \cup S_2| - |S_1 \cup S_2|) + 2(|S_1 \cup S_2| - 1) - r \\
&\quad + \sum_{H_i\overline{H}_j \in \mathcal{E}} 2(|H_i \cap \overline{H}_j| - |S_1 \cap S_2|) + 2(|S_1 \cap S_2| - 1) - |\mathcal{E}| \\
&\leq \sum_{i=1}^{k} 2(|H_i| - |S_1|) + \sum_{i=1}^{l} 2(|\overline{H}_i| - |S_2|) \\
&\quad + 2(|S_1 \cup S_2| - 1) + 2(|S_1 \cap S_2| - 1) - k - l \\
&= \sum_{i=1}^{k} 2(|H_i| - |S_1|) + \sum_{i=1}^{l} 2(|\overline{H}_i| - |S_2|) + 2|S_1| + 2|S_2| - 2 - 2 - k - l \\
&= \sum_{i=1}^{k}(2|H_i \setminus S_1| - 1) + 2(|S_1| - 1) + \sum_{i=1}^{l} 2|\overline{H}_i \setminus S_2| - 1 + 2(|S_2| - 1),
\end{aligned}$$

where the third inequality follows from (2.1) and (2.2), and the second last equality follows from the formula $|S_1 \cup S_2| + |S_1 \cap S_2| = |S_1| + |S_2|$. Therefore equality must hold throughout. Hence we can deduce that $\mathcal{H}_{\text{union}}$ and $\mathcal{H}_{\text{int}}$ are both tight. Clearly, we have $\mathrm{cov}(\mathcal{H}_1) \cup \mathrm{cov}(\mathcal{H}_2) \subseteq \mathrm{cov}(\mathcal{H}_{\text{union}})$. The lemma now follows by choosing a $T \subseteq \{u, v, w\}$ and a tight T-compatible family $\mathcal{H}_{\max}$ of G for which $\mathrm{cov}(\mathcal{H}_{\max})$ is maximal. ■

2.2.2 The Matroid and its Rank Function

Let $G = (V, E)$ be a graph and $u, v, w \in V$ be distinct vertices of G. We say G is *uvw-sparse* if it is S-sparse for all $S \subseteq \{u, v, w\}$ with $|S| \geq 2$. In this subsection we prove that the family

$$\mathcal{I}_G = \{F : F \subseteq E, H = (V, F) \text{ is } uvw\text{-sparse}\} \tag{2.3}$$

is a family of independent sets of a matroid on E. We need the following definition.

Let $\mathcal{H} = \{H_1, \ldots, H_t\}$ be an S-compatible family and let $X_1, \ldots, X_k$ be subsets of V of size at least two. Recall that the collection $\mathcal{K} = \{X_1, \ldots, X_k\}$ is 1-*thin* if
(i) $|X_i \cap X_j| \leq 1$ for all pairs $1 \leq i < j \leq k$.

Definition 2.2.2. The collection $\mathcal{L} = \{\mathcal{H}, X_1, \ldots, X_k\}$ where $\mathcal{H}$ is either the empty set, or an S-compatible family for some $S \subseteq \{u, v, w\}$ with $|S| \geq 2$ is 1-*thin* if (i) holds and
(ii) $H_i \cap H_j = S$ for all pairs $1 \leq i < j \leq t$, and
(iii) $|X_i \cap \bigcup_{j=1}^{t} H_j| \leq 1$ for all $1 \leq i \leq k$.

We define the *value* of $\mathcal{L}$ as

$$\text{val}(\mathcal{L}) := \begin{cases} \text{val}_S(\mathcal{H}) + \sum_{i=1}^{k} 2|X_i| - 3, & \text{if } \mathcal{H} \neq \emptyset \\ \sum_{i=1}^{k} 2|X_i| - 3, & \text{if } \mathcal{H} = \emptyset \end{cases}$$

It is clear that if G is uvw-sparse, then $i_G(\mathcal{L}) \leq \text{val}(\mathcal{L})$ holds for all 1-thin $\mathcal{L} = \{\mathcal{H}, X_1, \ldots, X_k\}$. For a graph $G = (V, E)$ with distinct vertices $u, v, w \in V$, we can now characterise the matroid $\mathcal{M}_{uvw}(G) = (E, \mathcal{I}_G)$. Note that after we prove $\mathcal{M}_{uvw}(G) \cong \mathcal{R}_{uvw}(G)$, this characterisation will be the uvw-coincident counterpart of Theorem 0.2.3.

Theorem 2.2.6. *Let $G = (V, E)$ be a graph and $u, v, w \in V$ be distinct vertices of G. Then $\mathcal{M}_{uvw}(G) = (E, \mathcal{I}_G)$ is a matroid on the ground set E, where $\mathcal{I}_G$ is defined by (2.3). The rank of a set $E' \subseteq E$ in $\mathcal{M}_{uvw}(G)$ is equal to*

$$\min\{\text{val}(\mathcal{L}) : \mathcal{L} \textit{ is a } 1\textit{-thin cover of } E' \setminus E(\{u, v, w\})\}.$$

Proof: We will proceed by showing that $\mathcal{I}_G$ satisfies the conditions (I1), (I2) and (I3$'$) of Definition 0.1.16. As (I1) and (I2) are trivial, we will only show (I3$'$) holds. Let $\mathcal{I} = \mathcal{I}_G$, $E' \subseteq E \setminus E(\{u,v,w\})$ and $F \subseteq E'$ be a maximal subset of E' in $\mathcal{I}$. Since $F \in \mathcal{I}$ we have $|F| \leq \text{val}(\mathcal{K})$ for all covers $\mathcal{K}$ of E'. We will show that there is a 1-thin cover $\mathcal{K}$ of E' with $|F| = \text{val}(\mathcal{K})$, from which the theorem will follow.

Let $J = (V, F)$ denote the subgraph induced by the edge set F. First suppose that there is no tight S-compatible family for all $S \subseteq \{u,v,w\}$ with $|S| \geq 2$ in J and consider the following 1-thin cover of F:

$$\mathcal{K}_1 = \{X_1, X_2, \ldots, X_k\},$$

where $X_1, X_2, \ldots, X_k$ are all of the maximal tight sets in J. Since every edge $f \in E \setminus E(\{u,v,w\})$ induces a tight set in J, $\mathcal{K}_1$ is a cover of F. It is 1-thin by Lemma 2.2.4. Thus

$$|F| = \sum_{j=1}^{k} |E_J(X_j)| = \sum_{j=1}^{k} (2|X_j| - 3) = \text{val}(\mathcal{K}_1)$$

follows. We claim that $\mathcal{K}_1$ is a cover of E'. To see this consider an edge $ab = e \in E' - F$. Since F is a maximal subset of E' in $\mathcal{I}$ we have $F + e \notin \mathcal{I}$. By our assumption there is no tight S-compatible family in J, and hence there must be a tight set X in J with $a, b \in X$. Hence $X \subseteq X_i$ for some $1 \leq i \leq k$ which implies $\mathcal{K}_1$ covers e, too.

Next suppose there is a tight S-compatible family for some $S \subseteq \{u,v,w\}$ with $|S| \geq 2$ in J and consider the following cover of F:

$$\mathcal{K}_2 = \{\mathcal{H}_{\max}, X_1, X_2, \ldots, X_k\},$$

where $\mathcal{H}_{\max} = \{H_1, H_2, \ldots, H_l\}$ is the tight T-compatible family of G for which $\text{cov}(\mathcal{H}_{\max})$ is maximal (c.f. Lemma 2.2.5) and $X_1, X_2, \ldots, X_k$ are maximal tight sets of $J' = (V, F - E(\mathcal{H}_{\max}))$. We see that $\mathcal{K}_2$ is indeed a cover of F. Lemma 2.2.4 implies $|X_i \cap X_j| \leq 1$, Lemma 2.2.1 implies $H_i \cap H_j = T$ for all $i \neq j$, and Lemmas 2.2.2 and 2.2.3 imply that $|X_i \cap \bigcup_{j=1}^{l} H_j| \leq 1$ for all $1 \leq i \leq k$. Hence the cover $\mathcal{K}_2$

is 1-thin and we have

$$\begin{aligned}|F| &= \sum_{i=1}^{l} |E_J(H_i)| + \sum_{j=1}^{k} |E_J(X_j)| \\ &= \sum_{i=1}^{l} (2|H_i \setminus T| - 1) + 2(|T| - 1) + \sum_{j=1}^{k} (2|X_j| - 3) = \mathrm{val}_T(\mathcal{K}_2).\end{aligned}$$

We will show that $\mathcal{K}_2$ is a cover of E'. As above, let $ab = e \in E' - F$ be an edge. By the maximality of F we have $F + e \notin \mathcal{I}$. Thus either there is a tight set $X \subseteq V$ in J with $a, b \in X$ or there is a tight S-compatible family $\overline{\mathcal{H}} = \{Y_1, \ldots, Y_t\}$ for some $S \subseteq \{u, v, w\}$ with $|S| \geq 2$ in J and $a, b \in Y_i$ for some $1 \leq i \leq t$.

In the latter case Lemma 2.2.5 implies that $\mathrm{cov}(\overline{\mathcal{H}}) \subseteq \mathrm{cov}(\mathcal{H}_{\max})$ and hence e is covered by $\mathcal{K}_2$. In the former case, when $a, b \in X$ for some tight set X in J we have two possibilities. First suppose that $|X \cap \bigcup_{i=1}^{l} H_i| \geq 2$. Then we can deduce that $X \subseteq H_i$ for some $1 \leq i \leq l$ by using Lemma 2.2.2 or 2.2.3 and the maximality of $\mathcal{H}_{\max}$ which implies that $\mathcal{K}_2$ covers e. Next suppose that $|X \cap \bigcup_{i=1}^{l} H_i| \leq 1$. Then $E(X) \subseteq E(J')$ and hence $X \subseteq X_i$ for some $1 \leq i \leq k$, since every edge of J' induces a tight set and every tight set is contained in a maximal tight set. Hence e is covered by $\mathcal{K}_2$, as claimed. ■

2.2.3 Independence in $\mathcal{R}_{uvw}$ and $\mathcal{M}_{uvw}$

Let $G = (V, E)$ be a graph and let $u, v, w \in V$ be distinct vertices and $S \subseteq \{u, v, w\}$ with $|S| \geq 2$. Let G_S denote the graph obtained from G by contracting the vertices in S into a new vertex z_S (and deleting the resulting loops and parallel edges). Given a realisation (G_S, p_S), we obtain an S-coincident realisation (G, p) of G by putting $p(x) = p_S(z_S)$ if $x \in S$ and $p(x) = p_S(x)$ if $x \notin S$. Furthermore, each vector q_S in the kernel of $R(G_S, p_S)$ (an infinitesimal motion q_S of (G_S, p_S)) determines a vector q in the kernel of $R(G, p)$ (an infinitesimal motion q of (G, p)) by setting $q(x) = q_S(z_S)$ if $x \in S$ and $q(x) = q_S(x)$ if $x \notin S$. It follows that

$$\dim\ker R(G, p) \geq \dim\ker R(G_S, p_S) \tag{2.4}$$

We can use this fact to prove that independence in $\mathcal{R}_{uvw}(G)$ implies independence in $\mathcal{M}_{uvw}(G)$.

Lemma 2.2.7. *Let $G = (V, E)$ be a graph and let $u, v, w \in V$ be distinct vertices. Suppose E is independent in $\mathcal{R}_{uvw}(G)$. Then E is independent in $\mathcal{M}_{uvw}(G)$.*

Proof: Let (G, p) be an independent uvw-coincident realisation of G. Independence implies that $i(H) \le \text{val}(H)$ holds for all $H \subseteq V$ with $|H| \ge 2$. Since $p(x) = p(y)$ when $x, y \in S \subseteq \{u, v, w\}$, we see that there is no edge between any two members of S.

Let $\mathcal{H} = \{H_1, \dots, H_k\}$ be an S-compatible family for some $S \subseteq \{u, v, w\}$ with $|S| \ge 2$ and consider the subgraph $F = (\bigcup_{i=1}^k H_i, \bigcup_{i=1}^k E(H_i))$. By contracting S into one vertex in F we obtain the graph F_S, in which $\mathcal{X} = \{H_1/S, \dots, H_k/S\}$ is a cover. Thus $r_2(F_S) \le \sum_{i=1}^k \big(2(|H_i| - (|S| - 1)) - 3\big)$ by Theorem 0.2.3. This bound and (2.4) imply that

$$\begin{aligned}\dim\ker R(F, p) \ge& \dim\ker R(F_S, p_S)\\ &\ge 2\big(|\bigcup_{i=1}^k H_i| - (|S| - 1)\big) - \sum_{i=1}^k \big(2|H_i| - 2(|S| - 1) - 3\big).\end{aligned}$$

Since (G, p) is an independent uvw-coincident realisation, we have

$$\begin{aligned}i_F(\mathcal{H}) = |E(F)| &\le 2\left|\bigcup_{i=1}^k H_i\right| - \left(2\left(\left|\bigcup_{i=1}^k H_i\right| - (|S| - 1)\right) - \sum_{i=1}^k (2|H_i| - 2(|S| - 1) - 3)\right)\\ &= \sum_{i=1}^k (2|H_i \setminus S| - 1) + 2(|S| - 1) = \text{val}_S(\mathcal{H}).\end{aligned}$$

Since $S \subseteq \{u, v, w\}$ with $|S| \ge 2$ is arbitrary, E is independent in $\mathcal{M}_{uvw}(G)$. ■

We next show that independence in $\mathcal{M}_{uvw}$ implies independence in $\mathcal{R}_{uvw}$. We will define some special operations that are based on Henneberg's 0- and 1-extension operations. Let $G = (V, E)$ be a graph and $S \subseteq \{u, v, w\} \subseteq V$ with $|S| \ge 2$. The 0-S-*extension* operation is a 0-extension operation on a pair a, b with $\{a, b\} \not\subset S$. The 1-S-*extension* operation is a 1-extension operation on some edge ab and vertex

c for which $|S \cap \{a, b, c\}| \leq 1$. The inverse operations are called 0-S-*reduction* and 1-S-*reduction*, respectively.

Lemma 2.2.8. *Let $G = (V, E)$ be a graph, (G, p_S) be an S-coincident realisation of G for some $S \subseteq \{u, v, w\}$ with $|S| \geq 2$ and suppose that G' is obtained from G by a 0-S-extension that adds the new vertex x. Let (G', q_S) be an S-coincident realisation with $q_S|_{V(G)} = p_S$ and $q_S(x)$ is not on the line through the positions of the neighbours of x. Then the rows of $R(G', q_S)$ are linearly independent if and only if the rows of $R(G, p_S)$ are linearly independent.*

Proof: Immediately follows from Lemma 0.1.7 with $d = 2$. ■

Lemma 2.2.9. *Let (G, p_S) be a generic S-coincident realisation of G for some $S \subseteq \{u, v, w\}$ with $|S| \geq 2$. Let G' be a graph obtained from G by a 1-S-extension operation and (G', q_S) be a generic S-coincident realisation with $q_S|_{V(G)} = p_S$. Suppose the rows of $R(G, p_S)$ are linearly independent. Then the rows of $R(G', q_S)$ are linearly independent.*

Proof: Immediately follows from Lemma 0.1.8 with $d = 2$. ■

Lemma 2.2.10 below is called the vertex splitting lemma, see [31]. We give its proof here because in [31], only the generic version is stated and no proof is given.

Lemma 2.2.10. *Let G be a graph with edges $zz_1, zz_2 \ldots, zz_k, \ldots, zz_m$. Let G' be the graph obtained from G by deleting the edges $zz_3, \ldots, zz_k$ and adding a new vertex z' incident with new edges $z'z_1, z'z_2, \ldots, z'z_k$. Let (G, p) be a realisation of G in $\mathbb{R}^2$. Suppose the rows of $R(G, p)$ are linearly independent, and $p(z), p(z_1)$ and $p(z_2)$ are not collinear. Define $q : V(G') \to \mathbb{R}^2$ by $q(x) = p(x)$ if $x \in V(G)$ and $q(z') = p(z)$. Then the rows of $R(G', q)$ are linearly independent.*

Proof: Let us relabel the vertices z, z' of G' by setting $z = y$ and $z' = x$. Note that we did not relabel z in G. The matrix below is the rigidity matrix $R(G,p)$.

$$
\begin{array}{cc}
 & \begin{array}{cccc} z & z_1 & z_2 & \end{array} \\
\begin{array}{c} zz_1 \\ zz_2 \\ \vdots \\ zz_k \\ \vdots \\ zz_m \\ \end{array} &
\begin{bmatrix}
p(z)-p(z_1) & p(z_1)-p(z) & (0,0) & \cdots \\
p(z)-p(z_2) & (0,0) & p(z_2)-p(z) & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
p(z)-p(z_k) & (0,0) & (0,0) & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
p(z)-p(z_m) & (0,0) & (0,0) & \cdots \\
0 & \multicolumn{3}{c}{R(G-z,p)}
\end{bmatrix}
\end{array}
$$

The matrix $R(G',q)$ below is the rigidity matrix of (G',q).

$$
\begin{array}{cc}
 & \begin{array}{ccccc} x & y & z_1 & z_2 & \end{array} \\
\begin{array}{c} xz_1 \\ yz_1 \\ xz_2 \\ yz_2 \\ xz_3 \\ \vdots \\ xz_k \\ yz_{k+1} \\ \vdots \\ yz_m \\ \end{array} &
\begin{bmatrix}
p(x)-p(z_1) & (0,0) & p(z_1)-p(x) & (0,0) & \cdots \\
(0,0) & p(y)-p(z_1) & p(z_1)-p(y) & (0,0) & \cdots \\
p(x)-p(z_2) & (0,0) & (0,0) & p(z_2)-p(x) & \cdots \\
(0,0) & p(y)-p(z_2) & (0,0) & p(z_2)-p(y) & \cdots \\
p(x)-p(z_3) & (0,0) & (0,0) & & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
p(x)-p(z_k) & (0,0) & (0,0) & (0,0) & \cdots \\
p(y)-p(z_k) & (0,0) & (0,0) & (0,0) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
p(y)-p(z_m) & (0,0) & (0,0) & (0,0) & \cdots \\
0 & 0 & \multicolumn{3}{c}{R(G-z,p)}
\end{bmatrix}
\end{array}
$$

Let $[e]_p$ and $[e]_q$ denote the row corresponding to the edge e in $R(G,p)$ and $R(G',q)$, respectively. If the rows of $R(G',q)$ are not linearly independent then there exists scalars $t,s,l,n,\ a_3,\dots,a_k,b_{k+1},\dots,b_m,c_1,\dots,c_r$ not all zero such that

$$t[xz_1]_q+s[yz_1]_q+l[xz_2]+n[yz_2]_q+\sum_{i=3}^{k}a_i[xz_i]_q+\sum_{i=k+1}^{m}b_i[yz_i]_q+\sum_{i=1}^{r}c_i[e_i]_q=0. \quad (2.5)$$

Moreover we see that at least one of a_i, b_j or c_k is non-zero, since otherwise we would have $t[xz_1]_q + s[yz_1]_q + l[xz_2]_q + n[yz_2]_q = 0$. This corresponds to a linear dependence in $R(G'', q)$ where G'' is the graph obtained from G by deleting the edges $zz_3, \ldots, zz_k, \ldots, zz_m$ and adding a new vertex z' incident with z_1 and z_2 by a 0-extension. As edge deletion and 0-extension preserve independence (by Lemma 0.1.7, since $q(z'), q(z_1)$ and $q(z_2)$ are not collinear), we have a contradiction.

Now by using (2.5) we can deduce that

$$(s+t)[zz_1]_p + (l+n)[zz_2]_p + \sum_{i=3}^{k} a_i[zz_i]_p + \sum_{i=k+1}^{m} b_i[zz_i]_p + \sum_{i=1}^{r} c_i[e_i]_p = 0$$

as $p(z) = p(x) = q(y)$. Clearly, not all of the scalars are zero. This contradicts the fact that (G, p) is independent. ■

Lemma 2.2.11. *Let $G = (V, E)$ be a graph and let $u, v, w \in V$ be distinct vertices. Suppose that E is independent in $\mathcal{M}_{uvw}(G)$ and $z \in V \setminus \{u, v, w\}$ is a vertex with $d(z) = 3$ and $|N(z) \cap \{u, v, w\}| \leq 1$. Then there is a 1-uvw-reduction at z which leads to an independent graph G' in $\mathcal{M}_{uvw}(G')$.*

Proof: Let $F = \{ab \notin E : a, b \in N(z)\}$, $G_1 = G - z + F$ and $G_2 = G + F$. Let $r_{\mathcal{M}}$ denote the rank function of $\mathcal{M}_{uvw}$. Suppose that the statement is false and we have $r_{\mathcal{M}}(G_1) \leq r_{\mathcal{M}}(G) - 3$. Take a base B_1 of $\mathcal{M}_{uvw}(G_1)$ that contains the triangle on $N(z)$ and extend it to a base B_2 of $\mathcal{M}_{uvw}(G_2)$. Since K_4 is a circuit of $\mathcal{M}_{uvw}(G_2)$ when $E(K_4) \cap E(\{u, v, w\}) = \emptyset$, we have $r_{\mathcal{M}}(G_2) \leq r_{\mathcal{M}}(G_1) + 2$. Hence $r_{\mathcal{M}}(G) \leq r_{\mathcal{M}}(G_2) \leq r_{\mathcal{M}}(G) - 1$, a contradiction. ■

Lemma 2.2.12. *Let (G, p) be a framework. Suppose that $(G[U], p|_U)$ is infinitesimally rigid for some $U \subseteq V(G)$. Let Y be the set of vertices in U which are adjacent to vertices in $V(G) \setminus U$ with $|Y| \geq 2$ and $p|_Y$ is generic. Let G' be a graph whose vertex set is Y for which $(G', p|_Y)$ is infinitesimally rigid. Let G'' be the graph $((V(G) \setminus U) \cup Y, E_G(V(G) \setminus U) \cup E_G(Y, V(G) \setminus U) \cup E(G'))$, and $q = p|_{(V(G) \setminus U) \cup Y}$. Then (G, p) is infinitesimally rigid if and only if (G'', q) is infinitesimally rigid.*

Proof: Note that an infinitesimal motion t of the whole $\mathbb{R}^2$ can be written as $t = aT_x + bT_y + cT_r$, where $T_x = (1,0)$ (infinitesimal translation along x-axis), $T_y = (0,1)$ (infinitesimal translation along y-axis), $T_r((s_1,s_2)) = (-s_2, s_1)$ (counterclockwise infinitesimal rotation about the origin) for a point $(s_1,s_2) \in \mathbb{R}^2$, and a,b,c are scalars.

First suppose (G'', q) is infinitesimally rigid. Let H be the graph obtained from G by adding the edges in $E_(G')$. Consider the frameworks (G'', q) and $(H[U], p|_U)$. Note that as $(G[U], p|_U)$ is infinitesimally rigid and G is a subgraph of H, we have $(H[U], p|_U)$ is infinitesimally rigid. We first show that (H,p) is infinitesimally rigid. Since $H = G'' \cup H[U]$, and (G'', q) and $(H[U], p|_U)$ are infinitesimally rigid, every infinitesimal motion of H must induce a trivial infinitesimal motion (an infinitesimal motion of the whole $\mathbb{R}^2$) on (G'', q) and $(H[U], p|_U)$. Let t be an infinitesimal motion of (H,p). Then we have $t|_{V(G'')} = a_1T_x + b_1T_y + c_1T_r$ and $t|_U = a_2T_x + b_2T_y + c_2T_r$ for some scalars $a_1, b_1, c_1, a_2, b_2, c_2$. Since $V(G'') \cap U = Y$ has size at least two, $t|_{V(G'')}$ and $t|_U$ must agree on at least two vertices $x, y \in Y$. Applying the velocities $t|_{V(G'')}$ and $t|_U$ to $p(x)$ and $p(y)$ and using the fact that $p|_Y$ is generic, we obtain $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$. This implies that t is a trivial infinitesimal motion. Since t is arbitrary, we conclude that (H,p) is infinitesimally rigid. Now the fact that $(G[U], p)$ is infinitesimally rigid, and $(E(H) \setminus E(G)) \subseteq E_H(U)$, every edge in $E(H) \setminus E(G)$ is contained in a different circuit in (H,p). Hence we can remove these edges and preserve being infinitesimally rigid, that is (G,p) is infinitesimally rigid.

Now suppose (G,p) is infinitesimally rigid, but (G'', q) is not. Then there exists a non-trivial infinitesimal motion t of (G'', q). Since $(G', p|_Y)$ is infinitesimally rigid and $q|_Y = p|_Y$, we see that $(G''[Y], q|_Y)$ is infinitesimally rigid and hence $t|_Y$ corresponds to an infinitesimal motion of the whole $\mathbb{R}^2$, that is, $t|_Y = aT_x + bT_y + cT_r$ for some scalars a, b, c. Since $Y \subseteq U$ and $(G[U], p|_U)$ is infinitesimally rigid, we can extend t to an infinitesimal motion t' of (G,p) by setting $t'|_U = aT_x + bT_y + cT_r$. The fact that t is a non-trivial infinitesimal motion of (G'', q) implies that t' is a non-trivial infinitesimal motion of (G,p), contradicting the fact that (G,p) is infinitesimally rigid. ■

Lemma 2.2.13. *Let $G = (V,E)$ be a graph and $u, v, w \in V$ be distinct. Let N^* be the set of vertices having at least two neighbours in $\{u,v,w\}$, $X = N^* \cup \{u,v,w\}$,* *and $Y = V \setminus X$. Suppose $G[Y]$ is minimally rigid in $\mathcal{R}_2$, $|E| = 2|V| - 3$, $G[X] = C_6$, $d_G(x) = 3$ for all $x \in X$, and $d_G(y, X) \geq 1$ for all $y \in Y$. Then E is independent in $\mathcal{R}_{uvw}(G)$.*

Proof: We proceed by induction on $|Y|$. First note that as $d_G(x) = 3$ for all $x \in X$ and $G[X] = C_6$, every vertex in X has a neighbour in Y. Therefore $|E(Y, X)| = 6$. The fact that $Y \cap N^* = \emptyset$ implies every vertex in $\{u, v, w\}$ has a different neighbour in Y. Therefore we have $|Y| \geq 3$. The facts $d_G(y, X) \geq 1$ and $|E(Y, X)| = 6$ imply that $|Y| \leq 6$.

The seven base cases when $|Y| = 3$ are drawn with an independent uvw-coincident realisation in Figure 2.2.

Now suppose $4 \leq |Y| \leq 6$. As G has $2|V| - 3$ edges, uvw-coincident rigidity and uvw-coincident independence are equivalent for G. By Lemma 2.2.12 (by taking $U = Y$), we may substitute some other minimally rigid graph with vertex set Y for $G[Y]$ without changing uvw-coincident rigidity of G. Since $|Y| \geq 4$ and u, v, w have different neighbours in Y, there exist distinct vertices $y_1, y_2 \in Y$ with $d_G(y_1, X) = 1$ and $d_G(y_2, \{u, v, w\}) = 0$. For the minimally rigid graph with vertex set Y that we will use instead of $G[Y]$, first choose an arbitrary minimally rigid graph with vertex set $Y - y_1$. Then add y_1 by a 0-extension operation such that y_1y_2 is an edge of the resulting graph. Replace $G[Y]$ by this graph within G and preserve the edges $E \setminus E_G(Y)$, say the resulting graph is G'. Note that $d_{G'}(y_1) = 3$. Apply a 1-reduction at y_1 in G' such that x_1y_2 is the added edge of this operation where x_1 is the unique neighbour of y_1 in X in G'. Say the graph we obtain after this operation is G''. Then G'' satisfies the induction hypotheses. As $V(G'') \setminus X$ has $|Y| - 1$ vertices, the set $E(G'')$ is independent in $\mathcal{R}_{uvw}(G'')$. We can now add y_1 back by a 1-$\{u, v, w\}$-extension to obtain G'. As this operation preserves independence by Lemma 2.2.9, $E(G')$ is independent in $\mathcal{R}_{uvw}(G')$. Since G' is a graph obtained from G by replacing $G[Y]$ by another minimally rigid graph in $\mathcal{R}_2$, by Lemma 2.2.12, we have E as an independent set in $\mathcal{R}_{uvw}(G)$. ■

The matroid $\mathcal{M}_{uv}(G)$ of a graph $G = (V, E)$ is the matroid whose independent sets are the subsets $E' \subseteq E$ such that there exists a $\{u, v\}$-sparse subgraph H of G with $E(H) = E'$. We will use the following result which is due to Fekete, Jordán, Kaszanitzky [6] to show that independence in $\mathcal{M}_{uvw}$ implies independence in $\mathcal{R}_{uvw}$.

Theorem 2.2.14. *[6] Let $G = (V, E)$ be a graph and $u, v \in V$ be distinct vertices. Then $\mathcal{M}_{uv}(G) \cong \mathcal{R}_{uv}(G)$.*

Theorem 2.2.15. *Let $G = (V, E)$ be a graph and $u, v, w \in V$ be distinct vertices. Suppose E is independent in $\mathcal{M}_{uvw}(G)$. Then E is independent in $\mathcal{R}_{uvw}(G)$.*

Proof: We proceed by induction on $|V|$. If $|V| \leq 4$, then G is a subgraph of the bipartite graph $K_{1,3}$, where u, v, w belong to the same part. As $E(K_{1,3})$ is independent in $\mathcal{R}_{uvw}(K_{1,3})$, we can assume that $|V| \geq 5$. Moreover, by extending E to a base of $\mathcal{M}_{uvw}(G)$ we may assume $|E| = 2|V| - 3$. We split the proof into three cases.

Case 1. There exists a vertex $a \in V$ with $d(a) = 2$.

Suppose $a \notin \{u, v, w\}$. If $N(a) \subseteq \{u, v, w\}$, consider $\mathcal{H} = \{\{a, u, v, w\}, V - \{a\}\}$. We have

$$2|V| - 3 = |E| = i_E(\mathcal{H}) \leq \mathrm{val}_{\{u,v,w\}}(\mathcal{H}) = 1 + 2|V| - 8 - 1 + 4 = 2|V| - 4,$$

a contradiction.

On the other hand if $|N(a) \cap \{u, v, w\}| \leq 1$, then $E(G - a)$ is independent in $\mathcal{M}_{uvw}(G)$. By the induction hypothesis $E(G - a)$ is independent in $\mathcal{R}_{uvw}(G)$. Now we apply a 0-uvw-extension to $G - a$ and get G back. By Lemma 2.2.8, we have E as an independent set of $\mathcal{R}_{uvw}(G)$.

Now suppose that $a \in \{u, v, w\}$. We may assume $a = w$. Consider the graph $G - w$. This graph is $\{u, v\}$-sparse as it is a subgraph of G. By using Theorem 2.2.14, $G - w$ has an independent uv-coincident realisation p, that is $p(u) = p(v)$. Then we can add w back at $p(v) = p(u)$ by a 0-extension. By Lemma 2.2.8, this preserves independence and hence E is independent in $\mathcal{R}_{uvw}(G)$.

We now consider the remaining two cases. Suppose $\delta(G) \geq 3$. Let N^* be the set of vertices having at least two neighbours in $\{u, v, w\}$ and $X = N^* \cup \{u, v, w\}$.

Case 2. There exists a vertex $a \in V \setminus X$ of degree three.

We apply Lemma 2.2.11 on a and obtain a graph G' that is independent in $\mathcal{M}_{uvw}(G')$. As G' has fewer vertices than G, the graph G' is also independent in $\mathcal{R}_{uvw}(G')$, by the induction hypothesis. Then we can obtain G from G' by a 1-$\{u, v, w\}$-extension. Thus by Lemma 2.2.9, E is independent in $\mathcal{R}_{uvw}(G)$.

Case 3. All the vertices of degree three are in X.

If $|N^*| \geq 5$, for the $\{u, v, w\}$-compatible family $\mathcal{H} = \{\{a, u, v, w\} : a \in N^*\}$ we would have $\mathrm{val}_{\{u,v,w\}}(\mathcal{H}) \leq i_G(\mathcal{H})$, contradicting the fact that E is independent in $\mathcal{M}_{uvw}(G)$. Hence $|N^*| \leq 4$ and so $|X| \leq 7$. Let $U = V \setminus X$. If we count the degrees of the vertices in U, we get

$$4|U| \leq \sum_{u \in U} d(u) = 2|E(U)| + |E(U, X)| \tag{2.6}$$

Since E is independent in $\mathcal{M}_{uvw}(G)$ we have $|E(U)| \leq 2|U| - 3$. This together with (2.6) imply $|E(U, X)| \geq 6$. Since the degree sum for G is $4|V| - 6$ and $\delta(G) \geq 3$, G has exactly six vertices of degree three and all other vertices have degree four. Hence

$$X \text{ has six vertices of degree three and } |X| - 6 \text{ vertices of degree four} \tag{2.7}$$

by the main assumption of Case 3. Then

$$\begin{aligned} 2|E(X)| + |E(U, X)| = \sum_{x \in X} d(x) &= 6 \cdot 3 + 4(|X| - 6) \\ &= 18 + 4(|N^*| + |\{u, v, w\}| - 6) \\ &= 4|N^*| + 6 \end{aligned} \tag{2.8}$$

Since every vertex in N^* has at least two neighbours in $\{u, v, w\}$, we have $|E(X)| \geq 2|N^*|$. This together with (2.8) imply that $|E(U, X)| \leq 6$. Therefore we have $|E(U, X)| = 6$ and $|E(X)| = 2|N^*|$. Since

$$\begin{aligned} |E(U)| + |E(U, X)| + |E(X)| = |E| = 2|V| - 3 &= 2|U| - 3 + 2|X| \\ &= 2|U| - 3 + 2|N^*| + 6 \end{aligned}$$

this gives $|E(U)| = 2|U| - 3$. This implies that the graph $G[U]$ is minimally rigid. That is the set X is attached to a minimally rigid graph. The possibilities for $G[X]$ are shown in Figure 2.3.

We now split this case into two sub-cases.

Case 3.1. There is no copy of C_4 in $G[X]$.

Then $G[X]$ is a copy of C_6. By using the fact that uvw-coincident independence is equivalent to uvw-coincident rigidity for G as G has $2|V|-3$ edges and Lemma 2.2.12 we can take an arbitrary minimally rigid graph on $Y \subseteq U$, where Y is the set of vertices that are adjacent to vertices in X. We can now apply Lemma 2.2.13 and obtain the result.

Case 3.2. There exists a copy of C_4 in $G[X]$.

Then $G[X]$ is one of the three graphs on the right in Figure 2.3. Pick one of u, v, w that is on a C_4 in $G[X]$ and has degree three in G, say w. We may assume that v is another vertex of this C_4. Let the vertex set of this C_4 be $\{v, a, w, b\}$. Now contract v and w into one vertex z_{vw} and delete the multiple edges and say the resulting graph is G'. We will show that G' is independent in $\mathcal{M}_{uz_{vw}}(G')$. Suppose not. Let $C \subseteq E(G')$ be a minimal dependent set, that is a circuit, in $\mathcal{M}_{uz_{vw}}(G')$. Then either there exists a $\{u, z_{vw}\}$-compatible family $\mathcal{H}$ in G' such that $E(\mathcal{H}) = C$ and $i(\mathcal{H}) = \mathrm{val}_{\{u,z_{vw}\}}(\mathcal{H})+1$ or there exists a subgraph H of G' such that $E(H) = C$, $u \notin V(H)$ and $i(H) = 2|V(H)| - 2$.

Suppose the second alternative holds. The minimality of C implies that $\delta(H) \geq 3$, and the fact that G is $\mathcal{M}_{uvw}$-independent tells us that $z_{vw} \in V(H)$. The fact that $u \notin V(H)$ and the definition of N^* imply that $d_H(y) \leq d_G(y) - 1$ for all $y \in N^* \cap V(H)$. Since $X = N^* \cup \{u, v, w\}$ has size at most seven (as $3 \leq |N^*| \leq 4$) there is at most one $y \in N^*$ with $d_G(y) > 3$ by (2.7). This tells us that $|N^* \cap V(H)| \leq 1$ (as $\delta(H) \geq 3$).

By examining the alternatives in Figure 2.3, we see that there are at least $|N^*|+2$ edges in G from $\{v, w\}$ to N^*. Combining this with the fact that $|N^* \cap V(H)| \leq 1$ (with equality only if $d_G(y) = 4$ for some $y \in N^*$), we obtain

$$d_H(z_{vw}) \leq d_G(v) + d_G(w) - (|N^*| + 1)$$

with equality only if $d_G(y) = 4$ for some $y \in N^*$. Since $d_G(v) + d_G(w) \leq 7$ with equality only if $|N^*| = 4$ and $d_G(y) = 3$ for all $y \in N^*$ (by (2.7)), we have $d_H(z_{vw}) \leq 2$. This contradicts the fact that $\delta(H) \geq 3$.

Hence there exists a $\{u, z_{vw}\}$-compatible family $\mathcal{H}$ in G' such that $E(\mathcal{H}) = C$ and $i_{G'}(\mathcal{H}) = \mathrm{val}_{\{u,z_{vw}\}}(\mathcal{H}) + 1$. Note that since at most one vertex in X is of degree four, we may assume a or b, say b, is of degree three in G. Then $d_{G'}(b) = 2$ and hence b cannot be in a member of $\mathcal{H}$. We define $\{u, v, w\}$-compatible families

$$\mathcal{H}_1 := \{H_i - z_{vw} + v + w : H_i \in \mathcal{H}\} \cup \{\{a, u, v, w\}, \{b, u, v, w\}\}$$

and

$$\mathcal{H}_2 := \{H_i - z_{vw} + v + w : H_i \in \mathcal{H}\} \cup \{\{b, u, v, w\}\}.$$

Let us consider $\mathcal{H}_1$ for the case a is not contained in a member of $\mathcal{H}$ and $\mathcal{H}_2$ for the case a is contained in a member of $\mathcal{H}$. Then we see that

$$\mathrm{val}_{\{u,v,w\}}(\mathcal{H}_1) + 1 = (\mathrm{val}_{\{u,z_{vw}\}}(\mathcal{H}) + 1) + 4 = i_{G'}(\mathcal{H}) + 4 = i_G(\mathcal{H}_1)$$

and

$$\mathrm{val}_{\{u,v,w\}}(\mathcal{H}_2) + 1 = (\mathrm{val}_{\{u,z_{vw}\}}(\mathcal{H}) + 1) + 3 = i_{G'}(\mathcal{H}) + 3 = i_G(\mathcal{H}_2),$$

contradicting the fact that G is $\{u, v, w\}$-sparse.

Therefore $E(G')$ is independent in $\mathcal{R}_{uz_{vw}}(G')$ by Theorem 2.2.14. Take an independent uz_{vw}-coincident realisation p of G'. Then applying Lemma 2.2.10 at z_{vw} and relabelling gives us G with the property that $p(u) = p(v) = p(w)$. Hence E is independent in $\mathcal{R}_{uvw}(G)$. ■

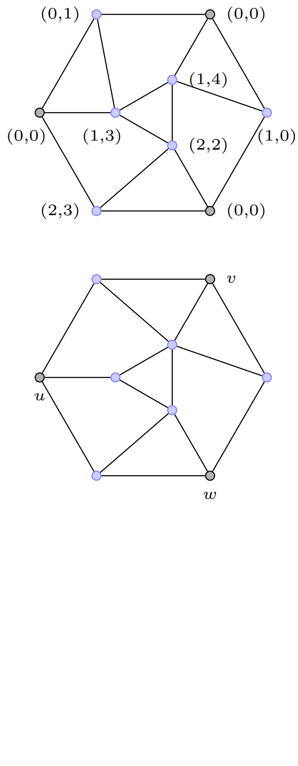

Figure 2.2: Base cases. The realisation of the framework on the top-left gives a uvw-rigid realisation for all seven frameworks. Their ranks were calculated by a computer program. The vertices in Y are drawn inside the outer six-cycle which corresponds to $G[X]$. To see that these are all the cases and they are all different first note that since $G[X] = C_6$, the outer six-cycle is fixed; and since $G[Y]$ is rigid, the inner triangle is fixed. The fact that $d_G(x) = 3$ for all $x \in X$ implies each vertex in X is adjacent to exactly one vertex in Y. Since $Y \cap N^* = \emptyset$, any two of u, v, w have distinct neighbours in Y.

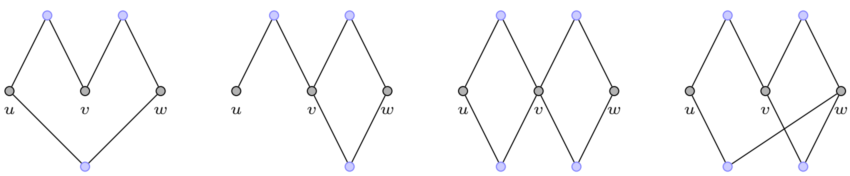

Figure 2.3: Possible alternatives for $G[X]$. Note that as $|E(X)| = 2|N^*|$, there are no edges with both endpoints in N^*.

2.3 Main Results

Theorem 2.3.1. *Let $G = (V, E)$ be a graph and u, v, w be distinct vertices. Then E is independent in $\mathcal{M}_{uvw}(G)$ if and only if E is independent in $\mathcal{R}_{uvw}(G)$.*

Proof. Immediately follows from Lemma 2.2.7 and Theorem 2.2.15. ∎

Theorem 2.3.2. *Let $G = (V, E)$ be a graph and let $u, v, w \in V$ be distinct vertices and $G' = G - uv - uw - vw$. Then G is uvw-rigid in $\mathbb{R}^2$ if and only if G' is rigid in $\mathbb{R}^2$ and G'_S is rigid in $\mathbb{R}^2$ for all $S \subseteq \{u, v, w\}$ with $|S| \geq 2$.*

Proof: Necessity is implied by (2.4) as an infinitesimal uvw-rigid realisation of G gives rise to infinitesimally rigid realisations of G' and G'_S for all $S \subseteq \{u, v, w\}$ with $|S| \geq 2$.

For sufficiency suppose G' and G'_S are rigid in $\mathbb{R}^2$ for all $S \subseteq \{u, v, w\}$ with $|S| \geq 2$, but G is not uvw-rigid in $\mathbb{R}^2$. Then by Theorems 2.2.6 and 2.3.1, there exist either a 1-thin cover $\mathcal{K}$ of G for which $\mathrm{val}(\mathcal{K}) \leq 2|V| - 4$ or a T-thin cover $\mathcal{L}$ of G for some $T \subseteq \{u, v, w\}$ with $|T| \geq 2$ for which $\mathrm{val}(\mathcal{L}) \leq 2|V| - 4$.

Case 1. A 1-thin cover $\mathcal{K}$ of G for which $\mathrm{val}(\mathcal{K}) \leq 2|V| - 4$ exists.

Then the fact that $\mathcal{K}$ also covers the graph G' implies that $r_2(G') \leq 2|V| - 4$, by Theorem 0.2.3, contradicting the fact that G' is rigid in $\mathbb{R}^2$.

Case 2. A T-thin cover $\mathcal{L}$ of G for some $T \subseteq \{u, v, w\}$ with $|T| \geq 2$ for which $\mathrm{val}(\mathcal{L}) \leq 2|V| - 4$ exists.

Let $\mathcal{L} = \{\mathcal{H}, X_1, \ldots, X_l\}$ where $\mathcal{H} = \{H_1, \ldots, H_k\}$ is a T-compatible family, and $X_1, \ldots, X_k$ are tight subsets of V. If we contract the vertices in T in G' into a new vertex z_T, we have a graph G'_T and a 1-thin cover $\mathcal{L}' = \{H'_1, \ldots, H'_k, X_1, \ldots, X_l\}$ of G'_T, where $H'_i = H_i/T$, $1 \leq i \leq k$. Then we have

$$\begin{aligned}
\sum_{i=1}^{l}(2|X_i| - 3) + \sum_{i=1}^{k}(2|H'_i| - 3) &= \sum_{i=1}^{l}(2|X_i| - 3) + \sum_{i=1}^{k}(2|H_i \setminus T| - 1) \\
&= \mathrm{val}_T(\mathcal{L}) - 2(|T| - 1) \\
&\leq 2|V| - 4 - 2(|T| - 1) \\
&= 2(|V| - (|T| - 1)) - 4,
\end{aligned}$$

contradicting the fact that G'_T is rigid in $\mathbb{R}^2$ by Theorem 0.2.3. ■

Example 2.3.1. One may think that for Theorem 2.3.2, rigidity of $G-uv-uw-vw$ and G_{uvw} would be enough. However, this is not the case and an example is given in Figure 2.4.

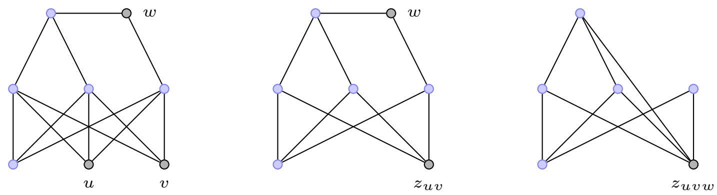

Figure 2.4: The graph on the left is G. The graph in the middle is G_{uv} and the graph on the right is G_{uvw}. Both G and G_{uvw} are rigid in $\mathbb{R}^2$, but G_{uv} is not. Hence G is not uvw-rigid in $\mathbb{R}^2$ by Theorem 2.3.2.

Theorem 2.3.3. *Let $G=(V,E)$ be a graph, $u,v,w \in V$ be distinct vertices and $G' = G-uv-uw-vw$. Then $r_{uvw}(G) = \min\{r_2(G'), r_2(G'_{uv})+2, r_2(G'_{uw})+2, r_2(G'_{vw})+2, r_2(G'_{uvw})+4\}$.*

Proof: Let $m := \min\{r_2(G'), r_2(G'_{uv})+2, r_2(G'_{uw})+2, r_2(G'_{vw})+2, r_2(G'_{uvw})+4\}$. Inequality (2.4) and the fact that $r_{uvw}(G) \leq r_2(G')$ imply that $r_{uvw}(G) \leq m$. Hence we only need to show that $m \leq r_{uvw}(G)$. By Theorems 2.2.6 and 2.3.1, there exist either a 1-thin cover $\mathcal{K}$ of G for which $r_{uvw}(G) = \text{val}(\mathcal{K})$ or a T-thin cover $\mathcal{L}$ of G for some $T \subseteq \{u,v,w\}$ with $|T| \geq 2$ for which $r_{uvw}(G) = \text{val}(\mathcal{L})$.

Case 1. There exists either a 1-thin cover $\mathcal{K}$ of G for which $r_{uvw}(G) = \text{val}(\mathcal{K})$.

As $\mathcal{K}$ also covers G', we have $m \leq r_2(G') \leq \text{val}(\mathcal{K}) = r_{uvw}(G)$ by Theorem 0.2.3.

Case 2. There exists a T-thin cover $\mathcal{L}$ of G for some $T \subseteq \{u,v,w\}$ with $|T| \geq 2$ for which $r_{uvw}(G) = \text{val}(\mathcal{L})$.

Let $\mathcal{L} = \{\mathcal{H}, X_1, \dots, X_l\}$ where $\mathcal{H} = \{H_1, \dots, H_k\}$ is a T-compatible family for a $T \subseteq \{u,v,w\}$ with $|T| \geq 2$, and $X_1, \dots, X_l$ are tight subsets of V. If we contract the vertices in T in G' into a new vertex z_T, we have a graph G'_T and a 1-thin cover $\mathcal{L}' = \{H'_1, \dots, H'_k, X_1, \dots, X_l\}$ of G'_T, where $H'_i = H_i/T$, $1 \leq i \leq k$. Since $\mathcal{L}'$ is a

1-thin cover of G'_T, we have $r_2(G'_T) \leq \mathrm{val}(\mathcal{L}')$ by Theorem 0.2.3. Then

$$\begin{aligned} r_2(G'_T) \leq \mathrm{val}(\mathcal{L}') &= \sum_{i=1}^{l} (2|X_i| - 3) + \sum_{i=1}^{k} (2|H'_i| - 3) \\ &= \sum_{i=1}^{l} (2|X_i| - 3) + \sum_{i=1}^{k} (2|H_i \setminus T| - 1) \\ &= \mathrm{val}_T(\mathcal{L}) - 2(|T| - 1) = r_{uvw}(G) - 2(|T| - 1). \end{aligned}$$

Hence $m \leq r_2(G'_T) + 2(|T| + 1) \leq r_{uvw}(G)$. ■

2.4 Further Remarks

We have a characterisation for the generic uv- and uvw-coincident rigidity of a framework in $\mathbb{R}^2$ by Theorems 2.1.1 and 2.3.2. It is natural to ask whether the analogues of these results hold in $\mathbb{R}^3$. That is, are the statements

$$G \textit{ is } uv-\textit{rigid in } \mathbb{R}^3 \textit{ if and only if } G_{uv} \textit{ and } G - uv \textit{ are rigid in } \mathbb{R}^3 \tag{S1}$$

and

$$\begin{gathered} G \textit{ is } uvw\textit{-rigid in } \mathbb{R}^3 \textit{ if and only if } G' \textit{ is rigid in } \mathbb{R}^3 \textit{ and } G'_S \textit{ is rigid in } \mathbb{R}^3 \\ \textit{for all } S \subseteq \{u, v, w\} \textit{ with } |S| \geq 2 \end{gathered} \tag{S2}$$

true?

We will show that these statements do not hold by giving counter-examples. Let us first give a result of Whiteley that will help us build the counter-examples.

Lemma 2.4.1. *[29] Let $(K_{m,n}, p)$ be a bar-and-joint framework in $\mathbb{R}^d$ where $K_{m,n}$ is the complete bipartite graph with parts of size m and n. Then $(K_{m,n}, p)$ is infinitesimally rigid if and only if each part of $K_{m,n}$ affinely spans $\mathbb{R}^d$, and the complete set of vertices do not lie on a quadric surface in $\mathbb{R}^d$.*

It is known that any set of nine points lie on a quadric surface (satisfy a quadric equation) in $\mathbb{R}^3$. We also know that $K_{5,5}$ is a rigid circuit in $\mathbb{R}^3$. We will use these facts to build the counter-examples. The construction of these counter-examples below is due to Jackson and Tanigawa [14].

Example 2.4.1. Consider the graphs $K_{5,5}$ and $K_{5,5}+w$ in Figure 2.5. Since $K_{5,5}$ is a rigid circuit, we see that $K_{5,5}-uv$ is rigid. It can also be shown by Henneberg moves and edge deletions that $(K_{5,5})_{uv}$ is rigid in $\mathbb{R}^3$. Therefore if statement (S1) above was true, then we would get $K_{5,5}$ is uv-rigid. Take a generic uv-coincident realisation $(K_{5,5}, p)$ Since $K_{5,5}$ has ten vertices and $p(u) = p(v)$, there are nine distinct points in $\mathbb{R}^3$. Thus there exists a quadric surface that contains all these nine points. Hence, by Lemma 2.4.1, $(K_{m,n}, p)$ is not infinitesimally rigid. That is, $K_{m,n}$ is not uv-rigid.

Now let $(K_{m,n}+w, p')$ be the framework obtained from $(K_{m,n}, p)$ by a 0-extension with $p'(w) = p'(u) = p'(v)$ and $p'|_{V(K_{m,n})} = p$. We can now use the facts that $(K_{m,n}, p)$ is not infinitesimally rigid and 0-extension preserves infinitesimal flexibility of frameworks to obtain a counter-example for statement (S2).

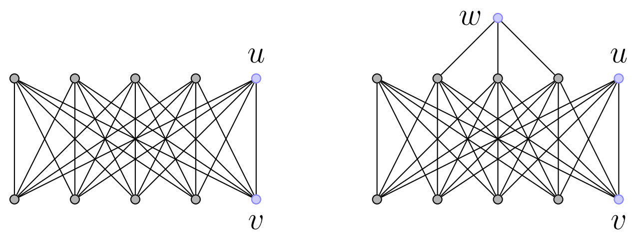

Figure 2.5: $K_{5,5}$ on the left and $K_{5,5}+w$ on the right.

Chapter 3

Coincident Rigidity in $\mathbb{R}^2$ with More Vertices

3.1 Introduction

One may ask whether we can characterise coincident infinitesimal rigidity with more than three vertices being coincident in $\mathbb{R}^2$. Given a framework (G, p) in $\mathbb{R}^2$ and a set $U \subseteq V(G)$ with $|U| \geq 2$, we say (G, p) is U-*coincident* if $p(x) = p(y)$ for all $x, y \in U$. We also say the framework (G, p) is *generic* U-coincident, if $p|_{(V \setminus U) \cup \{x\}}$ is generic for some (and hence for all) $x \in U$. Let $\mathcal{R}_U(G)$ denote the generic U-coincident rigidity matroid of G, that is, $\mathcal{R}_U(G)$ is the matroid obtained from the rigidity matrix of a generic U-coincident framework (G, p). In this chapter we will prove the following two results.

Theorem 3.1.1. *Let $G = (V, E)$ be a graph and $U \subseteq V$. The family*

$$\mathcal{I}_G = \{F : F \subseteq V \textit{ and } (V, F) \textit{ is } S\textit{-sparse for all } S \subseteq U \textit{ with } |S| \geq 2\}$$

is the family of independent sets of a matroid, $\mathcal{M}_U(G)$.

Theorem 3.1.2. *Let $G = (V, E)$ be a graph and $U \subseteq V$. Suppose E is independent in $\mathcal{R}_U(G)$. Then E is independent in $\mathcal{M}_U(G)$.*

The proof methods of both theorems will be an analogue of the corresponding theorems in the previous chapter. However, the tools to prove Theorem 3.1.1 will be a bit more complicated in this chapter due to the fact that we may need more than one compatible family in the cover whose value gives the rank in $\mathcal{M}_U(G)$. Let us show this with an example.

Example 3.1.1. Consider the graph $G = (V, E)$ drawn in Figure 3.1, with $U = \{u_1, u_2, u_3, u_4\} \subseteq V$. We see that G consists of the disjoint union of two copies of $K_{2,3}$. The copy of $K_{2,3}$ on the left is not $\{u_1, u_2\}$-sparse as for the $\{u_1, u_2\}$-compatible family $\mathcal{H}_1 = \{\{x_1, u_1, u_2\}, \{x_2, u_1, u_2\}, \{x_3, u_1, u_2\}\}$, we have $\mathrm{val}_{\{u_1,u_2\}}(\mathcal{H}_1) = 5 < 6 = i(\mathcal{H}_1)$. Similarly, the copy of $K_{2,3}$ on the right is not $\{u_3, u_4\}$-sparse as for the $\{u_3, u_4\}$-compatible family $\mathcal{H}_2 = \{\{y_1, u_3, u_4\}, \{y_2, u_3, u_4\}, \{y_3, u_3, u_4\}\}$, we have $\mathrm{val}_{\{u_3,u_4\}}(\mathcal{H}_2) = 5 < 6 = i(\mathcal{H}_2)$. These observations imply that the rank of G in $\mathcal{M}_U(G)$ is at most 10. However, there is no cover $\mathcal{K}$ of G containing at most one S-compatible family for an $S \subseteq U$ with $|S| \geq 2$ satisfying $\mathrm{val}(\mathcal{K}) \leq 10$. This is why we need more than one compatible family. We will explain why this difference arises in more detail later in the chapter.

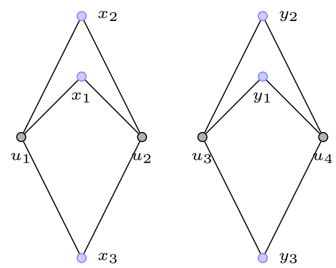

Figure 3.1: Disjoint union of two copies of $K_{2,3}$.

3.2 The Count Matroid

Most of the terminology we use in this chapter is from Chapter 2. We will give more definitions later that are special to this chapter.

3.2.1 Properties of Compatible Families

Lemmas 3.2.1, 3.2.2, 3.2.3 and 3.2.4 are the same Lemmas with the same proofs as Lemmas 2.2.1, 2.2.2, 2.2.3 and 2.2.4, respectively.

Lemma 3.2.1. *Let $G = (V, E)$ be a graph and $\mathcal{H} = \{H_1, \ldots, H_k\}$ be an S-compatible family for some $S \subseteq V$ with $|S| \geq 2$.*

(i) If $|H_i \cap H_j| \geq |S|+1$ for some pair $1 \leq i < j \leq k$, then there is an S-compatible family $\overline{\mathcal{H}}$ with $\mathrm{cov}(\mathcal{H}) \subseteq \mathrm{cov}(\overline{\mathcal{H}})$ for which $\mathrm{val}_S(\overline{\mathcal{H}}) \leq \mathrm{val}_S(\mathcal{H}) - 1$.

(ii) If G is S-sparse and $\mathcal{H}$ is S-tight, then $H_i \cap H_j = S$ for all $1 \leq i < j \leq k$.

Proof: We may assume that $i = k-1$ and $j = k$. Let $\overline{\mathcal{H}} = \{H_1, \ldots, H_{k-2}, (H_{k-1} \cup H_k)\}$. Then we have

$$\begin{aligned}
\mathrm{val}_S(\mathcal{H}) &= \sum_{l=1}^{k} (2|H_l \setminus S| - 1) + 2(|S| - 1) \\
&= \sum_{l=1}^{k-2} (2|H_l \setminus S| - 1) + 2(|S| - 1) \\
&\quad + (2|H_{k-1} \setminus S| - 1) + (2|H_k \setminus S| - 1) \\
&= \sum_{l=1}^{k-2} (2|H_l \setminus S| - 1) + 2(|S| - 1) + (2|(H_{k-1} \cup H_k) \setminus S| - 1) \\
&\quad + (2|(H_{k-1} \cap H_k) \setminus S| - 1) \\
&\geq \mathrm{val}_S(\overline{\mathcal{H}}) + 1.
\end{aligned}$$

Clearly, $\mathrm{cov}(\mathcal{H}) \subseteq \mathrm{cov}(\overline{\mathcal{H}})$ holds. This completes the proof of (i). It is easy to see that (ii) immediately holds from (i). ■

Lemma 3.2.2. *Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be an S-compatible family for some $S \subseteq V$ with $|S| \geq 2$ and $H_i \cap H_j = S$ for all $1 \leq i < j \leq k$. Let $Y \subseteq V$ be a set of vertices with $|Y \cap S| \leq 1$ and $|Y \cap H_i| \geq 2$ for some $1 \leq i \leq k$. Then there is an S-compatible family $\overline{\mathcal{H}}$ with $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\overline{\mathcal{H}})$ for which $\mathrm{val}_S(\overline{\mathcal{H}}) \leq \mathrm{val}_S(\mathcal{H}) + \mathrm{val}_S(Y)$. Furthermore, if G is S-sparse and $\mathcal{H}$ and Y are both tight, then $\overline{\mathcal{H}}$ is also tight.*

Proof: By renumbering the sets of $\mathcal{H}$, if necessary, we may assume that $|Y \cap H_i| \geq 2$ if $i \geq j$ for some $j \leq k$, and $|Y \cap H_i| \leq 1$ for all $i \leq j - 1$. Let $X = Y \cup \bigcup_{i=j}^{k} H_i$

and $\overline{\mathcal{H}} = \{H_1, \ldots, H_{j-1}, X\}$. Then we have $\operatorname{cov}(\mathcal{H}) \cup \operatorname{cov}(Y) \subseteq \operatorname{cov}(\overline{\mathcal{H}})$, and

$$
\begin{aligned}
&\mathrm{val}_S(\mathcal{H}) + \mathrm{val}_S(Y) = \sum_{i=1}^{k} (2|H_i \setminus S| - 1) + 2(|S| - 1) + (2|Y| - 3) \\
&= \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + 2(|S| - 1) + \sum_{i=j}^{k} (2|H_i \setminus S| - 1) + (2|Y| - 3) \\
&= \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + 2(|S| - 1) + (2|X \setminus S| - 1) \\
&\quad + 2|Y \cap S| - (k - j) + 2\sum_{i=j}^{k} |Y \cap (H_i \setminus S)| - 3 \\
&= \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + (2|X \setminus S| - 1) + 2(|S| - 1) \\
&\quad + 2|Y \cap S| - (k - j) + 2\sum_{i=j}^{k} |Y \cap H_i| - 2\sum_{i=j}^{k} |Y \cap S| - 3 \\
&= \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + (2|X \setminus S| - 1) + 2(|S| - 1) \\
&\quad - (k - j) + 2\sum_{i=j}^{k} |Y \cap (H_i)| - 2|Y \cap S|(k - j) - 3 \\
&\geq \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + (2|X \setminus S| - 1) + 2(|S| - 1) + 2\sum_{i=j}^{k} |Y \cap H_i| - 3(k - j + 1) \\
&= \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + (2|X \setminus S| - 1) + 2(|S| - 1) + \sum_{i=j}^{k} (2|Y \cap H_i| - 3) \\
&= \mathrm{val}_S(\overline{\mathcal{H}}) + \sum_{i=j}^{k} \mathrm{val}_S(Y \cap H_i)
\end{aligned}
$$

where for the inequality step we use $|Y \cap S| \leq 1$.

Now suppose that $\mathcal{H}$ and Y are tight. Then we have

$$
i(\overline{\mathcal{H}}) + \sum_{i=j}^{k} i(Y \cap H_i) \geq i(\mathcal{H}) + i(Y) = \mathrm{val}_S(\mathcal{H}) + \mathrm{val}_S(Y)
$$

$$\geq \mathrm{val}_S(\overline{\mathcal{H}}) + \sum_{i=j}^{k} \mathrm{val}_S(Y \cap H_i) \geq i(\overline{\mathcal{H}}) + \sum_{i=j}^{k} i(Y \cap H_i),$$

where the first inequality follows since the edges spanned by $\mathcal{H}$ or Y are spanned by $\overline{\mathcal{H}}$ and if some edge is spanned by both $\mathcal{H}$ and Y, then it is spanned by $Y \cap H_i$ for some i. The first equality holds because $\mathcal{H}$ and Y are tight, and the second inequality holds by our calculations above. The last inequality holds because G is S-sparse. Hence equality must hold everywhere, which implies that $\overline{\mathcal{H}}$ is also tight.

■

Lemma 3.2.3. *Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be an S-compatible family for some $S \subseteq V$ with $|S| \geq 2$ and $H_i \cap H_j = S$ for all $1 \leq i < j \leq k$. Let $Y \subseteq V$ be a set of vertices with $Y \cap S = \emptyset$ and $|Y \cap H_i| \leq 1$ for all $1 \leq i \leq k$, for which $|Y \cap H_i| = |Y \cap H_j| = 1$ for some pair $1 \leq i < j \leq k$. Then there is an S-compatible family $\overline{\mathcal{H}}$ with $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\overline{\mathcal{H}})$ for which $\mathrm{val}_S(\overline{\mathcal{H}}) = \mathrm{val}_S(\mathcal{H}) + \mathrm{val}_S(Y)$. Furthermore, if G is S-sparse and $\mathcal{H}$ and Y are both tight, then $\overline{\mathcal{H}}$ is also tight.*

Proof: We may assume that $i = k-1$ and $j = k$. Let $\overline{\mathcal{H}} = \{H_1, \ldots, H_{k-2}, (H_{k-1} \cup H_k \cup Y)\}$. Then we have

$$\begin{aligned}
&\mathrm{val}_S(\mathcal{H}) + \mathrm{val}_S(Y) = \sum_{i=1}^{k} (2|H_i \setminus S| - 1) + 2(|S| - 1) + (2|Y| - 3) \\
&= \sum_{i=1}^{k-2} (2|H_i \setminus S| - 1) + 2(|S| - 1) + (2|H_{k-1} \setminus S| - 1) + (2|H_k \setminus S| - 1) + (2|Y| - 3) \\
&= \sum_{i=1}^{k-2} (2|H_i \setminus S| - 1) + 2(|S| - 1) + (2(|H_{k-1} \setminus S| + |H_k \setminus S| + |Y|) - 1) - 4 \\
&= \sum_{i=1}^{k-2} (2|H_i \setminus S| - 1) + (2|(H_{k-1} \cup H_k \cup Y) \setminus S| - 1) + 2(|S| - 1) \\
&\quad + 2|Y \cap (H_{k-1} \setminus S)| + 2|Y \cap (H_k \setminus S)| - 4 \\
&= \mathrm{val}_S(\overline{\mathcal{H}}).
\end{aligned}$$

Clearly, we have $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\overline{\mathcal{H}})$. Now suppose that G is S-sparse and $\mathcal{H}$ and Y are tight. then we have

$$i(\mathcal{H}) + i(Y) = \mathrm{val}_S(\mathcal{H}) + \mathrm{val}_S(Y) = \mathrm{val}_S(\overline{\mathcal{H}}) \geq i(\overline{\mathcal{H}}) \geq i(\mathcal{H}) + i(Y)$$

where the last inequality follows since $|Y \cap H_i| \leq 1$ for all $1 \leq i \leq k$. Hence equality must hold everywhere which implies that $\overline{\mathcal{H}}$ is also tight. ■

Lemma 3.2.4. *Let $G = (V, E)$ be S-sparse for some $S \subseteq V$ with $|S| \geq 2$ and let $X, Y \subseteq V$ be S-tight sets in G with $|X \cap Y| \geq 2$ and $X, Y \nsubseteq S$. Then $X \cap Y \nsubseteq S$, and $X \cup Y$ and $X \cap Y$ are S-tight.*

Proof: First note that as G is S-sparse we have

$$\begin{aligned} 2|X| - 3 + 2|Y| - 3 = \mathrm{val}_S(X) + \mathrm{val}_S(Y) &= i(X) + i(Y) \\ &\leq i(X \cap Y) + i(X \cup Y) \\ &\leq \mathrm{val}_S(X \cap Y) + \mathrm{val}_S(X \cup Y) \\ &= \mathrm{val}_S(X \cap Y) + 2|X \cup Y| - 3 \end{aligned}$$

Suppose $X \cap Y$ is a subset of S. Then $\mathrm{val}_S(X \cap Y) = 0$ and putting this in the above equations gives $2|X| - 3 + 2|Y| - 3 \leq 2|X \cup Y| - 3 = 2|X| + 2|Y| - 2|X \cap Y| - 3 \leq 2|X| + 2|Y| - 7$, a contradiction.

Hence $X \cap Y$ is not a subset of S. Then we have $\mathrm{val}_S(X \cap Y) = 2|X \cap Y| - 3$ and hence equality holds throughout. In particular, $\mathrm{val}_S(X \cup Y) = i(X \cup Y)$ and $\mathrm{val}_S(X \cap Y) = i(X \cap Y)$, so $X \cup Y$ and $X \cap Y$ are S-tight. ■

We next choose a set U of vertices in a graph G and suppose that G is S-sparse for all $S \subseteq U$.

Lemma 3.2.5. *Let $G = (V, E)$ be S-sparse for all $S \subseteq U$ with $|S| \geq 2$. Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be an S-compatible family for some $S \subseteq U$ with $|S| \geq 2$ and $u \in H_j$ for some $u \in U \setminus S$ and $1 \leq j \leq k$. Define an $(S \cup \{u\})$-compatible family $\overline{\mathcal{H}} := \{H_1 \cup \{u\}, H_2 \cup \{u\}, H_3 \cup \{u\}, \ldots, H_k \cup \{u\}\}$. Then $\mathrm{val}_{S \cup \{u\}}(\overline{\mathcal{H}}) \leq \mathrm{val}_S(\mathcal{H})$ and $\mathrm{cov}(\mathcal{H}) \subseteq \mathrm{cov}(\overline{\mathcal{H}})$. Moreover, if $\mathcal{H}$ is S-tight, then $\overline{\mathcal{H}}$ is $(S \cup \{u\})$-tight and $\mathrm{val}_S(\mathcal{H}) = \mathrm{val}_{S \cup \{u\}}(\overline{\mathcal{H}})$.*

Proof: We may assume $j = 1$. Then we have

$$\begin{aligned}
\mathrm{val}_S(\mathcal{H}) &= \sum_{i=1}^{k} (2|H_i \setminus S| - 1) + 2(|S| - 1) \\
&= (2|H_1 \setminus S| - 1) - 2 + \sum_{i=2}^{k} (2|H_i \setminus S| - 1) + 2(|S| - 1) + 2 \\
&\geq (2|H_1 \setminus (S \cup \{u\})| - 1) + \sum_{i=2}^{k} (2|(H_i \cup \{u\}) \setminus (S \cup \{u\})| - 1) \\
&\quad + 2(|S \cup \{u\}| - 1) \\
&= \mathrm{val}_{S \cup \{u\}}(\overline{\mathcal{H}}).
\end{aligned}$$

It is clear that $\mathrm{cov}(\mathcal{H}) \subseteq \mathrm{cov}(\overline{\mathcal{H}})$. Now suppose $\mathcal{H}$ is S-tight. Then we have

$$i(\overline{\mathcal{H}}) \leq \mathrm{val}_{S\cup\{u\}}(\overline{\mathcal{H}}) \leq \mathrm{val}_S(\mathcal{H}) = i(\mathcal{H}) \leq i(\overline{\mathcal{H}})$$

where the first inequality follows from $(S \cup \{u\})$-sparsity and the last inequality follows from $\mathrm{cov}(\mathcal{H}) \subseteq \mathrm{cov}(\overline{\mathcal{H}})$. Hence we have $i(\overline{\mathcal{H}}) = \mathrm{val}_{S\cup\{u\}}(\overline{\mathcal{H}})$ implying that $\overline{\mathcal{H}}$ is $(S \cup \{u\})$-tight and $\mathrm{val}_S(\mathcal{H}) = \mathrm{val}_{S\cup\{u\}}(\overline{\mathcal{H}})$. ∎

Lemma 3.2.6. *Let $G = (V, E)$ be S-sparse for all $S \subseteq U$ with $|S| \geq 2$. Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be an S-compatible family for some $S \subseteq U$ with $|S| \geq 2$. Suppose $\mathcal{H}$ is S-tight. Then $H_i \not\subseteq U$ for all $1 \leq i \leq k$.*

Proof: Suppose not. We may assume $H_1 \subseteq U$. First note that $i(U) = 0$ by the sparsity conditions. If $k = 1$, we have $\mathcal{H} = \{H_1\}$ and $H_1 \subseteq U$. Then $\mathrm{val}_S(\mathcal{H}) = 2|H_1 \setminus S| - 1 + 2(|S| - 1) = 2|H_1| - 3 > 0$ holds. This implies $0 < \mathrm{val}_S(\mathcal{H}) = i(\mathcal{H}) = i(H_1) \leq i(U) = 0$, a contradiction.

Similarly, if $k \geq 2$, consider the S-compatible family $\mathcal{H}' = \mathcal{H} \setminus \{H_1\}$. Since $H_1 \subseteq U$ and $i(U) = 0$, we have $i(\mathcal{H}') = i(\mathcal{H})$. Then we have $\mathrm{val}_S(\mathcal{H}') < \mathrm{val}_S(\mathcal{H}) = i(\mathcal{H}) = i(\mathcal{H}')$, a contradiction. ∎

Lemma 3.2.7. *Let $G = (V, E)$ be S-sparse for all $S \subseteq U$ with $|S| \geq 2$. Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be an S-compatible family for some $S \subseteq U$ with $|S| \geq 2$. Let*

$F = \mathrm{cov}(\mathcal{H})$. *Suppose the property* $H_i \cap H_j = S$ *holds for all* $1 \leq i < j \leq k$. *Then* $\mathcal{H}$ *is the unique compatible family with this property whose cover set is* F.

Proof: Suppose the contrary and let $\overline{\mathcal{H}} = \{\overline{H}_1, \ldots, \overline{H}_l\} \neq \mathcal{H}$ be a T-compatible family with $\overline{H}_i \cap \overline{H}_j = T$, for all $1 \leq i < j \leq l$, satisfying $\mathrm{cov}(\overline{\mathcal{H}}) = F$.

Claim 3.2.7.1. *We have* $S \subseteq \overline{H}_i$, *for all* $1 \leq i \leq l$.

Proof of Claim: Suppose not. Let us take a vertex $s \in S \setminus \overline{H}_i$ for some $1 \leq i \leq l$ and another vertex $x \in \overline{H}_i \setminus T$. Since $\mathrm{cov}(\mathcal{H}) = F = \mathrm{cov}(\overline{\mathcal{H}})$, we have $x \in H_j$ for some $1 \leq j \leq k$. The fact that $S \subseteq H_j$ implies $(x, s) \in F$. Hence there exists a set $\overline{H}_t \in \overline{\mathcal{H}}$ with $x, s \in \overline{H}_t$ for some $1 \leq t \leq l$ and $t \neq i$ as $s \notin \overline{H}_i$. We also know that $x \notin T$ as $x \in \overline{H}_i \setminus T$. Combining these we have $\overline{H}_i \cap \overline{H}_t = T \cup \{x\} \neq T$, a contradiction. •

Claim 3.2.7.1 implies $S \subseteq T$ as $\overline{H}_i \cap \overline{H}_j = T$ for all $1 \leq i < j \leq l$. By the same technique, we can show $T \subseteq H_i$ for all $1 \leq i \leq k$, implying that $T \subseteq S$ as $H_i \cap H_j = S$ for all $1 \leq i < j \leq k$. Therefore we have $T = S$. That is $\overline{\mathcal{H}}$ is S-compatible and $\overline{H}_i \cap \overline{H}_j = S$ for all $1 \leq i < j \leq l$.

Since $\mathcal{H} \neq \overline{\mathcal{H}}$, we may assume by symmetry that there exists a set $H_i \in \mathcal{H}$ with $H_i \neq \overline{H}_j$ for all $1 \leq j \leq l$. Choose $x, y \in H_i$ with $x \notin S$. Since $\mathrm{cov}(\mathcal{H}) = F = \mathrm{cov}(\overline{\mathcal{H}})$, there exists a set $\overline{H}_j \in \overline{\mathcal{H}}$ with $x, y \in \overline{H}_j$ for some $1 \leq j \leq l$.

Then either $H_i \setminus \overline{H}_j$ or $\overline{H}_j \setminus H_i$ is non-empty. If $H_i \setminus \overline{H}_j \neq \emptyset$, we pick a vertex $z \in (H_i \setminus \overline{H}_j)$. Since $x, z \in H_i$, we have $(x, z) \in F$. Therefore, there exists a set $\overline{H}_t$ with $x, z \in \overline{H}_t$ for some $1 \leq t \leq l$ with $t \neq j$, since $z \notin \overline{H}_j$. This implies that $(S \cup \{x\}) \subseteq \overline{H}_j \cap \overline{H}_t$, contradicting the fact that $\overline{H}_i \cap \overline{H}_j = S$ for all $1 \leq i < j \leq l$. Similarly, if $\overline{H}_j \setminus H_i \neq \emptyset$, we pick a vertex $z \in \overline{H}_j \setminus H_i$. Since $x, z \in \overline{H}_j$, we have $(x, z) \in F$. Therefore there exists a set H_t with $x, z \in H_t$ for some $1 \leq t \leq k$ with $t \neq i$, since $z \notin H_i$. This implies that $(S \cup \{x\}) \subseteq H_t \cap H_i$, contradicting the fact that $H_i \cap H_j = S$ for all $1 \leq i < j \leq l$. Hence $\mathcal{H}$ is the unique S-compatible family with the property $H_i \cap H_j = S$, for all $1 \leq i < j \leq k$ and satisfying $\mathrm{cov}(\mathcal{H}) = F$. ■

Definition 3.2.1. An S-compatible family $\mathcal{H} = \{H_1, \ldots, H_k\}$ for some $S \subseteq U$ with $|S| \geq 2$ is (U, S)-*compatible* if $H_i \cap U = S$ for all $1 \leq i \leq k$.

If we are given a tight S-compatible family $\mathcal{H}$ for some $S \subseteq U$ with $|S| \geq 2$, then we can obtain a tight (U, T)-compatible family $\mathcal{H}'$, for some $U \supseteq T \supseteq S$, by applying Lemma 3.2.5 recursively. In addition we will have $\operatorname{cov}(\mathcal{H}) \subseteq \operatorname{cov}(\mathcal{H}')$. Since we will be considering tight compatible families with maximal cover sets, working with $\mathcal{H}'$ will be more helpful than working with $\mathcal{H}$.

We need Lemmas 3.2.8 and 3.2.9 to obtain a new compatible family with a larger cover set and smaller value from two distinct compatible families. Lemma 2.2.5 in Chapter 2 plays the same role as the combination of Lemmas 3.2.8 and 3.2.9. When U has size three any two subsets S_1, S_2 of U of size at least two have a non-empty intersection. However, if U has more than three vertices, then this does not hold. Therefore, we only need Lemma 2.2.5 in Chapter 2, because the only case in Chapter 2 is $S_1 \cap S_2 \neq \emptyset$. We need Lemmas 3.2.8 and 3.2.9 in this chapter because one deals with the case when $S_1 \cap S_2 \neq \emptyset$ and the other deals with the case when $S_1 \cap S_2 = \emptyset$.

Lemma 3.2.8. *Let $G = (V, E)$ be S-sparse for all $S \subseteq U$ with $|S| \geq 2$, and suppose that there are tight (U, S_1)- and (U, S_2)-compatible families $\mathcal{H}_1$ and $\mathcal{H}_2$ in G for some $S_1, S_2 \subseteq U$ with $|S_i| \geq 2$ and $S_1 \cap S_2 \neq \emptyset$. Then there is a $(U, S_1 \cup S_2)$-compatible family $\mathcal{H}_{\text{union}}$ in G with the properties*

(i) $\mathcal{H}_{\text{union}}$ *is $(S_1 \cup S_2)$-tight,*

(ii) $\operatorname{cov}(\mathcal{H}_1) \cup \operatorname{cov}(\mathcal{H}_2) \subseteq \operatorname{cov}(\mathcal{H}_{\text{union}})$, *and*

(iii) Either

$$\operatorname{val}_{S_1}(\mathcal{H}_1) + \operatorname{val}_{S_2}(\mathcal{H}_2) > \operatorname{val}_{S_1 \cup S_2}(\mathcal{H}_{\text{union}}) \tag{3.1}$$

or both

$$\operatorname{val}_{S_1}(\mathcal{H}_1) + \operatorname{val}_{S_2}(\mathcal{H}_2) = \operatorname{val}_{S_1 \cup S_2}(\mathcal{H}_{\text{union}}) \tag{3.2}$$

and

$$\operatorname{cov}(\mathcal{H}_1) \cup \operatorname{cov}(\mathcal{H}_2) \subsetneq \operatorname{cov}(\mathcal{H}_{\text{union}}) \tag{3.3}$$

hold.

Proof: Let $\mathcal{H}_1 = \{H_1, \ldots, H_k\}$ and $\mathcal{H}_2 = \{\overline{H}_1, \ldots, \overline{H}_l\}$. Since $\mathcal{H}_i$ is a tight S_i-compatible family for $i = 1, 2$, Lemma 3.2.1 implies that $H_i \cap H_j = S_1$ for all $1 \leq i < j \leq k$ and $\overline{H}_i \cap \overline{H}_j = S_2$ for all $1 \leq i < j \leq l$.

Since $\mathcal{H}_1$ and $\mathcal{H}_2$ are (U, S_1)- and (U, S_2)-compatible, we have $H_i \cap U = S_1$ for all $1 \le i \le k$ and $\overline{H}_j \cap U = S_2$ for all $1 \le i \le l$.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the bipartite graph with bipartition $(\mathcal{H}_1, \mathcal{H}_2)$, and edge set

$$\mathcal{E} := \{H_i\overline{H}_j : |(H_i \setminus S_1) \cap (\overline{H}_j \setminus S_2)| \ge 1, 1 \le i \le k, 1 \le j \le l\}.$$

Let $(\mathcal{V}_i, \mathcal{F}_i)$, $1 \le i \le r$ be the connected components of $\mathcal{G}$. Define $V_i = \bigcup_{H \in \mathcal{V}_i} H$ and put

$$\mathcal{H}_{\text{union}} := \{V_i \cup S_1 \cup S_2 : 1 \le i \le r\},$$

$$\mathcal{H}_{\text{int}} := \{H_i \cap \overline{H}_j : H_i\overline{H}_j \in \mathcal{E}\}.$$

Note that $\mathcal{H}_{\text{union}}$ and $\mathcal{H}_{\text{int}}$ are $(U, S_1 \cup S_2)$- and $(U, S_1 \cap S_2)$-compatible, respectively. We see that every edge in E which is covered by either $\mathcal{H}_1$ or $\mathcal{H}_2$ is covered by $\mathcal{H}_{\text{union}}$ and every edge covered by both $\mathcal{H}_1$ and $\mathcal{H}_2$ is covered by $\mathcal{H}_{\text{int}}$. This implies that $i(\mathcal{H}_1) + i(\mathcal{H}_2) \le i(\mathcal{H}_{\text{union}}) + i(\mathcal{H}_{\text{int}})$. Since $|\mathcal{V}| = k + l$ and r is the number of connected components of $\mathcal{G}$,

$$r + |\mathcal{E}| \ge k + l. \tag{3.4}$$

We also have

$$\begin{aligned}
&\sum_{i=1}^{r} (|V_i \cup S_1 \cup S_2| - |S_1 \cup S_2|) + \sum_{H_i\overline{H}_j \in \mathcal{E}} (|H_i \cap \overline{H}_j| - |S_1 \cap S_2|) \\
=&\sum_{i=1}^{k} (|H_i| - |S_1|) + \sum_{i=1}^{l} (|\overline{H}_i| - |S_2|)
\end{aligned} \tag{3.5}$$

as a vertex $x \notin S_1 \cup S_2$ contributes the same amount (one or two) to both sides, and a vertex $s \in S_1 \cup S_2$ contributes zero to both sides of (3.5).

Then we have

$$\begin{aligned}
&\sum_{i=1}^{k} (2|H_i \setminus S_1| - 1) + 2(|S_1| - 1) + \sum_{i=1}^{l} 2|\overline{H}_i \setminus S_2| - 1 + 2(|S_2| - 1) \\
&= \operatorname{val}_{S_1}(\mathcal{H}_1) + \operatorname{val}_{S_2}(\mathcal{H}_2) \\
&= i(\mathcal{H}_1) + i(\mathcal{H}_2) \\
&\le i(\mathcal{H}_{\text{union}}) + i(\mathcal{H}_{\text{int}})
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathrm{val}_{S_1\cup S_2}(\mathcal{H}_{\mathrm{union}}) + \mathrm{val}_{S_1\cap S_2}(\mathcal{H}_{\mathrm{int}})\\
&= \sum_{i=1}^{r} (2|(V_i \cup S_1 \cup S_2) \setminus (S_1 \cup S_2)| - 1) + 2(|S_1 \cup S_2| - 1)\\
&\quad + \sum_{H_i\overline{H}_j \in \mathcal{E}} (2|(H_i \cap \overline{H}_j) \setminus (S_1 \cap S_2)| - 1) + 2(|S_1 \cap S_2| - 1)\\
&= \sum_{i=1}^{r} 2(|V_i \cup S_1 \cup S_2| - |S_1 \cup S_2|) + 2(|S_1 \cup S_2| - 1) - r\\
&\quad + \sum_{H_i\overline{H}_j \in \mathcal{E}} 2(|H_i \cap \overline{H}_j| - |S_1 \cap S_2|) + 2(|S_1 \cap S_2| - 1) - |\mathcal{E}|\\
&\leq \sum_{i=1}^{k} 2(|H_i| - |S_1|) + \sum_{i=1}^{l} 2(|\overline{H}_i| - |S_2|)\\
&\quad + 2(|S_1 \cup S_2| - 1) + 2(|S_1 \cap S_2| - 1) - k - l\\
&= \sum_{i=1}^{k} 2(|H_i| - |S_1|) + \sum_{i=1}^{l} 2(|\overline{H}_i| - |S_2|) + 2|S_1| + 2|S_2| - 2 - 2 - k - l\\
&= \sum_{i=1}^{k} (2|H_i \setminus S_1| - 1) + 2(|S_1| - 1) + \sum_{i=1}^{l} 2|\overline{H}_i \setminus S_2| - 1 + 2(|S_2| - 1),
\end{aligned}
$$

where the third inequality follows from (3.4) and (3.5), and the second last equality follows from the formula $|S_1 \cup S_2| + |S_1 \cap S_2| = |S_1| + |S_2|$. Hence equality must hold throughout. In particular $\mathcal{H}_{\mathrm{union}}$ is $(S_1 \cup S_2)$-tight, so (i) holds. It is clear that $\mathrm{cov}(\mathcal{H}_1) \cup \mathrm{cov}(\mathcal{H}_2) \subseteq \mathrm{cov}(\mathcal{H}_{\mathrm{union}})$, so (ii) holds.

For the proof of (iii) we will show that if $\mathcal{H}_{\mathrm{int}}$ is non-empty, then (3.1) holds, and if it is empty (3.2) and (3.3) hold.

We obtain $\mathrm{val}_{S_1}(\mathcal{H}_1) + \mathrm{val}_{S_2}(\mathcal{H}_2) = \mathrm{val}_{S_1\cup S_2}(\mathcal{H}_{\mathrm{union}}) + \mathrm{val}_{S_1\cap S_2}(\mathcal{H}_{\mathrm{int}})$ from the (in)equalities above. If $\mathcal{H}_{\mathrm{int}}$ is non-empty, then $\mathrm{val}_{S_1\cap S_2}(\mathcal{H}_{\mathrm{int}}) > 0$, implying that $\mathrm{val}_{S_1}(\mathcal{H}_1) + \mathrm{val}_{S_2}(\mathcal{H}_2) > \mathrm{val}_{S_1\cup S_2}(\mathcal{H}_{\mathrm{union}})$. Now suppose $\mathcal{H}_{\mathrm{int}}$ is empty. Then we have $\mathrm{val}_{S_1\cap S_2}(\mathcal{H}_{\mathrm{int}}) = 0$, implying that $\mathrm{val}_{S_1}(\mathcal{H}_1) + \mathrm{val}_{S_2}(\mathcal{H}_2) = \mathrm{val}_{S_1\cup S_2}(\mathcal{H}_{\mathrm{union}})$, so (3.2) holds. It remains to show that (3.3) holds.

Claim 3.2.8.1. *$S_1 \not\subseteq S_2$ and $S_2 \not\subseteq S_1$.*

Proof of Claim: For a contradiction, assume $S_1 \subseteq S_2$, and hence $\mathcal{H}_{\mathrm{union}}$ is S_2-compatible. Since $\mathcal{H}_{\mathrm{int}}$ is empty, the connected components of $\mathcal{G}$ are $H_1, \ldots, H_k$,

$\overline{H}_1, \ldots, \overline{H}_l$. This implies that $\mathcal{H}_{\text{union}} = \{H_1 \cup S_2, \ldots, H_k \cup S_2, \overline{H}_1, \ldots, \overline{H}_l\}$. Note that we already have $S_2 \subseteq \overline{H}_j$ for all $1 \leq j \leq l$. Then

$$\begin{aligned}
&\text{val}_{S_1}(\mathcal{H}_1) + \text{val}_{S_2}(\mathcal{H}_2) \\
&= \sum_{i=1}^{k} (2|H_i \setminus S_1| - 1) + 2(|S_1| - 1) + \sum_{i=1}^{l} (2|\overline{H}_i \setminus S_2| - 1) + 2(|S_2| - 1) \\
&= \sum_{i=1}^{k} (2|(H_i \cup S_2) \setminus (S_1 \cup S_2)| - 1) + 2(|S_1| - 1) + \sum_{i=1}^{l} (2|\overline{H}_i \setminus S_2| - 1) + 2(|S_2| - 1) \\
&= \sum_{i=1}^{k} (2|(H_i \cup S_2) \setminus S_2| - 1) + \sum_{i=1}^{l} (2|\overline{H}_i \setminus S_2| - 1) + 2(|S_2| - 1) + 2(|S_1| - 1) \\
&= \text{val}_{S_2}(\mathcal{H}_{\text{union}}) + 2(|S_1| - 1) \\
&> \text{val}_{S_2}(\mathcal{H}_{\text{union}}).
\end{aligned}$$

This contradicts the fact that $\text{val}_{S_1}(\mathcal{H}_1) + \text{val}_{S_2}(\mathcal{H}_2) = \text{val}_{S_2}(\mathcal{H}_{\text{union}})$. •

Now pick a pair of vertices (s_1, s_2) with $s_1 \in S_1 \setminus S_2$ and $s_2 \in S_2 \setminus S_1$. Since $\mathcal{H}_1$ and $\mathcal{H}_2$ are (U, S_1)- and (U, S_2)-compatible, respectively, no set in $\mathcal{H}_1$ contains s_2 and no set in $\mathcal{H}_2$ contains s_1. This implies $(s_1, s_2) \notin \text{cov}(\mathcal{H}_1) \cup \text{cov}(\mathcal{H}_2)$. It is easy to see that $(s_1, s_2) \in \text{cov}(\mathcal{H}_{\text{union}})$ as $\mathcal{H}_{\text{union}}$ is $S_1 \cup S_2$-compatible. Hence we have $\text{cov}(\mathcal{H}_1) \cup (\mathcal{H}_2) \subsetneq \text{cov}(\mathcal{H}_{\text{union}})$. ■

Lemma 3.2.9. *Let $G = (V, E)$ be S-sparse for all $S \subseteq U$ with $|S| \geq 2$. Let $\mathcal{H}_1 = \{H_1, \ldots, H_k\}$ and $\mathcal{H}_2 = \{\overline{H}_1, \ldots, \overline{H}_l\}$ be tight (U, S_1)- and (U, S_2)-compatible families in G for some $S_1, S_2 \subseteq U$ with $|S_1|, |S_2| \geq 2$ and $S_1 \cap S_2 = \emptyset$. Suppose $|(\bigcup_{i=1}^{k} H_i) \cap (\bigcup_{j=1}^{l} \overline{H}_j)| \geq 2$. Then there is a $(U, S_1 \cup S_2)$-compatible family $\mathcal{H}_{\text{union}}$ in G with the properties*

(i) *$\mathcal{H}_{\text{union}}$ is $(S_1 \cup S_2)$-tight,*

(ii) *$\text{val}_{S_1}(\mathcal{H}_1) + \text{val}_{S_2}(\mathcal{H}_2) \geq \text{val}_{S_1 \cup S_2}(\mathcal{H}_{\text{union}})$, and*

(iii) *$\text{cov}(\mathcal{H}_1) \cup \text{cov}(\mathcal{H}_2) \subsetneq \text{cov}(\mathcal{H}_{\text{union}})$.*

Proof: First note that since $\mathcal{H}_1$ and $\mathcal{H}_2$ are S_1- and S_2-tight, respectively, we have $H_i \cap H_j = S_1$ and $\overline{H}_p \cap \overline{H}_q = S_2$ for all $1 \le i < j \le k$ and for all $1 \le p < q \le l$, by Lemma 3.2.1.

Note also that since $\mathcal{H}_1$ and $\mathcal{H}_2$ are (U, S_1)- and (U, S_2)-compatible respectively and $S_1 \cap S_2 = \emptyset$, we have $H_i \cap S_2 = \emptyset$ for all $1 \le i \le k$ and $\overline{H}_j \cap S_1 = \emptyset$ for all $1 \le j \le l$. Consider the bipartite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex bipartition $(\mathcal{H}_1, \mathcal{H}_2)$ and edge bipartition $(\mathcal{E}_1, \mathcal{E}_2)$ where

$$\mathcal{E}_1 = \{H_i\overline{H}_j : |H_i \cap \overline{H}_j| = 1, 1 \le i \le k, 1 \le j \le l\},$$

$$\mathcal{E}_2 = \{H_i\overline{H}_j : |H_i \cap \overline{H}_j| \ge 2, 1 \le i \le k, 1 \le j \le l\}.$$

Let $(\mathcal{V}_i, \mathcal{F}_i)$, $1 \le i \le r$ be the connected components of $\mathcal{G}$. Define $V_i = \bigcup_{H \in \mathcal{V}_i} H$ and put

$$\mathcal{H}_{\text{union}} := \{V_i \cup S_1 \cup S_2 : 1 \le i \le r\},$$

$$\mathcal{H}_{\text{int}} := \{H_i \cap \overline{H}_j : H_i\overline{H}_j \in \mathcal{E}_2\}.$$

Then $\mathcal{H}_{\text{union}}$ is $(U, S_1 \cup S_2)$-compatible. We see that every edge in E which is covered by either $\mathcal{H}_1$ or $\mathcal{H}_2$ is covered by $\mathcal{H}_{\text{union}}$ and every edge covered by both $\mathcal{H}_1$ and $\mathcal{H}_2$ is covered by $\mathcal{H}_{\text{int}}$. This implies that $i(\mathcal{H}_1) + i(\mathcal{H}_2) \le i(\mathcal{H}_{\text{union}}) + i(\mathcal{H}_{\text{int}})$. Note that $\mathcal{H}_{\text{int}}$ is not an $(S_1 \cap S_2)$-compatible family as $S_1 \cap S_2 = \emptyset$. It is just a family of subsets of V. Since $|(\bigcup_{i=1}^{k} H_i) \cap (\bigcup_{j=1}^{l} \overline{H}_j)| \ge 2$, either $|\mathcal{E}_1| \ge 2$ or $|\mathcal{E}_2| \ge 1$. From this we obtain

$$|\mathcal{E}_1| + 2|\mathcal{E}_2| \ge 2. \tag{3.6}$$

Note that since r is the number of connected components, $|\mathcal{E}|$ is the number of edges and $k + l$ is the number of vertices of $\mathcal{G}$,

$$r + |\mathcal{E}| \ge k + l. \tag{3.7}$$

Note also that

$$\begin{aligned}
&\sum_{i=1}^{r}(|V_i \cup S_1 \cup S_2| - |S_1 \cup S_2|) + \sum_{H_i\overline{H}_j \in \mathcal{E}_2} |H_i \cap \overline{H}_j| + |\mathcal{E}_1| \\
&= \sum_{i=1}^{k}(|H_i| - |S_1|) + \sum_{i=1}^{l}(|\overline{H}_i| - |S_2|)
\end{aligned} \tag{3.8}$$

as for every pair of distinct edges $H_i\overline{H}_j$ and $H_t\overline{H}_s$ of $\mathcal{G}$, the corresponding intersections $H_i \cap \overline{H}_j$ and $H_t \cap \overline{H}_s$ are disjoint. This fact is a consequence of the facts that $H_i \cap H_t = S_1$, $\overline{H}_j \cap \overline{H}_s = S_2$ and $S_1 \cap S_2 = \emptyset$ for all $1 \le i < t \le k$ and for all $1 \le j < s \le l$. Then we have

$$\begin{aligned}
&\sum_{i=1}^{k}(2|H_i \setminus S_1| - 1) + 2(|S_1| - 1) + \sum_{i=1}^{l} 2|\overline{H}_i \setminus S_2| - 1 + 2(|S_2| - 1) \\
&= \mathrm{val}_{S_1}(\mathcal{H}_1) + \mathrm{val}_{S_2}(\mathcal{H}_2) \\
&= i(\mathcal{H}_1) + i(\mathcal{H}_2) \\
&\le i(\mathcal{H}_{\text{union}}) + i(\mathcal{H}_{\text{int}}) \\
&\le \mathrm{val}_{S_1 \cup S_2}(\mathcal{H}_{\text{union}}) + \mathrm{val}(\mathcal{H}_{\text{int}}) \\
&= \sum_{i=1}^{r}(2|(V_i \cup S_1 \cup S_2) \setminus (S_1 \cup S_2)| - 1) + 2(|S_1 \cup S_2| - 1) + \sum_{H_i\overline{H}_j \in \mathcal{E}_2}(2|H_i \cap \overline{H}_j| - 3) \\
&= \sum_{i=1}^{r} 2(|V_i \cup S_1 \cup S_2| - |S_1 \cup S_2|) + \sum_{H_i\overline{H}_j \in \mathcal{E}_2} 2|H_i \cap \overline{H}_j| + 2(|S_1 \cup S_2| - 1) - 3|\mathcal{E}_2| - r \\
&= \sum_{i=1}^{k} 2(|H_i| - |S_1|) + \sum_{i=1}^{l} 2(|\overline{H}_i| - |S_2|) - 2|\mathcal{E}_1| + 2(|S_1 \cup S_2| - 1) - 3|\mathcal{E}_2| - r \\
&= \sum_{i=1}^{k} 2(|H_i| - |S_1|) + \sum_{i=1}^{l} 2(|\overline{H}_i| - |S_2|) + 2|S_1| + 2|S_2| - 2 - (|\mathcal{E}_1| + 2|\mathcal{E}_2|) - (|\mathcal{E}_1| + |\mathcal{E}_2|) - r \\
&\le \sum_{i=1}^{k} 2(|H_i| - |S_1|) + \sum_{i=1}^{l} 2(|\overline{H}_i| - |S_2|) + 2|S_1| + 2|S_2| - 2 - 2 - |\mathcal{E}| - r \\
&\le \sum_{i=1}^{k} 2(|H_i| - |S_1|) + \sum_{i=1}^{l} 2(|\overline{H}_i| - |S_2|) + 2|S_1| + 2|S_2| - 2 - 2 - k - l
\end{aligned}$$

$$= \sum_{i=1}^{k} (2|H_i \setminus S_1| - 1) + 2(|S_1| - 1) + \sum_{i=1}^{l} 2|\overline{H}_i \setminus S_2| - 1 + 2(|S_2| - 1),$$

where: the fifth equality follows from (3.8); the sixth equality follows from the fact that $|S_1 \cup S_2| = |S_1| + |S_2|$ as $S_1 \cap S_2 = \emptyset$; the third inequality follows from (3.6) and $|\mathcal{E}| = |\mathcal{E}_1| + |\mathcal{E}_2|$; the fourth inequality follows from (3.7). Hence equality holds throughout and we can deduce that $\mathcal{H}_{\text{union}}$ is $(S_1 \cup S_2)$-tight so (i) holds. We obtain $\text{val}_{S_1}(\mathcal{H}_1) + \text{val}_{S_2}(\mathcal{H}_2) = \text{val}_{S_1 \cup S_2}(\mathcal{H}_{\text{union}}) + \text{val}(\mathcal{H}_{\text{int}})$ from the (in)equalities above. This implies $\text{val}_{S_1}(\mathcal{H}_1) + \text{val}_{S_2}(\mathcal{H}_2) \geq \text{val}_{S_1 \cup S_2}(\mathcal{H}_{\text{union}})$, so (ii) holds. It is clear that we have $\text{cov}(\mathcal{H}_1) \cup \text{cov}(\mathcal{H}_2) \subseteq \text{cov}(\mathcal{H}_{\text{union}})$. Pick a pair of vertices (s_1, s_2) with $s_1 \in S_1$ and $s_2 \in S_2$. Since no member of $\mathcal{H}_1$ contains s_2 and no member of $\mathcal{H}_2$ contains s_1, $(s_1, s_2) \notin \text{cov}(\mathcal{H}_1) \cup \text{cov}(\mathcal{H}_2)$. It is easy to see that $(s_1, s_2) \in \text{cov}(\mathcal{H}_{\text{union}})$ as $\mathcal{H}_{\text{union}}$ is $(S_1 \cup S_2)$-compatible. Hence, $\text{cov}(\mathcal{H}_1) \cup \text{cov}(\mathcal{H}_2) \subsetneq \text{cov}(\mathcal{H}_{\text{union}})$, so (iii) holds. ■

3.2.2 Systems of Compatible Families

Let $G = (V, E)$ be a graph and $U \subseteq V$. Suppose that G is S-sparse for all $S \subseteq U$.

Definition 3.2.2. An $(S_1, S_2, \ldots, S_k)$-*compatible system* of G is a collection $\mathcal{K} = \{\mathcal{H}_1, \ldots, \mathcal{H}_k\}$ such that each $\mathcal{H}_i$ is an (U, S_i)-compatible family for some $S_i \subseteq U$ with $|S_i| \geq 2$, for all $1 \leq i \leq k$. We will also refer to $\mathcal{K}$ as a U-*system* when we do not want to specify the sets $S_1, \ldots, S_k$.

We define the *value* of the U-system $\mathcal{K}$ as

$$\text{val}(\mathcal{K}) := \sum_{i=1}^{k} \text{val}_{S_i}(\mathcal{H}_i).$$

We say $\mathcal{K}$ is $(S_1, \ldots, S_k)$-*tight*, or just *tight* if it is clear what S_i we are referring to, if $\text{val}(\mathcal{K}) = i_G(\mathcal{K}) := |\bigcup_{i=1}^{k} E_G(\mathcal{H}_i)|$. The cover set of $\mathcal{K}$ is defined as $\text{cov}(\mathcal{K}) := \bigcup_{i=1}^{k} \text{cov}(\mathcal{H}_i)$.

Definition 3.2.3. A U-system $\mathcal{K} = \{\mathcal{H}_1, \ldots, \mathcal{H}_k\}$ of (U, S_i)-compatible families for $1 \leq i \leq k$, is 1-*thin* if

- (T1) $H \cap H' = S_i$ for all distinct $H, H' \in \mathcal{H}_i$, $1 \leq i \leq k$,

- (T2) $S_i \cap S_j = \emptyset$ for all $1 \leq i < j \leq k$,
- (T3) $|(\bigcup_{H \in \mathcal{H}_i} H) \cap (\bigcup_{H' \in \mathcal{H}_j} H')| \leq 1$ for all $1 \leq i < j \leq k$.

Lemma 3.2.10. *Let $G = (V, E)$ be S-sparse for all $S \subseteq U$ with $|S| \geq 2$. Suppose that $\mathcal{K} = \{\mathcal{H}_1, \ldots, \mathcal{H}_k\}$ is a tight $(S_1, \ldots, S_k)$-compatible system. Then $\mathcal{H}_i$ is S_i-tight for all $1 \leq i \leq k$ and $E(\mathcal{H}_i) \cap E(\mathcal{H}_j) = \emptyset$ for all $1 \leq i < j \leq k$.*

Proof: Since $\mathcal{K}$ is $(S_1, \ldots, S_k)$-tight we have $\mathrm{val}(\mathcal{K}) = i(\mathcal{K})$. Then

$$\sum_{j=1}^{k} \mathrm{val}_{S_j}(\mathcal{H}_j) = \mathrm{val}(\mathcal{K}) = i(\mathcal{K}) = |\bigcup_{j=1}^{k} E(\mathcal{H}_j)| \leq \sum_{j=1}^{k} i(\mathcal{H}_j) \leq \sum_{j=1}^{k} \mathrm{val}_{S_j}(\mathcal{H}_j),$$

where the last inequality follows from the fact that G is S-sparse for all $S \subseteq U$ with $|S| \geq 2$. Hence equality holds throughout. In particular, we have $\mathrm{val}_{S_j}(\mathcal{H}_j) = i(\mathcal{H}_j)$ for all $1 \leq j \leq k$, since $\sum_{j=1}^{k} \mathrm{val}_{S_j}(\mathcal{H}_j) = \sum_{j=1}^{k} i(\mathcal{H}_j)$ and $\mathrm{val}_{S_j}(\mathcal{H}_j) \geq i(\mathcal{H}_j)$ for all $1 \leq j \leq k$. We also have $E(\mathcal{H}_i) \cap E(\mathcal{H}_j) = \emptyset$ for all $1 \leq i < j \leq k$ as $|\bigcup_{j=1}^{k} E(\mathcal{H}_j)| = \sum_{j=1}^{k} i(\mathcal{H}_j)$. ■

Lemma 3.2.11. *Let $G = (V, E)$ be S-sparse for all $S \subseteq U$ with $|S| \geq 2$. Suppose that $\mathcal{K} = \{\mathcal{H}_1, \ldots, \mathcal{H}_k\}$ is a 1-thin $(S_1, \ldots, S_k)$-compatible system of S_1-,…,S_k-tight compatible families, respectively. Then $\mathcal{K}$ is $(S_1, \ldots, S_k)$-tight.*

Proof: Since $\mathcal{K}$ is 1-thin and $\mathcal{H}_i$ is S_i-tight for all $1 \leq i \leq k$, we have $i(\mathcal{K}) = \sum_{j=1}^{k} i(\mathcal{H}_i)$ and $\mathrm{val}_{S_i}(\mathcal{H}_i) = i(\mathcal{H}_i)$, for all $1 \leq i \leq k$. Hence

$$\mathrm{val}(\mathcal{K}) = \sum_{i=1}^{k} \mathrm{val}_{S_i}(\mathcal{H}_i) = \sum_{i=1}^{k} i(\mathcal{H}_i) = i(\mathcal{K}),$$

where the second equality follows from the fact that $\mathcal{H}_i$ is S_i-tight and the third inequality follows from the fact that $\mathcal{K}$ is 1-thin. Hence $\mathcal{K}$ is $(S_1, \ldots, S_k)$-tight. ■

Lemma 3.2.12. *Let $G = (V, E)$ be an S-sparse graph for all $S \subseteq U$ with $|S| \geq 2$. Let $\mathcal{K} = \{\mathcal{H}_1, \ldots, \mathcal{H}_k\}$ be a tight $(S_1, \ldots, S_k)$-compatible system with maximal cover set, $\mathrm{cov}(\mathcal{K})$, over all tight U-systems, and subject to this condition $\sum_{\mathcal{H} \in \mathcal{K}} |\mathrm{cov}(\mathcal{H})|$ is maximum. Then $\mathcal{K}$ is 1-thin.*

Proof: Suppose $\mathcal{K} = \{\mathcal{H}_1, \dots, \mathcal{H}_k\}$ is a tight $(S_1, \dots, S_k)$-compatible system such that $\mathrm{cov}(\mathcal{K})$ is maximal, and subject to this condition, $\sum_{\mathcal{H}\in\mathcal{K}} |\,\mathrm{cov}(\mathcal{H})|$ is maximum. By Lemmas 3.2.10 and 3.2.1, (T1) holds. Suppose (T2) or (T3) does not hold.

We may assume $\mathcal{H}_{k-1}$ and $\mathcal{H}_k$ are the compatible families for which (T2) or (T3) fails. Then we apply Lemma 3.2.8 (if (T2) fails) or Lemma 3.2.9 (if (T2) holds but (T3) fails) and obtain an $(S_{k-1} \cup S_k)$-tight compatible family $\mathcal{H}_{\mathrm{union}}$ for which either

$$\mathrm{val}_{S_{k-1}}(\mathcal{H}_{k-1}) + \mathrm{val}_{S_k}(\mathcal{H}_k) > \mathrm{val}_{S_{k-1}\cup S_k}(\mathcal{H}_{\mathrm{union}}) \tag{3.9}$$

or both

$$\mathrm{val}_{S_{k-1}}(\mathcal{H}_{k-1}) + \mathrm{val}_{S_k}(\mathcal{H}_k) = \mathrm{val}_{S_{k-1}\cup S_k}(\mathcal{H}_{\mathrm{union}}) \tag{3.10}$$

and

$$\mathrm{cov}(\mathcal{H}_{k-1}) \cup \mathrm{cov}(\mathcal{H}_k) \subsetneq \mathrm{cov}(\mathcal{H}_{\mathrm{union}}) \tag{3.11}$$

holds. Now consider $\mathcal{K}' := \mathcal{K} \setminus \{\mathcal{H}_{k-1}, \mathcal{H}_k\} \cup \{\mathcal{H}_{\mathrm{union}}\}$. If (3.9) holds, then we would have $i(\mathcal{K}') \le \mathrm{val}(\mathcal{K}') < \mathrm{val}(\mathcal{K}) = i(\mathcal{K}) \le i(\mathcal{K}')$, a contradiction. Hence (3.10) and (3.11) hold. Then we obtain $i(\mathcal{K}') \le \mathrm{val}(\mathcal{K}') = \mathrm{val}(\mathcal{K}) = i(\mathcal{K}) \le i(\mathcal{K}')$, implying $\mathcal{K}'$ is tight. By the maximality of $\mathrm{cov}(\mathcal{K}')$ and (3.11), we may assume $\mathrm{cov}(\mathcal{K}') = \mathrm{cov}(\mathcal{K})$. Then, for every pair in $(x, y) \in \mathrm{cov}(\mathcal{H}_{\mathrm{union}}) \setminus (\mathrm{cov}(\mathcal{H}_{k-1}) \cup \mathrm{cov}(\mathcal{H}_k))$, there exists a compatible family $\mathcal{H}_j$ with $(x, y) \in \mathrm{cov}(\mathcal{H}_j)$ for some $1 \le j \le k-2$. However, then we have $\sum_{\mathcal{H}\in\mathcal{K}} |\,\mathrm{cov}(\mathcal{H})| < \sum_{\mathcal{H}\in\mathcal{K}'} |\,\mathrm{cov}(\mathcal{H})|$, contradicting the the fact that $\sum_{\mathcal{H}\in\mathcal{K}} |\,\mathrm{cov}(\mathcal{H})|$ is maximum. Hence our assumption is wrong and both (T2) and (T3) hold for $\mathcal{K}$, so $\mathcal{K}$ is 1-thin. ■

Lemma 3.2.13. *Let $G = (V, E)$ be S-sparse for all $S \subseteq U$ with $|S| \ge 2$. Suppose G has a tight U-system. Then G has a unique tight U-system $\mathcal{K}_{\max}$ with the property that $\mathrm{cov}(\mathcal{K}_{\max})$ is maximal over all tight U-systems, and subject to this condition $\sum_{\mathcal{H}\in\mathcal{K}} |\,\mathrm{cov}(\mathcal{H})|$ is maximum.*

Proof: Let $\mathcal{K}_1 = \{\mathcal{H}_1, \dots, \mathcal{H}_k\}$ and $\mathcal{K}_2 = \{\overline{\mathcal{H}}_1, \dots, \overline{\mathcal{H}}_l\}$ be two distinct tight $(S_1, \dots, .S_k)$- and $(T_1, \dots, T_l)$-compatible systems which are both maximal with respect to cover sets, and subject to this $\sum_{i=1}^{k} |\,\mathrm{cov}(\mathcal{H}_k)|$ and $\sum_{j=1}^{l} |\,\mathrm{cov}(\overline{\mathcal{H}}_j)|$ are maximum. Then by Lemma 3.2.12, $\mathcal{K}_1$ and $\mathcal{K}_2$ are 1-thin. Moreover, $\mathcal{H}_i$ is S_i-tight for all $1 \le i \le k$, and $\overline{\mathcal{H}}_j$ is T_j-tight for all $1 \le j \le l$, by Lemma 3.2.10.

Claim 3.2.13.1. $\mathrm{cov}(\mathcal{K}_1) \neq \mathrm{cov}(\mathcal{K}_2)$.

Proof of Claim: Suppose the contrary and set $F := \mathrm{cov}(\mathcal{K}_1) = \mathrm{cov}(\mathcal{K}_2)$. Then for any two distinct vertices $u, u' \in U$ with $(u, u') \in F$, there exists an S_i with $u, u' \in S_i$ for some $1 \leq i \leq k$ as $\mathcal{K}_1$ consists of (U, S_i)-compatible families. Since $\mathcal{K}_1$ is 1-thin, the S_i are disjoint by (T2). This implies there exists a unique S_i for each such pair. Similarly, there exists a unique T_j with $u, u' \in T_j$, for some $1 \leq j \leq l$. Let us set $F_U := F \cap U \times U$. Then the facts that $F_U = \bigcup_{i=1}^{k} \mathrm{cov}(S_i) = \bigcup_{j=1}^{l} \mathrm{cov}(T_j)$ and that the S_i and the T_j are disjoint imply $\{S_1, S_2, \ldots, S_k\} = \{T_1, T_2, \ldots, T_l\}$.

By relabelling, if necessary, we may assume $\mathcal{K}_2 = \{\overline{\mathcal{H}}_1, \ldots, \overline{\mathcal{H}}_k\}$ and $\overline{\mathcal{H}}_i$ is S_i-compatible. If $\mathrm{cov}(\mathcal{H}_i) = \mathrm{cov}(\overline{\mathcal{H}}_i)$ for all $1 \leq i \leq k$, then by Lemma 3.2.7 we would have $\mathcal{H}_i = \overline{\mathcal{H}}_i$ for all $1 \leq i \leq k$, contradicting the fact that $\mathcal{K}_1 \neq \mathcal{K}_2$. Hence we may assume $\mathrm{cov}(\mathcal{H}_i) \neq \mathrm{cov}(\overline{\mathcal{H}}_i)$ for some $1 \leq i \leq k$. Then either $\mathrm{cov}(\mathcal{H}_i) \setminus \mathrm{cov}(\overline{\mathcal{H}}_i)$ or $\mathrm{cov}(\overline{\mathcal{H}}_i) \setminus \mathrm{cov}(\mathcal{H}_i)$ is non-empty. By symmetry we may assume $\mathrm{cov}(\mathcal{H}_i) \setminus \mathrm{cov}(\overline{\mathcal{H}}_i) \neq \emptyset$ and pick a pair $(x, y) \in (\mathrm{cov}(\mathcal{H}_i) \setminus \mathrm{cov}(\overline{\mathcal{H}}_i))$. Since $(x, y) \in \mathrm{cov}(\mathcal{H}_i)$, there exists a set $H \in \mathcal{H}_i$ with $x, y \in H$. The fact that $S_i \subset H$ implies $(u, x), (u, y) \in F$ for all $u \in S_i$. Then since $S_i \cap S_j = \emptyset$ for all $i \neq j$, we must have $(u, x), (u, y) \in \mathrm{cov}(\overline{\mathcal{H}}_i)$. Therefore there exist sets $\overline{H}_1, \overline{H}_2 \in \overline{\mathcal{H}}_i$ with $x \in \overline{H}_1$ and $y \in \overline{H}_2$. We also know there exists $\overline{\mathcal{H}}_j$ with $i \neq j$ for which $(x, y) \in \mathrm{cov}(\overline{\mathcal{H}}_j)$ as $(x, y) \in F$. Combining these we have $|(\bigcup_{\overline{H} \in \overline{\mathcal{H}}_i} \overline{H}) \cap (\bigcup_{\overline{H} \in \overline{\mathcal{H}}_j} \overline{H})| \geq 2$, contradicting the fact that $\mathcal{K}_2$ is 1-thin. •

Now Claim 3.2.13.1 implies that we have either $\mathrm{cov}(\mathcal{K}_1) \setminus \mathrm{cov}(\mathcal{K}_2) \neq \emptyset$ or $\mathrm{cov}(\mathcal{K}_2) \setminus \mathrm{cov}(\mathcal{K}_1) \neq \emptyset$. We may assume $\mathrm{cov}(\mathcal{K}_2) \setminus \mathrm{cov}(\mathcal{K}_1) \neq \emptyset$ by symmetry. Pick a pair of vertices x, y with $(x, y) \in \mathrm{cov}(\mathcal{K}_2) \setminus \mathrm{cov}(\mathcal{K}_1)$. Then there exists a set in $\overline{\mathcal{H}}_j \in \mathcal{K}_2$ containing x and y for some $1 \leq i \leq l$. Consider the U-system $\mathcal{K}' = \mathcal{K}_1 \cup \{\overline{\mathcal{H}}_j\}$. Clearly, we have $\mathrm{cov}(\mathcal{K}_1) \subsetneq \mathrm{cov}(\mathcal{K}')$. Suppose $\mathcal{K}'$ is 1-thin. Since $\mathcal{H}_i$ is S_i-tight for all $1 \leq i \leq k$, and $\overline{\mathcal{H}}_j$ is T_j-tight, $\mathcal{K}'$ is $(S_1, \ldots, S_k, T_j)$-tight, by Lemma 3.2.11. However, this contradicts the maximality of $\mathrm{cov}(\mathcal{K}_1)$.

Hence we may assume $\mathcal{K}'$ is not 1-thin. Then either (T2) or (T3) does not hold. Note that (T1) holds by Lemma 3.2.1 as the compatible families in $\mathcal{K}'$ are $S_1, \ldots, S_k, T_j$-tight, respectively. Then we apply Lemma 3.2.8 (if (T2) fails) or Lemma 3.2.9 (if (T2) holds but (T3) fails) on $\mathcal{K}'$, recursively and within $\mathcal{K}'$, replace the corresponding compatible families by $\mathcal{H}_{\mathrm{union}}$ that we obtain after applying

Lemma 3.2.8 or Lemma 3.2.9 at each recursion step. Since $\mathcal{H}_{\text{union}}$ is tight for every recursion step, we preserve the fact that every member of $\mathcal{K}'$ is tight. Hence this process turns $\mathcal{K}'$ into a 1-thin U-system $\mathcal{K}''$ of tight compatible families with $\text{cov}(\mathcal{K}') \subseteq \text{cov}(\mathcal{K}'')$. Then by Lemma 3.2.11, $\mathcal{K}''$ is tight. However, this contradicts the maximality of $\text{cov}(\mathcal{K}_1)$ over tight U-systems as we have $\text{cov}(\mathcal{K}_1) \subsetneq \text{cov}(\mathcal{K}') \subseteq \text{cov}(\mathcal{K}'')$. ■

3.2.3 The Matroid $\mathcal{M}_U(G)$ and its Rank Function

Let $G = (V, E)$ be a graph and $U \subseteq V$. Let us say G is *U-sparse* if it is S-sparse for all $S \subseteq U$ with $|S| \geq 2$. In this subsection we prove that the family

$$\mathcal{I}_G = \{F : F \subseteq E, H = (V, F) \text{ is } U\text{-sparse}\} \tag{3.12}$$

is a family of independent sets of a matroid on E. We need the following definition. We say a system $\mathcal{L} = \mathcal{K} \cup \{X_1, \ldots, X_l\}$, where $\mathcal{K}$ is either empty or an $(S_1, \ldots, S_k)$-compatible system $\{\mathcal{H}_1, \ldots, \mathcal{H}_k\}$ and $X_1, \ldots, X_l \subseteq V$ are of size at least two, is a *cover* of $E' \subseteq E$ if $E' \subseteq \text{cov}(\mathcal{K}) \cup \bigcup_{i=1}^{l} \text{cov}(X_i)$. We define $\text{cov}(\mathcal{L}) := \text{cov}(\mathcal{K}) \cup \bigcup_{i=1}^{l} \text{cov}(X_i)$. We say that the cover $\mathcal{L}$ is 1-*thin* if

- $\mathcal{K}$ is 1-thin,
- (T4) $|X_i \cap X_j| \leq 1$ for all $1 \leq i < j \leq l$,
- (T5) $|X_i \cap \bigcup_{H \in \mathcal{H}_j} H| \leq 1$ for all $1 \leq j \leq k$ and $1 \leq i \leq l$.

We define the value of $\mathcal{L}$ as

$$\text{val}(\mathcal{L}) := \text{val}(\mathcal{K}) + \sum_{j=1}^{l} (2|X_j| - 3) = \sum_{i=1}^{k} \text{val}_{S_i}(\mathcal{H}_i) + \sum_{j=1}^{l} (2|X_j| - 3).$$

Let us also define $i_G(\mathcal{L}) := |E \cap \text{cov}(\mathcal{L})|$. It is clear that if G is U-sparse, then $i_G(\mathcal{L}) \leq \text{val}(\mathcal{L})$ holds for all 1-thin covers $\mathcal{L}$.

Theorem 3.2.14. *Let $G = (V, E)$ be a graph and $U \subseteq V$. Then $\mathcal{M}_U(G) := (E, \mathcal{I}_G)$ is a matroid on ground set E, where $\mathcal{I}_G$ is defined by (3.12). The rank of a set $E' \subseteq E$ in $\mathcal{M}_U(G)$ is equal to*

$$\min\{\text{val}(\mathcal{L}) : \mathcal{L} \textit{ is a } 1\textit{-thin cover of } E' \setminus E(U)\}.$$

Proof: We will proceed by showing that $\mathcal{I}_G$ satisfies the conditions (I1), (I2) and (I3$'$) of Definition 0.1.16. As (I1) and (I2) are trivial, we will only show (I3$'$) holds. Let $\mathcal{I} = \mathcal{I}_G$, $E' \subseteq E \setminus E(U)$ and $F \subseteq E'$ be a maximal subset of E' in $\mathcal{I}$. Since $F \in \mathcal{I}$ we have $|F| \leq \text{val}(\mathcal{L})$ for all covers $\mathcal{L}$ of E'. We will show that there is a 1-thin cover $\mathcal{L}$ of E' with $|F| = \text{val}(\mathcal{L})$, from which the theorem will follow.

Let $J = (V, F)$ denote the subgraph induced by the edge set F. First suppose that, for all $S \subseteq U$ with $|S| \geq 2$, there is no tight S-compatible family in J. Consider the following cover of F:

$$\mathcal{L}_1 = \{X_1, X_2, \ldots, X_l\},$$

where $X_1, X_2, \ldots, X_l$ are maximal tight sets in J. Since every edge $f \in F$ induces a tight set in J, $\mathcal{L}_1$ is a cover of F. It is 1-thin by Lemma 3.2.4. Thus

$$|F| = \sum_{j=1}^{k} |E_J(X_j)| = \sum_{j=1}^{l} (2|X_j| - 3) = \text{val}(\mathcal{L}_1)$$

follows. We claim that $\mathcal{L}_1$ is a cover of E'. To see this consider an edge $ab = e \in E' - F$. Since F is maximal subset of E' in $\mathcal{I}$ we have $F + e \notin \mathcal{I}$. By our assumption that there is no tight S-compatible family in J, there must be a tight set X in J with $a, b \in X$. Hence $X \subseteq X_i$ for some $1 \leq i \leq k$ which implies $\mathcal{L}_1$ covers e, too.

Next suppose there is a tight S-compatible family $\mathcal{H}$ for some $S \subseteq U$ with $|S| \geq 2$ in J. Then there must be a tight (U, T)-compatible family $\overline{\mathcal{H}}$ for which $\text{cov}(\mathcal{H}) \subseteq \text{cov}(\overline{\mathcal{H}})$ for some $T \supseteq U$, by Lemma 3.2.5. Hence there exists a tight U-system in J. Consider the following cover of F:

$$\mathcal{L}_2 = \{\mathcal{H}_1, \ldots, \mathcal{H}_k\} \cup \{X_1, X_2, \ldots, X_l\},$$

where $\{\mathcal{H}_1, \ldots, \mathcal{H}_k\} = \mathcal{K}_{\max}$ is the unique U-system of J for which $\text{cov}(\mathcal{K}_{\max})$ is maximal and subject to this $\sum_{i=1}^{k} |\text{cov}(\mathcal{H}_k)|$ is maximum (c.f. Lemma 3.2.13) and $X_1, X_2, \ldots, X_k$ are maximal tight sets of $J' = (V, F - E(\mathcal{K}_{\max}))$. We see that $\mathcal{L}_2$ is indeed a cover of F. Lemma 3.2.12 implies $\mathcal{K}_{\max}$ is 1-thin, Lemma 3.2.4 implies $|X_i \cap X_j| \leq 1$, and Lemmas 3.2.2 and 3.2.3 imply that $|X_i \cap \bigcup_{H \in \mathcal{H}_j} H| \leq 1$ for all

$1 \leq j \leq k$ and $1 \leq i \leq l$. Hence the cover $\mathcal{L}_2$ is 1-thin and we have

$$\begin{aligned}|F| &= \sum_{i=1}^{k} |E_J(\mathcal{H}_i)| + \sum_{j=1}^{k} |E_J(X_j)| \\ &= \text{val}(\mathcal{K}_{\max}) + \sum_{j=1}^{k} (2|X_j| - 3) = \text{val}(\mathcal{L}_2).\end{aligned}$$

We will show that $\mathcal{L}_2$ is a cover of E'. As above, let $ab = e \in E' - F$ be an edge. By the maximality of F we have $F + e \notin \mathcal{I}$. Thus either there is a tight set $X \subseteq V$ in J with $a, b \in X$ or there is a tight S-compatible family $\mathcal{H} = \{Y_1, \ldots, Y_t\}$ for some $S \subseteq U$ with $|S| \geq 2$ in J and $a, b \in Y_i$ for some $1 \leq i \leq t$.

In the latter case recursive applications of Lemma 3.2.5 imply that there exists a tight (U, T)-compatible family $\overline{\mathcal{H}}$ with $\text{cov}(\mathcal{H}) \subseteq \text{cov}(\overline{\mathcal{H}})$ in J for some $T \supseteq S$. Since $\{\overline{\mathcal{H}}\}$ is a tight U-system, we see that $\text{cov}(\overline{\mathcal{H}}) \subseteq \text{cov}(\mathcal{K}_{\max})$ by the maximality of $\text{cov}(\mathcal{K}_{\max})$. Combining these we have $\text{cov}(\mathcal{H}) \subseteq \text{cov}(\overline{\mathcal{H}}) \subseteq \text{cov}(\mathcal{K}_{\max}) \subseteq \text{cov}(\mathcal{L}_2)$, hence e is covered by $\mathcal{L}_2$.

In the former case, when $a, b \in X$ for some tight set X in J we have two possibilities. First suppose that $|X \cap \bigcup_{H \in \mathcal{H}_i} H| \geq 2$ for some $1 \leq i \leq k$. Then we can deduce that $X \subseteq H$ for some $H \in \mathcal{H}_i$ by using Lemma 3.2.2 or 3.2.3 and the maximality of $\mathcal{K}_{\max}$ which implies that $\mathcal{L}_2$ covers e. Next suppose that $|X \cap \bigcup_{H \in \mathcal{H}_i} H| \leq 1$ for all $1 \leq i \leq k$. Then $E(X) \subseteq E(J')$ and hence $X \subseteq X_j$ for some $1 \leq j \leq l$, since every edge of J' induces a tight set and every tight set is contained in a maximal tight set. Hence e is covered by $\mathcal{L}_2$, as claimed. ■

3.3 The U-coincident Matroid $\mathcal{R}_U(G)$

Let $G = (V, E)$ be a graph and $U \subseteq V$. Let $S \subseteq U$ with $|S| \geq 2$ and let G_S denote the graph obtained from G by contracting the vertices in S into a new vertex z_S (and deleting the resulting loops and parallel edges). Given a realisation (G_S, p_S), we obtain an S-coincident realisation (G, p) of G by putting $p(x) = p_S(z_S)$ if $x \in S$ and $p(x) = p_S(x)$ if $x \notin S$. Furthermore, each vector q_S in the kernel of $R(G_S, p_S)$ (an infinitesimal motion q_S of (G_S, p_S)) determines a vector q in the kernel of $R(G, p)$ (an infinitesimal motion q of (G,p)) by setting $q(x) = q_S(z_S)$ if $x \in S$ and $q(x) = q_S(x)$ if $x \notin S$. It follows that

$$\dim \ker R(G,p) \geq \dim \ker R(G_S, p_S) \tag{3.13}$$

We can use this fact to prove that independence in $\mathcal{R}_U G$ implies independence in $\mathcal{M}_U(G)$.

Theorem 3.3.1. *Let $G = (V,E)$ be a graph and $U \subseteq V$. Suppose E is independent in $\mathcal{R}_U(G)$. Then E is independent in $\mathcal{M}_U(G)$.*

Proof: Let (G,p) be an independent generic U-coincident realisation of G. Independence implies that $i(H) \leq \mathrm{val}_U(H) \leq 2|H| - 3$ holds for all $H \subseteq V$ with $|H| \geq 2$. Since $p(x) = p(y)$ if $x, y \in S$, there is no edge between any two members of S.

Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be an S-compatible family for some $S \subseteq U$ with $|S| \geq 2$ and consider the subgraph $F = (\bigcup_{i=1}^k H_i, \bigcup_{i=1}^k E(H_i))$. By contracting S into one vertex in F we obtain the graph F_S, in which $\mathcal{X} = \{H_1/S, \ldots, H_k/S\}$ is a cover. Thus $r_2(F_S) \leq \sum_{i=1}^k \big(2(|H_i| - (|S| - 1)) - 3\big)$. This bound and (3.13) imply that

$$\begin{aligned}\dim \ker R(F,p) \geq& \dim \ker R(F_S, p_S) \\ &\geq 2\big(|\bigcup_{i=1}^k H_i| - (|S| - 1)\big) - \sum_{i=1}^k \big(2|H_i| - 2(|S| - 1) - 3\big).\end{aligned}$$

Since (G,p) is S-independent, we have

$$\begin{aligned} i_F(\mathcal{H}) = |F| &\leq 2\Big|\bigcup_{i=1}^k H_i\Big| - \left(2\left(\Big|\bigcup_{i=1}^k H_i\Big| - (|S| - 1)\right) - \sum_{i=1}^k (2|H_i| - 2(|S| - 1) - 3)\right) \\ &= \sum_{i=1}^k (2|H_i \setminus S| - 1) + 2(|S| - 1) = \mathrm{val}_S(\mathcal{H}). \end{aligned}$$

Thus E is independent in $\mathcal{M}_U(G)$, since $S \subseteq U$ with $|S| \geq 2$ is arbitrary. ■

3.4 Further Remarks

Let $G = (V, E)$ be a graph. By Theorem 0.1.2, we know that $|E| = 2|V| - 3$ must hold in order for G be minimally U-coincident rigid in $\mathbb{R}^2$. This implies that such a graph has minimum degree at most three. It was proved that independence in $\mathcal{M}_U$ implies independence in $\mathcal{R}_U$ when $|U| = 2$ by Fekete, Jordán and Kaszanitzky in [6], and when $|U| = 3$ in Chapter 2. When U has size two, we can use 0-U- and 1-U-reduction operations to show that independence in $\mathcal{M}_U(G)$ implies independence in $\mathcal{R}_U(G)$. When U has size three, we can still use 0-U- and 1-U-reduction operations for some cases to show that independence in $\mathcal{M}_U(G)$ implies independence in $\mathcal{R}_U(G)$. For the cases we cannot use these moves we have a special property that the graph $G[V \setminus N[U]]$ is $(2, 3)$-tight hence minimally rigid in $\mathbb{R}^2$, where $N[U]$ is the closed neighbourhood of the vertices in U. This allows us to apply induction on $G[V \setminus N[U]]$.

When $|U| \geq 4$ we lose these properties and therefore cannot apply the same arguments to prove that independence in $\mathcal{M}_U$ implies independence in $\mathcal{R}_U$. However, we still believe this is true and state the following conjecture.

Conjecture 3.4.1. *Let $G = (V, E)$ be a graph and $U \subseteq V$ with $|U| \geq 4$. Then $\mathcal{M}_U(G) \cong \mathcal{R}_U(G)$.*

Chapter 4

Rigidity of Transitioned Body-Bar Frameworks in $\mathbb{R}^3$

4.1 Introduction to Body-Bar Frameworks

In the previous chapters we studied bar-and-joint frameworks in which each vertex of the underlying graph corresponds to a single point in the ambient space. In a body-bar framework, each vertex will correspond to a general 3-dimensional rigid body in $\mathbb{R}^3$. Since a 3-dimensional rigid body has six degrees of freedom (whereas a point has three) in $\mathbb{R}^3$, we will have to modify the definitions of rigidity matrices and infinitesimal motions.

More precisely, the rigidity matrix of a body-bar framework in $\mathbb{R}^3$ will have $6|V|$ columns and each instantaneous velocity (assigned by infinitesimal motions) will be a vector in $\mathbb{R}^6$ rather than a vector in $\mathbb{R}^3$.

A rotation about an axis in $\mathbb{R}^3$ is given by the angular velocity vector $A = (a_1, a_2, a_3)$ and a point $Q = (q_1, q_2, q_3)$ on the axis of the rotation. The velocity vector $W = (w_1, w_2, w_3)$ at a point $P = (p_1, p_2, p_3)$ is given by $W = A \times (P - Q)$. We can identify such a rotation with a vector

$$\left(\begin{vmatrix} a_1 & q_1 \\ a_2 & q_2 \end{vmatrix}, \begin{vmatrix} a_1 & q_1 \\ a_3 & q_3 \end{vmatrix}, \begin{vmatrix} a_2 & q_2 \\ a_3 & q_3 \end{vmatrix}, \begin{vmatrix} a_1 & q_1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} a_2 & q_2 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} a_3 & q_3 \\ 0 & 1 \end{vmatrix} \right) = (r_1, r_2, r_3, r_4, r_5, r_6) \in \mathbb{R}^6,$$

since

$$\begin{bmatrix} 0 & -r_6 & r_5 & -r_3 \\ r_6 & 0 & -r_4 & r_2 \\ -r_5 & r_4 & 0 & -r_1 \\ r_3 & -r_2 & r_1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ -W \cdot P \end{bmatrix}$$

holds. Similarly, when we consider a translation t, the velocity vector at each point $P = (p_1, p_2, p_3)$ is a constant vector $W = (w_1, w_2, w_3) = (-t_3, t_2, -t_1)$. Hence we can identify t with a vector $(t_1, t_2, t_3, 0, 0, 0) \in \mathbb{R}^6$, since

$$\begin{bmatrix} 0 & 0 & 0 & -t_3 \\ 0 & 0 & 0 & t_2 \\ 0 & 0 & 0 & -t_1 \\ t_3 & -t_2 & t_1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ -W \cdot P \end{bmatrix}$$

holds. The two 4×4 matrices above are called *action matrices* M_r and M_t of the rotation and translation, respectively. Similarly, if we consider a rotation r and a translation t simultaneously, then the instantaneous velocity W at a point $P = (p_1, p_2, p_3)$ is given by $(M_r + M_t)(P, 1) = (W, -W \cdot P)$. The resulting infinitesimal motion is called a *screw motion* S. The action matrix M of S is given by $M_r + M_t$ and $S = r + t$. For two simultaneous screw motions S_1 and S_2, the resulting screw motion is $S = S_1 + S_2$ and hence we can regard the space of screw motions as a 6-dimensional real vector space. For more detail on screw motions, see for example [3].

Now take two points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ on a line l in $\mathbb{R}^3$ and coordinatise l as

$$\left(\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} a_3 & b_3 \\ 1 & 1 \end{vmatrix} \right) = (l_1, l_2, l_3, l_4, l_5, l_6).$$

We call this six-tuple the *Plücker coordinates* of the line l. We can now formally define body-bar frameworks in $\mathbb{R}^3$.

Definition 4.1.1. A *body-bar framework* in $\mathbb{R}^3$ is a pair (G, p) where $G = (V, E)$ is a multigraph without loops and $p : E \to \mathbb{R}^6$ is a map such that the image of an edge under this map is a representative of the Plücker coordinates of a line.

We regard each edge $e \in E$ as a bar on the line corresponding to Plücker coordinates $p(e)$ of e. We regard each vertex $v \in V$ as an arbitrary 3-dimensional rigid body B_v such that it intersects every bar incident to B_v (corresponding to an edge e incident with v) at a single point. We assume that bodies B_v for $v \in V$ are disjoint.

We will assign some screw motions to rigid bodies and as in the previous chapters we want these motions to preserve bar lengths. For an infinitesimal motion q of a bar-and-joint framework (G, p) the fact that $q(u) - q(v)$ is orthogonal to $p(u) - p(v)$ for an edge uv keeps bar lengths fixed. Similarly, in order to fix the length of a bar e incident with bodies B_v and B_u in a body-bar framework (G, p), the difference of the velocities at points Q and T must be orthogonal to the bar e, where Q and T are the intersection points in $\mathbb{R}^3$ of the bar e and the bodies B_v, and B_u, respectively. If we assign screw motions $m(u)$ and $m(v)$ to bodies B_u and B_v, then the length of the bar e is fixed if and only if $[m(u) - m(v), p(e)] = 0$, where $[m(u) - m(v), p(e)]$ is as defined in (4.1).

$$[Q, K] := q_1k_6 - q_2k_5 + q_3k_4 + q_4k_3 - q_5k_2 + q_6k_1, \text{ for } Q, K \in \mathbb{R}^6. \tag{4.1}$$

We can now give a formal definition of an infinitesimal motion of a body-bar framework.

Definition 4.1.2. An *infinitesimal motion* of a body-bar framework (G, p) in $\mathbb{R}^3$ is a function $m : V \to \mathbb{R}^6$ that assigns a screw motion to each vertex (body) such that $[m(u) - m(v), p(e)] = 0$ for all edges e with endpoints u and v.

The *(body-bar) rigidity matrix* $R(G, p)$ of a body-bar framework (G, p) is a $|E| \times 6|V|$ matrix whose rows are indexed by E and of the form

$$\begin{array}{cc} & \begin{array}{ccccccccccc} & & & v_i & & & & v_j & & & \end{array} \\ e = v_iv_j & \begin{pmatrix} \mathbf{0} & \dots & \mathbf{0} & \mathbf{p(e)} & \mathbf{0} & \dots & \mathbf{0} & -\mathbf{p(e)} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix} \end{array}$$

with $i < j$, where $\mathbf{p(e)}$ is the Plücker coordinates corresponding to e, and $\mathbf{0} = (0, 0, 0, 0, 0, 0)$.

For a vector $C = (c_1, c_2, c_3, c_4, c_5, c_6) \in \mathbb{R}^6$ define $C^* := (c_6, -c_5, c_4, c_3, -c_2, c_1)$. It follows that m is an infinitesimal motion of a body-bar framework (G, p) if and only if

$$R(G,p)\begin{bmatrix} m^*(v_1) \\ m^*(v_2) \\ \vdots \\ m^*(v_n) \end{bmatrix} = 0.$$

Therefore the space of infinitesimal motions of (G,p) is isomorphic to the kernel of $R(G,p)$. Since the screw motions corresponding to infinitesimal rotations and translations of the whole $\mathbb{R}^3$ generate a 6-dimensional vector space, we see that $\operatorname{rank} R(G,p) \leq 6|V| - 6$.

Definition 4.1.3. The body-bar framework (G,p) is *infinitesimally rigid* if $R(G,p)$ has rank $6|V| - 6$.

We also say a multigraph G is *infinitesimally rigid* (as a body-bar framework in $\mathbb{R}^3$), if there exists a p such that (G,p) is an infinitesimally rigid body-bar framework in $\mathbb{R}^3$. A *minimally infinitesimally rigid* body-bar framework (or graph) is an infinitesimally rigid body-bar framework (or graph) such that removing an arbitrary bar (or edge) results in a non-rigid body-bar framework (or graph).

A body-bar framework (G,p) in $\mathbb{R}^3$ is *generic*, if $R(G,p)$ and its all edge induced submatrices have maximum rank, taken over all realisations of G.

From the rigidity matrix $R(G,p)$ of a body-bar framework we can construct a matroid $\mathcal{R}(G,p)$ on $E(G)$, the *(body-bar) rigidity matroid* of (G,p), by defining a subset F of E to be independent if the set of rows of $R(G,p)$ corresponding to F is linearly independent. If (G,p) and (G,q) are two generic body-bar frameworks, then they give rise to the same rigidity matroid $\mathcal{R}(G)$, *the generic rigidity matroid* of the graph G.

The following lemma is due to Nash-Williams [19].

Lemma 4.1.1. *[19] Let $G = (V,E)$ be a multigraph such that G is the union of six edge-disjoint spanning trees. Then we have*

(a) There is a vertex $v \in V$ with $d(v) = 6 + k$, where $0 \leq k < 6$.

(b) There are $2k$ distinct edges incident with a vertex v satisfying part (a) $e_1 = vu_1, f_1 = vw_1, \ldots, e_k = vu_k, f_k = vw_k$ such that the graph H obtained from G by

deleting v and its incident edges and inserting k new edges $g_i = u_i w_i$, $1 \leq i \leq k$, is the union of six edge-disjoint spanning trees.

Tay gave the following characterisation for generic body-bar frameworks for which Lemma 4.1.1 is a key step in the proof.

Theorem 4.1.2. *[27] A generic body-bar framework (G, p) is minimally rigid if and only if G is the union of six edge-disjoint spanning trees.*

We will characterise the rigidity of a class of non-generic body-bar frameworks in $\mathbb{R}^3$, where we allow sets of bars corresponding to two or three parallel edges of the underlying multigraph to intersect in a common point. See Figure 4.1.

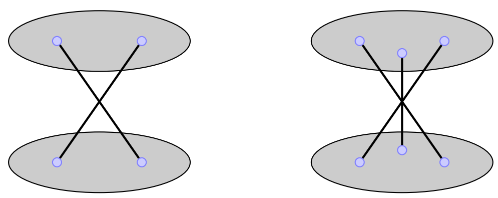

Figure 4.1: Variations of intersection points of bars we will be focusing on(ellipses correspond to bodies).

When we have an intersection point for the bars corresponding to two edges as in Figure 4.1 on the left, we say those edges are *concurrent*. When we have a common point for the bars corresponding to three edges as in Figure 4.1 on the right, we say those edges are a *pin*.

In the literature a pin refers to the unique intersection point of two distinct objects in space. This motivates us to use this name for three bars intersecting at a point. To be more explicit, let us consider two disjoint bodies B_u and B_v such that there are three bars joining these bodies and these bars intersect at a point. Let $P \in \mathbb{R}^3$ denote the intersection point of these bars and let $U_1, U_2, U_3 \in \mathbb{R}^3$ denote the common intersection points of the bars with the body B_u. We can think of P being attached to B_u with three bars $U_i P$ with endpoints U_i and P for $1 \leq i \leq 3$. Since a point in $\mathbb{R}^3$ has three degrees of freedom, if U_1, U_2, U_3, P are not coplanar, we can regard $B_u \cup \{P\}$ with the bars $U_i P$, $1 \leq i \leq 3$ as a rigid body. Similarly, we can regard $B_v \cup \{P\}$ as another rigid body and these bodies intersect at a unique point P. This is similar to the idea we used when we showed 0-extension preserves rigidity for bar-and-joint frameworks in Lemma 0.1.7. In fact, we can replace each rigid body B_v by a sufficiently large complete graph and obtain a bar-and-joint framework whose infinitesimal rigidity is equivalent to the infinitesimal rigidity of the body-bar framework we started with. However, in this chapter we will not use this idea of transforming a body-bar framework to a bar-and-joint framework.

Let us formally define the frameworks we are interested in. Let $G = (V, E)$ be a multigraph and p be a realisation of G as a body-bar framework such that a bar intersects parallel bars to itself at at most one point, and each intersection point can have at most three bars going through it. Apart from these types of intersections we assume the framework is as generic as possible. To control these intersections we use a family X of pairs of multiple edges of G. We only allow single bars, concurrent pairs of bars and pins. To distinguish a pair of concurrent bars e_1, e_2, we add $\{e_1, e_2\}$ to X. We distinguish a pin consisting of the edges e_1, e_2, e_3 by adding $\{e_1, e_2\}$, $\{e_1, e_3\}$ and $\{e_2, e_3\}$ to X. We call a set in X a *transition*.

A *transitioned multigraph* is a pair (G, X) where G is a multigraph and X is a set of transitions of G. A *transitioned framework* $(G, X; p)$ is a body-bar framework (G, p) such that the lines assigned by p satisfy the relations given by X. We say $(G, X; p)$ is *generic*, if $R(G, X; p)$ and its all edge induced submatrices have maximum rank, taken over all realisations of (G, X). Note that if $X = \emptyset$, then a generic transitioned framework is a generic body-bar framework. We will say a transitioned multigraph (G, X) is *rigid* if there exists a p such that $(G, X; p)$ is infinitesimally rigid. We use $r(G, X)$ to denote the rank of the rigidity matrix $R(G, X; p)$ of a generic transitioned framework $(G, X; p)$. In this chapter we will prove the following result.

Theorem 4.1.3. *Let (G, X) be a transitioned graph with only pairwise concurrences. Then (G, X) is minimally rigid if and only if G is the union of six edge-disjoint spanning trees.*

4.2 Examples and Tools

First note that by the definitions of a body-bar framework and its rigidity matrix, the actual shape and positions of the bodies are irrelevant for infinitesimal rigidity.

The only properties we need are that the bodies are disjoint and that every body intersects each incident bar at a single point.

Example 4.2.1. Let (G, X) be the transitioned multigraph in Figure 4.2 on the left. The edges between v_1 and v_3 are e_1, e_2, e_3, e_4, the edges between v_1 and v_2 are f_1, f_2, f_3, and the edges between v_2 and v_3 are g_1, g_2. Then we see that the set $X = \{\{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}, \{f_1, f_2\}\}$, that is, e_1, e_2, e_3 (red edges) belong to a pin and f_1, f_2 (blue edges) are concurrent. A realisation of (G, X) as a transitioned framework is shown in Figure 4.2 on the right.

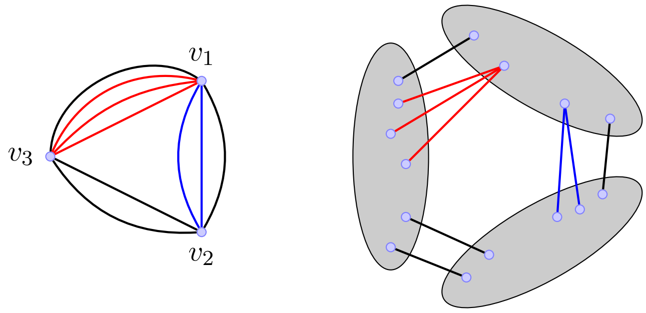

Figure 4.2: A transitioned multigraph and its realisation.

Lemma 4.2.1. *Let G be a multigraph and X be a set of transitions of G. If (G, X) is (minimally) rigid, then so is (G, S), for all $S \subseteq X$.*

Proof: The proof is straightforward as a generic realisation of (G, X) is a non-generic realisation of (G, S). ■

Let us use notations $\mathbf{p_0} = (0, 0, 0)$, and $\mathbf{p_i}$ for the i^{th} vector of the standard basis for $1 \leq i \leq 3$ in $\mathbb{R}^3$. Let $(\mathbf{p}_i * \mathbf{p}_j)$ denote the Plücker coordinates of the line segment $\overline{\mathbf{p}_i \mathbf{p}_j}$. The following result for body-bar frameworks will be crucial for our proofs.

Lemma 4.2.2. *[13] Let $G = (V, E)$ be a multigraph and $F \subseteq E$. Suppose that F can be partitioned into 6 forests $F_{i,j}$, $0 \leq i < j \leq 3$. Let (G, q) be a body-bar realisation of G in $\mathbb{R}^3$ with the property that $q(e) = (\mathbf{p}_i * \mathbf{p}_j)$ when $e \in F_{i,j}$, for all $0 \leq i < j \leq 3$. Then the rows of $R(G, q)$ indexed by F are linearly independent.*

We need to define some more terminology. A multigraph G is called $(2,2)$-*sparse* if $i(X) \leq 2|X| - 2$ for all $X \subseteq V(G)$ with $|X| \geq 2$. Nash-Williams' characterisation says that a graph G is $(2,2)$-sparse if and only if it can be partitioned into two forests. It is known that for an arbitrary graph G, if we put every edge set that induces a $(2,2)$-sparse graph in a family, then this family satisfies the independent set axioms of a matroid (see for example [7]). Let us denote this matroid by $M_{2,2}(G)$.

Let (G, X) be a transitioned graph and suppose that G is the union of six edge-disjoint spanning trees. We want to decompose G into three $(2,2)$-sparse subgraphs H_1, H_2, H_3 such that H_i does not contain two parallel edges e_1, e_2 with $\{e_1, e_2\} \in X$. First note that if we define a set of edges as being independent if it is $(2,2)$-sparse and does not contain a transition, then we get another matroid $\tilde{M}_{2,2}(G, X)$. To see this, choose a representative edge for each concurrent pair and for each pin. Remove the non-representative edges of all concurrent pairs and pins from the graph and the corresponding transitions from X. We are left with a transitioned graph $(G', \emptyset)$. Since there are no transitions in $(G', \emptyset)$, being independent in $M_{2,2}(G)$ and being independent in $\tilde{M}_{2,2}(G', \emptyset)$ are equivalent. Then add the deleted edges of the concurrent pairs and pins and define each pair of edges in X as a circuit. This operation is called *parallel extension* in matroid theory and it gives a new matroid $\tilde{M}_{2,2}(G, X)$.

Lemma 4.2.3. *Let (G, X) be a transitioned graph and G be the union of six edge-disjoint spanning trees. Then the edge set of G can be partitioned into three bases of $\tilde{M}_{2,2}(G, X)$.*

Proof: Let (G, X) be a counter-example with X being minimal. We will apply induction by taking a pin. If there are no pins, then one can prove it by taking a pair of concurrent bars in the same way. Suppose e_1, e_2, e_3 are the edges corresponding to a pin in (G, X). Consider $X' = X - \{e_1, e_2\} - \{e_1, e_3\}$ and the transitioned graph (G, X'). By the minimality of X, (G, X') can be partitioned into three bases H_1, H_2, H_3. We may assume $e_2 \in H_2$ and $e_3 \in H_3$. If $e_1 \in H_1$, then H_1, H_2, H_3 would be the required decomposition of (G, X). Hence $e_1 \in H_2$ or $e_1 \in H_3$. We may assume $e_1 \in H_2$. Then apply a basis exchange on e_1 between H_1 and H_2. That is, $H_1 - f + e_1, H_2 - e_1 + f, H_3$ are three disjoint bases of $\tilde{M}_{2,2}(G, X')$ for some $f \in H_1$. Then $H_1 - f + e_1, H_2 - e_1 + f, H_3$ are disjoint bases of $\tilde{M}_{2,2}(G, X)$. ■

4.3 Main Results

Theorem 4.3.1. *Let (G, X) be a transitioned graph with only pairwise concurrences. Then (G, X) is minimally rigid if and only if G is the union of six edge-disjoint spanning trees.*

Proof: If (G, X) is minimally rigid, then applying Lemma 4.2.1 (with $S = \emptyset$) and Theorem 4.1.2 we obtain that G is the union of six edge-disjoint spanning trees.

For the other direction suppose G is the union of six edge-disjoint spanning trees. By Lemma 4.2.3 we can decompose G into three bases H_1, H_2, H_3 of $\tilde{M}_{2,2}(G, X)$. Let $H_i = T_i \cup T_i'$ for $1 \leq i \leq 3$, where T_i, T_i' are spanning trees. We can map T_i, T_i' for $1 \leq i \leq 3$ to the edge set of K_4 whose vertices are labelled by $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ as defined above such that the image of T_i and T_i' are not incident for $1 \leq i \leq 3$. See Figure 4.3. The fact that the image of T_i is incident with the image of T_j and T_j', $1 \leq i < j \leq 3$ under this mapping implies that all pairs of concurrent bars defined by X have a common endpoint at $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ or $\mathbf{p}_3$. Now we apply Lemma 4.2.2 to get the desired result. ■

Consider a transitioned graph (G, X) for which G is the union of six edge-disjoint spanning trees and a partition H_1, H_2, H_3 of $\tilde{M}_{2,2}(G, X)$ into bases. Let $H_i = T_i \cup T_i'$, $1 \leq i \leq 3$, where T_i and T_i' are spanning trees. Consider also the graph K_4 whose vertices are p_i, $0 \leq i \leq 3$ as defined earlier. We can map each of these 6 spanning trees T_i and T_i' to an edge of this K_4 such that T_i and T_i' are not mapped to edges having a common endpoint. See Figure 4.3.

We would like to use Lemma 4.2.2 to obtain the required rigid realisation of (G, X). For this to work we need the above map to map every concurrent pair to a path of length two and every pin to a 3-star in the K_4. As H_i does not contain a transition and T_i intersects all other trees except T_i', every pair of concurrent bars are mapped to a path of length two. However, a pin can be mapped to either a 3-star or a triangle. Let us call a pin which is mapped to a triangle a *misplaced pin.*

For the rest of this chapter we will give some classes of transitioned graphs such

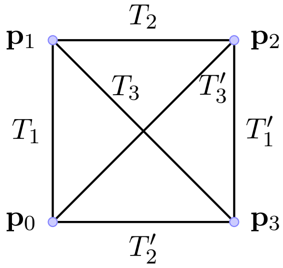

Figure 4.3: A mapping of T_i and T_i' 's to the edge set of K_4.

that each pin can be mapped to a 3-star. Then we will apply Lemma 4.2.2 in order to characterise the rigidity of such transitioned graphs.

For a multigraph G and vertices u and v the *multiplicity* of uv is the number of edges $e = uv$ in G and is denoted by $\mu(uv)$. We will use the notation $\mathcal{P}(G, X)$ (or $\mathcal{P}$ when it is clear) as the set of all pins of (G, X). Let us denote the set of all pins $\{e_1, e_2, e_3\}$ incident with the vertices u and v with $\mu(uv) = 3$ by $\mathcal{P}_0(G, X)$ (or $\mathcal{P}_0$ when it is clear). Let us also denote the set of all pins $\{e_1, e_2, e_3\}$ incident with the vertices u and v with $4 \leq \mu(uv) \leq 6$ and such that $\{e_1, e_2, e_3\}$ is the only pin between u and v by $\mathcal{P}_1(G, X)$ (or $\mathcal{P}_1$ when it is clear). Finally let us denote the set of all pins $\{e_1, e_2, e_3\}$ incident with the vertices u and v with $\mu(uv) = 6$ and such that there is another pin $\{f_1, f_2, f_3\}$ incident with u and v by $\mathcal{P}_2(G, X)$ (or $\mathcal{P}_2$ when it is clear). See Figure 4.4.

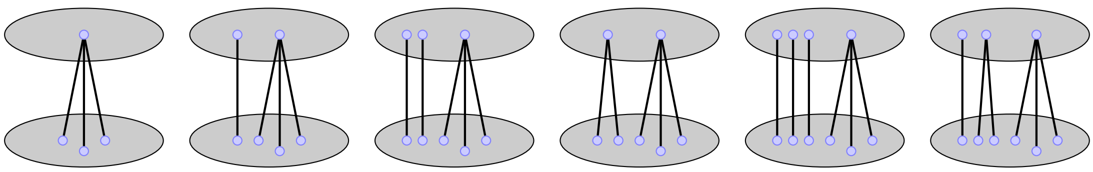

Figure 4.4: The pin on the far left belongs to $\mathcal{P}_0$ whereas all other pins belong to $\mathcal{P}_1$. Note that if there are two pins between two vertices, then neither of those pins belongs to $\mathcal{P}_0$ or $\mathcal{P}_1$.

Theorem 4.3.2. *Let (G, X) be a transitioned graph such that $|\mathcal{P}_0| \leq 1$ and $\mathcal{P}_2 = \emptyset$. Then (G, X) is minimally rigid if and only if G is the union of six edge-disjoint spanning trees.*

Proof: If (G, X) is minimally rigid, then applying Lemma 4.2.1 (with $S = \emptyset$) and Theorem 4.1.2 we obtain that G is the union of six edge-disjoint spanning trees.

For the other direction suppose G is the union of six edge-disjoint spanning trees. We assume $|\mathcal{P}_0| = 1$ and the unique pin in $\mathcal{P}_0$ is $\{e_1, e_2, e_3\}$. By Lemma 4.2.3, we can partition $\tilde{M}_{2,2}(G, X)$ into three bases H_1, H_2, H_3. Let $H_i = T_i \cup T_i'$ for $1 \leq i \leq 3$ where T_i, T_i' are spanning trees. Relabelling if necessary, we may assume that $e_i \in T_i$ for $1 \leq i \leq 3$. Use the mapping in Figure 4.3 to make sure that $\{e_1, e_2, e_3\}$ is not a misplaced pin. Therefore if there is a misplaced pin consisting of the edges f_1, f_2, f_3 with $f_i \in H_i$ between the vertices u and v, it must be in $\mathcal{P}_1$. This implies $\mu(uv) \geq 4$. Take an edge f with endpoints u and v such that $f \neq f_i$ for all $1 \leq i \leq 3$. We may assume that $f \in H_1$. Since f and f_1 are multiple edges within H_1, we must have $f \in T_1'$ and $f_1 \in T_1$ (or $f \in T_1$ and $f_1 \in T_1'$). If we exchange f and f_1 of T_1' and T_1 this will give us a basis exchange of spanning trees and the triangle corresponding to the misplaced pin will become a 3-star. Then we apply Lemma 4.2.2 to get the desired result. ■

We can obtain the following result with the same method.

Theorem 4.3.3. *Let (G, X) be a transitioned graph such that*
- *G is the union of six edge-disjoint spanning trees T_i, T_i', $1 \leq i \leq 3$, and for each pin $P = \{e_1, e_2, e_3\} \in \mathcal{P}_0$ we have $|P \cap T_i| = 1$, for all $1 \leq i \leq 3$.*
- *If $\{e, f\} \in X$ and $e \in T_1 \cup T_2 \cup T_3$, then $f \in T_1 \cup T_2 \cup T_3$*
- *$\mathcal{P}_2 = \emptyset$.*

Then (G, X) is minimally rigid.

Proof: If we let $H_i = T_i \cup T_i'$, $1 \leq i \leq 3$, then we obtain a partition of (G, X) into bases of $\tilde{M}_{2,2}$, as every transition occurs within $T_1 \cup T_2 \cup T_3$ or $T_1' \cup T_2' \cup T_3'$. If we use the mapping in Figure 4.3, then all misplaced pins belong to $\mathcal{P}_1$. Now we can proceed as in the proof of Theorem 4.3.2. ■

Theorem 4.1.2 together with Lemma 4.2.1 implies that if a transitioned graph (G, X) is rigid, then G contains the union of six edge-disjoint spanning trees. We now give an example that shows the converse is not true.

Example 4.3.1. Consider a body-bar framework (G, p) consisting of two bodies and two distinct pins, see Figure 4.5. We can fix the body in the bottom and assign an infinitesimal screw motion to the body on the top that corresponds to a rotation about the line going through v_1 and v_2. Since this does not correspond to an isometry of $\mathbb{R}^3$, we have $\dim\ker R(G,p) \geq 7$, and so $\operatorname{rank} R(G,p) < 6$. Hence, even though its transitioned graph is the union of six edge-disjoint spanning trees, the framework is not rigid in $\mathbb{R}^3$.

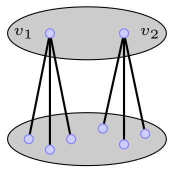

Figure 4.5: A framework with an underlying graph having six edge-disjoint spanning trees that is not infinitesimally rigid.

We end this chapter by giving a conjecture for transitioned graphs due to Jackson and Jordán. Let us first define some tools. For a transitioned graph (G,X) we define the *mixed graph* $H = (V; B, C, P)$ which is a graph on V with a three-partition of its edges such that B is the set of edges that do not belong to a transition in (G,X), C is the set of representatives for every pair of concurrent bars of (G,X) and P is the set of representatives for every pin of (G,X).

Let $H = (V; B, C, P)$ be a mixed graph and Q be a partition of V. For $R \subseteq Q$, we use $E_H(R)$ and $I_H(R)$ to denote the set of edges of H which join two vertices in different sets, respectively the same set, in R. A *hinge* of Q in H is a pair of pins $e, f \in P$ such that $E_H(X_1, X_2) = \{e, f\}$ for some $X_1, X_2 \in Q$. We denote the number of hinges of R in H by $h_H(R)$. Let $e_H(R) = |E_H(R) \cap B| + 2|E_H(R) \cap C| + 3|E_H(R) \cap P|$. Given a mixed graph $H = (V; B, C, P)$, a partition Q of V, and $R \subseteq Q$, we will refer to the number $6(|R| - 1) + h_H(R) + e_H(R)$ as the *deficiency* of R in H and denote it by $\operatorname{def}_H(R)$. The *deficiency* of H, $\operatorname{def}(H)$, is the maximum value $\operatorname{def}_H(Q)$ taken over all partitions Q of V.

Conjecture 4.3.4. *[11] Let $H = (V; B, C, P)$ be a mixed graph of some transitioned graph (G,X). Then $r(G,X) = 6(|V| - 1) - \operatorname{def}(H)$.*

Chapter 5

A Necessary Condition for Generic Rigidity of Bar-and-Joint Frameworks in d-Space

5.1 Introduction

In this chapter we will give an upper-bound on the rank function of generic d-dimensional bar-and-joint frameworks for all $d \leq 11$. Before stating the main result of this chapter let us first give some definitions and known results.

Recall that a graph G is *rigid*, respectively *independent*, or *dependent* in $\mathbb{R}^d$ if there exists a rigid, respectively independent, or dependent d-dimensional framework (G, p) for some generic realisation p. The independence of a graph G is closely related to the $(d, \binom{d+1}{2})$-sparsity of G. The following which is a restatement of Lemma 0.1.6 is due to Maxwell [18].

Theorem 5.1.1. *[18] If G is an independent graph in $\mathbb{R}^d$, then G is $(d, \binom{d+1}{2})$-sparse.*

It is known that the converse of Theorem 5.1.1 does not hold in d-dimensions for $d \geq 3$. The graph B_3 is an example for this. Note that, in Chapter 0, we have shown that B_3 is a non-rigid circuit in $\mathbb{R}^3$.

By Theorems 0.2.1 and 0.2.2, we can deduce that, for $d = 1, 2$, the size of a maximal independent set of edges ($r_d(G)$) in $\mathcal{R}_d(G)$ is equal to the number of edges of a maximal $(d, \binom{d+1}{2})$-sparse subgraph of G.

Since bases of a matroid have the same size, the maximal independent sets of $\mathcal{R}_d(G)$ have the same size. However, it is not true that all maximal $(d, \binom{d+1}{2})$-sparse subgraphs of G have the same number of edges when $d \geq 3$. On the other hand Cheng and Sitharam [5] have recently shown that the number of edges in any maximal $(3,6)$-sparse subgraph of G does at least give an upper bound on $r_3(G)$.

Theorem 5.1.2. *[5] Let $G = (V, E)$ be a graph and $H = (V, F)$ be a maximal $(3,6)$-sparse subgraph of G. Then $r_3(G) \leq |F|$.*

Jackson [10] extended 5.1.2 to all values $d \leq 5$.

Theorem 5.1.3. *[10] Let $G = (V, E)$ be a graph, $d \leq 5$ be an integer and $H = (V, F)$ be a maximal $(d, \binom{d+1}{2})$-sparse subgraph of G. Then $r_d(G) \leq |F|$.*

In this chapter we will extend this result to all values of $d \leq 11$.

Theorem 5.1.4. *Let $G = (V, E)$ be a graph, $d \leq 11$ be an integer and $H = (V, F)$ be a maximal $(d, \binom{d+1}{2})$-sparse subgraph of G. Then $r_d(G) \leq |F|$.*

5.2 Non-rigid Circuits

Jackson [10] used the minimum number of edges a circuit can have in order to obtain the result for $d \leq 5$. The minimum number of edges a circuit has in $\mathbb{R}^d$ is $\binom{d+2}{2}$ and the corresponding circuit is K_{d+2}. It is known that K_{d+2} is a rigid circuit. Instead of considering the minimum number of edges a circuit can have in $\mathbb{R}^d$, we will consider the minimum number of edges a non-rigid circuit can have in $\mathbb{R}^d$. We will show that the minimum number of edges necessary for a non-rigid circuit in $\mathbb{R}^d$ is $\frac{d^2+9d}{2}$ when $d \leq 12$ (Lemma 5.2.7). We will then proceed as Jackson did in [10].

We know a rigid circuit on n vertices has $dn - \binom{d+1}{2} + 1$ edges in $\mathbb{R}^d$. However, we do not even know a lower bound for the number of edges in a non-rigid circuit on n vertices in $\mathbb{R}^d$.

In this section we will introduce some basic results about non-rigid circuits. Let us first define some graph operations. Let H and G be graphs satisfying $H = G - v + uw$ for some vertex v of degree $d+1$ and non-adjacent neighbours u, w of v.

Recall that we say H is a (d-dimensional) 1-*reduction* of G. Now suppose we have $H = G - v + u_1u_2 + w_1w_2$ for some vertex v of degree $d+2$ and disjoint non-adjacent pairs of neighbours u_1, u_2 and w_1, w_2 of v. Then we say H is a (d-dimensional) 2-*reduction* of G, see Figure 5.1.

The two lemmas below will be useful tools for proving our results.

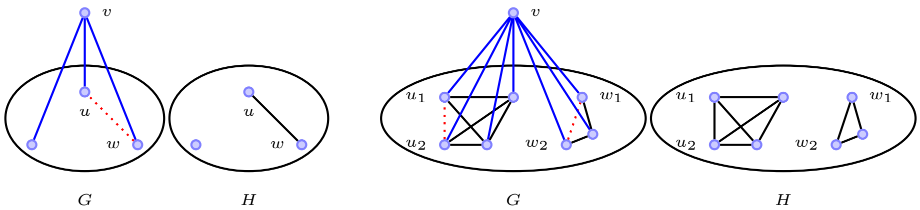

Figure 5.1: On the left hand side H is a 2-dimensional 1-reduction of G and on the right hand side H is a 5-dimensional 2-reduction of G. Missing edges are denoted by dotted red lines. The edges which are not drawn may or may not exist.

Lemma 5.2.1. *The* 1*-reduction operation preserves dependence in* $\mathbb{R}^d$*. For the* 2*-reduction operation defined above, if* H *also has disjoint copies of cliques* K_m *and* K_n *with* $u_1u_2 \in K_m$*,* $w_1w_2 \in K_n$ *and* $V(K_m) \cup V(K_n) = N_G(v)$*, then this operation preserves dependence in* $\mathbb{R}^d$*.*

Proof: Lemma 0.1.8 implies the proof of the 1-reduction part of the statement as 1-reduction is the inverse operation of 1-extension. We will prove the contrapositive of the 2-reduction part of the lemma. Suppose (H, p) is independent for a generic p. Consider the two cliques K_m and K_n with $V(K_m) = U = \{u_1, \ldots, u_m\}$, $V(K_n) = W = \{w_1, \ldots, w_n\}$ and $m + n = d + 2$ in H. Then $p(u_i)$, $1 \leq i \leq m$, and $p(w_j)$, $1 \leq j \leq n$, span $(m-1)$- and $(n-1)$-dimensional affine subspaces of $\mathbb{R}^d$, respectively. Since $m - 1 + n - 1 = d$, these subspaces have an intersection point Q.

Let (G, p') be a non-generic framework with $p'(v) = Q$ and $p'|_H = p$ in $\mathbb{R}^d$. The framework $(G + u_1u_2 + w_1w_2 - vu_1 - vw_1, p')$ is independent, since it is obtained from H, which is independent, by adding a d-valent vertex whose neighbours do not lie on a $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$, by Lemma 0.1.7. Consider the framework $(G + u_1u_2 + w_1w_2 - vw_1, p')$. Since $\{u_1, \ldots, u_m, v\}$ induces a copy of K_{m+1} and $p'(u_i)$, $1 \leq i \leq m$, together with $p'(v)$ span an $(m-1)$-dimensional affine subspace of $\mathbb{R}^d$, we see that the sub-framework $(K_{m+1}, p'|_{U \cup \{v\}})$ of $(G + u_1u_2 +$

$w_1w_2 - vw_1, p')$ is dependent. Hence $(G + u_1u_2 + w_1w_2 - vw_1, p')$ is dependent. The fact that $(G + u_1u_2 + w_1w_2 - vu_1 - vw_1, p')$ independent now implies that $(G+u_1u_2+w_1w_2-vw_1, p')$ has a unique circuit. Since K_{m+1} is a circuit in $\mathbb{R}^{m-1}$, this unique circuit must be $E(K_{m+1})$. Therefore we can delete an edge from the K_{m+1} and obtain an independent framework, that is $(G + w_1w_2 - vw_1, p')$ is independent.

Now consider the framework $(G+w_1w_2, p')$. Since $\{w_1, \dots, w_n, v\}$ induces a copy of K_{n+1} and $p'(w_j)$, $1 \leq j \leq n$, together with $p'(v)$ span an $(n-1)$-dimensional affine subspace of $\mathbb{R}^d$, by similar arguments as in the previous paragraph there exists a unique circuit in $(G + w_1w_2, p')$ that contains the edge w_1w_2. Hence (G, p') is independent in $\mathbb{R}^d$, and so G is independent in $\mathbb{R}^d$. ■

Let $G = (V, E)$ be a graph. Let H be the graph obtained from G by adding a new vertex z, and an edge vz for each $v \in V$. We say H is the *cone* of G. The following lemma is due to Schulze and Whiteley [24], and will be useful to construct non-rigid circuits in $\mathbb{R}^d$, $d \geq 4$, by using the fact that B_3 is a non-rigid circuit in $\mathbb{R}^3$.

Lemma 5.2.2. *[24] Let G be a graph and G^* its cone. Then G is independent (dependent, rigid, a circuit) in $\mathbb{R}^d$ if and only if G^* is independent (dependent, rigid, a circuit) in $\mathbb{R}^{d+1}$.*

We will prove that the minimum number of vertices on which there exists a non-rigid circuit in $\mathbb{R}^d$ is $d + 5$. Let us first introduce a non-rigid circuit on $d + 5$ vertices in $\mathbb{R}^d$. Let $B_d = G_1 \cup G_2$ where G_1 and G_2 are distinct copies of $K_{d+2} - e$ and $G_1 \cap G_2 = K_{d-1} - e$. Then G is flexible in $\mathbb{R}^d$ since we can rotate G_1 about the $(d-2)$-dimensional affine subspace spanned by the vertices of $G_1 \cap G_2 = K_{d-1} - e$ while fixing G_2 in any generic realization of B_d in $\mathbb{R}^d$. The graph B_d is the cone of the graph B_{d-1}. The facts that the cone of a circuit in $\mathbb{R}^d$ is a circuit in $\mathbb{R}^{d+1}$ by Lemma 5.2.2 and that the graph B_3 in Figure 12 is a circuit in $\mathbb{R}^3$ imply the graph B_d is a circuit in $\mathbb{R}^d$ for $d \geq 3$. The graph B_d has $\frac{d^2+9d}{2}$ edges.

We can decompose a graph into rigid subgraphs, since an edge is rigid in $\mathbb{R}^d$ for all d.

Definition 5.2.1. We say that a rigid subgraph H of a graph G is a *rigid component* of G if there is no rigid subgraph of G properly containing H.

Lemma 5.2.3. *The minimum number of vertices on which there exists a non-rigid circuit in* $\mathbb{R}^d$ *is* $d+5$*, for all* $d \geq 3$.

Proof: Since B_d is a non-rigid circuit on $d+5$ vertices, we only need to show that such a circuit on less than $d+5$ vertices cannot exist. Let G be a non-rigid circuit with the minimum number of vertices in $\mathbb{R}^d$ and suppose that $|V(G)| \leq d+4$.

First consider the case when $\delta(G) = d+1$. Take a vertex v with $d(v) = d+1$. Since G is a non-rigid circuit in $\mathbb{R}^d$ it is $(d, \binom{d+1}{2})$-sparse which implies that we must have a missing edge u_1u_2 between two neighbours of v. Otherwise we would have a copy of K_{d+2} whose vertices are v and the neighbours of v. Since K_{d+2} is a rigid circuit in $\mathbb{R}^d$, this would be a contradiction. We can perform a 1-reduction on the missing edge u_1u_2 and v. Let H be the resulting graph, that is $H = G - v + u_1u_2$. Since 1-reduction preserves dependence in $\mathbb{R}^d$ by Lemma 5.2.1, H is dependent in $\mathbb{R}^d$ and so it contains a circuit H' with $u_1u_2 \in E(H')$. Since G is a non-rigid circuit on the minimum number of vertices, H' must be a rigid circuit implying that there is a rigid component H'' containing $H' - u_1u_2$ with at least $d+2$ vertices in G. We have at most 2 vertices outside H'' each with at least d neighbours in H''. Thus if we add them one by one to H'' we preserve the rigidity in each step by Lemma 0.1.7. This implies G is rigid, a contradiction.

Now, suppose $\delta(G) = d+2$. Since G is $(d, \binom{d+1}{2})$-sparse we have $|V(G)| = d+4$. Let v be a vertex with $d(v) = \delta(G)$. Then there exist at least two non-incident missing edges in $G[N(v)]$ since G is a non-rigid circuit. We will try to perform a 2-reduction on these edges with the vertex v. Note the facts $\delta(G) = d+2$ and $|V(G)| = d+4$ imply that every induced subgraph of G is a copy of $K_r - F$ where F is a set of pairwise non-adjacent edges. In particular $N(v)$ induces a copy of $K_{d+2} - F'$ where F' is a set of pairwise non-adjacent edges. It is now straightforward to find two disjoint subgraphs isomorphic to $K_{\frac{d+2}{2}} - e$ in $G[N(v)]$ if d is even. Otherwise d is odd and we can find two disjoint copies of $K_{\frac{d+1}{2}} - e$ and $K_{\frac{d+3}{2}} - e$ in $G[N(v)]$. By Lemma 5.2.1 we can perform a 2-reduction on the missing edges e and the vertex v without changing the dependency. Then there must be a rigid circuit in the resulting graph H. Since $|E(H) \setminus E(G)| = 2$, we see that adding an edge to G gives a rigid subgraph H with at least $d+2$ vertices. If H has $d+2$ vertices, then G has at least $\frac{d^2+3d}{2} - 1 + 1 + 2(d+1) = \frac{d^2+7d}{2} + 2$ edges where the first two terms come from the deleting the added edge from H. The third term corresponds to the edge between the two vertices outside H and the last term is the number of edges between the vertices in H and the vertices outside H. By a similar counting we can show if H has $d+3$ vertices then G has at least $\frac{d^2+7d}{2}+1$ edges. Since G is $(d, \binom{d+1}{2})$-sparse, both cases give a contradiction. ■

The following lemma, which is referred to as the *glueing lemma*, tells us we can obtain a larger rigid graph in $\mathbb{R}^d$ from two rigid graphs in $\mathbb{R}^d$ when they have at least d common vertices, and can be proven by a simple observation, see for example [31].

Lemma 5.2.4. *Let G be a graph that is obtained by glueing together two subgraphs $H_1 = (V_1, E_1)$ and $H_2(V_2, E_2)$. Suppose H_1 and H_2 are both rigid in $\mathbb{R}^d$ and that $|V_1 \cap V_2| \geq d$. Then G is rigid in $\mathbb{R}^d$.*

Lemma 5.2.5. *The only non-rigid circuit on $d+5$ vertices with a vertex v of degree $d+1$ in $\mathbb{R}^d$ is B_d.*

Proof: We will prove this by showing there are two distinct rigid components with $d+2$ vertices in G, implying that $G = B_d$. Perform a 1-reduction on non-adjacent neighbours u_1, u_2 of v. Then, by Lemmas 5.2.1 and 5.2.3, in the resulting graph H we must have a rigid circuit C_1. This implies that there exists a rigid component $H' \supseteq C_1$ with at least $d+2$ vertices in G. Since G is non-rigid, H' has exactly $d+2$ vertices and $C_1 - u_1u_2 = H' = K_{d+2} - e$. Otherwise we can sequentially add the vertices which are not in H' to H' to obtain G and preserve rigidity. Therefore H' has $\binom{d+2}{2} - 1 = \frac{d^2+3d}{2}$ edges. Since G is $(d, \binom{d+1}{2})$-sparse it can have at most $\frac{d^2+9d}{2}$ edges implying that there are at most $3d$ edges incident to the three vertices v, v_1, v_2 outside H'. The maximality of H' implies that v, v_1, v_2 are adjacent to each other and each has $d-1$ neighbours in H'. See Figure 5.2 for an illustration in $\mathbb{R}^5$.

Suppose we do not have $N[v] = N[v_1] = N[v_2]$. Say $N[v_1] \neq N[v]$ and let $y \in N[v_1]$, $y \notin N[v]$. We have such a vertex, since $d(v) = \delta(G) = d+1$. Then we can do another 1-reduction on the missing edge vy and the vertex v_1 and obtain a rigid circuit in $G - v_1 + vy$ and a corresponding rigid subgraph H'' in with at least $d+2$ vertices in $G - v_1$. However, since $v_1 \notin H''$ and there are two vertices other than v_1 outside H' (as $|V(G)| = d+5$), we must have $|V(H'') \cap V(H')| \geq d$, implying that their union $H' \cup H''$ is rigid by Lemma 5.2.4. The fact that $|V(G)| = d+5$ and $|V(H'') \cup V(H')| \geq d+3$ implies there are at most two vertices left in $V(G) \setminus (V(H') \cup V(H''))$. Since each such vertex has at least d neighbours in $V(H') \cup V(H'')$ (as $\delta(G) = d+1$), we can add these vertices to $H' \cup H''$ by 0-extensions and preserve rigidity. This implies that G is rigid, a contradiction.

If we have $N[v] = N[v_1] = N[v_2]$, then clearly we have $G = B_d$. ■

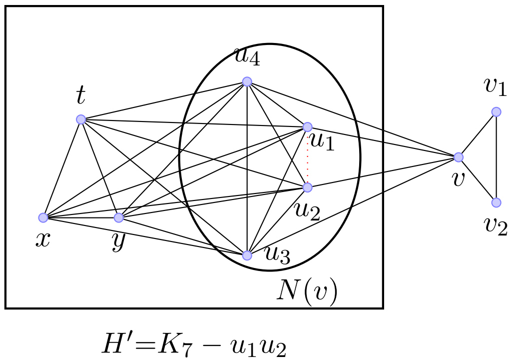

Figure 5.2: If we have $y \in N(v_1)$, $y \notin N(v)$, then we can perform a 1-reduction on v_1 and the missing edge vy to obtain a rigid subgraph H'' with at least $d+2$ vertices of $G - v_1$.

Lemma 5.2.6. *B_d is the unique non-rigid circuit on $d+5$ vertices in $\mathbb{R}^d$, $d \geq 3$.*

Proof: We will proceed by induction on d. The base case is $d = 3$. Since a non-rigid circuit in 3-dimensions is $(3,6)$-sparse, it can have at most 18 edges and hence we always have a vertex of degree 4. Then by Lemma 5.2.5 $G = B_3$. Suppose the statement holds for all dimensions less than d and let G be a non-rigid circuit on $d+5$ vertices in $\mathbb{R}^d$. By Lemma 5.2.5, we may assume that $\delta(G) \geq d+2$. Pick a vertex $v \in V(G)$ and add all the missing edges adjacent to v to obtain the cone, G', of $G - v$. First note that $|E(G') \setminus E(G)| \leq 2$ since $|V(G)| = d+5$, $\delta(G) \geq d+2$ and G' is obtained from G by adding all missing edges incident with a vertex v. Since G' is dependent in $\mathbb{R}^d$, $G - v$ is dependent in $\mathbb{R}^{d-1}$ by Lemma 5.2.2. Then there is a circuit C within $G - v$ in $\mathbb{R}^{d-1}$. We also have $\delta(G - v) \geq d+1$.

Suppose C is a non-rigid circuit. Then $C = B_{d-1}$ by the induction hypothesis. The facts that $\delta(G - v) \geq d+1$, $\delta(B_{d-1}) = d$ and there are six vertices in B_{d-1} of degree d imply that $|E(G-v)\setminus E(B_{d-1})| \geq 3$. Since B_d is the cone of B_{d-1} and G' is the cone of $G-v$, we see that there is a copy of B_d within G' and $|E(G')\setminus E(B_d)| \geq 3$. However, we know that $|E(G')\setminus E(G)| \leq 2$. This implies that $|E(G)| \geq |E(B_d)|+1$. This is a contradiction, since G is $(d, \binom{d+1}{2})$-sparse.

Thus we can assume that C is a rigid circuit in $\mathbb{R}^{d-1}$. Then $|V(C)| \geq d+1$ implying that $|V(G-v)\setminus V(C)| \leq 3$. Combining this and the fact that $\delta(G-v) \geq d+1$, we can sequentially add the vertices in $V(G-v)\setminus V(C)$ to C and preserve rigidity in $\mathbb{R}^{d-1}$. Therefore we get $G-v$ is rigid in $\mathbb{R}^{d-1}$. Since taking the cone preserves generic rigidity and dependency of a graph by Lemma 5.2.2, G' is rigid and contains a rigid circuit in $\mathbb{R}^d$. Let us keep the fact $|E(G')\setminus E(G)| \leq 2$ in mind and consider G and G'. The fact that G is a non-rigid circuit and G' is rigid and contains a rigid circuit implies that when we delete edges from G' to obtain G we must decrease the rank and destroy the rigid circuit. Since deletion of an edge either decreases the rank or destroys a circuit (not both), we have $|E(G')\setminus E(G)| = 2$ and hence $G' = G + u_1v + u_2v$ for some vertices $u_1, u_2 \in V$. We also have that deletion of one of u_1v and u_2v, say u_1v, destroys rigidity (decreases the rank) and deletion of u_2v destroys all rigid circuits. Then $G + u_2v$ still has some rigid circuits implying that G has a rigid component H with at least $d+2$ vertices. Since $\delta(G) \geq d+2$ and we have at most three vertices outside H, such vertices have at least d neighbours in H. Thus we can add those vertices to H and preserve rigidity, a contradiction. ■

Lemma 5.2.7. *The minimum number of edges necessary for a non-rigid circuit in* $\mathbb{R}^d$ *is* $\frac{d^2+9d}{2}$ *when* $d \leq 12$.

Proof: Let G be a non-rigid circuit with the minimum number of edges in $\mathbb{R}^d$. If G has a vertex v with $d(v) = d+1$ we can perform a 1-reduction on v and two of its neighbours without changing the dependency of G. This implies that G has a rigid component H with at least $d+2$ vertices and there are at least 3 vertices outside this component since G is non-rigid. Then we have at least $\binom{d+2}{2}-1+3+3(d-1) = \frac{d^2+9d}{2}$ edges in G where the first two terms are the number of edges in H. The third term is the number of edges which do not have an endpoint in H and the last term is the number of edges having exactly one endpoint in H.

Hence we may suppose that $\delta(G) \geq d+2$. Then the result holds for $d+6 \leq |V(G)|$

when $d \leq 12$, since such a graph has at least $\frac{(d+2)(d+6)}{2} = \frac{d^2+8d+12}{2}$ edges. Therefore we only need to consider the case when $|V(G)| = d+5$ by Lemma 5.2.3. However, this case is not possible, since B_d is the unique non-rigid circuit on $d+5$ vertices by Lemma 5.2.6 and $\delta(B_d) = d+1$. ■

It may be feasible to characterise non-rigid circuits with at most $2d+3$ vertices. However, there exist some strange non-rigid circuits on at least $2d+4$ vertices, e.g. $K_{d+2,d+2}$ which is a circuit for all $d \geq 3$ and non-rigid for all $d \geq 4$.

5.3 Sparse subgraphs

A subgraph $H = (U, F)$ of a $(d, \binom{d+1}{2})$-sparse graph G is *d-critical* if either $|U| = 2$ and $|F| = 1$, or $|U| \geq d$ and H is $(d, \binom{d+1}{2})$-tight. The assumption that G is $(d, \binom{d+1}{2})$-sparse implies that every d-critical subgraph of G is an induced subgraph. A *d-critical component* of G is a d-critical subgraph which is not properly contained in any other d-critical subgraph of G. The following results are due to Jackson [10].

Lemma 5.3.1. *[10] Let $G = (V, E)$ be a $(d, \binom{d+1}{2})$-sparse graph and $H_1 = (U_1, F_1)$, $H_2 = (U_2, F_2)$ be distinct critical components of G. Then $|U_1 \cap U_2| \leq d-1$ and, if equality holds, then $i_G(U_1 \cap U_2) = \binom{d-1}{2}$.*

Proof: Suppose that $|U_1 \cap U_2| \geq d-1$. When $|U_1 \cap U_2| \geq d$ we have $i(U_1 \cap U_2) \leq d|U_1 \cap U_2| - \binom{d+1}{2}$ since G is $(d, \binom{d+1}{2})$-sparse. When $|U_1 \cap U_2| = d-1$, we have $i(U_1 \cap U_2) \leq \binom{d-1}{2} = d|U_1 \cap U_2| - \binom{d+1}{2} + 1$ trivially. The maximality of H_1, H_2 and the definition of a d-critical component imply that $|U_1|, |U_2| \geq d$, and $d(|U_1| + |U_2|) - 2\binom{d+1}{2} = i_G(U_1) + i_G(U_2) \leq i_G(U_1 \cup U_2) + i_G(U_1 \cap U_2) \leq d|U_1 \cup U_2| - \binom{d+1}{2} - 1 + d|U_1 \cap U_2| - \binom{d+1}{2} + 1 = d(|U_1| + |U_2|) - 2\binom{d+1}{2}$. Equality must hold throughout. In particular we have $i_G(U_1 \cap U_2) = d|U_1 \cap U_2| - \binom{d+1}{2} + 1$. This implies that $|U_1 \cap U_2| = d-1$ and $i_G(U_1 \cap U_2) = \binom{d-1}{2}$. ■

Let k, t be non-negative integers, $G = (V, E)$ be a graph and $\mathcal{X}$ be a family of subsets of V. Recall that $\mathcal{X}$ is *t-thin* if every pair of sets in $\mathcal{X}$ intersect in at most t vertices. A *k-hinge* of $\mathcal{X}$ is set of k vertices which lie in the intersection of at least two sets in $\mathcal{X}$. A k-hinge U of $\mathcal{X}$ is *closed in G* if $G[U]$ is a complete graph. We use

$\Theta_k(\mathcal{X})$ to denote the set of all k-hinges of $\mathcal{X}$. For $U \in \Theta_k(\mathcal{X})$, let $d_{\mathcal{X}}(U)$ denote the number of sets in $\mathcal{X}$ which contain U. Note that if G is t-thin then $\Theta_k(\mathcal{X}) = \emptyset$ for all $k \geq t+1$. Note also that $\Theta_0(\mathcal{X}) = \{\emptyset\}$ and $d_{\mathcal{X}}(\emptyset) = |\mathcal{X}|$.

Lemma 5.3.2. *[10] Let $H = (V, E)$ be a $(d, \binom{d+1}{2})$-sparse graph, $\mathcal{X}$ be a family of subsets of V such that $H[V_i]$ is d-critical for all $V_i \in \mathcal{X}$, and let $W \in \Theta_k(\mathcal{X})$ for some $0 \leq k \leq d-1$. Suppose that $|V_i| \geq d$ for all $V_i \in \mathcal{X}$ with $W \subseteq V_i$. Then*

$$(d-k) \sum_{\substack{U \in \Theta_{k+1}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) - \sum_{\substack{U \in \Theta_{k+2}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) \leq \binom{d+1-k}{2} (d_{\mathcal{X}}(W) - 1).$$

Proof: Let $d_{\mathcal{X}}(W) = t$ and let $V_1, V_2, \ldots, V_t$ be the sets in $\mathcal{X}$ which contain W. Let $H_i = (V_i, E_i) = H[V_i]$ for $1 \leq i \leq t$. Let $H' = \bigcup_{i=1}^t H_i$ and put $H' = (V', E')$. Then

$$|V'| = \sum_{i=1}^t |V_i| - k(t-1) - \sum_{\substack{U \in \Theta_{k+1}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) \tag{5.1}$$

since, for $v \in V'$, if $v \in W$ then v is counted t times in $\sum_{i=1}^t |V_i|$, if $v \in U \setminus W$ for some $U \in \Theta_{k+1}$ with $W \subset U$ then v is counted $d_{\mathcal{X}}(U)$ times in $\sum_{i=1}^t |V_i|$, and all other vertices of V' are counted exactly once in $\sum_{i=1}^t |V_i|$.

Similarly,

$$|E'| \geq \sum_{i=1}^t |E_i| - \binom{k}{2}(t-1) - k \sum_{\substack{U \in \Theta_{k+1}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) - \sum_{\substack{U \in \Theta_{k+2}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) \tag{5.2}$$

since, for $e = uv \in E'$, if $u, v \in W$ then e is counted t times in $\sum_{i=1}^t |E_i|$ and there are at most $\binom{k}{2}$ such edges, if $u \in W$ and $v \in U \setminus W$ for some $U \in \Theta_{k+1}$ with $W \subset U$ then e is counted $d_{\mathcal{X}}(U)$ times in $\sum_{i=1}^t |E_i|$ and for each such v there are at most k choices for u, if $u, v \in U \setminus W$ for some $U \in \Theta_{k+2}$ with $W \subset U$ then e is counted $d_{\mathcal{X}}(U)$ times in $\sum_{i=1}^t |E_i|$, and all other edges of E' are counted exactly once in $\sum_{i=1}^t |E_i|$.

Since $H' \subseteq H$, H' is $(d, \binom{d+1}{2})$-sparse. Hence $|E'| \leq d|V'| - \binom{d+1}{2}$. We may substitute equations (5.1) and (5.2) into this inequality and use the fact that $|E_i| =$

$d|V_i| - \binom{d+1}{2}$ for all $1 \leq i \leq t$ to obtain

$$\begin{aligned}
& (d-k) \sum_{\substack{U \in \Theta_{k+1}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) - \sum_{\substack{U \in \Theta_{k+2}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) \\
\leq \; & \left[\binom{d+1}{2} + \binom{k}{2} - dk \right] (t-1) \\
= \; & \binom{d+1-k}{2} (t-1).
\end{aligned}$$

■

Lemma 5.3.3. *[10] Let $H = (V, E)$ be a $(d, \binom{d+1}{2})$-sparse graph, $\mathcal{X}$ be a family of subsets of V such that $H[V_i]$ is d-critical and $|V_i| \geq d$ for all $V_i \in \mathcal{X}$. Put $a_k = \sum_{U \in \Theta_k(\mathcal{X})} (d_{\mathcal{X}}(U) - 1)$ for $0 \leq k \leq d$. Then for all $0 \leq k \leq d-2$ we have:*

(i) $(d-k)(k+1)a_{k+1} - \binom{k+2}{2} a_{k+2} \leq \binom{d+1-k}{2} a_k$;

(ii) $(d-k)a_{k+1} - (k+1)a_{k+2} \leq \binom{d+1}{k+2}(|\mathcal{X}| - 1)$;

(iii) if $\mathcal{X}$ is $(d-1)$-thin, $d(d-k)a_{k+1} \leq (k+2)(d-k-1)\binom{d+1}{k+2}(|\mathcal{X}| - 1)$.

Proof: Part (i) follows by summing the inequality in Lemma 5.3.2 over all $W \in \Theta_k$, and using the facts that

$$\sum_{W \in \Theta_k(\mathcal{X})} \sum_{\substack{U \in \Theta_{k+1}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) = (k+1) \sum_{U \in \Theta_{k+1}(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) = (k+1)a_{k+1}$$

and

$$\sum_{W \in \Theta_k(\mathcal{X})} \sum_{\substack{U \in \Theta_{k+2}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) = \binom{k+2}{2} \sum_{U \in \Theta_{k+2}(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) = \binom{k+2}{2} a_{k+2}\,.$$

We prove (ii) by induction on k. When $k = 0$, (ii) follows by putting $k = 0$ in (i), and using the fact that $a_0 = |\mathcal{X}| - 1$. Hence suppose that $k \geq 1$. Then (i) gives

$$2(d-k)a_{k+1} - 2(k+1)a_{k+2} \leq \frac{(d-k+1)(d-k)}{k+1} a_k - k a_{k+2}\,. \tag{5.3}$$

We may also use (i) to obtain

$$ka_{k+2} \geq \frac{k(d-k)}{k+2}\left(2a_{k+1} - \frac{d-k+1}{k+1}a_k\right). \tag{5.4}$$

Substituting (5.4) into (5.3) and using induction we obtain

$$\begin{aligned}(d-k)a_{k+1} - (k+1)a_{k+2} &\leq \tfrac{d-k}{k+2}\left[(d-k+1)a_k - ka_{k+1}\right] \\ &\leq \tfrac{d-k}{k+2}\tbinom{d+1}{k+1}(|\mathcal{X}|-1) \\ &= \tbinom{d+1}{k+2}(|\mathcal{X}|-1).\end{aligned}$$

We prove (iii) by induction on $d-k$. When $d-k=2$, (iii) follows by putting $k=d-2$ in (ii) and using the fact that $a_d=0$ since $\mathcal{X}$ is $(d-1)$-thin. Hence suppose that $d-k\geq 3$. Then (ii) gives

$$d(d-k)a_{k+1} \leq d\tbinom{d+1}{k+2}(|\mathcal{X}|-1) + d(k+1)a_{k+2}.$$

We may now apply induction to a_{k+2} to obtain

$$\begin{aligned}d(d-k)a_{k+1} &\leq \left[d\tbinom{d+1}{k+2} + \tfrac{(k+1)(k+3)(d-k-2)}{d-k-1}\tbinom{d+1}{k+3}\right](|\mathcal{X}|-1) \\ &= (k+2)(d-k-1)\tbinom{d+1}{k+2}(|\mathcal{X}|-1).\end{aligned}$$

■

Theorem 5.3.4. *[10] Let $H=(V,E)$ be a $(d,\binom{d+1}{2})$-sparse graph, $\mathcal{X}$ be a $(d-1)$-thin family of subsets of V such that $H[V_i]$ is d-critical and $|V_i|\geq d$ for all $V_i\in\mathcal{X}$. For each $V_i\in\mathcal{X}$ let $\theta_k(H_i)$ be the number of k-hinges of $\mathcal{X}$ contained in V_i. Then:*

(i) $\theta_1(V_1)\leq 2d-1$ for some $V_1\in\mathcal{X}$;

(ii) $\theta_2(V_2)\leq (d-2)(d+1)-1$ for some $V_2\in\mathcal{X}$;

(iii) $\theta_{d-1}(V_3)\leq d$ for some $V_3\in\mathcal{X}$.

Proof:

We first prove (i). Putting $k = 0$ in Lemma 5.3.3(iii) we obtain

$$d \sum_{U \in \Theta_1(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) \leq (d-1)(d+1)(|\mathcal{X}| - 1)\,. \tag{5.5}$$

Since $d_{\mathcal{X}}(U) \geq 2$ for all $U \in \Theta_1(\mathcal{X})$ we have $d_{\mathcal{X}}(U) - 1 \geq d_{\mathcal{X}}(U)/2$ and hence (5.5) gives

$$\sum_{U \in \Theta_1(\mathcal{X})} d_{\mathcal{X}}(U) < 2d\,|\mathcal{X}|\,.$$

This tells us that the average number of 1-hinges in a set in $\mathcal{X}$ is strictly less that $2d$.

We next prove (ii). Putting $k = 1$ in Lemma 5.3.3(iii) we obtain

$$\sum_{U \in \Theta_2(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) \leq (d-2)(d+1)(|\mathcal{X}| - 1)/2\,. \tag{5.6}$$

We can now proceed as in (i).

Finally we prove (iii). Putting $k = d - 2$ in Lemma 5.3.3(iii) gives

$$2 \sum_{U \in \Theta_{d-1}(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) \leq (d+1)(|\mathcal{X}| - 1)\,. \tag{5.7}$$

We can now proceed as in (i).

5.4 An upper bound on the rank

Let $G = (V, E)$ be a graph and $\mathcal{X}$ be a family of subsets of V. Recall that $\mathcal{X}$ is a *cover* of G if every set in $\mathcal{X}$ contains at least two vertices, and every edge of G is induced by at least one set in $\mathcal{X}$.

Lemma 5.4.1. *[10] Let $G = (V, E)$ be a graph, $H = (V, F)$ be a maximal $(d, \binom{d+1}{2})$-sparse subgraph of G, and $H_1, H_2, \ldots, H_m$ be the d-critical components of H. Let X_i be the vertex set of H_i for $1 \leq i \leq m$. Then $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ is a $(d-1)$-thin cover of G and each $(d-1)$-hinge of $\mathcal{X}$ is closed in H.*

Proof: The definition of a d-critical subgraph implies that each H_i has at least two vertices and that every edge of H belongs to at least one H_i. Thus $\mathcal{X}$ is a cover of H. To see that $\mathcal{X}$ also covers G we choose $e = uv \in E \setminus F$. The maximality of H implies that $H + e$ is not $(d, \binom{d+1}{2})$-sparse. Hence $\{u, v\}$ is contained in some d-critical subgraph of H. Thus $\mathcal{X}$ also covers G. The facts that $\mathcal{X}$ is $(d-1)$-thin and that each $(d-1)$-hinge of $\mathcal{X}$ is closed follow from Lemma 5.3.1. ∎

We refer to the closed $(d-1)$-thin cover of G described in Lemma 5.4.1 as the *H-critical cover* of G. Note that the definition of a d-critical set implies that each set in a d-critical cover has size two or has size at least d.

Theorem 5.4.2. *Let $G = (V, E)$ be a graph, $d \leq 11$ be an integer and $H = (V, F)$ be a maximal $(d, \binom{d+1}{2})$-sparse subgraph of G. Then $r_d(G) \leq |F|$.*

Proof: We proceed by contradiction. Suppose the theorem is false and choose a counterexample (G, H) such that $|E|$ is as small as possible. Let $H_1, H_2, \ldots, H_m$ be the d-critical components of H where $H_i = (V_i, F_i)$ for $1 \leq i \leq m$. Then $\mathcal{X}_0 = \{V_1, V_2, \ldots, V_m\}$ is the H-critical cover of G.

Choose a cover $\mathcal{X}$ of G such that $\mathcal{X} \subseteq \mathcal{X}_0$ and $|\mathcal{X}|$ is as small as possible. Note that $\mathcal{X}_0$, and hence also $\mathcal{X}$, are $(d-1)$-thin. For each $V_i \in \mathcal{X}$, let F_i^* be the set of all edges $uv \in F_i$ such that $\{u, v\}$ is a 2-hinge of $\mathcal{X}$, and let E_i be the set of edges of G induced by V_i.

Claim 5.4.2.1. *If $e = uv \in E$ satisfies $r_d(G) = r_d(G - e)$, then $\{u, v\}$ is a 2-hinge of $\mathcal{X}$.*

Proof: First suppose that $e \in E \setminus F$. Then since H is a maximal $(d, \binom{d+1}{2})$-sparse subgraph of $G - e$, by using the minimality of $|E|$ and $r_d(G) = r_d(G - e)$ we get a contradiction.

Thus we can assume that $e \in F$. Let $h(e)$ be the number of $V_i \in \mathcal{X}$ such that $e \in F_i$. We know that $H - e$ is a $(d, \binom{d+1}{2})$-sparse subgraph of $G - e$. Let $H' = (V, F') \supseteq H - e$ be a maximal $(d, \binom{d+1}{2})$-sparse subgraph of $G - e$. If $e \notin F_i$, then no edge of $E_i \setminus F_i$ can be in F', since F_i is d-critical and if $e \in F_i$, then at most one edge of $E_i \setminus F_i$ can be in F', since $|F_i - e| = d|V_i| - \binom{d+1}{2} - 1$. Then we see that $|F'| \leq |F| - 1 + h(e)$. By the minimality of $|E|$ we have $r_d(G - e) \leq |F'|$, and hence $r_d(G) \leq |F| - 1 + h(e)$. Combining this and $r_d(G) > |F|$ we get $h(e) \geq 2$.•

Note that this claim implies that F_i^* is dependent for all i. Suppose this is not the case. Then we have E_i as an independent edge set by Claim 5.4.2.1. Since E_i can have at most $d|V_i| - \binom{d+1}{2}$ edges and $F_i \subseteq E_i$ is d-critical, we have $E_i = F_i$. Either $E_i = F_i = F_i^*$ or $E_i = F_i \neq F_i^*$ holds. The former case contradicts the minimality of $\mathcal{X}$. The latter case contradicts the minimality of $|E|$. To see this consider $H - e$ and $G - e$ for an edge $e \in F_i \setminus F_i^*$. Since $F_i = E_i$ all edges of $G - e$ which are induced by V_i are in $H - e$, and those V_j with $j \neq i$, are already d-critical in $H - e$, we conclude $H - e$ is a maximal $(d, \binom{d+1}{2})$-sparse subgraph of $G - e$. Then we have $r_d(G - e) = r_d(G) - 1 > |F| - 1 = |F - e|$, contradicting the minimality of $|E|$, where the first equality is by Claim 5.4.2.1.

Since F_i^* is dependent it contains a circuit of $\mathcal{R}_d(G)$. This circuit cannot be rigid, since it is $(d, \binom{d+1}{2})$-sparse. By Lemma 5.2.7, $|F_i^*| \geq \frac{d^2+9d}{2}$ for all $V_i \in \mathcal{X}$. This contradicts Theorem 5.3.4 (ii) when $d \leq 11$. ■

5.5 Closing remarks

An improved upper bound on the rank

Given a graph G, let $s_d(G)$ be the minimum number of edges in a maximal $(d, \binom{d+1}{2})$-sparse subgraph of G. Theorem 5.4.2 tells us that $r_d(G) \leq s_d(G)$ when $d \leq 11$. It is not difficult to construct graphs for which strict inequality holds.

Example 5.5.1. Consider the graph B_3 in Figure 12. We see that $s_3(B_3) = |E(B_3)| = 18 > 17 = r_3(B_3)$. On the other hand we may improve the upper bound on $r_3(B_3)$ in this example by considering the graph $B_3^* = B_3 + uv$. A maximal $(3, 6)$-sparse subgraph of B_3^* which contains uv has 17 edges. Thus we have $17 = r_3(B_3) \leq r_3(B_3^*) \leq s_3(B_3^*) = 17$ by Theorem 5.4.2.

More generally, for any graph G we have the improved upper bound

$$r_d(G) \leq \min\{s_d(G^*) \ : \ G \subseteq G^*\} =: s_d^*(G) \tag{5.8}$$

for all $d \leq 11$.

The following example shows that strict inequality can also hold in (5.8).

Example 5.5.2. Let G be obtained from K_5 by taking parallel connections with 10 different K_5 along each of the edges of the original K_5. We will first show that $r_3(G) = 89$. We remove an edge from the original K_5 and an edge e for each of other copies of K_5 such that e is not an edge of the original K_5. Let us say the resulting graph is G'. Then we have $|E(G')| = |E(G)| - 11 = 100 - 11 = 89$. Since each of the edges we removed sequentially cancels a distinct copy of K_5 in G and K_5 is dependent in $\mathbb{R}^3$, we see that $r_3(G') = r_3(G)$. Since the edges we removed from the copies of non-original K_5 leaves a vertex of degree three in G', we can now sequentially remove the vertices of degree at most three in G' and obtain the empty graph. As this operation corresponds to a 0-reduction, and 0-reduction and 0-extension preserve independence, we conclude that $r_3(G) = r_3(G') = |E(G')| = 89$. On the other hand, $s_3(G) = 90$ (obtained by taking a maximal $(3, 6)$-sparse subgraph which contains 9 of the edges of the original K_5). Note that the non-trivial motions of G are the ones corresponding to the rotation of a copy of non-original K_5 about its common edge with the original K_5. Hence adding an edge to G, which must connect two distinct copies of non-original K_5, will make the motions of the copies these K_5 dependent on each other. This implies that adding an edge to G will increase the rank. Therefore we have $s_3(G^*) \geq r_3(G^*) > r_3(G)$ for all graphs G^* which properly contain G. Thus $s_3^*(G) = 90 > r_3(G)$.

Algorithmic considerations

For fixed d, we can use network flow algorithms to test whether a graph is $(d, \binom{d+1}{2})$-sparse in polynomial time, see for example Berg and Jordán [4]. This means we can greedily construct a maximal d-sparse subgraph H of a graph G in polynomial time and hence obtain an upper bound on $r_d(G)$ via Theorem 5.4.2. We do not know whether $s_d(G)$ or $s_d^*(G)$ can be determined in polynomial time.

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