Stochastic Complementarity*

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Abstract

The Hicksian definition of complementarity and substitutability may not apply in contexts in which agents are not utility maximisers or where price or income variations, whether implicit or explicit, are not available. We look for tools to identify complementarity and substitutability satisfying the following criteria: they are *behavioural* (based only on observable choice data); *model-free* (valid whether the agent is rational or not); and they do not rely on price or income variation. We uncover a conflict between properties that it is arguably reasonable for a complementarity notion to possess. We discuss three different possible resolutions of the conflict.
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1 Introduction

1.1 Motivation

In this paper we take a fresh look at the notion of complementarity. Intuitively, complements are goods that 'go together'. However, this can be specified in several different ways and on the basis of different primitives. Price (or income) variations are just a *tool* to detect complementarity: the notion of complementarity itself is not *intrinsically* related to such variations. Furthermore, we also take the view that complementarity is meaningful independently of whether decision makers maximise utility or not. After all, goods may or may not go together whatever the cognitive process that leads to choice, including boundedly rational ones. For example, we would like to be able to talk about complementarity for a 'satisficer' à la Simon [28] or an 'eliminator by aspects' à la Tversky [30] (who does not even 'have' a utility, let alone maximise it). For these reasons we look for definitions of complementarity (and substitutability) between goods that complement those of the standard approach. In sum, we look for a notion that is:

(1) priceless: it is not based on price (nor income) variations;

(2) behavioural: it just uses choice data as inputs;

(3) model free: it does not commit to any specific mechanism underlying the choice data.

To motivate this program, suppose that a local government wants to know whether the uses of two free public attractions, say a museum and a park, 'go together'. An economist will translate this question as the question of whether the park and the museum are complements. However, this situation does not fit directly the textbook criteria for complementarity, which are based on price elasticities while in this case both prices are fixed at zero. The authority may also be surprised to hear that the gold standard concept of Hicksian complementarity is embedded in a rational utility maximisation framework.[1]

[1]It merits noting in this respect that goverments are more and more exposed to the idea that citizens are not necessarily fully rational utility maximisers. They are increasingly being advised by 'behavioural economics' units. These include the US Office of Information and Regulatory Affairs, which led to the institution of the US government's Social and Behavioral Sciences Team in July 2015; the Behaviour Insights team in the UK, established in 2010 and then spun off into a separate company in 2014; the

Of course, a well-bred economist could explain to the authority that it is always possible to retrieve 'implicit' prices. One could for instance check the impact of having the park located at different distances from the museum, observe the resulting effect on the demand for the museum, impute a value to the time needed to move between the two attractions, and then calculate an elasticity based on such values. Because it may be somewhat impractical to experiment with moving the park at various distances, the impact would be estimated from the observations of analogous effects relating to other museums and other parks, and these effects would be adjusted for the various factors affecting demand in the different locations. But, while this standard type of methodology has merits[2], it is fair to say that it is very indirect and thus its validity necessarily rests on several assumptions (regarding the imputed value of time, the comparisons across locations, the specification of the model that generates demand, etcetera). And the fact remains that Hicksian compensated elasticities are guaranteed to be meaningful only within a constrained utility maximisation framework.[3]

Some recent econometric approaches, pioneered by Gentzkow [18], deal with the zero price problem. They typically assume an additive Random Utility Model (RUM) such as the multinomial logit and variations thereof (see section 2 for a discussion). However, the recent wave of abstract works on stochastic choice (e.g. Aguiar, Boccardi and Dean [1], Brady and Rehbeck [11], Caplin and Dean [13], Echenique, Saito and Tserenjigmid [17], Gül, Natenzon and Pesendorfer [19], Kovach and Ülkü [21], Manzini and Mariotti [22], among others) has highlighted a wide variety of possible 'choice errors' and choice procedures, and so a number of reasons why agents' behaviour might fail to be described by a logit model, and indeed even by the much larger class of RUMs.

Behavioural Insights Unit established in 2012 as part of the New South Wales Premier and Cabinet's office. The German Chancellery is in the process of setting up a similar unit. Several other governmental units and groups exist in various countries and in the European Union Directorates General.

[2]A neat early example of the methodology is Becker and Murphy's [6] analysis of the complementarity between advertising and advertised goods based on the implicit price of commercials, which are shown on television without a price. If networks stopped showing commercials, the public would have to pay for television content. The change in the price of content is the implicit price of commercials.

[3]In principle, one might be able to perform a Hicksian type of compensation (when price variations are available) even when the agent is not a utility maximiser, provided there is some obvious way of defining 'revealed indifference'. This is not always the case. Also, we note that different tests of complementarity might use *income* rather than price variations (see Quah [26] and our literature review section 2), but still in a framework of constrained utility maximisation.

We take seriously the multiplicity of plausible models and the consequent difficulty of model selection.

Zero or hard-to-define prices are observed for many other goods beside public attractions: online newspapers, reviews/advice (e.g. financial) on social networks, public radio broadcasts, file sharing are often free. Another leading example is that of complementarity in business practices, such as training the workforce and allowing it more decisional discretion (Brynjolfson and Milgrom [12]). More abstractly, the 'goods' may be characteristics embodied in the objects of choice, so that any price variation is perfectly correlated between the goods. In some cases, prices may be especially difficult to conceptualise: is beauty a complement or a substitute of wealth in a partner? Finally, interest in complementarity may concern non-market individual activities and behaviours, such as voting participation in local and national elections.

1.2 The basic ideas

How can the intuitive concept of goods 'going together' be made operational?

The first idea in our analysis is to use *stochastic* choice data as a primitive. Because we are not going to exploit responses to price variations, and more in general the variation of choices across menus, we lose some information compared to the standard approach. To obviate this, we consider instead a multiplicity of choices from a fixed menu, in the form of choice frequencies. As in standard stochastic choice models, such frequencies admit multiple interpretations: they may express repeated individual choices; fractions of time an individual spends performing an activity or using a durable good; or the proportion of a population purchasing a good. The rich structure of this type of data is an alternative source of information about underlying complementarities.

The second idea is to examine two principles that we consider as basic:

a) *Statistical Association*: Association is exactly what it means to 'go together' in statistical language. Therefore it seems that if choice data come in the form of frequencies then the consumption of complementary goods should exhibit some kind of statistical association. Indeed, in the literature about complementarities in business practices, the positive *correlation* (or clustering) of practices is the most common complementarity test[4].

[4]See e.g. Brynjolfson and Milgrom [12], p.33.

b) '*Revealed preference':* Complementarity between goods, unlike natural phenomena for which 'going together' is exhausted by statistical association, has the peculiarity that it is the expression of an act of choice. What information do choices convey about complementarity? At the very least it seems that consuming the goods jointly should not *in itself* be evidence of substitutability, and consuming the goods individually should not *in itself* be evidence of complementarity. Note well: this principle is *not saying that joint consumption per se is evidence of complementarity* - there could be complementary goods whose joint consumption is low.

The revealed preference principle is also fundamental from a welfare perspective. Interest in complementarity is often a consequence of a welfare question, such as "is it welfare enhancing to build a park next to the existing museum?" or "would introducing a new product be welfare enhancing for the consumers (so that he would be willing to pay more)?". We take the classical view that an agent's choices encode welfare information, and that, as articulated by Bernheim and Rangel [8], they do so whether the agent is rational or not. It would be odd indeed if the local authority, having decided that the construction of the park is in the interest of the community, was dissuaded after learning that the joint consumption of park and museum in a similar location has increased.

1.3 Preview of results

Consider two goods, say the online and the print versions of a newspaper. As in Gentzkow [18], the data come in the form $(p_{OP}, p_O, p_P, p_\varnothing)$, where p_O and p_P denote the consumption frequency of the online version only and of the print version only, respectively; p_{OP} denotes the frequency of joint consumption; and $p_\varnothing$ denotes the frequency with which neither version is read.

Let's consider the statistical principle discussed above. For simplicity, let's also focus in this section on association as correlation (this is just for concreteness: later on we will use a much more abstract concept). Then we would say the two versions are complementary whenever they are positively correlated, that is, when the posterior probability of reading one version conditional on reading the other version, $\frac{p_{OP}}{p_{OP}+p_P}$, is greater than the prior probability, $p_{OP} + p_O$.

It is easily shown, however, that this property flatly contradicts the intuitive revealed preference view that increases in joint consumption should not by themselves constitute evidence of substitutability. Consider the data in the following table:

	Read Print	Did not read print
Read Online	0.3	0.2
Did not read online	0.2	0.3

That is, $(p_{OP}, p_O, p_P, p_\varnothing) = (0.3, 0.2, 0.2, 0.3)$. Then the data indicate a positive correlation ($\frac{p_{OP}}{p_{OP}+p_P} = 0.6 > 0.5 = p_{OP} + p_O$). Suppose now that joint consumption rises to $p'_{OP} = 0.55$ while single good consumptions stay the same. Then the correlation turns negative ($\frac{p'_{OP}}{p'_{OP}+p'_P} = 0.73 < 0.75 = p'_{OP} + p'_O$). An increase in joint consumption has transformed the goods from complementary to substitutes!

Our first main contribution is to show that this simple example illustrates a deeper conflict between two natural properties that criteria for complementarity should satisfy. One is *monotonicity*, embodying the revealed preference principle: an increase in joint consumption accompanied by (weak) decreases in single good consumption should not overturn an existing complementarity (and analogously for substitutability).

The second property is *duality*: if in a dataset O and P are complementary, then they are substitutes in the 'opposite' dataset in which the instances of consumption of P are switched with the instances of non-consumption of P (holding fixed the consumption/non-consumption of O). Duality is evidently satisfied by all common measures of correlation and association. It is intrinsic to the nature of association that 'inverting' behaviour changes the sign of the association.

The conflict in general is somewhat more subtle than in the simple example above. The two properties do not flatly contradict each other. However, theorem 1 shows that any concept of complementarity that satisfies monotonicity and duality must be also *unresponsive*, in the sense that the level of non-consumption $p_\varnothing$ *on its own* determines whether the goods are complements or substitutes, irrespective of the distribution between single and joint consumption. This is a very undesirable feature, and for this reason we interpret the result as one of conflict between the the statistical association and the revealed preference principles.[5]

[5] Furthermore, a second impossibility result (theorem 6) shows that duality and monotonicity are in outright conflict if it is also assumed that the frontier between complementarity and independence is thin, as is the case for the standard elasticity-based criteria.

We then look for ways out of the impossibility (Section 5), focussing on symmetric criteria (i.e., such that complementarity, or lack of it, is bidirectional among the two goods). Correlation turns out to be the only symmetric criterion of complementarity that satisfies both duality and a modified monotonicity condition, which embodies only a 'proportional' version of the revealed preference principle.

Next, we examine monotonic criteria that satisfy modified notions of duality, based on alternative interpretations of what constitutes the 'opposite' of a given behaviour. One criterion is economically intuitive if the numbers p_{OP}, p_O and p_P are taken as expressing the values of the respective options: O and P are complementary (resp., substitutes) if $p_{OP} > p_O + p_P$ (resp., $p_{OP} < p_O + p_P$). This criterion satisfies a duality property based on exchanging joint consumption with total single good consumption.

The third criterion for complementarity to be considered says that O and P are complementary (resp., substitutes) if $p_{OP} > \max\{p_O, p_P\}$ (resp., $p_{OP} < \min\{p_O, p_P\}$). This criterion satisfies a notion of duality based on exchanging joint consumption with one type of single good consumption. We consider these as the three main candidate criteria of model-free stochastic complementarity.

In the concluding discussion we argue the Hicksian complementarity criterion may not be always appropriate for the type of data we are considering.

2 Related literature

Samuelson [27] contains an erudite discussion of the subtleties of the concept of complementarity, with an exhaustive review of the classical literature.

The work by Gentzkow [18] we have already mentioned pioneers the approach to the zero price problem. He asks the question of whether the online and print versions of a newspaper are complements or substitutes. The main difficulty to be solved in this case is that the observed correlation in consumption may partly reflect correlated unobservable tastes for the goods, rather than 'true' complementarity: for instance, a news junkie may consume both paper and online versions even when there is no 'true' complementarity, which *in a model of (random) utility maximisation* means a positive difference between the value of joint consumption and the sum of the values of single good consumptions.[6] Gentzkow finds sufficient conditions under which correlation in choice data is indicative of Hicksian complementarity, and analyses the identification of complementarity (as opposed to taste correlation) in the data by using exogenous variations in factors that do not interact with preferences. This requires the development of an innovative econometric identification technique which, however, is meaningful only within the random utility model. Our approach, in contrast, is to investigate whether complementarity or substitution can be identified in a model-free fashion. Our agents may not even 'have' a utility function.

It is a surprising fact that the full behavioural implications of the classical definitions of complementarity and substitutability, based on cross price elasticities, have only recently been uncovered, in two papers by Chambers, Echenique and Shmaya ([14] and [15]). The key difference between their work and ours is that their hypothetical data include observations of consumption decisions for different prices (as the classical definition requires), whereas ours are based on consumption decisions alone.

A large literature exists in which supermodularity of a utility function gives, by definition, a complementarity relationship between the goods, and likewise submodularity is equivalent to substitutability (see for example Bikhchandani and Mamer [9], Gül and Stachetti [20] for applications of submodularity and related notions of substitutability in general equilibrium models with indivisibilities.[7]) These definitions are cardinal. Complementarity in the form of supermodularity is also the bread and butter of modern monotone comparative static techniques as surveyed by Topkis [29]. Our approach differs from this line of work, in that our axiomatic analysis relies on the data, rather than on the underlying preference.

Finally, an interesting line of research concerns the relation between a supermodularity-based notion of complementarity and *income* (rather than price) variations in constrained utility maximisation consumer problems. Chipman [16] was the first to prove that the so-called Auspitz-Lieben-Edgeworth-Pareto notion of complementarity (positive cross partials of the utility function), together with strong concavity of the utility, implies

[6] As Gentzkow shows, in the two good model this is equivalent to a positive compensated cross price elasticity of demand.

[7] See Baldwin and Klemperer [5] for an innovative approach (based on tools from tropical geometry) that yields complements/substitutes types of conditions for the existence of equlibria with discrete goods.

that all goods are normal (so that in other words, their demands are correlated if the data is generated by income variations), and the demand for each goods is a decreasing function of its own price. Quah [26] provides the modern general theory that implies this insight using the tools of supermodularity theory.

3 Preliminaries

There are two goods, x and y. A *datapoint* is an ordered four-tuple $p = (p_{xy}, p_x, p_y, p_\varnothing)$ with $p_k \in (0,1)$ for $k \in \{xy, x, y, \varnothing\}$ and $\sum_{k\in\{xy,x,y,\varnothing\}} p_k = 1$. The interpretation is that p_{xy} denotes the probability (or frequency) of joint consumption of x and y, p_x and p_y denote the probabilities of consumption of x but not y and of y but not x, respectively, and $p_\varnothing$ denotes the probability of consuming neither x nor y. Let T be the set of datapoints, i.e.,

$$T = \left\{(a,b,c,d) \in (0,1)^4 : a+b+c+d = 1\right\}.$$

Definition 1 A *criterion* is any triple (C, I, S) where C, I and S are pairwise disjoint subsets of T, and C and S are nonempty.

Hence a criterion partitions datapoints into three distinct regions: a *complementarity region* C, a *substitution region* S and an *independence region* I. If $p \in C$ (resp. $p \in S$, resp. $p \in I$) we say that x and y are complements (resp. substitutes, resp. independent) at p. We next define some criteria of interest.

Definition 2 A criterion (C, I, S) is the *correlation criterion* if

$$\begin{aligned} C &= \left\{(p_{xy}, p_x, p_y, p_\varnothing) \in T : \frac{p_{xy}}{p_{xy}+p_y} > p_{xy} + p_x\right\} \text{ and} \\ S &= \left\{(p_{xy}, p_x, p_y, p_\varnothing) \in T : \frac{p_{xy}}{p_{xy}+p_y} < p_{xy} + p_x\right\} \end{aligned}$$

According to the correlation criterion a datapoint is in C (resp., S) if and only if the information that one of the goods is consumed increases (resp., decreases) the probability the other good is also consumed.

Definition 3 A criterion (C, I, S) is the *additivity criterion* if

$$\begin{aligned} C &= \left\{ \left(p_{xy}, p_x, p_y, p_\varnothing\right) \in T : p_{xy} > p_x + p_y \right\} \text{ and} \\ S &= \left\{ \left(p_{xy}, p_x, p_y, p_\varnothing\right) \in T : p_{xy} < p_x + p_y \right\}. \end{aligned}$$

The additivity criterion is natural whenever one thinks of the probabilities as expressing 'values' (as is the case in the logit model). Then it says that x and y are complements whenever the value of joint consumption is greater than the sums of the values of the goods when consumed singly. This is in fact the notion of complementarity used in many applications, e.g. the literature on bundling (e.g. Armstrong [4]).

Definition 4 A criterion (C, I, S) is the *maxmin criterion* if

$$\begin{aligned} C &= \left\{ \left(p_{xy}, p_x, p_y, p_\varnothing\right) \in T : p_{xy} > \max\left\{p_x, p_y\right\} \right\} \text{ and} \\ S &= \left\{ \left(p_{xy}, p_x, p_y, p_\varnothing\right) \in T : p_{xy} < \min\left\{p_x, p_y\right\} \right\} \end{aligned}$$

The maxmin criterion fits, for instance, the situation in which one good is an 'accessory' and only the 'dominant' single good consumption is relevant in comparison with joint consumption to declare complementarity. To check whether steak and pepper are complementary you may want to compare the probability of consumption of steak with that of steak and pepper, rather than with that of pepper alone. Substitution is declared symmetrically.

Definition 5 A criterion (C, I, S) is the *supermodularity criterion* if

$$\begin{aligned} C &= \left\{ \left(p_{xy}, p_x, p_y, p_\varnothing\right) \in T : p_{xy} + p_\varnothing > p_x + p_y \right\} \text{ and} \\ S &= \left\{ \left(p_{xy}, p_x, p_y, p_\varnothing\right) \in T : p_{xy} + p_\varnothing < p_x + p_y \right\} \end{aligned}$$

Here the goods are declared complementary if a supermodularity condition on p is satisfied (with p seen as a function defined on the set of consumption bundles $\{xy, x, y, \varnothing\}$). Supermodularity-type conditions capture complementarity when imposed on an objective function to be maximised (Topkis [29], Milgrom and Roberts [24], Milgrom and Shannon [25]). Note that the condition is equivalent to $p_{xy} + p_\varnothing > \frac{1}{2}$.

For illustration, consider table 1, calculated on the basis of Gentzkow's [18] data on 5-day readership of the online and print version of the Washington Post:

	Read print	Did not read print
Read online	0.137	0.043
Did not read online	0.447	0.373

Table 1: Washington Post, 5-day readership of online and print version (Gentzkow [18]).

In this case the above criteria are in deep conflict: the correlation and supermodularity criteria indicate that the two versions are complementary, the additivity criterion indicates that they are substitutes, and the maximin criterion indicates that they are independent. Therefore, in order to assess the different criteria, we propose an axiomatic analysis, examining the properties that criteria should possess given the interpretation.

4 Impossibilities

In this section we uncover the core conflict between the association and the revealed preference principle discussed in the introduction. We view duality as a prominent feature of all association-based definitions of complementarity and substitution:

Duality

1) If $(a, b, c, d) \in C$ then $(b, a, d, c) \in S$ and $(c, d, a, b) \in S$.

2) If $(a, b, c, d) \in S$ then $(b, a, d, c) \in C$ and $(c, d, a, b) \in C$.

Suppose that you have two datapoints p and q. Suppose that, whether x is consumed or not, y is consumed at q with the same frequency with which it is not consumed at p. If a datapoint were presented in table form, as in the introduction, q would be obtained from p by switching the rows. For example, q could be obtained when y is consumed only in weekends while p is obtained when y is consumed only in weekdays (assuming for simplicity that y's consumption pattern is the same whether x is consumed or not). In this sense q expresses a behaviour that is the 'opposite' of the behaviour at p. Then duality says that x and y are complements at p only if they are substitutes at q, and vice-versa. The same transition between C and S follows when the columns, rather than the rows, are switched.

Note that the two duality conditions imply a third one concerning the independence region, namely that if $(a, b, c, d) \in I$, then $(b, a, d, c) \in I$ and $(c, d, a, b) \in I$.

As noted in the introduction, duality is satisfied by all common measures of statistical association (like risk, relative risk, the odds ratio or the phi coefficient). It is important to emphasise that any given measure does not necessarily preserve the magnitude of association (while changing the sign) when a datapoint is transformed into its opposite in the sense described above. For example, applying the phi coefficient to the datapoints (a,b,c,d) and (b,a,d,c) gives, respectively, $\frac{ad-bc}{(a+d)(c+d)(a+c)(b+d)}$ and $\frac{bc-ad}{(b+c)(c+d)(b+d)(a+c)}$ which differ in general in absolute value. The common feature across measures is instead the sign change of the association following the row or column change in the contingency table, and this is precisely what the Duality axiom captures. It is this particular property that we take to be at the heart of the notion of association.

The revealed preference principle is formalised as follows:

Monotonicity

1) If $(a,b,c,d) \in C$, $(a',b',c',d') \in T$, $a' \geq a$, $b \geq b'$ and $c \geq c'$ then $(a',b',c',d') \in C$.
2) If $(a,b,c,d) \in S$, $(a',b',c',d') \in T$, $a \geq a'$, $b' \geq b$ and $c' \geq c$ then $(a',b',c',d') \in S$.

Monotonicity says that, if goods are complements, then an increase in joint consumption without an increase in single consumption cannot transform them into substitutes or render them independent, and vice-versa.

There do exist criteria that satisfy Duality and Monotonicity: for example, the supermodularity criterion above. However, this criterion is highly unsatisfactory, because it declares the goods complementary at any datapoint for which $p_\varnothing > \frac{1}{2}$, for *all* possible values of p_{xy}, p_x and p_y. This looks 'wrong' on two counts. First, it is desirable that no individual component of (a,b,c,d), should be decisive *by itself* to declare either complementarity or substitutability: the criterion should also respond to variations in the other components. Secondly, even granting the possibility of one component dictating the criterion, it is hard to justify the fact that it is a high value of $p_\varnothing$ on its own to mandate complementarity. The following axiom rules out this possibility.

Responsiveness

a) If $(a,b,c,d) \in C$, then $(a',b'c',d) \notin C$ for some a',b' and c' which sum up to $1-d$.
b) If $(a,b,c,d) \in S$, then $(a',b'c',d) \notin S$ for some a',b' and c' which sum up to $1-d$.

Responsiveness says that keeping the no-consumption frequency constant, some reshuffling of the consumption frequencies should be able to change the judgement of the criterion. The axiom is completely agnostic as to which operations on the first three coordinates of a datapoint should turn complements into non-complements and substitutes into non-substitutes.[8]

The supermodularity criterion fails Responsiveness as discussed above. The correlation criterion satisfies Responsiveness. It is also satisfies Duality, but, as noted in the introduction, it is not monotonic. The additivity criterion satisfies Monotonicity but fails part (2) of Duality. The maxmin criterion fails Duality.

It turns out that all possible criteria that satisfy Duality and Monotonicity must fail Responsiveness:

Theorem 1 *There exists no criterion that satisfies Monotonicity, Duality and Responsiveness.*

All proofs are in Appendix A, but the examples given previously show that the impossibility result of theorem 1 is tight, as no axiom among the three is implied by the remaining two.

The clash between association and monotonicity properties can also be observed from a different angle. In Appendix C we show that a variation of Monotonicity is also incompatible with Duality.

5 Possibilities

While we consider the two general principles (revealed preference and association) as basic minimal features of the notion of complementarity, we are not asserting that the specific axioms with which they can be expressed have universal validity. So we take the impossibility results we have obtained only as indicating a general tension between the principles, not a universal incompatibility. Weaker or different forms of the axioms may not only resolve the tension, but even be more appropriate in some contexts. For example, we could require in the Monotonicity axiom that the non-consumption component is held fixed across the two datapoints being compared. This would 'neutralise' (in the population interpretation of the data) accidental inclusions or exclusions

[8]In the next section, we will study two axioms in (T,J)-Duality and (S,J)-Duality which specify such operations.

of datapoints from individuals who just are not interested in the goods. Only shifts in the behaviour of people who actively consume the goods would be relevant for revealed preference. Then, the correlation criterion would become acceptable, satisfying all properties.

In this spirit, we now turn our attention to three arguably plausible criteria that meet the desired principles in a qualified or different specification of the axioms.[9] The *correlation* criterion is obtained by preserving Duality and weakening the monotonicity properties. For the other two criteria, *additivity and maxmin*, we instead retain monotonicity but vary the notion of duality. A duality operation produces the behaviour that is the 'opposite' of the one to which the operation is applied. A duality property in our context asserts that if a datapoint is classified in a certain way, then its dual is classified in the opposite way. This is an intuitive requirement but, as we will see, there are other reasonable ways to interpret the concept of 'opposite' behaviour beside the one we have considered, hence other reasonable versions of duality. In Appendix B we also offer a characterisation of the supermodularity criterion. However, we find this criterion conceptually implausible because of its lack of responsiveness discussed before.

Before moving on to the characterisations, note the following axiom which is satisfied by all three criteria of our interest (and, as shown in Appendix D, is implied by Monotonicity and Duality):

Symmetry:

1) If $(a,b,c,d) \in C$ then $(a,c,b,d) \in C$.

2) If $(a,b,c,d) \in S$ then $(a,c,b,d) \in S$.

Symmetry says that exchanging the amounts of single good consumptions is immaterial for the purpose of classifying goods into complementary or substitutes. Samuelson [27] considers its symmetry as one of the two major improvements of the Slutsky-Hicks-Allen-Schultz 'compensated' definitions compared to the 'uncompensated' one.

Note that, as for Duality, the two Symmetry conditions imply an analogous property for I: if $(a,b,c,d) \in I$, then $(a,c,b,d) \in I$. For if $(a,c,b,d) \notin I$, then one of the two conditions would yield $(a,b,c,d) \notin I$.

[9] These are the only criteria to have been suggested in a random survey of colleagues.

5.1 Correlation

Recall that according to the correlation criterion two goods are complements (substitutes) if their consumption is positively (negatively) correlated. While, as we have seen, the criterion fails Monotonicity, it satisfies a different monotonicity condition.

Scale Monotonicity
1) For any $(a, b, c, d) \in C$ and $(a', b', c', d') \in T$ such that $\frac{c'}{c} = \frac{d'}{d}$, if $\frac{a'}{a} \geq \frac{b'}{b}$, then $(a', b', c', d') \in C$.
2) For any $(a, b, c, d) \in S$ and $(a', b', c', d') \in T$ such that $\frac{c'}{c} = \frac{d'}{d}$, if $\frac{a'}{a} \leq \frac{b'}{b}$, then $(a', b', c', d') \in S$.
3) For any $(a, b, c, d) \in I$ and $(a', b', c', d') \in T$ such that $\frac{c'}{c} = \frac{d'}{d}$, if $\frac{a'}{a} > (=, <) \frac{b'}{b}$, then $(a', b', c', d') \in C (\in I, \in S)$.

Suppose that the total time spent reading the online version (alone or together with the print version) changes, but the time spent reading the online version alone decreases (resp., increases) as a proportion of the time spent reading both versions. Suppose also that the time left is allocated exactly in the same proportion as before between reading the print version and not reading either version. Parts (1) and (2) of Scale Monotonicity say that if the initial consumption pattern indicated complementarity (resp., substitutability), then the new consumption pattern should also indicate complementarity (resp., substitutability). Part (3) of the axiom states that if the goods are independent and joint consumption is increased, then the goods will remain independent only if single good consumption is rescaled by the same amount, and otherwise they will be classified in the obvious way.

In general, this version of the criterion fits situations where what is of interest is the comparison between fractions of time or of populations engaged in certain activities. For example, in the face of a declining trend in museum attendance, the local authority of the introduction may judge that stronger complementarity between a park and a museum has been revealed even after a decrease in joint consumption, provided this decrease is less than that of attendance of the museum alone.

Theorem 2 *A criterion satisfies Symmetry, Duality and Scale Monotonicity if and only if it is the correlation criterion.*

5.2 Additivity

As noted before, the additivity criterion is symmetric and monotonic. It also satisfies a notion of duality based on the operation of exchanging *Total* single good consumption with *Joint* consumption (with the joint consumption allocated to the two goods in proportion to the amounts that were consumed singly).

(T,J)-Duality For $\alpha = \frac{b}{b+c}$:
1) If $(a, b, c, d) \in C$, then $(b + c, \alpha a, (1 - \alpha)\, a, d) \in S$.
2) If $(a, b, c, d) \in S$, then $(b + c, \alpha a, (1 - \alpha)\, a, d) \in C$.

(T,J)-Duality says that the duality operation above transforms complementarity into substitution and viceversa. For example, if online and print newspapers are complements for a consumer who reads both versions two thirds of the time and a single version (either print or online) one fourth of the time, then they must be substitutes for a consumer who reads both versions one fourth of the time and the single versions two thirds of the time.

Note that if (a', b', c', d') is a (T,J)-dual to (a, b, c, d) (in the sense that $a' = b + c$, $b' = ab/(b + c)$, $c' = ac/(b + c)$ and $d' = d$), then (a, b, c, d) is dual to (a', b', c', d') in the same way as well. Consequently, (T,J)-Duality implies: if $(a, b, c, d) \in I$, then $\left(a + b, \frac{ab}{a+b}, \frac{ac}{a+b}, d\right) \in I$.

Also observe that (T,J)-Duality implies Responsiveness. Indeed, for any datapoint in region C or S, (T,J)-Duality gives an operation on the first three coordinates of the datapoint which will lead to a departure from the original region.

This notion of duality may be of relevance, for instance, for a firm considering whether to sell goods in a bundled or separated format: from its perspective, the 'opposite' of a consumption pattern is one that switches the two types of consumptions in the manner indicated above.

Next, consider the following variant of the Monotonicity property:

$I-$**Monotonicity**
1) If $(a, b, c, d) \in I$, $(a', b', c', d') \in T$, and $a' \geq a$, $b' + c' \leq b + c$, with at least one of the two inequalities strict, then $(a', b', c', d') \in C$.
2) If $(a, b, c, d) \in I$, $(a', b', c', d') \in T$, and $a' \leq a$, $b' + c' \geq b + c$, with at least one of the two inequalities strict, then $(a', b', c', d') \in S$.

Loosely, $I-$Monotonicity says that, if the goods are independent, then increasing joint consumption while decreasing single good consumption makes them complementary. This monotonicity property incorporates a responsiveness requirement: essentially, it implies that the Independence area is thin, as is the case for all standard definitions of complementarity/substitutability.

With these two additional properties we can state the following:

Theorem 3 *A criterion satisfies Monotonicity, $I-$Monotonicity and (T,J)-Duality if and only if it is the additivity criterion.*

5.3 Maxmin

The Maxmin criterion is a monotonic criterion that differs structurally from the other two because it has a thick independence region (so that it will not satisfy $I-$Monotonicity). It expresses yet a different notion of duality, according to which the choice behaviour opposite to a given one is defined by exchanging joint consumption with *only one* of the single good consumptions.

Ideally, we would like to impose a property of the following type. Suppose that online and print newspapers are complements for a consumer who, when he reads the print version, also reads the online version α% of the time; then, they must be substitutes for a consumer who, when he reads the print version, also reads the online version $(1-\alpha)$% of the time (and analogously starting from substitutability). It is a consequence of our characterisation below that this type of duality together with Symmetry and Monotonicity leads to another impossibility. So we use a weakened version of the property, which settles for merely switching out of the initial region after the duality operation.

(S,J)-Duality
1) If $(a,b,c,d) \in C$ then $(b,a,c,d) \notin C$ and $(c,b,a,d) \notin C$.
2) If $(a,b,c,d) \in S$ then $(b,a,c,d) \notin S$ and $(c,b,a,d) \notin S$.
3) If $(a,b,c,d) \in I$ and $a \neq b$ (resp. $a \neq c$) then $(b,a,c,d) \notin I$ (resp. $(c,b,a,d) \notin I$).

Note that, like (T,J)-Duality, (S,J)-Duality also implies Responsiveness. This notion of duality may be of relevance, for instance, for a firm considering whether or not to include an accessory with its main product.

Theorem 4 *A criterion* (C, I, S) *satisfies Symmetry, (S,J)-Duality and Monotonicity if and only if it is the maxmin criterion.*

The characterisations in theorems 2-4 show that one cannot 'take a stance' in the abstract regarding the three criteria. The relative appeal of the criteria will depend on the context where they are applied and on the objectives of the user. By singling out a handful of characterising properties our analysis may help in this kind of evaluations by making *all* the implications of each criterion transparent. Moreover, it forces an economist to make up his mind on exactly which 'consistency' principles he is willing to defend when making certain evaluations.

For example, suppose that an economist accepts, beside the symmetry of complementarity:

(1) if two online newspapers are complementary when the data show x% joint paper readers and y% single paper readers, then they must be substitutes if the percentages x and y were reversed.

Then our economist must also accept one the following two propositions:

(2) complementarity is indicated by the additivity criterion; or

(3) there exist datasets from which he would infer complementarity or independence but will infer substitutability after a conversion of single paper readers to joint reading.

Also the economist cannot simultaneously accept (1), reject (3), and at the same time defend:

(4) if two online newspapers are considered complementary for a dataset, then they must be substitutes if the percentages of joint paper readers and readers of one of the papers were reversed.

The net of such logical constraints uncovered in our results is non-obvious, and understanding it seems important both at the theoretical and the empirical level.

6 Concluding remarks

Complementarity in general is such a central concept in Economics that its study hardly needs to be motivated. Complementarity has deeply engaged at the theoretical level some of the giants of the profession (see the historical overview in Samuelson [27]).

Knowing whether goods are complements or substitutes (or neither) is of major practical importance in disparate areas: for example, suppliers must have information about complementarity when introducing new products or when pricing existing products; so do regulators to evaluate the competitiveness of a market; businesses may be reluctant to change a practice because of its complementarity with another; and so on. Finally, the concept of complementarity can be imaginatively applied in non-obvious contexts: for example, Becker and Murphy [6] introduced the idea of assimilating the theory of advertising into the theory of complementarity.

While statistical association is intuitively part of what it means for goods to 'go together' (and sheer correlation in consumption or usage data is often taken as a behavioural indicator of complementarity), we have shown that, in general, criteria for complementarity based on statistical association alone conflict with a basic monotonicity requirement that captures the 'revealed preference' aspect of complementarity. Our axiomatic analysis suggests that if monotonicity is considered primarily important, then different criteria (additivity and maxmin) may be preferable.

We have illustrated that the theoretical distinction between criteria is also relevant in practice, since correlation, additivity and maxmin give strongly contrasting indications using the data found in a leading application (Gentzkow [18]).

6.1 'True complementarity' vs. taste correlation

One criticism that could be made of our approach runs along the following lines.[10] Suppose that we observe a daily series of online/print news consumptions for an individual at fixed prices. Then each of our definitions will declare whether or not the two versions are complements or substitutes. A definition of complementarity based on random utility, on the other hand, will distinguish between the case in which the individual derives more utility from joint consumption than from the sum of the utilities of single-version consumption ('true complementarity'), and the case in which whenever the individual wakes up in a mood for reading online news he is also likely to be in the mood for reading the print version - he may simply wake up sometimes in the mood for news and some other times not in the mood for news, independently of the form in which they come ('correlation in taste'). Now, we could hold the choices constant and

[10] We thank Matthew Gentzkow for raising this important issue.

vary whether or not they indicate 'true complementarity' by changing the correlation in taste, whereas our definitions will fail to record these changes. But the key point to understand here is that the kind of complementarity we are trying to capture with our definitions is a related but separate concept from the random utility based concept of complementarity. The latter concept is tied to a specific assumption on the process that drives behaviour. But one could make different assumptions. Suppose for example that behaviour was driven instead by 'random consideration set' mechanisms of the type discussed in Brady and Rehbeck [11], Kovach and Ülkü [21], and Manzini and Mariotti [22]) in which preferences are deterministic but the subset of the feasible set that is actually considered by (or available to) the agent is random. Then we should separate 'true complementarity' from correlation in consideration rather than from correlation in taste, as our individual's mood is now expressed by a shock in consideration and not by a shock in taste: he wakes up sometimes considering both types of news media and sometimes not considering either, while always deriving utility from either in the same way. This leads to a different identification problem and likely to a different measurement of complementarity. If we only observe behaviour, which of the two measurements of 'true complementarity' should we regard as 'truer'?[11]

Our behavioural, model-free approach is designed *precisely* to cut through this type of modelling dilemmas. It takes choice data at face value. It serves a different purpose from 'standard' definitions: it suits the researcher or user who wishes to be noncommittal as to the mechanism that generates behaviour, which is treated as unknown and unknowable. We view this approach as a complement, rather than as a substitute, of the standard one.

6.2 If you could, should you use Hicks complementarity with stochastic choice data?

An interesting by-product of our approach is its implication that *if* prices were available and consumers were utility maximisers, and therefore the standard Hicks criterion of complementarity could be applied, it would conflict with Monotonicity.[12] This can be

[11]Note that some specifications of the consideration set model can be rewritten as RUMs while others cannot.

[12]We thank Jean-Pierre Dubé for pointing this out to us.

quickly seen through the following reasoning. Define Hicksian complementarity with stochastic demand in the standard way using expected demand. Gentzkow [18] shows that, for the simple two-good logit model, Hicksian complementarity thus defined is equivalent to the supermodularity of the utility function. On the other hand, we have shown that the correlation criterion fails Monotonicity.

Because Monotonicity seems such a fundamental principle, we argue that this reasoning shows that the Hicks criterion may be ill-applied to the context of random utility. That is, even when it *could* be applied because prices are available, the Hicks criterion *should not* be uncritically applied to stochastic choice data. To refer to table 1 again, suppose that the data changed from those in that table, to the following:

	Read print	Did not read print
Read online	0.487	0.043
Did not read online	0.447	0.023

This can mean for example that for every 1000 people, 350 of the non-readers converted to reading. Each of these new readers reads both versions and none of them reads just one version. Now the previously positive consumption correlation has turned negative ($\frac{0.487}{0.487+0.447} = 0.521 < 0.530 = 0.487 + 0.043$). On the assumption that consumption is generated by a logit random utility model, negative (resp., positive) consumption correlation is, as noted before, equivalent to Hicksian substitutability (resp., complementarity) using expected demand. But how can the conversion of over one third of the population to reading jointly the print and the online version be taken as diagnostic of a switch from complementarity to substitution in the nature of these goods? This seems a perverse conclusion.

The paradoxical behaviour of the Hicks criterion in this circumstance stems from the often neglected fact that a parameter (in this case the sign of the Hicksian elasticity) that is meaningful in the deterministic utility model may not carry the same meaning when applied to the perturbed utility version of the model. For an analogy, think of the fact that the Arrow-Pratt risk aversion coefficient of a deterministic utility cannot be taken as a measure of risk aversion in the random utility version of the model (Apesteguia and Ballester [2], Blavatsky [10], Wilcox [31]): in that case, an individual with a higher Arrow-Pratt coefficient of risk aversion may be more likely to accept risk than one with a lower coefficient. We hope that our analysis helps to reinforce the notion that the interpretation of certain features of a utility function (such as supermodularity) is not necessarily inherited by the stochastic version of that utility. Complementarity criteria for stochastic choice data should, even if based on utility maximisation, address directly the stochastic nature of the primitives.

6.3 More than two goods

Various additional complementarity questions arise if we have access to consumption frequencies with more than two goods. Let X be a finite set of goods with at least three elements and suppose that consumption data comes in the form of a probability distribution p over all subsets S of X. If we are interested in the complementarity between disjoint sets of goods, say S and T, then we can define the following cumulative probabilities: $p_{ST} := \sum_{A:S\cup T\subseteq A} p(A)$, $p_S := \sum_{A:S\subseteq A,\, T\cap A=\varnothing} p(A)$, $p_T := \sum_{A:T\subseteq A,\, S\cap A=\varnothing} p(A)$ and $p_\varnothing := \sum_{A:(S\cup T)\cap A=\varnothing} p(A)$. Hence p_{ST} is the probability that all goods in $S \cup T$ are consumed (with or without other goods) and $p_\varnothing$ is the probability that no good in $S \cup T$ is consumed. Now we can use any one of the criteria we have discussed on the 4-tuple $(p_{ST}, p_S, p_T, p_\varnothing)$. Likewise, if we are interested in the complementarity between goods x and y conditional on the consumption of good z, then we can use our criteria on the 4-tuple $(p_{xy|z}, p_{x|z}, p_{y|z}, p_{\varnothing|z})$ of conditional probabilities where $p_{xy|z} := \left(\sum_{A:\{x,y,z\}\subseteq A} p(A)\right) / \left(\sum_{A:z\in A} p(A)\right)$, $p_{x|z} := \left(\sum_{A:\{x,z\}\subseteq A,\, y\notin A} p(A)\right) / \left(\sum_{A:z\in A} p(A)\right)$, $p_{y|z} := \left(\sum_{A:\{y,z\}\subseteq A,\, x\notin A} p(A)\right) / \left(\sum_{A:z\in A} p(A)\right)$ and $p_{\varnothing|z} := \left(\sum_{A:z\in A,\, x,y\notin A} p(A)\right) / \left(\sum_{A:z\in A} p(A)\right)$.

A distinct possibility that arises with multiple goods is when complementarity is betwen more than two goods. For instance, Aral et al. [3] discuss a three-way complementarity between performance pay, human resource analytics and information technology. Our methods can be used to capture such complementarity, say between x, y and z, by checking that any two of the goods are complements conditional on the third good.

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Appendices

A Proofs

Theorem 1 *There exists no criterion that satisfies Monotonicity, Duality and Responsiveness.*

Proof. We start by proving:

Claim: Let (C, I, S) be a criterion that satisfies Duality and Monotonicity. If $(a, b, c, d) \in C$ then $(d, b, c, a) \in C$.

To prove the Claim, suppose $(a, b, c, d) \in C$ but $(d, b, c, a) \notin C$. By Monotonicity, then, $a > d$. Using Duality twice on (a, b, c, d), we get $(d, c, b, a) \in C$, and using it twice on (d, b, c, a) we get $(a, c, b, d) \notin C$. Again, by Monotonicity, $d > a$, a contradiction. $\square$

Returning to the proof of the main result, suppose that a criterion (C, I, S) satisfies Monotonicity, Duality and Responsiveness. Take $p = (a, b, c, d)$ such that $p \in C$. Let $\theta = \min\{a, b, c, d\}$, and note in particular that it must be $a < 1 - \theta$.

We will now show that for all $q = (a', b', c', d') \in T$, if $d' > 1 - \theta$ then $q \in C$. This contradicts Responsiveness and thus proves the impossibility. Take such a q, and let $r = (d', b', c', a')$. Note that $b' < \theta$ (otherwise, if $b' \geq \theta$, then $d' > 1 - b'$ and thus $b' + d' > 1$), and similarly $c' < \theta$. Then $d' > 1 - \theta > a$, $b' < \theta \leq b$ and $c' < \theta \leq c$. By Monotonicity, $r \in C$ and by the Claim above, we conclude that $q \in C$. $\blacksquare$

Theorem 2 *A criterion satisfies Symmetry, Duality and Scale Monotonicity if and only if it is the correlation criterion.*

Proof. That the three axioms are necessarily satisfied by the correlation definition is trivial. Suppose that a criterion (C, I, S) satisfies the three axioms. Begin by noting that

$$\begin{aligned}
(p_{xy}, p_x, p_y, p_\varnothing) &\in C \Leftrightarrow \frac{p_{xy}}{p_{xy} + p_y} > p_{xy} + p_x \\
&\Leftrightarrow p_{xy}\left(1 - p_{xy} - p_x - p_y\right) > p_x p_y \\
&\Leftrightarrow p_{xy} p_\varnothing > p_x p_y
\end{aligned}$$

and similarly $(p_{xy}, p_x, p_y, p_\varnothing) \in S \Leftrightarrow p_{xy}p_\varnothing < p_x p_y$. Then, since C, I and S form a partition, the result follows from the following three claims.

Claim 1: $C \subseteq \{(a,b,c,d) \in T : ad > bc\}$. Take $(a,b,c,d) \in C$ and suppose towards a contradiction that $ad \leq bc$. It follows that $\min\{a,d\} \leq \max\{b,c\}$. Symmetry and Duality imply that w.l.o.g. we can assume $d \leq a$ and $b \leq c$ so that $d \leq c$.

We will show that $b < a$. First note that $(d,c,b,a) \in C$ by Symmetry and Duality. By Scale Monotonicity (recall $d \leq c$) $(c,d,b,a) \in C$. Now using Symmetry and Duality again we get $(a,b,d,c) \in C$. Duality gives $(b,a,c,d) \in S$. Now if $b \geq a$, applying Scale Monotonicity, $(a,b,c,d) \in S$, a contradiction. Hence $b < a$ as we wanted to show.

In the rest of the proof, for any $q = (q_1, q_2, q_3, q_4) \in \mathbb{R}^4_{++}$, let $q^* = \frac{1}{\sum_{i=1}^4 q_i} q$, so that $q^* \in T$. We have $(a, \frac{ad}{bc}b, c, d)^* = (a, \frac{ad}{c}, c, d)^* \in C$ by Scale Monotonicity since $\frac{ad}{bc} \leq 1$. By Symmetry and Duality $(d, \frac{ad}{c}, c, a) \in C$. Applying Scale Monotonicity again, $(\frac{c}{d}d, \frac{c}{d}\frac{ad}{c}, c, a)^* = (c,a,c,a)^* \in C$. By Duality $(c,a,c,a)^* \in S$, a contradiction.

Claim 2: $S \subseteq \{(a,b,c,d) \in T : ad < bc\}$. Take $(a,b,c,d) \in S$ and suppose towards a contradiction that $ad \geq bc$. It follows that $\min\{b,c\} \leq \max\{a,d\}$. Symmetry and Duality say that w.l.o.g. we can assume $a \leq d$ and $c \leq b$ so that $c \leq d$.

We will show that $a < b$. First note that $(d,c,b,a) \in S$ by Symmetry and Duality. By Scale Monotonicity (recall $c \leq d$) $(c,d,b,a) \in S$. Now Using Symmetry and Duality again we get $(a,b,d,c) \in S$. Duality gives $(b,a,c,d) \in C$. Now if $a \geq b$, applying Scale Monotonicity, $(a,b,c,d) \in C$, a contradiction. Hence $a < b$ as we wanted to show.

We have $(a, \frac{ad}{bc}b, c, d)^* = (a, \frac{ad}{c}, c, d)^* \in S$ by Scale Monotonicity since $\frac{ad}{bc} \geq 1$. By Symmetry and Duality $(d, \frac{ad}{c}, c, a) \in S$. Applying Scale Monotonicity again, $(\frac{c}{d}d, \frac{c}{d}\frac{ad}{c}, c, a)^* = (c,a,c,a)^* \in S$. By Duality $(c,a,c,a)^* \in C$, a contradiction.

Claim 3: $I \subseteq \{ad = bc\}$. Take $(a,b,c,d) \in I$ and suppose towards a contradiction that $ad < bc$. Then set w.l.o.g. $d < c$ and consequently, using part (3) of Scale Monotonicity in an exact adaptation of Claim 1, $a > b$. The rest of the argument mirrors that in Claim 1. Similarly follow, with the obvious necessary modifications, the proof of Claim 2 if $ad > bc$. ■

Theorem 3 *A criterion satisfies Monotonicity, I−Monotonicity and (T,J)-Duality if and only if it is the additivity criterion.*

Proof. It is straightforward to show that the additivity criterion satisfies the three axioms. Suppose that (C, I, S) satisfies the three axioms. The result follows from the following three claims.

Claim 1: $C \subseteq \{(a,b,c,d) \in T : a > b + c\}$.

Suppose towards a contradiction that $(a,b,c,d) \in C$ and $a \leq b + c$. By (T,J)-Duality, $(b + c, ab/(b+c), ac/(b+c), d) \in S$. Since $a/\,(b+c) \leq 1$ this contradicts Monotonicity.

Claim 2: $S \subseteq \{(a,b,c,d) \in T : a < b + c\}$.

The proof is symmetric to that of Claim 1.

Claim 3: $I \subseteq \{(a,b,c,d) \in T : a = b + c\}$.

Suppose that $(a,b,c,d) \in I$ but $a < b + c$. (T,J)-Duality yields $(b + c, ab/(b + c), ac/(b+c), d) \in I$, which contradicts I-Monotonicity. Similarly if $a > b + c$. ∎

Theorem 4 *A criterion (C, I, S) satisfies Symmetry, (S,J)-Duality and Monotonicity if and only if is the maxmin criterion.*

Proof. Necessity is easily checked. In the other direction, suppose that (C, I, S) satisfies these three axioms. We will first show that $C \subseteq \{(a,b,c,d) : a > b, c\}$. Suppose, towards a contradiction, that $(a,b,c,d) \in C$ but $a \leq \max\{b,c\}$. By (S,J)-Duality $(b,a,c,d) \notin C$ and $(c,b,a,d) \notin C$, and this contradicts Monotonicity. Hence $a > b, c$. Similarly if $(a,b,c,d) \in S$ but $a \geq \min\{b,c\}$, then (S,J)-Duality yields $(b,a,c,d) \notin S$ and $(c,b,a,d) \notin S$, again contradicting Monotonicity. Hence $S \subseteq \{(a,b,c,d) : a < b, c\}$.

It remains to show that $I \subseteq \{\min\{b,c\} \leq a \leq \max\{b,c\}\}$ and the proof will follow the fact that (C, I, S) is a partition. To this end take some $(a,b,c,d) \in I$. There are three cases to consider regarding where the dual datapoint (b,a,c,d) lies.

Case 1: $(b,a,c,d) \in I$. Then by (S,J)-Duality (part 3), $a = b$ and therefore $\min\{b,c\} \leq a \leq \max\{b,c\}$.

Case 2: $(b,a,c,d) \in S$. By Monotonicity we must have $a > b$, giving $a \geq \min\{b,c\}$. By Symmetry on the other hand, $(a,c,b,d) \in I$. By (S,J)-Duality applied to (b,a,c,d), $(c,a,b,d) \notin S$. Now either $(c,a,b,d) \in I$, in which case $a = c$ by (S,J)-Duality (part 3), or $(c,a,b,d) \in C$, in which case $c > a$ by Monotonicity. Hence $a \leq c$ and therefore $a \leq \max\{b,c\}$.

Case 3: $(b,a,c,d) \in C$. By Monotonicity $b > a$ and therefore $a \leq \max\{b,c\}$. By Symmetry $(a,c,b,d) \in I$. By (S,J)-Duality $(c,a,b,d) \notin C$. Either $(c,a,b,d) \in I$, in which case $a = c$ by (S,J)-Duality (part 3), or $(c,a,b,d) \in S$, in which case $c < a$ by Monotonicity. Hence $a \geq c$ and therefore $a \geq \min\{b,c\}$. ∎

B A characterisation of the supermodularity criterion

T-Monotonicity For any two datapoints (a, b, c, d) and (a', b', c', d'):

1. if $(a, b, c, d) \in C$ and $b' + c' \leq b + c$, then $(a', b', c', d') \in C$
2. if $(a, b, c, d) \in S$ and $b' + c' \geq b + c$, then $(a', b', c', d') \in S$
3. if $(a, b, c, d) \in I$ and $b' + c' < (>) b + c$, then $(a', b', c', d') \in C (\in S)$

That is, a decrease in *Total* single-good consumption cannot change complements into non-complements. Likewise, an increase in total single-good consumption cannot change substitutes into non-substitutes.

Theorem 5 *: A criterion (C, I, S) satisfies T-Monotonicity and Duality if and only if it is the supermodularity criterion.*

Proof. Necessity of the axioms is obvious. Suppose (C, I, S) satisfies the axioms. The following three steps establish that (C, I, S) is the supermodularity criterion.

Step 1. Take $(a, b, c, d) \in C$. Suppose $a + d \leq b + c$. By S-Monotonicity, $(b, a, d, c) \in C$ which violates Duality. Hence $C \subseteq \{(a, b, c, d) : a + d > b + c\}$.

Step 2. Take $(a, b, c, d) \in S$. Suppose $a + d \geq b + c$. By S-Monotonicity $(b, a, d, c) \in S$ which violates Duality. Hence $S \subseteq \{(a, b, c, d) : a + d < b + c\}$.

Step 3. Take $(a, b, c, d) \in I$. By Duality $(b, a, d, c) \in I$. If $a + d < b + c$, then $(b, a, d, c) \in S$ by S-Monotonicity and if $a + d > b + c$, then $(b, a, d, c) \in C$ by S-Monotonicity. Hence $a + d = b + c$ and $I \subseteq \{(a, b, c, d) : a + d = b + c\}$. ■

C Incompatibility between $I-$Monotonicity and Duality

In this Appendix we show that $I-$Monotonicity and Duality are incompatible.

Theorem 6 *There exists no criterion that satisfies $I-$Monotonicity and Duality.*

Proof: Suppose that (C, I, S) satisfies the axioms. Take $p = (a, a, b, b)$ with $b > a$. It cannot be $(a, a, b, b) \in S$, for then by Duality $(a, a, b, b) \in C$, a contradiction. Similarly, it cannot be $(a, a, b, b) \in C$. Then $(a, a, b, b) \in I$. By Duality $(b, b, a, a) \in I$. But this contradicts $I-$Monotonicity. ■

D Monotonicity and Duality imply Symmetry

Proposition 1 *If a criterion satisfies Duality and Monotonicity, then it satisfies Symmetry.*

Proof: Suppose that a criterion (C, I, S) satisfies Duality and Monotonicity but fails Symmetry. We consider four cases.

Case 1: $(a, b, c, d) \in C$ but $(a, c, b, d) \in S$. By Duality $(b, a, d, c) \in S$ and $(c, a, d, b) \in C$. By Monotonicity $c > b$. Applying Duality to the first two datapoints we get $(c, d, a, b) \in S$ and $(b, d, a, c) \in C$, and then by Monotonicity $b > c$, a contradiction.

Case 2: $(a, b, c, d) \in C$ but $(a, c, b, d) \in I$. By Duality $(b, a, d, c) \in S$ and $(c, a, d, b) \in I$. By Monotonicity $c > b$. Applying Duality to the first two datapoints we get $(c, d, a, b) \in S$ and $(b, d, a, c) \in I$, and then by Monotonicity $b > c$, a contradiction.

Cases 3 and 4 where $(a, b, c, d) \in S$ and $(a, c, b, d) \notin S$ are similar. ■