Quasitriangular structure and twisting of the 3D bicrossproduct model

S. Majid and P. K. Osei

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Abstract

We show that the bicrossproduct model $C[SU_2^*]{\blacktriangleright\!\!<}U(su_2)$ quantum Poincaré group in 2+1 dimensions acting on the quantum spacetime $[x_i,t]=\imath\lambda x_i$ is related by a Drinfeld and module-algebra twist to the quantum double $U(su_2){>\!\!\triangleright}C[SU_2]$ acting on the quantum spacetime $[x_\mu,x_\nu]=\imath\lambda\epsilon_{\mu\nu\rho}x_\rho$. We obtain this twist by taking a scaling limit as $q\to 1$ of the q-deformed version of the above, where it corresponds to a previous theory of q-deformed Wick rotation from q-Euclidean to q-Minkowski space. We also recover the twist result at the Lie bialgebra level.

1 Introduction and Motivation

It is now widely accepted that quantum gravity effects may plausibly lead to spacetime, even flat spacetime, being better modelled by noncommuting coordinates than classical ones. One such model that is is clearly related to 3D quantum gravity (without cosmological constant and with point sources) is the angular momentum algebra $[x_\mu,x_\nu]=\imath\lambda\epsilon_{\mu\nu\rho}x_\rho$ as spacetime, as first proposed by 't Hooft in the mid 1990s[1, 2, 3, 4]. Another well-known model from the mid 1990s is the Majid-Ruegg 'bicrossproduct' model[5, 6] with spacetime $[x_i,t]=\imath\lambda x_i$, which we take in the 3D case. Both models are notably for having Poincaré quantum group symmetries of interest in their own right[2, 7]. In the present paper we show that these well-known models are in fact related and in some sense equivalent via a module-algebra (or Drinfeld-type) twist.

Previously it was explained at the q-deformed level in [8] that these two models are quantum Born reciprocal or 'semidual' aspects of 3d quantum gravity and that at this q-deformed level they are also related by twisting and hence in some sense self-dual up to twisting equivalence. However, the isomorphisms used were highly singular as $q\to 1$, so only apply strictly with cosmological constant. The interpretation of q-deformation here as introducing a cosmological constant is clear from the close link between the quantisation of the relevant Chern-Simons theory (giving the Turaev-Viro invariant), the relevant q-deformation quantum group and the relevant WZNW model of conformal field theory[9, 10]. In this context, quantum Born reciprocity interchanges the cosmological and Planck scales for a fixed value of q-deformation parameter and the quantum double $D(U_q(su_2))=U_q(so_{1,3})$ with the quantum group $U_q(su_2)^{\rm cop}{\blacktriangleright\!\!<}U_q(su_2)=U_q(so_4)$ at the

level of isometry quantum group, so these are twisting equivalent, a result first introduced in [11] as 'quantum Wick rotation', see [12]. Our surprising new result is that by working out the structures in great detail and carefully taking the $q \to 1$ limit while at the same time scaling the generators, i.e in a contraction limit, a remnant of the result survives in the form a module algebra twist between the above two quantum spacetimes and their Poincaré quantum groups $D(U(su_2))$ and $C[SU_2^*]{\blacktriangleright\!\!\triangleleft}U(su_2)$ respectively. The role of $D(U(su_2))$ in particular for constructing the states of 3d quantum gravity with point sources is well established and we refer to [4, 8] for an introduction. That a scaling 'contraction' limit of $U_q(so_4)$ gives a quantum Euclidean group was first pointed out in [13] and this is presumably isomorphic to $C[SU_2^*]{\blacktriangleright\!\!\triangleleft}U(su_2)$ in the same way as the 4d quantum Poincaré quantum group proposed in [7] by contraction of $U_q(so_{2,3})$ was shown in [5] to be a bicrossproduct $C[\mathbb{R} \ltimes \mathbb{R}^3]{\blacktriangleright\!\!\triangleleft}U(so_{1,3})$.

Our scaling limit result is striking because the two quantum spacetimes models appear very different and have always been treated as such; one quantum spacetime is the enveloping algebra of a simple Lie algebra and the other of a solvable one. One Poincaré quantum group is quasitriangular while bicrossproducts are not usually quasitriangular, although the 3d one in [13] is, a result which in our version is now explained by twisting as this preserves quasitriangularity. Moreover, whereas the quasitriangular structure of the double exists formally, it does not take a usual algebraic form as the exponential of generators, whereas our universal R-matrix on the bicrossproduct does and this implies such a form also for the double by twisting. Similarly, when quantum spacetimes are related by a module algebra twist then their covariant noncommutative differential geometry is related by twisting[14, 15] and hence that must also be the case here: For the spin model the smallest covariant calculus is known to be 4D [2] and for the standard bicrossproduct models it is known to be one dimension higher than classical [16], so again 4D but now this is explained by our twisting result. Similarly, the construction of particle state representations of $D(U(su_2))$ by the Wigner little group method in [8] should have a parallel on the bicrossproduct model side via twisting. Such possible applications will be considered elsewhere.

The paper starts in Section 2 with some general Hopf algebra constructions which underly the quantum Wick rotation[11] and semidualisation in [17, 8] but which were not given so explicitly before. We carefully specialise these to $U_q(su_2)$, again giving all constructions in explicit detail in Sections 3.1–3.5. These exact formulae then allow us in Section 3.6 to take the $q \to 1$ limit with suitably scaled generators. This is a rather tricky process due to $1/(1-q^{-2})$ singularities but we remarkably do obtain finite results, which we then verify explicitly, see Corollary 3.1. Section 4 rounds off the paper with the Poisson-Lie or semiclassical level version of our results in line with [19] and mainly as a further check of our calculations (notably, we show that we recover the expected Lie bialgebra double r-matrices). Our results relate to a different Lie bialgebra contraction than [20] but the latter may emerge as a different limit of our results. Another direction for future work is that $U_q(su_2)$ as quantum spacetime is a unit hyperboloid in q-Minkowski space and as such its constant-time slices give the 2-parameter Podles spheres [21], all of which may have a parallel on the bicrossproduct model side of the twisting. Another topic for further study is to look carefully at the different signatures (so the spin model

is the version for Euclideanised 3D quantum gravity) and the relationship between the different real forms as reflected in the applicable $*$-structures.

2 Explicit Hopf algebra isomorphisms

This section brings together two different contexts in the book [12]. The first, about semidualisation, was explained in [8] in the present context of 3d quantum gravity while the second about twisting was explained in [11] in the context of quantum Wick rotation. It was also outlined in [8] how to bring these together but now we need to work out the underlying isomorphisms rather explicitly, which is not easy from the literature.

We use the conventions for Hopf algebras in [12] namely a Hopf algebra or 'quantum group' H is both an algebra and a coalgebra, with 'coproduct' $\Delta : H \to H \otimes H$ which is an algebra homomorphism. There is also a counit $\epsilon : H \to k$ if we work over k and an 'antipode' $S : H \to H$ defined by $(Sh_{(1)})h_{(2)} = h_{(1)}Sh_{(2)} = \epsilon(h)$ for all $h \in H$ and notation $\Delta h = h_{(1)} \otimes h_{(2)}$. We shall refer to a *covariant system* (H, A) meaning a Hopf algebra H acting on an algebra A as a module algebra, i.e., in the left handed case,

$$h{\triangleright}(ab) = (h_{(1)}{\triangleright}a)(h_{(2)}{\triangleright}b), \quad h{\triangleright}1 = \epsilon(h).$$

where $\triangleright$ is a left action. There is then a left cross-product algebra $A{\rtimes}H$. We refer to [12] for details. We denote by H^* a suitable dual Hopf algebra with dual pairing given by a non degenerate bilinear map $\langle\,,\,\rangle$ and $H^{\rm cop}, H^{\rm op}$ denote taking the flipped coproduct or flipped product. As an easy exercise, if H acts covariantly on A from the right then

$$h{\triangleright}a = a{\triangleleft}S^{-1}h \tag{2.1}$$

is a left action of H on $A^{\rm op}$ as another covariant system.

2.1 Semidualisation and the quantum double

(i) A *double crossproduct* Hopf algebra $H_1{\bowtie}H_2$ can be thought of as a Hopf algebra H which factorises into two sub-Hopf algebras built on $H_1 \otimes H_2$ as a vector space. By factorisation, we mean a map $H_1 \otimes H_2 \to H$ as an isomorphism of linear spaces. One can then naturally extract the actions $\triangleright : H_2 \otimes H_1 \to H_1$ and $\triangleleft : H_2 \otimes H_1 \to H_2$ of each Hopf algebra on the vector space of the other defined by $(1 \otimes a).(h \otimes 1) = a_{(1)}{\triangleright}h_{(1)} \otimes a_{(2)}{\triangleleft}h_{(2)}$ for the product viewed on $H_1 \otimes H_2$ obeying some further compatibility properties (one says that one has a matched pair of interacting Hopf algebras). Conversely given such data one can reconstruct the algebra of $H_1{\bowtie}H_2$ from these actions as a double (both left and right) cross product. The coproduct of $H_1{\bowtie}H_2$ is the tensor one given by the coproduct of each factor and there is a canonical right action of this Hopf algebra on the vector space of H_2 which respects the coalgebra structure of H_2 and thus provides in a canonical way a covariant left action of $H_1{\bowtie}H_2$ on H_2^* as an algebra. Here H_1 acts on H_2^* by dualising the above right action $\triangleleft$ on H_2, and H_2 acts on H_2^* by the coregular action $a{\triangleright}\phi = \phi_{(1)}\langle a, \phi_{(2)}\rangle$. Hence we have a covariant system $(H_1{\bowtie}H_2, H_2^*)$ and an associated cross product $H_2^*{\rtimes}(H_1{\bowtie}H_2)$. Further details are in [12] and earlier works by the first author.

(ii) The *semidual* of this picture associated to the same matched pair data was introduced by the first author, see [12] for details, and is constructed by dualising the data involving H_2 to give a bicrossproduct Hopf algebra $H_2^*{\blacktriangleright\!\!<}H_1$ which then acts covariantly on H_2 from the right as an algebra as the semidual covariant system $(H_2^*{\blacktriangleright\!\!<}H_1, H_2)$. The remarkable fact is that $(H_2^*{\blacktriangleright\!\!<}H_1){>\!\!\!\triangleleft}H_2 = H_2^*{>\!\!\!\triangleleft}(H_1{\triangleright\!\!<}H_2)$ as algebras, i.e. the combined system is the same actual algebra but its interpretation is different in that the role of spacetime coordinates H_2 and momentum H_2^* coordinates in the first case is reversed in the other, with rotations H_1 the same. This is the B-model semidualisation referred to in [8]. There is equally well an A-model semidualisation where we dualise H_1 to obtain $H_2{\triangleright\!\!<}H_1^*$ acting on the left on H_1 while $H_1{\triangleright\!\!<}H_2$ acts on the right on H_1^* and the two covariant systems again have the same cross products, $H_1{>\!\!\!\triangleleft}(H_2{\triangleright\!\!<}H_1^*) = (H_1{\triangleright\!\!<}H_2){>\!\!\!\triangleleft}H_1^*$. These ideas go back to the first author as a new foundation ('quantum born reciprocity') proposed for quantum gravity namely that one can swap position and momentum generators in the algebraic structure [17].

We will particularly need details of the B-model which were not provided explicitly in [8]. Starting with a matched pair H_1, H_2 acting on each other the left action $\triangleright : H_1 \otimes H_2^* \to H_2^*$ of H_1 on H_2^* and a right coaction $\Delta_R : H_1 \to H_1 \otimes H_2^*$ of H_2^* on H_1 are define by

$$(h{\triangleright}\phi)(a) := \phi(a{\triangleleft}h), \quad \phi \in H_2^*, \quad a \in H_2 \quad h \in H_1$$

$$h^0\langle h^1, a\rangle = a{\triangleright}h, \quad h \in H_1, \quad a \in H_2, \quad \Delta_R h = h^0 \otimes h^1 \in H_1 \otimes H_2^*.$$

These define the bicrossproduct $H_2^*{\blacktriangleright\!\!<}H_1$ by a left handed cross product $H_2^*{>\!\!\!\triangleleft}H_1$ as an algebra and a right handed cross coproduct $H_2^*{\blacktriangleright\!\!<}H_1$ as coalgebra:

$$(\phi \otimes h)(\psi \otimes g) = \phi(h_{(1)}{\triangleright}\psi) \otimes h_{(2)}g, \qquad h \in H_1, \quad \phi, \psi \in H_2^* \tag{2.2}$$

$$\Delta(\phi \otimes h) = (\phi_{(1)} \otimes h^0{}_{(1)}) \otimes (\phi_{(2)} h^1_{(1)} \otimes h_{(2)}) \tag{2.3}$$

The canonical right action of $H_2^*{\blacktriangleright\!\!<}H_1$ on H_2 is

$$a{\triangleleft}(\phi \otimes h) = a_{(2)}{\triangleleft}h\langle \phi, a_{(1)}\rangle, \quad \forall h \in H_1, \quad a \in H_2, \quad \phi \in H_2^*. \tag{2.4}$$

Note that $H_2^* \otimes 1$ and $1 \otimes H_1$ appear as subalgebras with cross relations

$$h\psi = (1 \otimes h)(\psi \otimes 1) = h_{(1)}{\triangleright}\psi \otimes h_{(2)} = (h_{(1)}{\triangleright}\psi \otimes 1)(1 \otimes h_{(2)}) = (h_{(1)}{\triangleright}\psi)h_{(2)}$$

where we identify $h = 1 \otimes h$ and $\psi = \psi \otimes 1$.

(iii) We now apply the above construction to the specific case of the Drinfeld quantum double $D(H) = H{\triangleright\!\!<}H^{*\mathrm{op}}$ due to [18] and viewed as an example of a double crossproduct from work of the first author, see [12] for details. Here the right action of H on $H^{*\mathrm{op}}$ and the left action of $H^{*\mathrm{op}}$ on H are given respectively by

$$a{\triangleleft}h = a_{(2)}\langle h_{(1)}, a_{(1)}\rangle\langle Sh_{(2)}, a_{(3)}\rangle, \quad a{\triangleright}h = h_{(2)}\langle h_{(1)}, a_{(1)}\rangle\langle Sh_{(3)}, a_{(2)}\rangle, \ h \in H, \ a \in H^{*\mathrm{op}}. \tag{2.5}$$

The double cross product $H{\triangleright\!\!<}H^{*\mathrm{op}}$ then comes out as

$$(h \otimes a).(g \otimes b) = hg_{(2)} \otimes ba_{(2)}\langle g_{(1)}, a_{(1)}\rangle\langle Sg_{(3)}, a_{(3)}\rangle, \quad h, g \in H, \ a, b \in H^*, \tag{2.6}$$

with the tensor product coproduct. This Hopf algebra canonically acts on $(H^{*\mathrm{op}})^* = H^{\mathrm{cop}}$ from the left as an algebra. The action is

$$(h\otimes a)\triangleright\phi = \langle\phi_{(1)}, a\rangle h\triangleright\phi_{(2)}, \quad \phi\in H^{\mathrm{cop}} \tag{2.7}$$

in terms of the coproduct of H and the action $\triangleright$ in (2.8).

Semidualising, the left action of H on H^{cop} already referred to and the right coaction of H^{cop} on H are respectively

$$h\triangleright\phi = h_{(1)}\phi Sh_{(2)} = \mathrm{Ad}_h(\phi), \quad \Delta_R h = h_{(2)}\otimes h_{(1)}Sh_{(3)} \tag{2.8}$$

so that the product of $H^{\mathrm{cop}}{\blacktriangleright\!\!\triangleleft}H$ is therefore

$$(\phi\otimes h)(\psi\otimes g) = \phi h_{(1)}\triangleright\psi\otimes h_{(2)}g, \quad h\in H,\ \ \phi,\psi\in H^{\mathrm{cop}} \tag{2.9}$$

as the standard cross product $H_{\mathrm{Ad}}{\triangleright\!\!<}H$ and the coproduct is

$$\Delta(\phi\otimes h) = \phi_{(2)}\otimes h_{(2)}\otimes\phi_{(1)}h_{(1)}Sh_{(3)}\otimes h_{(4)} \tag{2.10}$$

in terms of coproducts of H. This Hopf algebra acts covariantly on $H^{*\mathrm{op}}$ from the right according to

$$a\triangleleft(\phi\otimes h) = \langle\phi h_{(1)}, a_{(1)}\rangle a_{(2)}\langle Sh_{(2)}, a_{(3)}\rangle,\ h\in H,\ \phi\in H^{\mathrm{cop}},\ a\in H^*. \tag{2.11}$$

Using (2.1), and correctly using the inverse antipode of the bicrossproduct determined by the coproduct (2.10) gives the covariant left action of the bicrossproduct quantum group on H^* as

$$(\phi\otimes h)\triangleright a = \langle Sh_{(1)}S\phi, a_{(1)}\rangle a_{(2)}\langle h_{(2)}, a_{(3)}\rangle. \tag{2.12}$$

In summary, the semidual of the left covariant system $(D(H), H^{\mathrm{cop}})$ is the right covariant system $(H^{\mathrm{cop}}{\blacktriangleright\!\!\triangleleft}H, H^{*\mathrm{op}})$, which is $(H^{\mathrm{cop}}{\blacktriangleright\!\!\triangleleft}H, H^*)$ as a left covariant system with action (2.12). This is essentially as in [12] where we denoted $H^{\mathrm{cop}}{\blacktriangleright\!\!\triangleleft}H = M(H)$ the 'mirror product' but now in our current conventions and, critically, keeping track of algebras on which our Hopf algebras act.

(iv) Finally, we observe as a right-left flipped version of [12, Prop. 6.2.9] that there is a Hopf algebra isomorphism

$$\theta_1 : H^{\mathrm{cop}}\otimes H\to H^{\mathrm{cop}}{\blacktriangleright\!\!\triangleleft}H,\ \theta_1(\phi\otimes h) = \phi Sh_{(1)}\otimes h_{(2)}, \quad \theta_1^{-1}(\phi\otimes h) = \phi h_{(1)}\otimes h_{(2)} \tag{2.13}$$

under which the right action of $H^{\mathrm{cop}}{\blacktriangleright\!\!\triangleleft}H$ on $H^{*\mathrm{op}}$ by (2.11) is isomorphic to a right action of $H^{\mathrm{cop}}\otimes H$ on $H^{*\mathrm{op}}$ by

$$a\triangleleft(\phi\otimes h) = a\triangleleft_{H^{\mathrm{cop}}{\blacktriangleright\!\!\triangleleft}H}\theta_1(\phi\otimes h) = a_{(2)}\langle\phi, a_{(1)}\rangle\langle Sh, a_{(3)}\rangle.$$

and by observation (2.1) this is equivalent to $H^{\mathrm{cop}}\otimes H$ acting on the left on H^* by

$$(\phi\otimes h)\triangleright a = a\triangleleft S^{-1}(\phi\otimes h) = a\triangleleft(S\phi\otimes S^{-1}h) = a_{(2)}\langle S\phi, a_{(1)}\rangle\langle h, a_{(3)}\rangle \tag{2.14}$$

and this is also $\theta_1(\phi\otimes h)\triangleright a$ acting by (2.12).

In summary, the semidual of the left covariant system $(D(H), H)$ acting by (2.7) is isomorphic to the left covariant system $(H^{\mathrm{cop}}\otimes H, H^*)$ acting by (2.14). This action is equivalent to a left action of H and a right action of another copy of H it is H^* with a natural H-bimodule structure afforded by the coproduct (the Hopf algebra version of left and right derivatives on H^*).

2.2 Twisting of module algebras and quantum Wick rotation

(i) We recall following Drinfeld that a quasitriangular Hopf algebra is a pair $(H,\mathcal{R})$, where H is a Hopf algebra and $\mathcal{R}$ is an invertible element of $H\otimes H$ satisfying

$$(\Delta\otimes\mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23},\quad (\mathrm{id}\otimes\Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$$
$$\Delta^{\mathrm{cop}}(h) = \mathcal{R}(\Delta h)\mathcal{R}^{-1},\quad h\in H.$$

In this case $\mathcal{R}$ obeys

$$(\epsilon\otimes\mathrm{id})(\mathcal{R}) = (\mathrm{id}\otimes\epsilon)(\mathcal{R}) = 1,$$
$$(S\otimes\mathrm{id})\mathcal{R} = \mathcal{R}^{-1},\quad (\mathrm{id}\otimes S)\mathcal{R}^{-1} = \mathcal{R},$$
$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \tag{2.15}$$

where we write $\mathcal{R} = \mathcal{R}^{[1]}\otimes\mathcal{R}^{[2]}$ with the notation that

$$\mathcal{R}_{ij} = 1\otimes ...\otimes\mathcal{R}^{[1]}\otimes 1\otimes ...\otimes\mathcal{R}^{[2]}\otimes ...\otimes 1$$

is the element of $H\otimes H...\otimes H$ which is $\mathcal{R}$ in the i'th and j'th factors. The identity (2.15) is known as the *quantum Yang-Baxter Equation* (QYBE) and on account of this $\mathcal{R}$ is also called a Universal R-matrix.

Next, an element $\chi\in H\otimes H$ for any Hopf algebra H is called a twisting 2-cocycle [12] if

$$\chi_{12}(\Delta\otimes\mathrm{id})\chi = \chi_{23}(\mathrm{id}\otimes\Delta)\chi,\quad (\epsilon\otimes\mathrm{id})\chi = 1$$

and in this case there is a new Hopf algebra H_χ with the same algebra and [12],

$$\Delta_\chi h = \chi(\Delta h)\chi^{-1}\quad \mathcal{R}_\chi = \chi_{21}\mathcal{R}\chi^{-1},\quad S_\chi h = U(Sh)U^{-1}\ \ \forall h\in H_\chi, \tag{2.16}$$

where $U = \cdot(\mathrm{id}\otimes S)\chi$ is invertible. Moreover, if H acts covariantly on A from the left then H_χ acts covariantly on a new algebra A_χ with product

$$a\cdot_\chi b = \cdot(\chi^{-1}\triangleright(a\otimes b)). \tag{2.17}$$

This cocycle twisting theory was introduced by the first author in [22, 11] and other works from that era (Drinfeld did not consider 2-cocycles or module algebra twists but rather conjugation by general elements χ in the category of quasi-Hopf algebras). Clearly, if H is quasitriangular and we take 2-cocycle $\chi=\mathcal{R}$, then $H_\chi = H^{\mathrm{cop}}$.

(ii) Following [11], we similarly see that $H^{\mathrm{cop}}\otimes H$ acting on H^* by (2.14) twists via $\chi_1 = \mathcal{R}_{13}^{-1}$ to $H\otimes H$ acting on a new algebra, which we will denote $H^{\boxdot}$, with product

$$a\boxdot b = \cdot\mathcal{R}\triangleright(a\otimes b) = ((\mathcal{R}^{[1]}\otimes 1)\triangleright a)((\mathcal{R}^{[2]}\otimes 1)\triangleright b) = \mathcal{R}(a_{(1)}\otimes b_{(1)})a_{(2)}b_{(2)}$$

Thus the covariant system $(H^{\mathrm{cop}}\otimes H, H^*)$ at the end of Section 2.1 twists to $(H\otimes H, H^{\boxdot})$ by χ_1.

Moreover, the further twist of $H\otimes H$ by the 2-cocycle $\chi_2 = \mathcal{R}_{23}^{-1}$ gives a Hopf algebra which we will denote $H{\blacktriangleright\!\!\blacktriangleleft}_{\mathcal{R}}H$ (it is technically a double cross coproduct) acting

covariantly on an algebra $\underline{H^*}$ with

$$\begin{aligned} a\underline{\cdot}b &= \boxdot(\mathcal{R}_{23}\triangleright(a\otimes b)) \\ &= a_{(1)}\langle\mathcal{R}^{[1]},a_{(2)}\rangle \boxdot b_{(2)}\langle S\mathcal{R}^{[2]},b_{(1)}\rangle \\ &= a_{(2)}b_{(3)}\mathcal{R}(a_{(1)}\otimes b_{(2)})\mathcal{R}(a_{(2)}\otimes Sb_{(1)}) \\ &= a_{(2)}b_{(3)}\mathcal{R}(Sa_{(1)}\otimes Sb_{(2)})\mathcal{R}(a_{(3)}\otimes Sb_{(1)}) \\ &= a_{(2)}b_{(2)}\mathcal{R}(Sa_{(1)}a_{(3)}\otimes Sb_{(1)}) \end{aligned}$$

for all $a,b\in H^*$, where we view $\mathcal{R}$ by evaluation as a map on $H^{*\otimes 2}$ and use the axioms of $\mathcal{R}$ in dual form. This product makes $\underline{H^*}$ with its unchanged coproduct into a braided-Hopf algebra as part of the theory of transmutation so the result in [11] was that this can be seen as a twist (namely by $\chi=\chi_2\chi_1=\mathcal{R}_{23}^{-1}\mathcal{R}_{13}^{-1}=(\Delta\otimes\mathrm{id})\mathcal{R}^{-1}$).

(iii) Moreover it is known [12, Thm 7.3.5] that there is a Hopf algebra map

$$\theta_2: D(H)\to H{\blacktriangleright\!\!\blacktriangleleft}_{\mathcal{R}}H,\quad \theta_2(h\otimes a)=h_{(1)}\mathcal{R}^{-[2]}\otimes h_{(2)}\mathcal{R}^{[1]}\langle a,\mathcal{R}^{-[1]}\mathcal{R}^{[2]}\rangle \tag{2.18}$$

according to inclusions $i=\Delta$ and $j(a)=(\mathrm{id}\otimes\mathrm{id}\otimes a)(\mathcal{R}_{31}^{-1}\mathcal{R}_{23})$ of $H,H^{*\mathrm{op}}$ in $H{\blacktriangleright\!\!\blacktriangleleft}_{\mathcal{R}}H$. Note also that the latter has at least a couple of interesting quasitriangular structures built from $\mathcal{R}$ namely,

$$\mathcal{R}_D=\mathcal{R}_{41}^{-1}\mathcal{R}_{31}^{-1}\mathcal{R}_{24}\mathcal{R}_{23}=(\chi_2)_{21}\mathcal{R}_{13}^{-1}\mathcal{R}_{24}\chi_2^{-1},\ \ \mathcal{R}_L=\mathcal{R}_{41}^{-1}\mathcal{R}_{13}\mathcal{R}_{24}\mathcal{R}_{23}=(\chi_2)_{21}\mathcal{R}_{31}\mathcal{R}_{24}\chi_2^{-1} \tag{2.19}$$

with $\mathcal{R}_D$ the image under θ_2 of the canonical quasitriangular structure of $D(H)$ (at least if H is finite dimensional so that the latter is defined). In the factorisable case the map θ_2 is an isomorphism of Hopf algebras, where 'factorisable' means $Q=\mathcal{R}_{21}\mathcal{R}$ viewed by evaluation as a map $Q:H^*\to H$ by $Q(a)=\langle a,Q^{[1]}\rangle Q^{[2]}$ is an isomorphism. This holds formally for the standard quantum groups associated to semisimple Lie algebras.

Pulling back under θ_2 we compute using the Hopf algebra and quasitriangularity axioms that $D(H)=H\bowtie H^{*\mathrm{op}}$ acts covariantly on $\underline{H^*}$ by

$$(h\otimes a)\triangleright b=\theta_2(h\otimes a)\triangleright_{H^{\mathrm{cop}}\otimes H}b=b_{(3)}\langle h,(Sb_{(2)})b_{(4)}\rangle\mathcal{R}(a_{(1)},b_{(1)})\mathcal{R}(b_{(5)},a_{(2)}) \tag{2.20}$$

This is also the action on of $\theta_1\theta_2(h\otimes a)\in H^{\mathrm{cop}}{\blacktriangleright\!\!\blacktriangleleft}H$ on b for the action (2.12).

Lemma 2.1 *$Q:\underline{H^*}\to H$ is a map of covariant algebras, intertwining the action of $D(H)$ in (2.20) with its action on H in (2.7).*

Proof It is known [12, Prop 7.4.3] that Q is a homomorphism of braided-Hopf algebras where $\underline{H^*}$ is as above with unchanged coproduct and H has an unchanged product and modified coproduct $\underline{\Delta}$. In particular, it maps the algebras and $Q(b_{(2)})\otimes\langle h,(Sb_{(1)})b_{(3)}\rangle = h\triangleright Q(b)$ as it intertwines the left action given by evaluating with the right adjoint coaction, with the left adjoint action of H. Hence

$$\begin{aligned} Q((h\otimes a)\triangleright b) &= \mathcal{R}(a_{(1)},b_{(1)})\mathcal{R}(b_{(3)},a_{(2)})h\triangleright Q(b_{(2)}) \\ &= \langle Q(b_{(1)}),a_{(1)}\rangle\mathcal{R}((Sb_{(2)})b_{(4)},a_{(2)})h\triangleright Q(b_{(2)}) \\ &= \langle Q(b_{(1)})\mathcal{R}^{[2]},a\rangle(h\mathcal{R}^{[1]})\triangleright Q(b_{(2)}) \\ &= =\langle Q(b)_{\underline{(1)}}\mathcal{R}^{[2]},a\rangle(h\mathcal{R}^{[1]})\triangleright Q(b)_{\underline{(2)}} \\ &= \langle Q(b)_{(1)},a\rangle h\triangleright Q(b)_{(2)}=(h\otimes a)\triangleright Q(b) \end{aligned}$$

where we used that $Q(b_{(1)})\otimes Q(b_{(2)})=\underline{\Delta}Q(b)=Q(b)_{(1)}S\mathcal{R}^{[2]}\otimes\mathcal{R}^{[1]}\triangleright Q(b)_{(2)}$ and indicated the braided coproduct by the underlining the numerical suffices. The 2nd equality is easily proven by breaking down the 3rd expressions in terms of parings of H with H^* and using the quasitriangular and Hopf algebra pairing axioms. $\square$

Putting all the above together, we arrive at our main result:

Theorem 2.2 *If H is factorisable then the covariant system $(D(H),H)$ in Section 2.1 viewed via Q as a covariant system $(D(H),\underline{H^*})$ is isomorphic to a twisting of its semidual $(H^{cop}\blacktriangleright\!\!\blacktriangleleft H,H^*)$. Here we twist by $(\theta_1\otimes\theta_1)(\chi)=\mathcal{R}_{23}^{-1}$ and the isomorphism is given by $\theta=\theta_1\theta_2:D(H)\to H^{cop}\blacktriangleright\!\!\blacktriangleleft H$, where*

$$\theta(h\otimes a)=h_{(1)}Q^{-[2]}Sh_{(2)}\otimes h_{(3)}\mathcal{R}^{[1]}\langle a,Q^{-[1]}\mathcal{R}^{[2]}\rangle$$

Moreover, $H^{cop}\blacktriangleright\!\!\blacktriangleleft H$ has two quasitriangular structures given by $\theta_1\otimes\theta_1$ of $\mathcal{R}_{13}^{-1}\mathcal{R}_{24}$ and $\mathcal{R}_{31}\mathcal{R}_{24}$.

Proof We combine the results above together with an straightforward computation for θ. We recognise $\chi=\chi_2\chi_1=\mathcal{R}_{23}^{-1}\mathcal{R}_{13}^{-1}=(\Delta\otimes\mathrm{id})\mathcal{R}=(\theta_1^{-1}\otimes\theta_1^{-1})\mathcal{R}_{23}^{-1}$ so under θ_1 this maps over to $\mathcal{R}_{23}^{-1}\in(H^{\mathrm{cop}}\blacktriangleright\!\!\blacktriangleleft H)^{\otimes 2}$. Because the action of the double on the vector space H^* in Lemma 2.1 agrees via θ with the action (2.12), it means that Q at the algebra level with the transmuted product and θ at the quantum symmetry level together form an isomorphism of the covariant systems as stated. $\square$

We can also compute the quasitriangular structures in $H^{\mathrm{cop}}\blacktriangleright\!\!\blacktriangleleft H$ explicitly in terms of $\mathcal{R}$'s using the axioms of a quasitriangular structure as

$$\begin{aligned}
\mathcal{R}_{B_D}=(\theta_1\otimes\theta_1)(\mathcal{R}_{13}^{-1}\mathcal{R}_{24}) &= \mathcal{R}_{13}^{-1}(S\otimes\mathrm{id}\otimes S\otimes\mathrm{id})(\Delta\otimes\Delta)\mathcal{R}\\
&= \mathcal{R}_{13}^{-1}(S\otimes\mathrm{id}\otimes S\otimes\mathrm{id})\mathcal{R}_{14}\mathcal{R}_{24}\mathcal{R}_{13}\mathcal{R}_{23}\\
&= (\mathrm{id}\otimes S^{-1}\otimes\mathrm{id}\otimes\mathrm{id})(\mathcal{R}_{13}^{-1}\mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{14}^{-1}\mathcal{R}_{24}^{-1})\\
\mathcal{R}_{B_L}=(\theta_1\otimes\theta_1)(\mathcal{R}_{31}\mathcal{R}_{24}) &= \mathcal{R}_{31}(S\otimes\mathrm{id}\otimes S\otimes\mathrm{id})(\Delta\otimes\Delta)\mathcal{R}\\
&= \mathcal{R}_{31}(S\otimes\mathrm{id}\otimes S\otimes\mathrm{id})\mathcal{R}_{14}\mathcal{R}_{24}\mathcal{R}_{13}\mathcal{R}_{23}\\
&= (\mathrm{id}\otimes S^{-1}\otimes\mathrm{id}\otimes\mathrm{id})(\mathcal{R}_{31}\mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{14}^{-1}\mathcal{R}_{24}^{-1})
\end{aligned}$$

where all expressions are reduced to tensor products of H.

3 Computations for $H=U_q(su_2)$ and $q\to 1$ scaling limit

Here we obtain the main result, starting with explicit formulae in the q-deformed case.

3.1 The Hopf algebra $U_q(su_2)=C_q[SU_2^*]$

We recall that the Hopf algebra $U_q(su_2)$ is defined over formal power series $\mathbb{C}[[t]]$ with generators H, $X_\pm$, where $q=e^{\frac{t}{2}}$, say. The relations are defined by

$$[H,H]=0,\quad [H,X_\pm]=\pm 2X_\pm,\quad [X_+,X_-]=\frac{q^H-q^{-H}}{q-q^{-1}}. \tag{3.1}$$

The coproduct, counit and antipode are given by

$$\begin{aligned} \Delta H = H\otimes 1 + 1\otimes H \qquad & \Delta(X_\pm) = q^{-\frac{H}{2}}\otimes X_\pm + X_\pm \otimes q^{\frac{H}{2}}, \\ \epsilon(H) = 0 \qquad & \epsilon(X_\pm) = 0, \\ S(H) = -H, \qquad & S(X_\pm) = -q^{\pm 1} X_\pm, \end{aligned} \tag{3.2}$$

For q real, the $\star$-structure takes the form $H^\star = H \quad X_\pm^\star = X_\mp$. The Hopf algebra $U_q(su_2)$ is called the q-deformation of the universal enveloping algebra $U(su_2)$. It is quasitriangular with real-type universal R-matrix

$$\mathcal{R} = q^{\frac{H\otimes H}{2}} e^{(1-q^{-2})\, q^{\frac{H}{2}} X_+ \otimes q^{-\frac{H}{2}} X_-}_{q^{-2}}, \tag{3.3}$$

where $e^z_{q^{-2}}$ is the q-exponential $e^z_{q^{-2}} = \sum_{k=0}^\infty \frac{z^k}{[k;q^{-2}]!}$, with $[k;q^{-2}] = \frac{1-q^{-2k}}{1-q^{-2}}$ and $[k;q^{-2}]! = [k;q^{-2}][k-1;q^{-2}]...[1;q^{-2}]$. This means that $e^z_{q^{-2}} e^{-z}_{q^2} = 1$ and that if $AB = q^2 BA$, then $e^{A+B}_{q^{-2}} = e^A_{q^{-2}} e^B_{q^{-2}}$.

Next, unusually, we write $U_q(su_2)$ as $C_q[SU_2^*]$ where latter has $\alpha, \beta, \gamma, \delta$ generators of $B_q[SU_2]$ related via the map Q. The coordinate algebra $B_q[M_2]$ is the space of 2×2 braided Hermitian matrices [29, 12], or q-Minkowski space, with generators $\mathbf{u} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ satisfying the relations

$$\begin{aligned} &\beta\alpha = q^2\alpha\beta, \quad \gamma\alpha = q^{-2}\alpha\gamma, \quad \delta\alpha = \alpha\delta, \\ &[\beta,\gamma] = (1-q^{-2})\alpha(\delta-\alpha), \quad [\delta,\beta] = (1-q^{-2})\alpha\beta, \quad [\gamma,\delta] = (1-q^{-2})\gamma\alpha, \end{aligned} \tag{3.4}$$

and real form $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^\star = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$. Its quotient by the braided-determinant relation $\det(\mathbf{u}) = \alpha\delta - q^2\gamma\beta = 1$ gives the braided group $B_q[SU_2]$ or q-hyperboloid. When $q \neq 1$ this algebra with α^{-1} adjoined provides a version $U_q(su_2)$ via the map Q the 'quantum Killing form'[12] as

$$Q\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} q^H & q^{-\frac{1}{2}}(q-q^{-1}) q^{\frac{H}{2}} X_- \\ q^{-\frac{1}{2}}(q-q^{-1}) X_+ q^{\frac{H}{2}} & q^{-H} + q^{-1}(q-q^{-1})^2 X_+ X_- \end{pmatrix} \tag{3.5}$$

which we regard as an identification. If we assume α is invertible then the element δ is determined by the braided-determinant relation and not regarded as a generator. This map can also be viewed as essentially an isomorphism between the braided enveloping algebra $BU_q(su_2)$ (which has the same algebra as $U_q(su_2)$) and its dual which is the braided function algebra $B_q[SU_2]$. Here, the *unbraided* coproduct of $B_q[SU_2]$ as inherited from that of $U_q(su_2)$ is

$$\begin{aligned} &\Delta\alpha = \alpha\otimes\alpha \quad \Delta\beta = 1\otimes\beta + \beta\otimes\alpha \quad \Delta\gamma = 1\otimes\gamma + \gamma\otimes\alpha, \\ &S\alpha = \alpha^{-1} \quad S\beta = -q^{-2}\alpha^{-1}\beta \quad S\gamma = -\gamma\alpha^{-1}, \quad \epsilon(\alpha^{\pm 1}) = 1, \quad \epsilon(\beta) = \epsilon(\gamma) = 0. \end{aligned} \tag{3.6}$$

The R-matrix becomes

$$\mathcal{R} = q^{\frac{H\otimes H}{2}} e^{(1-q^{-2})^{-1}\,\gamma\otimes\alpha^{-1}\beta}_{q^{-2}}; \quad \alpha = q^H. \tag{3.7}$$

We denote $U_q(su_2)$ in the form of the algebra of $B_q[SU_2]$ with α invertible and the coproduct in (3.6) as the Hopf algebra $C_q[SU_2^*]$.

3.2 The Hopf algebra $C_q[SU_2] = U_q(su_2^*)$

The well-known Hopf algebra $C_q[SU_2]$ is the dual of $U_q(su_2)$ and can be viewed as the quantum deformation of the algebra of functions of $SU(2)$. A set of generators for $C_q[SU_2]$ is given by the matrix elements $t^i{}_j : U_q(su_2) \to \mathbb{C}$ in the defining representation of $U_q(su_2)$ where

$$\langle h, t^i{}_j \rangle = \rho(h)^i_j, \quad h \in U_q(su_2), \quad t^i{}_j \in C_q[SU_2] \tag{3.8}$$

and ρ is in the spin-$\frac{1}{2}$ representation. As usual we write $t^i{}_j = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which we recall have the relations

$$\begin{aligned} ba =& qab, \quad bc = cb, \quad bd = q^{-1}db, \\ ca =& qac, \quad cd = q^{-1}dc, \quad da = ad + (q - q^{-1})bc. \end{aligned} \tag{3.9}$$

The coproduct, counit and antipode are given by

$$\begin{aligned} \Delta a =& a \otimes a + b \otimes c, \ \Delta b = a \otimes b + b \otimes d, \ \Delta c = c \otimes a + d \otimes c, \ \ \Delta d = c \otimes b + d \otimes d, \\ \epsilon a =& \epsilon d = 1, \quad \epsilon b = \epsilon c = 0, \quad Sa = d, \quad Sd = a, \quad Sb = -qb, \ \ Sc = -q^{-1}c, \end{aligned} \tag{3.10}$$

and the real form by $a^\star = d, b^\star = -q^{-1}c$ for q real. The duality pairing takes the form

$$\langle q^{\pm\frac{H}{2}}, a \rangle = q^{\pm\frac{1}{2}}, \quad \langle q^{\pm\frac{H}{2}}, d \rangle = q^{\mp\frac{1}{2}}, \quad \langle X_+, b \rangle = 1, \quad \langle X_-, c \rangle = 1. \tag{3.11}$$

Applying a representation (3.8) to one half of the R-matrix leads to the definition of the well-known L-matrices

$$(L^+)^i_j = \mathcal{R}^{[1]} \rho(\mathcal{R}^{[2]}), \quad (L^-)^i_j = \rho(\mathcal{R}^{-[1]}) \mathcal{R}^{-[2]}, \tag{3.12}$$

where

$$L^+ = \begin{pmatrix} q^{\frac{H}{2}} & 0 \\ q^{\frac{1}{2}}\mu X_+ & q^{-\frac{H}{2}} \end{pmatrix} \quad L^- = \begin{pmatrix} q^{-\frac{H}{2}} & -q^{\frac{3}{2}}\mu X_- \\ 0 & q^{\frac{H}{2}} \end{pmatrix}, \quad \mu = 1 - q^{-2}.$$

We also, unusually, write $C_q[SU_2]$ with new generators $z, x_\pm$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^z & q^{\frac{1}{2}}\mu x_- \\ q^{\frac{3}{2}}\mu x_+ & q^{-z}(1 + q\mu^2 x_+ x_-) \end{pmatrix}. \tag{3.13}$$

If we assume a is invertible, the element d is not regarded as a generator as it is fixed by the q-determinant relation $\det_q(\mathbf{t}) = ad - q^{-1}bc = 1$. The algebra then takes the form

$$[x_\pm, z] = x_\pm, \qquad [x_+, x_-] = 0. \tag{3.14}$$

The coproduct, counit and antipode can then be translated as

$$\begin{aligned} \Delta(q^z) &= q^z \otimes q^z + q^2\mu^2 x_- \otimes x_+, \quad \Delta(x_-) = q^z \otimes x_- + x_- \otimes q^{-z} + q\mu^2 x_- \otimes q^{-z} x_+ x_-, \\ \Delta(x_+) &= x_+ \otimes q^z + q^{-z} \otimes x_+ + q\mu^2 x_+ x_- q^{-z} \otimes x_+, \quad \epsilon(z) = 0, \quad \epsilon(x_\pm) = 0, \\ S(q^z) &= q^{-z}(1 + q\mu^2 x_+ x_-), \quad S(x_\pm) = -q^{\mp 1} x_\pm. \end{aligned} \tag{3.15}$$

We denote $C_q[SU_2]$ with a invertible as the Hopf algebra $U_q(su_2^*)$. The corresponding $*$-structure on $U_q(su_2^*)$ is given by $x_-^\star = -x_+$, $(q^z)^\star = q^{-z}(1 + q\mu^2 x_+ x_-)$.

3.3 The quantum double covariant system ($D(U_q(su_2))$, $U_q(su_2)$)

The quantum double of $U_q(su_2)$ is the double cross product Hopf algebra $D(U_q(su_2)) = U_q(su_2){\bowtie}C_q[SU_2]^{\rm op}$, with algebra structure given by (3.1), the opposite algebra to (3.9) together with cross relations obtained from (2.6) as

$$[q^{\frac{H}{2}},a]=0,\quad q^{\frac{H}{2}}b=q^{-1}bq^{\frac{H}{2}},\quad q^{\frac{H}{2}}c=qcq^{\frac{H}{2}},\quad [q^{\frac{H}{2}},d]=0,$$
$$X_-a=q^{-1}aX_-+bq^{\frac{H}{2}},\quad [X_-,b]=0,\quad [X_-,c]=q(q^{\frac{H}{2}}d-q^{-\frac{H}{2}}a),\quad dX_-=q^{-1}X_-d+q^{\frac{H}{2}}b,$$
$$aX_+=qX_+a+q^{-\frac{H}{2}}c,\quad [X_+,c]=0,\quad [X_+,b]=q^{-1}(q^{\frac{H}{2}}a-q^{-\frac{H}{2}}d),\quad X_+d=qdX_++cq^{\frac{H}{2}}. \tag{3.16}$$

The coproduct, counit and antipode are given by (3.2) for the Hopf subalgebra $U_q(su_2)$ and the coproduct, the counit and inverse of the antipode in (3.10) for the Hopf subalgebra $C_q[SU_2]^{\rm op}$. The quantum double $D(U_q(su_2))$ canonically acts on $U_q(su_2)$ from the left with (3.1) as algebra, resulting in the covariant system $(D(U_q(su_2)),\, U_q(su_2))$. The left covariance action is given by (2.7) as

$$\begin{aligned}
H{\triangleright}H &= 0,\quad H{\triangleright}X_\pm=\pm 2X_\pm,\quad X_\pm{\triangleright}H=-2q^{\pm1}q^{-\frac{H}{2}}X_\pm,\quad X_\pm{\triangleright}X_\pm=(q^{\pm2}-q^{\pm1})q^{-\frac{H}{2}}X_\pm^2,\\
X_\pm{\triangleright}X_\mp &= q^{-\frac{H}{2}}(X_\pm X_\mp - q^{\pm1}X_\mp X_\pm),\quad a{\triangleright}H=1+H,\quad a{\triangleright}q^{\frac{H}{2}}=q^{\frac12}q^{\frac{H}{2}},\quad a{\triangleright}X_\pm=q^{-\frac12}X_\pm,\\
b{\triangleright}H &= 0,\quad b{\triangleright}q^{\frac{H}{2}}=0,\quad b{\triangleright}X_+=q^{\frac{H}{2}},\quad b{\triangleright}X_-=0,\quad c{\triangleright}H=0,\quad c{\triangleright}q^{\frac{H}{2}}=0,\\
c{\triangleright}X_+ &= 0,\quad c{\triangleright}X_-=q^{\frac{H}{2}},\quad d{\triangleright}H=-1+H,\quad d{\triangleright}q^{\frac{H}{2}}=q^{-\frac12}q^{\frac{H}{2}},\quad d{\triangleright}X_\pm=q^{\frac12}X_\pm.
\end{aligned} \tag{3.17}$$

This is the standard q-deformed quantum double system. This $q\neq 1$ corresponds to a cosmological constant in 3d quantum gravity.

3.4 The bicrossproduct covariant system $(C_q[SU_2^*]^{\rm cop}{\blacktriangleright\!\!\triangleleft}U_q(su_2),\, U_q(su_2^*))$

Here we use the alternative description of one of the $U_q(su_2)$ as $C_q[SU_2^*]$ and of $C_q[SU_2]$ as $U_q(su_2^*)$ as explained above. From (2.8), we obtain the left action of $U_q(su_2)$ on $C_q[SU_2^*]$ as

$$H{\triangleright}\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}=\begin{pmatrix}0&-2\beta\\ 2\gamma&0\end{pmatrix},\quad q^{\pm\frac{H}{2}}{\triangleright}\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}=\begin{pmatrix}\alpha&q^{\mp1}\beta\\ q^{\pm1}\gamma&\delta\end{pmatrix},$$
$$X_+{\triangleright}\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}=\begin{pmatrix}-q^{\frac32}\gamma&-q^{\frac12}(\delta-\alpha)\\ 0&q^{-\frac12}\gamma\end{pmatrix},\quad X_-{\triangleright}\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}=\begin{pmatrix}q^{\frac12}\beta&0\\ q^{-\frac12}(\delta-\alpha)&-q^{-\frac32}\beta\end{pmatrix}.$$

Writing $\alpha\equiv\alpha\otimes 1$, $\beta\equiv\beta\otimes 1$, $\gamma\equiv\gamma\otimes 1$, in the subalgebra $C_q[SU_2^*]^{\rm cop}\otimes 1$ and $H\equiv 1\otimes H$, $X_\pm\equiv 1\otimes X_\pm$, in $1\otimes U_q[su_2]$ of the 'mirror product' $M(U_q(su_2)) = C_q[SU_2^*]^{\rm cop}{\blacktriangleright\!\!\triangleleft}U_q(su_2)$, we obtain its cross relations from (2.9) as

$$\begin{aligned}
[H,\alpha] &= [H,\delta]=0,\quad [H,\beta]=-2\beta,\quad [H,\gamma]=2\gamma,\\
[X_+,\alpha] &= -q^{\frac32}\gamma q^{\frac{H}{2}},\quad X_+\beta=q\beta X_+-q^{\frac12}(\delta-\alpha)q^{\frac{H}{2}},\quad X_+\gamma=q^{-1}\gamma X_+,\ [X_+,\delta]=q^{-\frac12}\gamma q^{\frac{H}{2}},\\
[X_-,\alpha] &= q^{\frac12}\beta q^{\frac{H}{2}},\quad X_-\beta=q\beta X_-,\quad X_-\gamma=q^{-1}\gamma X_-+q^{-\frac12}(\delta-\alpha)q^{\frac{H}{2}},\quad [X_-,\delta]=-q^{-\frac32}\beta q^{\frac{H}{2}}.
\end{aligned} \tag{3.18}$$

The coproduct is given by (2.10) as

$$\begin{aligned}\Delta\alpha &= \alpha\otimes\alpha \quad \Delta\beta=\beta\otimes 1+\alpha\otimes\beta \quad \Delta\gamma=\gamma\otimes 1+\alpha\otimes\gamma, \quad \Delta H=1\otimes H+H\otimes 1,\\ \Delta X_+ &= q^{-\frac{H}{2}}\otimes X_+ + X_+\otimes\alpha^{-1}q^{\frac{H}{2}}+q^{-\frac{1}{2}}\mu^{-1}(q^{\frac{H}{2}}-q^{-\frac{H}{2}})\otimes\gamma\alpha^{-1}q^{\frac{H}{2}},\\ \Delta X_- &= q^{-\frac{H}{2}}\otimes X_- + X_-\otimes\alpha^{-1}q^{\frac{H}{2}}+q^{-\frac{3}{2}}\mu^{-1}(q^{\frac{H}{2}}-q^{-\frac{H}{2}})\otimes\alpha^{-1}\beta q^{\frac{H}{2}}.\end{aligned} \tag{3.19}$$

This Hopf algebra covariantly acts from the left on $U_q(su_2^*)$ with (3.14) as algebra giving, the covariant system $(C_q[SU_2^*]^{\rm cop}\blacktriangleright\!\!\blacktriangleleft U_q(su_2),\, U_q(su_2^*))$. From (2.12), we obtain this left action on $C_q[SU_2^*]$ as

$$\begin{aligned}&H\triangleright\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}0&-2b\\2c&0\end{pmatrix},\quad q^{\pm\frac{H}{2}}\triangleright\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a&q^{\mp1}b\\q^{\pm1}c&d\end{pmatrix},\\ &X_+\triangleright\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}-q^{\frac{3}{2}}c&q^{\frac{1}{2}}(a-d)\\0&q^{-\frac{1}{2}}c\end{pmatrix},\ X_-\triangleright\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}q^{\frac{1}{2}}b&0\\q^{-\frac{1}{2}}(d-a)&-q^{-\frac{3}{2}}b\end{pmatrix},\\ &\alpha\triangleright\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}q^{-1}a&q^{-1}b\\qc&qd\end{pmatrix},\quad \beta\triangleright\begin{pmatrix}a&b\\c&d\end{pmatrix}=-q\mu\begin{pmatrix}0&0\\a&b\end{pmatrix},\\ &\gamma\triangleright\begin{pmatrix}a&b\\c&d\end{pmatrix}=-q^{-1}\mu\begin{pmatrix}c&d\\0&0\end{pmatrix}.\end{aligned} \tag{3.20}$$

This then translates to the left action on $U_q(su_2^*)$ given by

$$\begin{aligned}&H\triangleright\begin{pmatrix}q^z\\x_+\\x_-\end{pmatrix}=\begin{pmatrix}0\\2x_+\\-2x_-\end{pmatrix},\quad q^{\frac{H}{2}}\triangleright\begin{pmatrix}q^z\\x_+\\x_-\end{pmatrix}=\begin{pmatrix}q^z\\qx_+\\q^{-1}x_-\end{pmatrix},\\ &X_+\triangleright\begin{pmatrix}q^z\\x_+\\x_-\end{pmatrix}=\begin{pmatrix}-q^3\mu x_+\\0\\\mu^{-1}\left(q^z-q^{-z}(1+q\mu^2x_+x_-)\right)\end{pmatrix},\\ &X_-\triangleright\begin{pmatrix}q^z\\x_+\\x_-\end{pmatrix}=\begin{pmatrix}q^2\mu x_-\\-q^{-2}\mu^{-1}\left(q^z-q^{-z}(1+q\mu^2x_+x_-)\right)\\0\end{pmatrix},\\ &\alpha\triangleright\begin{pmatrix}q^z\\x_+\\x_-\end{pmatrix}=\begin{pmatrix}q^{-1}q^z\\qx_+\\q^{-1}x_-\end{pmatrix},\quad \beta\triangleright\begin{pmatrix}q^z\\x_+\\x_-\end{pmatrix}=\begin{pmatrix}0\\-q^{-\frac{1}{2}}q^z\\0\end{pmatrix},\\ &\gamma\triangleright\begin{pmatrix}q^z\\x_+\\x_-\end{pmatrix}=\begin{pmatrix}-q^{\frac{1}{2}}\mu^2x_+\\0\\-q^{-\frac{3}{2}}q^{-z}(1+q\mu^2x_+x_-)\end{pmatrix}.\end{aligned} \tag{3.21}$$

This is the 'mirror product' covariant system semidual to the quantum double one. The mirror product Hopf algebra[12] for generic $q\neq 1$ here is isomorphic to a tensor product so it not usually considered of interest, though it is for us.

3.5 Twisting equivalence of the q-deformed covariant systems

In this section we work out the algebra isomorphism and the twisting of the preceding two covariant systems as established in Theorem 2.2. Here the algebra isomorphism

$$\theta : D(U_q(su_2))\to C_q[SU_2^*]^{\rm cop}\blacktriangleright\!\!\blacktriangleleft U_q(su_2)$$

is defined in Theorem 2.2 by

$$\theta(h\otimes \mathbf{t}) = h_{(1)}Q^{-[2]}Sh_{(2)}\otimes h_{(3)}\mathcal{R}^{[1]}\langle \mathbf{t}, Q^{-[1]}\mathcal{R}^{[2]}\rangle, \quad h\in U_q(su_2),\ \mathbf{t}\in C_q[SU_2].$$

If $\mathcal{R}=\mathcal{R}^{[1]}\otimes\mathcal{R}^{[2]}$, then $Q=\mathcal{R}_{21}\mathcal{R}=\mathcal{R}^{[2]}\mathcal{R}'^{[1]}\otimes\mathcal{R}^{[1]}\mathcal{R}'^{[2]}$, so that $Q^{-1}=\mathcal{R}'^{-[1]}\mathcal{R}^{-[2]}\otimes\mathcal{R}'^{-[2]}\mathcal{R}^{-[1]}$. Therefore

$$\begin{aligned} Q^{-[2]}\langle t^i{}_k, Q^{-[1]}\rangle &= \mathcal{R}'^{-[2]}\mathcal{R}^{-[1]}\langle t^i{}_k, \mathcal{R}'^{-[1]}\mathcal{R}^{-[2]}\rangle \\ &= \mathcal{R}'^{-[2]}\mathcal{R}^{-[1]}\langle t^i{}_k, \mathcal{R}'^{-[1]}\mathcal{R}^{-[2]}\rangle \\ &= \mathcal{R}'^{-[2]}\mathcal{R}^{-[1]}\langle t^i{}_m, \mathcal{R}'^{-[1]}\rangle\langle t^m{}_k, \mathcal{R}^{-[2]}\rangle \\ &= \mathcal{R}'^{[2]}S\mathcal{R}^{[1]}\langle t^i{}_m, S\mathcal{R}'^{[1]}\rangle\langle t^m{}_k, \mathcal{R}^{[2]}\rangle \\ &= L^{-i}{}_m SL^{+m}{}_k. \end{aligned}$$

This combination is a conjugate map Q to the one we used before. Hence

$$\begin{aligned} \theta(1\otimes t^i{}_j) &= Q^{-[2]}\otimes\mathcal{R}^{[1]}\langle t^i{}_j, Q^{-[1]}\mathcal{R}^{[2]}\rangle = Q^{-[2]}\otimes\mathcal{R}^{[1]}\langle t^i{}_k, Q^{-[1]}\rangle\langle t^k{}_j, \mathcal{R}^{[2]}\rangle \\ &= L^{-i}{}_m SL^{+m}{}_k\otimes L^{+k}{}_j. \end{aligned}$$

Also, for $h\otimes 1\in U_q(su_2)\otimes 1$, we have

$$\theta(h\otimes 1) = h_{(1)}Q^{-[2]}Sh_{(2)}\otimes h_{(3)}\mathcal{R}^{[1]}\langle 1, Q^{-[1]}\mathcal{R}^{[2]}\rangle = h_{(1)}Sh_{(2)}\otimes h_{(3)} = 1\otimes h. \tag{3.22}$$

In terms of our generators this means that θ identifies the $q^{\frac{H}{2}}, X_\pm$ generators of the two quantum groups and

$$\begin{aligned} \theta\begin{pmatrix} a & b\\ c & d\end{pmatrix} &= \begin{pmatrix} q^{-H}+q^3\mu^2X_-X_+ & -q^{\frac{3}{2}}\mu X_-q^{\frac{H}{2}}\\ -q^{\frac{3}{2}}\mu q^{\frac{H}{2}}X_+ & q^H\end{pmatrix}\otimes\begin{pmatrix} q^{\frac{H}{2}} & 0\\ q^{\frac{1}{2}}\mu X_+ & q^{-\frac{H}{2}}\end{pmatrix} \\ &= \begin{pmatrix} q^2(\delta-\mu\alpha) & -q^2\beta\\ -q^2\gamma & \alpha\end{pmatrix}\otimes\begin{pmatrix} q^{\frac{H}{2}} & 0\\ q^{\frac{1}{2}}\mu X_+ & q^{-\frac{H}{2}}\end{pmatrix} \end{aligned} \tag{3.23}$$

where $\delta=\alpha^{-1}(1+q^2\beta\gamma)$ and in the 2nd expression we replace by α,β,γ for the generators of $C_q[SU_2^*]^{\rm cop}$. One can check that this is indeed an algebra isomorphism as dictated by the theorem.

The Drinfeld twist of the bicrossproduct $C_q[SU_2^*]^{\rm cop}{\blacktriangleright\!\!\triangleleft}U_q(su_2)$, defined in Theorem 2.2 as $\chi_B=(\theta_1\otimes\theta_1)\chi=\mathcal{R}_{23}^{-1}\in C_q[SU_2^*]^{\rm cop}{\blacktriangleright\!\!\triangleleft}U_q(su_2)\otimes C_q[SU_2^*]^{\rm cop}{\blacktriangleright\!\!\triangleleft}U_q(su_2)$ is given by

$$\chi_B = e_{q^2}^{-q^{-\frac{1}{2}}1\otimes q^{\frac{H}{2}}X_+\otimes\alpha^{-1}\beta\otimes 1}q^{-\frac{1}{2}1\otimes H\otimes H\otimes 1} = e_{q^2}^{-q^{-\frac{1}{2}}KX_+\otimes\alpha^{-1}\beta}q^{-\frac{1}{2}H\otimes\tilde{H}}. \tag{3.24}$$

where $q^{\tilde{H}}=\alpha$ when viewed in $C_q[SU_2^*]^{\rm cop}$ and $K=q^{\frac{H}{2}}$ in $U_q(su_2)$ and the second equality uses the identifications $C_q[SU_2^*]^{\rm cop}\equiv C_q[SU_2^*]^{\rm cop}\otimes 1$ and $U_q(su_2)\equiv 1\otimes U_q(su_2)$ in $C_q[SU_2^*]^{\rm cop}{\blacktriangleright\!\!\triangleleft}U_q(su_2)$. One can check that $\chi_B(\Delta\ )\chi_B^{-1}$ where Δ is the coproduct of $C_q[SU_2^*]^{\rm cop}{\blacktriangleright\!\!\triangleleft}U_q(su_2)$, gives us a coalgebra isomorphic by θ to the coalgebra of the quantum double (so θ is not just an algebra isomorphism if we take this twisted coproduct) as per the theorem. For example,

$$(\theta\otimes\theta)\Delta d = (\theta\otimes\theta)(d\otimes d+c\otimes b) = \alpha K^{-1}\otimes\alpha K^{-1}+(-q^2\gamma K+\alpha q^{\frac{1}{2}}\mu X_+)\otimes(-q^2\beta K^{-1})$$

$$\chi_B(\Delta\theta(d))\chi_B^{-1} = e_{q^2}^{-q^{-\frac{1}{2}}\,KX_+\otimes\alpha^{-1}\beta}(\alpha K^{-1}\otimes\alpha K^{-1})e_{q^{-2}}^{q^{-\frac{1}{2}}\,KX_+\otimes\alpha^{-1}\beta}$$

which gives the same answer on writing $A = -q^{-\frac{1}{2}}\,KX_+\otimes\alpha^{-1}\beta$ and $B = \alpha K^{-1}\otimes\alpha K^{-1}$ and $C = q^3\gamma K\otimes K^{-1}\beta$ so that $AB = q^2BA + C$, $AC = CA$ using the commutation relations in $C_q[SU_2^*]^{\rm cop}{\blacktriangleright\!\!\blacktriangleleft}U_q(su_2)$. Operators with these relations formally obey

$$e_{q^2}^A Be_{q^{-2}}^{-A} = B + (q^2-1)BA + C,$$

which we use.

The map of covariant algebras provided by Lemma 2.1 is computed from $Q(\mathbf{t}) = \langle \mathbf{t}, Q^{[1]}\rangle Q^{[2]} = L^+SL^-$ and was already given in (3.5) when one notes that $B_q[SU_2]$ and $C_q[SU_2]$ have the same coalgebra and the same generators (but different algebra relations). From this point of view $Q : U_q(su_2^*)\to U_q(su_2)$ is not an algebra map (we would have to use the transmuted or twisted product on the first algebra) and obeys

$$Q\begin{pmatrix} q^z & q^{\frac{1}{2}}\mu x_- \\ q^{\frac{3}{2}}\mu x_+ & q^{-z}(1+q\mu^2x_+x_-)\end{pmatrix} = \begin{pmatrix} q^H & q^{\frac{1}{2}}\mu q^{\frac{H}{2}}X_- \\ q^{\frac{1}{2}}\mu X_+q^{\frac{H}{2}} & q^{-H}+\mu^2X_+X_-\end{pmatrix}.$$

Here $a^n = a\underline{\cdot}a\underline{\cdot}a\cdots$ (n times) when one looks carefully at the transmuted product $\underline{\cdot}$ on the generator $a = q^z$, which implies

$$Q(z) = H,\quad Q(x_-) = q^{\frac{H}{2}}X_-,\quad Q(x_+) = q^{-1}X_+q^{\frac{H}{2}}. \tag{3.25}$$

at the level of $U_q(su_2^*)$ generators.

We also obtain two quasitriangular structures for the bicrossproduct $C_q[SU_2^*]^{\rm cop}{\blacktriangleright\!\!\blacktriangleleft}U_q(su_2)$ defined in Theorem 2.2. Then we find expressions for

$$\mathcal{R}_{B_D},\mathcal{R}_{B_L}\in C_q[SU_2^*]^{\rm cop}{\blacktriangleright\!\!\blacktriangleleft}U_q(su_2)\otimes C_q[SU_2^*]^{\rm cop}{\blacktriangleright\!\!\blacktriangleleft}U_q(su_2)$$

as follows: From Theorem 2.2, we have $\mathcal{R}_{B_D} = (\mathrm{id}\otimes S^{-1}\otimes\mathrm{id}\otimes\mathrm{id})(\mathcal{R}_{13}^{-1}\mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{14}^{-1}\mathcal{R}_{24}^{-1})$ and similarly for $\mathcal{R}_L$ with $\mathcal{R}_{31}$ in place of $\mathcal{R}_{13}^{-1}$. Writing $\mathcal{R}$ for $U_q(su_2)$, this is

$$\begin{aligned}\mathcal{R}_{B_D} &= (\mathrm{id}\otimes S^{-1}\otimes\mathrm{id}\otimes\mathrm{id})\big(q^{\frac{1}{2}1\otimes H\otimes H\otimes 1}e_{q^2}^{-\mu KX_+\otimes q^H\otimes K^{-1}X_-\otimes 1}e_{q^{-2}}^{\mu q^H\otimes KX_+\otimes K^{-1}X_-\otimes 1}\\
&\quad e_{q^{-2}}^{\mu KX_+\otimes 1\otimes K^{-1}X_-\otimes 1}e_{q^2}^{-\mu KX_+\otimes 1\otimes 1\otimes K^{-1}X_-}e_{q^2}^{-\mu q^H\otimes KX_+\otimes 1\otimes K^{-1}X_-}q^{-\frac{1}{2}(H\otimes 1\otimes 1\otimes H+1\otimes H\otimes 1\otimes H)}\big)\\
&= (\mathrm{id}\otimes S^{-1}\otimes\mathrm{id}\otimes\mathrm{id})\big(q^{\frac{1}{2}1\otimes H\otimes\tilde H\otimes 1}e_{q^{-2}}^{q^{-\frac{1}{2}}\alpha\otimes KX_+\otimes\alpha^{-1}\beta\otimes 1}e_{q^2}^{-\frac{1}{\mu}\gamma\otimes q^H\otimes\alpha^{-1}\beta\otimes 1}\\
&\quad e_{q^{-2}}^{\frac{1}{\mu}\gamma\otimes 1\otimes\alpha^{-1}\beta\otimes 1}e_{q^2}^{-q^{\frac{1}{2}}\gamma\otimes 1\otimes 1\otimes K^{-1}X_-}e_{q^2}^{-\mu\alpha\otimes KX_+\otimes 1\otimes K^{-1}X_-}q^{-\frac{1}{2}(\tilde H\otimes 1\otimes 1\otimes H+1\otimes H\otimes 1\otimes H)}\big)\end{aligned} \tag{3.26}$$

where $K = q^{\frac{H}{2}}$ and $\alpha = q^{\tilde H}$ viewed in $C_q[SU_2^*]^{\rm cop}$. Here, we have written $\mathcal{R}_{B_D}$ as an element of $U_q(su_2){\blacktriangleright\!\!\blacktriangleleft}U_q(su_2)\otimes U_q(su_2){\blacktriangleright\!\!\blacktriangleleft}U_q(su_2)$ in the first equality. In the second equality, we view the first and the third legs in $C_q[SU_2^*]^{\rm cop}$ using the map Q in (3.5) and used the fact that $e_{q^{-2}}^{q^{-\frac{1}{2}}\alpha\otimes KX_+\otimes\alpha^{-1}\beta\otimes 1}$ commutes with $e_{q^2}^{-\frac{1}{\mu}\gamma\otimes q^H\otimes\alpha^{-1}\beta\otimes 1}$. Note that

S^{-1} reverses order, resulting in more complicated expressions if we apply this. Similarly, Theorem 2.2 gives

$$\mathcal{R}_{B_L} = (\mathrm{id}\otimes S^{-1}\otimes\mathrm{id}\otimes\mathrm{id})\big(q^{\frac{1}{2}H\otimes 1\otimes H\otimes 1}e_{q^{-2}}^{\mu K^{-1}X_-\otimes 1\otimes KX_+\otimes 1}q^{\frac{1}{2}1\otimes H\otimes H\otimes 1}e_{q^{-2}}^{\mu 1\otimes KX_+\otimes K^{-1}X_-\otimes 1}$$
$$q^{\frac{1}{2}H\otimes 1\otimes H\otimes 1}e_{q^{-2}}^{\mu KX_+\otimes 1\otimes K^{-1}X_-\otimes 1}e_{q^2}^{-\mu KX_+\otimes 1\otimes 1\otimes K^{-1}X_-}q^{-\frac{1}{2}H\otimes 1\otimes 1\otimes H}$$
$$e_{q^2}^{-\mu 1\otimes KX_+\otimes 1\otimes K^{-1}X_-}q^{-\frac{1}{2}(1\otimes H\otimes 1\otimes H)}\big)$$
$$= (\mathrm{id}\otimes S^{-1}\otimes\mathrm{id}\otimes\mathrm{id})\big(q^{\frac{1}{2}\tilde{H}\otimes 1\otimes\tilde{H}\otimes 1}e_{q^{-2}}^{\frac{1}{\mu}\alpha^{-1}\beta\otimes 1\otimes\gamma\otimes 1}q^{\frac{1}{2}1\otimes H\otimes\tilde{H}\otimes 1}e_{q^{-2}}^{q^{-\frac{1}{2}}1\otimes KX_+\otimes\alpha^{-1}\beta\otimes 1}$$
$$q^{\frac{1}{2}\tilde{H}\otimes 1\otimes\tilde{H}\otimes 1}e_{q^{-2}}^{\frac{1}{\mu}\gamma\otimes 1\otimes\alpha^{-1}\beta\otimes 1}e_{q^2}^{-q^{\frac{1}{2}}\gamma\otimes 1\otimes 1\otimes K^{-1}X_-}q^{-\frac{1}{2}\tilde{H}\otimes 1\otimes 1\otimes H}$$
$$e_{q^2}^{-\mu 1\otimes KX_+\otimes 1\otimes K^{-1}X_-}q^{-\frac{1}{2}(1\otimes H\otimes 1\otimes H)}\big). \tag{3.27}$$

3.6 Limiting twist between the spin model and the bicrossproduct model

We are now in position to consider the degenerations of the two covariant systems by scaling the various generators appropriately to recover the results of Theorem 2.2 in the limit $q\to 1$.

(i) For the quantum double $D(U_q(su_2)) = U_q(su_2)\bowtie C_q[SU_2]^{\mathrm{op}}$, the $U_q(su_2)$ part has no problem with the limit $q\to 1$ and so $U_q(su_2)\mapsto U(su_2)$, with the standard Lie brackets

$$[H,H]=0,\quad [H,X_\pm]=\pm 2X_\pm,\quad [X_+,X_-]=H, \tag{3.28}$$

and cocommutative coalgebra

$$\begin{aligned}\Delta H = H\otimes 1+1\otimes H \qquad & \Delta(X_\pm)=1\otimes X_\pm + X_\pm\otimes 1,\\ \epsilon(H)=0\quad \epsilon(X_\pm)=0, \qquad & S(H)=-H,\quad S(X_\pm)=-X_\pm,\end{aligned} \tag{3.29}$$

Similarly, $C_q[SU_2]^{\mathrm{op}}\mapsto C(SU_2)$, the commutative algebra of functions on SU_2 as $q\to 1$. The cross relations in the limit can easily be extracted from (3.16) as

$$\begin{aligned}[H,a]=0,\quad [H,b]=-b,\quad [H,c]=2c,\quad [H,d]=0,\\ [X_-,a]=b,\quad [X_-,b]=0,\quad [X_-,c]=d-a,\quad [X_-,d]=-b,\\ [X_+,a]=-c,\quad [X_+,c]=0,\quad [X_+,b]=a-d),\quad [X_+,d]=c,\end{aligned} \tag{3.30}$$

and the coproduct is the tensor product one. The quantum group $U_q(su_2)$ is also the quantum spacetime algebra for the covariant system $(D(U_q(su_2)), U_q(su_2))$. In the limit the covariant action of $D(U(su_2)) = U(su_2)\ltimes C(SU_2)$ on $U(su_2)$ is given by the $q\to 1$ limit of (3.31) as

$$\begin{aligned}H\triangleright H &= 0,\quad H\triangleright X_\pm = \pm 2X_\pm,\quad X_\pm\triangleright H = -2X_\pm,\quad X_\pm\triangleright X_\pm = 0,\\ X_\pm\triangleright X_\mp &= H,\quad a\triangleright H = 1+H,\quad a\triangleright X_\pm = X_\pm,\quad b\triangleright H = 0,\quad b\triangleright X_+ = 1,\\ b\triangleright X_- &= 0,\quad c\triangleright H = 0,\quad c\triangleright X_+ = 0,\quad c\triangleright X_- = 1,\quad d\triangleright H = -1+H,\quad d\triangleright X_\pm = X_\pm.\end{aligned} \tag{3.31}$$

This limit action was first computed in [2] and to match standard conventions, we choose generators

$$H = 2J_0, \quad X_\pm = J_1 \pm \imath J_2, \quad t^i_j = \begin{pmatrix} e^{\lambda\mathcal{P}_3} + \imath\frac{\lambda}{2}\mathcal{P}_0 & \imath\frac{\lambda}{2}(\mathcal{P}_1 - \imath\mathcal{P}_2) \\ \imath\frac{\lambda}{2}(\mathcal{P}_1 + \imath\mathcal{P}_2) & e^{\lambda\mathcal{P}_3} - \imath\frac{\lambda}{2}\mathcal{P}_0 \end{pmatrix}, \tag{3.32}$$

where $\lambda \in \mathbb{R}$ is a deformation parameter and $\mathcal{P}_3$ not regarded as a generator but is determined by the $\det(t) = 1$ condition. This gives the algebra as

$$[J_a, J_b] = \imath\epsilon_{abc}J_c, \quad [\mathcal{P}_a, J_b] = \imath\epsilon_{abc}\mathcal{P}_c, \quad [\mathcal{P}_a, \mathcal{P}_b] = 0, \tag{3.33}$$

and the coproducts turns out to be

$$\Delta J_a = J_a \otimes 1 + 1 \otimes J_a, \quad \Delta\mathcal{P}_a = \mathcal{P}_a \otimes 1 + 1 \otimes \mathcal{P}_a - \frac{\lambda}{2}\epsilon_{abc}\mathcal{P}_b \otimes \mathcal{P}_c. \tag{3.34}$$

For the quantum spacetime algebra for the covariant system, we write

$$\lambda H = 2x_0, \quad \lambda X_\pm = x_1 \pm \imath x_2 \tag{3.35}$$

and then the limit of relations (3.28) gives the spin model spacetime algebra

$$[x_\mu, x_\nu] = \imath\lambda\epsilon_{\mu\nu\rho}x_\rho. \tag{3.36}$$

This is the $q \to 1$ limit $(D(U(su_2)), U(su_2))$ as a deformation of $U(iso(3))$ on $\mathbb{R}^3$[2].

(ii) In the covariant system $(C_q[SU_2^*]^{\rm cop}\blacktriangleright\!\!\blacktriangleleft U_q(su_2), U_q(su_2^*))$, we have $C_q[SU_2^*]^{\rm cop} \mapsto C[SU_2^*]^{\rm cop}$ with commutative algebra and $U_q(su_2) \mapsto U_(su_2)$ with the standard algebra (3.28). From (3.18), the cross relations becomes

$$\begin{aligned} [H,\alpha] &= [H,\delta] = 0, \quad [H,\beta] = -2\beta, \quad [H,\gamma] = 2\gamma, \\ [X_+,\alpha] &= -\gamma, \ \ [X_+,\beta] = -(\delta-\alpha), \ \ [X_+,\gamma] = 0, \ [X_+,\delta] = \gamma, \\ [X_-,\alpha] &= \beta, \ \ [X_-,\beta] = 0, \ \ [X_-,\gamma] = (\delta-\alpha), \ \ [X_-,\delta] = -\beta. \end{aligned} \tag{3.37}$$

The coproduct is obtained from (3.19) as

$$\begin{aligned} \Delta\alpha &= \alpha\otimes\alpha \quad \Delta\beta = \beta\otimes 1 + \alpha\otimes\beta \quad \Delta\gamma = \gamma\otimes 1 + \alpha\otimes\gamma, \quad \Delta H = 1\otimes H + H\otimes 1 \\ \Delta X_+ &= 1\otimes X_+ + X_+\otimes\alpha^{-1} + \tfrac{H}{2}\otimes\gamma\alpha^{-1}, \ \ \Delta X_- = 1\otimes X_- + X_-\otimes\alpha^{-1} + \tfrac{H}{2}\otimes\alpha^{-1}\beta. \end{aligned} \tag{3.38}$$

In this limit the covariant action of $C[SU_2^*]^{\rm cop}\blacktriangleright\!\!\blacktriangleleft U(su_2)$ on $U(su_2^*)$ from (3.21) is

$$\begin{aligned} &H\triangleright z = 0, \ \ H\triangleright x_\pm = \pm 2x_\pm, \ \ X_\pm\triangleright x_\pm = 0, \ \ X_\pm\triangleright z = \mp 2x_\pm, \ \ X_\pm\triangleright x_\mp = \pm z \\ &\alpha^{\pm 1}\triangleright z = z \mp 1, \quad \alpha\triangleright x_\pm = x_\pm, \quad \beta\triangleright z = 0, \ \ \beta\triangleright x_+ = -1, \ \ \beta\triangleright x_- = 0, \ \ \gamma\triangleright z = 0 \\ &\gamma\triangleright x_+ = 0, \ \ \gamma\triangleright x_- = -1. \end{aligned} \tag{3.39}$$

To match standard conventions for the bicrossproduct quantum group $C[SU_2^*]^{\rm cop}\blacktriangleright\!\!\blacktriangleleft U(su_2)$, we now identify new generators

$$\begin{aligned} \alpha &= e^{\lambda p_0}, \quad \beta = \lambda P_+, \quad \gamma = \lambda P_-, \quad P_\pm = p_2 \pm \imath p_1, \\ H &= 2M, \quad X_+ = N_2 - \imath N_1, \quad X_- = N_2 + \imath N_1. \end{aligned} \tag{3.40}$$

Then the relations (3.37) become

$$\begin{aligned}
&[p_a,p_b] = 0, \quad [M,N_1]=N_2, \quad [M,N_2]=-N_1, \quad [N_1,N_2]=-M,\\
&[M,p_0] = 0 \quad [M,p_i]=\imath\epsilon_{ij}p_j, \quad [N_i,p_0,] = -\imath\epsilon_{ij}p_j e^{-\lambda p_0}\\
&[N_i,p_j] = \tfrac{\imath}{2}\epsilon_{ij}\, e^{-\lambda p_0}\left(\tfrac{e^{2\lambda p_0}-1}{\lambda} - \lambda\vec{p}^{\,2}\right), \quad i,j=1,2,
\end{aligned} \tag{3.41}$$

where $\vec{p}^{\,2}=p_1^2+p_2^2$ and coproduct (3.38) gives

$$\begin{aligned}
\Delta p_0 &= p_0\otimes 1 + 1\otimes p_0, \quad \Delta M = 1\otimes M + M\otimes 1,\\
\Delta p_i &= p_i\otimes 1 + e^{\lambda p_0}\otimes p_i,\\
\Delta N_i &= 1\otimes N_i + N_i\otimes e^{-\lambda p_0} + \lambda M\otimes p_i e^{-\lambda p_0}, \quad i=1,2.
\end{aligned} \tag{3.42}$$

For the model spacetime $U_q(su_2^*)$, we set

$$x_0=\imath\lambda z, \quad x_1=-\imath\lambda(x_+ + x_-), \quad x_2=\lambda(x_+-x_-) \tag{3.43}$$

and then take the limit $q\to 1$ to get from (3.14), the bicrossproduct model spacetime

$$[x_i,x_0]=\imath\lambda x_i. \tag{3.44}$$

In terms of these standard generators (3.40), the covariant actions (3.39) can be translated as

$$\begin{aligned}
&M\triangleright x_0=0, \quad M\triangleright x_i=-\epsilon_{ij}x_j, \quad N_i\triangleright x_0=-\imath x_i, \quad N_i\triangleright x_j=-\imath\delta_{ij}x_0,\\
&p_0\triangleright x_0=-\imath, \quad p_0\triangleright x_i=0, \quad p_i\triangleright x_0=0, \quad p_i\triangleright x_j=i\epsilon_{ij}, \quad i,j=1,2.
\end{aligned} \tag{3.45}$$

This is the bicrossproduct model covariant system $(C[SU_2^*]^{\rm cop}\blacktriangleright\!\!\blacktriangleleft U(su_2), U(su_2^*))$ as a quantum Poincare group in three dimensions acting on the Majid-Ruegg quantum spacetime as a 3d version of [5].

(iii) We now look at the $q\to 1$ limit of the twist between the two covariant systems. We remind the reader that for handling of the cocycle and R-matrices we reduced expressions to the tensor product of the underling Hopf algebras. In effect in what follows we equip the vector space of $C[SU_2^*]^{\rm cop}\blacktriangleright\!\!\blacktriangleleft U(su_2)$ with two products, one is the cross product algebra as part of the bicrossproduct $\blacktriangleright\!\!\blacktriangleleft$ construction and the other is the tensor product $\otimes$ algebra. In the following, we use the convention that all exponentials are multiplied in the tensor product $\otimes$ algebra, which does not impact (3.46) but is important for the correct reading of (3.47).

Corollary 3.1 *From the above analysis, we arrive at the degeneration limit $q\to 1$ of our result that the covariant system $(D(U(su_2))=U(su_2)\bowtie C[SU_2]^{op}$, $U(su_2))$ is isomorphic to a twisting of the covariant system $(C[SU_2^*]^{cop}\blacktriangleright\!\!\blacktriangleleft U(su_2)$, $U(su_2^*))$. The twist is derived from* (3.24) *in the limit $q\to 1$ as*

$$\chi_{B_0}=e^{-X_+\otimes\alpha^{-1}\beta}\, e^{-\frac{1}{2}H\otimes(\alpha-1)}\in (C[SU_2^*]^{cop}\blacktriangleright\!\!\blacktriangleleft U(su_2))^{\otimes 2}. \tag{3.46}$$

The degeneration limit $q \to 1$ of (3.26) also provides an R-matrix for the bicrossproduct $C[SU_2^*]^{cop} \blacktriangleright\!\!\triangleleft U(su_2)$ given by

$$\mathcal{R}_{B_0} = e^{\frac{1}{2}\gamma H \otimes \alpha^{-1}\beta} : e^{-\alpha X_+ \otimes \alpha^{-1}\beta} : e^{-\frac{1}{2}H\otimes(\alpha-1)} e^{-\gamma \otimes X_-} e^{-\frac{1}{2}(\alpha-1)\otimes H}, \tag{3.47}$$

where

$$: e^{-\alpha X_+ \otimes \alpha^{-1}\beta} := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \alpha^n X_+^n \otimes (\alpha^{-1}\beta)^n.$$

Proof We write $\alpha \equiv \alpha \otimes 1$, $\beta \equiv \beta \otimes 1$, $\gamma \equiv \gamma \otimes 1$, in the subalgebra $C_q[SU_2^*]^{\rm cop} \otimes 1$ and $H \equiv 1 \otimes H$, $X_\pm \equiv 1 \otimes X_\pm$, in $1 \otimes U_q(su_2)$ of $C_q[SU_2^*]^{\rm cop} \blacktriangleright\!\!\triangleleft U_q(su_2)$, with $q = e^t$. Then in the limit $t \to 0$, the algebra isomorphism becomes $\theta : D(U(su_2)) \to C[SU_2^*]^{\rm cop} \blacktriangleright\!\!\triangleleft U(su_2)$ given by

$$\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}, \quad \theta(h) = h,\ h \in U(su_2) \tag{3.48}$$

It is easy to check that θ is indeed an algebra isomorphism. For example $[\theta(X_+), \theta(b)] = \theta(a) - \theta(d)$, etc. Now we write $\alpha = e^{\frac{t}{2}\tilde{H}} = 1 + \frac{t}{2}\tilde{H} + O(t^2)$, so that

$$q^{-\frac{1}{2}H\otimes\tilde{H}} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \frac{H^n \otimes \tilde{H}^n}{4^n} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{H}{2}\right)^n \otimes (\alpha-1)^n + O(t^2) = e^{-\frac{1}{2}H\otimes(\alpha-1)} + O(t^2),$$

and therefore from (3.24), we see that $\chi_B \to \chi_{B_0} = e^{-X_+ \otimes \alpha^{-1}\beta}\, e^{-\frac{1}{2}H\otimes(\alpha-1)}$ as $t \to 0$. To obtain the limit for the R-matrix, we first note that for X commuting with H,

$$\lim_{t\to 0} \left(e_{q^2}^{-\frac{X}{\mu}} e^{\frac{t}{2}H} e_{q^{-2}}^{\frac{X}{\mu}} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{X}{2}\right)^n \prod_{i=1}^{n-1} (H - 2i).$$

Then form (3.26), the limit of $\mathcal{R}_{B_D}$ as $q \to 1$ becomes

$$\begin{aligned} \mathcal{R}_{B_0} =& (\mathrm{id}\otimes S \otimes \mathrm{id} \otimes \mathrm{id}) \Big(e^{\frac{1}{2}1\otimes H\otimes(\alpha-1)\otimes 1} e^{\alpha \otimes X_+ \otimes \alpha^{-1}\beta \otimes 1} \\ & \sum_{n=0}^{\infty} \frac{(-1)^n}{n! 2^n} \left(\gamma \otimes 1 \otimes \alpha^{-1}\beta \otimes 1\right)^n (1 \otimes \prod_{i=1}^{n-1} (H - 2i) \otimes 1 \otimes 1) \\ & e^{-\gamma\otimes 1\otimes 1\otimes X_-} e^{-\frac{1}{2}(\alpha-1)\otimes 1\otimes 1\otimes H} \Big), \end{aligned} \tag{3.49}$$

where we used that $X = \gamma \otimes 1 \otimes \alpha^{-1}\beta \otimes 1$ commutes with $1 \otimes H \otimes 1 \otimes 1$ and also that $S^2 = \mathrm{id}$ on $U(su_2)$. Next, we also note that for any elements

$$\begin{aligned} (\mathrm{id}\otimes S\otimes \mathrm{id}\otimes\mathrm{id})\big((a\otimes b\otimes c\otimes d)\cdot(A\otimes B\otimes C\otimes D)\big) &= (\mathrm{id}\otimes S\otimes\mathrm{id}\otimes\mathrm{id})(aA\otimes bB\otimes cC\otimes dD) \\ &= aA\otimes S(bB)\otimes cC\otimes dD = Aa\otimes(SB)(Sb)\otimes Cc\otimes Dd \\ &= (\mathrm{id}\otimes S\otimes\mathrm{id}\otimes\mathrm{id})(A\otimes B\otimes C\otimes D)\cdot(\mathrm{id}\otimes S\otimes\mathrm{id}\otimes\mathrm{id})(a\otimes b\otimes c\otimes d) \end{aligned}$$

provided $dD = Dd$, since the first and third legs are in $C[SU_2^*]^{\rm cop}$ which is already commutative. Here, $\cdot$ indicates that the product is in the tensor product one of the Hopf

algebra. Using this observations, (3.49) becomes

$$\begin{aligned}\mathcal{R}_{B_0} =&(\mathrm{id}\otimes S\otimes \mathrm{id}\otimes \mathrm{id})\big(\sum_{n=0}^{\infty}\frac{1}{n!2^n}\gamma^n\otimes\prod_{i=1}^{n-1}(H+2i)\otimes(\alpha^{-1}\beta)^n\otimes 1\big)\\ &(\mathrm{id}\otimes S\otimes \mathrm{id}\otimes \mathrm{id})\big(e^{\frac{1}{2}1\otimes H\otimes(\alpha-1)\otimes 1}e^{\alpha\otimes X_+\otimes\alpha^{-1}\beta\otimes 1}\big)\\ &e^{-\gamma\otimes 1\otimes 1\otimes X_-}e^{-\frac{1}{2}((\alpha-1)\otimes 1\otimes 1\otimes H)}\end{aligned}\tag{3.50}$$

because the first two lines of (3.49) are of the form $A\otimes B\otimes C\otimes 1$ and the last line is unchanged under the action of $(\mathrm{id}\otimes S\otimes \mathrm{id}\otimes \mathrm{id})$. We evaluate the first two lines of the above equation as follows: In the first line, we have

$$\begin{aligned}&(\mathrm{id}\otimes S\otimes \mathrm{id}\otimes \mathrm{id})\sum_{n=0}^{\infty}\frac{(-1)^n}{n!2^n}\gamma^n\otimes\prod_{i=1}^{n-1}(H-2i)\otimes(\alpha^{-1}\beta)^n\otimes 1\\ &=\sum_{n=0}^{\infty}\frac{1}{n!2^n}\gamma^n\otimes\prod_{i=1}^{n-1}(H+2i)\otimes(\alpha^{-1}\beta)^n\otimes 1=e^{\frac{1}{2}\gamma H\otimes\alpha^{-1}\beta}.\end{aligned}$$

The second line of (3.50) is evaluated as

$$\begin{aligned}&(\mathrm{id}\otimes S\otimes \mathrm{id}\otimes \mathrm{id})\big(e^{\frac{1}{2}1\otimes H\otimes(\alpha-1)\otimes 1}e^{\alpha\otimes X_+\otimes\alpha^{-1}\beta\otimes 1}\big)\\ &=e^{-\alpha\otimes X_+\otimes\alpha^{-1}\beta\otimes 1}e^{-\frac{1}{2}1\otimes H\otimes(\alpha-1)\otimes 1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\alpha^nX_+^n\otimes(\alpha^{-1}\beta)^ne^{-\frac{1}{2}H\otimes(\alpha-1)}\\ &=:e^{-\alpha X_+\otimes\alpha^{-1}\beta}:e^{-\frac{1}{2}H\otimes(\alpha-1)}.\end{aligned}$$

Finally, we check that $U(su_2)$ is a module algebra twist by χ_{B_0} of $U(su_2)$, i.e. $U(su_2^*)_{\chi_{B_0}}=U(su_2)$, where the twisted product $a\cdot_{\chi_{B_0}}b$ is given by (2.17). With $\chi_{B_0}^{-1}=e^{\frac{H}{2}\otimes(\alpha-1)}\,e^{X_+\otimes\alpha^{-1}\beta}$, we have that for example

$$\begin{aligned}z\cdot_{\chi_{B_0}}x_+&=\cdot(\chi_{B_0}^{-1}\triangleright(z\otimes x_+))\\ &=\sum\sum\frac{1}{m!}\frac{1}{n!}\left(\left(\frac{H}{2}\right)^m\triangleright X_+^n\triangleright z\right)\left((\alpha-1)^m\triangleright(\alpha^{-1}\beta)^n\triangleright x_+\right)\\ &=\sum\frac{1}{m!}\left(\left(\frac{H}{2}\right)^m\triangleright z\right)((\alpha-1)^m\triangleright x_+)+\sum\frac{1}{m!}\left(\left(\frac{H}{2}\right)^m\triangleright(-2x_+)\right)((\alpha-1)^m\triangleright(-1))\\ &=zx_++2x_+,\end{aligned}$$

and

$$\begin{aligned}x_+\cdot_{\chi_{B_0}}z&=\sum\sum\frac{1}{m!}\frac{1}{n!}\left(\left(\frac{H}{2}\right)^m\triangleright X_+^n\triangleright x_+\right)\left((\alpha-1)^m\triangleright(\alpha^{-1}\beta)^n\triangleright z\right)\\ &=\sum\frac{1}{m!}\left(\left(\frac{H}{2}\right)^m\triangleright x_+\right)((\alpha-1)^m\triangleright z)\\ &=x_+z-x_+.\end{aligned}$$

Computing all possible combination of products, we obtain the following twisted algebra

$$[x_+,x_-]\cdot_{\chi_{B_0}}=[x_+,x_-]+z,\quad [x_\pm,x_\pm]\cdot_{\chi_{B_0}}=[x_\pm,x_\pm],\quad [z,z]\cdot_{\chi_{B_0}}=[z,z]$$
$$[x_+,z]\cdot_{\chi_{B_0}}=[x_+,z]-3x_+,\quad [x_-,z]\cdot_{\chi_{B_0}}=[x_-,z]+x_-=2x_-$$

which on evaluating the product in $U(su_2^*)$ gives

$$[x_+,x_-]\cdot_{\chi_{B_0}}=z,\quad [x_\pm,x_\pm]\cdot_{\chi_{B_0}}=0,\quad [z,z]\cdot_{\chi_{B_0}}=0,\quad [x_\pm,z]\cdot_{\chi_{B_0}}=\mp 2x_\pm.$$

We see that

$$\Phi(H)=z,\quad \Phi(X_\pm)=x_\pm \tag{3.51}$$

defines an isomorphism of $U(su_2)$ with $U(su_2^*)$ after twisting. This is manifestly the inverse of $Q:U(su_2^*)\to U(su_2)$ of covariant algebras in Lemma 2.1 which in our case by (3.25) is just $Q(z)=H$, $Q(x_\pm)=X_\pm$ in the $q\to 1$ limit. By construction, the identification must be covariant but it is a useful check to see this directly. For example,

$$\theta(X_+)\triangleright\Phi(H)=X_+\triangleright z=-2x_+=\Phi(X_+\triangleright H)$$

$$\theta(a)\triangleright\Phi(H)=\delta\triangleright z=z+1=\Phi(1+H)=\Phi(a\triangleright H)$$
$$\theta(d)\triangleright\Phi(X_-)=\alpha\triangleright x_-=\Phi(X_-)=\Phi(d\triangleright X_-).$$

4 Semiclassical limit of results

In this section describe our twisting result at the infinitesimal Lie bialgebra level, i.e. the classical double as a Lie bialgebra twist of the bicross sum, both in the case when q is switched on and in the scaling limit $q\to 1$ with a parameter λ. We begin with a brief review of Lie bialgebras and classical r-matrices and refer the reader to [12, 23] and references therein for details.

4.1 Double and Semidual Lie Bialgebras and classical r-matrices

A Lie bialgebra in the sense of Drinfeld provides a semiclassical or infinitesimal notion of a Hopf algebra. A Lie bialgebra $(\mathfrak{g},[\ ,\ ],\delta)$ is a Lie algebra $(\mathfrak{g},[\ ,\ ])$ over a field k equipped with a cocommutator $\delta:\mathfrak{g}\mapsto\mathfrak{g}\otimes\mathfrak{g}$ is a skew-symmetric linear map, i.e. $\delta:\mathfrak{g}\mapsto\wedge^2\mathfrak{g}$ satisfying the coJacobi identity

$$(\delta\otimes\mathrm{id})\circ\delta(\xi)+\mathrm{cyclic}=0,\quad \forall\xi\in\mathfrak{g}$$

and that for all $\xi,\eta\in\mathfrak{g}$,

$$\delta([\xi,\eta])=(\mathrm{ad}_\xi\otimes 1+1\otimes\mathrm{ad}_\xi)\delta(\eta)-(\mathrm{ad}_\eta\otimes 1+1\otimes\mathrm{ad}_\eta)\delta(\xi).$$

There exist a Lie bialgebra version of the quasitriangular Hopf algebra which arise naturally in the following way: Since δ is a 1-cocycle, an element $r=r^{[1]}\otimes r^{[2]}\in\mathfrak{g}\otimes\mathfrak{g}$ provides a coboundary structure for the Lie bialgebra $(\mathfrak{g},[\ ,\ ],\delta)$ if $\delta=\partial r$, i.e. $\delta(\xi)=$

$\mathrm{ad}_\xi(r) = [\xi \otimes 1 + 1 \otimes \xi, r]$. This requires that $\mathrm{ad}_\xi(r + r_{21}) = 0$ for all $\xi \in \mathfrak{g}$ to have δ antisymmetric. For any Lie algebra $\mathfrak{g}$, we define the map

$$\mathfrak{g}^{\otimes 2} \to \mathfrak{g}^{\otimes 3}, \quad r \mapsto [[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]. \tag{4.1}$$

This map restricts to the map $\wedge^2\mathfrak{g} \to \wedge^3\mathfrak{g}$. The equation

$$[[r, r]] = 0 \tag{4.2}$$

is called the classical Yang-Baxter equation(CYBE) and any solution of the CYBE in $\mathfrak{g} \otimes \mathfrak{g}$ is called a classical r-matrix. The classical r-matrix provides a quasitriangular structure for the Lie bialgebra. It is triangular if it satisfies the CYBE and $r_{21} = -r$ and called factorisable if it satisfies the CYBE and $r + r_{21} : \mathfrak{g}^* \to \mathfrak{g}$ is a linear surjection. The classical r-matrix therefore provides a natural infinitesimal version of the universal R-matrix while the factorisable case correspond to $\mathcal{R}$ factorisable.

If $(\mathfrak{g}, [\ ,\ ], r)$ is a quasitriangular Lie bialgebra and $\chi^c \in \mathfrak{g} \otimes \mathfrak{g}$ obeys

$$[[r, \chi^c]] + [[\chi^c, r]] + [[\chi^c, \chi^c]] = 0, \quad \mathrm{ad}_\xi(\chi^c + \chi^c_{21}) = 0, \quad \forall \xi \in \mathfrak{g}, \tag{4.3}$$

then $(\mathfrak{g}, [\ ,\ ], r + \chi^c)$ is also a quasitriangular Lie bialgebra. The element χ is called a Lie bialgebra twist and $\delta_{\chi^c} = \delta + \partial\chi^c$ is also a Lie bialgebra.

A quantised enveloping algebra roughly speaking means a Hopf algebra over $\mathbb{C}[[t]]$ generated by a vector space $\mathfrak{g}$, with relations and coproduct of the form

$$\xi\eta - \eta\xi = [\xi, \eta] + O(t), \quad (\Delta - \Delta^{\mathrm{cop}})\xi = t\delta\xi + O(t^2), \tag{4.4}$$

where t is a formal deformation parameter. Further, if the Hopf algebra has a quasitriangular structure of the form

$$\mathcal{R} = 1 + tr + O(t^2), \tag{4.5}$$

then $\delta = \partial r$, and our Lie bialgebra is quasitriangular. This interpretation is also compatible with twisting. More explicitly, if $\chi = 1 + tf + O(t^2)$, then from (2.16), we have

$$\chi^c = f_{21} - f \tag{4.6}$$

Classical double and bicross sum Lie bialgebras provide a semiclassical version of the quantum doubles and bicrossproduct quantum groups respectively, described in Section 2.1. For any finite dimensional Lie bialgebra $\mathfrak{g}$ with dual $\mathfrak{g}^*$, there is a quasitriangular Lie bialgebra, $D(\mathfrak{g})$, the classical double of $\mathfrak{g}$ built on $\mathfrak{g} \oplus \mathfrak{g}^*$ as a vector space, with

$$\begin{aligned}
[\xi \oplus \phi, \eta \oplus \psi] &= ([\xi, \eta] + \sum \xi_{[1]} \langle \psi, \xi_{[2]} \rangle - \eta_{[1]} \langle \phi, \eta_{[2]} \rangle) \\
&\quad \oplus ([\psi, \phi] + \sum \psi_{[1]} \langle \psi_{[2]}, \xi \rangle - \phi_{[1]} \langle \phi_{[2]}, \eta \rangle), \\
\delta(\xi \oplus \phi) &= \sum (0 \oplus \phi_{[1]}) \otimes (0 \oplus \phi_{[2]}) + \sum (\xi_{[1]} \oplus 0) \otimes (\xi_{[2]} \oplus 0), \\
r &= \sum_a (0 \oplus f^a) \otimes (e_a \oplus 0).
\end{aligned}$$

Here $\mathfrak{g}^{*\mathrm{op}}, \mathfrak{g}$, appear as sub-Lie bialgebras, where $()^{\mathrm{op}}$ denotes the opposite (negated) Lie bracket. The set $\{e_a\}$ is a basis of $\mathfrak{g}$ and $\{f^a\}$, a dual basis. Moreover, if $(\mathfrak{g}, r)$ is factorisable then $D(\mathfrak{g}) \cong (\mathfrak{g} \oplus \mathfrak{g})_{\chi_2^c}$ where we mean twisting by a certain cocycle χ_2^c, which as we saw amounts to adding χ_2^c to the $(-r_{21}) \oplus r$ if we want r_D and to $r \oplus r$ for another r-matrix r_L. We can further view this as a twisting of $\mathfrak{g}^{\mathrm{cop}} \oplus \mathfrak{g}$ by a certain other cocycle χ_1^c built from r.

Semidualisation can also be defined for Lie bialgebras which are double cross sums. Given a matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{m})$, one can define the double cross sum $\mathfrak{g} \bowtie \mathfrak{m}$ as the vector space $\mathfrak{g} \oplus \mathfrak{m}$. The semidual gives the the bicross sum Lie bialgebra $\mathfrak{m}^* \blacktriangleright\!\!\blacktriangleleft \mathfrak{g}$ built on $\mathfrak{m}^* \oplus \mathfrak{g}$ with

$$[f \oplus \xi, h \oplus \eta] = (\xi \triangleright h - \eta \triangleright f) \oplus [\xi, \eta],$$

$$\begin{aligned} \delta(f \oplus \xi) &= \sum_a (0 \oplus e_a \triangleright \xi) \otimes (f^a \oplus 0) - (f^a \oplus 0) \otimes (0 \oplus e_a \triangleright \xi) \\ &\quad + \sum (f_{[1]} \oplus 0) \otimes (f_{[2]} \oplus 0), \end{aligned}$$

for all $f \oplus \xi, h \oplus \eta \in \mathfrak{m}^* \blacktriangleright\!\!\blacktriangleleft \mathfrak{g}$, where $\delta(f) = \sum f_{[1]} \otimes f_{[2]}$ is the Lie coalgebra given by the dualisation of the Lie bracket of $\mathfrak{m}$. For a detailed account of these constructions, we refer to [12]. In particular, the splitting from the double semidualises to a bicross-sum $\mathfrak{g}^{\mathrm{cop}} \blacktriangleright\!\!\blacktriangleleft \mathfrak{g} \cong \mathfrak{g}^{\mathrm{cop}} \oplus \mathfrak{g}$ as Lie bialgebras.

Putting these facts together gives an isomorphism of quasitriangular Lie bialgebras $\theta^c : D(\mathfrak{g}) \to (\mathfrak{g}^{\mathrm{cop}} \blacktriangleright\!\!\blacktriangleleft \mathfrak{g})_{\chi_B^c}$, with

$$\theta^c(\xi) = \xi, \quad \theta^c(\phi) = -2\tilde{r}_+(\phi) + (\mathrm{id} \otimes \phi) r, \quad \chi_B^c = r_{23} - r_{41}, \tag{4.7}$$

for all $\xi \in \mathfrak{g}$ and $\phi \in \mathfrak{g}^*$, where $r_+ = (r + r_{21})/2$ is viewed as a map $\mathfrak{g}^* \to \mathfrak{g}$ and tilde indicates that the result is viewed in the $\mathfrak{g}^{\mathrm{cop}}$ copy. As a check, one has

$$(\theta^c \otimes \theta^c)(f^a \otimes e_a) = r_{24} - r_{14} - r_{41} = r_{B_D} + \chi_B^c, \quad r_{B_D} = r_{24} - r_{23} - r_{14}$$

so the canonical r-matrix for the double when mapped over under the isomorphism is the twist of r_{B_D}. There is also

$$r_{B_L} = r_{31} - r_{23} + r_{13} - r_{14} + r_{24}$$

which twists to the image under $\theta^c \otimes \theta^c$ of the other r-matrix r_L on $D(\mathfrak{g})$. This is the Lie bialgebra version of the general theory in Section 2. We now verify everything on our examples as a check.

4.2 Infinitesimal limit of results in the limit $q \to 1$

In the infinitesimal Lie bialgebra limit, the quantum double $D(U(su_2))$ becomes the Lie bialgebra double $D(su_2) = su_2 \bowtie su_2^{*\mathrm{op}} = su_2 \ltimes \mathbb{R}^3$. Here, the su_2 parts has its standard Lie bracket and $su_2^{*\mathrm{op}} = \mathbb{R}^3$ has a commutative algebra. The relations are given by (3.33)

$$[J_a, J_b] = \imath \epsilon_{abc} J_c, \quad [\mathcal{P}_a, J_b] = \imath \epsilon_{abc} \mathcal{P}_c, \quad [\mathcal{P}_a, \mathcal{P}_b] = 0. \tag{4.8}$$

The bicrossproduct quantum group $C[SU_2^*]^{\rm cop}\blacktriangleright\!\!\blacktriangleleft U(su_2)$ described in Section 3.6 in terms of the basis (3.40) can be viewed as a deformation of $U(su_2^{\rm cop}\blacktriangleright\!\!\blacktriangleleft su_2)$, where λ is the deformation parameter. In the semiclassical limit, $C[SU_2^*]^{\rm cop}\blacktriangleright\!\!\blacktriangleleft U(su_2)$ becomes the bicross sum $su_2^{\rm cop}\blacktriangleright\!\!\blacktriangleleft su_2$, where su_2 has its standard Lie bracket and $su_2^{\rm cop}=\mathbb{R}^3$ has a commutative Lie bracket. From (3.41) the relations are given by

$$\begin{aligned}
&[p_a,p_b] \;=0, \quad [M,N_1]=N_2, \quad [M,N_2]=-N_1, \quad [N_1,N_2]=-M,\\
&[M,p_0] \;=0 \quad [M,p_i]=\imath\epsilon_{ij}p_j, \quad [N_i,p_0,]=-\imath\epsilon_{ij}p_j, \quad [N_i,p_j]=\imath\epsilon_{ij}p_0 \quad i,j=1,2,
\end{aligned} \tag{4.9}$$

In terms of the basis (3.40), the R-matrix becomes

$$\mathcal{R}_{B_0}=e^{\lambda P_- M\otimes \lambda P_+ e^{-\lambda p_0}}:e^{-e^{\lambda p_0}X_+\otimes \lambda P_+ e^{-\lambda p_0}}:e^{-M\otimes(e^{\lambda p_0}-1)}e^{-\lambda P_-\otimes X_-}e^{-(e^{\lambda p_0}-1)\otimes M}, \tag{4.10}$$

where we have kept the $X_\pm$ for simplicity. Then the semiclassical limit of the bicrossproduct R-matrix (4.10) gives a classical r-matrix for the bicross sum as

$$\begin{aligned}
r_{B_0} &= -M\otimes p_0-p_0\otimes M-P_-\otimes X_- - X_+\otimes P_+\\
&= -M\otimes p_0-p_0\otimes M-(p_2\otimes N_2+p_1\otimes N_1+N_2\otimes p_2+N_1\otimes p_1)\\
&\quad -\imath(p_2\otimes N_1-p_1\otimes N_2+N_2\otimes p_1-N_1\otimes p_2)
\end{aligned} \tag{4.11}$$

It is interesting to see that if we set $M=J_0$, $J_i=N_i$ and $P_a=-p_a$, to match standard notation, we get an r-matrix for the bicross sum $su_2^{\rm cop}\blacktriangleright\!\!\blacktriangleleft su_2$ as

$$\begin{aligned}
r_{B_0} &= P_a\otimes J_a+J_a\otimes P_a-\imath(P_1\wedge J_2-P_2\wedge J_1)\\
&= P_a\otimes J_a+J_a\otimes P_a+\imath m_a\epsilon^{abc}P_b\wedge J_c, \quad \mathbf{m}^2=1,
\end{aligned} \tag{4.12}$$

where $\mathbf{m}$ is a unit time-like vector.

We observe that the symmetric part of the r-matrix (4.12) is equal to the Casimir associated to the invariant, non-degenerate symmetric bilinear form used in the Chern-Simons action [24, 25] and therefore suitable for constructing the Poisson structure on the classical phase space via the Fock-Rosly construction [26]. This shows that the bicrossproduct with r-matrix depending on a time-like deformation vector and with complex antisymmetric part is compatible with 3d gravity via the Fock-Rosly compatibility condition. This is different from the family of classical bicross sum r-matrices associated to 3d gravity with vanishing cosmological constant obtained in [27, 19]. In the later, the r-matrices are real and depend on space-like deformations vectors. See also [20] where a complete classification of all r-matrices compatible to 3d gravity with vanishing cosmological is constructed via semidualisation of Lie bialgebras which are double cross sums.

Now, rewriting the twist in Corollary 3.1 in terms of the basis (3.40), we get

$$\chi_{B_0}=e^{-X_+\otimes\lambda P_+e^{-\lambda p_0}}\,e^{-M\otimes(e^{\lambda p_0}-1)}, \tag{4.13}$$

and the semiclassical limit for the twist gives the Lie bialgebra twist by (4.6) as

$$\begin{aligned}
\chi^c_{B_0} &= X_+\otimes P_+ + M\otimes p_0 - P_+\otimes X_+ - p_0\otimes M\\
&= M\otimes p_0-p_0\otimes M+N_2\otimes p_1+N_1\otimes p_1-p_2\otimes N_2-p_1\otimes N_1\\
&\quad +\imath(N_2\otimes p_1-N_1\otimes p_2-p_2\otimes N_1-p_1\otimes N_2)
\end{aligned} \tag{4.14}$$

The above considerations leads to the semiclassical limit of the results in Corollary 3.1 that the double Lie bialgebra $D(su_2)$ is a Lie bialgebra twisting of bicross sum Lie bialgebra $su_2^{\rm cop}{\blacktriangleright\!\!\blacktriangleleft} su_2$. The isomorphism (3.48) becomes $\theta^c : D(su_2) \to su_2^{\rm cop}{\blacktriangleright\!\!\blacktriangleleft} su_2$, where

$$\theta^c(J_0) = M,\ \ \theta^c(J_i) = N_i, \quad \theta^c(\mathcal{P}_a) = -2p_a,\ \ i = 1, 2,\ \ a = 0, 1, 2. \tag{4.15}$$

Twisting the bicrossproduct r-matrix by $\chi^c_{B_0}$ and using the isomorphism (4.15) gives the r-matrix for a classical double $D(su_2)$ as

$$r_{D_0} = r_{B_0} + \chi^c_{B_0} \ \ = \ \ -2p_0 \otimes M - P_- \otimes X_- - P_+ \otimes X_+ = \mathcal{P}_a \otimes J_a \tag{4.16}$$

on using the identification θ^c for the last step, in agreement with the general theory in Section 4.1.

4.3 Infinitesimal limit of the q-deformed results

If we take the semiclassical limit of Section 3.5 without sending $q \to 1$, we have on the one side the Lie bialgebra double $D(su_2) = su_2{\bowtie\!\!\triangleleft} su_2^{*op}$ where $su_2^{*op} = an_2$ is the Lie algebra of the Lie group AN_2 of 2×2 matrices of the form

$$\begin{pmatrix} e^{\phi} & \xi + i\eta \\ 0 & e^{-\phi} \end{pmatrix}, \quad \phi, \xi, \eta \in \mathbb{R},$$

and the notation refers the abelian and the nilpotent parts of this group. The vector space splitting expressed in the Lie double cross sum is the lie version of the Iwasawa factorisation of $SL(2, \mathbb{C}) = SU_2.AN_2$. This result from [28] is the reason that the quantum double $D(U_q(su_2))$ can be regarded as the q-Lorentz quantum group. Our result is that this $D(su_2)$ is a twist of the double cross sum $su_2{\blacktriangleright\!\!\blacktriangleleft} su_2$ as quasi-triangular Lie bialgebras. The latter is known to be isomorphic to $su_2^{\rm cop} \oplus su_2$ recovering the (complexified) Lorentz Lie bialgebra as twist of a direct sum. This fact is essentially known[12] but it is a nice check of our formulae to check this from the semiclassical limit of Section 3.5.

To this end, the quantum double is given by (3.1), (3.14), with cross relations in (3.16) written conveniently as

$$\begin{aligned} &[q^{\frac{H}{2}}, q^z] = 0, \quad q^{\frac{H}{2}} x_\pm = q^{\pm 1} x_\pm q^{\frac{H}{2}}, \quad [X_\pm, x_\pm] = 0, \\ &[X_-, x_+] = q^{-\frac{1}{2}}\mu^{-1}\left(q^{\frac{H}{2}} q^{-z}(1 + q\mu^2 x_+ x_-) - q^{-\frac{H}{2}} q^z\right), \ \ X_- q^z = q^{-1} q^z X_- + q^{\frac{1}{2}}\mu x_- q^{\frac{H}{2}}, \\ &[X_+, x_-] = q^{-\frac{3}{2}}\mu^{-1}\left(q^{\frac{H}{2}} q^{-z}(1 + q\mu^2 x_+ x_-) - q^{-\frac{H}{2}} q^z\right), \quad q^z X_+ = q X_+ q^z + q^{\frac{3}{2}}\mu q^{-\frac{H}{2}} x_+. \end{aligned} \tag{4.17}$$

The coproduct, counit and antipode are given by (3.2) for $U_q(su_2)$ and the opposite of the coproduct, the counit and inverse of the antipode in (3.15) for $U_q(su_2^*)^{\rm op}$. The bicrossproduct $U_q(su_2)^{\rm cop}{\blacktriangleright\!\!\blacktriangleleft} U_q(su_2)$ is from (3.18) and (3.19). The twisting (3.24) takes the form

$$\chi'_B = e_{q^2}^{-(1-q^{-2})\, 1 \otimes q^{\frac{H}{2}} X_+ \otimes q^{-\frac{\tilde{H}}{2}} \tilde{X}_- \otimes 1} q^{-\frac{1}{2} 1 \otimes H \otimes \tilde{H} \otimes 1} = e_{q^2}^{-(1-q^{-2})\, q^{\frac{H}{2}} X_+ \otimes q^{-\frac{\tilde{H}}{2}} \tilde{X}_-} q^{-\frac{1}{2} H \otimes \tilde{H}}, \tag{4.18}$$

where $\tilde{H}, \tilde{X}_\pm$ are the generators for $U_q(su_2)^{\rm cop}$.

For the semiclassical or infinitesimal regime, in (3.26) and (3.27), we write $q = e^{\frac{t}{2}}$ and use (4.5) to get classical r-matrices for the bicross sum as

$$r_{B_D} = \frac{1}{4}(H\otimes H - \tilde{H}\otimes H - H\otimes\tilde{H}) + X_+\otimes X_- - \tilde{X}_+\otimes X_- - X_+\otimes\tilde{X}_-, \qquad (4.19)$$

and

$$\begin{aligned} r_{B_L} &= \frac{1}{4}(2\tilde{H}\otimes\tilde{H} - \tilde{H}\otimes H - H\otimes\tilde{H} + H\otimes H) \\ &+ \tilde{X}_-\otimes\tilde{X}_+ + \tilde{X}_+\otimes\tilde{X}_- + X_+\otimes X_- - X_+\otimes\tilde{X}_- - \tilde{X}_+\otimes X_-. \end{aligned} \qquad (4.20)$$

It is easy to check that r_{B_D} and r_{B_L} satisfy the CYBE (4.2). In this semiclassical regime, the twisting (4.18) becomes

$$\chi_B^c = \frac{1}{4}(H\otimes\tilde{H} - \tilde{H}\otimes H) + X_+\otimes\tilde{X}_- - \tilde{X}_-\otimes X_+. \qquad (4.21)$$

Thus the two classical r-matrices for the corresponding classical double are

$$r_D = \chi_B^c + r_{B_D} = \frac{1}{4}(H\otimes H - 2\tilde{H}\otimes H) + X_+\otimes X_- - \tilde{X}_+\otimes X_- - \tilde{X}_-\otimes X_+, \qquad (4.22)$$

$$\begin{aligned} r_L = \chi_B^c + r_{B_L} &= \frac{1}{4}(H\otimes H - 2\tilde{H}\otimes H - 2\tilde{H}\otimes\tilde{H}) \\ &+ \tilde{X}_-\otimes\tilde{X}_+ + \tilde{X}_+\otimes\tilde{X}_- + X_+\otimes X_- - \tilde{X}_+\otimes X_- - \tilde{X}_-\otimes X_+. \end{aligned} \qquad (4.23)$$

Now for the Lie algebra su_2 with basis $\{H, X_\pm\}$, the standard Drinfeld-Sklyanin r-matrix is $r = \frac{1}{4}H\otimes H + X_+\otimes X_-$ so that $r_+ = \frac{1}{4}H\otimes H + \frac{1}{2}(X_+\otimes X_- + X_-\otimes X_+)$. We let $su_2^* = \text{span}\{\phi, \psi_\pm\}$ be the dual Lie algebra with relations

$$[\psi_\pm, \phi] = \frac{1}{2}\psi_\pm, \quad [\psi_+, \psi_-] = 0,$$

and dual pairing

$$\langle\phi, H\rangle = 1, \quad \langle\psi_+, X_+\rangle = 1, \quad \langle\psi_-, X_-\rangle = 1.$$

Then from (4.7) we get

$$\theta^c(H) = H,\ \theta^c(X_\pm) = X_\pm,\ \theta^c(\phi) = -\frac{\tilde{H}}{2} + \frac{H}{4},\ \theta^c(\psi_+) = -\tilde{X}_-,\ \theta^c(\psi_-) = -\tilde{X}_+ + X_+.$$

which one can check is in agreement with semiclassicalising (3.23). Then

$$\begin{aligned} &(\theta^c\otimes\theta^c)(\phi\otimes H + \psi_-\otimes X_- + \psi_+\otimes X_+) \\ &\qquad = (\frac{H}{4} - \frac{\tilde{H}}{2})\otimes H + (X_+ - \tilde{X}_+)\otimes X_- - \tilde{X}_-\otimes X_+ = r_D \end{aligned}$$

in agreement with the general theory in Section 4.1.

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